(No. of Printed Pages: 2)

Indian Statistical Institute
B. STAT. Second Year (2024-25)
Mid Semester Examination 2025

Paper: **Differential Equations** Date: **28/03/2025**

Time : **2 Hrs.** Full Marks : **30**

Note: Symbols used have their usual meaning. *The figures in the margin indicate full marks.*

Answer any FOUR from the following questions. $4 \times 7\frac{1}{2} = 30$

1. (a) Let $G(x, y)$ be an integrating factor of the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$. If $F(x, y) = c$ represents the general solution, then prove or disprove that $G(x, y)\chi(F)$ is an integrating factor of the above differential equation. Here $\chi(F)$ is some integrable function.

   (b) Solve the differential equation

   $$xyp^2 + (x + y)p + 1 = 0$$

   and find a particular solution that is valid for all x and passes through the point $(-1/2, 2)$.

   $[4 + 3\frac{1}{2}]$

2. (a) Consider the initial value problem

   $$\frac{dy}{dx} = \frac{10}{3}xy^{2/5}, \quad y(0) = -1.$$

   i. Determine the range of x for which the solution to the differential equation exists.

   ii. Find a solution and determine the largest open interval containing $x_0 = 0$ on which the solution is unique.

   (b) It has been observed through empirical studies that during an epidemic outbreak, the infection rate is directly proportional to the product of the number of infected and susceptible persons. Suppose a single infected person triggers the outbreak in a town with a total population of P_0, and half of this population is initially susceptible to the disease. Find the number of infected persons at any time t.

   $[(1 + 3) + 3\frac{1}{2}]$

3. (a) Let ϕ_1 and ϕ_2 be two linearly independent solutions of the second order homogeneous linear differential equation

   $$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

   on $a \le x \le b$. If $\phi_1''(x_0) = \phi_2''(x_0) = 0$ at some $x_0 \in (a, b)$ then prove or disprove that $a_1(x_0) = a_2(x_0) = 0$.

   (b) Find the general solution of the following differential equation

   $$\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1 + x)^3.$$

   $[3\frac{1}{2} + 4]$

P.T.O.

4. (a) Let $\Phi_k(t) = \begin{pmatrix} \phi_{1k}(t) \\ \phi_{2k}(t) \end{pmatrix}$, $k = 1, 2$ be two solutions of the homogeneous linear system of differential equation

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{on} \quad a \le t \le b, \text{ and } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a real 2×2 constant matrix. Show that the Wronskian $W(\Phi_1, \Phi_2)(t) = 0$ for all $t \in [a, b]$ or $W(\Phi_1, \Phi_2)(t)$ in nowhere zero on $[a, b]$.

(b) Find the general solution of the following system of homogeneous linear differential equations

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$[4 + 3\frac{1}{2}]$

5. Let an undamped forced oscillation of a mass m, suspended from a vertical spring, is governed by the differential equation

$$m\ddot{x} + kx = f_0 \cos \omega t,$$

where k is a positive constants and $(f_0 \cos \omega t)$ is the periodic forcing term, with f_0 and w being constants. Find the solution corresponding to the initial conditions $x(0) = 0$ and show that the solution approaches the value $\frac{f_0}{2m\omega_0} t \sin(\omega_0 t)$ as $\omega \to \omega_0$, where $\omega_0 = \sqrt{\frac{k}{m}}$. $[7\frac{1}{2}]$

6. (a) Find the indicial polynomial $q(r)$ for the regular singular point $x = 0$ of the second order differential equation

$$L(y) = x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0.$$

(b) Show that the equation $L(y) = x^k$ has a solution ψ of the form $\psi(x) = cx^k$ if $q(k) \neq 0$. Compute c.

(c) Suppose k is a root of the indicial polynomial (q) of multiplicity one. Show that there is a solution ψ of $L(y) = x^k$ of the form

$$\psi(x) = cx^k \log x.$$

Compute c.

$[2\frac{1}{2} + 2 + 3]$