Probability Theory III
B. Stat. IInd Year Semester 1
Indian Statistical Institute

Semestral Examination

Date: November 13, 2025 Time: 3 hours

This exam is of 60 marks.

Answer as many as you can.

Maximum you can score is 50.

1. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers with $a_n \geq 1$ for all $n \geq 1$. State whether the following statements are true or false with appropriate reasons:
   (a) $\limsup [0, a_n) = [0, \limsup a_n)$.
   (b) $\limsup [0, a_n] = [0, \limsup a_n]$. [3+3=6]

2. For any $0 < \alpha < 1$, show that
$$\phi_\alpha(t) := \frac{1}{1 + |t|^\alpha}, \quad t \in \mathbb{R},$$
is a characteristic function. [7]

3. Let $\mathcal{S}$ be the set of all distribution functions corresponding to non-negative integer valued random variables. Show that $\mathcal{S}$ is a closed set under weak convergence. [7]

4. Let $\mathcal{T}$ be the set of all real-valued random variables defined on a given probability space (where two random variables are considered equivalent, if they are equal with probability one). Define
$$d(X, Y) := \inf \{\epsilon \geq 0 : \mathbb{P}(|X - Y| > \epsilon) \leq \epsilon\}, \quad X, Y \in \mathcal{T}.$$
   (a) Show that the infimum in the definition of d is attained.
   (b) Show that d is a metric on $\mathcal{T}$.
   (c) Let $X \in \mathcal{T}$ and $\{X_n\}_{n\geq 1}$ be a sequence with $X_n \in \mathcal{T}$ for all $n \geq 1$. Show that X_n converges in probability to X if and only if $d(X_n, X) \to 0$.
   (d) Show that $\mathcal{T}$ is complete with respect to the metric d. [3+5+4+4=16]

5. Let $\{X_n : n \geq 1\}$ be an i.i.d. sequence of standard Cauchy random variables. For $n \geq 1$, define $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Show that $\overline{X}_n$ converges weakly but does not converge in probability. [3+5=8]

6. Let $\{Y_n : n \geq 1\}$ be an i.i.d. sequence of Poisson random variables with mean 3. Consider a sequence of independent random variables $\{X_n : n \geq 1\}$ (defined on the same probability space as the sequence $\{Y_n\}$), such that
$$\mathbb{P}(X_n = Y_n) = 1 - \frac{1}{n^3}, \quad \mathbb{P}(X_n = -n^3) = \frac{1}{n^3}, \quad n \geq 1.$$
Show that $\frac{1}{n}\sum_{i=1}^n X_i \to 3$ almost surely. [8]

7. Using Central Limit Theorem or otherwise, show that the limit
$$\lim_{n\to\infty} \frac{1}{n!} \int_n^\infty x^n e^{-x}\, dx$$
exists and find its value. [8]