CONTRIBUTION TO THE LINEAR COMPLEMENTARITY PROBLEM AND COMPLETELY MIXED GAMES

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To

My co-author

late Prof. T. Parthasarathy

and

my late grandfather

Master Balbir Singh

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ABSTRACT

This dissertation focuses on the linear complementarity problem (LCP), two-person zero-sum matrix games, and Q-tensors. A matrix game is considered completely mixed if all the optimal pairs of strategies in the game are completely mixed. In this thesis, we provide new characterizations of Kaplansky's results (1945 and 1995) on completely mixed games.

Pang proved that within the class of semimonotone matrices, R_0-matrices are Q-matrices and conjectured that the converse is also true. Gowda proved that the conjecture is true for symmetric matrices. We prove that semimonotone Q-matrices are R_0-matrices up to order 3 and provide a counterexample to show that this statement does not hold for matrices of order 4 and higher. We also provide an application of this result using completely mixed games. Stone proposed that fully semimonotone Q_0-matrices are P_0-matrices. In this thesis, we establish that this conjecture holds true for matrices with certain sign patterns. Since fully semimonotone matrices are semimonotone and Z-matrices are Q_0, we demonstrate that semimonotone Z-matrices are P_0. Gowda proved that a Z-matrix with value zero is completely mixed if and only if it is irreducible. We provide new equivalent conditions for this statement. Additionally, we present results on completely mixed games, exploring their connection to various classes of matrices. We also extend some results of Q-matrices to Q-tensors.

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GLOSSARY OF NOTATION

Symbols	**Description**
Spaces	
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^n$	real n-dimensional Euclidean space
$\mathbb{R}^n_+$	the nonnegative orthant of $\mathbb{R}^n$
$\mathbb{R}^n_{++}$	the positive orthant of $\mathbb{R}^n$
$\mathbb{R}^{m\times n}$	the space of all $m \times n$ real matrices
Sets	
$\bar{n}$ *or* $[n]$	the set $\{1, 2, \ldots, n\}$
n^*	all nonempty subsets of $\bar{n}$
α, β, γ	subsets of $\bar{n}$
$\lvert\alpha\rvert$	cardinality of the set α
$\bar{\alpha}$ *or* α^c	Complement of the set α in $\bar{n}$
$\alpha \setminus \beta$	the set $\alpha \cap \beta^c$
$\alpha \triangle \beta$	the set $(\alpha \cup \beta) \setminus (\alpha \cap \beta)$
Vectors	
z^t	transpose of z
$x^t y$	the standard inner product between x and y
$x \geq y$	$x_i \geq y_i$ for all $i \in \bar{n}$
$x > y$	$x_i > y_i$ for all $i \in \bar{n}$

supp(z)	$\{i \in \bar{n} : z_i \neq 0\}$
probability vector	$x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0$, for all $i \in \bar{n}$
e	$(1, 1, \ldots, 1)^t$

Matrices

$A = (a_{ij})$	a matrix with a_{ij} as its i^{th}-row, j^{th}-column entry
$A \geq B$	$a_{ij} \geq b_{ij}$ for all i, j
$A > B$	$a_{ij} > b_{ij}$ for all i, j
$det(A)$ *or* $\|A\|$	determinant of a square matrix A
$adj\ A$	adjoint of a square matrix A
I	the identity matrix
$A_{\alpha\beta}$	the submatrix of A obtained by dropping rows and columns of A corresponding $\bar{\alpha}$ and $\bar{\beta}$ respectively
A_{ij}	the submatrix of A obtained by dropping i^{th}-row and j^{th}-column
$A_{\alpha.}$	stands for submatrix $A_{\alpha\bar{n}}$ of matrix A
$A_{.\beta}$	stands for submatrix $A_{\bar{n}\beta}$ of matrix A
$A_{i.}$	i^{th}-row of matrix A
$A_{.j}$	j^{th} column of matrix A
diag$(a_1, a_2, ..., a_3)$	diagonal matrix with $a_{ii} = a_i$
$val(A)$	value of the game associated with matrix A

Sign Symbols

$+$	positive real number
$-$	negative real number
$\oplus$	nonnegative real number
$\ominus$	nonpositive real number
$*$	real number

LCP Notations

$LCP(q, A)$	Linear complementarity problem associated to vector q and matrix A
$F(q, A)$	set of all feasible solutions to $LCP(q, A)$
$S(q, A)$	set of all solutions to $LCP(q, A)$
$pos(A)$	cone generated by columns of A
$C_A(\alpha)$	complementary submatrix $\alpha \subseteq \bar{n}$

Matrix Classes

C_0	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : x^tAx \geq 0 \; \forall x \in \mathbb{R}^n_+\}$,
C_0^+	$\bigcup_n\{A \in \mathbb{R}^{n\times n} \cap C_0 : [x^tAx = 0 \; x \in \mathbb{R}^n_+] \Rightarrow [(A + A^t)x = 0]\}$,
E	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall 0 \neq x \in \mathbb{R}^n_+, \; \exists k \in \bar{n} \ni x_k > 0$ and $(Ax)_k > 0\}$,
E_0	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall 0 \neq x \in \mathbb{R}^n_+, \; \exists k \in \bar{n} \ni x_k > 0$ and $(Ax)_k \geq 0\}$,
E_0^f	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall \alpha \subseteq \bar{n}, \; det(A_{\alpha\alpha}) \neq 0 \Rightarrow ppt(A, \alpha) \in E_0\}$,
P	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall \alpha \in n^*, det \; A_{\alpha\alpha} > 0\}$,
P_0	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall \alpha \in n^*, det \; A_{\alpha\alpha} \geq 0\}$,
N	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall \alpha \in n^*, det \; A_{\alpha\alpha} < 0\}$,
N_0	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall \alpha \in n^*, det \; A_{\alpha\alpha} \leq 0\}$,
Q	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : S(q, A) \neq \phi, \forall q \in \mathbb{R}^n\}$,
Q_0	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : \forall q \in \mathbb{R}^n, F(q, A) \neq \phi \Rightarrow S(q, A) \neq \phi\}$,
R_0	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : LCP(q, A)$ has a unique solution for $q = 0\}$,
R	$\bigcup_n\{A \in R_0 : LCP(d, A)$ has a unique solution for some $d > 0\}$,
S	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : val(A) > 0\}$,
S_0	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : val(A) \geq 0\}$,
Z	$\bigcup_n\{A \in \mathbb{R}^{n\times n} : a_{ij} \leq 0 \; \forall i \neq j\}$,

CHAPTER 1

Introduction

The Linear Complementarity Problem (LCP) was first introduced in the early 1940s by mathematicians and economists studying equilibrium properties in economic models. Over time, the problem has undergone several name changes, including the "composite problem", the "fundamental problem", and the "complementary pivot problem". The term "linear complementarity problem" was introduced by Cottle in 1965. Today, LCP remains an active area of research, with new solution methods and applications continually being developed. Its significance across various fields ensures that it will continue to be an important topic of study for years to come.

Matrix classes play an important role in the study of the Linear Complementarity Problem because the properties of matrices can affect the solvability and solution methods of the corresponding LCP. In particular, there are several important matrix classes that have been studied extensively in the literature, such as Z-matrices, P-matrices, Q-matrices, R-matrices, etc.

In this thesis, we obtain new characterization of Kaplansky's result on completely mixed games and provide a connection to Q-matrices. In the following sections, we introduce the linear complementarity problem, study some properties of matrix classes and their principal pivot transforms. We also introduce two-person zero-sum matrix games.

1.1 The Linear Complementarity Problem

The linear complementarity problem associated with a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, denoted by $LCP(q, A)$, is to find a vector $x \in \mathbb{R}^n$ that satisfies the following conditions:

$$x \geq 0, \tag{1.1}$$

$$Ax + q = w \geq 0, \tag{1.2}$$

$$x^t w = 0. \tag{1.3}$$

A vector x that satisfies conditions (1.1) and (1.2) is referred to as a feasible solution, and the set of feasible solutions is given by

$$F(q, A) := \{x \in \mathbb{R}^n : x \geq 0 \ \&\ Ax + q \geq 0\}.$$

In the above problem, (x, w) is called a complementary pair. A vector that satisfies (1.1), (1.2), and (1.3) is called a complementary solution, or simply a solution to $LCP(q, A)$. The set of all solutions is denoted by

$$S(q, A) := \{x \in F(q, A) : x^t(Ax + q) = 0\}.$$

The linear complementarity problem has been applied in many areas, such as determining equilibrium prices in economics, solving optimization problems in engineering and computer science, and more. LCP unifies several optimization problems; in particular, linear programming and quadratic programming problems can be formulated as an LCP [25, 26]. When Lemke and Howson [24] gave an algorithm for finding the equilibrium strategies for a bimatrix game, it brought a new dimension to LCP. For more details, refer to [6, 35].

The matrix associated with an LCP can be classified into various classes based on its properties and characteristics, including symmetric matrices, positive definite matrices,

M-matrices, P-matrices, Q-matrices etc. In the next section, we will discuss several matrix classes.

1.2 Matrix Classes and Their Inheritance Properties

In 1962, Fiedler and Ptak [10] introduced P-matrices, which have since found numerous applications, particularly in optimization [9]. P-matrices are matrices that have positive principal minors. The class of P-matrices is another important matrix class for solving LCPs, since they have well-defined inverses and can be solved using iterative methods [35].

Definition 1.2.1. A matrix $A \in \mathbb{R}^{n\times n}$ is called a P_0-matrix if every principal minor is nonnegative, and a P-matrix if every principal minor is positive. The classes of P_0-matrices and P-matrices are denoted by P_0 and P, respectively. A matrix is called an almost P-matrix if all its proper principal minors are positive, but its determinant is negative.

Murty [34] established that the linear complementarity problem $LCP(q, A)$ associated with a P-matrix A has a unique solution.

Definition 1.2.2. A matrix $A \in \mathbb{R}^{n\times n}$ is called a positive semi-definite matrix if, for all $x \in \mathbb{R}^n, \;\; x^tAx \geq 0.$

Remark 1.2.3. [6] It is known that the class of positive semi-definite matrices is a subset of the class of P_0-matrices. A matrix A is called a skew symmetric matrix if $A = -A^t$. A skew symmetric matrix is a positive semi-definite matrix.

Definition 1.2.4. A matrix A is said to be a S-matrix if there exists a vector $x > 0$ such that $Ax > 0$.

Definition 1.2.5. A matrix is said to be a Z-matrix if all of its off diagonal entries are nonpositive.

M-matrices were first studied by Minkowski [29] and later by Ostrowski [36]. An M-matrix is a matrix that has non-positive off-diagonal elements and satisfies certain additional conditions. The class of M-matrices forms a subset of the class of Z-matrices. M-matrices have several important properties, including the fact that non-singular M-matrices are inverse positive. Furthermore, if a Z-matrix is inverse positive, then it is a non-singular M-matrix. For more details, one may refer to [28].

1.2.1 Principal Pivot Transforms

The principal pivot transforms (PPTs) were first introduced by A.W. Tucker [57]. Principal pivot transforms play an important role in the study of linear complementarity problems.

Consider any matrix $A \in \mathbb{R}^{n\times n}$ and an index set $\alpha \subseteq \{1, 2, \ldots n\}$, such that $A_{\alpha\alpha}^{-1}$ exists. Then the principal pivot transform of A with respect to α, denoted by $ppt(A, \alpha)$, is obtained by modifying A as follows:

$$A_{\alpha\alpha} \to A_{\alpha\alpha}^{-1}, \qquad A_{\alpha\bar{\alpha}} \to -A_{\alpha\alpha}^{-1}A_{\alpha\bar{\alpha}},$$
$$A_{\bar{\alpha}\alpha} \to A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1}, \qquad A_{\bar{\alpha}\bar{\alpha}} \to A/A_{\alpha\alpha}$$

where $A/A_{\alpha\alpha} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1}A_{\alpha\bar{\alpha}}$ is called the Schur complement of $A_{\alpha\alpha}$. Whenever $A_{\alpha\alpha}$ is invertible for any α, the corresponding PPT exists, and we call those legitimate PPTs.

Remark 1.2.6. Let A be a matrix such that $A_{\alpha\alpha}^{-1}$ exists for some α. Suppose that B is obtained by applying the principal pivot transform to A with respect to α, i.e., $B = ppt(A, \alpha)$. Then applying the same principal pivot transform to B with respect to α recovers the original matrix A, i.e., $A = ppt(B, \alpha)$.

Definition 1.2.7. A matrix $A \in \mathbb{R}^{n\times n}$ is called a (strictly) semimonotone matrix, also called an $E_0(E)$-matrix, if for every vector $x \geq 0$ and $x \neq 0$, there exists an index k such that $x_k > 0$ and $(Ax)_k \geq 0(> 0$ in the strict case). The corresponding class of matrices is denoted by $E_0(E)$. A matrix A is called a fully semimonotone matrix, denoted by E_0^f, if A and all its legitimate principal pivotal transforms belong to E_0.

Definition 1.2.8. A matrix $A \in \mathbb{R}^{n\times n}$ is called a (strictly) copositive matrix, if for all (nonzero) $x \in \mathbb{R}^n_+,\ x^tAx(> 0) \geq 0$. We denote the class of (strictly) copositive matrices by $(C)C_0$. A matrix A is called a fully copositive matrix, denoted by C_0^f, if A and all its legitimate principal pivotal transforms belong to C_0.

Remark 1.2.9. The class of semimonotone matrices includes both the class of P_0-matrices and the class of copositive matrices.

Definition 1.2.10. A matrix $A \in \mathbb{R}^{n\times n}$ is called a copositive-plus matrix, whenever A is copositive and $[x^tAx = 0, x \geq 0]$ implies $[(A + A^t)x = 0]$. We denote the class of copositive-plus matrices by C_0^+.

Remark 1.2.11. For copositive plus matrices Lemke's algorithm always gives a solution if it exists.

1.2.2 Matrix Classes Related to the Linear Complementarity Problem

Next we present some classes of matrices relating to $LCP(q, A)$ and their properties.

Definition 1.2.12. A matrix $A \in \mathbb{R}^{n\times n}$ is called a Q_0-matrix if, for any vector q, whenever $LCP(q, A)$ has a feasible solution, then $LCP(q, A)$ also has a solution. A matrix A is called a Q-matrix if $LCP(q, A)$ has a solution for all $q \in \mathbb{R}^n$.

Remark 1.2.13. The class consisting of Q-matrices (Q_0-matrices respectively,) is denoted by Q (Q_0 respectively). It is well known that $Q \subseteq Q_0$.

Remark 1.2.14. For any matrix $A \in Q$, $LCP(q, A)$ has at least one solution for every vector q. However, if $A \in P$, then $LCP(q, A)$ has a unique solution for every vector q. Consequently, it follows that $P \subseteq Q$.

The class of regular matrices was introduced by Karamardian [23]. Garcia [12] considered the class of R_0-matrices, which was initially denoted by $E^*(0)$. This class guarantees the uniqueness of solution to the LCP associated to a zero vector.

Definition 1.2.15. A matrix $A \in \mathbb{R}^{n\times n}$ is called a regular matrix if there exists a positive vector $d \in \mathbb{R}^n_{++}$ such that $LCP(\lambda d, A)$ has a unique solution, namely the vector zero, for all

$\lambda \geq 0$. The class of regular matrices is denoted by R. If $LCP(0, A)$ has a unique solution, namely the vector zero, then A is called an R_0-matrix.

Definition 1.2.16. Let $P \in \mathbb{R}^{n \times n}$ be a permutation matrix, and let $A \in \mathbb{R}^{n \times n}$. Then PAP^t is called a principal rearrangement of A.

Remark 1.2.17. All the principal submatrices of the matrices belonging to the classes P, P_0, Z, E_0, skew symmetric, etc. remain in the same class.

If a $Q(Q_0)$-matrix has all its principal submatrices in the same class, then the matrix is called a completely $Q(Q_0)$-matrix. These classes are denoted by $\overline{Q}(\overline{Q_0})$ respectively.

1.3 Cones

Definition 1.3.1. A nonempty set $S \in \mathbb{R}^n$ is called a cone if, for any $x \in S$ and any $\alpha \geq 0$, we have $\alpha x \in S$. The origin is an element of every cone.

A cone is called a convex cone if it is also convex.

Definition 1.3.2. Let A be a matrix in $\mathbb{R}^{m \times n}$. The set $A(\mathbb{R}^n_+) = \{q \in \mathbb{R}^m : q = Ax$ for some $x \in \mathbb{R}^n_+\}$ is called the convex cone generated by the columns of A. This cone is denoted by $pos\ A$.

The interpretation of the LCP in terms of complementary cones appeared in Samuelson, Thrall, and Wesler [49]. Further research on complementary cones was carried out by Murty [34], who studied them in greater depth and obtained some remarkable results concerning the number and parity of solutions to the linear complementarity problem.

Definition 1.3.3. Let $A \in \mathbb{R}^{n \times n}$ and α be a subset of $\bar{n}$. We define a matrix $C_A(\alpha)$ as follows:

$$C_A(\alpha)_{.j} = \begin{cases} -A_{.j}, & \text{if } j \in \alpha; \\ I_{.j}, & \text{if } j \in \bar{\alpha}. \end{cases}$$

The matrix $C_A(\alpha)$ is called a complementary submatrix of $[I : -A]$ with respect to α. Further, if $det(A_{\alpha\alpha}) \neq 0$, then $C_A(\alpha)$ is called a complementary basis.

Definition 1.3.4. Let $A \in \mathbb{R}^{n \times n}$ and α be a subset of $\bar{n}$. Then $pos\ C_A(\alpha)$ is referred to as the complementary cone of $[I : -A]$ with respect to α.

Remark 1.3.5. Let $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. $LCP(q, A)$ has a solution if and only if q belongs to the complementary cone $pos\ C_A(\alpha)$ for some $\alpha \subseteq \bar{n}$. Further, if $q \in pos\ C_A(\alpha)$ for some $\alpha \subseteq \bar{n}$ and $det(C_A(\alpha)) \neq 0$, then the solution $z \in S(q, A)$ satisfies $z_\alpha = -(A_{\alpha\alpha})^{-1} q_\alpha$ and $z_{\bar{\alpha}} = 0$.

Definition 1.3.6. Let $A \in \mathbb{R}^{n \times n}$ and some $\alpha \subseteq \bar{n}$. The complementary cone with respect to α is called nondegenerate (or full) if $detA_{\alpha\alpha} \neq 0$. Otherwise, it is called a degenerate complementary cone.

Definition 1.3.7. Let $A \in \mathbb{R}^{n \times n}$. The matrix A is said to be nondegenerate if $det\ A_{\alpha\alpha} \neq 0$ for all $\alpha \subseteq \bar{n}$.

Definition 1.3.8. Let $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. Any solution z of $LCP(q, A)$ is said to be nondegenerate if $z + (Az + q) > 0$. Otherwise, it is called a degenerate solution. A vector q is said to be nondegenerate with respect to A if every solution of $LCP(q, A)$ is nondegenerate.

1.4 Pfaffians

In 1849, Arthur Cayley [3] proved that the determinant of an even-ordered skew symmetric matrix is the square of a polynomial of its elements with integer coefficients. In 1852, Cayley [4] named that polynomial the Pfaffian, in honor of Johann Friedrich Pfaff. The Pfaffian of any odd-ordered skew symmetric matrix is zero. If the Pfaffian of a matrix A is defined, it is denoted by $Pf(A)$. It can be defined as follows.

Definition 1.4.1. Let A be a $2n \times 2n$ skew symmetric matrix. We partition the set $\{1, 2, \ldots, 2n\}$ into pairs $(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)$ such that $i_1 < i_2 < \cdots < i_n$ and $i_k < j_k$ for all $1 \leq k \leq n$. Let

$$\Pi_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \ldots & i_n & j_n \end{pmatrix}$$

be the permutation corresponding to the partition $\alpha = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$. Let Π be the set of all partitions of {1,2,...,2n} into pairs without regard to order. Then, the Pfaffian of matrix A would be

$$Pf(A) = \sum_{\alpha \in \Pi} sgn(\Pi_\alpha) a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_n j_n}.$$

The following two examples show the way to find the Pfaffians for $n = 1$ and 2 respectively:

Example 1.1. Let A be a skew symmetric matrix of order 2, that is $n = 1$, as follows:

$$A = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}.$$

Observe that $\Pi = \{(1, 2)\}$. Then the Pfaffians of A is $Pf(A) = a_{12} = p$.

Example 1.2. Let A be a skew symmetric matrix of order 4, that is $n = 2$, as follows:

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & e & f \\ -b & -e & 0 & h \\ -c & -f & -h & 0 \end{pmatrix}.$$

Observe that $\Pi = \{\{(1, 2)(3, 4)\}, \{(1, 3)(2, 4)\}, \{(1, 4)(2, 3)\}\}$. Then the Pfaffians of A is

$$\begin{aligned} Pf(A) &= \sum_{\alpha \in \Pi} sgn(\Pi_\alpha) a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_n j_n} \\ &= (-1)^0 a_{12} a_{34} + (-1)^1 a_{13} a_{24} + (-1)^2 a_{14} a_{23} \\ &= ah - bf + ce. \end{aligned}$$

The principal Pfaffians p_i, for $i = 1, 2, \ldots, n$, of a matrix of order n, is the Pfaffian of matrix of order $(n - 1)$ obtained by deleting i^{th}-row and i^{th}-column of given matrix. Thus, principal Pfaffians may be non-zero only for odd ordered matrix. For more results refer to [45].

We provide the following examples from [21] to illustrate the principal Pfaffians.

Example 1.3. Let A be a skew symmetric matrix of order 3 as follows:

$$A = \begin{pmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{pmatrix}.$$

Then the principal Pfaffians are $p_1 = a,\ p_2 = b$ and $p_3 = c$.

Remark 1.4.2. Pfaffian is also multiplicative, meaning that $Pf(AB) = Pf(A).Pf(B)$ for any two compatible skew symmetric matrices A and B.

1.5 Two-Person Zero-Sum Game and Value

Consider a two-person zero-sum matrix game with payoff matrix A of order m by n as follows: Let Player 1 choose an integer $i = 1, 2, \ldots, m$, and Player 2 choose an integer $j = 1, 2, .., n$. Then Player 1 pays an amount a_{ij} to Player 2, which may be positive, negative, or zero. We call a probability vector $x = (x_1, x_2, \ldots, x_m)$, a mixed strategy for Player 1 if $\sum_{i=1}^{m} x_i = 1$ and $x_i \geq 0$ for all i. Similarly, a mixed strategy for Player 2 is defined as $y = (y_1, y_2, \ldots, y_n)$ and $\sum_{j=1}^{n} y_j = 1$ and $y_j \geq 0$ for all j. A pair of strategies (x^*, y^*) is said to be optimal for Player 1 and Player 2, respectively, if

$$\sum_i x_i^* a_{ij} \leq v \qquad \text{for } j = 1, 2, \ldots, n, \tag{1.4}$$

$$\sum_j y_j^* a_{ij} \geq v \qquad \text{for } i = 1, 2, \ldots, m. \tag{1.5}$$

Here, $v = val(A)$ is called the value of the game associated with matrix A. In the game described here, we assume Player 1 as the minimizer and Player 2 as the maximizer; that is, A is a payoff matrix for player 2. The game described above is called a matrix game A, or a game associated with matrix A. For more details, one may refer to [37, 40].

Definition 1.5.1. If all entries $(x_1, x_2, \ldots, x_m)$ of a mixed strategy are positive, then such a strategy is called a completely mixed strategy.

Definition 1.5.2. If all the optimal pairs of strategies of the matrix game A are completely mixed, then the game is called a completely mixed game. In this case, we simply say that A is completely mixed.

Remark 1.5.3. A game whose pay-off matrix is skew symmetric is called a symmetric game.

1.6 Organization of the Thesis

In Chapter 2, we introduce fundamental concepts and present key results on Q-matrices, semimonotone matrices, P-matrices. Additionally, we provide some results relating these matrix classes to the value of games.

Chapter 3 deals with Pang's conjecture. Aganagic and Cottle [1] proved that within the class of P_0-matrices, a matrix is a Q-matrix if and only if it is an R_0-matrix. It is known that the class of P_0-matrices is a subset of the class of semimonotone matrices. Pang proved that within the class of semimonotone matrices, every R_0-matrix is a Q-matrix and conjectured that the converse also holds. Gowda [14] proved that Pang's conjecture is true for symmetric matrices. In Chapter 3, we prove that the above Pang's conjecture is true for semimonotone matrices of order $n \leq 3$ and provide a counterexample for $n > 3$, demonstrating the sharpness of the result.

In Chapter 4, we present new characterizations for completely mixed matrix games with value zero. Kaplansky introduced the concept of completely mixed games. He proved that a game associated with a matrix $A \in \mathbb{R}^{n \times n}$ with value zero is completely mixed if and only if the matrix is a square matrix of rank $n-1$ and all cofactors of A are nonzero and have same sign. In section 4.3, we establish that a game associated with matrix $A \in \mathbb{R}^{n \times n}$ having value zero is completely mixed if and only if $val(A+D_i) > 0$, where D_i is a diagonal matrix whose i^{th} diagonal entry is 1 and else 0. In section 4.4, we prove that the game associated with an odd-ordered skew symmetric matrix is completely mixed if and only if $A + D_i$ is a

Q-matrix for all i. In section 4.5, we show that for an almost skew symmetric matrix A, if $val(A)$ is positive, then $A + D_i \in Q$ for all i. Further we provide a counterexample showing that the converse does not hold.

Chapter 5 deals with Stone's conjecture and the relationship between completely mixed games and various matrix classes. In 1981 Stone conjectured that $E_0^f \cap Q_0 \subseteq P_0$. Since, $Z \subseteq Q_0$, Murthy proved that $E_0^f \cap Z \subseteq P_0$. In section 5.3 we prove that $E_0 \cap Z \subseteq P_0$. In section 5.4, we prove that Stone's conjecture holds for some specific sign patterns. In section 5.5, we explore the connection between completely mixed games and different matrix classes. We establish that for $A \in Z$ with $val(A) = 0$, A is completely mixed if and only if $A + D_i \in Q$, for all i, which is further equivalent to A being irreducible. We also prove that A is completely mixed if and only if $A + D_i \in Q$ for fully copositive or semimonotone matrix.

Chapter 6 of this thesis focuses on tensors, the tensor complementarity problem, and tensor classes. We extend some of the known matrix results to tensors. In section 6.3, we derive some results on nonnegative tensors, which are needed for proving our results. In section 6.4, we prove some results for Q-tensors. In section 6.5, we prove that a rank-one Q-tensor has all positive entries.

1.7 Publications from the Thesis

1. Parthasarathy Thiruvankatachari, Ravindran Gomatam, and Sunil Kumar: "On Semimonotone Matrices, R_0-Matrices and Q-Matrices", Journal of Optimization Theory and Applications, vol. 195(1), pages 131-147, (2022). 10.1007/s10957-022-02066-3 [Out of Chapter 3 and section 5.5.3]

2. Kumar Sunil and Ravindran Gomatam: "On Semimonotone Z-Matrices", Advances

in Mathematical Modelling, Applied Analysis and Computation, Lecture Notes in Networks and Systems, vol. 666, pages 110-120, Springer, Cham., (2023). 10.1007/978-3-031-29959-9_7 [Out of Section 5.3 and 5.4]

3. Parthasarathy Thiruvankatachari, Ravindran Gomatam, and Sunil Kumar: “On Copositive Matrices and Completely Mixed Games”, Advances in Mathematical Modelling, Applied Analysis, and Computation, Lecture Notes in Networks and Systems, vol. 952, pages 31-37, Springer, Cham., (2024). https://doi.org/10.1007/978-3-031-56307-2_3 [Out of Section 5.5.2]

4. Parthasarathy Thiruvankatachari, Ravindran Gomatam, and Sunil Kumar: “On Completely Mixed Games”, Journal of Optimization Theory and Applications, vol. 201, pages 313-322, (2024). 10.1007/s10957-024-02395-5 [Out of Chapter 4]

5. Parthasarathy Thiruvankatachari, Ravindran Gomatam, and Sunil Kumar: “On Q-tensors”, Communicated [Out of Chapter 6]

CHAPTER 2

Preliminaries

In this chapter, we list some known results on different matrix classes. We also provide some relationship between different matrix classes and the value of the game.

2.1 Some Basic Results

In this section, we provide some fundamental results on matrix classes, covering important properties and theorems that are specific to certain classes of matrices. These concepts and theorems are used in further chapters.

2.1.1 $Q(Q_0)$-Matrices

The classes of Q-matrices and Q_0-matrices were introduced by Murty [34] and Parsons [39], respectively. A Q-matrix ensures the existence of a solution to the LCP for any vector. A large number of subclasses of Q have been identified. Positive definite matrices, regular matrices, and P-matrices are some subclasses of Q. Here, we state some known results related to Q-matrices.

The following two results are due to Murty and Yu [35].

Theorem 2.1.1. *If $A \in \mathbb{R}^{n\times n}$ is a Q-matrix having a nonnegative row, then the matrix obtained by deleting that (nonnegative) row and its corresponding column is also a Q-matrix.*

Theorem 2.1.2. *If $A \in \mathbb{R}^{n\times n}$ is a nonnegative Q-matrix, then all its diagonal entries must be positive.*

Remark 2.1.3. If $A \in \mathbb{R}^{n\times n} \cap Q$, then any row of the matrix A cannot be nonpositive. Note that if the i^{th} row of A is nonpositive, then for any vector $q \in \mathbb{R}^n$ with $q_i < 0$, $LCP(q, A)$ will not have a solution.

The following result is due to Ingleton [18] that provides a sufficient condition for any matrix to be a Q-matrix.

Theorem 2.1.4. *Let $A \in \mathbb{R}^{n\times n}$. If for some $d > 0$, $LCP(0, A)$ and $LCP(d, A)$ have a unique solution $x = 0$, then $A \in Q$.*

2.1.2 Semimonotone Matrices

Eaves [8] introduced the class of semimonotone matrices and denoted it as L_1. Later, in [6], the class of semimonotone matrices was denoted by E_0. The name 'semimonotone' was proposed by Karamardian. The class of semimonotone matrices includes the class of P_0-matrices and the copositive matrices. Semimonotone matrices have important properties related to LCPs, including the existence and uniqueness of solutions.

The following theorem summarizes some known facts for semimonotone matrices.

Theorem 2.1.5. *[6] The semimonotone matrices have the following properties:*

1. *Any principal submatrix of an E_0-matrix is also an E_0-matrix.*

2. *If $A \in E_0$, then $A^t \in E_0$.*

3. *If $A \in E_0$, then diagonal entries of A are nonnegative.*

4. *If $A \in E_0$, $LCP(q, A)$ will have a unique solution for any vector $q > 0$.*

Tsatsomeros and Wendler [56] proved the following theorem, which relates the determinant and entries of a semimonotone matrix of order 2.

Theorem 2.1.6. *Let A be a 2×2 real matrix with nonnegative diagonal entries. Then A is a semimonotone matrix if and only if either all entries of A are nonnegative or the determinant is nonnegative.*

We can conclude the following corollary from the above theorem.

Corollary 2.1.7. *Let $A \in \mathbb{R}^{n\times n} \cap E_0$. If a principal submatrix of A of order 2×2 has the sign pattern*

$$\begin{pmatrix} \oplus & - \\ c & 0 \end{pmatrix},$$

then c must be nonnegative.

Proof. Since $A \in E_0$, each principal submatrix is a semimonotone matrix. Hence, the given sign pattern is also a semimonotone matrix. Since the given sign pattern has a negative entry, using Theorem 2.1.6, determinant of the given sign pattern has to be nonnegative. Therefore, c must be nonnegative. □

Tsatsomeros and Wendler [56, 59] have provided several results related to semimonotone matrices.

2.1.3 $P(P_0)$-Matrices

The class of P (P_0)-matrices is an important class of matrices that arise in the study of linear algebra, linear complementarity problems (LCPs), and related fields. Recall A is called a P (P_0)-matrix if all of its principal minors are positive (nonnegative).

Definition 2.1.8. Let x be a vector in $\mathbb{R}^n$. Then A is said to reverse the sign of x if $(Ax)_i x_i \leq 0$ for every $i = 1, 2, ..., n$.

The following theorem provides a characterization of P-matrix. Gale and Nikaido [11] established the following two results.

Theorem 2.1.9. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Then $A \in P$ if and only if $\{x \in \mathbb{R}^n : x_i(Ax)_i \leq 0$ for all $i = 1, 2 \ldots, n\} = \{0\}$.*

Theorem 2.1.10. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Then A is a P-matrix if and only if it does not reverse the sign of any nonzero vector.*

Theorem 2.1.11. *[6] Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Then A is a P-matrix if and only if all real eigenvalues of A, as well as those of its principal submatrices, are positive.*

2.1.4 Principal Pivotal Transforms

Principal pivotal transforms have been defined in section 1.2.1, here we explore some additional properties. The following theorems can be found in [5, 39, 57].

Theorem 2.1.12. *Let A be any square matrix and M be any PPT of A. Then*

1. *A is an R_0-matrix if and only if M is an R_0-matrix.*
2. *$A \in Q$ if and only if $M \in Q$.*

Theorem 2.1.13. *[30] Let $A \in \mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{n\times n}$ be a PPT of A with respect to some $\alpha \subseteq \{1, 2, \ldots, n\}$. Then $val(M) > 0$ if and only if $val(A) > 0$.*

Tsatsomeros [55] provided an extensive treatment of principal pivot transforms. The following formula from [6] is used in proving our next result.

Theorem 2.1.14. *Let M be a matrix obtained from the square matrix A by a principal pivot on the submatrix $A_{\alpha\alpha}$. Then for any submatrix $M_{\beta\beta}$ of M,*

$$det(M_{\beta\beta}) = \frac{det(A_{\gamma\gamma})}{det(A_{\alpha\alpha})}$$

where $\gamma = \alpha \bigtriangleup \beta$.

A P_0-matrix need not necessarily be a Q-matrix or an R_0-matrix. To analyze the PPT of P_0 and skew symmetric matrices, we require the following results. Theorem 2.1.15 and 2.1.17 establish that the PPT of P_0 and skew symmetric matrix remains P_0 and skew symmetric respectively.

Theorem 2.1.15. *Let A be a P_0-matrix. Then, any principal pivot transform of A is also a P_0-matrix.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a P_0-matrix and α be some subset of $\{1, 2, \ldots, n\}$ such that $det(A_{\alpha\alpha}) \neq 0$. Hence, ppt of A with respect to α exists. Let us call it M. Now for any $\beta \subseteq \{1, 2, \ldots, n\}$, using Theorem 2.1.14, we have

$$det(M_{\beta\beta}) = \frac{det(A_{\gamma\gamma})}{det(A_{\alpha\alpha})} \tag{2.1}$$

where $\gamma = \alpha \triangle \beta$. Observe that $A_{\alpha\alpha}$ and $A_{\gamma\gamma}$ are principal submatrices of A. Since $A \in P_0$ and $det(A_{\alpha\alpha}) \neq 0$, we have $det(A_{\alpha\alpha}) > 0,\ det(A_{\gamma\gamma}) \geq 0$. From (2.1), it follows that $det(M_{\beta\beta}) \geq 0$. Since β is arbitrary, we conclude that $M \in P_0$. Therefore, any ppt of a P_0-matrix is a P_0-matrix. □

Remark 2.1.16. Let A be a non-singular skew symmetric matrix of even order. Then the inverse of A is also skew symmetric.

Theorem 2.1.17. *Let $A \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix. Then any principal pivot transform of A is also a skew symmetric matrix.*

Proof. Let A be a skew symmetric matrix. Let M be a ppt of A with respect to some α. Let us assume $A_{\alpha\alpha} = D$ is non-singular. Then we can write A as:

$$A = \begin{pmatrix} D & B \\ -B^t & C \end{pmatrix},$$

where $D \in \mathbb{R}^{|\alpha|\times|\alpha|}$, $B \in \mathbb{R}^{|\alpha|\times|\bar{\alpha}|}$, and $C \in \mathbb{R}^{|\bar{\alpha}|\times|\bar{\alpha}|}$. Observe that D and C are also skew symmetric, i.e., $D^t = -D$ and $C^t = -C$. The ppt of A with respect to α will be

$$M = \begin{pmatrix} D^{-1} & -D^{-1}B \\ -B^tD^{-1} & C - (-B^t)D^{-1}B \end{pmatrix}.$$

To prove this matrix M to be skew symmetric, we need to check the following:

1. D^{-1} *is a skew symmetric matrix*: It is true due to remark 2.1.16.

2. $(-B^tD^{-1})^t = D^{-1}B$: Taking the transpose, we obtain:

$$(-B^tD^{-1})^t = -(D^{-1})^t(B^t)^t = -(D^{-1})^tB = -(-D^{-1})B = D^{-1}B.$$

3. *$C-(-B^t)D^{-1}B$ is skew symmetric, that is $(C-(-B^t)D^{-1}B)^t = -(C-(-B^t)D^{-1}B)$*:

   Taking the transpose, we obtain:

$$\begin{aligned}
(C-(-B^t)D^{-1}B)^t &= C^t - ((-B^t)D^{-1}B)^t \\
&= -C + (B^tD^{-1}B)^t \qquad \text{[Because } C^t = -C\text{]} \\
&= -C + (D^{-1}B)^t(B^t)^t \\
&= -C + B^t(D^{-1})^tB \\
&= -C + B^t(-D^{-1})B \qquad \text{[Because } D^{-1} \text{ is skew symmetric]} \\
&= -C - B^tD^{-1}B \\
&= -(C + B^tD^{-1}B) \\
&= -(C-(-B^t)D^{-1}B)
\end{aligned}$$

Hence, M is a skew symmetric matrix. Since, α was arbitrary, every PPT of a skew symmetric matrix is also skew symmetric. □

2.2 Relationship Between Matrix Classes

In this section, we present some known results for subclasses of Q and Q_0. We need the following definitions.

Definition 2.2.1. A matrix $A \in \mathbb{R}^{n\times n} \cap P_0$ is said to be column adequate if for each $\alpha \subseteq \{1, 2, \ldots, n\}$, $detA_{\alpha\alpha} = 0$ implies $A_{.\alpha}$ has linearly dependent columns. A is said to be row adequate if A^t is column adequate. A is called adequate if A is both row and column adequate.

Definition 2.2.2. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be column sufficient if it satisfies the following implication:

$$[x_i(Ax)_i \leq 0 \text{ for all } i] \Rightarrow [x_i(Ax)_i = 0 \text{ for all } i].$$

The matrix A is said to be row sufficient if A^t is column sufficient. A is called sufficient if A is both row and column sufficient.

Definition 2.2.3. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a hidden Z-matrix if there exist $X, Y \in Z$ and $r, s \in \mathbb{R}^n_+$, such that $AX = Y$ and $r^tX + s^tY > 0$.

The following theorem presents several subclasses of Q.

Theorem 2.2.4. *[30] Let $A \in \mathbb{R}^{n\times n}$. Then $A \in Q$, if A belongs to any of the following classes of matrices:*

1. *positive definite matrices,*
2. *positive matrices,*
3. *P-matrices,*
4. *R-matrices,*
5. *E-matrices,*
6. *$E_0 \cap R_0$-matrices,*
7. *$C_0 \cap R_0$-matrices,*
8. *$Q_0 \cap S$-matrices.*

In the following theorem, it provides several subclasses of Q_0.

Theorem 2.2.5. *[30] Let $A \in \mathbb{R}^{n\times n}$. Then $A \in Q_0$, if A belongs to any of the following classes of matrices:*

1. *positive semi definite matrices,*
2. *nonpositive matrices,*
3. *copositive plus matrices,*
4. *Z-matrices,*
5. *hidden Z-matrices,*

6. *sufficient matrices,*

7. *adequate matrices.*

Remark 2.2.6. If A is a Q-matrix, then there exists a vector x such that $Ax > 0$. This implies A is an S-matrix, hence $Q \subseteq S$. Also we know that $Q \subseteq Q_0$. Using Theorem 2.2.4, we can say that

$$Q = Q_0 \cap S.$$

2.3 Relationship in Matrix Classes and Value of a Game

In this section, we discuss about a matrix A and the value of the game associated with it. Whenever we refer to the value of A or $val(A)$, we mean the value of the two-person zero-sum game associated with matrix A. It is easy to check that the value of A is positive (nonnegative) if and only if there exists a nonnegative, nonzero vector x such that $Ax > 0 (\geq 0)$. From the definition of S-matrix, we can conclude that if A is an S-matrix, then $val(A)$ is positive.

Remark 2.3.1. [21] Every skew symmetric matrix A is a positive semi-definite matrix, and the $val(A)$ is zero.

Lemma 2.3.2. *If $A \in \mathbb{R}^{n\times n} \cap Q$, then $val(A) > 0$.*

Proof. Let $val(A) \not> 0$. Then, there cannot exist a vector $x \in \mathbb{R}^n$ such that $Ax > 0$. That means for each nonnegative vector x, some coordinate of Ax is nonpositive (say $(Ax)_j \leq 0$). Then consider a vector $q \in \mathbb{R}^n$ such that $q < 0$. Now for $LCP(q, A)$, we cannot find a solution, because $(Ax + q)_j < 0$. This contradicts the hypothesis that $A \in Q$. Therefore, $A \in Q$ implies $val(A) > 0$. □

Remark 2.3.3. In the above theorem, we have $A \in Q$. It is known that $Q \subseteq S$. Hence, $val(A) > 0$.

The converse of the above lemma is not true. The following example demonstrates that $val(A) > 0$ does not necessarily imply that $A \in Q$.

Example 2.1. *Consider the matrix*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is straightforward to verify that the value of A is $\frac{1}{2}$. However, consider $q = (-1, 1)^t$. The linear complementarity problem $LCP(q, A)$ does not have a solution. Hence, $A \notin Q$.

However, the condition $A \in Q_0$ with $val(A) > 0$ ensures that A is a Q-matrix.

Lemma 2.3.4. *If $A \in \mathbb{R}^{n\times n} \cap Q_0$ and $val(A) > 0$, then $A \in Q$.*

Proof. Since $val(A) > 0$, there exists a probability vector x such that $Ax > 0$. Let $q \in \mathbb{R}^n$ be any vector. We define $y = \lambda x$ for sufficiently large $\lambda > 0$. Then,

$$Ay + q = A(\lambda x) + q = \lambda(Ax) + q$$

Since $Ax > 0$, choosing λ large enough, ensures that $Ay + q \geq 0$. This implies y is a feasible solution. Since q was an arbitrary vector, it follows that $F(q, A)$ is non-empty for every $q \in \mathbb{R}^n$. Since $A \in Q_0$, we conclude that $A \in Q$. □

Remark 2.3.5. [33] It is known that, if $A \in \mathbb{R}^{n\times n} \cap E_0$, then $val(A) \geq 0$. Moreover, since $P_0 \subseteq E_0$, we conclude that the value is nonnegative for P_0-matrices.

CHAPTER 3

Semimonotone, Q-matrices

3.1 Introduction

In 1979, Aganagic and Cottle [1] proved that within the class of P_0-matrices, any matrix is a Q-matrix if and only if it is an R_0-matrix. It is known that the class of P_0-matrices is a subset of the class of semimonotone matrices. With this in mind, it is natural to ask whether Q-matrices and R_0-matrices are equivalent within the class of E_0-matrices. Pang [38] proved that within the class of semimonotone matrices, R_0-matrices are Q-matrices and conjectured the converse. However, Jeter and Pye [20] refuted the conjecture by providing a counterexample of order 5; that is, they demonstrated that a matrix belonging to E_0 and Q is not necessarily in R_0.

The class of copositive matrices is a subset of the class of semimonotone matrices. Jeter and Pye conjectured that the equivalence of Q and R_0 holds within the class of copositive matrices. In 1990, Gowda [14] proved this equivalence for symmetric matrices. However, in 1993, Murthy, Parthasarathy, and Ravindran provided a counterexample, showing that a general copositive Q-matrix is not R_0 for order 4 and above.

Jeter and Pye [19] proved that within the class of copositive matrices, Q-matrices are equivalent to R_0-matrices for orders up to 3. A similar equivalence result has not been established for the class of E_0-matrices. In this chapter, we show that if $A \in E_0 \cap Q$, then

$A \in R_0$ for matrices of order up to 3 and provide a counterexample for matrices of order 4 and above.

3.2 Preliminaries

A Q-matrix assures the existence of solution to the corresponding linear complementarity problem. However, still there is no efficient way to identify whether a matrix is a Q-matrix. Murthy, Parthasarathy, and Ravindran in [33] have proved results that help establish a matrix of order n as a Q-matrix if we know that two principal submatrices of order $n-1$ are Q-matrices under specific conditions. The next two theorems from [33] provide a way to determine whether a matrix is a Q-matrix based on the properties of its principal submatrices.

Theorem 3.2.1. *Suppose $A \in \mathbb{R}^{n\times n}$. Let the corresponding elements of row 1 and row 2 are equal. Further assume that $A_{\alpha\alpha}, A_{\beta\beta} \in Q$, where $\alpha = \{1, 2, \ldots, N\} \setminus \{1\}$ and $\beta = \{1, 2, \ldots, N\} \setminus \{2\}$. Then $A \in Q$.*

Theorem 3.2.2. *Suppose $A \in \mathbb{R}^{n\times n}$. Let $a_{11} \leq a_{21}$, $a_{22} \leq a_{12}$ and $(a_{13}, \ldots, a_{1n})^t = (a_{23}, \ldots, a_{2n})^t$. Further assume that $A_{\alpha\alpha}, A_{\beta\beta} \in Q$, where $\alpha = \{1, 2, \ldots, N\} \setminus \{1\}$ and $\beta = \{1, 2, \ldots, N\} \setminus \{2\}$. Then $A \in Q$.*

Thus, the above theorems provide a way to determine whether a matrix is a Q-matrix using the property of its principal submatrices being Q-matrices. There is also a result for matrices being positive when their principal submatrices are positive. In particular, if all the proper principal submatrices of a matrix are positive, then the whole matrix is a positive matrix. However, this result only holds for matrices of order 3 and above. For order 2, we can see the following counterexample whose all proper principal submatrices are positive but the matrix has negative entries.

Example 3.1. *Let A be the matrix given below*

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}.$$

It is easy to see that its proper principal submatrices are positive. The matrix A has negative entries. Hence, the above result does not hold for $n = 2$.

Furthermore, using this result, we can prove the following theorem for rank-one matrices that are Q as well. The following theorem is due to Sushmitha [54]. For the sake of completeness, we are also providing a proof here.

Theorem 3.2.3. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix of rank one. If $A \in Q$, then all the entries of matrix A are positive.*

Proof. We prove the result using mathematical induction. It is known that each row of a rank one matrix is some constant multiple of the first row.

For $n = 1$, it is obvious as the matrix A will have only one entry. Since $A \in Q$, hence this unique entry must be positive.

For $n = 2$, on the contrary, let us suppose that $A \not> 0$. Without loss of generality, assume $a_{11} \leq 0$.

1. Let a_{11} is zero. Then A cannot have a nonpositive row, since $A \in Q$. Hence, $a_{12} > 0$. Since, row 2 of A is constant multiple of row 1, that is, $a_{21} = 0$. Also, $A \in Q$ implies that $a_{22} > 0$. Let $q = (q_1, q_2)^t$ be a vector such that $q_1 < 0$ and $q_2 > 0$. Observe that, $LCP(q, A)$ has no solution, which contradicts the fact that A is a Q-matrix. Therefore, A having some zero entry is not possible.

2. Let a_{11} is negative. Since $A \in Q$, $a_{12} > 0$. Let the first row of A be $[-a \quad b]$, where $a, b > 0$. Since A is of rank 1, the second row is α multiple of the first row. Since, $A \in Q$, $\alpha \neq 0$. Now, for $\alpha > 0$, consider the $LCP(q, A)$ for some vector

$q = (-q_1, q_2)^t$ such that $q_1, q_2 > 0$. Then $w = Ax + q$ gives us:

$$-ax_1 + bx_2 - q_1 = w_1 \tag{3.1}$$

$$-\alpha ax_1 + \alpha bx_2 + q_2 = w_2 \tag{3.2}$$

$$\alpha[-ax_1 + bx_2] + q_2 = w_2 \tag{3.3}$$

Now, using equation (3.1), $x_2 = 0$ gives us $w_1 < 0$. But this leads to a contradiction. Hence, $x_2 > 0$. If $x_1 = 0$, then using equation (3.2), $w_2 > 0$. But $w_2 > 0$ and $x_2 > 0$ will contradict the condition of complementarity. Now, if both x_1 and x_2 are positive, it implies that $w_1 = 0$ and $w_2 = 0$. From equation (3.1), we have:

$$-ax_1 + bx_2 = q_1 > 0.$$

Put this in equation (3.3), we have:

$$\alpha q_1 + q_2 = w_2.$$

This leads to a contradiction because $\alpha > 0,\ q_1 > 0,\ q_2 > 0$, but $w_2 = 0$. Hence, A cannot have any negative entry. Similarly, for $\alpha < 0$, we choose $q = (-q_1, -q_2)^t$, where $q_1, q_2 > 0$ and show that $LCP(q, A)$ has no solution. This is a contradiction to $A \in Q$.

Therefore, for $n = 2$, A is a positive matrix.

Now, let us assume that the statement is true for matrices of order $n - 1$. We need to prove that it is also true for n.

Consider a vector q with some $q_j = 0$ and other entries of q are arbitrary. If all entries are nonnegative, then it has a trivial solution. Without loss of generality, let us assume that the first coordinate of q is negative, that is, $q = (-q_1, q_2, \ldots, q_j = 0, \ldots, q_n)^t$ such that $q_1 > 0$. Since $A \in Q$, consider $LCP(q, A)$. Let $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$ be the solution of $LCP(q, A)$. We have

$$w_1 = A_1 x - q_1 \geq 0$$
$$\Rightarrow A_1 x = w_1 + q_1 > 0.$$

Since A is of rank 1, let the j^{th}-row of A be β times of the first row of A. Then, for w_j, we have:

$$0 \leq w_j = A_j x + q_j = A_j x + 0 = \beta A_1 x.$$

Observe that $\beta \geq 0$. If $\beta = 0$, then j^{th}-row of A is zero, which is not possible for a Q-matrix. Hence, $\beta > 0$. Since $A_1 x > 0$ and $\beta > 0$, we have $w_j > 0$. Hence, from the complementarity, $x_j = 0$.

Now, $x = (x_1, x_2, \ldots, x_j = 0, \ldots, x_n)^t$ be a solution to $LCP(q, A)$. If B is a principal submatrix of A obtained by deleting the j^{th}-row and j^{th}-column, then observe that $x = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)^t$ is a solution to $LCP(\bar{q}, B)$, where $\bar{q} = (q_1, q_2, .., q_{j-1}, q_{j+1}, .., q_n)^t$. Since each entry of $\bar{q}$ is arbitrary, $B \in Q$. Since the matrix B of order $n-1$ is a principal submatrix of the rank-one matrix A, hence B has also rank one. Therefore, using induction hypothesis, $B > 0$.

Similarly, each of the principal submatrices of A is positive. Hence, A is a positive matrix. Therefore, all rank-one Q-matrices are positive matrices. □

3.3 Results Relating Q-Matrices, E_0-Matrices and R_0-Matrices

In 1979, Pang [38] proved several results regarding semimonotone Q-matrices. He demonstrated that for any semimonotone Q-matrix A, if $LCP(0, A)$ has a solution, then that solution cannot have all the coordinates positive, nor can it have exactly one coordinate positive. These two results are stated as follows:

Theorem 3.3.1. *If $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Q$, then $x > 0$ cannot be a solution of $LCP(0, A)$.*

Theorem 3.3.2. *Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Q$. If $x(\neq 0)$ is a solution of $LCP(0, A)$, then at least two components of x are nonzero.*

The subsequent section of this chapter discusses the semimonotone Q-matrices of order 3, which are R_0-matrices. In that case, we assume $A \in \mathbb{R}^{3\times 3} \cap E_0 \cap Q$, further if $LCP(0, A)$

has a nonzero solution x, then from the above two results, exactly two coordinates of x are nonzero. This fact is used throughout the proof of our results.

We can rephrase Theorem 3.3.2 as follows. This helps in obtaining the sign pattern of elements in matrix A. We provide the proof of the theorem for completeness.

Theorem 3.3.3. *Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Q$. Furthermore, assume that A has some diagonal entry equal to 0. Then the column with diagonal entry 0 cannot be nonnegative.*

Proof. Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Q$ with some diagonal entry zero. Without loss of generality, let $a_{11} = 0$. We want to prove that the first column has some negative entry. Suppose, on the contrary, that all the entries of the first column are nonnegative. Then, the matrix A has the following sign pattern:

$$\begin{pmatrix} 0 & * & * & * \\ \oplus & \oplus & * & * \\ \oplus & * & \oplus & * \\ \oplus & * & * & \oplus \end{pmatrix}.$$

Consider a vector $q = (-q_1, q_2, \ldots, q_n)^t$, where $q_i > 0$ for all $i = 1, 2, \ldots, n$. Since $A \in Q$, $LCP(q, A)$ must have a solution. Let $x \in \mathbb{R}^n$ be a solution. Then, $w_1 = (Ax+q)_1 \geq 0$ gives us:

$$w_1 = a_{12}x_2 + \cdots + a_{1n}x_n - q_1 \geq 0.$$

Since $q_1 > 0$, observe that for some i, $x_i > 0$ for $i \in \{2, 3, \ldots, n\}$. Hence, $w_i = 0$. On expanding, we have:

$$w_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + q_i = 0,$$
$$a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = -(q_i + a_{i1}x_1).$$

Since $q_i > 0$ and $a_{i1} \geq 0$ for all $i = 2, 3, \ldots, n$, this implies that $a_{i2}x_2+a_{i3}x_3+\cdots+a_{in}x_n < 0$ for all $i = 2, 3, \ldots, n$.

Consider a vector $\bar{x} = (0, x_2, x_3, \ldots, x_n)^t$. Then, for any of $\bar{x}_k > 0$, we have $(A\bar{x})_k <$

0. This leads to a contradiction to $A \in E_0$. Hence, our assumption is incorrect. Therefore, any semimonotone Q-matrix cannot have a diagonal entry zero and the corresponding column nonnegative. $\square$

Aganagic and Cottle proved that within the class of P_0-matrices, Q-matrices are equivalent to R_0-matrices. Later in 1992, Pye proved that Q-matrices are equivalent to R_0-matrices whenever it is an almost P_0-matrix. The following theorems provide necessary and sufficient condition for a Q-matrix to be an R_0-matrix, as established by Aganagic and Cottle, and Pye.

Theorem 3.3.4. *[1] Let $A \in \mathbb{R}^{n\times n}$ be a P_0-matrix. Then A is a Q-matrix if and only if A is an R_0-matrix.*

Theorem 3.3.5. *[46] Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that $|A_{\alpha\alpha}| \geq 0$ for $|\alpha| \leq n-1$ and $|A| < 0$. Then A is a Q-matrix if and only if A is an R_0-matrix.*

Since P_0 is a subset of E_0, one can ask whether this equivalence holds true for semimonotone matrices. In order to that, Murthy, Parthasarathy, and Ravindran proved that Q-matrices are equivalent to R_0-matrices within the class of semimonotone matrices, subject to certain conditions. The following theorem states their result:

Theorem 3.3.6. *Let $A \in \mathbb{R}^{n\times n} \cap E_0$, where $n \geq 3$. Suppose any one of the following conditions holds:*

1. *Every principal submatrix of A of order $(n-1)$ is an R_0-matrix.*

2. *$A_{\alpha\alpha} \in R_0$ for all $\alpha \subseteq \{1, 2, \ldots, n\}$ with $|\alpha| \leq n-2$.*

Then, $A \in Q$ if and only if $A \in R_0$.

3.4 Semimonotone Q-Matrices

The following example is due to Murthy, Parthasarathy, and Ravindran [32]. It served as a counterexample to the conjecture due to Jeter and Pye, that is, Q and R_0 are equivalent within the class of copositive matrices. Since $C_0 \subseteq E_0$, we have the following example of a C_0-matrix. Hence, it is a semimonotone matrix and we will also show that it is a Q-matrix.

Example 3.2. Let A be the matrix given below

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

Notice that

$$x^t Ax = x_3^2 + 2x_1x_3 + 2x_2x_4 + x_4^2 \geq 0, \ \forall x \in \mathbb{R}^4_+.$$

Thus, $A \in C_0$. It can be observed that row 1 and row 2 of A are identical. Further A_{11} and A_{22} are $C_0 \cap R_0$. Hence, from Pang's result, $A_{11}, A_{22} \in Q$. Using the Theorem 3.2.1, $A \in Q$. But any nonzero vector $x = (k, k, 0, 0)^t \geq 0$, is a solution to $LCP(0, A)$. Therefore, $A \notin R_0$.

The following example is due to Jeter and Pye [20].

Example 3.3. Let A be the matrix given below

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix}.$$

Jeter and Pye proved that the above matrix is a semimonotone matrix and Q-matrix but it is not an R_0-matrix.

3.5 A Semimonotone Q-Matrix that is an R_0-matrix

Pang proved that semimonotone R_0-matrices are Q-matrices and conjectured that the converse is true. However, Jeter and Pye provided a counterexample of order 5 (example 3.3), which shows that a semimonotone Q-matrix may not necessarily be R_0. They conjectured that the same equivalence may hold for C_0-matrices instead of semimonotone. But, Murthy, Parthasarathy, and Ravindran provided a counterexample (example 3.2) demonstrating that this new conjecture is not valid for matrices of order 4 and above. On the other hand, Jeter and Pye [19] proved that copositive Q-matrices are R_0-matrices for order 3. Their result is stated as the following theorem.

Theorem 3.5.1. *If $A \in \mathbb{R}^{3\times 3} \cap C_0 \cap Q$, then $A \in R_0$.*

It is known that a symmetric matrix is copositive if and only if it is a semimonotone matrix. Gowda [14] proved that copositive Q-matrices are R_0-matrices, for symmetric matrices.

Theorem 3.5.2. *Let A be a symmetric matrix and $A \in \mathbb{R}^{n\times n} \cap C_0$. Then $A \in R_0$ if and only if $A \in Q$.*

These two theorems raised the following question: Does an $E_0 \cap Q$ matrix of order up to three give an R_0-matrix? In this section, we provide an affirmative answer to this question.

Theorem 3.5.3. *If $A \in \mathbb{R}^{n\times n} \cap Q$, then $A \in R_0$ for $n \leq 2$.*

Proof. When $n = 1$, the matrix A consists of a single positive entry since it is a Q-matrix and hence $A \in R_0$.

Now, let $n = 2$. Suppose A does not belong to R_0. Then, $LCP(0, A)$ has a nonzero solution $x \geq 0$.

Assuming one coordinate of x to be zero, without loss of generality, let $x_1 > 0$ and $x_2 = 0$. Then, $Ax = w \geq 0$ and $x_i w_i = 0$ gives us $w_1 = 0$. This implies that $a_{11} = 0$ because $x_1 > 0$. Since $A \in Q$, using Remark 2.1.3, we can conclude that $a_{12} > 0$. Now, since the first row is nonnegative, Theorem 2.1.1 implies that on omitting the first row and the first column, the remaining matrix must be a Q-matrix. Therefore, $a_{22} > 0$. Since we have assumed $x_2 = 0$, this implies $w_2 \geq 0$ and hence, $a_{21} \geq 0$. Now, the second row becomes nonnegative. Again, using Theorem 2.1.1, on omitting the second row and the second column, a_{11} has to be positive, which contradicts $a_{11} = 0$. Hence, we conclude that it is not possible for one coordinate of x to be zero.

Now, assume that no coordinate of x is zero. This means $Ax = 0$. Therefore, the rank of A is 1. Since $A \in Q$, then by Theorem 3.2.3, $A > 0$. However, this contradicts the assumption that no coordinate of x is zero. Therefore, $A \in R_0$. □

It is important to note that the result in Theorem 3.5.3 is true only for matrices up to order two. The following example shows that for $n = 3$, $A \in Q$ does not imply $A \in R_0$.

Example 3.4. *Let*

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Proof. Notice that the entries of matrix A satisfy $a_{11} = -1 < 1 = a_{21}$, $a_{22} = -1 < 1 = a_{12}$, and $a_{13} = a_{23}$. We can see that

$$A_{11} = A_{22} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

It is easy to check that $A_{11}, A_{22} \in Q$. Using Theorem 3.2.2, we conclude that $A \in Q$. However, we can find a nonzero solution $(1, 1, 0)$ for $LCP(0, A)$, which implies that $A \notin R_0$. □

Hence, we need some additional assumption for $n = 3$. Specifically, we assume that $A \in E_0 \cap Q$ and show that A is in R_0. Before proving Theorem 3.5.8, we provide several lemmas for matrices with various sign patterns, which will be useful in the proof..

Lemma 3.5.4. *Let $A \in \mathbb{R}^{3\times3} \cap E_0 \cap Q$. Assume A has the following sign pattern*

$$A = \begin{pmatrix} + & + & - \\ + & + & - \\ + & 0 & 0 \end{pmatrix}.$$

Then $A \in R_0$.

Proof. Consider the PPT of A with respect to the set $\alpha = \{1, 3\}$. Let $M = ppt(A, \alpha)$. We have the sign patterns as follows:

$$A_{\alpha\alpha} = \begin{pmatrix} + & - \\ + & 0 \end{pmatrix}, A_{\alpha\bar{\alpha}} = \begin{pmatrix} + \\ 0 \end{pmatrix}, A_{\bar{\alpha}\alpha} = \begin{pmatrix} + & - \end{pmatrix}, A_{\bar{\alpha}\bar{\alpha}} = \begin{pmatrix} + \end{pmatrix}$$

With these sign patterns, we can determine the sign patterns for the entries of the PPT as follows:

$$A_{\alpha\alpha}^{-1} = \begin{pmatrix} 0 & + \\ - & + \end{pmatrix}, -A_{\alpha\alpha}^{-1}A_{\alpha\bar{\alpha}} = \begin{pmatrix} 0 \\ + \end{pmatrix}, A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1} = \begin{pmatrix} + & * \end{pmatrix},$$

$$A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1}A_{\alpha\bar{\alpha}} = \begin{pmatrix} * \end{pmatrix}$$

Therefore, the PPT M of A with respect to α has the following sign pattern:

$$M = \begin{pmatrix} 0 & 0 & + \\ + & * & * \\ - & + & + \end{pmatrix}.$$

Since $A \in Q$, using Theorem 2.1.12, we observe that $M \in Q$. Since the first row of M is nonnegative, using Theorem 2.1.1, the matrix obtained by omitting the first row and first column of M must be a Q-matrix. We denote this matrix as B, which has the following sign pattern:

$$B = \begin{pmatrix} * & * \\ + & + \end{pmatrix}.$$

Note that the second row of B is positive. Then, again using Theorem 2.1.1, omitting the second row and second column of B, the remaining matrix must be a Q-matrix. Hence, we conclude that $b_{11} > 0$. That is, $m_{22} > 0$. Hence, we obtain the following sign pattern for M:

$$\begin{pmatrix} 0 & 0 & + \\ + & + & * \\ - & + & + \end{pmatrix}.$$

Next, we check whether the matrix M with this sign pattern is R_0 or not. Suppose $M \notin R_0$. Then, there exists a nonzero solution vector $x \in \mathbb{R}^3_+$, such that $Mx \geq 0$ and $x_i(Mx)_i = 0$. Let $x = (x_1, x_2, x_3)^t$. First, we consider the cases when exactly one coordinate of x is positive:

1. If $x_1 > 0$, then we have $(Mx)_3 < 0$, which is a contradiction. Hence, $x_1 > 0$ is not possible.

2. If $x_2 > 0$, then we have $(Mx)_2 > 0$, which implies $x_2(Mx)_2 > 0$. This is a contradiction to complementarity condition. Therefore, $x_2 > 0$ is not possible.

3. Similarly, if $x_3 > 0$, then we have $(Mx)_3 > 0$, which again gives a contradiction. Therefore, $x_3 > 0$ is not possible.

Now, we consider the cases when exactly two coordinates of x are positive:

1. If $x_1, x_2 > 0$, then we have $(Mx)_2 > 0$. This implies that $x_2(Mx)_2 > 0$. Therefore, $x_1, x_2 > 0$ is not possible.

2. If $x_2, x_3 > 0$, then we have $(Mx)_3 > 0$. This implies that $x_3(Mx)_3 > 0$. Therefore, $x_2, x_3 > 0$ is not possible.

3. Similarly, if $x_1, x_3 > 0$, then we have $(Mx)_1 > 0$. This implies that $x_1(Mx)_1 > 0$. Therefore, $x_1, x_3 > 0$ is not possible.

If all the three coordinates of x are positive, then $(Mx)_1 > 0$. This implies that $x_1(Mx)_1 > 0$, which is a contradiction. Therefore, all three coordinates positive is also not possible. Hence, no nonzero solution x is possible, which implies that $M \in R_0$. Therefore, using Theorem 2.1.12, we can conclude that $A \in R_0$. □

Lemma 3.5.5. *Let $A \in \mathbb{R}^{3\times 3} \cap E_0 \cap Q$. Suppose A has the following sign pattern:*

$$A = \begin{pmatrix} + & + & - \\ + & + & - \\ 0 & + & 0 \end{pmatrix}.$$

Then $A \in R_0$.

Proof. We consider M as the PPT of A with respect to $\alpha = \{2, 3\}$ and proceed with the proof similar to the proof of Lemma 3.5.4. □

Lemma 3.5.6. *Let $A \in \mathbb{R}^{3\times 3} \cap E_0 \cap Q$. Assume that A has the following sign pattern:*

$$\begin{pmatrix} 0 & + & 0 \\ + & + & - \\ - & - & + \end{pmatrix}.$$

Then $A \in R_0$.

Proof. Let M be the PPT of A with $\alpha = \{3\}$. Then, M has the following sign pattern:

$$\begin{pmatrix} 0 & + & 0 \\ * & * & - \\ + & + & + \end{pmatrix}.$$

Observe that the third row of M is positive. Since $M \in Q$, using Theorem 2.1.1, the matrix obtained after omitting the third row and third column of M, say B, is a Q-matrix. Note that the first row of B is nonnegative; hence matrix obtained after omitting the first row and first column of B would be a Q-matrix. Therefore, we have $b_{22} > 0$. If $b_{21} \geq 0$, then the second row of B becomes nonnegative. Then the matrix obtained, by omitting the second row and second column of B, must be a Q-matrix. Since $b_{11} = 0$, this leads to a contradiction that $B \in Q$. Therefore, $b_{21} < 0$, and we have the following sign pattern for M:

$$\begin{pmatrix} 0 & + & 0 \\ - & + & - \\ + & + & + \end{pmatrix}.$$

It can be verified that any matrix with this sign pattern is an R_0-matrix. Therefore, using Theorem 2.1.12, since $M \in R_0$, we conclude that $A \in R_0$. □

Lemma 3.5.7. *Let $A \in \mathbb{R}^{3\times 3} \cap E_0 \cap Q$. Assume A has the following sign pattern:*

$$\begin{pmatrix} 0 & + & 0 \\ + & + & - \\ - & \oplus & + \end{pmatrix}.$$

Then $A \in R_0$.

Proof. Let us put some arbitrary values in A with the given sign pattern. Then, we have

$$A = \begin{pmatrix} 0 & a & 0 \\ b & c & -d \\ -e & f & g \end{pmatrix}.$$

Note that f is nonnegative and $a, b, c, d, e, g, > 0$. Suppose, on the contrary, $A \notin R_0$. Let $x = (x_1, x_2, x_3)$ be a nonzero solution to $LCP(0, A)$. That is, $x \neq 0$, $x_i \geq 0$, $w_i = (Ax)_i \geq 0$ and $x_i w_i = 0$ for $i = 1, 2, 3$. Using Theorem 3.3.1 and Theorem 3.3.2, we can conclude that exactly two coordinates of x can be nonzero. It can be observed that for $A \notin R_0$, the only possibility is $(x_1, x_3) > 0$ when $bg - ed \geq 0$. Consider a vector $q = (-q_1, q_2, q_3)$, where $q_i > 0$ for $i = 1, 2, 3$. It can be verified that $LCP(q, A)$ has no solution along with condition $bg - ed \geq 0$. This leads to a contradiction to $A \in Q$. Therefore, $A \in R_0$. □

Now, we are ready to state and prove our main theorem of this chapter:

Theorem 3.5.8. *Let $A \in \mathbb{R}^{3\times 3} \cap E_0$. Then $A \in Q$ if and only if $A \in R_0$.*

Proof. First, let us assume $A \in E_0 \cap R_0$. Since $A \in R_0$, zero is the only solution to $LCP(0, A)$. Since $A \in E_0$, Theorem 2.1.5 implies that for any $d > 0$, $LCP(d, A)$ has only the unique trivial solution. Hence, using Theorem 2.1.4, $A \in Q$.

Let $A \in E_0 \cap Q$, and we will show that $A \in R_0$. Suppose, on the contrary, $A \notin R_0$; that is, $LCP(0, A)$ has a nonzero solution x. Let $x = (x_1, x_2, x_3)$. It can be concluded from Theorem 3.3.1 and Theorem 3.3.2 that x can have exactly two nonzero coordinates.

Suppose $A \in E_0 \cap Q$. We want to show that $A \in R_0$. Notice if $A \in P_0$, then the result holds by Theorem 3.3.4. Therefore, suppose $A \notin P_0$. If $|A| < 0$ and $|A_{\alpha\alpha}| \geq 0$ for $|\alpha| \leq 2$, then $A \in R_0$ by Theorem 3.3.5. Therefore, suppose $|A_{\alpha\alpha}| \geq 0$ for $|\alpha| \leq 2$ is not true. Then it must be the case that $|A_{\alpha\alpha}| < 0$ for some α such that $|\alpha| = 2$. (Notice that since $A \in E_0$, the diagonal entries are nonnegative, so it is not possible for $|A_{\alpha\alpha}| < 0$ where $|\alpha| = 1$.) Then Theorem 2.1.6 implies that corresponding to that α, $A_{\alpha\alpha} \geq 0$. Hence, from $|A_{\alpha\alpha}| < 0$ for $|\alpha| = 2$ and $A_{\alpha\alpha} \geq 0$, we conclude that off-diagonal entries of $A_{\alpha\alpha}$ are strictly positive.

Without loss of generality, let us assume that $A_{\alpha\alpha}$, $A_{\beta\beta}$, and $A_{\gamma\gamma}$ are the principal submatrices of A of order 2, where $\alpha = \{1, 2\}, \beta = \{2, 3\}$, and $\gamma = \{1, 3\}$. Using Theorem 2.1.5, each of $A_{\alpha\alpha}$, $A_{\beta\beta}$, and $A_{\gamma\gamma}$ are semimonotone matrices.

Without loss of generality, let us assume that $|A_{\alpha\alpha}| < 0$. Then $A_{\alpha\alpha} \geq 0$. $|A_{\alpha\alpha}| < 0$ and $A_{\alpha\alpha} \geq 0$ implies that a_{12} and a_{21} are positive, and the diagonal entries a_{11} and a_{22} are nonnegative. Accordingly, we have the following subcases.

1. *Both diagonals are positive* ($a_{11} > 0$ and $a_{22} > 0$): If $a_{33} > 0$, then $A \in R_0$ follows from Theorem 3.3.6. Suppose $a_{33} = 0$. We consider the following two such cases:

   If $|A_{\beta\beta}| < 0$, then we have $a_{23} > 0$ and $a_{32} > 0$. Hence, the second row of A becomes nonnegative. Using Theorem 2.1.1, omitting the second row and second column, the remaining matrix is Q only if $a_{31} > 0$ (using Remark 2.1.3, any row of a Q-matrix cannot be nonpositive). Hence, the sign pattern of A is as follows:

$$\begin{pmatrix} + & + & * \\ + & + & + \\ + & + & 0 \end{pmatrix}.$$

Observe that the matrix A with such a sign pattern is R_0. (As already seen, if x is a nonzero solution to $LCP(0, A)$, then only two coordinates of x can be nonzero. With the above sign pattern we observe that $(x_1, x_2) > 0$ is not possible from the first row of A, $(x_1, x_3) > 0$ and $(x_2, x_3) > 0$ are not possible from the third row.)
If $|A_{\beta\beta}| \geq 0$, then $a_{23}a_{32} \leq 0$. We have the following sign pattern for A,

$$\begin{pmatrix} + & + & * \\ + & + & * \\ * & * & 0 \end{pmatrix}.$$

Consider the different cases according to the sign of a_{32}.

(a) Let $a_{32} < 0$. Then $a_{23} \geq 0$. Since $A \in Q$, we must have $a_{31} > 0$, because any row of a Q-matrix cannot be nonpositive. It can be verified that matrix A, with this sign pattern, is an R_0-matrix. (As already seen, exactly two coordinates of x can be nonzero. With the above sign pattern we observe that $(x_1, x_2) > 0$ is not possible from the first row, and $(x_2, x_3) > 0$ and $(x_1, x_3) > 0$ are not possible from the third row.)

(b) Let $a_{32} = 0$. Since $A \in Q$, using Remark 2.1.3, we must have $a_{31} > 0$. If $a_{23} \geq 0$, then we can verify that the corresponding matrix A is an R_0-matrix. (Since we cannot get a nonzero solution for $LCP(0, A)$ with exactly two positive coordinates).

Suppose $a_{23} < 0$. If $a_{13} \geq 0$, then the first row becomes nonnegative. Hence, Theorem 2.1.1 leads to a contradiction that $A \in Q$ (because omitting the first row and first column gives us a matrix having a zero row, which cannot be a

Q-matrix). Hence, $a_{13} < 0$, and we have the following sign pattern for A:

$$\begin{pmatrix} + & + & - \\ + & + & - \\ + & 0 & 0 \end{pmatrix}.$$

Using Lemma 3.5.4, we conclude that $A \in R_0$.

(c) Let $a_{32} > 0$. Then $a_{23} \leq 0$. If $a_{13} \geq 0$, then the corresponding matrix A is an R_0-matrix.

Let us assume $a_{13} < 0$. If $a_{23} = 0$, then the second row becomes nonnegative. Using Theorem 2.1.1 and Remark 2.1.3, we conclude that $a_{31} > 0$. Thus, we have the following sign pattern for A:

$$\begin{pmatrix} + & + & - \\ + & + & 0 \\ + & + & 0 \end{pmatrix}.$$

It can be easily verified that such a matrix A is an R_0-matrix.

Now, let $a_{23} < 0$. If $a_{31} \neq 0$, then $LCP(0, A)$ has no solution with two nonzero coordinates, that is, $A \in R_0$. Suppose $a_{31} = 0$. We arrive at a sign pattern which is R_0 by Lemma 3.5.5.

2. *One diagonal is zero and one positive* (say $a_{11} = 0$ and $a_{22} > 0$): Since $A \in E_0 \cap Q$, Theorem 3.3.3 implies that $a_{31} < 0$. Additinally, corollary 2.1.7 implies that $a_{13} \geq 0$. If $|A_{\beta\beta}| < 0$, then $a_{23} > 0$ and $a_{32} > 0$. Hence, the second row becomes nonnegative. Using Theorem 2.1.1, we further conclude that $a_{13} > 0$ and $a_{33} > 0$. With this sign pattern for A, we can establish that A is an R_0-matrix.

Now, suppose $|A_{\beta\beta}| \geq 0$. If $a_{33} = 0$, then $a_{32} > 0$ (since any row of a Q-matrix cannot be nonpositive). We have the following sign pattern for A:

$$\begin{pmatrix} 0 & + & \oplus \\ + & + & * \\ - & + & 0 \end{pmatrix}.$$

It can be verified that with this sign pattern, $LCP(0, A)$ has no nonzero solution with exactly two nonzero coordinates. Therefore, $A \in R_0$.
If $a_{33} > 0$, then we consider the cases for the sign of a_{13}:

(a) Suppose $a_{13} > 0$. Thus, we have the following sign pattern for A:

$$\begin{pmatrix} 0 & + & + \\ + & + & * \\ - & * & + \end{pmatrix}.$$

Observe that the first row of A is nonnegative. Using Theorem 2.1.1, we can conclude that the matrix (say B) obtained by omitting the first row and first column must be a Q-matrix. Using Theorem 3.5.3, we can further conclude that this matrix (B of order 2) is an R_0-matrix. We can easily verify that the remaining principal submatrices of order 2 are also R_0. Therefore, using Theorem 3.3.6, $A \in R_0$.

(b) Suppose $a_{13} = 0$. If $a_{23} \geq 0$, then the second row becomes nonnegative. On omitting the second row and second column, we obtain a matrix that is not Q. This contradicts Theorem 2.1.1. Hence, $a_{23} < 0$. Therefore, we have the following sign pattern for A

$$\begin{pmatrix} 0 & + & 0 \\ + & + & - \\ - & * & + \end{pmatrix}.$$

If $a_{32} \geq 0$, then Lemma 3.5.7 implies that $A \in R_0$. If $a_{32} < 0$, then Lemma 3.5.6 implies that $A \in R_0$. Therefore, $A \in R_0$.

3. *Both diagonals are zero* ($a_{11} = 0$ and $a_{22} = 0$): Since a_{12} and a_{21} are positive, Theorem 3.3.3 implies $a_{32} < 0$ and $a_{31} < 0$ respectively. Since $A \in Q$, using Remark 2.1.3, we can conclude that $a_{33} > 0$. Since $A \in E_0$ and $a_{32} < 0$, Corollary 2.1.7 implies $a_{23} \geq 0$. Hence, the second row becomes nonnegative. Now, on omitting the second row and second column, the remaining matrix will be a Q-matrix only if $a_{13} > 0$. Moreover, since the first row is nonnegative, the resulting matrix, on omitting the first row and first column is a Q-matrix only if $a_{23} > 0$. Hence, we have the following sign pattern for A:

$$\begin{pmatrix} 0 & + & + \\ + & 0 & + \\ - & - & + \end{pmatrix}.$$

It can be easily verified that such a matrix is an R_0-matrix.

Thus, we have proved that if $A \in \mathbb{R}^{n \times n} \cap E_0$, then $A \in Q$ if and only if $A \in R_0$ for $n \leq 3$. □

3.6 A Counterexample

Jeter and Pye [20] provided an example of order 5 (Example 3.3), which shows that the above result does not hold. We now observe that the above theorem is also not valid for matrices of order 4. The following example demonstrates this.

Example 3.5. *Consider the matrix*

$$A = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

Proof. Let $x = (x_1, x_2, x_3, x_4)$ be any nonzero nonnegative vector. If either x_1 or x_2 is positive, then $(Ax)_1$ and $(Ax)_2$ are nonnegative (as the first and second rows are nonnegative). If x_1 and x_2 are both zero, and either x_3 or x_4 is positive, then we have $(Ax)_3$ and $(Ax)_4$ to be nonnegative (as $A_{\alpha\alpha}$ is a nonnegative matrix for $\alpha = \{3, 4\}$). Hence, in each case, there exists some i such that $x_i > 0$ and $(Ax)_i \geq 0$. Therefore, $A \in E_0$.

Since $A \in E_0$, every principal submatrix $A_{\alpha\alpha} \in E_0$. Consider $\alpha = \{2, 3, 4\}$. Then we have

$$A_{\alpha\alpha} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Let $y = (y_1, y_2, y_3)^t$ be a solution of $LCP(0, A_{\alpha\alpha})$. We observe the following cases:

1. Exactly one coordinate of y cannot be positive as $y_1 > 0$ gives $(A_{\alpha\alpha}y)_2 < 0$, $y_2 > 0$ gives $y_2(A_{\alpha\alpha}y)_2 > 0$, and $y_3 > 0$ gives $y_3(A_{\alpha\alpha}y)_3 > 0$. These contradicts the conditions of LCP.

2. Exactly two coordinates of y cannot be positive as $y_1, y_2 > 0$ or $y_1, y_3 > 0$ gives $y_1(A_{\alpha\alpha}y)_1 > 0$, and $y_2, y_3 > 0$ gives $y_2(A_{\alpha\alpha}y)_2 > 0$. These contradicts the complementarity condition.

3. $(y_1, y_2, y_3)^t > 0$ is not possible, because it contradicts the complementarity condition, as it gives $y_1(A_{\alpha\alpha}y)_1 > 0$.

Therefore, y has to be a zero vector. Hence, $A_{\alpha\alpha} \in R_0$, and Theorem 3.5.8 implies $A_{\alpha\alpha} \in Q$. Similarly, $A_{\beta\beta} \in Q$ for $\beta = \{1, 3, 4\}$. Since the first two rows of matrix A

are identical, using Theorem 3.2.1, we can conclude that $A \in Q$. Therefore, $A \in E_0 \cap Q$. However, $A \notin R_0$ because we have a nonzero solution $z = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ for $LCP(0, A)$. $\square$

In fact, we can construct an example to show that Theorem 3.5.8 may not hold for matrices of order 5 or more.

Example 3.6. *Consider*

$$M = \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0}^t & I_{n-4} \end{pmatrix},$$

where A is the matrix given in Example 3.6, $\boldsymbol{0}$ is a null matrix of order $4 \times n-4$ and I_{n-4} is the identity matrix of order $n-4$; $(n \geq 5)$. It can be shown that $M \in E_0 \cap Q$, but $M \notin R_0$. $\square$

CHAPTER 4

Completely Mixed Games

4.1 Introduction

In 1945, Kaplansky introduced the concept of completely mixed games. In [21], he provided many characterizations for a matrix game to be completely mixed. Furthermore, he raised a question as to when a game associated with a skew symmetric matrix of odd order can be completely mixed. Almost half a century later, he himself provided a condition for an odd-ordered skew symmetric matrix to be completely mixed using the concept of Pfaffian of a matrix. In this chapter, we revisit Kaplansky's question and provide a new characterization for an odd-ordered symmetric game to be completely mixed, using the concept of the linear complementarity problem.

Completely mixed games have numerous applications in the literature. In [41], Parthasarathy and Ravindran used the concept of completely mixed games in proving global univalence theorems. Parthasarathy, Sharma, and Sricharan, [45], have extended the results of Kaplansky to skew symmetric bimatrix games. Gowda and Ravindran [13, 15] generalized these concepts to completely mixed games on self dual cones.

In this chapter, we aim to provide new characterizations for a game to be completely mixed. Characterizing a Q-matrix that arises in the linear complementarity problem, has evoked a lot of interest among the researchers. In this chapter, we provide another characterization of Kaplansky's result for symmetric games that connects Q-matrices and completely mixed games. In particular, we prove that a symmetric game associated with matrix A of odd-order is completely mixed if and only if, for all $i = 1, 2, \ldots, n$, $A + D_i$ is a Q-matrix, where D_i is a diagonal matrix whose i^{th} diagonal entry is 1, else zero. For results related to LCP, one may refer to [6, 14, 38, 56].

This chapter is organized as follows: In Section 2, we provide some basic definitions and results that are needed in the sequel. In Section 3, we describe our result for a game with value zero to be completely mixed. In Section 4, we prove another characterization for a symmetric game to be completely mixed. In Section 5 and 6, we expand our results of section 4 for almost skew symmetric and semimonotone matrices, respectively.

4.2 Preliminaries

In this section, we present a few definitions and related results that are used in later sections.

Definition 4.2.1. A matrix A is said to be an S-matrix if there exists a vector $x > 0$ such that $Ax > 0$. If A is an S-matrix, then $val(A)$ is positive.

The following are some results from Kaplansky [21]:

- A game with matrix $A \in \mathbb{R}^{m\times n}$ and $val(A) = 0$ is completely mixed if and only if $m = n$, the rank of A is $n - 1$, and all cofactors of A are nonzero and have same sign.
- If A is any square matrix and the game is not completely mixed, then both players have optimal strategies that are not completely mixed.
- If player 2 has a completely mixed optimal strategy, then $p^tA = ve^t$ for every optimal strategy p of player 1.
- Kaplansky also gave conditions for symmetric games of order 3 and order 5 to be completely mixed.

From Kaplansky's results in [21], the following remark can be concluded:

Remark 4.2.2. Let $A \in \mathbb{R}^{n\times n}$ and the game associated with A is completely mixed. Then the games associated with $-A$ and A^t are also completely mixed.

The question of characterizing symmetric games of odd order to be completely mixed remained open until 1995, when Kaplansky [22], proved the following theorem:

Theorem 4.2.3. *Let $A \in \mathbb{R}^{n \times n}$, where n is odd. If A is a skew symmetric matrix, then following conditions are equivalent:*

1. *The game associated with matrix A is completely mixed.*
2. *The principal Pfaffians $p_1, p_2, \ldots, p_n$ are all non-zero and alternate in sign.*

It is known that even-ordered skew symmetric matrices can never be completely mixed. This follows from a remark in [21] on symmetric games. An example can be seen as follows:

Example 4.1. Let A be a skew symmetric matrix of even order as follows:

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Here, $x = (1, 0)$ and $y = (1, 0)$ form an optimal strategy pair. Hence, the game associated with matrix A is not completely mixed.

Remark 4.2.4. [6] It is known that the skew symmetric matrices are positive semi-definite matrices, hence are in $Q_0 \cap P_0$.

The following results were stated in [6] with the concept of S-matrix. But here we restate a few results needed in our proof using the concept of value of game. We need the following lemma, due to Murthy, Parthasarathy, and Ravindran [33].

Lemma 4.2.5. *If $A \in \mathbb{R}^{n \times n} \cap Q$, then $val(A) > 0$.*

Lemma 4.2.6. *If $A \in \mathbb{R}^{n \times n} \cap Q_0$ and $val(A) > 0$, then $A \in Q$.*

Remark 4.2.7. It is known that, if $A \in \mathbb{R}^{n \times n} \cap Q_0 \cap E_0$, then $val(A) \geq 0$.

Lemma 4.2.8. If $A \in \mathbb{R}^{n \times n} \cap P_0 \cap Q_0$ and $A \notin Q$, then $val(A) = 0$.

Proof. Let $A \in \mathbb{R}^{n \times n} \cap P_0 \cap Q_0 \setminus Q$. Since the class of P_0-matrices is contained in the class of semimonotone matrices. Hence, $A \in Q_0 \cap E_0$. Then, $val(A) \geq 0$. Since $A \notin Q$, using Lemma 4.2.6, we have $val(A) \not> 0$. Therefore, $val(A) = 0$. □

4.2.1 Some Results on Pfaffians of a Matrix

From the definition of Pfaffian in section 1.4, we know that for any skew symmetric matrix $A \in \mathbb{R}^{2n\times 2n}$, the Pfaffian is given by

$$Pf(A) = \sum_{\alpha\in\Pi} sgn(\Pi_\alpha) a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n}, \text{ where } i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n \in \overline{2n}.$$

The set {1,2,...,2n} is partitioned into pairs $\{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$ such that $i_1 < i_2 < \cdots < i_n$ and $i_k < j_k$ for all $1 \leq k \leq n$. Let

$$\Pi_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \dots & i_n & j_n \end{pmatrix}$$

be the permutation corresponding to the partition $\alpha = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$. Let Π be the set of all partitions of {1,2,...,2n} into pairs regardless of order.

A diagonal matrix S with entries either 1 or -1 is known as a signature matrix. If a signature matrix S has i^{th} diagonal entry negative and else positive, then SAS changes the sign of each entry in i^{th} row and i^{th} column, except the sign of a_{ii}. Now, the following observations for Pfaffians are noteworthy:

1. Since $j_k > i_k$, every entry in each term of the Pfaffian expression is from the upper triangular.

2. Consider any row l. Then each term in the summation of the Pfaffian expression has one element either from l^{th} row or from l^{th} column. That is, each term of $Pf(A)$ has a_{il} for $i < l$ or a_{lj} for $j > l$ as an element for some i or j.

3. Consider SAS, where S is a signature matrix that has l^{th} diagonal entry as -1 and else 1. Then the Pfaffian of the resulting matrix is negative of the Pfaffian of A. That is, $Pf(SAS) = -Pf(A)$.

Remark 4.2.9. Let $A \in \mathbb{R}^{n\times n}$, where n is odd. Principal Pfaffians are defined for odd-ordered skew symmetric matrices. The principal Pfaffian p_i is the Pfaffian of matrix obtained by deleting i^{th} row and i^{th} column, where $i = 1, 2, \ldots, n$. Then, we have the following observations:

1. $p_k = Pf(A_{kk}) = \sum_{\alpha \in \Pi} sgn(\Pi_\alpha) a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_n j_n}$, where $i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n \in \bar{n} \setminus \{k\}$.

2. Changing any element in the k^{th} row or k^{th} column does not affect the principal Pfaffian p_k.

3. Consider SAS, where S has l^{th} diagonal entry as -1 and else 1. Then it will change the nature of each principal Pfaffian except P_l. That is, $Pf((SAS)_{ii}) = -Pf(A_{ii})$ for each $i \neq l$ and $Pf((SAS)_{ll}) = Pf(A_{ll})$.

4.3 A Completely Mixed Zero-Value Game

Kaplansky [21] proved the following theorem:

Theorem 4.3.1. *A game associated with a matrix $A \in \mathbb{R}^{m \times n}$ with value zero is completely mixed if and only if A is a square matrix with rank $n - 1$ and all cofactors are nonzero and have same sign.*

Here, we provide another characterization for a game with value zero to be completely mixed [42].

Theorem 4.3.2. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix. Further, suppose that the game associated with A has the value zero. Then the following conditions are equivalent:*

1. *The game associated with matrix A is completely mixed.*

2. *For all $i = 1, 2, \ldots, n$, $val(A + D_i) > 0$, where D_i is a diagonal matrix whose i^{th} diagonal entry is 1 and else 0.*

Proof. $(1 \Rightarrow 2)$ Let the game associated with matrix A be completely mixed. Let $A = [a_{ij}]$ and $A + D_i = [b_{ij}]$ where $i, j \in \{1, 2, \ldots, n\}$. Observe that $a_{ij} \leq b_{ij}$ for all $i, j = 1, 2, \ldots, n$. That is, entrywise $A \leq A + D_i$. Hence, $0 = val(A) \leq val(A + D_i)$. We need to show

$val(A+D_i) > 0$ for all $i = 1, 2, \ldots, n$. On the contrary, WLOG suppose $val(A+D_1) = 0$. Let $u = (u_1, u_2, \ldots, u_n)^t$ be a strategy such that $u^t(A+D_1) \leq 0$. Then

$$u^t(A+D_1)_{.k} = \begin{cases} u^t A_{.k} + u_k, & \text{for } k = 1; \\ u^t A_{.k}, & \text{for } k \neq 1. \end{cases} \tag{4.1}$$

1. From (4.1), if $u_1 > 0$, then we have $u^t A_{.1} < 0$ and $u^t A_{.j} \leq 0$ for $j \neq 1$. Since the game A is completely mixed with value zero, for any optimal strategy u, $u^t A = 0$. This leads to a contradiction.

2. If $u_1 = 0$, then $u^t A \leq 0$. This also contradicts that A is completely mixed.

Therefore, $val(A+D_1) > 0$. Similarly, we can show that $val(A+D_i) > 0$ for all $i = 1, 2, \ldots, n$.

$(2 \Rightarrow 1)$ Let us assume $val(A+D_i) > 0$ for all $i = 1, 2, \ldots, n$. We show that the game associated with matrix A is completely mixed.

Suppose the game associated with matrix A is not completely mixed, then there exists a probability vector x with some coordinates zero, say $x_{i_0} = 0$ for some i_0. Since $val(A) = 0$, we have $x^t A \leq 0$.

Consider $x^t(A+D_{i_0})$. We have $x^t(A+D_{i_0}) = x^t A \leq 0$. Hence, $val(A+D_{i_0}) = 0$. This is a contradiction to our hypothesis that $val(A+D_i) > 0$ for all $i = 1, 2, \ldots, n$. Therefore, the game associated with A is completely mixed.

□

Remark 4.3.3. The above result provides a new characterization to Kaplansky's result from 1945, nearly 80 years later, which can be very useful. This result uses the diagonal perturbation. Here, in particular, a singular P_0-matrix whose proper principal minors are positive and with value zero will also satisfy this property, namely $val(A+D_i) > 0$ for all $i = 1, 2, \ldots, n$.

It is known that the game associated with a Q-matrix has positive value. The above theorem states that if the value of a game associated with a matrix is zero and the game is completely mixed, then $val(A + D_i) > 0$ for all i. Can we say that $A + D_i \in Q$?

The following example shows, in general, the answer is negative.

Example 4.2. Let

$$A = \begin{pmatrix} -2 & 0 & 2 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

It is easy to verify that the matrix game A is completely mixed with a value of zero. In fact, the optimal strategies $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$ and $y = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})^t$ yield $Ax = 0$ and $y^tA = 0$. Hence, the value of the game associated with A is zero.

Moreover, all the cofactors of A are positive, indicating that the game associated with A is completely mixed. However, it should be observed that for $q = (-2, -3, 3)^t$, there exists no solution for $LCP(q, A + D_3)$. Therefore, $A + D_3 \notin Q$. □

4.4 A Completely Mixed Symmetric Game

Kaplansky proved that a game associated with an even-ordered skew symmetric matrix cannot be completely mixed. However, for games associated with odd-ordered skew symmetric matrices, Kaplansky proved that the game is completely mixed if and only if the principal Pfaffians are alternate in sign. In the following theorem, we provide another characterization for a symmetric game to be completely mixed.

Theorem 4.4.1. *Let $A \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix, where n is odd. Then the following conditions are equivalent:*

1. *The principal Pfaffians $p_1, p_2, \ldots, p_n$ are all non-zero and alternate in sign.*

2. *The game associated with matrix A is completely mixed.*

3. *For all $i = 1, 2, \ldots, n$, $val(A + D_i) > 0$, where D_i is a diagonal matrix whose i^{th} diagonal entry is 1 and else 0.*

4. *For all* $i = 1, 2, \ldots, n$*,* $A + D_i \in Q$*, where* D_i *is a diagonal matrix whose* i^{th} *diagonal entry is 1 and else 0.*

Proof. $(1 \Leftrightarrow 2)$ The proof is by Kaplansky [22].

$(2 \Rightarrow 3)$ Follows from Theorem 4.3.2.

$(3 \Rightarrow 4)$ Let $val(A + D_i) > 0$, for all $i = 1, 2, \ldots, n$. Let $x = (x_1, x_2, \ldots, x_n)^t$ be any vector. Since A is a skew symmetric matrix, $x^t Ax = 0$. Consider $x^t(A + D_i)x$, and observe that

$$x^t(A + D_i)x = x_i^2 + x^t Ax = x_i^2 \geq 0.$$

Hence, $A + D_i$ is a positive semi definite matrix, and $A + D_i \in Q_0$. Since, $val(A + D_i) > 0$, using Lemma 4.2.6, $A + D_i \in Q$ for all $i = 1, 2, .., n$.

$(4 \Rightarrow 2)$ Let us assume $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$. We show that the game associated with matrix A is completely mixed.

On the contrary, suppose that the game associated with matrix A is not completely mixed. Then, there exists a probability vector x with some coordinates equal to zero, such that $Ax \geq 0$ (because for skew symmetric matrix A, $val(A) = 0$). Let $x_{i_0} = 0$ for some i_0. Then, consider $LCP(0, A + D_{i_0})$,

$$w = (A + D_{i_0})x = Ax + D_{i_0}x = Ax \geq 0. \quad \text{(because } D_{i_0}x = 0\text{)}$$
$$\Rightarrow \; x^t w = x^t(A + D_{i_0})x = x^t Ax = 0.$$

Hence, x is a non-zero solution to $LCP(0, A + D_{i_0})$. Therefore, $A + D_{i_0} \notin R_0$. Since $A + D_{i_0} \in P_0$, Theorem 3.3.4 implies that $A + D_{i_0}$ cannot be a Q-matrix, but this is a contradiction to our assumption. Since i_0 was arbitrary, x_{i_0} has to be positive for all $i_0 = 1, 2, \ldots, n$. Therefore, the game associated with matrix A is completely mixed. □

In Theorem 4.2.3, it was shown that, for the symmetric game associated with odd-ordered matrix A to be completely mixed if and only if the principal Pfaffians of A must be nonzero and alternate in sign. The next theorem states that if the principal Pfaffians of A are nonzero, then there exists a signature matrix S such that the game associated with SAS is completely mixed.

Theorem 4.4.2. *Let $A \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix of odd-order. If all the principal Pfaffians of A are nonzero, then there exists a signature matrix S such that the game associated with the matrix SAS is completely mixed.*

Proof. Let A be a skew symmetric matrix of odd-order and $p_1, p_2, \dots, p_n$ be the principal Pfaffians, such that $p_i \neq 0$ for all $i = 1, 2, \dots, n$.

If $p_1, p_2, \dots, p_n$ are alternate in sign, then using Theorem 4.2.3, we can conclude that the game associated with SAS is completely mixed, where S is an identity matrix.

Now, suppose that some of the consecutive principal Pfaffians have the same sign. Starting from p_1, suppose up to p_{i-1} are alternate in sign, and p_{i-1} and p_i have the same sign. Then consider the signature matrix S' that has i^{th} entry as -1 and else 1. Using remark 4.2.9, observe that $Pf((S'AS')_{jj}) = -p_j$ for all $j \neq i$, and $Pf((S'AS')_{ii}) = p_i$. Now, the first i principal Pfaffians of $S'AS'$ are alternate in sign.

Now, again observe the principal Pfaffians of $S'AS'$. If all are alternate in sign, then using Theorem 4.2.3, we can conclude that the game associated with $S'AS'$ is completely mixed. Otherwise, there is some j such that $Pf((S'AS')_{j-1\, j-1})$ and $Pf((S'AS')_{jj})$ have same sign for $j > i$. Now change the j^{th} entry of S' from 1 to -1 (resulting in a new matrix say S''). Using remark 4.2.9, observe that $Pf((S''AS'')_{kk}) = -Pf((S'AS')_{kk})$ for all $k \neq j$, and $Pf((S''AS'')_{jj}) = Pf((S'AS')_{jj})$. Now, the first j principal Pfaffians of $S''AS''$ are alternate in sign.

Proceeding in the same way, we obtain a signature matrix S, such that all the principal Pfaffians of SAS are alternate in sign. Hence the game associated with SAS is a completely mixed game. □

4.5 Almost Skew Symmetric Matrices with Positive Value

We know that if A is a skew symmetric matrix, then $A + A^t = 0$. It is also known that a skew symmetric matrix A is positive semi-definite matrix and $A \in P_0 \cap Q_0$. Now, consider a matrix A with $rank(A + A^t) = 1$; such a matrix is called an almost skew symmetric matrix. For further results on almost skew symmetric matrices, refer to McDonald, Psarrakos and Tsatsomeros [27].

Lemma 4.5.1. *Let* $A \in \mathbb{R}^{n\times n}$ *such that* $rank(A + A^t) = 1$ *and* $val(A) > 0$. *Then* $A \in P_0 \cap Q$.

Proof. Since $rank(A+A^t) = 1$ and $val(A) > 0$, it is clear that there exists an $x \in \mathbb{R}^n \setminus \{0\}$ such that $A + A^t = xx^t$.

Hence, for all $y \in \mathbb{R}^n$, $y^t(A+A^t)y = y^t(xx^t)y = y^tx(y^tx)^t \geq 0$. Therefore, $A+A^t$ is a positive semi-definite matrix, and hence A is positive semi definite. Therefore, $A \in P_0 \cap Q_0$. Since $val(A) > 0$, using Lemma 4.2.6, we can conclude that $A \in Q$. □

However, if $val(A) \not> 0$, then we cannot conclude that $A + A^t = xx^t$, as shown in the following example:

Example 4.3.

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Observe that $val(A) = 0$ and $A + A^t = -xx^t$, where $x = (-\sqrt{2}, \sqrt{2})$.

In the above lemma, we have $A \in Q$. In general, $A \in Q$ does not imply $A + D_i \in Q$, which can be seen from the following example:

Example 4.4.

$$A = \begin{bmatrix} -1/2 & 1 \\ 1 & -1 \end{bmatrix}$$

Here we can easily check that $A \in Q$. But $A + D_1 \notin Q$.

We know that the games associated with Q-matrices have positive values. In the next theorem, we prove that for almost skew symmetric matrices, $val(A) > 0$ implies that $A + D_i \in Q$ for all i.

Theorem 4.5.2. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that $rank(A + A^t) = 1$. If $val(A) > 0$, then $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$.*

Proof. Let $rank(A + A^t) = 1$ and $val(A) > 0$. Using Lemma 4.5.1, we observe that A is a positive semi definite matrix and $A \in P_0 \cap Q$. Using Theorem 3.3.4, we conclude $A \in R_0$. Notice that $A + D_i \in P_0$, since $A \in P_0$.

On the contrary, suppose $A + D_i \notin Q$ for some i. Then, by Theorem 3.3.4, $A + D_i \notin R_0$. Hence, there exists a nonzero vector $x \geq 0$, such that

$$(A + D_i)x = w \geq 0 \text{ and } x^t w = 0.$$

Two cases, x_i may be zero or positive, arise. We will see these cases one by one:

(i) If $x_i = 0$, then $w_i = (A + D_i)_{i.}x = A_{i.}x + D_{i_{i.}}x_i = A_{i.}x \geq 0$. For all $j \neq i$, we have $w_j = (A + D_i)_{j.}x = A_{j.}x + D_{i_{j.}}x_j = A_{j.}x$, because $D_{i_{j.}} = 0$ (j^{th} row of matrix D_i is zero). This implies that the nonzero vector x is a solution to $LCP(0, A)$. This leads to a contradiction to $A \in R_0$. Hence, $x_i \neq 0$.

(ii) If $x_i > 0$,then $w_i = (A + D_i)_{i.}x = A_{i.}x + D_{i_{i.}}x_i = A_{i.}x + x_i = 0$, which implies that $A_{i.}x = -x_i < 0$.

For all k, $k \neq i$ and $x_k > 0$, we have $w_k = A_{k.}x = 0$. Hence, for $\alpha = supp(x)$, we have

$$A_{\alpha\alpha}x_\alpha = \begin{pmatrix} 0, & 0, & \ldots & -x_i, & \ldots & 0 \end{pmatrix}^t.$$

Therefore, $x_\alpha^t A_{\alpha\alpha} x_\alpha = 0 + 0 + .. + (x_i)(-x_i) + .. + 0 = -x_i^2 < 0$. Let us write A and x in block form:

$$A = \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{pmatrix}, \; x^t = (x_\alpha^t, x_{\bar{\alpha}}^t) \text{ and } x_{\bar{\alpha}} = 0.$$

This implies $x^tAx = -x_i^2 < 0$, which means A is not positive semi-definite, which leads to a contradiction. Hence, $A + D_i \in R_0$. Therefore, Theorem 3.3.4 implies $A + D_i \in Q$. □

However, the converse of the above theorem is not true. The following example shows that an almost skew symmetric matrix A such that $A + D_i \in Q$ for all i does not imply that value of A is positive.

Example 4.5.

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

Here, we can easily check that rank of $A + A^t$ is one and $A + D_i \in Q$ for $i = 1, 2$. Observe that the value of A is zero.

4.6 Completely Mixed Games Associated to E_0-Matrices

We know that skew symmetric matrices are P_0, hence, semimonotone. In Theorem 4.4.1, we established the connection between completely mixed symmetric games of odd order and Q-matrices. In this section, we aim to extend Theorem 4.4.1 for semimonotone matrices. We know that the game associated with an E_0-matrix has a nonnegative value and the game associated with a Q-matrix has a positive value. But the value of a game associated with a matrix being positive does not imply that the matrix is a Q-matrix. From Theorem 4.3.2, we can conclude that if a game associated with a semimonotone matrix is completely mixed, then adding 1 to any of its diagonal entries results in a matrix whose associated game has a positive value. The following theorem further establishes that the resulting matrix is a Q-matrix.

Theorem 4.6.1. *Let $A \in \mathbb{R}^{3\times 3} \cap E_0$. Assume that the value of game associated with A is zero. Then the game associated with A is completely mixed if and only if $A + D_i \in Q$ for all $i = 1, 2, 3$, where D_i is a diagonal matrix whose i^{th} entry is 1 and else are zero.*

Proof. Let's assume, if possible, $A+D_1 \notin Q$. Since $A \in E_0$, $A+D_1$ is also a semimonotone matrix. Using Theorem 3.5.8, we can conclude that $A + D_1 \notin R_0$. Then, there exists some nonzero vector $z \geq 0$, such that $(A + D_1)z = w \geq 0$ and $z^t w = 0$.

Case-1) When only one coordinate of z is positive:

1. When $z_1 > 0$,

$$(A + D_1) \begin{pmatrix} z_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \oplus \\ \oplus \end{pmatrix}.$$

   Here, we can observe from the first row that

$$a_{11} z_1 + z_1 = 0.$$

   Since, $z_1 > 0$, it implies that $a_{11} < 0$, but this is not possible for $A \in E_0$.

2. When either z_2 or z_3 is positive, we have $(A + D_1)z = Az \geq 0$. It is a contradiction to the fact that A is completely mixed.

Therefore, one coordinate of z cannot be positive.

Case-2) When two coordinates are positive.

1. When $z_1 > 0$ and one of z_2 or z_3 are positive (let's say z_2). Then, we have

$$(A + D_1)z = \begin{pmatrix} 0 \\ 0 \\ \oplus \end{pmatrix} = Az + \begin{pmatrix} z_1 \\ 0 \\ 0 \end{pmatrix}.$$

   Hence, the sign pattern of Az is as follows:

$$Az = \begin{pmatrix} - \\ 0 \\ \oplus \end{pmatrix}.$$

Now, assume $w_3 = 0$. Then, we have $Az = (-, 0, 0)^t$, which implies that $-Az = (+, 0, 0)^t$. Therefore, $-A$ is not completely mixed. This is a contradiction to remark 4.2.2. Therefore, $w_3 > 0$. Hence, we have

$$\begin{pmatrix} a_{11}+1 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We observe that $a_{11}z_1 + z_1 + a_{12}z_2 = 0$, which implies that $a_{12} < 0$, because $a_{11} \geq 0, z_1 > 0, z_2 > 0$. We have $a_{22} + A_{33} = 0$; where A_{33} is a cofactor of matrix A, obtained by omitting the third row and third column of A. Here, we observe that $a_{22} \neq 0$; otherwise, $A_{33} = 0$. This is a contradiction to Theorem 4.3.1. Hence, $a_{22} > 0$. Using $a_{22} > 0$, and $a_{21}z_1 + a_{22}z_2 = 0$, we can conclude that $a_{21} < 0$. Therefore, the submatrix has the following sign-pattern:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \oplus & - \\ - & + \end{pmatrix}$$

and its determinant is negative, that is $A_{33} < 0$. This is a contradiction to the Theorem 2.1.6. Therefore, $z_1 > 0$ and one of z_2 or z_3 being positive is not possible.

2. When $z_1 = 0$, and both z_2 and z_3 are positive. Observe that $(A + D_1)z = Az = (\oplus, 0, 0)^t$. This is a contradiction to A being a completely mixed game with value zero.

Case-3) When all three coordinates are positive. It results in $(A + D_1)z = (0, 0, 0)^t$, that is $Az = (-, 0, 0)^t$, which implies $z^tA^t = (-, 0, 0)$. Since A is completely mixed with value zero, using remark 4.2.2, A^t is also completely mixed with value zero. This is a contradiction to A^t being completely mixed with value zero.

Therefore, $A + D_1 \in R_0$ and hence $A + D_1 \in Q$. Similarly, we can show that $A + D_i \in Q$ for all $i = 1, 2, 3$.

Now for the converse, let us assume $A \in \mathbb{R}^{3\times 3} \cap E_0$, $val(A) = 0$, and $A + D_i \in Q$ for all $i = 1, 2, 3$. We want to show that A is completely mixed. Since $A + D_i \in Q$, $val(A + D_i)$

is positive. Notice that $val(A) = 0$, A cannot be a Q-matrix. From Theorem 3.5.8, we can conclude that $A \notin R_0$. That is, there exists a nonzero vector $z = (z_1, z_2, z_3) \geq 0$, such that $Az = w \geq 0$ and $z^t w = 0$.

1. Let z be positive. Suppose A is not completely mixed. Since value of A is zero, there exists an optimal strategy u that is not completely mixed. Hence, $u^t A = (0, 0, 0)$. Let $u = (u_1, u_2, u_3)^t$ with $u_k = 0$ for some $k = 1, 2, 3$. Suppose $k = 1$, then $u^t(A + D_1) = (0, 0, 0)$ implies that $val(A + D_1) = 0$. This leads to a contradiction.

2. Let z has some entry zero, that is, z is not completely mixed. Since A is not completely mixed and $val(A) = 0$, there exists a strategy u that is not completely mixed, say $u = (u_1, u_2, 0)^t$. Hence, $u^t A \geq 0$. Observe that $u^t(A + D_3) \geq 0$. Therefore, $val(A + D_3) \geq 0$, which leads to a contradiction to $A + D_i \in Q$ for all $i = 1, 2, 3$. Hence, A is completely mixed.

□

The following three examples show the sharpness of the hypothesis in Theorem 4.6.1. If the game is not completely mixed, then the result does not hold true.

Example 4.6. *Let A be the matrix given by*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that $A \in E_0$, $val(A) = 0$. We observe that for any $i = 1, 2, 3$, $A + D_i$ is not a Q-matrix, since there exists no solution for any vector $q \in \mathbb{R}^n$, such that $q_i > 0$, and $q_j < 0$ for $j \neq i$. □

If the value of the game is not zero, then result in Theorem 4.6.1 does not hold.

Example 4.7. *Let*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Observe that $A \in E_0$, $val(A) > 0$, and $A + D_i \in Q$ for all $i = 1, 2, 3$. Here, the game associated with matrix A is not completely mixed. □

If the matrix is not an E_0-matrix, then Theorem 4.6.1 does not hold true.

Example 4.8. *Let*

$$A = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{pmatrix}.$$

Observe that A is completely mixed and $val(A) = 0$ (for a strategy $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$, we have $Ax = 0$, and for a strategy $y = (\frac{4}{7}, \frac{1}{7}, \frac{2}{7})^t$, we have $y^t A = 0$, hence the value of game is 0). Since the diagonal entries are negative, A is not E_0-matrix. Observe that $LCP(q, A+D_1)$ has no solution for a vector $q = (1, -2, -1)^t$. Therefore, $A+D_1 \notin Q$. □

The result in Theorem 4.6.1 does not apply to the matrices of order 4. The following example serves the purpose as a counterexample.

Example 4.9. *Let A be the matrix given by*

$$A = \begin{pmatrix} 0 & \frac{-1}{5} & \frac{2}{5} & \frac{-1}{5} \\ 0 & 0 & \frac{1}{5} & \frac{-9}{5} \\ -1 & \frac{1}{5} & \frac{1}{5} & \frac{-1}{5} \\ \frac{3}{5} & 0 & \frac{-2}{5} & \frac{3}{5} \end{pmatrix}.$$

Proof. Observe that A is a semimonotone matrix. Let there are some vectors $x = (\frac{5}{32}, \frac{17}{32}, \frac{9}{32}, \frac{1}{32})^t$ and $y = (\frac{3}{12}, \frac{1}{12}, \frac{3}{12}, \frac{5}{12})^t$. Observe that $Ax = 0$ and $y^t A = 0$. Hence, the value of the game associated with A is zero.

Now, the cofactor of each element of the matrix A are as follows:

$$Cofactor(A) = \begin{pmatrix} \frac{3}{25} & \frac{51}{125} & \frac{27}{125} & \frac{3}{125} \\ \frac{1}{25} & \frac{17}{125} & \frac{9}{125} & \frac{1}{125} \\ \frac{3}{25} & \frac{51}{125} & \frac{27}{125} & \frac{3}{125} \\ \frac{1}{5} & \frac{17}{25} & \frac{9}{25} & \frac{1}{25} \end{pmatrix}.$$

All cofactors are positive, hence the game associated with A is completely mixed.

However, for some vector $q = (10, -2, -3, -8)^t$, there exists no solution for $LCP(q, A+D_1)$. Therefore, $A + D_1 \notin Q$. $\square$

CHAPTER 5

Fully Semimonotone, Z-Matrices, and Completely Mixed Games

5.1 Introduction

The LCP associated with a P-matrix has a unique solution for every vector q. In 1983, Cottle and Stone introduced a new class of matrices called U-matrices (forming the class U). A matrix A belongs to this class U if the $LCP(q, A)$ has a unique solution whenever q is in the interior of the union of complementary cones. Moreover, they extended the class to include matrices for which $LCP(q, A)$ has a unique solution whenever q lies in the interior of any non-degenerate complementary cone. This class is known as the class of fully semimonotone matrices, denoted by E_0^f, which was also introduced by Stone.

In [7], Cottle and Stone proved that the class of P-matrices is a subset of the class of U-matrices, which is a subset of the class of fully semimonotone matrices (that is, $P \subseteq U \subseteq E_0^f$). In [53], Stone proved that $U \cap Q_0 \subseteq P_0$. Furthermore, he raised the conjecture that $E_0^f \cap Q_0 \subseteq P_0$.

From the definition of fully semimonotone matrix, it is understood that it is a semimonotone matrix, that is $E_0^f \subseteq E_0$. Additionally, it is known that the class of Z-matrices is a subset of the class of Q_0-matrices, that is $Z \subseteq Q_0$.

The next section presents some foundational results required later in this chapter. Section 3 and 4 of this chapter contribute to Stone's conjecture. In particular, in section

3, we prove that $E_0 \cap Z \subseteq P_0$. In section 4, we prove that the conjecture holds true with some specific sign patterns. In section 5, we prove some results that relate completely mixed games to the class of matrices based on LCP.

5.2 Preliminaries

This section contains some basic results for fully semimonotone and fully copositive matrices. First, we provide the definition for these classes.

Definition 5.2.1. A matrix A is called a fully semimonotone matrix, denoted by E_0^f, if A and all its legitimate principal pivot transforms are in E_0.

Definition 5.2.2. A matrix A is called a fully copositive matrix, denoted by C_0^f, if A and all its legitimate principal pivot transforms are in C_0.

In this chapter, we prove some results regarding Stone's conjecture, which states that $E_0^f \cap Q_0 \subseteq P_0$. Before proceeding, we present some known results on fully semimonotone Q_0 matrices. The class of fully semimonotone matrices includes the class of fully copositive matrices. In 1997, Murthy and Parthasarathy proved the following result.

Theorem 5.2.3. *Let $A \in \mathbb{R}^{n\times n} \cap C_0^f \cap Q_0$. Then $A \in P_0$.*

Next, we state a few known results for fully semimonotone Q_0 matrices. In 1995, Murthy and Parthasarathy [31], proved the following result.

Theorem 5.2.4. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^f \cap Q_0$. Suppose that every proper principal minor of A is non-negative. Then $A \in P_0$.*

Now, we have the following corollary.

Corollary 5.2.5. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^f \cap Q_0$. Suppose that every principal submatrix of A of order $n-2$ or less is P-matrix. Then $A \in P_0$.*

Proof. Let B be any principal submatrix of A of order $n-1$ such that $|B| < 0$. Since $A \in E_0^f$, we have $B \in E_0$ and $B^{-1} \in E_0$. Since each principal submatrix of A of order $\leq n-2$ is a P-matrix, we have each proper principal submatrix of B is P-matrix. Since

$|B| < 0$, there must be some diagonal entry of B^{-1} is negative, which is a contradiction to $B^{-1} \in E_0$ (diagonal entry of any E_0 matrix cannot be negative). Therefore, $|B| \geq 0$, and hence using Theorem 5.2.4, $A \in P_0$. □

It is known that principal pivot transforms of a semimonotone matrix are not necessarily semimonotone. However, if all the existing principal pivot transforms of a semimonotone matrix are semimonotone, then it is said to be a fully semimonotone matrix. From Theorems 2.1.12 and 2.1.15, we can say that PPTs of Q-,R_0-, and P_0-matrices are $Q, R_0,$ and P_0 respectively.

Remark 5.2.6. $P \subseteq P_0 \subseteq E_0^f \subseteq E_0$.

The following result due to Berman and Plemmons, which relates a Z-matrix to a P-matrix using the value of the game associated with that matrix.

Theorem 5.2.7. *[2] Let $A \in \mathbb{R}^{n\times n} \cap Z$ and $val(A) > 0$. Then $A \in P$.*

5.3 Semimonotone Z-Matrices

In 1995, Murthy and Parthasarathy [31] proved that $E_0^f \cap Z \subseteq P_0$. However, the condition of fully semimonotone is not necessary. Hence, we present the following theorem which exempts the condition of each PPT to be semimonotone. In [48], we proved that within the class of Z, it is sufficient for a matrix to be semimonotone in order to belong to P_0.

Theorem 5.3.1. *Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Z$, then $A \in P_0$.*

Proof. We prove this using mathematical induction. For $n = 1$. Since E_0 has a nonnegative diagonal, it has only one entry that is nonnegative. Hence, it is P_0.

For $n = 2$. Using Theorem 2.1.6, either each entry is nonnegative or the determinant of the matrix A is nonnegative. Since $A \in E_0$, diagonal entries are nonnegative. Hence, $det(A) \geq 0$ implies $A \in P_0$. Now, suppose $det(A) < 0$, then each entry of A is nonnegative.

Since $A \in Z$, off-diagonal entries will be zero. Therefore, $A \in P_0$.

Assume that it is true up to order $n-1$. Now, we show that it is true for n.

a.) Let all the coordinates of x be positive. Then, the game is completely mixed. Since $A \in E_0$, it follows that $val(A) \geq 0$. If $val(A) > 0$, then by Theorem 5.2.7, $A \in P$. Therefore, $A \in P_0$. If $val(A) = 0$, then using Theorem 4.3.1, $rank(A) = n-1$. That means one row is a linear combination of the others. Hence, $det(A) = 0$, and therefore $A \in P_0$.

b.) Let $k \leq n-1$ be the non-zero coordinates of x. WLOG, let the first k coordinates of x are positive. Since $A \in E_0 \cap Z$, $Ax \geq 0$ gives us the following matrix:

$$A = \begin{pmatrix} C & B \\ \mathbf{0} & D \end{pmatrix}$$

where $C \in \mathbb{R}^{k \times k}$, $B \in \mathbb{R}^{k \times n-k}$, $D \in \mathbb{R}^{n-k \times n-k}$ and **0** is null matrix of order $n-k \times k$. From the properties of partitioned matrices, we know that $det(A) = det(C).det(D)$. Since from induction we know that for any $k \leq n-1$, $det(C) \geq 0$, and $det(D) \geq 0$. Hence, $det(A) \geq 0$. Therefore, $A \in P_0$.

Hence, for any n, $A \in \mathbb{R}^{n \times n} \cap E_0 \cap Z$ implies $A \in P_0$. □

We can have the following alternate proof for the above theorem:

Proof. We have $A \in E_0 \cap Z$. Then, for every $\epsilon > 0$, $A + \epsilon I \in E_0 \cap Z$. We claim that $val(A + \epsilon I) > 0$. On the contrary, suppose $val(A + \epsilon I)$ is not positive. Hence, for every vector x, we have $x^t(A + \epsilon I) \leq 0$, which implies $x^t A \leq 0$, and $A^t x \leq 0$. Therefore, A^t is not an E_0-matrix, which leads to a contradiction. Hence, value of $A + \epsilon I$ is positive. Using Theorem 5.2.7, we have $A + \epsilon I \in P$. Therefore, $\epsilon \to 0$ implies $A \in P_0$. □

In the theorem mentioned above, it is crucial to note that both conditions, namely, that A being E_0 and $A \in Z$, are necessary. This fact becomes apparent through the examination of the following two examples.

Example 5.1. Let

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}.$$

It can be seen that $A \in Z$ and A is not an E_0-matrix (since $Ax < 0$ for some vector $x = (1,1)^t$). Notice that $det(A) = -2$. Therefore, it is not a P_0-matrix. □

Example 5.2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Since A is a nonnegative matrix, it can be easily verified that $A \in E_0$. Since off-diagonal entries are positive, $A \notin Z$. But the determinant of A is negative, hence A is not a P_0-matrix. □

Remark 5.3.2. It is known that $P_0 \subseteq E_0$. From Theorem 5.3.1, we can conclude that within the class of Z-matrices, P_0 is equivalent to E_0. From example 5.2, it can be seen that the Z-property is necessary for the equivalence to hold. From this result, we can conclude the next two theorems.

In Theorem 3.5.8, we showed that within the class of E_0, R_0-matrices are equivalent to Q for matrices of order up to 3. They provided counterexamples of matrices that are $E_0 \cap Q$ but not R_0 for order 4 and above. Here, we prove the equivalence with the additional assumption of Z for any order of matrices.

Theorem 5.3.3. *Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Z$. Then $A \in Q$ iff $A \in R_0$.*

Proof. Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Z$. By Theorem 5.3.1, it follows that $A \in P_0$. Then Theorem 3.3.4 states that within P_0, Q is equivalent to R_0. Therefore, within the class of E_0 and Z, $A \in Q$ if and only if $A \in R_0$. □

Theorem 5.3.4. *Let $A \in \mathbb{R}^{n\times n} \cap Z$. Then $A \in E_0$ if and only if $A \in E_0^f$.*

Proof. First, let us assume $A \in E_0^f$, which means that every PPT of A is an E_0-matrix. Let $\alpha = \phi$. Then $ppt(A, \alpha) = A$. Hence, $A \in E_0$.

Now, to prove the converse, let us assume $A \in E_0$. Using Theorem 5.3.1, we can conclude that $A \in P_0$. Using remark 5.2.6, it is known that $P_0 \subseteq E_0^f$. Therefore, $A \in E_0^f$.

Hence, within the class of Z-matrices, E_0 is equivalent to E_0^f. □

However, in general, E_0-matrices are not equivalent to E_0^f. Hence, the condition of matrix being a Z-matrix is necessary in Theorem 5.3.4. It can be seen by the example below.

Example 5.3. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since A is a nonnegative matrix, it is easy to verify that $A \in E_0$. Let $\alpha = \{1, 2\}$. The PPT of A with respect to α is as follow

$$ppt(A, \alpha) = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & 2 & -2 \end{pmatrix}.$$

Observe that the diagonal entries of the PPT corresponding to the above α are negative. Hence, $ppt(A, \alpha)$ is not an E_0-matrix. Therefore, $A \notin E_0^f$.

5.4 Contribution to Stone's Conjecture

5.4.1 Matrices with Specific Sign Patterns

First, recall Stone's conjecture, which states that fully semimonotone Q_0-matrices are P_0. Next, we describe some specific sign patterns and their properties.

Theorem 5.4.1. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that all its diagonal entries are positive, and each entry below the diagonal is non-negative. Then $A \in R$ and so $A \in Q$.*

Proof. For the given sign pattern of A, consider $LCP(q, A)$, where $q = 0 \in \mathbb{R}^n$. We have

$$Ax = \begin{pmatrix} + & * & * & \dots & * \\ \oplus & + & * & \dots & * \\ \oplus & \oplus & + & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \oplus & \oplus & \oplus & \dots & + \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ 0 \end{pmatrix} = w,$$

where $*$ is any real number and $w \in \mathbb{R}^n$. Here, for any k, such that $x_k > 0$ and each $x_m = 0$ for $m > k$, we have w_k positive. This contradicts the condition of complementarity. Hence, no x_k can be positive, that is, the only solution for $LCP(0, A)$ is zero.

Similarly, for $LCP(d, A)$ with $d > 0$, the only solution is zero. Hence, using Theorem 2.1.4, we can conclude that $A \in Q$. □

Next, we show another sign pattern and its property, and then we demonstrate that the conjecture holds for this sign pattern as well.

Theorem 5.4.2. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that all its diagonal entries are positive and upper triangular is non-negative. Then $A \in R_0$.*

Proof. For given A, consider $LCP(0, A)$. We have

$$Ax = \begin{pmatrix} + & \oplus & \oplus & \dots & \oplus \\ * & + & \oplus & \dots & \oplus \\ * & * & + & \dots & \oplus \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & + \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ x_k \\ \vdots \\ x_{k+l} \end{pmatrix} = w,$$

where $*$ is any real number and $w \in \mathbb{R}^n$. Here for least value of k such that $x_k > 0$, $(Ax)_k = w_k$ is also positive. This contradicts the condition of complementarity. Hence, no x_k is positive, which implies that $LCP(0, A)$ has the only trivial solution. Therefore, $A \in R_0$. □

5.4.2 Fully Semimonotone Q_0-Matrices

In [31], Murthy proved that if a matrix $A \in \mathbb{R}^{n\times n} \cap E_0^f \cap Q_0$ has positive diagonal entries, then $A \in P_0$ for $n = 5$. In this section, we show that for general n, matrices with sign patterns from last subsection, the result $E_0^f \cap Q_0 \subseteq P_0$ holds.

Theorem 5.4.3. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^f$ be a matrix with positive diagonal entries, nonnegative lower triangular, and nonpositive upper triangular. Then, $A \in P_0$.*

Proof. Since A has the same sign pattern as in Theorem 5.4.1, we have $A \in Q$ as well as $A \in Q_0$. Hence, with the given sign pattern, we have $A \in E_0^f \cap Q_0$.

Now, for $n = 2$, we have

$$A = \begin{pmatrix} + & \ominus \\ \oplus & + \end{pmatrix}$$

Here, $det(A) > 0$, hence $A \in P$ as well as $A \in P_0$.

For $n = 3$, since we have seen that it is true for $n = 2$. Using Theorem 5.2.4, we can conclude that $A \in P_0$.

Hence, whenever it is true for $n - 1$, it is also true for n. Therefore, such a matrix A is always a P_0-matrix. □

Theorem 5.4.4. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^f$ be a matrix such that all its diagonal entries are positive, and upper triangular entries are non-negative. Then $A \in P_0$.*

Proof. Let $A \in \mathbb{R}^{n\times n} \cap E_0^f$ be a matrix with positive diagonal entries, and nonnegative upper triangular entries. By Theorem 5.4.2, we have $A \in R_0$. In [30], Theorem 5.2.24 states that $A \in \mathbb{R}^{n\times n} \cap E_0^f \cap R_0$ implies $A \in P_0$. Therefore, $A \in P_0$. □

In fact, matrices with the above sign patterns are completely Q_0. Murthy [31] proved that if a matrix $A \in \mathbb{R}^{n\times n} \cap E_0^f$ and A is completely Q_0 , then $A \in P_0$. Using this we can directly say that the above two theorems hold true.

5.5 Matrix Classes Related to Completely Mixed Games

5.5.1 Z-Matrices

The next theorem states equivalent conditions for Z-matrices whose associated game has value zero. It establishes a connection between completely mixed games, irreducible matrices and Q-matrices. First, we present the definition of reducible matrix.

Definition 5.5.1. A matrix $P \in \mathbb{R}^{n\times n}$ is called a permutation matrix if P can be obtained from an identity matrix $I \in \mathbb{R}^{n\times n}$ by permuting its rows and columns.

Definition 5.5.2. A matrix $A \in \mathbb{R}^{n\times n}$, where $n \geq 2$, is called a reducible matrix if there exists a permutation matrix P such that

$$PAP^t = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

For $n = 1$, null matrix is reducible matrix. A matrix that is not reducible is called an irreducible matrix.

Theorem 5.5.3. *Let $A \in \mathbb{R}^{n\times n} \cap Z$ and $val(A) = 0$. Then the following conditions are equivalent:*

1. *The matrix game A is completely mixed.*
2. $val(A + D_i) > 0$ *for all* $i = 1, 2, \ldots, n$.
3. $A + D_i \in Q$ *for all* $i = 1, 2, \ldots, n$.
4. *A is irreducible.*

Proof. $(1 \Rightarrow 2)$ We are given that the value of the game associated with A is zero. The proof follows the same lines as Theorem 4.3.2.

$(2 \Rightarrow 3)$ Let $val(A + D_i) > 0$, for all $i = 1, 2, \ldots, n$. Since $A \in Z$, we have $A + D_i \in Z$ for all $i = 1, 2, \ldots, n$. It is known that $Z \subseteq Q_0$. Hence, $A + D_i \in Q_0$ for all

$i = 1, 2, \ldots, n$. Using Lemma 4.2.6, $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$.

$(3 \Rightarrow 4)$ Let $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$. Hence, $val(A + D_i) > 0$, for all $i = 1, 2, \ldots, n$. Since $A \in Z$, we have $A + D_i \in Z$ for all $i = 1, 2, \ldots, n$. Using Theorem 5.2.7, $A + D_i \in P$ for all $i = 1, 2, \ldots, n$, that is, all the principal minors of $A + D_i$ are positive for all $i = 1, 2, \ldots, n$. Hence, all the proper principal minors of A are positive.

Suppose, if possible, A is not irreducible. Then, WLOG, we can write A as :

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

Then $det(A) = det(B) \times det(D)$. B and D are proper principal matrices of A, and hence $det(B)$ and $det(D)$ are positive. Therefore, $det(A) = det(B) \times det(D) > 0$. Since we have shown that all the proper principal minors of A are positive, $A \in P$. Therefore $A \in Q$, and hence $val(A) > 0$. But it contradicts our hypothesis that $val(A) = 0$. Therefore, A is irreducible.

$(4 \Rightarrow 1)$ Let A be an irreducible matrix. Suppose A not completely mixed. Let x be a probability vector such that some coordinates of x are zero. Let the first k coordinates of x be nonzero and $x = (x_1, \ldots x_k, x_{k+1}, \ldots, x_n)^t = (y, z)^t$, where $y = (x_1, \ldots x_k)^t$ and $z = (0, 0, \ldots, 0)^t$. Let the corresponding partitions is as follows:

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}.$$

where $B \in \mathbb{R}^{k \times k}$, $C \in \mathbb{R}^{k \times n-k}$, $D \in \mathbb{R}^{n-k \times k}$, and $E \in \mathbb{R}^{n-k \times n-k}$. Then, for $Ax \geq 0$, we have:

$$By + Cz \geq 0 \tag{5.1}$$

$$Dy + Ez \geq 0 \tag{5.2}$$

Since $z = 0$, equation (5.2) gives us $Dy \geq 0$. Since A is a Z-matrix, $D, C \leq 0$. Hence, $y > 0$ implies $Dy \leq 0$. Therefore, from the conditions $y > 0$, $Dy \geq 0$ and $Dy \leq 0$, we have $D = 0$. This leads to a contradiction that A is irreducible. Hence, the game associated with A is completely mixed. □

Gowda [16] also proved the equivalence of the first and fourth parts of the above theorem, that is, he showed that for $A \in \mathbb{R}^{n\times n} \cap Z$ and $val(A) = 0$, the matrix game A is completely mixed if and only if A is irreducible. From the above theorem, we can conclude the following corollary:

Corollary 5.5.4. *Let $A \in \mathbb{R}^{n\times n} \cap Z$. Suppose the game associated to A is completely mixed and $val(A) = 0$. Then $A \in P_0$.*

Proof. Suppose that the game associated to A is completely mixed and $val(A) = 0$. Hence there exists a completely mixed strategy $y > 0$ such that $Ay = 0$. Therefore for all $\alpha > 0$, we have $(A + \alpha I)y > 0$. This implies that $val(A + \alpha I)$ is positive for all α. Since $A \in Z$, we have $A + \alpha I \in Z$ for all α. Therefore, using Theorem 5.2.7, $A + \alpha I \in P$ for all α. Now if α approaches zero, then $A \in P_0$. □

5.5.2 Copositive Q_0-Matrices

In Theorem 5.2.3, we saw that if $A \in \mathbb{R}^{n\times n} \cap C_0^f \cap Q_0$, then $A \in P_0$. We now relate this class of fully copositive Q_0-matrices with completely mixed games. The following result establishes this connection.

Theorem 5.5.5. *[43] Let $A \in \mathbb{R}^{n\times n} \cap C_0^f \cap Q_0$. If the game associated with matrix A is completely mixed, then $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$.*

Proof. Let $A \in \mathbb{R}^{n\times n} \cap C_0^f \cap Q_0$. Then, Theorem 5.2.3 implies that A is a P_0-matrix. Therefore, $A + D_i \in P_0$ for all $i = 1, 2, \ldots, n$.

Without loss of generality, we consider $A + D_1$. Suppose $A + D_1$ is not a Q-matrix. Since $A + D_1 \in P_0$ and $A + D_1$ is not a Q-matrix, using Theorem 3.3.4, we have $A + D_1$ is not an R_0-matrix. Therefore, there exists $x \geq 0$ $(x \neq 0)$ such that $(A + D_1)x = w \geq 0$

and $x^t w = 0$. Let $x = (x_1, x_2, \ldots, x_n)^t$ and $w = (w_1, w_2, \ldots, w_n)^t$. Then we consider the following two cases:

Case(i) Suppose $A \in Q_0 \setminus Q$. Since $A \in P_0$, by Lemma 4.2.8, $val(A) = 0$. Observe $x_1 = 0$ is not possible, otherwise

$$w = (A + D_1)x = Ax + D_1 x = Ax \geq 0.$$

Therefore, $Ax \geq 0$ with $x_1 = 0$, implies x is not completely mixed and this leads to a contradiction. Suppose $x_1 > 0$, then $w_1 = 0$. Therefore, considering the first row of $A + D_1$, we have

$$w_1 = (A + D_1)_{1.}x = A_{1.}x + D_{1_{1.}}x = 0, \Rightarrow \ A_{1.}x = -D_{1_{1.}}x = -x_1 < 0.$$

Here $D_{1_{1.}}, A_{1.}$ and $(A + D_1)_{1.}$ denotes the first row of matrix D_1, A and $A + D_1$ respectively.

Define $I_x = \{i : x_i > 0\}$. Then $A_{i.}x = 0, \ \forall\, i \in I_x \setminus \{1\}$ [because $x_i > 0$ implies that $w_i = 0$].

Now $A_{1.}x = -x_1 < 0$ and $A_{i.}x = 0, \forall\, i \in I_x \setminus \{1\}$.

Hence, if $\alpha = I_x = \{i : x_i > 0\}$, then $x_{\bar{\alpha}} = 0$ and

$$A_{\alpha\alpha}x_\alpha = \begin{pmatrix} -x_1, & 0, & 0, & \ldots, & 0 \end{pmatrix}^t.$$

Therefore, $x_\alpha^t A_{\alpha\alpha} x_\alpha = (x_1)(-x_1) + 0 + 0 + \cdots + 0 = -x_1^2 < 0.$

$$A = \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{pmatrix}, \ x^t = (x_\alpha^t, x_{\bar{\alpha}}^t).$$

Therefore, $x^t(Ax) = -x_1^2 < 0$. This leads to a contradiction. Hence, $A + D_1$ has to be in R_0. Therefore, $A + D_1 \in Q$.

Case(ii) Suppose $A \in Q$. Since $A \in P_0$, by Theorem 3.3.4, $A \in R_0$. Now, we consider two subcases:

(a) If $x_1 = 0$, then $D_1 x = 0$ and therefore, $(A + D_1)x = Ax = w$, $x^t w = 0$. This is not possible since $A \in R_0$.

(b) Suppose $x_1 > 0$, then $w_1 = 0$. Therefore, considering the first row of $A + D_1$, we have

$$w_1 = (A + D_1)_{1.}x = A_{1.}x + D_{1_{1.}}x = 0,$$
$$\Rightarrow \; A_{1.}x = -D_{1_{1.}}x = -x_1 < 0.$$

For $i \neq 1$, $w_i = (Ax)_i$ and $w_i x_i = 0$. Therefore, $x^t w = x^t(Ax) = -x_1^2 < 0$. This leads to a contradiction that A is fully copositive. Hence, $A + D_1$ has to be in R_0. Since $A + D_1$ is also a P_0-matrix, using Theorem 3.3.4, we can conclude that $A + D_1 \in Q$.

Similarly, we can show that $A + D_i \in Q$, for all $i = 1, 2, \ldots, n$. □

In general, we cannot say that the converse of the above theorem is true. The following example illustrates this fact.

Example 5.4. *Consider a matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Here, we can easily check that $A \in C_0^f \cap Q_0$ and $A + D_i \in Q$ for $i = 1, 2$. However, the game associated with matrix A is not completely mixed.

Note that in the above example, $A \in Q$; hence, the game associated with the matrix A is not completely mixed. Now, the following theorem provides a characterization of completely mixed games for fully copositive Q_0-matrices, but not Q.

Theorem 5.5.6. *Let $A \in \mathbb{R}^{n \times n} \cap C_0^f \cap Q_0$, and $A \notin Q$. Then the game associated with matrix A is completely mixed if and only if $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$.*

Proof. Suppose $A \in \mathbb{R}^{n \times n} \cap C_0^f \cap Q_0$. Then, Theorem 5.2.3 implies $A \in P_0$. If the game associated with matrix A is completely mixed, then $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$ follows from Theorem 5.5.5.

For the converse, suppose $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$. Since $A \notin Q$ and $A \in P_0 \cap Q_0$, using Lemma 4.2.8, we can conclude that $val(A) = 0$. Therefore, there exist probability vectors x and y such that $Ax \geq 0$ and $y^t A \leq 0$.

Suppose that the game associated with matrix A is not completely mixed. Then, there exists a probability vector y such that $y_j = 0$ and $y^t A \leq 0$. Now, consider the following:

$$\begin{aligned} y^t(A + D_j) &= y^t A + y^t D_j, \\ &= y^t A, \qquad \text{[because } y_j = 0\text{]} \\ &\leq 0. \end{aligned}$$

This means that $val(A + D_j) \leq 0$. Therefore, by Lemma 4.2.5, we have $A + D_j \notin Q$. This leads to a contradiction. Therefore, the game associated with matrix A is completely mixed. □

The following theorem unifies Theorem 5.5.5 and 5.5.6. While the overall proof follows similar ideas, we provide a slightly different proof for the converse part.

Theorem 5.5.7. *Let $A \in \mathbb{R}^{n\times n} \cap P_0 \cap C_0$, and $A \in Q_0 \setminus Q$. Then the game associated with A is completely mixed if and only if $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$.*

Proof. Suppose $A \in \mathbb{R}^{n\times n} \cap C_0 \cap P_0 \cap Q_0$ and $A \notin Q$. By Lemma 4.2.8, we have $val(A) = 0$. If the game associated with matrix A is completely mixed, then $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$, follows from Theorem 5.5.5.

To prove the converse, suppose $A + D_i \in Q$ for all $i = 1, 2, \ldots, n$. Since $A \in P_0$, observe that $A + D_i \in P_0$. Using Theorem 3.3.4, we conclude $A + D_i \in R_0$. Since $A \notin Q$, Theorem 3.3.4 implies that $A \notin R_0$, that is, there exists nonzero z such that $z \geq 0,\ Az = w \geq 0,\ z^t w = 0$.

Let us suppose $z_i = 0$, for some i. Then, $(A + D_i)z = Az + D_i z = Az = w \geq 0$ (because when $z_i = 0, D_i z = 0$). This implies that $A + D_i \notin R_0$, which leads to a contradiction. Therefore, $z_i \neq 0$, for all $i = 1, 2, \ldots, n$. This means that $(Az)_i = 0$, for all $i = 1, 2, \ldots, n$. Suppose A is not completely mixed. Since $val(A) = 0$, there exists an optimal strategy x with $x_j = 0$ for some j, such that $Ax \geq 0$. Note that there exists some k such that $(Ax)_k > 0$. (Otherwise, if $(Ax)_k = 0$ for all k, then $A + D_j$ will not be an R_0-matrix.)

Since $z > 0$, there exists some $\lambda > 0$ such that $z - \lambda x > 0$. This implies that $A(z - \lambda x) = Az - \lambda Ax \leq 0$, with at least one coordinate strictly less than zero (because $Az = 0$ and $Ax \geq 0$). Therefore, $(z - \lambda x)^t A(z - \lambda x) < 0$, which leads to a contradiction, since A is a copositive matrix. Therefore, the game associated with matrix A is completely mixed. □

5.5.3 An Application of Theorem 3.5.8

In section 4.6, we proved that a game associated with a matrix $A \in \mathbb{R}^{3\times3} \cap E_0$ having $val(A) = 0$ is completely mixed if and only if $A + D_i \in Q$ for all $i = 1, 2, 3$, where D_i is a diagonal matrix whose i^{th} entry is 1 and else are zero. Since A is a semimonotone matrix, the value of A must be nonnegative. We also provided an example showing that if the value of A is positive then this result does not hold true, i.e., the condition of $val(A) = 0$ in the above result is necessary. However, in [44], we proved that if the game associated with a matrix $A \in \mathbb{R}^{3\times3} \cap E_0 \cap Q$ is completely mixed, then $A + D_i$ is a Q-matrix for $i = 1, 2, 3$, where D_i is a diagonal matrix such that only i^{th} diagonal entry is 1, else 0.

Theorem 5.5.8. *Let $A \in \mathbb{R}^{3\times3} \cap E_0 \cap Q$. If the game associated with matrix A is completely mixed, then $A + D_i$ is a Q-matrix for $i = 1, 2, 3$, where D_i is a diagonal matrix such that only i^{th} diagonal entry is 1, else 0.*

Proof. Let $A \in \mathbb{R}^{3\times3} \cap E_0 \cap Q$. It follows from Theorem 3.5.8 that $A \in R_0$. Since A is an E_0-matrix, we observe that $A + D_i$ is also an E_0-matrix for $i = 1, 2, 3$.

Suppose $A + D_1 \notin Q$, by Theorem 3.5.8, $A + D_1 \notin R_0$. This implies that there exists a nonzero $x \geq 0$, such that

$$(A + D_1)x = w \geq 0 \text{ and } x_i w_i = 0 \text{ for } i = 1, 2, 3. \tag{5.3}$$

We claim that $x_1 > 0$. On the contrary, suppose $x_1 = 0$, then $(A + D_1)x = Ax + D_1x = Ax = w \geq 0$. This means that x is a nonzero solution of $LCP(0, A)$. This is a contradiction to $A \in R_0$. Hence, $x_1 > 0$ and $w_1 = 0$. Now, multiplying x with the first row of $A + D_1$, we have

$$(A + D_1)_{1.}x = A_{1.}x + (D_1)_{1.}x = A_{1.}x + x_1 = 0, \qquad \Rightarrow A_{1.}x = -x_1 < 0.$$

Now, if possible, let $x_2 > 0$ and $x_3 > 0$. This implies that $(A + D_1)x = 0$. Hence, we have

$$Ax = \begin{pmatrix} -x_1 \\ 0 \\ 0 \end{pmatrix}.$$

This implies that $x^t A^t \leq 0$, which means $val(A^t) \leq 0$. Since $A \in Q$, we must have $val(A) > 0$. Since A is completely mixed, we have $val(A) = val(A^t)$. This leads to a contradiction. Hence, all three coordinates of x cannot be positive and x_1 must be positive.

Therefore, only two coordinates of x are nonzero. Without loss of generality, we assume $x_1 > 0, x_2 > 0, x_3 = 0$. From (5.3), it follows that:

$$Ax = \begin{pmatrix} -x_1 \\ 0 \\ \oplus \end{pmatrix}.$$

If we denote A, B, and D as:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, D = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}.$$

Then,

$$B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}. \tag{5.4}$$

Observe that B is a principal submatrix of A. Since $A \in E_0$, we conclude that $B \in E_0$ as well as $B^t \in E_0$.

Hence, there exists $y_1 \geq 0,\ y_2 \geq 0$ and $y_1 + y_2 = 1$, satisfying the inequality

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{5.5}$$

If $y_1 > 0$, then using the inequality (5.5), we obtain:

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0. \tag{5.6}$$

On the other hand, inequality (5.4) implies:

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} < 0 \tag{5.7}$$

However, inequalities (5.6) and (5.7) contradicts. Therefore, $y_1 = 0$ and $y_2 = 1$.

Since $B \in E_0$ and using the inequalities (5.4) and (5.5), we can infer that the diagonals of matrix B must be nonnegative. Since $a_{11} \geq 0$, inequality (5.4) implies $a_{12} < 0$. If $a_{22} > 0$, then inequality (5.4) implies $a_{21} < 0$. But this is a contradiction to inequality (5.5), because $[0\ y_2]B = [0\ 1]B \ngeq 0$. Therefore, $a_{22} = 0 = a_{21}$. Therefore, we have the sign pattern for B

as:

$$B = \begin{pmatrix} \oplus & - \\ 0 & 0 \end{pmatrix}.$$

Now, we put these signs in matrix A. Observe that $a_{23} > 0$ (because $A \in Q$ and any row of Q-matrix cannot be nonpositive). Similarly, $a_{32} > 0$ (because if $a_{32} < 0$, the second column would be nonpositive which contradicts $val(A^t) > 0$). Hence, matrices A and D have the following sign patterns:

$$A = \begin{pmatrix} \oplus & - & * \\ 0 & 0 & + \\ * & + & \oplus \end{pmatrix}, D = \begin{pmatrix} \oplus & * \\ * & \oplus \end{pmatrix}. \tag{5.8}$$

Observe that the 2nd row of A is nonnegative. Therefore, after omitting the 2nd row and 2nd column of matrix A, the corresponding matrix $D \in Q$ by Theorem 2.1.1.

Now, claim $a_{11} > 0$. On the contrary, suppose $a_{11} = 0$. Since $D \in Q$, we must have $a_{13} > 0$. Now, we proceed with the following cases according to the sign of a_{31}.

1. If $a_{31} \geq 0$, then the 2nd row of D is nonnegative. Since $D \in Q$, after omitting the 2nd row and 2nd column of D, the remaining matrix $[a_{11}]$ must be a Q-matrix. Hence, $a_{11} \neq 0$.

2. If $a_{31} < 0$, then this implies that $a_{33} > 0$, since $D \in Q$. Hence, we arrive at the following sign pattern for A:

$$A = \begin{pmatrix} 0 & - & + \\ 0 & 0 & + \\ - & + & + \end{pmatrix}.$$

   Since the 1st column of A has nonpositive entries and the 2nd row has nonnegative entries, $val(A^t) = 0$. This leads to a contradiction to $val(A) = val(A^t) > 0$.

Therefore, $a_{11} > 0$.

Now, let $a_{11} > 0$. If $a_{31} \geq 0$, the 3rd row of A in (5.8) becomes nonnegative. However, after omitting the third row and third column, the remaining matrix of A is not a Q-matrix. This leads to a contradiction to Theorem 2.1.1. Hence, $a_{31} < 0$. Furthermore, since the second row of A is nonnegative and after omitting the second row and second column, the remaining matrix D is a Q-matrix only if $a_{33} > 0$. Hence, we obtain the following sign pattern of A:

$$\begin{pmatrix} + & - & * \\ 0 & 0 & + \\ - & + & + \end{pmatrix}.$$

We can easily obtain a probability vector $u = (u_1, u_2, 0)$, where $u_1 > 0$ and $u_2 > 0$ such that:

$$A \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}.$$

Since $A \in R_0$, $*$ will be negative. Therefore, $val(A^t) \not> 0$. This leads to a contradiction. Hence, $A + D_1 \in R_0$, and therefore, from Theorem 3.5.8, we have $A + D_1 \in Q$. Similarly, we can show that $A + D_2 \in Q$ and $A + D_3 \in Q$. □

Corollary 5.5.9. *Let $A \in \mathbb{R}^{3\times3} \cap E_0 \cap Q$. If $val(A^t) > 0$, then $A + D_i$ is a Q-matrix for $i = 1, 2, 3$, where D_i is a diagonal matrix such that only the i^{th} diagonal entry is 1, else 0.*

Proof. For completely mixed games, $val(A) = val(A^t)$. Since $A \in Q$, $val(A) = val(A^t) > 0$. In proving the above theorem, we only used $val(A^t) > 0$. Hence, the proof of this corollary is similar to the proof of Theorem 5.5.8. □

We provide the following two examples to show the sharpness of the above theorem.

Example 5.5. Let

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

One can easily verify that

1. $A \in E_0 \cap R_0$ and $val(A^t) = 0$.

2. Also, notice that $A + D_1 \notin Q$. For $q = (-4, -10, -2)$, $LCP(q, A + D_1)$ has no solution.

Here, the condition of Corollary 5.5.9, $val(A^t) > 0$, is not satisfied. □

The following example shows that if $A \in E_0 \cap R_0$ and $A + D_i \in E_0 \cap R_0$ for $i = 1, 2, 3$, then it does not necessarily imply A is completely mixed or in general, $val(A^t) > 0$.

Example 5.6. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

One can easily verify that

1. $A \in E_0 \cap R_0$, hence, $A \in Q$. $A + D_i \in E_0 \cap R_0$, hence $A + D_i \in Q$ for $i = 1, 2, 3$.

2. $val(A) = 1$ and $val(A^t) = 0$.

Clearly, A does not possess a completely mixed optimal strategy for the maximizer (the one who chooses the column), which means A is not completely mixed. □

CHAPTER 6

Q-Tensors

6.1 Introduction

A tensor is a hypermatrix $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$, where $i_j = 1, 2, \dots, n_j$ for $j = 1, 2, \dots, m$, that is, $(n_1, n_2, \dots, n_m)$ is the dimension of $\mathcal{A}$. When $n_1 = n_2 = \dots = n_m = n$, we call the tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is an m-ordered, n-dimensional tensor. We call $\mathcal{A}_{j_1 j_2 \dots j_m} = (a_{i_1 \dots i_m}) \in \mathbb{T}_{m,n-1}$ a subtensor of $\mathcal{A} \in \mathbb{T}_{m,n}$ if $i_k \in \{1, 2, \dots, n\} \setminus \{j_k\}$, where $j_k \in \{1, 2, \dots, n\}$. We call $\mathcal{A}_{ii \dots i} = (a_{i_1 \dots i_m}) \in \mathbb{T}_{m,n-1}$ a principal subtensor of $\mathcal{A} \in \mathbb{T}_{m,n}$ if $i_1, i_2, \dots, i_m \in \{1, 2, \dots, n\} \setminus \{i\}$.

Song and Qi extended the linear complementarity problem to the tensor complementarity problem (TCP), which is a special class of nonlinear complementarity problems. The tensor complementarity problem, $TCP(q, \mathcal{A})$, associated with a tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{T}_{m,n}$ and a vector $q \in \mathbb{R}^n$, is to find a vector $x \in \mathbb{R}^n$, if exists, that satisfies the following:

$$x \geq 0, \tag{6.1}$$

$$\mathcal{A}x^{m-1} + q \geq 0, \text{ and} \tag{6.2}$$

$$x^t(\mathcal{A}x^{m-1} + q) = 0, \tag{6.3}$$

where $(\mathcal{A}x^{m-1})_i = \sum\limits_{i_2, i_3, \dots, i_m = 1}^{n} a_{i i_2 i_3 \dots i_m} x_{i_2} x_{i_3} \dots x_{i_m}$. Any vector x satisfying the conditions (6.1) and (6.2) is called a feasible solution for $TCP(q, \mathcal{A})$. In addition, if condition (6.3) is also satisfied then we say $TCP(q, \mathcal{A})$ has a solution x. A tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is called a Q-tensor if $TCP(q, \mathcal{A})$ has a solution for every vector q. For extensive treatment of tensor, their product, rank, and related concepts, refer to [47].

Song and Qi [52], extended the concept of Q-matrices to Q-tensors. They also demonstrated that certain results of Q-matrices can be extended to Q-tensors. In this chapter, we will extend some results of Q-matrices to Q-tensors.

P_0-matrices and Q-matrices have broad applications in mathematical sciences. Aganagic and Cottle [1] showed that Q-matrices are R_0-matrices within the class of P_0-matrices. However, Huang, Suo and Wang [17] proved that this result does not hold in the case of tensors. They also proved that in the class of strong P_0-tensors or nonnegative tensors, the four classes Q, R, R_0 and ER are equivalent. Pang [38] showed that within the class of semimonotone matrices, each R_0-matrix is a Q-matrix and conjectured the converse. Song and Qi [52] proved that R-tensors are Q-tensors. In this chapter, we prove that rank-one Q-tensors are not only R-tensors, but also R_0-tensors and ER-tensors. Murty [35] showed that a nonnegative matrix is a Q-matrix if and only if all its diagonal entries are positive. Song and Qi [52] proved that this result also holds in the case of tensors. Furthermore, Murty [35] proved that if a matrix is a Q-matrix with some nonnegative row, then the matrix obtained by omitting the nonnegative row and the corresponding column would result in another Q-matrix. Q-matrices have the property that any row of a Q-matrix cannot be nonpositive. In this chapter, we prove that both of these results for Q-matrices due to Murty are true in the case of tensors. Also, we prove that rank-one Q-tensors are positive tensors.

The chapter is organized as follows: In Section 2, we present some basic definitions and results that are used in the subsequent sections. In Section 3, we prove some results on nonnegative tensors. In section 4, we provide some results which classify Q-tensor. Finally, Section 5 presents condition for a Q-tensor being positive.

6.2 Preliminaries

In this section, we present some definitions and results that are used in further sections.

Definition 6.2.1. A tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is called a Q-tensor if $TCP(q, \mathcal{A})$ has a solution for every q.

Definition 6.2.2. Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be a tensor. If there is no nonzero nonnegative vector x exists such that

1. for any $t \in \mathbb{R}_+$, $(\mathcal{A}x^{m-1})_i + t = 0$, for $x_i > 0$ and $(\mathcal{A}x^{m-1})_i + t \geq 0$, for $x_i = 0$. Then $\mathcal{A} \in R$.

2. In above case, if we put $t = 0$, then $\mathcal{A} \in R_0$.

3. for any $t \in \mathbb{R}_+$, $(\mathcal{A}x^{m-1})_i + tx_i = 0$, for $x_i > 0$ and $(\mathcal{A}x^{m-1})_i \geq 0$, for $x_i = 0$. Then $\mathcal{A} \in ER$.

Song and Qi [52] introduced the concepts of Q-tensors, R_0-tensor, and R-tensor. Bai, Huang, and Wang [58] introduced the concept of ER-tensors.

We know that a matrix $A \in \mathbb{R}^{n\times n}$ is called a semimonotone matrix, if for all $x(\neq 0) \geq 0$, there exists an index k, such that $x_k > 0$ and $(Ax)_k \geq 0$. The concept of semimonotone matrices is extended to semipositive tensors as follows.

Definition 6.2.3. A tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is said to be semipositive iff for each $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index $i \in [n]$, such that $x_i > 0$ and $(\mathcal{A}x^{m-1})_i \geq 0$.

Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$ be a tensor whose entries $a_{i_1\ldots i_m}$ are invariant under any permutation of indices. Then, $\mathcal{A}$ is called a symmetric tensor, denoted by $\mathcal{A} \in \mathbb{S}_{m,n}$.

Remark 6.2.4. [51] For symmetric tensor $\mathcal{A} \in \mathbb{S}_{m,n}$, the following results hold:

1. $\mathcal{A}$ is semi-positive if and only if $\mathcal{A}$ is copositive.

2. $\mathcal{A}$ is strictly semi-positive if and only if $\mathcal{A}$ is strictly copositive.

3. If $\mathcal{A}$ is strictly semi-positive, then $\mathcal{A} \in Q$.

Definition 6.2.5. A tensor $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$; such that $\mathcal{A} = y^{\otimes m} = y \otimes \ldots . \otimes y = (y_{i_1} \ldots y_{i_m})$, where $y \in \mathbb{R}^n \setminus \{0\}$, is called a symmetric rank-one tensor.

Example 6.1. Let $\mathcal{A} \in \mathbb{T}_{3,2}$ be a rank-one tensor associated to $y = (a, b)^t$. Then $a_{111} = a^3,\ a_{112} = a^2b,\ a_{121} = a^2b,\ a_{122} = ab^2,\ a_{211} = a^2b,\ a_{212} = ab^2,\ a_{221} = ab^2$, and $a_{222} = b^3$.

Remark 6.2.6. Let $\mathcal{A} \in \mathbb{T}_{m,2}$ be symmetric rank-one tensor corresponding to vector $y = (a, b)^t$. The following observations can be made easily:

1. $(\mathcal{A}x^{m-1})_1 = \sum_{i=0}^{m-1} {}^{m-1}C_i \; a^{m-i} b^i \; x_1^{m-1-i} x_2^i$.

2. $(\mathcal{A}x^{m-1})_2 = \sum_{i=0}^{m-1} {}^{m-1}C_i \; a^{m-1-i} b^{i+1} \; x_1^{m-1-i} x_2^i$.

3. $(\mathcal{A}x^{m-1})_2 = \frac{b}{a}(\mathcal{A}x^{m-1})_1 = \frac{y_2}{y_1}(\mathcal{A}x^{m-1})_1$.

Remark 6.2.7. It can be observed that for any symmetric rank-one tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ and any $i, j \in \{1, 2, \ldots, n\}$, there exists some $\alpha \in \mathbb{R}$ such that $(\mathcal{A}x^{m-1})_i = \alpha(\mathcal{A}x^{m-1})_j$.

6.3 Nonnegative Tensors

In real-life tensor problems, nonnegative tensors are of great interest, since they are commonly encountered. In hypergraph theory, nonnegative tensors are commonly used, for example, adjacency tensors, degree tensors, etc. In this section, we first present some known result on nonnegative tensors. Then we present some results regarding the relationship between nonnegative tensors and their subtensors.

The following equivalence result is known for nonnegative tensors [17].

Theorem 6.3.1. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ be a nonnegative tensor. Then we have the following equivalence relation*

$$\mathcal{A} \in Q \Leftrightarrow \mathcal{A} \in R \Leftrightarrow \mathcal{A} \in R_0 \Leftrightarrow \mathcal{A} \in ER$$

In matrices, it is known that for any matrix $A \in \mathbb{R}^{n \times n}$ for $n \geq 3$, if every principal submatrix of order $n-1$ is positive, then matrix A is positive. However, this result does not, in general, hold for tensors, as demonstrated in the following example:

Example 6.2. Let $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{T}_{3,3}$, where $i_1, i_2, i_3 = 1, 2, 3$. Further, a_{111}, a_{112}, a_{113}, a_{121}, a_{122}, a_{131}, a_{133}, a_{211}, a_{212}, a_{221}, a_{222}, a_{223}, a_{232}, a_{233}, a_{311}, a_{313}, a_{322}, a_{323}, a_{331}, a_{332}, $a_{333} > 0$ and $a_{123} < 0$.

Proof. It can be easily seen that $\mathcal{A}_{111} > 0$ as it have entries a_{222}, a_{223}, a_{232}, a_{233}, a_{322}, a_{323}, a_{332}, $a_{333} > 0$. Similarly, we can check that $\mathcal{A}_{222} > 0$ and $\mathcal{A}_{333} > 0$. Observe that $a_{123} < 0$. In this case, $\mathcal{A}$ is not positive. □

To establish a condition under which a similar result holds for tensors, we provide the following theorem:

Theorem 6.3.2. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ be a tensor with $n > m$. If all the principal subtensors $\mathcal{A}_{ii\ldots i} \in \mathbb{T}_{m,n-1}$, for $i = 1, 2, \ldots, n$, are positive (nonnegative) tensors, then $\mathcal{A}$ is a positive (nonnegative) tensor.*

Proof. Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ be a tensor with $n > m$. Assume all the principal subtensors $\mathcal{A}_{ii\ldots i} \in \mathbb{T}_{m,n-1}$, for $i = 1, 2, \ldots, n$, are positive. First, consider the principal subtensor $\mathcal{A}_{11\ldots 1} > 0$. By examining, we conclude that

$$a_{i_1 \ldots i_m} > 0 \text{ for } i_1, i_2, \ldots, i_m \in [n] \setminus \{1\}. \tag{6.4}$$

This leads to a condition where the elements $a_{i_1 \ldots i_m}$ may have any sign when any of $i_1, i_2, \ldots, i_m$ is 1.

Now, let $\mathcal{A}_{22\ldots 2} > 0$, which implies

$$a_{i_1 \ldots i_m} > 0 \text{ for } i_1, i_2, \ldots, i_m \in [n] \setminus \{2\}. \tag{6.5}$$

Hence, from equations (6.4) and (6.5), we can conclude that $a_{i_1 \ldots i_m} \leq 0$ may be possible when some of $i_1, i_2, \ldots, i_m$ must have values 1 and 2 both.

Similarly, proceeding in this way, let $\mathcal{A}_{mm\ldots m} > 0$, which implies

$$a_{i_1 \ldots i_m} > 0 \text{ for } i_1, i_2, \ldots, i_m \in [n] \setminus \{m\}. \tag{6.6}$$

Hence, using all such m conditions, we cannot fix the sign of remaining $m!$ entries $a_{i_1 \ldots i_m}$ when $i_1, i_2, \ldots, i_m$ are $1, 2, \ldots, m$ in any order.

Since we have $n > m$. Now, $\mathcal{A}_{m+1m+1\ldots m+1} > 0$ implies that

$$a_{i_1\ldots i_m} > 0 \text{ for } i_1, i_2, \ldots, i_m \in [n] \setminus \{m+1\}. \tag{6.7}$$

Using equations (6.7), we can observe that $a_{i_1\ldots i_m} > 0$ when $i_1, i_2, \ldots, i_m$ are $1, 2, \ldots, m$ in any order. Hence, observe that all the entries of $\mathcal{A}$ are positive, that is, $\mathcal{A} > 0$. □

For the following result, the condition $n > m$ given in the above theorem is not required. Instead, we assume that the tensor is a rank-one tensor associated with a vector.

Theorem 6.3.3. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}$ be a rank-one tensor associated with a vector. If all the principal subtensors $\mathcal{A}_{ii\ldots i} \in \mathbb{T}_{m,n-1}$, for $i = 1, 2, \ldots, n$, are positive (nonnegative) tensors, then $\mathcal{A}$ is a positive (nonnegative) tensor.*

Proof. Let $\mathcal{A}$ be a rank-one tensor associated with a vector $y = (y_1, y_2, \ldots, y_n)^t$. Each entry of tensor $\mathcal{A}$ can be expressed as $a_{i_1\ldots i_m} = y_1^{k_1} y_2^{k_2} \ldots y_n^{k_n}$ where $\sum\limits_{i=1}^{n} k_i = m$. If $\mathcal{A}_{ii\ldots i} > 0$, then this will imply that for indices $i_1, i_2, \ldots, i_m \in [n] \setminus \{i\}$, we have $a_{i_1\ldots i_m} = y_1^{k_1} y_2^{k_2} \ldots y_i^{k_i} \ldots y_n^{k_n} > 0$ when $k_i = 0$. Given that all the principal subtensors in $\mathbb{T}_{m,n-1}$ are positive. Hence, for any $k_i = 0$, where $i = 1, 2, \ldots, n$, we have:

$$y_1^{k_1} y_2^{k_2} \ldots y_n^{k_n} > 0. \tag{6.8}$$

1. Let $y_i > 0$ for all $i = 1, 2, \ldots, n$. It implies that $\mathcal{A}$ is positive.

2. Let any of the components $y_i = 0$. Then $\mathcal{A}_{jj\ldots j}$ must have at least one zero entry, where $j \neq i$. This contradict the assumption that all principal subtensors in $\mathbb{T}_{m,n-1}$ are positive.

3. Suppose $y_i < 0$ for some i. Without loss of generality, assume $y_1 < 0$.
   In the case of an odd m, $y_1^m < 0$. Hence, $a_{11..1} = y_1^m y_2^0 \ldots y_n^0 = y_1^m < 0$. This leads to a contradiction to equation (6.8). In the case of an even m, $y_1^{m-1} < 0$. Hence, utilizing equation (6.8), we find that
$$y_1^{m-1} y_i^1 \ldots y_j^0 > 0, \tag{6.9}$$

for all $i = 2, 3, \ldots, n$ and $j \neq i$. Since, $y_1^{m-1} < 0$ and $y_1^{m-1} y_i^1 > 0$ for all $i = 2, 3, \ldots, n$, we have $y_i < 0$ for all $i = 2, 3, \ldots, n$. Hence, $y_i < 0$ for all $i = 1, 2, ..n$.

Since, $\mathcal{A} = a_{i_1 \ldots i_m} = y_1^{k_1} y_2^{k_2} \ldots y_n^{k_n}$, where $y_i < 0$ for all $i = 1, 2, ..n$ and $\sum_{i=1}^{n} k_i = m$, where m is even, we conclude that $\mathcal{A} = a_{i_1 \ldots i_m} = y_1^{k_1} y_2^{k_2} \ldots y_n^{k_n} > 0$

Therefore, from all the cases, we can conclude that $\mathcal{A} > 0$. □

6.4 Characterizing Q-Tensors

Theorem 6.4.1. *Let* $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ *be a* Q*-tensor. Then,* $a_{i i_2 \ldots i_m} \leq 0$ *for all* $i_2, \ldots, i_m \in \{1, 2, \ldots, n\}$ *is not possible, where* $i \in \{1, 2, \ldots, n\}$*,.*

Proof. Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ be a Q-tensor, and assume that for some $i \in \{1, 2, \ldots, n\}$, $a_{i i_2 \ldots i_m} \leq 0$. Observe that $TCP(q, \mathcal{A})$ cannot have a solution for any $q \in \mathbb{R}^n$ such that $q < 0$, since $(\mathcal{A}x^{m-1})_i + q_i < 0$. This leads to a contradiction to $\mathcal{A} \in Q$. □

The following result on Q-tensors provides a sufficient condition for its subtensor to also be a Q-tensor.

Theorem 6.4.2. *Let* $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ *be a* Q*-tensor such that for any* $i \in \{1, 2, \ldots, n\}$, $a_{i i_2 \ldots i_m} \geq 0$*. Then the corresponding sub-tensor* $\mathcal{A}_{ii \ldots i}$ *is also a* Q*-tensor.*

Proof. Without loss of generality, let us consider the case where $i = 1$. We want to show that the subtensor $\mathcal{A}_{11 \ldots 1}$ is a Q-tensor, that is, for any arbitrary vector $\bar{q} = (q_2, q_3, \ldots, q_n)^t$, $TCP(\bar{q}, \mathcal{A}_{11 \ldots 1})$ has a solution.

Consider a vector $q = (q_1, \bar{q}^t)^t$, where $q_1 > 0$. Since $\mathcal{A}$ is a Q-tensor, $TCP(q, \mathcal{A})$ has a solution $x = (x_1, x_2, \ldots, x_n)^t = (x_1, \bar{x}^t)^t$. Since $a_{1 i_2 \ldots i_m} \geq 0$, we have $(\mathcal{A}x^{m-1})_1 + q_1 > 0$, which implies that $x_1 = 0$. When $x_1 = 0$, we observe that $(\mathcal{A}x^{m-1})_{k+1} = (\mathcal{A}_{11 \ldots 1} \bar{x}^{m-1})_k$ for $k = 1, 2, \ldots, n-1$. Therefore, $(\mathcal{A}_{11 \ldots 1} \bar{x}^{m-1} + \bar{q})_j \geq 0$ and $\bar{x}_j (\mathcal{A}_{11 \ldots 1} \bar{x}^{m-1} + \bar{q})_j = 0$ for all j. Hence, $\bar{x}$ is a solution to $TCP(\bar{q}, \mathcal{A}_{11 \ldots 1})$.

Similarly, we can show that $\mathcal{A}_{ii \ldots i} \in Q$ for all $i = 1, 2, .., n$. □

Huang, Suo, and Wang [17] extended the result on Q-matrices from Proposition 2.1 from [33] to tensors. They provided the following result, which gives a condition under which a tensor is a Q-tensor when its subtensors are Q-tensors.

Theorem 6.4.3. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ be a tensor. Let $\mathcal{A}_{11....1}$ and $\mathcal{A}_{22...2}$ are Q-tensors. Suppose that $a_{1i_2 \ldots i_m} = a_{2i_2 \ldots i_m}$ for all $i_2, \ldots, i_m \in [n]$. Then $\mathcal{A}$ is a Q-tensor.*

Example 6.3. *Let $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{T}_{3,2}$ be a tensor such that $a_{111} = 1, a_{211} = 1$ and $a_{122} = 1, a_{222} = 1$. Further assume $a_{112} = a, a_{121} = b, a_{212} = c, a_{221} = d$, where $a = c$ and $b = d$. Then $\mathcal{A} \in Q$.*

Proof. In this case, we have $a_{1i_2i_3} = a_{2i_2i_3}$ for $i_2, i_3 \in \{1, 2\}$. Observe that $\mathcal{A}_{111}, \mathcal{A}_{222} \in Q$. Using the above theorem, $\mathcal{A} \in Q$. □

In the subsequent theorem, we demonstrate that the assumptions in Theorem 6.4.3 can be relaxed to some extent. In fact, no conditions on a, b, c, d are required in the aforementioned example.

Theorem 6.4.4. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$, and let the principal subtensors $\mathcal{A}_{11...1}$ and $\mathcal{A}_{22...2}$ be Q-tensors. Further, suppose $a_{1i_2 \ldots i_m} \geq a_{2i_2 \ldots i_m}$ for $i_2, \ldots, i_m \in [n] \setminus \{1\}$ and $a_{1i_2 \ldots i_m} \leq a_{2i_2 \ldots i_m}$ for $i_2, \ldots, i_m \in [n] \setminus \{2\}$. Then $\mathcal{A}$ is a Q-tensor.*

Proof. Consider an arbitrary vector $q = (q_1, q_2, \ldots, q_n)^t$. Suppose $q_1 \geq q_2$. Let $\bar{x} = (x_2, x_3, \ldots, x_n)^t$ be a solution of $TCP(\bar{q}, \mathcal{A}_{11...1})$, where $\bar{q} = (q_2, q_3, \ldots, q_n)^t$. Let $x = (x_1, \bar{x}^t)^t$, note that $\bar{x}_i = x_{i+1}$ and $\bar{q}_i = q_{i+1}$, for all $i = 1, 2, \ldots, n-1$. We have:

$$(\mathcal{A}_{11...1}\bar{x}^{m-1} + \bar{q})_i = \sum_{i_2,\ldots,i_m \in [n]\setminus\{1\}} a_{i+1 i_2 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} + q_{i+1} \geq 0, \tag{6.10}$$

$$\begin{aligned} \bar{x}_i(\mathcal{A}_{11...1}\bar{x}^{m-1} + \bar{q})_i &= x_{i+1}(\mathcal{A}_{11...1}\bar{x}^{m-1})_i + x_{i+1}q_{i+1} \\ &= x_{i+1}\Big(\sum_{i_2,\ldots,i_m \in [n]\setminus\{1\}} a_{i+1 i_2 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}\Big) + x_{i+1}q_{i+1} \\ &= 0. \end{aligned} \tag{6.11}$$

Now, for $x = (0, \bar{x}^t)^t$, consider:

$$(\mathcal{A}x^{m-1})_1 = \sum_{i_2,\ldots,i_m \in [n]} a_{1i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}. \tag{6.12}$$

In equation (6.12), whenever any of $i_k = 1$, then $a_{1i_2\ldots i_k \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_k} \ldots x_{i_m}$ contributes zero to the total sum because $x_{i_k} = x_1 = 0$. When $i_k > 1$, where $i_k \in [n] \setminus \{1\}$ and $k = \{2, 3 \ldots, m\}$, we have $a_{1i_2\ldots i_m} \geq a_{2i_2\ldots i_m}$. Therefore, we can deduce that:

$$\begin{aligned}(\mathcal{A}x^{m-1})_1 &= \sum_{i_2,\ldots,i_m \in [n]} a_{1i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} \\ &= \sum_{i_2,\ldots,i_m \in [n]\setminus\{1\}} a_{1i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} \\ &\geq \sum_{i_2,\ldots,i_m \in [n]\setminus\{1\}} a_{2i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} \\ &= (\mathcal{A}_{11\ldots 1}\bar{x}^{m-1})_1. \end{aligned} \tag{6.13}$$

Since $q_1 \geq q_2$, and using equations (6.10) and (6.13), we have:

$$(\mathcal{A}x^{m-1} + q)_1 = (\mathcal{A}x^{m-1})_1 + q_1 \geq (\mathcal{A}_{11\ldots 1}\bar{x}^{m-1})_1 + q_2 = (\mathcal{A}_{11\ldots 1}\bar{x}^{m-1} + \bar{q})_1 \geq 0.$$

Now, for the other entries of $\mathcal{A}x^{m-1} + q$, we have:

$$\begin{aligned}(\mathcal{A}x^{m-1} + q)_{i+1} &= \sum_{i_2,\ldots,i_m \in [n]} a_{i+1i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} + q_{i+1} \\ &= \sum_{i_2,\ldots,i_m \in [n]\setminus\{1\}} a_{i+1i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} + q_{i+1} \\ &= (\mathcal{A}_{11\ldots 1}\bar{x}^{m-1} + \bar{q})_i \geq 0. \qquad \text{[from (6.10)]}\end{aligned}$$

Thus, we conclude that $(\mathcal{A}x^{m-1} + q)_i \geq 0$ for all i.

Since $x_1 = 0$, we can observe the following:

$$\sum_{i_2,\ldots,i_m \in [n]} a_{i+1i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m} = \sum_{i_2,\ldots,i_m \in [n]\setminus\{1\}} a_{i+1i_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m},$$

because $x_1 = 0$. Now, we check for complementarity. Since x_1 is zero, $x_1(\mathcal{A}x^{m-1} + q)_1 = 0$ and

$$\begin{aligned} x_{i+1}(\mathcal{A}x^{m-1} + q)_{i+1} &= x_{i+1}\Big(\sum_{i_2,\ldots,i_m \in [n]} a_{i+1 i_2 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}\Big) + x_{i+1} q_{i+1} \\ &= x_{i+1}\Big(\sum_{i_2,\ldots,i_m \in [n]\setminus\{1\}} a_{i+1 i_2 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}\Big) + x_{i+1} q_{i+1} \\ &= 0. \qquad\qquad \text{[from (6.11)]} \end{aligned}$$

Hence, $x_i(\mathcal{A}x^{m-1} + q)_i = 0$ for all $i = 1, 2, \ldots, n$. Therefore, $x = (0, \bar{x}^t)^t$ is the solution for $TCP(q, \mathcal{A})$.

Similarly, when $q_2 \geq q_1$, we use the fact that $\mathcal{A}_{22\ldots2}$ is a Q-tensor. □

We have the following example that illustrates the sharpness of our result.

Example 6.4. *Let $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{T}_{3,2}$ be the tensor such that $a_{111} = 1, a_{211} = 1 + \epsilon$, $a_{122} = 1 + \delta$, and $a_{222} = 1$, where $\epsilon, \delta \in \mathbb{R}_+$. Then $\mathcal{A} \in Q$.*

Proof. Here, $a_{1 i_2 i_3} \geq a_{2 i_2 i_3}$ for $i_2, i_3 \in [n] \setminus \{1\}$ and $a_{1 i_2 i_3} \leq a_{2 i_2 i_3}$ for $i_2, i_3 \in [n] \setminus \{2\}$. Observe that $\mathcal{A}_{111} = (a_{222}) \in Q$ and $\mathcal{A}_{222} = (a_{111}) \in Q$. Using above theorem, $\mathcal{A} \in Q$. □

6.5 Q-Tensor of Rank-One

It is known that rank-one Q-matrices are positive matrices [54]. We extend this result to tensors. It is a known fact that a positive tensor is a Q-tensor. The following theorem provides a condition for a Q-tensor to be positive. First, recall the definition of rank-one tensor $\mathcal{A}$ associated with a vector y, we have $a_{i_1 \ldots i_m} = y_{i_1} \ldots y_{i_m}$. The following result is also attributed to Sonali Sharma and K. Palpandi [50], here, we provide an alternative proof.

Theorem 6.5.1. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ be a rank-one tensor associated with a vector. If $\mathcal{A} \in Q$, then all entries of tensor $\mathcal{A}$ are positive.*

Proof. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{T}_{m,n}$ be a rank-one tensor corresponding to some vector $y \in \mathbb{R}^n$. Suppose $\mathcal{A} \in Q$. If $\mathcal{A}$ contains a zero entry, then at least one coordinate of y must be zero. Let $y_i = 0$, and this will imply that $a_{ii_2 \dots i_m} = 0$. This leads to a contradiction to Theorem 6.4.1. Therefore, $y_i \neq 0$ for all $i \in [n]$. We provide the proof using mathematical induction.

For $n = 1$, $\mathcal{A} = (a_{11\dots1})$. Since $\mathcal{A} \in Q$, we have $a_{11\dots1} > 0$. Hence, the statement holds for $n = 1$.

For $n = 2$, let us assume that $\mathcal{A}$ has some negative entry. Then, the vector y must have at least one negative coordinate. WLOG, let $y = (-a, b)^t$, where $a, b > 0$. Consider a vector $q = (-q_1, -q_2)^t$, where $q_1, q_2 > 0$. Let $x = (x_1, x_2)^t$ be a solution to $TCP(q, \mathcal{A})$. We consider the following cases:

1. It is clear that $x = (0, 0)^t$ is not a solution to $TCP(q, \mathcal{A})$.

2. Let $x = (x_1, 0)^t$, where $x_1 > 0$. If m is odd, we have $(\mathcal{A}x^{m-1} + q)_1 = (-a)^m x_1^{m-1} - q_1 < 0$, which is a contradiction. If m is even, we have $(\mathcal{A}x^{m-1} + q)_2 = (-a)^{m-1} b x_1^{m-1} - q_2 < 0$, which is a contradiction. Hence, $x = (x_1, 0)^t$ cannot be a solution to $TCP(q, \mathcal{A})$.

3. Let $x = (0, x_2)^t$, where $x_2 > 0$. Then $(\mathcal{A}x^{m-1} + q)_1 = -ab^{m-1} x_2^{m-1} - q_1 < 0$, which leads to a contradiction. Hence, exactly one coordinate of x cannot be positive.

4. Let $x = (x_1, x_2)^t > 0$. Therefore $(\mathcal{A}x^{m-1} + q)_1 = 0 = (\mathcal{A}x^{m-1} + q)_2$. Using Remark 6.2.6, we deduce that $(\mathcal{A}x^{m-1})_1 = \frac{-a}{b} (\mathcal{A}x^{m-1})_2$. Therefore, we have

$$\begin{aligned}
(\mathcal{A}x^{m-1} + q)_2 = (\mathcal{A}x^{m-1})_2 - q_2 &= 0 \\
\Rightarrow (\mathcal{A}x^{m-1})_2 &= q_2 \\
\text{Now, } (\mathcal{A}x^{m-1})_1 - q_1 &= 0 \\
\Rightarrow \frac{-a}{b} (\mathcal{A}x^{m-1})_2 - q_1 &= 0 \\
\Rightarrow \frac{-a}{b} q_2 - q_1 &= 0 \\
\Rightarrow q_1 &= \frac{-a}{b} q_2. \qquad (6.14)
\end{aligned}$$

Since $q_1, q_2, a, b > 0$, this contradicts equation (6.14).

Hence, for $\mathcal{A}$ having any nonpositive value, $TCP(q, \mathcal{A})$ have no solution. This leads to a contradiction to our hypothesis $A \in Q$. Therefore, $\mathcal{A} > 0$ for $n = 2$.

Let the result be valid for all $k \leq n-1$.

Consider $\mathcal{A}_{nn\ldots n} = (a_{i_1\ldots i_m})$ a principal subtensor of $\mathcal{A}$, where $i_1, \ldots, i_m \in [n-1]$. Let $q \in \mathbb{R}^{n-1}$ be an arbitrary vector. WLOG, let $q_1 < 0$ (since $q \geq 0$ gives us a trivial solution). Consider $\bar{q} = (q^t, 0)^t \in \mathbb{R}^n$. Since, $\mathcal{A} \in Q$, there exists some $x = (x_1, x_2, \ldots, x_n)^t \geq 0$ such that

$$\mathcal{A}x^{m-1} + \bar{q} \geq 0 \text{ and } x^t(\mathcal{A}x^{m-1} + \bar{q}) = 0. \tag{6.15}$$

We claim that $x_n = 0$. On the contrary, suppose that $x_n > 0$. Then $(\mathcal{A}x^{m-1} + \bar{q})_n = (\mathcal{A}x^{m-1})_n + 0 = 0$. Using Remark 6.2.7, for some α, we have $(\mathcal{A}x^{m-1})_1 = \alpha(\mathcal{A}x^{m-1})_n = 0$. Hence, we have $(\mathcal{A}x^{m-1} + \bar{q})_1 = (\mathcal{A}x^{m-1})_1 - q_1 < 0$. This leads to a contradiction to (6.15). Therefore, $x_n = 0$.

Consequently, $(x_1, \ldots, x_{n-1})^t$ is a solution to $TCP(q, \mathcal{A}_{nn\ldots n})$. Therefore, $\mathcal{A}_{nn\ldots n} \in Q$. Using the hypothesis, we have $\mathcal{A}_{nn\ldots n} > 0$.

Similarly, we can demonstrate that each subtensor $\mathcal{A}_{ii\ldots i} > 0$ for all $i \in [n]$. Hence, each principal subtensor of order $n-1$ of $\mathcal{A}$ is positive. Therefore, using Theorem 6.3.3, we conclude that $\mathcal{A} > 0$. □

In Theorem 6.5.1, both conditions, namely, $\mathcal{A}$ has rank-one and $\mathcal{A} \in Q$ are necessary, as demonstrated in the following examples.

Example 6.5. *Let* $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{T}_{3,2}$ *be the tensor with the following entries* $a_{111} = 1, a_{211} = 1 + \epsilon$ *and* $a_{122} = 1 + \delta, a_{222} = 1$, *where* $\epsilon, \delta \in \mathbb{R}_+$. *Further,* $a_{121} = -1$. *Then* $A \in Q$ *but* A *is not a positive tensor.*

Proof. We have $a_{1 i_2 i_3} \geq a_{2 i_2 i_3}$ for $i_2, i_3 \in [n] \setminus \{1\}$ and $a_{1 i_2 i_3} \leq a_{2 i_2 i_3}$ for $i_2, i_3 \in [n] \setminus \{2\}$. Observe that $\mathcal{A}_{111}, \mathcal{A}_{222} \in Q$. Using Theorem 6.4.4, we can conclude that $\mathcal{A} \in Q$. However,

$a_{121} = -1$ implies that A is not a positive tensor. Therefore, it is clear that $\mathcal{A} \in Q$ alone is not sufficient to establish $\mathcal{A} > 0$. □

Example 6.6. Let $\mathcal{A} \in \mathbb{T}_{3,2}$ be the tensor with $a_{111} = a_{122} = a_{212} = a_{221} = 1$ and $a_{112} = a_{121} = a_{211} = a_{222} = -1$.

Proof. We observe that $\mathcal{A}$ is a rank-one tensor associated with vector $y = (1, -1)^t$. Note that $\mathcal{A} \notin Q$, since $(\mathcal{A}x^{m-1})_2 = -(x_1 - x_2)^2 \leq 0$. Hence, there is no solution for any $TCP(q, \mathcal{A})$, where $q = (q_1, q_2)^t$ and $q_2 < 0$. However, the entries $a_{112} = a_{121} = a_{211} = a_{222} = -1$ indicate that $\mathcal{A}$ is not a positive tensor. Therefore, it is evident that rank-one of tensor $\mathcal{A}$ alone is not sufficient to establish that $\mathcal{A}$ is positive. □

Song and Qi [52] proved that R-tensors are Q-tensors, converse of this is not true in general. The following example is a Q-tensor but not an R-tensor.

Example 6.7. Let $\mathcal{A} \in \mathbb{T}_{3,2}$ be the tensor such that $a_{111} = 1,\ a_{112} = -1,\ a_{121} = -1,\ a_{122} = 1,\ a_{211} = 2,\ a_{212} = -3,\ a_{221} = 0$, and $a_{222} = 1$.

Proof. We observe that $\mathcal{A}_{111} \in Q$ and $\mathcal{A}_{222} \in Q$. Further, $a_{111} \leq a_{211}$ and $a_{222} \leq a_{122}$. Hence, using Theorem 6.4.4, $\mathcal{A} \in Q$. Notice that $x = (1, 1)^t$ is a solution for $TCP(q, \mathcal{A})$, for $q = (0, 0)^t$. Therefore, $\mathcal{A} \notin R_0$ as well as $\mathcal{A} \notin R$. □

In the following theorem, we provides a sufficient condition for a Q-tensor to be an R-tensor.

Theorem 6.5.2. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ be a rank-one tensor associated with a vector. If $\mathcal{A} \in Q$, then $\mathcal{A} \in R$. In fact, it is R_0 and ER.*

Proof. Consider $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{T}_{m,n}$ as a rank-one tensor associated with a vector. If $\mathcal{A} \in Q$, using Theorem 6.5.1, we have $\mathcal{A} > 0$. Hence, using Theorem 6.3.1, we can conclude that $\mathcal{A} \in R$, $\mathcal{A} \in R_0$, and $\mathcal{A} \in ER$. □

CHAPTER 7

Conclusion

In this concluding chapter, we summarize the contents of the thesis and list some of the questions raised during the process of this research. We also suggest some possible directions to extend our work.

7.1 Summary

This thesis has explored significant contributions to the study of the Linear Complementarity Problem (LCP) and completely mixed games, addressing open questions and establishing novel results within these domains. The primary focus has been on the interplay between matrix classes, their properties, and their implications for solving LCPs and understanding the structure of completely mixed games.

In Chapter 3, we settled Pang's conjecture by proving that semimonotone Q-matrices are R_0-matrices for orders up to 3, with a counterexample demonstrating the limitation of this result for higher orders.

In Chapter 4, we provided a new characterization for completely mixed games similar to that of Kaplansky. We also provide new characterizations for odd-ordered symmetric games. Furthermore, these results were generalized to include almost skew symmetric matrices.

In Chapter 5, we contributed to Stone's conjecture, focusing on matrices with specific sign patterns and extending results to Z-matrices. We also provide connections between completely mixed games and matrix classes such as semimonotone, copositive, and Z-matrices.

In Chapter 6, we generalize matrix results to tensors. We extend some known results for Q-matrices, nonnegative matrices to Q-tensor and nonnegative tensors, respectively.

7.2 Implications and Future Directions

The findings in this thesis have implications across optimization, game theory, and computational mathematics. By bridging concepts from linear algebra, game theory, and tensor analysis, the work lays a foundation for further exploration in these interconnected fields. We have the following problems for future work:

- Investigating semimonotone Q-matrices of higher orders with some additional condition to identify subclasses that might satisfy the equivalence with R_0-matrices.
- Let $A \in \mathbb{R}^{n\times n} \cap R_0$ where $n \geq 4$, and assume that the game associated with matrix A is completely mixed. Can we say, for all i, $A + D_i \in Q$?
- Stone's conjecture is still an open problem.

This thesis represents a step forward in understanding the fundamental properties of the Linear Complementarity Problem and completely mixed games. The theoretical contributions and connections established here underscore the rich interplay between algebraic structures and optimization problems, offering a fertile ground for further exploration and practical applications.

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