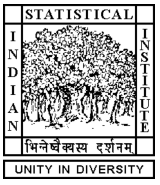
Subconvex Bounds for L-functions on $GL(3)$ and $GL(2) \times GL(2)$

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Subconvex Bounds for L-functions on $GL(3)$ and $GL(2) \times GL(2)$

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Chapter 1

Introduction

We give a brief introduction to the sub-convexity problem. Let π be an automorphic form or automorphic representation of $GL_n(\mathbb{A})$. One may then attach an automorphic L-function to π

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$$

for $\mathfrak{Re}(s) \gg 1$. This L-function admits a meromorphic continuation to the whole complex plane with a possible pole at $s = 0, 1$. The completed L-function $\Lambda(s,\pi)$ (see [17]) satisfies a functional equation

$$\Lambda(s,\pi) = \epsilon(\pi)\Lambda(1-s,\check{\pi})$$

where $\check{\pi}$ is the contra-gradient/dual of π.

The Generalised Riemann Hypothesis (GRH) predicts that all the "non-trivial" zeroes of $L(s,\pi)$ lie on the critical line $\mathfrak{Re}(s) = \frac{1}{2}$ from which it follows that

$$L(\frac{1}{2} + it, \pi) \ll_\epsilon C(\pi, \frac{1}{2} + it)^\epsilon$$

where $C(s,\pi)$ is the "analytic conductor" of the pair (s,π). Unconditionally, by the convexity principle, we know that

$$L(\frac{1}{2}+it,\pi) \ll C(\pi,\frac{1}{2}+it)^{1/4+\epsilon}$$

The sub-convexity problem is to beat the above bound.

Instead of talking about the sub-convexity problem in the entire conductor aspect, one can talk about it in particular special aspects. In this thesis, we shall be concerned with the level and t-aspects.

Theorem 1. *Let f be a holomorphic modular form of level p and nebentypus χ. Assume χ is not quadratic. Let $\mathrm{sym}^2 f$ be its symmetric square representation. Then*

$$L(\mathrm{sym}^2 f,\frac{1}{2}+it) \ll p^{1/2+\epsilon}t^{3/4-1/12+\epsilon}$$

Theorem 1 is joint work with Munshi.

Theorem 2. *Let p be an odd prime. Let π_1 and π_2 be two automorphic representations with level p^{l_1} and p^{l_2} respectively. Assume that the nebentypus of π_1, denoted ω_{π_1}, with conductor exponent $a(\omega_{\pi_1})$ satisfies $a(\omega_{\pi_1}) \leq \lfloor \frac{a(\pi_1)}{2} \rfloor - 1$. Assume that the nebentypus of π_2, denoted ω_{π_2}, with conductor exponent $a(\omega_{\pi_2})$ satisfies $a(\omega_{\pi_2}) \leq \frac{2l_1}{5}$ and $l_2 \leq \frac{4l_1}{5}$. Then we have the bound*

$$L\left(\pi_1 \otimes \pi_2,\frac{1}{2}\right) \ll p^{O(1)}p^{\frac{a(\pi_1\otimes\pi_2)}{4}\left(1-\frac{1}{10}\right)}.$$

We use the delta symbol, recently pioneered by Munshi to great effect in solving the sub-convexity problem in various aspects.

Chapter 2

Preliminaries

2.1 p-adic Characters and Sums

2.1.1 The p-adic Logarithm and Exponential Maps

Much of what's stated in this subsection can be found in [25] for properties of $\mathbb{Q}_p$ and chapter XI of [23] over number fields over $\mathbb{Q}_p$. We always assume that $p \neq 2$.

Denote the p-adic norm of $\mathbb{Q}_p$ by $|\cdot|_p$. Given $x \in \mathbb{Q}_p$ we define the formal power series $\exp_p(x)$ and $\log_p x$ defined by

$$\exp_p(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x)$$

$$\log_p(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} = \lambda(x)$$

$\epsilon_0(x)$ is well-defined for $|x|_p < r_p = p^{-1/(p-1)}$ while $\lambda(x)$ is well-defined for $|x|_p < 1$. If we set $B_r = \{x \in \mathbb{Q}_p : |x|_p < r\}$ then $\epsilon_0, \lambda : B_{r_p} \to B_{r_p}$ define isometries that is, $|\lambda(x)|_p = |\epsilon_0(x)|_p = |x|_p$ for $x \in B_{r_p}$. Furthermore, $\epsilon_0 \circ \lambda = \lambda \circ \epsilon_0 = id_{B_{r_p}}$.

One can define the same function over any finite extension $E/\mathbb{Q}_p$ by defining identically to the power series in the $\mathbb{Q}_p$ case. Let e be the ramification index of E and $\mathfrak{p}$ the maximal

ideal of E. Let $|\cdot|_{\mathfrak{p}}$ be the associated $\mathfrak{p}$-adic valuation. Then the $\mathfrak{p}$-adic logarithmic, defined identically as in the p-adic case, is well-defined for $|x|_{\mathfrak{p}} < 1$ while the $\mathfrak{p}$-adic exponential $exp_{\mathfrak{p}}$ also identically defined as in the p-adic case, is well-defined for $|x|_{\mathfrak{p}} < p^{-e/(p-1)}$. The same relations hold between the two functions, namely ϵ_0 and λ are inverses of each other where they are defined.

A useful consequence of these observations is the following. Let $j \geq s \geq 1$. Then for residue classes $k \bmod p^{j-s}$, the map $kp^s \bmod p^j \mapsto \epsilon_0(kp^s) \bmod p^j$ is a bijection.

2.1.2 Characters on $\mathbb{Q}_p$

Denote the set of characters on $\mathbb{Q}_p^\times$ with conductor exponent j and $\chi(p) = 1$ by $\mathfrak{X}_j$. These are (essentially) the primitive Dirichlet characters mod p^j and also denote such characters by $\mathfrak{X}_j$. There exists a constant α_χ such that given such a character, for any $s \geq 1$ and $k \in \mathbb{Z}_p$,

$$\chi(1 + kp^s) = e\left(\alpha_\chi \frac{\log_p(1 + kp^s)}{p^j}\right).$$

The constant $\alpha_\chi \in (\mathbb{Z}/p^j\mathbb{Z})^\times$. Denote the set of all characters mod p^s by G_s. Then if $s < j$, the set G_s has a group action on $\mathfrak{X}_j$. We denote $\mathfrak{X}_j/\sim s$ to be the quotient space $\mathfrak{X}_J/G_s$. Given two characters $\chi_1, \chi_2 \in \mathfrak{X}_j$ we say $\chi_1 =_{\sim s} \chi_2$ if there exists $\psi \in G_s$ such that $\chi_1 = \psi\chi_2$. We note that if $\chi_1, \chi_2 \in \mathfrak{X}_j$ and α_1 and α_2 are their corresponding constants, then this implies that $\alpha_1 - \alpha_2 \equiv 0 \bmod p^{j-s}$. We then observe that the set $\mathfrak{X}_j/G_s$ is in bijection with the set $(\mathbb{Z}/p^{j-s}\mathbb{Z})^\times$.

We use the above properties to prove the following.

Proposition 2.1.1. *Let χ be a character mod p^j. Let $j > s \geq 1$. Then the map*

$$k \mapsto \chi(\exp(kp^s))$$

defines a character on the additive group $(\mathbb{Z}/p^{j-s}\mathbb{Z})$.

Proof: Let $k_1, k_2 \bmod p^{j-s}$. We have

$$\begin{aligned}\chi(\exp((k_1+k_2)p^s)) &= e\left(\alpha_\chi \frac{k_1+k_2}{p^{j-s}}\right) \\ &= e\left(\alpha_\chi \frac{k_1}{p^{j-s}}\right) e\left(\alpha_\chi \frac{k_2}{p^{j-s}}\right) \\ &= \chi(\exp(k_1 p^s))\chi(\exp(k_2 p^s)).\end{aligned}$$

□

Lemma 2.1.2. *Let χ be a Dirichlet character mod p^j, not necessarily primitive. Let $j > s \geq 1$ be integers and let $n = n_0 p^r$. Assume $r \leq j - s$. Then*

$$\sum_{k \bmod p^{j-s}} \chi(1+kp^s) e\left(\frac{kn}{p^{j-s-r}}\right)$$

is 0 unless χ is induced from a character mod p^{j-r}. In that chase, the character sum becomes

$$p^r \sum_{k \bmod p^{j-s-r}} \chi(1+kp^s) e\left(\frac{kn_0}{p^{j-s-r}}\right)$$

where by an abuse of notation we've identified χ with its inducing character.

Proof: We apply the bijection $kp^s \mapsto \epsilon_0(kp^s)$. We then have that

$$\sum_{k \bmod p^{j-s}} \chi(1+kp^s) e\left(\frac{kn}{p^{j-s-r}}\right) = \sum_{k \bmod p^{j-s}} e\left(\alpha_\chi \frac{k}{p^{j-s}} + \frac{n_0 \epsilon_0(kp^s)}{p^{j-r}}\right).$$

We now write $k = k_0 + k_1 p^{j-s-r}$ with $k_0 \bmod p^{j-s-r}$ and $k_1 \bmod p^r$. We observe that $\epsilon_0(kp^s) \equiv \epsilon_0(k_0 p^s) \bmod p^{j-r}$. Thus the sum becomes

$$\sum_{k_0 \bmod p^{j-s-r}} e\left(\alpha_\chi \frac{k_0}{p^{j-s}} + \frac{n_0 \epsilon_0(kp^s)}{p^{j-r}}\right) \sum_{k_1 \bmod p^r} e\left(\alpha_\chi \frac{k_1}{p^r}\right).$$

We observe that the inner-sum is 0 unless $\alpha_\chi \equiv 0 \bmod p^r$. In that case, we deduce that χ is lifted from a character mod p^{j-r}. The lemma then follows upon reversing the bijection $\epsilon_0(kp^s) \mapsto k$ for $k \bmod p^{j-s-r}$. □

Lemma 2.1.3. *Let $j > s \geq 1$ be integers. Suppose $\chi \in \mathfrak{X}_j$. Let n be an integer. Then the sum*

$$\sum_{k \bmod p^{j-s}} \chi(1+kp^s) e\left(\frac{nk}{p^{j-s}}\right)$$

vanishes unless $(n,p) = 1$.

Proof: Suppose $(n,p) > 1$. Set $n = pn_1$. Then from the proof of the previous lemma we deduce that the sum is 0 unless $p|\alpha_\chi$. But since χ is primitive, $\alpha_\chi \in (\mathbb{Z}/p^j\mathbb{Z})^\times$. The lemma then follows. □

Let's denote the set of all characters mod p^j by G_j. We observe that for $s < j$ the action of multiplication defines a group action of G_s on $\mathfrak{X}_j$. We denote $\mathfrak{X}_j/_{\sim s}$ to be the quotient space $\mathfrak{X}_j/G_s$ respectively. The action is faithful since the kernel of the action is simply the trivial character. Give two characters χ_1 and χ_2 in $\mathfrak{X}_j$, we write $\chi_1 =_{\sim s} \chi_2$ if there exists a character χ_0 in G_s such that $\chi_1 = \chi_0\chi_2$. Equivalently, both χ_1 and χ_2 have the same image in $G_j/_{\sim s}$.

2.1.3 Characters on Extensions of $\mathbb{Q}_p$

Let $E/\mathbb{Q}_p$ be a degree 2 extension. Let e be its ramification index$E$. Let d be the order of its different. So when $e = 2$ then $d = 1$ and when $e = 0$ then $d = 0$. Let $\mathfrak{o}_E$ be its ring of integers. Let ω be a uniformizer of E. Let $\mathfrak{p}_E$ be its maximal ideal. Let $U_0 = \mathfrak{o}_E^\times$ and $U_r = 1 + \mathfrak{p}_E^r$ for $r \geq 1$. Denote by Nm the norm map $E \to \mathbb{Q}_p$ and tr the trace map $\mathrm{tr} : E \to \mathbb{Q}_p$. Let ψ_E be the character of order $-d$ that is, if $\psi_{\mathbb{Q}_p}$ is the standard additive character on $\mathbb{Q}_p$ then we define $\psi_E = \psi_{\mathbb{Q}_p} \circ \mathrm{tr}(\omega^{-d}(\cdot))$. With these normalizations we have the standard fact that

$$\psi_E(\omega^{-n}u) = \begin{cases} |\mathfrak{o}/\mathfrak{p}|^n & \text{if } u \bmod \mathfrak{p}^n \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$.

We can in fact extend the notation to the principal series representation as well. Let $E = \mathbb{Q}_p \times \mathbb{Q}_p$ be the split extension of $\mathbb{Q}_p$ with component-wise addition and multiplication. Note that E is not a field. We then have the norm map Nm_E and trace map tr_E given by

$$Nm_{E/\mathbb{Q}_p}(x_1, x_2) = x_1 x_2,$$

$$\mathrm{tr}_E(x_1, x_2) = x_1 + x_2.$$

A character on $E^\times$ is then given by a pair of characters (χ_1, χ_2) on $\mathbb{Q}_p^\times$ with action given by

$$(\chi_1, \chi_2)(x_1, x_2) = \chi_1(x_1)\chi_2(x_2)$$

for $x_1, x_2 \in \mathbb{Q}_p^\times$. The maximal ideal $\mathfrak{p}$ is defined as the ideal generated by the element (p, p) and the ring of integers $\mathfrak{o}_E$ is defined as $\mathbb{Z}_p \times \mathbb{Z}_p$. We then similarly define $U_0 = \mathfrak{o}_E^\times$ and $U_r = 1 + \mathfrak{p}_E^r$ for $r \geq 1$ as before.

Let $E/\mathbb{Q}_p$ be a degree 2 extension as given above. Given a character θ on $E^\times$, denote $a(\theta)$ to be the conductor exponent of θ that is, there exists a constant α_θ such that for any $s \geq e$ and $u \in \mathfrak{o}_E$ we have

$$\theta(1 + \omega^s u) = \psi_E(\omega^{-a(\theta)} \log_{\mathfrak{p}}(1 + \omega^s u))$$

or equivalently

$$\theta(\exp \omega^s u)) = \psi_E(\omega^{-a(\theta)+s} c_\theta u).$$

Observe that for an admissible pair (E, θ) attached to a supercuspidal representation π such that E is purely ramified, $a(\theta)$ is always even as long as $a(\pi) > 2$.

In particular for $j \geq 2$ if for $\chi \in \mathfrak{X}_j$ we have

$$\chi(1 + kp^s) = e\left(\alpha_\chi \frac{\log_p(1 + kp^s)}{p^j}\right),$$

then for $r \geq e$

$$\chi(1+\omega^r u) = \psi_E(\alpha_\chi \omega^{-a(j)} \log_p(1+\omega^r u))$$

where $a(j) := ej$.

We will use the following stabilization lemma due to Deligne.

Lemma 2.1.4. *Let θ_1 and θ_2 be two characters on $E^\times$ with conductor exponents $a(\theta_1)$ and $a(\theta_2)$. Assume that $a(\theta_1) \geq 2a(\theta_2)$. Let c_{θ_1} be the constant associated to θ_1. Then*

$$\epsilon(\theta_1\theta_2) = \theta_2(c_{\theta_1})\epsilon(\theta_1).$$

Proof: The proof can be found in [7]. □

2.1.4 Hensel's Lemma

We use the following versions of Hensel's Lemma. The proof is well known.

Lemma 2.1.5 (Multivariate Hensel's Lemma)**.** *Let $f = (f_1, ..., f_d) \in \mathbb{Z}_p[X_1, ..., X_d]^d$. Define the Jacobian of f by the matrix*

$$J_{f_1,...,f_d} = \left(\frac{\partial f_i}{\partial X_j}\right)_{1 \leq i,j \leq d}.$$

Let $a = (a_1, ..., a_d) \in \mathbb{Z}_p^d$. Set $||a|| = \sup_{1 \leq i \leq d} |a_i|_p$. Suppose a satisfies

$$||f(a)|| < |\det J_f(a)|_p^2.$$

Then there exists a unique point $\alpha \in \mathbb{Z}_p^d$ with $f(\alpha) = (0, ..., 0)$ and $||\alpha - a|| < J_f(a)$. In particular, if $||f(a)|| < 1$ and $|\det J_f(a)|_p = 1$ then there exists a unique α satisfying $f(\alpha) = 0$ and $||\alpha - a|| < 1$.

We will use this in the cases $d = 1$ and $d = 2$. When $d = 1$ we recover the classical Hensel's Lemma. When $d = 2$, we encounter the following problem - let F_1 and F_2 be

two polynomial functions in two variables, say α_1 and α_2. We then require to count the number of solutions to the simultaneous equations

$$F_1(\alpha_1, \alpha_2) \equiv F_2(\alpha_1, \alpha_2) \equiv 0 \bmod p^j$$

for $j \geq 2$. We will in fact show that mod p there are only $O(1)$ many solutions, and then check that the Jacobian is non-singular mod p. This guarantees that the lift mod p^j is unique per root, ergo concluding that there are only $O(1)$ many solutions. It is important that the count of the solutions doesn't grow with p^j and forms the basis of proving "square-root cancellation" via p-adic stationary phase.

2.2 Quadratic Character Sums

We evaluate certain character sums that occur in the proofs of Theorem 1 and Theorem 2.

Define

$$G(a, b; c) = \sum_{x \bmod c} e\left(\frac{ax^2}{c} + \frac{bx}{c}\right)$$

to be the quadratic Gauss sum. We always assume $(a, c) = 1$.

Lemma 2.2.1. *(1) If c is even, say $c = 2r$ then $G(a, b, 2r)$ is 0 unless $ar + b$ is even.*

(2) If $(r, s) = 1$ then $G(a, b, rs) = G(ar, b, s)G(as, b, r)$.

(3) We have

$$G(a,b,c) = \begin{cases} e\left(-\frac{\overline{4a}b^2}{c}\right)G(a,0,c) & \text{if } c \text{ is odd} \\ 2e\left(-\frac{\overline{8a_r}b^2}{r}\right)G(2a,0,r) & \text{if } 2||c, c = 2r \text{ and } b \text{ is odd} \\ e\left(-\frac{\overline{a}(b/2)^2}{c}\right)G(a,0,c) & \text{if } 4|c \text{ and } b \text{ is even} \end{cases} \tag{2.1}$$

where a_r is to denote the inverse mod r.

Proof: This is contained in Lemma 5.4.5 of [15]. □

Lemma 2.2.2. *Assume c is odd and let $(a,c)=1$. We have*

$$\frac{G(a,0,c)}{c^{1/2}} = \begin{cases} \left(\frac{a}{c}\right) & \text{if } c \equiv 1 \bmod 4 \\ i\left(\frac{a}{c}\right) & \text{if } c \equiv 3 \bmod 4 \end{cases} \tag{2.2}$$

Proof: This is Theorem 1.3.4 of [1]. □

We denote

$$\epsilon_c = \begin{cases} 1 & \text{if } c \equiv 1 \bmod 4 \\ i & \text{if } c \equiv 3 \bmod 4 \end{cases}$$

Lemma 2.2.3. *Suppose $c = 2^k c_0$ with $k \geq 2$ and c_0 odd. Assume $(a,c)=1$. Then*

$$\frac{G(a,0,c)}{c^{1/2}} = \epsilon_{c_0}\left(\frac{a}{c_0}\right) \begin{cases} 1+e\left(\frac{ac_0}{4}\right) & \text{if } k \equiv 0 \bmod 2 \\ 2^{1/2}e\left(\frac{ac_0}{8}\right) & \text{if } k \equiv 1 \bmod 2 \end{cases} \tag{2.3}$$

Proof: This is Lemma 5.1 of [20]. □

Now let p be an odd prime. In the proof of Theorem 1 we encounter the following character sum. Define

$$\mathcal{C}(m,n,q) = \frac{1}{q}\sideset{}{^*}\sum_{a \bmod q}\sum_{x \bmod q} e\left(\frac{ax^2+xn+\overline{a}\overline{p}m}{q}\right) \tag{2.4}$$

We first assume that q is odd. Applying Lemma (2.2.1) and Lemma (2.2.2) and a change of variable $a \mapsto \overline{a} \bmod q$ we get

$$\frac{1}{q^{1/2}}\sideset{}{^*}\sum_{a \bmod q}\left(\frac{a}{q}\right)e\left(\frac{(\overline{p}m-\overline{4}n^2)a}{q}\right).$$

Next make the change of variable $a \mapsto 4ap$ giving

$$\frac{1}{q^{1/2}}\left(\frac{p}{q}\right)\sum_{a \bmod q}\left(\frac{a}{q}\right)e\left(\frac{a(4m-pn^2)}{q}\right).$$

Denote the sum over $a \bmod q$ as $\mathcal{C}_1(4m-pn^2,q)$. For notational purposes, let's assume $l = 4m - pn^2$.

First, decompose $q = q^*q_0^2$ with q^* square-free. Then q^* is the conductor of the Legendre symbol $\left(\frac{\cdot}{q}\right)$. Next, we split $q_0^2 = q_1q_2$ where $q_1|(q^*)^\infty$ and $(q_2, q^*) = 1$. We apply the Chinese Remainder Theorem getting

$$\begin{aligned}
\mathcal{C}_1(l,q) &= \sum_{a \bmod q_1q^*}\left(\frac{a}{q^*}\right)e\left(\frac{a\overline{q_2}l}{q_0}\right)\sideset{}{^*}\sum_{b \bmod q_2}e\left(\frac{b\overline{q_0}l}{q_2}\right)\\
&= S(l,0,q_2)\sum_{a \bmod q_1q^*}\left(\frac{a}{q^*}\right)e\left(\frac{al}{q_0}\right)\\
&= S(l,0,q_2)\sum_{a \bmod q^*}\left(\frac{a}{q^*}\right)\sum_{k=1}^{q_1}e\left(\frac{l(a+kq^*)}{q_0}\right)\\
&= \begin{cases} S(l,0,q_2)q_1\epsilon_{q^*}\sqrt{q^*}\left(\frac{\frac{l}{q_1}}{q^*}\right) & \text{if } q_1|l\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

Let us state this calculation as a lemma.

Lemma 2.2.4. *Let m and n be integers and q an odd positive integer. Let $l = 4m - pn^2$. Assume $q = q^*q_1q_2$ with q^* the square-free part of q, $q_1|(q^*)^\infty$ and $(q_2, q^*) = 1$. Then*

$$\mathcal{C}(m,n,q) = \frac{1}{q^{1/2}}\left(\frac{p}{q}\right)S(l,0,q_2)q_1\epsilon_q^*\sqrt{q^*}\left(\frac{\frac{l}{q_1}}{q^*}\right) \tag{2.5}$$

if $q_1|l$ and is 0 otherwise.

Next we assume $2||q$, say $q = 2q_0$ with q_0 odd. Then the character sum is given by

$$\frac{1}{q_0^{1/2}}\sideset{}{^*}\sum_{a \bmod 2q_0}\left(\frac{2a}{r}\right)e\left(\frac{\overline{ap}m}{2q_0} - \frac{\overline{8a_{q_0}}n^2}{q_0}\right)$$

where a_{q_0} denote the residue class of $a \bmod q_0$. We note that $\overline{a} \equiv 2\overline{2a_r} + r \bmod 2r$. Applying a suitable change of variables gives

$$\frac{2}{q_0^{1/2}}\left(\frac{p}{q_0}\right)e\left(\frac{pm}{2}\right)\mathcal{C}_1(4m - pn^2, q_0)$$

The character sum $\mathcal{C}_1$ can then be evaluated as above.

Now assume $4|q$. Suppose $q = 2^k q_0$ with q_0 odd and $k \geq 2$. First we assume $k \equiv 0 \bmod 2$. Then by Lemma 2.2.3 we have

$$q^{1/2}\epsilon_{q_0}\sideset{}{^*}\sum_{a \bmod q}\left(\frac{a}{q_0}\right)e\left(\frac{(\overline{p}m - (n/2)^2)\overline{a}}{q}\right)\left(1 + e\left(\frac{aq_0}{4}\right)\right).$$

We apply a change of variable $a \mapsto ap$ to get

$$q^{1/2}\epsilon_{q_0}\left(\frac{p}{q_0}\right)\sideset{}{^*}\sum_{a \bmod q}\left(\frac{a}{q_0}\right)e\left(\frac{(m - p(n/2)^2)\overline{a}}{q}\right)\left(1 + e\left(\frac{apq_0}{4}\right)\right)$$

Next we make the change of variable $a \mapsto \overline{a} \bmod c$ and note that $a \equiv \overline{a}_c \bmod 4$. Thus we get

$$q^{1/2}\epsilon_{q_0}\left(\frac{p}{q_0}\right)\sideset{}{^*}\sum_{a \bmod q}\left(\frac{a}{q_0}\right)e\left(\frac{(m - p(n/2)^2)a}{q}\right)\left(1 + e\left(\frac{apq_0}{4}\right)\right).$$

Applying the Chinese Remainder Theorem gives

$$q^{1/2}\epsilon_{q_0}\left(\frac{p}{q_0}\right)\sum_{a \bmod q_0}\left(\frac{a}{q_0}\right)e\left(\frac{\overline{2^k}a(m - p(n/2)^2)}{q_0}\right)$$
$$\times \sideset{}{^*}\sum_{b \bmod 2^k} e\left(\frac{b\overline{q_0}(m - p(n/2)^2)}{2^k}\right)\left(1 + e\left(\frac{bpq_0}{4}\right)\right).$$

We now calculate the $b \bmod 2^k$ sum. For notation, we set $l = m - p(n/2)^2$. We have

$$\begin{aligned}\sum_{b \bmod 2^k}^{*} e\left(\frac{b\overline{q_0}l}{2^k}\right)\left(1+e\left(\frac{bpq_0}{4}\right)\right) &= \sum_{b \bmod 2^k}^{*} e\left(\frac{bl}{2^k}\right)\left(1+e\left(\frac{bp}{4}\right)\right)\\ &= S(l,0,2^k) + \sum_{b \bmod 2^k}^{*} e\left(\frac{bl}{2^k}\right)e\left(\frac{bp}{4}\right)\\ &= S(l,0,2^k) + \sum_{a \bmod 4}^{*} e\left(\frac{ap}{4}\right)\sum_{r=1}^{2^{k-2}} e\left(\frac{(a+4r)l}{2^k}\right)\\ &= S(l,0,2^k) + S(p+l,0,4)2^{k-2}\delta(2^{k-2}|l).\end{aligned}$$

Note that $S(p+l,0,4) = 0$ if $2|l$.

Similarly evaluating in the case $k \equiv 1 \bmod 2$ and noting that $a \equiv \overline{a_q} \bmod 8$, we get

$$\mathcal{C}_1(l,q_0)S(p+l,0,8)2^{k-3}\delta(2^{k-3}|l).$$

Similarly, $S(p+l,0,8) = 0$ when $2|l$. We then evaluate $\mathcal{C}_1(l,q_0)$ from the previous lemma. We collect these calculations in the following lemma.

Lemma 2.2.5. *Assume $q = 2^k q_0$ with $k \geq 1$ and q_0 odd. Assume that $q_0 = q^* q_1 q_2$ with q^* square-free part of q_3, $q_1|(q^*)^\infty$ and $(q_2, q^*) = 1$.*

If $k = 1$, set $l = 4m - pn^2$. Then if $q_1|l$

$$\mathcal{C}(m,n,q) = \frac{2^{1/2}}{q^{1/2}}\left(\frac{p}{q_3}\right)e\left(\frac{pm}{2}\right)S(l,0,q_2)q_1\epsilon_{q^*}\sqrt{q^*}\left(\frac{\frac{l}{q_1}}{q^*}\right) \tag{2.6}$$

and is 0 otherwise.

If $k \geq 2$ then $\mathcal{C}(m,n,q)$ vanishes unless $2|n$. Assume $2|n$ and set $l = m - p(n/2)^2$. Assume $k \equiv 0 \bmod 2$. Then if $q_1|l$

$$\mathcal{C}(m,n,q) = \frac{1}{q^{1/2}}\left\{S(l,0,2^k) + S(p+l,0,4)2^{k-2}\delta(2^{k-2}|l)\right\}S(l,0,q_2)q_1\epsilon_{q^*}\sqrt{q^*}\left(\frac{\frac{l}{q_1}}{q^*}\right) \tag{2.7}$$

and is 0 otherwise.

If $k = 3$ then

$$\mathcal{C}(m,n,q) = \frac{1}{q^{1/2}} S(p+l,0,8)S(l,0,q_2)q_1\epsilon_{q^*}\sqrt{q^*}\left(\frac{\frac{l}{q_1}}{q^*}\right). \tag{2.8}$$

If $k \equiv 1 \bmod 2$ and $k > 3$ then the sum vanishes.

2.3 Quadratic Large Sieve

We state a version of the large sieve when the characters run over only quadratic characters, a result due to Heath-Brown.

Theorem 2.3.1 (Heath-Brown)**.** *Let $Q, N \geq 1$ and let a_n be a sequence of complex numbers for $1 \leq n \leq N$. Then*

$$\sum_{q\leq Q}^{*}\left|\sum_{n\leq N} a_n\left(\frac{n}{q}\right)\right|^2 \ll (QN)^{\epsilon}(Q+N)\sum_{n_1n_2=\square}|a_{n_1}a_{n_2}|$$

where $\sum^*$ indicates that the sum is taken over square-free integers and $n_1n_2 = \square$ indicates that n_1n_2 is an integer squared.

Proof: This is Corollary 2 of [11]. □

2.4 Stationary Phase Analysis

We require a study of oscillatory integrals of the form

$$I = \int_{\mathbb{R}} g(x)e(f(x))dx$$

with g and f smooth functions on $\mathbb{R}$. We have the following general bound.

Theorem 2.4.1 (Second Derivative Bound)**.** *Suppose $|f''| \geq C$ on $supp(g)$. Let $Var(g)$ denote the variation of g. Then*

$$I \ll \frac{Var(g)}{C^{1/2}}.$$

We in fact have a general asymptotic expansion of the integral. We state a version of a theorem of McKee, Sun and Ye. The original version is more general.

Theorem 2.4.2 (McKee-Sun-Ye)**.** *Let $f(x)$ and $g(x)$ be real-valued smooth functions on $\mathbb{R}$. Define*

$$H_1(x) = \frac{g(x)}{2\pi i f'(x)},$$

$$H_i(x) = \frac{H'_{i-1}(x)}{2\pi i f'(x)}$$

for $i \geq 2$.

Let $[\alpha, \beta]$ be an interval. Assume there are parameters M, N, T, U such that

$$M \geq \beta - \alpha$$

and positive constants C_k such that for $\alpha \leq x \leq \beta$ such that

$$|f^{(r)}(x)| \leq C_r \frac{T}{M^r} \quad r \geq 2$$

$$f''(x) \geq \frac{T}{C_2 M^2}$$

and

$$|g^{(s)}(x)| \leq C_s \frac{U}{N^s}, \quad s \geq 0.$$

Suppose $f'(x)$ changes sign only at $x = \gamma$ from negative to positive with $\alpha < \gamma < \beta$. Let

$$\Delta_n = \inf \left\{ \frac{\log 2}{C_2}, \frac{1}{C_2^2 \sup\limits_{2 \leq k \leq 2n+3} \{C_k\}} \right\}.$$

If $T\Delta_n > 1$ then for $n \geq 2$ we have

$$\begin{aligned}
&\int_\alpha^\beta g(x)e(f(x))dx \\
=&\frac{e(f(\gamma)+1/8)}{\sqrt{f''(\gamma)}}\left(g(\gamma)+\sum_{j=1}^{n}\omega_{2j}\frac{(-1)^j(2j-1)!}{(4\pi i\lambda_2)^j}\right)+\left[e(f(x))\sum_{i=1}^{n+1}H_i(x)\right]_\alpha^\beta \\
&+O\left(\frac{UM^{2n+5}}{T^{n+2}N^{n+2}}\left(\frac{1}{(\gamma-\alpha)^{n+2}}+\frac{1}{(\beta-\gamma)^{n+2}}\right)\right) \\
&+O\left(\frac{UM^{2n+4}}{T^{n+2}}\left(\frac{1}{(\gamma-\alpha)^{2n+3}}+\frac{1}{(\beta-\gamma)^{2n+3}}\right)\right) \\
&+O\left(\frac{UM^{2n+4}}{T^{n+2}N^{2n}}\left(\frac{1}{(\gamma-\alpha)^{3}}+\frac{1}{(\beta-\gamma)^{3}}\right)\right)+O\left(\frac{U}{T^{n+1}}\left(\frac{M^{2n+2}}{N^{2n+1}}+M\right)\right)
\end{aligned} \tag{2.9}$$

where

$$\lambda_k = \frac{f^{(k)}(\gamma)}{k!}, \quad k \geq 2$$

$$\eta_k = \frac{g^{(k)}(\gamma)}{k!}, \quad k \geq 1$$

and

$$\omega_k = \eta_k + \sum_{l=0}^{k-1}\eta_l \sum_{j=1}^{k-l}\frac{C_{k,l,j}}{\lambda_2^j} \sum_{\substack{3\leq n_1,\ldots,n_j\leq 2n+3 \\ n_1+n_2+\ldots+n_j=k-l+2j}} \lambda_{n_1}\lambda_{n_2}\ldots\lambda_{n_j}$$

with $C_{k,l,j}$ constants.

Proof: This follows immediately from Theorem 1.2 of [24]. □

We will choose α and β away from the support of $g(x)$ so that $H_i(\alpha) = H_i(\beta) = 0$ for all i.

In Theorem 1, had we only been interested in just the t-aspect, we would only need to expand up to $n = 3$ with a sufficient error term. However, due to the presence of the level in the $GL(2)$ form, we require more terms in the expansion so that the error term is sufficiently small in both the level and t-aspects.

2.5 Automorphic Forms and Representations on $GL(2)$

Most of this subsection has been taken from [9], where additional details can be found.

For p a prime, let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the p-adic norm. If $p = \infty$ then $\mathbb{Q}_\infty = \mathbb{R}$. Set $G_p = GL_2(\mathbb{Q}_p)$ for all p prime. If $p = \infty$, we set $G_\infty = GL_2(\mathbb{R})$. If p is finite let $K_p = GL_2(\mathbb{Z}_p)$. If $p = \infty$ then $K_\infty = SO_2(\mathbb{R})$. Let $\mathbb{A}_\mathbb{Q}$ denote ring of adeles over $\mathbb{Q}$ and $\mathbb{Z}_\mathbb{A}^\times$ the group of ideles. We set

$$G_\mathbb{A} := GL_2(\mathbb{A}_\mathbb{Q}) := \prod_p{}' G_p \tag{2.10}$$

which is the direct product of G_p restricted along K_p, that is, if $\gamma = (\gamma_\infty, \gamma_2, ..., \gamma_p, ...) \in G(\mathbb{A}_\mathbb{Q})$ then almost all of γ_p belong in K_p. The centre is given by

$$Z_\mathbb{A} = \left\{ \begin{pmatrix} t & \\ & t \end{pmatrix} : t \in \mathbb{A}^\times \right\}.$$

Let $G_\mathbb{Q}$ be the diagonal embedding of $GL_2(\mathbb{Q})$ in $G_\mathbb{A}$. The centre of $G_\mathbb{Q}$ is given by

$$Z_\mathbb{Q} = \left\{ \begin{pmatrix} t & \\ & t \end{pmatrix} : t \in \mathbb{Q}^\times \right\}.$$

Thus we have a natural isomorphism $Z_\mathbb{Q}\backslash Z_\mathbb{A} \simeq \mathbb{Q}^\times\backslash\mathbb{A}^\times$ which is also known as the idele class group. The measure of the space $Z_\mathbb{A}G_\mathbb{Q}\backslash G_\mathbb{A}$ is finite.

We may embed G_∞ in $GL_2(\mathbb{A}_\mathbb{Q})$ as $(g, I, I, ...)$. Let $\mathfrak{g}$ be the Lie algebra of $GL_2(\mathbb{R})$ and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Let $Z(U(\mathfrak{g}))$ be the centre of $U(\mathfrak{g})$.

Let $N \geq 1$ be an integer. Define

$$K_0(N)_p := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p : c \equiv 0 \bmod N \right\}$$

and define

$$K_0'(N) = SO_2(\mathbb{R}) \times \prod_{p<\infty} K_0(N)_p.$$

By the strong approximation theorem (see Theorem 4.11.5 in [9]) we have

$$G_{\mathbb{A}} = G_{\mathbb{Q}} G_{\infty}^{+} K_0'(N).$$

Let $\mathfrak{X}_p$ denote the set of all purely ramified continuous characters on $\mathbb{Q}_p^{\times}$. These are characterised by characters $\chi : \mathbb{Q}_p^{\times} \to \mathbb{C}$ such that $\chi(p) = 1$. This is in bijection with the set of Dirichlet characters to p-prime power modulus. The bijection is given as follows.

If ψ is a primitive Dirichlet character mod p^{μ} then we define a character $\tilde{\chi}$ on $\mathbb{Z}_p^{\times}/U_{\mu} \cong (\mathbb{Z}/p^{\mu}\mathbb{Z})^{\times}$ by

$$\tilde{\chi}(a + p^{\mu}\mathbb{Z}_p) = \psi(a)$$

and then lift this to $\chi \in \mathfrak{X}_p$. This lift is unique since any character is completely determined by its action on $p \in \mathbb{Q}_p$ and $\mathbb{Z}_p^{\times}$.

We define an idele character χ to be a unitary character on the idele class group, that is, a character on $\mathbb{A}^{\times}$ that is trivial on $\mathbb{Q}^{\times}$. This lifts to a map on $\chi : \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{C}^{\times}$ that is trivial on $\mathbb{Q}^{\times}$. This decomposes to an infinite product $\chi = \prod_p \chi_p$ where χ_p is a unitary character on $\mathbb{Q}_p^{\times}$ where χ_p is unramified for almost all p.

Let dx_p denote the Haar measure on $\mathbb{Q}_p$ normalised so that $dx_p(\mathbb{Z}_p) = 1$ and $d^{\times}x_p$ the Haar measure on $\mathbb{Q}_p^{\times}$ normalised so that $d^{\times}x_p(\mathbb{Z}_p^{\times}) = 1$.

Definition 2.5.1. *Let $\omega : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \to \mathbb{C}^{\times}$ be a unitary idele character on $\mathbb{A}_{\mathbb{Q}}^{\times}$ trivial on $\mathbb{Q}^{\times}$. Let $K = K_0(N)'$. An automorphic form on $GL(2, \mathbb{A}_{\mathbb{Q}})$ is a smooth function $\phi : GL(2, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ which satisfies the following properties -*

(1) $\phi(\gamma g) = \phi(g)$ for all $g \in G_{\mathbb{A}}$ and $\gamma \in G_{\mathbb{Q}}$.

(2) $\phi(zg) = \omega(z)\phi(g)$ for all $g \in G_{\mathbb{A}}$ and $z \in Z_{\mathbb{A}}$.

(3) ϕ is right K-finite.

(4) ϕ is $Z(U(\mathfrak{g}))$ finite.

(5) ϕ is of moderate growth.

The definitions of smooth, right K-finite and $Z(U(\mathfrak{g}))$-finite can be found in Sections 4.5 - 4.7 in [9].

If in addition

$$\int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} \phi\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right) = 0$$

for all $g \in \mathbb{A}_{\mathbb{Q}}$ then we say that ϕ is a cusp form.

Given an automorphic form ϕ, one can attach an automorphic representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ and conversely. It is known that

$$\pi = \otimes'_v \pi_v$$

where the $'$ indicates that this is a restricted tensor product (see Chapter 10 of [9]). We shall denote the dual form of ϕ by $\check{\phi}$ and the contra-gradient representation by $\check{\pi}$. Let v be either a prime p or ∞. Let $a(\pi_v)$ denote the conductor exponent of π_v. Then $a(\pi_v) = 0$ for almost all v. Let $L(\pi_v, s)$ denote the local L-factor at v. Let

$$L(s, \pi) = \prod_{v<\infty} L(\pi_v, s) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$$

denote the standard L-function attached to π, normalized so that $\lambda_\pi(1) = 1$. We shall often call this $L(s, \phi)$ as well where ϕ is attached to π.

Let p be a prime. For $r \geq 0$ an integer, set $U_0 = \mathbb{Z}_p^\times$ and

$$U_r = 1 + p^r \mathbb{Z}_p$$

for $r \geq 1$. For a character χ on $\mathbb{Q}_p$, define the conductor exponent $a(\chi)$ to be

$$a(\chi) = \inf\{r \geq 0 : U_r \subseteq \ker(\chi)\}.$$

For each place v, let ψ_v denote the standard additive character on $\mathbb{Q}_v$. Let $\epsilon(s, \pi_v, \psi_v)$ denote the local root number at v. We have the formula

$$\epsilon(s, \pi_v, \psi_v) = \epsilon(1/2, \pi_v, \psi_v) q^{a(\pi_v)\left(\frac{1}{2}-s\right)}.$$

We define the local gamma factor to be

$$\gamma(s, \pi_v, \psi_v) = \epsilon(s, \pi_v, \psi_v) \frac{L(1-s, \check{\pi}_v)}{L(s, \pi_v)}$$

where $\check{\pi}_v$ is the contra-gradient representation of π_v. Now let ϕ be an automorphic form on $GL(2)$. If ϕ is a holomorphic modular form then Deligne [6] proved that

$$\lambda_\pi(n) \ll n^\epsilon \tag{2.11}$$

This is known for the Eisenstein series with Fourier coefficients $d(n)$ by elementary means. For Maass forms, the best known bound is due to Kim-Sarnak (see [21])

$$\lambda_\pi(n) \ll n^{7/64+\epsilon}.$$

On the other hand we have the general l^4-Rankin-Selberg estimate (see Lemma 2.2 of [13])

$$\sum_{n \leq X} |\lambda_\pi(n)|^4 \ll C(\pi)^\epsilon X^{1+\epsilon} \tag{2.12}$$

which follows from properties of the adjoint L-function of π. By Cauchy-Schwartz this implies the classical l^2 Rankin-Selberg estimate

$$\sum_{n \leq X} |\lambda_\pi(n)|^2 \ll C(\pi)^\epsilon X^{1+\epsilon}. \tag{2.13}$$

2.5.1 Local Representations of $GL_2(\mathbb{Q}_p)$

The classification of local representations on $GL_2(\mathbb{Q}_p)$ is well known. These are the special or the Steinberg representations and all its twists, the principal series representations and the supercuspidal representations. We briefly go through their descriptions.

We denote the space of principal series representations as $\pi(\chi_1, \chi_2)$. This space consists of the space of all local constant functions $f : GL_2(\mathbb{Q}_p) \to \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)|a/d|^{-1/2} f(g)$$

for all $g \in GL_2(\mathbb{Q}_p)$, $a, d \in \mathbb{Q}_p^\times$ and $b \in \mathbb{Q}_p$. It is known that this representation is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$.

In case $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$ that is, $\chi_1 = \chi|\cdot|^{-1/2}$ and $\chi_2 = \chi|\cdot|^{1/2}$, then there is a unique irreducible sub-representation denoted χStn called the twisted Steinberg representation.

Any irreducible representation that doesn't arise as some sub-representation of the principal series are called supercuspidal representations. These are parametrized by admissible pairs (E, θ) where E is a degree 2 extension of $\mathbb{Q}_p$ and θ is an admissible character which we now define when $p \neq 2$. Let Nm_E be the norm map. A character θ is said to be admissible if

(1) θ doesn't factor through the norm map and

(2) if $\theta|_{U_1}$ does factor through the norm map, then $E/\mathbb{Q}_p$ is unramified.

This is in fact the definition of a tame admissible pair and holds when $p \neq 2$. When $p = 2$ the picture is more complicated. (see [2] for the complete picture, including the case $p = 2$). Since we never work in the case $p = 2$, we shall ignore it.

Let $\mathfrak{D}$ be the different of E and d the order of $\mathfrak{D}$. Let e be the ramification index of E. Then when $e = 2$, $d = 1$ and when $e = 1$, $d = 0$.

Now let π be an automorphic representation on $GL_2(\mathbb{A}_{\mathbb{Q}})$. Assume that π_p is ramified only at p. Let $a(\pi_p)$ be its conductor exponent of π_p. This is the smallest integer $a \geq 0$ such that π_p admits a $K_0(p^a)$ fixed vector. We then define the conductor to be $C(\pi_p) = p^{a(\pi_p)}$.

We now assume further that the central character ω_π with conductor exponent $a(\omega_\pi)$ satisfies $a(\omega_\pi) \leq \frac{a(\pi_p)}{2}$. Then this forces π_p to be

(1) a principal series representation $\pi(\chi_1, \chi_2)$ where $a(\chi_1) = a(\chi_2)$ and $\omega_\pi = \chi_1\chi_2$.

(2) a twisted Steinberg representation χStn with $a(\pi_p) = 2a(\chi)$ and $\omega_\pi = \chi^2$.

(3) a supercuspidal representation associated to an admissible pair (E, θ) consisting of a quadratic extension $E/\mathbb{Q}_p$ and a character θ on $E^\times$. Let $a(\theta)$ be the conductor exponent of θ. We have $a(\pi_p) = \frac{2a(\theta)}{e} + d$ and $\omega_\pi = \theta|_{\mathbb{Q}_p^\times}$.

It is known that all the L-functions attached to the above representations are trivial, that is, $L(\pi_p, s) = 1$ for all $s \in \mathbb{C}$ but we in fact need more additional datum. We will later discuss the L- and ϵ-factors attached to these representations under all twist twists $\psi \in \mathfrak{X}$. We shall ignore the twisted Steinberg representation. It's L- and ϵ-factors can be found in [29] where additional information can also be found.

(1) Let π_p be a supercuspidal representation. Let (E, θ) be the datum associated to π_p. Then

$$L(s, \pi_p\chi) = 1, \quad \epsilon(\pi_p\psi) = \epsilon(\theta\psi_E)$$

for all $\psi \in \mathfrak{X}_j$, $\psi_E = \psi \circ Nm_{E/\mathbb{Q}_p}$ is the induced character of χ on $E^\times$.

(2) Let $\pi_p = \pi(\chi_1, \chi_2)$ with both χ_i ramified and $a(\chi_1) = a(\chi_2)$. Then

$$L(s, \pi_p\psi) = 1$$

$$\epsilon(\pi_p\psi) = \epsilon(\chi_1\psi)\epsilon(\chi_2\psi)$$

for all $\psi \in \mathfrak{X}$ with $a(\psi) \neq a(\chi_1)$. The above properties remain true even for ψ with $a(\psi) = a(\chi_1)$ except when one, or both, of $\psi\chi_1$ and $\psi\chi_2$ become unramified. In that case

$$L(s, \pi_p\psi) = \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1}, \quad \epsilon(\pi_p\psi) = \epsilon(\chi_2\psi)$$

if $\psi\chi_1$ is unramified and $\psi\chi_2$ is ramified,

$$L(s, \pi_p\psi) = \left(1 - \frac{\chi_2(p)}{p^s}\right)^{-1}, \quad \epsilon(\pi_p\psi) = \epsilon(\chi_1\psi)$$

if $\psi\chi_2$ is unramified and $\psi\chi_1$ is ramified and

$$L(s, \pi_p\psi) = \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1}\left(1 - \frac{\chi_2(p)}{p^s}\right)^{-1}, \quad \epsilon(\pi_p\psi) = 1$$

if both $\psi\chi_2$ and $\psi\chi_1$ are unramified. Note that

$$a(\pi_p\chi) = a(\chi_1\psi) + a(\chi_2\psi)$$

for all choice of $\psi \in \mathfrak{X}_p$.

2.5.2 $GL(2) \times GL(2)$ Rankin-Selberg L-functions

Let π_1 and π_2 be two cuspidal automorphic representations of level N and M respectively. Let ω_{π_1} and ω_{π_2} be their central character/nebentypus respectively. Let

$$L(\pi_i, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi_i}(n)}{n^s}$$

be their corresponding L-functions. Let $L(\omega_{\pi_1}\omega_{\pi_2}, s)$ denote the Dirichlet L-function of $\omega_{\pi_1}\omega_{\pi_2}$. One then defines the Rankin-Selberg L-function of π_1 and π_2 as

$$L(s, \pi_1 \otimes \pi_2) = L(\omega_{\pi_1}\omega_{\pi_2}, 2s) \prod_{p|MN} L_p(s, \pi_1 \otimes \pi_2) \sum_{(n,MN)=1} \frac{\lambda_{\pi_1}(n)\lambda_{\pi_2}(n)}{n^s}$$

for $\mathfrak{Re}(s) > 1$. Here $L_p(\pi_1 \otimes \pi_2, s)$ are local L-factors that depend on both π_1 and π_2. This admits meromorphic continuation to the entire complex plane, with a pole at $s = 1$ if and only if $\pi_1 = \check{\pi}_2$. If $\Lambda(\pi_1 \otimes \pi_2, s)$ denotes its completed L-function then it admits a functional equation

$$\Lambda(\pi_1 \otimes \pi_2, s) = \epsilon(\pi_1 \otimes \pi_2)\Lambda(\check{\pi}_1 \otimes \check{\pi}_2, 1 - s)$$

where $\check{\pi}_i$ denotes the contra-gradient representation of π_i. This L-function is known to be attached to a cuspidal automorphic representation on $GL_4(\mathbb{A})$. (see [28]).

2.5.3 Lifts of $GL(2)$ Forms to $GL(3)$

The full details for what we discuss in this section can be found in [8], but the only relevant details we'd like are about their L-functions from which we quote all results.

From Section 1 of *loc cit.* we deduce the following. Let f be a $GL(2)$ automorphic form of level p and nebentypus χ. It's attached L-function is given by

$$L(s, f) = \prod_q L_p(f, s)$$

where

$$L_q(f, s) = \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(q)}{p^{2s}}\right)^{-1} = \left(1 - \frac{\alpha_{1,q}}{q^s}\right)^{-1}\left(1 - \frac{\alpha_{2,q}}{q^s}\right)^{-1}$$

if $q \neq p$. Over here the $\alpha_{i,q}$ denote the Satake parameters at the prime q and satisfy $\alpha_{1,q}(q)\alpha_{2,q}(q) = \chi(q)$. If $p = q$ then

$$L_p(f, s) = \left(1 - \frac{\lambda_f(p)}{p^s}\right)^{-1} = \left(1 - \frac{\alpha_p(p)}{p^s}\right)^{-1}.$$

Note that when χ is trivial then $\alpha_p(p) = \pm p^{-1/2}$. The sign is determined as follows. The local representation at p is χStn where $\chi^2 = 1$ and χ is unramified. If $\chi = 1$ then $\alpha_p(p) = p^{-1/2}$. If $\chi \neq 1$ then $\alpha_p(p) = -p^{-1/2}$.

We now define the local L-factors of the symmetric square lift of f, denoted $L_q(\mathrm{sym}^2 f, s)$. The local L-factor at q for the $\mathrm{sym}^2 f$ is given by

$$L_q(\mathrm{sym}^2 f, s) = \left(1 - \frac{\alpha_{1,q}^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{1,q}\alpha_{2,q}}{q^s}\right)^{-1} \left(1 - \frac{\alpha_{2,q}^2}{q^s}\right)^{-1}$$

if $q \neq p$.

If $\chi = \chi_0$ the principal character, then

$$L_p(\mathrm{sym}^2 f, s) = \left(1 - \frac{1}{p^{s+1}}\right)^{-1}.$$

If χ is the quadratic character then

$$L_p(\mathrm{sym}^2 f, s) = \left(1 - \frac{1}{p^s}\right)^{-1}.$$

If $\chi^2 \neq 1$ then

$$L_p(\mathrm{sym}^2 f, s) = \left(1 - \frac{\alpha_p(p)^2}{p^s}\right)^{-1}.$$

Putting this together we obtain the symmetric square L-function of f

$$L(\mathrm{sym}^2 f, s) = \prod_q L_q(\mathrm{sym}^2 f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f}(n)}{n^s} = L(\chi^2, 2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}$$

Alternatively, one can define this as

$$L(f \otimes f, s) = \frac{L(\mathrm{sym}^2 f, s)}{L(s, \chi)}$$

This L-function is known to be an entire function of s. If $\Lambda(\mathrm{sym}^2 f, s)$ denotes the completed L-function then it admits a functional equation

$$\Lambda(\mathrm{sym}^2 f.s) = \epsilon(\mathrm{sym}^2 f)\Lambda(\mathrm{sym}^2 \check{f}, 1 - s)$$

where $\check{f}$ denotes the dual cusp form of f.

2.5.4 Conductor Formulas for $GL(2) \times GL(2)$ L-functions

Given two cuspidal automorphic representations π_1 on π_2 on $GL_2(\mathbb{A})$ we now state the conductor of their Rankin-Selberg convolution $\pi_1 \otimes \pi_2$. Given a local representation π_p of $GL_2(\mathbb{Q}_p)$, we define the conductor exponent $a(\pi_{1,p})$ to be the smallest non-negative integer n such that π_p admits a non-zero $K(p^n)$ fixed vector.

We shall now calculate the conductor exponent of $\pi_1 \otimes \pi_2$ in terms of the conductor exponents of π_1 and π_2. It is enough to calculate this for the local representations $\pi_{1,p} \otimes \pi_{2,p}$ for all p prime since $C(\pi_1 \otimes \pi_2) = \prod_p C(\pi_{1,p} \otimes \pi_{2,p})$. All calculations can be deduced by the Local Langlands Correspondence, which is now a theorem (see [10], [12], [30]) and can be found in Section 4.3 of [14].

If both $\pi_{1,p}$ and $\pi_{2,p}$ are unramified then $C(\pi_{1,p} \otimes \pi_{2,p}) = 1$. If one of the representations, say $\pi_{1,p}$ is ramified and the other is unramified then $C(\pi_{1,p} \otimes \pi_{2,p}) = C(\pi_1)^2$.

Now assume that both $\pi_{i,p}$ are ramified and we assume that they're either both supercuspidal or principal series representations. If $\pi_{1,p}$ is supercuspidal and $\pi_{2,p} = \pi(\chi_1, \chi_2)$ is a principal series then $C(\pi_{1,p} \otimes \pi_{2,p}) = C(\pi_{1,p}\chi_1)C(\pi_{1,p}\chi_2)$. If in addition $\pi_{1,p}$ is twist-minimal then $C(\pi_{1,p}\chi) = \sup\{C(\pi_{1,p}), C(\chi)^2\}$ (see 2.6.2).

We now state the conductor when both $\pi_{1,p}$ are of the same type, that is, $\pi_{i,p}$ are both supercuspidal or both ramified principal series.

(1) First suppose $\pi_{i,p}$ are supercuspidal associated to the datum (E_i, θ_i). By conductor formulas of Bushnell, Henniart and Kutzko (see [3]) we deduce that $a(\pi_{1,p} \otimes \pi_{2,p}) = 2\sup\{a(\pi_{1,p}), a(\pi_{2,p})\}$, so $C(\pi_{1,p} \otimes \pi_{2,p}) = \sup\{C(\pi_{1,p})^2, C(\pi_{2,p})^2\}$ unless $E_1 = E_2$ and $a(\theta_1) = a(\theta_2)$. In that case, we have $C(\pi_{1,p} \otimes \pi_{2,p}) = C(\pi_{\theta_1\theta_2}) \times C(\pi_{\theta_1\overline{\theta_2}})$. Here π_θ is the supercuspidal representation associated to (E, θ).

(2) The formula in (1) remains true for a pair of principal series in the following sense. Suppose $\pi_{i,p} = \pi(\chi_i, \psi_i)$. Then we set $\theta_i = (\chi_i, \psi_i)$ a character on $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$ and write $\pi(\theta_i) = \pi(\chi_i, \psi_i)$. Then $C(\pi_{1,p} \otimes \pi_{2,p}) = C(\pi(\theta_1\theta_2)) \times C(\pi(\theta_1\overline{\theta_2}))$.

2.5.5 Local L-factors for $GL(2) \times GL(2)$ L-functions

Having studied the conductors, we now study the local L-factor $L_p(\pi_{1,p} \otimes \pi_{2,p}, s)$.

If one of the representations, say $\pi_{1,p}$, is supercuspidal, then the local L-factor is trivial unless $\pi_{2,p}$ is the contra-gradient representation of $\pi_{1,p}$. This follows from a combination of Corollary (1.3) and Proposition (1.4) of [8].

So now we assume that $\pi_{1,p}$ and $\pi_{2,p}$ are both principal series representations, say $\pi_{1,p} = \pi(\chi_1, \chi_2)$ and $\pi_{2,p} = \pi(\psi_1, \psi_2)$. Then by the same Proposition (1.4) of *loc. cit.*

$$L_p(\pi_{1,p} \otimes \pi_{2,p}, s) = \prod_{i,j=1}^{2} L_p(\chi_i \psi_j, s).$$

One then observes that each factor in the product is trivial unless $\chi_i \psi_j$ is unramified.

2.5.6 Approximate Functional Equations

Let's briefly discuss the conductor of an L-function $L(s, \pi)$. Given any degree d automorphic L-function $L(s, \pi)$ with level q, one defines the conductor of an such an L-function as

$$q_\infty(1/2 + it, \pi) = q \times \prod_{i=1}^{d} (1 + |\kappa_i + it|)$$

where κ_i are the Langlands/spectral parameters of π. Given these parameters one also defines

$$\gamma(s, \pi) = \pi^{-ds/2} \prod_{i=1}^{d} \Gamma\left(\frac{s + \kappa_i}{2}\right)$$

where $s \in \mathbb{C}$ and Γ is the usual Gamma function. For general definitions and properties of such L-functions, see Chapter 5 of [17].

Given any such automorphic L-functions above, one can use the functional equation to deduce an "approximate" version of the same.

Theorem 2.5.2. *Let $L(s, \pi)$ be an L-function with conductor q. Let $G(u)$ be a holomorphic function of $-A < \mathfrak{Re}(s) < A$, even and normalized so that $G(0) = 1$. Let $X > 1$ be a*

parameter of our choice. Then for $0 \leq \mathfrak{Re}(s) \leq 1$ *we have*

$$L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} V_s\left(\frac{n}{X\sqrt{q}}\right) + \epsilon(s,\pi) \sum_n \frac{\lambda_{\check{\pi}}(n)}{n^{1-s}} V_{1-s}\left(\frac{nX}{\sqrt{q}}\right)$$

where

$$V_s(x) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma(s+u,\pi)}{\gamma(s,\pi)} \frac{du}{u}$$

and

$$\epsilon(s,\pi) = \epsilon(f) q^{1/2-s} \frac{\gamma(\pi, 1-s)}{\gamma(s,\pi)}.$$

Proof: This is Theorem 5.3 in [17]. □

One in fact observes that

$$V_s\left(\frac{x}{\sqrt{q}}\right) \ll \left(1 + \frac{x}{q_\infty(1/2+it,\pi)}\right)^{-A}$$

and

$$V_s\left(\frac{x}{\sqrt{q}}\right) = 1 + O\left(\left(\frac{x}{\sqrt{q_\infty(1/2+it,\pi)}}\right)^{\alpha}\right)$$

where $0 < 3\alpha \leq \mathfrak{Re}(s+\kappa_j)$ for all $1 \leq j \leq d$. (see Proposition 5.4 in [17]).

Thus one deduces that

$$L(\frac{1}{2}+it,\pi) \approx \sum_{n \ll \sqrt{q_\infty(1/2+it,\pi)}} \frac{\lambda_\pi(n)}{n^{1/2+it}} + \epsilon(\pi) \sum_{n \ll \sqrt{q_\infty(1/2+it,\pi)}} \frac{\lambda_{\check{\pi}}(n)}{n^{-it}}.$$

In Theorem 1 we have $q_\infty(\mathrm{sym}^2 f, it) = O(pt^{3/2})$ with constants depending on the spectral parameter of f. In Theorem 2 we have $q_\infty(0, \pi_1 \otimes \pi_2) = O(p^{a(\pi_1 \otimes \pi_2)})$ with constants depending on the spectral parameters of $\pi_1 \otimes \pi_2$.

2.6 Summation Formulae

2.6.1 Voronoi Summation

We briefly recall the Whittaker and Kirillov models attached to local and global automorphic representations. We only require these to state the Voronoi summation formula and calculations thereafter but they are important objects in of itself.

Let F be a number field. let $\pi = \otimes_v \pi_v$ be an irreducible, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, realised in a space of automorphic forms V_π. There is a $\mathrm{GL}_n(\mathbb{A})$-intertwining map from V_π to the space of functions $W\colon \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ that satisfy $W(ug) = \psi(u)W(g)$ for each $u \in U_n(\mathbb{A})$ carrying the right regular representation ρ of $\mathrm{GL}_n(\mathbb{A})$. Explicitly, one maps $\varphi \in V_\pi$ to the element

$$W_\varphi\colon g \longmapsto \int_{U_n(F)\backslash U_n(\mathbb{A})} \varphi(ug)\overline{\psi(u)}du, \tag{2.14}$$

where we choose du to be the invariant probability measure on $U_n(F)\backslash U_n(\mathbb{A})$. Let $W(\pi, \psi)$ denote the image of V_π under (2.14), another irreducible $GL_n(\mathbb{A})$-module. $W(\pi, \psi)$ is called the $(\psi-)$ Whittaker model of π.

Similarly, at each place v of F there is a notion of a ψ_v-Whittaker model; this is the space $W(\pi_v, \psi_v)$ of functions $W\colon GL_n(F_v) \to \mathbb{C}$ satisfying $W_v(ug) = \psi_v(u)W_v(g)$ for each $u \in U_n(F_v)$. The uniqueness of Whittaker models follows by theorems of Jacquet and Shalika (see [18] and [19]). It also implies that $W(\pi, \psi)$ is given by the restricted tensor product $W(\pi, \psi) = \otimes'_v W(\pi_v, \psi_v)$.

If v is non-Archimedean, then we may restrict $W_v \in W(\pi_v, \psi_v)$ to the 'mirabolic subgroup' $P_n(F_v)$ of $GL_n(F_v)$. P_n is defined as the matrices in GL_n stabilizing $(0, ..., 0, 1)$ on the right.

It is known that $W_{\pi_v}|_{P_n}$ contains all Bruhat-Schwartz functions ϕ. In particular we thus can identify functions ϕ on P_n to some function $W \in W(\pi)$ such that $W\begin{pmatrix} p & & \\ & & \\ & & 1 \end{pmatrix} = \phi(p)$. Note that Φ_p is only supported on $\mathbb{Z}_p$ in the non-archimedean case.

For $y \in F_v$ define $a(y) = \begin{pmatrix} y & \\ & I \end{pmatrix} \in GL_{n-1}$. Now given ϕ_∞ of Bruhat-Schwartz type on $\mathbb{R}^\times$ then $\Phi_\infty(x) = W_{\pi_\infty}\left(\begin{pmatrix} a(x) & \\ & 1 \end{pmatrix}\right)$ for some $W \in W(\pi_\infty)$. We set $\Phi_\infty(x) = W_{\pi_\infty}\left(\begin{pmatrix} a(x) & \\ & 1 \end{pmatrix}\right)$. Note that this depends on W. Similarly, let $\Phi_p(x) = W_{\pi_p}\left(\begin{pmatrix} a(x) & \\ & 1 \end{pmatrix}\right) = \Phi_p(x)$ for some $W \in W(\pi_\infty)$ for all $p|M$. Define its twist by ζ_p by $\Phi_p^{\zeta_p}(y) := \psi_p(y\zeta_p)\Phi_p(y)$.

We now state a a Voronoi Summation formula specialized to forms on $GL(2)$ over $\mathbb{Q}$ due to Corbett [5]. His summation formula holds for all cusp forms on $GL(n)$ with $n \geq 2$ and over any number field with suitable restrictions. We refer to *loc. cit.* for the general statement and further discussions.

Theorem 2.6.1 (Corbett)**.** *Let $N \geq 1$ be an integer. Let ψ be a Dirichlet character mod N. Let f be a cuspidal automorphic from on $\Gamma_0(N)$with nebentypus χ. Let $M \geq 1$ be an integer such that $N|M$. Assume that f is an eigenform for all Hecke operators T_n with $(n, N) = 1$. For each $p|M$ let $\phi_p \in C_c^\infty(\mathbb{Q}_p^\times)$. Let $\phi_\infty \in C_c^\infty(\mathbb{R}^\times)$. Let For $a, l, q \in \mathbb{Z}$ with $a \neq 0$, $l, q \geq 1$, $(a, lq) = (q, M) = 1$ and $l|M^\infty$ we have the following formula*

$$\sum_{m\in\mathbb{Z}_{\neq 0}} \frac{\lambda_\pi(m)}{|m|^{1/2}} e\left(\frac{am}{lq}\right)\phi_\infty(m)\prod_{p|M}\phi_p(m) = \sum_{\substack{m\in\mathbb{Z}_{\neq 0}\\(m,M)=1}}\sum_{r|M^\infty}\frac{\lambda_{\check{\pi}}(m)}{|m|^{1/2}} e\left(-\frac{\overline{a\lambda_l}lrm}{q}\right) \times \chi(\overline{m}q)^{-1}\mathcal{B}_{\pi_\infty,\phi_\infty}\left(\frac{rm}{\lambda_l q^2}\right)\prod_{p|M}\mathcal{B}_{\pi_p,\Phi_p^{a/lq}}\left(\frac{rm}{\lambda_l q^2}\right)$$

where

$$\mathcal{B}_{\pi_\infty,\phi_\infty}(y) = \frac{1}{4\pi i}\sum_{r=0}^{1}\mathrm{sgn}(y)^r \int\limits_{\mathfrak{Re}(s)=\sigma} \gamma(1-s, \mathrm{sgn}^r\pi_\infty, \psi_\infty)|y|^{\frac{1}{2}-s}\int\limits_{\mathbb{R}^\times}\Phi_\infty(x)\mathrm{sgn}(x)^r|x|^{\frac{1}{2}-s}d^\times x ds \tag{2.15}$$

$$\mathcal{B}_{\pi_p,\phi_p^{a/lq}}(y) = \frac{\log p}{2\pi}\sum_{\chi\in\mathfrak{X}_p}^{*}\chi(y)\int\limits_{\frac{1}{2}-i\pi/\log p}^{\frac{1}{2}+i\pi/\log p}\gamma(1-s,\chi^{-1}\pi_p,\psi_p)|y|_p^{\frac{1}{2}-s}\int\limits_{\mathbb{Q}_p^\times}\Phi_p^{a/lq}(x)\chi(x)|x|_p^{\frac{1}{2}-s}d^\times x ds \tag{2.16}$$

with $\lambda_l = l[l, N]L^2$ with $L = rad(N)$, the radical of N.

Corbett's theorem builds upon an adelic version of the Voronoi summation due to Ichino and Templier (see [16]). He develops it when both the modulus and the representation are ramified at the same place. The version of Ichino and Templier comes with important restrictions - one of which makes it unusable in general for all number fields. These restrictions were removed in Appendix A of [27]. In our situation, the restrictions don't play an important role since we always work over $\mathbb{Q}$. With more work, one can deduce that the formulation of Ichino and Templier and Corbett works for any number field (see the comments in Appendix A of [27]).

In the rest of this thesis, we shall only be interested in the case when $N = p^l$ a prime power and $M = p$. We shall in fact always pick $\phi_p = \mathbb{I}_{\mathbb{Z}_p^\times}$.

2.6.2 Evaluation of Bessel Transforms

Archimedean Bessel Transforms

For the Archimedean Bessel transform we have the following. When f is a holomorphic modular form, for $y > 0$,

$$\mathcal{B}_{\pi_\infty, f, \phi_\infty}(y) = y^{1/2} \int_{\mathbb{R}} \frac{\phi_\infty(x)}{x^{1/2}} J_{k-1}(4\pi\sqrt{xy})dx.$$

If f is a Maass form of tpye ν then for $|y| > 0$

$$\mathcal{B}_{\pi_\infty, f, \phi_\infty}(y) = |y|^{1/2} \sum_{\epsilon \in \{\pm\}} \int_{\mathbb{R}} \frac{\phi_\infty(x)}{x^{1/2}} \mathcal{M}_f^\epsilon(y)dy$$

where

$$\mathcal{M}_f^\epsilon(y) = \begin{cases} \{J_{2i\nu} - J_{-2i\nu}\}(y) & \text{if } \epsilon = -1 \\ 4\epsilon_f cosh(\pi\nu) K_{2i\nu}(y) & \text{if } \epsilon = 1 \end{cases}. \tag{2.17}$$

We require the following asymptotic behaviour of the Bessel functions. Let $k \geq 2$ and $\nu \in \mathbb{R}$. Then

$$J_k(y) = W_k^+(y)e(y) + W_k^-(y)e(-y), \quad J_{2i\nu}(y) = W_{2i\nu}(y)e(y) + W_{2i\nu}^-(y)e(-y), \tag{2.18}$$

with $W^{\pm}(y)$ real functions satisfying the bound

$$W_k^{\pm(j)}(y) \ll_{k,j} (1+y)^{-1/2}, \quad W_\nu^{\pm(j)}(y) \ll_{\nu,j} (1+y)^{-1/2} \tag{2.19}$$

for all $j \geq 0$.

Let $\nu \in \mathbb{R}$. Then

$$y^k K_{2i\nu}^{(k)}(y) \ll_{k,\nu} \frac{e^{-y}}{\sqrt{y}} \tag{2.20}$$

Non-Archimedean Bessel Transforms

Throughout, we assume that ψ_p is the standard additive character on $\mathbb{Q}_p$ and drop it from further notation.

By definition, we have

$$\gamma(s, \chi^{-1}\pi_p) = \epsilon(s, \chi^{-1}\pi)\frac{L(1-s, \chi\check{\pi})}{L(s, \chi^{-1}\pi)}$$

where $L(s, \pi\chi)$ and $L(1-s, \chi^{-1}\check{\pi})$ are defined in Section 2.5.1 and

$$\epsilon(s, \pi\chi) = \epsilon(\frac{1}{2}, \pi\chi)p^{a(\pi\chi)(\frac{1}{2}-s)}.$$

Using the fact that Φ_p is only supported on $\mathbb{Z}_p$ we have the expansion

$$\begin{aligned}&\int_{\mathbb{Q}_p^\times} \Phi(x)\chi(x)|x|_p^{\frac{1}{2}-s} \\ &= \sum_{r\geq 0} W_p(a(p^r)) \int_{\mathbb{Z}_p^\times} \psi_p(\zeta_p p^r u)\chi(u)p^{-r(\frac{1}{2}-s)}\phi_p(p^r u)d^\times u.\end{aligned}$$

In the special case that $\phi_p = \mathbb{I}_{\mathbb{Z}_p^\times}$, we see that all terms except $r = 0$ vanish. Define

$$\mathfrak{C}(x, \chi) := \int_{\mathbb{Z}_p^\times} \psi_p(xu)\chi(u)d^\times u.$$

We have the following lemma, which is Lemma 4.5 in [5].

Lemma 2.6.2. *First assume $a(\chi) \geq 1$. Then $\mathfrak{C}(x, \chi)$ vanishes unless $|x|_p = p^{a(\chi)}$ in which case*

$$\mathfrak{C}(x, \chi) = (1 - 1/p)^{-1} p^{a(\chi)/2} \chi^{-1}(x)\epsilon(\chi)$$

where $\epsilon(\chi)$ is the usual sign of the Gauss sum of χ. If $a(\chi) = 0$ then

$$\mathfrak{C}(x, \chi) = \begin{cases} 1 & \text{if } |x|_p \leq 1 \\ \frac{p^{1/2}}{1-p} & \text{if } |x|_p = p \\ 0 & \text{if } |x|_p > p \end{cases}.$$

The crucial observation the lemma makes is that $|\zeta_p|_p = p^{a(\chi)}$ or else the character sum vanishes. Next we observe that

$$\frac{\log p}{2\pi} \int_{-\pi i/\log p}^{\pi i/\log p} p^{ks} ds = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}.$$

We say a representation π_p is twist-minimal if and only if

$$a(\pi_p) = \inf_{\chi \in \mathfrak{X}_p} a(\pi_p \chi).$$

Now suppose π_p is twist-minimal supercuspidal. We have $L(s, \pi\chi) = 1$ for all purely ramified twists χ. We thus see that

$$\gamma(1 - s, \chi^{-1}\pi) = \epsilon(\frac{1}{2}, \chi^{-1}\pi)p^{a(\chi^{-1}\pi)(s-\frac{1}{2})}.$$

By conductor formulas in [4] we have

$$a(\pi\chi) = \sup\{a(\pi), 2a(\chi)\}.$$

Taking $\sigma = 1/2$ and evaluating the s integral, we thus deduce the following.

Proposition 2.6.3. *Let π_p be a twist-minimal supercuspidal representation. Let $\zeta_p \in \mathbb{Q}_p^\times$. Let $\phi_p = \mathbb{I}_{\mathbb{Z}_p^\times}$. Then $\mathcal{B}_{\pi_p,\phi_p^{\zeta_p}}(y) = 0$ if $|y|_p \neq p^{\sup\{a(\pi_p),-2\mathrm{ord}_p(\zeta_p)\}}$.*

Now suppose $|y|_p = p^{\sup\{a(\pi_p),-2\mathrm{ord}_p(\zeta_p)\}}$.

If $|\zeta_p|_p \leq 1$ then

$$\mathcal{B}_{\pi_p,\phi_p^{\zeta_p}}(y) = \epsilon(1/2, \pi_p, \psi_p).$$

If $|\zeta_p|_p = p$ then

$$\begin{aligned}\mathcal{B}_{\phi_p,\phi_p^{\zeta_p}}(y) =& \epsilon(1/2, \pi_p, \psi_p)\frac{1}{1-p} \\ &+ \frac{p^{-\frac{1}{2}}}{1-1/p}\sum_{\substack{\chi\in\mathfrak{X}_p \\ a(\chi)=1}}^{*} \chi(-\zeta_p^{-1}y)\epsilon(1/2, \overline{\chi}\pi_p, \psi_p)\epsilon(1/2, \chi, \psi_p).\end{aligned}$$

If $|\zeta_p|_p > p$ then

$$\begin{aligned}\mathcal{B}_{\phi_p,\phi_p^{\zeta_p}}(y) =& \frac{1}{1-p}p^{\frac{\mathrm{ord}_p(\zeta_p)}{2}} \\ &\times \sum_{\substack{\chi\in\mathfrak{X}_p \\ a(\chi)=-v_p(\zeta_p)}}^{*} \chi(-\zeta_p^{-1}y)\epsilon(1/2, \overline{\chi}\pi_p, \psi_p)\epsilon(1/2, \chi, \psi_p).\end{aligned}$$

Applying the correspondence between Dirichlet characters and the set $\mathfrak{X}_p$ we get the following.

Corollary 2.6.3.1. *Assume the same as in the above proposition. Further assume that $\zeta_p = a_1/q_1p^{k_1}$ and $y = a_2/q_2p^{k_2}$ are both rational with $k_1, k_2 \in \mathbb{Z}$ and $(a_1a_2q_1q_2, p) = (a_1, q_1) = (a_2, q_2) = 1$.*

Suppose $k_2 = \sup\{a(\pi_p), 2k_1\}$ so that $\mathcal{B}_{\pi_p,\phi_p^{\zeta_p}}(y) \neq 0$.

If $k_1 \leq 0$ *then*

$$\mathcal{B}_{\pi_p,\phi_p^{\zeta_p}}(y) = \epsilon(\pi_p).$$

If $k_1 = 1$ *then*

$$\begin{aligned}\mathcal{B}_{\phi_p,\phi_p^{\zeta_p}}(y) =& \epsilon(\pi_p)\frac{1}{1-p} \\ &+ \frac{p^{-\frac{1}{2}}}{1-1/p}\sideset{}{^*}\sum_{\chi \bmod p} \chi(-q_1 a_2 \overline{a_1 q_2})\epsilon(\overline{\chi}\pi_p)\epsilon(\chi).\end{aligned}$$

If $k_1 > 1$ *then*

$$\begin{aligned}\mathcal{B}_{\phi_p,\phi_p^{\zeta_p}}(y) =& \frac{1}{1-1/p}p^{-\frac{k_1}{2}} \\ &\times \sideset{}{^*}\sum_{\chi \bmod p^{k_1}} \chi(-q_1 a_2 \overline{a_1 q_2})\epsilon(\overline{\chi}\pi_p)\epsilon(\chi)\end{aligned}$$

Using Corollary 2.6.3.1 we see that Theorem 2.6.1 simplifies to the following statement.

Theorem 2.6.4. *Assume that* π_p *is supercuspidal and twist-minimal. Set* $\phi_p = \mathbb{I}_{Z_p^\times}$ *and* $\phi_\infty \in \mathcal{C}_c(\mathbb{R}^+)$*. Then*

$$\begin{aligned}\sum_{(m,p)=1}\frac{\lambda_\pi(m)}{|m|^{1/2}}\phi_\infty(m)e\left(\frac{am}{p^j q}\right) =& \sum_{(m,p)=1}\frac{\lambda_{\check{\pi}}(m)}{|m|^{1/2}}e\left(-\frac{\overline{a[p^{2j},p^{a(\pi_p)}]}p^j m}{q^2}\right)\chi(\bar{m}q) \\ &\times \mathcal{B}_{\pi_\infty,\phi_\infty}\left(\frac{m}{[p^{2j},p^{a(\pi_p)}]q^2}\right)\mathcal{B}_{\pi_p}^{a/lq}\left(\frac{m}{[p^{2j},p^{a(\pi_p)}]q^2}\right).\end{aligned}$$

Theorem 2.6.4 is in fact stated as Corollary 3 in [5] with the additional assumption that $j \geq 2$ (which was done for aesthetic purposes but note that the definition of $S_f\left(m, \frac{a}{lq}\right)$ needs to be adjusted for $n \geq 3$).

Now let's assume that $\pi_p = \pi(\chi_1, \chi_2)$. We observe that calculations for the non-Archimedean Bessel transforms remain the same as before as long as $v_p(\zeta_p) \neq -a(\chi_1), -a(\chi_2)$. We thus conclude the following.

Proposition 2.6.5. *Let $\pi_p = \pi(\chi_1, \chi_2)$ be a principal series representation. Let $\zeta_p \in \mathbb{Q}_p^\times$. Let $\phi_p = \mathrm{Char}_{\mathbb{Z}_p^\times}$. Assume $v_p(\zeta_p) \neq -a(\chi_1), -a(\chi_2)$. Then $\mathcal{B}_{\pi_p, \phi_p^{\zeta_p}}(y) = 0$ if $|y|_p \neq p^{\sup\{a(\chi_1), -\mathrm{ord}_p(\zeta_p)\} + \sup\{a(\chi_2), -\mathrm{ord}_p(\zeta_p)\}}$.*

Now suppose $|y|_p = p^{\sup\{a(\chi_1), -\mathrm{ord}_p(\zeta_p)\} + \sup\{a(\chi_2), -\mathrm{ord}_p(\zeta_p)\}}$.

If $|\zeta_p|_p \leq 1$ then

$$\mathcal{B}_{\pi_p, \phi_p^{\zeta_p}}(y) = \epsilon(1/2, \pi_p, \psi_p)$$

If $|\zeta_p|_p = p$ then

$$\begin{aligned}\mathcal{B}_{\phi_p, \phi_p^{\zeta_p}}(y) =& \epsilon(1/2, \pi_p, \psi_p) \frac{1}{1-p} \\ &+ \frac{p^{-\frac{1}{2}}}{1 - 1/p} \sum_{\substack{\chi \in \mathfrak{X}_p \\ a(\chi) = 1}}^{*} \chi(-\zeta_p^{-1} y) \epsilon(1/2, \overline{\chi}\pi_p, \psi_p) \epsilon(1/2, \chi, \psi_p).\end{aligned}$$

If $|\zeta_p|_p > p$ then

$$\begin{aligned}\mathcal{B}_{\phi_p, \phi_p^{\zeta_p}}(y) =& \frac{1}{1-p} p^{\frac{\mathrm{ord}_p(\zeta_p)}{2}} \\ &\times \sum_{\substack{\chi \in \mathfrak{X}_p \\ a(\chi) = -v_p(\zeta_p)}}^{*} \chi(-\zeta_p^{-1} y) \epsilon(1/2, \overline{\chi}\pi_p, \psi_p) \epsilon(1/2, \chi, \psi_p).\end{aligned}$$

The following corollary then immediately follows.

Corollary 2.6.5.1. *Assume the same as in the above proposition. Further assume that $\zeta_p = a_1/q_1 p^{k_1}$ and $y = a_2/q_2 p^{k_2}$ are both rational with $k_1, k_2 \in \mathbb{Z}$ and $(a_1 a_2 q_1 q_2, p) = (a_1, q_1) = (a_2, q_2) = 1$. Suppose $k_1 \neq a(\chi_1), a(\chi_2)$. Suppose $k_2 = \sup\{\sup\{a(\chi_1), k_1\} + \sup\{a(\chi_2), k_1\}$ so that $\mathcal{B}_{\pi_p, \phi_p^{\zeta_p}}(y) \neq 0$.*

If $k_1 \leq 0$ then

$$\mathcal{B}_{\pi_p, \phi_p^{\zeta_p}}(y) = \epsilon(\pi_p).$$

If $k_1 = 1$ then

$$\begin{aligned}\mathcal{B}_{\phi_p,\phi_p^{\zeta_p}}(y) =&\epsilon(\pi_p)\frac{1}{1-p}\\ &+\frac{p^{-\frac{1}{2}}}{1-1/p}\sideset{}{^*}\sum_{\chi \bmod p}\chi(-q_1a_2\overline{a_1q_2})\epsilon(\overline{\chi}\pi_p)\epsilon(\chi).\end{aligned}$$

If $k_1 > 1$ then

$$\begin{aligned}\mathcal{B}_{\phi_p,\phi_p^{\zeta_p}}(y) =&\frac{1}{1-1/p}p^{-\frac{k_1}{2}}\\ &\times\sideset{}{^*}\sum_{\chi \bmod p^{k_1}}\chi(-q_1a_2\overline{a_1q_2})\epsilon(\overline{\chi}\pi_p)\epsilon(\chi).\end{aligned}$$

The following then follows.

Theorem 2.6.6. *Assume that $\pi_p = \pi(\chi_1, \chi_2)$. Set $\phi_p = \mathbb{I}_{\mathbb{Z}_p^\times}$ and $\phi_\infty \in \mathcal{C}_c^\infty(\mathbb{R}^+)$. Let π_0 be the twist-minimal representation associated to π. Let ω be the central character of π. Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying $(a, qp) = (q, p) = 1$. Assume that $j \neq a(\chi_1), a(\chi_2)$. Then*

$$\begin{aligned}&\sum_{(m,p)=1}\frac{\lambda_\pi(m)}{|m|^{1/2}}\phi_\infty(m)e\left(\frac{am}{p^jq}\right) = \sum_{(m,p)=1}\frac{\lambda_{\check{\pi}}(m)}{|m|^{1/2}}e\left(-\frac{\overline{ap^{\sup\{(a(\chi_1),j)+\sup\{a(\chi_2),j\}}}p^jm}{q}\right)\\ &\times\mathcal{B}_{\pi_\infty,\phi_\infty}\left(\frac{m}{p^{\sup\{(a(\chi_1),j)+\sup\{a(\chi_2),j\}}q^2}\right)\\ &\times\left(1-\frac{1}{p}\right)^{-1}p^{-j/2}\left\{\delta_{j=1}\frac{\epsilon(\pi_p)}{1-p}+\sum_{\chi\in\mathfrak{X}_j}\epsilon(\pi\chi^{-1})\epsilon(\chi)\chi(-\overline{aq}m)\right\}.\end{aligned}$$

Now suppose $v_p(\zeta_p) = -a(\chi_1)$ or $v_p(\zeta_p) = -a(\chi_2)$. We additionally assume that $a(\chi_1\chi_2^{-1}) \neq 0$. Then $a(\pi\chi^{-1}) = a(\chi_1\chi^{-1}) + a(\chi_2\chi^{-1})$ is still true but we can no longer replace $a(\chi_i\chi^{-1})$ by $\sup\{a(\chi_1), a(\chi)\}$. We observe that from Section 2.5.1 that the L-factor remains trivial unless $a(\chi_i\chi^{-1}) = 0$ for at-least one of the χ_i. We see from Section 2.5.1 that if $a(\chi_1\chi^{-1}) = 0$ and $a(\chi_2\chi^{-1}) \neq 0$

$$\frac{L(s,\chi\check{\pi})}{L(1-s,\chi^{-1}\pi)} = -p^{-1}\chi_1(p)p^s + (1-1/p)^{-1}\sum_{k=0}^{\infty}\frac{\chi_1(p^k)^{-1}}{p^{ks}}$$

and if $a(\chi_2\chi^{-1}) = 0$ and $a(\chi_1\chi^{-1}) \neq 0$,

$$\frac{L(s, \chi\check{\pi})}{L(1-s, \chi^{-1}\pi)} = -p^{-1}\chi_2(p)p^s + (1-1/p)^{-1}\sum_{k=0}^{\infty}\frac{\chi_2(p^k)^{-1}}{p^{ks}}.$$

Note that from the assumption $a(\chi_1\chi_2^{-1}) \neq 0$, $a(\chi_1\chi^{-1})$ and $a(\chi_2\chi^{-1})$ cannot both be simultaneously 0.

We now need information about $\chi_1(p)$ and $\chi_2(p)$. If f is an automorphic form with local representation $\pi_p(\chi_1, \chi_2)$, then this representation is not twist-minimal unless at-least one of χ_1 and/or χ_2 is unramified.

Assume that $a(\chi_1\chi_2^{-1}) \neq 0$. Let $\chi \in \mathfrak{X}_p$ be the unique Dirichlet character such that $a(\chi^{-1}\chi_1) = 0$. Then $f_0 = f \otimes \chi^{-1}$ is twist-minimal. Note that $\chi_1\chi^{-1}(p) = \chi_1(p)$ and the local factor at p for f_0 is given by

$$L_p(f_0, s) = \left(1 - \frac{\chi_1(p)}{p^s}\right)^{-1}.$$

We therefore deduce that $\chi_1(p) = \lambda_{f_0}(p)$ and $\lambda_{f\otimes\chi^{-1}}(p^k) = \lambda_{f_0}(p^k)$ for all $k \in \mathbb{N}$.

Similarly, if $\chi \in \mathfrak{X}_p$ is the unique Dirichlet character such that $a(\chi^{-1}\chi_2) = 0$. Then $f_0 = f \otimes \chi^{-1}$ is also twist-minimal. Since $\chi_1\chi_2(p) = 1$ we also deduce that if $\chi \in \mathfrak{X}_p$ is the unique Dirichlet character such that $a(\chi^{-1}\chi_2) = 0$ then $\chi_2(p) = \lambda_{\check{f}_0}(p)$ and $\lambda_{f\otimes\chi^{-1}}(p^k) = \lambda_{\check{f}_0}(p^k)$ for all $k \in \mathbb{N}$.

We thus see that if $a(\chi_1\chi^{-1}) \neq 0$ and $a(\chi_2\chi^{-1}) \neq 0$ then

$$\gamma(1-s, \chi^{-1}\pi) = \epsilon(\frac{1}{2}, \chi^{-1}\pi)p^{a(\chi^{-1}\pi)(s-\frac{1}{2})}.$$

If $a(\chi_1\chi^{-1}) = 0$ and $a(\chi_2\chi^{-1}) \neq 0$ then

$$\gamma(1-s, \chi^{-1}\pi) = \epsilon(\frac{1}{2}, \chi^{-1}\pi)p^{a(\chi^{-1}\pi)(s-\frac{1}{2})}\left(-p^{-1}\chi_1(p)p^s + (1-1/p)^{-1}\sum_{k=0}^{\infty}\frac{\chi_2(p^k)}{p^{ks}}\right).$$

If $a(\chi_2\chi^{-1}) = 0$ and $a(\chi_1\chi^{-1}) \neq 0$ then

$$\gamma(1-s,\chi^{-1}\pi) = \epsilon(\frac{1}{2},\chi^{-1}\pi)p^{a(\chi^{-1}\pi)(s-\frac{1}{2})}\bigg(- p^{-1}\chi_2(p)p^s + (1-1/p)^{-1}\sum_{k=0}^{\infty}\frac{\chi_1(p^k)}{p^{ks}}\bigg).$$

Setting $\sigma = 1/2$ and evaluating the s-integral we deduce the following.

Theorem 2.6.7. *Assume that $\pi_p = \pi(\chi_1,\chi_2)$. Set $\phi_p = \mathbb{I}_{\mathbb{Z}_p^\times}$ and $\phi_\infty \in \mathcal{C}_c^\infty(\mathbb{R}^+)$. Let π_0 be the twist-minimal representation associated to π. Let ω be the central character of π. Let $k = a(\chi_1\chi_2^{-1})$. Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying $(a,qp) = (q,p) = 1$. Assume that $j = a(\chi_1)$ or $j = a(\chi_2)$. Then*

$$\begin{aligned}\sum_{(m,p)=1}\frac{\lambda_\pi(m)}{|m|^{1/2}}\phi_\infty(m)e\bigg(\frac{am}{p^jq}\bigg) =& \sum_{r=1}^{j}\sum_{(m,p)=1}\frac{\lambda_{\check{\pi}}(m)}{|m|^{1/2}}e\bigg(-\frac{\overline{ap^{k+r}}p^jm}{q}\bigg)\mathcal{B}_{\pi_\infty,\phi_\infty}\bigg(\frac{m}{p^{k+r}q^2}\bigg)\\ &\times\bigg(1-\frac{1}{p}\bigg)^{-1}p^{-j/2}\sum_{\substack{\chi\in\mathfrak{X}_j\\ a(\pi\chi^{-1})=k+r}}\epsilon(\pi\chi^{-1})\epsilon(\chi)\chi(-\overline{aq}m)\\ &+ S_1 + S_2\end{aligned}$$

where the term S_1 appears if $a(\chi) = j$ and the term S_2 appears if $a(\chi_2) = j$. They are given by

$$\begin{aligned}S_1 =& \sum_{r=1}^{\infty}{}^*\frac{\epsilon(\pi_0)\lambda_{\pi_0}(p^r)+\epsilon(\pi_0)^{-1}\lambda_{\check{\pi}_0}(p^r)}{p^{r/2}}\sum_{(m,p)=1}\frac{\lambda_{\check{\pi}}(m)}{|m|^{1/2}}e\bigg(-\frac{\overline{ap^k}p^jm}{q}\bigg)\omega^{-1}(\overline{m}q)\\ &\times\mathcal{B}_{\pi_\infty,\phi_\infty}\bigg(\frac{p^rm}{p^kq^2}\bigg)p^{-j/2}\bigg(1-\frac{1}{p}\bigg)^{-1},\end{aligned}$$

$$\begin{aligned}S_2 =& \sum_{r=1}^{\infty}{}^*\frac{\epsilon(\pi_0)^{-1}\lambda_{\pi_0}(p^r)+\epsilon(\pi_0)\lambda_{\check{\pi}_0}(p^r)}{p^{r/2}}\sum_{(m,p)=1}\frac{\lambda_{\check{\pi}}(m)}{|m|^{1/2}}e\bigg(-\frac{\overline{ap^k}p^jm}{q}\bigg)\omega^{-1}(\overline{m}q)\\ &\times\mathcal{B}_{\pi_\infty,\phi_\infty}\bigg(\frac{p^rm}{p^kq^2}\bigg)p^{-j/2}\bigg(1-\frac{1}{p}\bigg)^{-1}\end{aligned}$$

where the $\sum^*$ indicates that $r = 1$ is weighted by $(1-1/p)$.

The case $a(\chi_1\chi_2^{-1}) = 0$ corresponds to a twist of an unramified representation π_0 by a character χ with $a(\chi) = a(\chi_1) = a(\chi_2)$. Here one can use the classical Voronoi summation

formula on arithmetic progressions to get an explicit formula but we do not pursue this.

We next state a Voronoi summation formula for holomorphic modular forms of level p.

Theorem 2.6.8 (Kowalski-Michel-VanderKam)**.** *Let $g(x)$ be a smooth function on $\mathbb{R}_{>0}$ with compact support. Let $(a,q)=1$ and let $\overline{a}$ denote the multiplicative inverse of a mod q. Let p be a prime and assume f is a newform of level p and nebentypus χ. Set $p_2 = p/(p,q)$. Write $\chi = \chi_1\chi_2$ for χ_2 the induced character on p_2. Then there exists a newform $\check{f}$ of the same level and weight and a complex number η of modulus 1 (which depends on f and p) such that*

$$\sum_{n=1}^{\infty} e\left(\frac{an}{q}\right)\lambda_f(n)g(n) = \chi_1(\overline{a})\chi_2(-q)\frac{2\pi i^k\eta}{q\sqrt{p_2}}\sum_n \lambda_{\check{f}}(n)e\left(-\frac{n\overline{ap_2}}{q}\right)\int_0^{\infty} g(y)J_{k-1}\left(\frac{4\pi\sqrt{ny}}{q\sqrt{p_2}}\right)dy.$$

Proof: See the appendix to [22]. □

2.6.3 Poisson Summation

We use the following forms of the Poisson summation.

Theorem 2.6.9. *Let $V \in C_c^{\infty}(\mathbb{R}^+)$ and $N > 0$ be a parameter. Let $q \in \mathbb{N}$ and $a \in \mathbb{Z}$. Then*

$$\sum_{n\in\mathbb{Z}} V\left(\frac{n}{N}\right)e\left(\frac{an}{q}\right) = N\sum_{\substack{n\in\mathbb{Z}\\ a\equiv n \bmod q}}\int_{\mathbb{R}} V(y)e\left(-\frac{Nny}{q}\right)dy$$

Theorem 2.6.10. *Let $V \in C_c^{\infty}(\mathbb{R}^+)$ and $N > 0$ a parameter. Let $q \in \mathbb{N}$ and $a \in \mathbb{Z}$. Suppose $(a,q)=1$. Then*

$$\sum_{n} V\left(\frac{n}{N}\right)e\left(\frac{an^2}{q}\right) = \frac{N}{q}\sum_{n\in\mathbb{Z}} G(a,n,q)\int_{\mathbb{R}} V(y)e\left(-\frac{Nny^2}{q}\right)dy$$

2.7 The Delta Symbol

We use the following delta symbol expansion due to Duke,Friedlander and Iwaniec (see Chapter 20 of [17])

$$\delta(n=0)=\frac{1}{Q}\sum_{q\leq Q}\int\frac{g(q,x)}{q}\sideset{}{^*}\sum_{a \bmod q}e\left(\frac{an}{q}+\frac{nx}{qQ}\right)dx \tag{2.21}$$

valid for any $Q>2$.

The function $g(q,x)$ has the following properties.

$$g(q,x)=1+O_A\left(\frac{1}{qQ}\left(\frac{q}{Q}+|x|\right)^A\right), \tag{2.22}$$

$$g^{(k)}(q,x)\ll|x|^{-A} \tag{2.23}$$

for any $A>0$ and $k\geq 1$. Note that Equation (2.22) is incorrectly stated in [17].

(2.23) says for all q, the integral in the x-variable is supported on $[-Q^\epsilon,Q^\epsilon]$ and (2.22) says can replace $g(q,x)$ by 1 at the cost of a negligible error term for $q\ll Q^{1-\epsilon}$.

For $q\gg Q^{1-\epsilon_0}$, while we don't know that $g(q,x)$ is 1 we have

$$g^{(k)}(q,x)\ll_{k,\epsilon_0}Q^{\epsilon_0(k+1)} \tag{2.24}$$

which says that for $q\gg Q^{1-\epsilon}$, $g(q,x)$ has essentially no oscillation. To prove this, by definition we have

$$g(q,x)=\int\Delta_q(u)f(u)e\left(-\frac{ux}{qQ}\right)du$$

where $\Delta_q(u)$ satisfies

$$\Delta_q(u)\ll\frac{1}{Q(q+Q)}+\frac{1}{|u|+qQ} \tag{2.25}$$

and f is a smooth function supported in $[-Q^2,Q^2]$ with $f(0)=||f||_{L^\infty}=1$ and $f^j(u)\ll Q^{-2j}$. Then by differentiating under the integral sign we have

$$g^{(k)}(q,x)=\left(-\frac{2\pi i}{qQ}\right)^k\int\Delta_q(u)f(u)u^ke\left(-\frac{ux}{qQ}\right)du \tag{2.26}$$

Applying (2.25), (2.26) and assuming $q\gg Q^{1-\epsilon_0}$ we have

$$
\begin{aligned}
g^{(k)}(q,x) &\ll \frac{Q^{\epsilon_0 k}}{Q^{2(k+1)}}\left(\int\limits_{|u|\ll Q^2} |u|^k du + Q^2 \int\limits_{|u|\ll Q^2} |u|^{k-1} du\right) \\
&\ll Q^{\epsilon_0 k}
\end{aligned} \tag{2.27}
$$

Chapter 3

Proof of Theorem 1

By the approximate functional equation and a smooth partition of unity, we have (see [17])

$$L\left(\frac{1}{2}+it,\mathrm{sym}^2 f\right)\ll (pt)^{\epsilon}\sup_{N\ll p^{1+\epsilon}t^{3/2+\epsilon}}\frac{S(N)}{\sqrt{N}}$$

where

$$S(N)=\sum_n \lambda_f(n^2)n^{-it}V\left(\frac{n}{N}\right) \tag{3.1}$$

where $V(x)$ is a smooth function with compact support in $[1/2,5/2]$, is identically 1 on $[1,2]$ and satisfies the bounds $V^{(j)}(x)\ll_j 1$.

We now wish to separate the oscillation of $\lambda_f(n^2)$ from n^{-it}. Applying (2.21) to (3.1) we get

$$\begin{aligned}S(N)=\sum_{q\leq Q}\int_{\mathbb{R}}\frac{g(q,x)}{qQ}\sideset{}{^*}\sum_{a \bmod q}\sum_m \lambda_f(m)U\left(\frac{m}{N^2}\right)e\left(-\frac{am}{q}-\frac{mx}{qQ}\right)\\ \times\sum_n V\left(\frac{n}{N}\right)n^{-it}e\left(\frac{an^2}{q}+\frac{n^2x}{qQ}\right)dxdv\end{aligned} \tag{3.2}$$

with

$$Q=\frac{N}{\sqrt{K}}.$$

Here U is another smooth function with compact support in $[1/16,15/2]$, is identically 1 on $[1/8,13/2]$ and satisfies $U^{(j)}(x)\ll_j 1$. The goal is to prove the following main theorem.

Theorem 3.0.1. *For $pt \ll N \ll p^{1+\epsilon}t^{3/2+\epsilon}$ and $t^\epsilon \ll K \ll t^{1-\epsilon}$ we have*

$$S(N) \ll N^{1/2}p^{1/2+\epsilon}t^{1/2+\epsilon}\left(K^{1/4} + \frac{t}{K^{5/4}}\right).$$

3.1 Proof Sketch

For simplicity let's assume that we're in the generic situation that is, $N = pt^{3/2}$ and $q \sim Q$, dropping the ϵs. Then we're looking at an expression (roughly) of the form

$$\sum_{q\sim Q}\sideset{}{^*}\sum_{a \bmod q}\sum_{m\sim N^2}\lambda_f(m)e\left(-\frac{am}{q}\right)\sum_{n\sim N}n^{-it}e\left(\frac{an^2}{q}\right)$$

where we're ignoring the analytic oscillation coming from the integral over x. Notice that by applying the delta method we've obtained N^3 terms and therefore gambited N^2. So to beat the trivial convexity estimate we need to save N^2 and more.

We apply Voronoi summation to the m sum which gives us a sum of length $m \sim pK$ which saves us $N/\sqrt{pK}$. We apply Poisson summation to the n sum which gives us a sum of length $n \sim t/\sqrt{K}$ which saves us $\sqrt{N/(t/\sqrt{K})}$. The integral over x saves us an extra $\sqrt{K}$ and the resulting character sum arising from the sum over a mod q saves us another $\sqrt{q} \sim \sqrt{Q}$. So far we have saved

$$\frac{N}{\sqrt{pK}} \times \sqrt{\frac{N}{t/\sqrt{K}}} \times \sqrt{K} \times \sqrt{Q} = \frac{N}{\sqrt{pt}}$$

It remains to save $\sqrt{pt}$.

We've arrived at the expression (after simplifying the sum a mod q)

$$\sum_{q\leq Q}\int_x \frac{g(q,x)}{qQ}\sum_{m\sim pK}\sum_{n\sim t/\sqrt{K}}\lambda_f(m)\left(\frac{4m-pn^2}{q}\right)\int_v(...)$$

Applying Cauchy-Schwartz in the q variable,we get the following expression

$$L = \sum_{q\sim Q} \left| \sum_{d\sim pt^2/\sqrt{K}} \left(\frac{d}{q}\right)\alpha(d) \right|^2$$

where we've set $d = pn^2 - 4m$ and

$$\alpha(d) = \sum_{\substack{m\sim pK \\ n\sim t/\sqrt{K} \\ d=pn^2-4m}}\sum \lambda_f(m) \int_v (...) \ll (pt)^\epsilon \sum_{\substack{m\sim pK \\ n\sim t/\sqrt{K} \\ d=pn^2-4m}}\sum 1.$$

Since we applied Cauchy-Schwartz we now require a saving of pt. Trivially, we see that

$$L \ll Q(pK)^2(t/\sqrt{K})^2 = p^3t^{7/2}\sqrt{K}.$$

We now apply quadratic large sieve which gives (ignoring the square-free conditions on q and d)

$$L \ll (pt)^\epsilon \bigg(Q + pt^2/K\bigg)||\alpha||^2$$

where

$$||\alpha||^2 = \sum_d |\alpha(d)|^2 = \sum_{\substack{m_1,m_2\sim pK \\ n_1,n_2\sim t/\sqrt{K} \\ pn_1^2-4m_1=pn_2^2-4m_2}}\sum\sum\sum\sum 1.$$

We get $n_1^2 - n_2^2 = 4(m_2 - m_1)/p \ll K$ which implies $n_1 - n_2 \ll K/(n_1 + n_2) \ll K^{3/2}/t$. If $t^{2/3} \ll K \ll t$ then the number of pairs of (n_1, n_2) is $\ll K$. It follows that the total number of terms (n_1, n_2, m_1, m_2) is $\ll pK^2$. So $||\alpha||^2 \ll pK^2$ and hence $L \ll (pt)^\epsilon p^2t^2K$. This saves us $pt^{3/2}/\sqrt{K}$. This is better than pt as long as $K < t$. Choosing $K \sim t^{2/3}$ gives the best possible result.

3.2 Application of Summation Formulae

3.2.1 Poisson Summation

The n-sum is given by

$$\sum_n V\left(\frac{n}{N}\right) n^{-it} e\left(\frac{an^2}{q} + \frac{n^2 x}{qQ}\right).$$

We set $g(y) = y^{it-2iv} V\left(\frac{y}{N}\right) e\left(\frac{y^2 x}{qQ}\right)$. We apply Theorem 2.6.10 to the n-sum after making the change of variable $y \mapsto yN$ we obtain

$$\sum_{n\geq 1} g(n) e\left(\frac{an^2}{q}\right) = \frac{N^{1-it}}{q} \sum_{|n|\ll N_0} G(a,n,q) I_1(n,q) \tag{3.3}$$

where

$$I_1(n,q) = \int_{\mathbb{R}} y_1^{-it} V(y_1) e\left(\frac{N^2 y_1^2 x}{qQ} - \frac{nNy_1}{q}\right) dy_1.$$

By repeated integration by parts we see that the integral is negligibly small

$$n \gg N^{\epsilon}\left(\frac{qt}{N} + K^{1/2}\right). \tag{3.4}$$

3.2.2 Voronoi Summation

Here we set our $g(y) = U(\frac{y}{N^2}) e(-\frac{xy}{qQ})$. Then by Theorem 2.6.8

$$\begin{aligned} &\sum_{m=1}^{\infty} \lambda_f(m) U\left(\frac{m}{N^2}\right) e\left(-\frac{am}{q} - \frac{mx}{qQ}\right) \\ =&\frac{1}{q\sqrt{p_2}} \sum_{m=1}^{\infty} \lambda_{\check{f}}(m) e\left(\frac{m\overline{ap_2}}{q}\right) \int_{\mathbb{R}} U\left(\frac{y_2}{N^2}\right) e\left(\frac{y_2 x}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{my_2}}{q\sqrt{p_2}}\right) dy_2. \end{aligned} \tag{3.5}$$

We apply (2.18) and (2.19) so that the integral as a linear combination of integrals of the form

$$\int_{\mathbb{R}} U\left(\frac{y_2}{N^2}\right) W_{k-1}\left(\frac{4\pi\sqrt{my_2}}{q\sqrt{p_2}}\right) e\left(-\frac{y_2 x}{qQ}\right) e\left(\pm\frac{2\sqrt{my_2}}{q\sqrt{p_2}}\right) dy_2 \tag{3.6}$$

Recall that $p_2 = p/(p,q)$. We from now on assume $p_2 = p$ that is, $(p,q) = 1$. The case $p|q$ gives a considerably shorter sum and saves an extra $\sqrt{p}$ over the complimentary case.

Let us make the change of variable $y \mapsto yN^2$ to obtain

$$I_2(m,q) = \sum_{\pm} \int_{\mathbb{R}} U(y_2) W_{k-1}\left(\frac{4\pi N\sqrt{my_2}}{q\sqrt{p}}\right) e\left(-\frac{N^2 y_2 x}{qQ} \pm \frac{2N\sqrt{my_2}}{q\sqrt{p}}\right) dy_2$$

and by repeated integration by parts we have that the integral is negligible if

$$m \gg N^{\epsilon}\frac{q^2 p}{N^2}\left(1 + \frac{N^4 x^2}{q^2 Q^2}\right) =: M_0. \tag{3.7}$$

Note that $|x| \ll N^{\epsilon}$ and $q \leq Q = \frac{N}{\sqrt{K}}$ so that

$$M_0 \ll N^{\epsilon} pK.$$

We get that the m-sum is transformed into

$$\frac{N^2}{p^{1/2}q} \sum_{m \ll M_0} \lambda_f(m) e\left(\frac{\overline{ap}m}{q}\right) I_2(m,q).$$

3.3 Simplification

3.3.1 The Integral

Now we attempt to simplify the threefold integral. We follow the treatment of the integral as in [26]. Recall this is given by

$$\begin{aligned} I(m,n,q) = \int_{\mathbb{R}} g(q,x) \int_{\mathbb{R}} U(y_1) y_1^{-it} \int_0^{\infty} V(y_2) W_{k-1}\left(\frac{4\pi N\sqrt{my_2}}{q\sqrt{p}}\right) \\ \times e\left(\frac{N^2 x(y_1^2 - y_2)}{qQ} - \frac{nNy_1}{q} \pm \frac{2N\sqrt{my_2}}{q\sqrt{p}}\right) dy_1 dy_2. \end{aligned}$$

Now we assume that $q \sim C$ and $m \sim M_1$ with $C \ll Q$ and $M_1 \ll M$. First we assume that $N^2 M_1/C^2 p \gg N^\epsilon$. Then we note that looking at the y_2 integral $I_2(m,q)$ we see that by repeated integration by parts,$I(m,n,q)$ is negligible unless $t^{-\epsilon}\sqrt{\frac{M_1}{pK}} \ll |x| \ll t^\epsilon \sqrt{\frac{M_1}{pK}}$.

Next assume that $M_1 \ll C^2 p/N^{2-\epsilon}$. Then looking at the y_2 integral once again, we see that the integral is negligible unless $|x| \ll CQ/N^{2-\epsilon}$.

Therefore we may control the domain of integration of the x-integral as a function of M_1, say $I(M_1)$ with (here μ is the Lebesgue measure on $\mathbb{R}$)

$$\mu(I(M_1)) \ll \begin{cases} CQ/N^{2-\epsilon} & \text{if } M_1 \ll C^2 p/N^{2-\epsilon} \\ t^\epsilon \sqrt{\frac{M_1}{pK}} & \text{if } M_1 \gg C^2 p/N^{2-\epsilon}. \end{cases} \tag{3.8}$$

Now we assume $C \ll Q^{1-\epsilon}$. By looking at the x-integral and using (2.22) we conclude that $|y_1^2 - y_2| \ll N^{2\epsilon} C/QK$ which implies $|y_1 - \sqrt{y_2}| \ll N^{2\epsilon} C/QK$. Next, assume $q \gg Q^{1-\epsilon}$. Then we use (2.24) and repeated integration by parts to conclude that $|y_1 - \sqrt{y_2}| \ll N^{2\epsilon} C/QK$. So for all $q \leq Q$ we conclude that $|y_1 - \sqrt{y_2}| \ll N^{2\epsilon} C/QK$.

We set $\sqrt{y_2} = y_1 + u$ with $|u| \ll N^{2\epsilon} C/QK$ and insert this into the integral to obtain

$$\int_{\mathbb{R}} g(q,x) \int_{\mathbb{R}} U(y_1) y_1^{-it} \int_0^\infty V((y_1+u)^2)\sqrt{y_1+u} W_{k-1}\left(\frac{4\pi N\sqrt{m}(y_1+u)}{q\sqrt{p}}\right)$$
$$\times e\left(\frac{N^2 x(y_1^2 - (y_1+u)^2)}{qQ} - \frac{nNy_1}{q} \pm \frac{2N\sqrt{m}(y_1+u)}{q\sqrt{p}}\right) du dy_1 dx.$$

Now we want to perform stationary phase analysis on the y_1 integral. The weight function is given by

$$G(z) = U(z)V((z+u)^2)\sqrt{z+u}W_{k-1}\left(\frac{4\pi N\sqrt{m}(y_1+u)}{q\sqrt{p}}\right).$$

From (2.19) we see that

$$G^{(j)}(z) \ll_{j,k} \left(1 + \frac{4\pi N\sqrt{m}}{q\sqrt{p}}\right)^{-1/2}, \quad j \geq 0.$$

The phase function is given by

$$P(z) = -\frac{t}{2\pi}\log z - \frac{2N^2uxz}{qQ} - \frac{nNz}{q} \pm \frac{2N\sqrt{m}z}{q\sqrt{p}}$$

while it's derivatives satisfy

$$P'(z) = -\frac{t}{2\pi z} - \frac{2N^2ux}{qQ} - \frac{nN}{q} \pm \frac{2N\sqrt{m}}{q\sqrt{p}},$$

$$P^{(j)}(z) = t\frac{(-1)^j(j-1)!}{2\pi z^j}, \quad j \geq 2.$$

Therefore we see that the stationary point,which is the point z_0 satisfying

$$P'(z_0) = 0,$$

is given by

$$z_0 = \frac{qt}{2\pi N}p(m,n,u,x)^{-1}$$

where

$$p(m,n,u,x) = \frac{2Nux}{Q} - n \pm \sqrt{\frac{m}{p}}.$$

For notational purposes, we'll drop the dependence on u and x. We apply Theorem 2.4.2 with $\alpha = 1/32$, $\beta = 7$, $M = \beta - \alpha$, $T = t$ and $U = N = 1$. We thus see that all the H_i are 0.

Due to support conditions on the weight function, we must have $z_0 \asymp 1$ otherwise the integral is negligibly small. This implies

$$p(m,n) \asymp \frac{Ct}{N}. \tag{3.9}$$

Note that

$$\frac{2Nux}{Q} \ll \frac{1}{N}. \tag{3.10}$$

If in addition, $qt/N \gg t^\epsilon K^{1/2}$ then $n \sim Ct/N$ from (3.10) and (3.7).

3.3.2 Separation of Variables

We now require a bilinear structure in the q and (m,n) variable, that is, we require that if $(b_{m,n,q})$ is a sequence depending on m,n and q, then

$$b_{m,n,q} = a_{m,n}c_q \tag{3.11}$$

for some sequences $(a_{m,n})$ and (c_q). We choose n in Theorem 2.4.2, (not to be confused with the variable n that we seek a bilinear structure for) sufficiently large so that the error term is negligible. We then see that a typical term in the expansion is given by

$$e\frac{(P(z_0)+1/8)}{\sqrt{P''(z_0)}}\omega_{2j}\frac{(-1)(2j-1)!}{(4\pi\lambda_2)^j}$$

with

$$\lambda_k = \frac{P^{(k)}(z_0)}{k!}, \quad k \geq 2$$

$$\eta_k = \frac{G^{(k)}(z_0)}{k!}, \quad k \geq 1$$

and

$$\omega_{2j} = \eta_{2j} + \sum_{l=0}^{2j-1}\eta_l \sum_{i=1}^{2j-l}\frac{C_{k,l,j}}{\lambda_2^j} \sum_{\substack{3\leq n_1,\ldots,n_i,2n+3 \\ n_1+\ldots+n_i=2j-l+2i}} \lambda_{n_1}\ldots\lambda_{n_j}.$$

By definition of z_0, we see that the λ_k and $e(P(z_0))$ are bilinear in q and (m,n). To get the bilinear structure in η_k we apply Mellin inversion. Therefore

$$G^{(k)}(z_0) = \int_0^\infty \tilde{G^{(k)}}(s)(z_0)^{s-1}ds.$$

Here, the $\tilde{G^{(k)}}(s)$ depends on m and C. Note that the integral can be truncated upto $|s| \ll N^\epsilon$ and

$$\tilde{G^{(k)}}(s) \ll_k \left(1+\frac{N\sqrt{m}}{C\sqrt{p}}\right)^{-1/2}.$$

We therefore have a bilinear structure in η_k as well. The bilinear structure for the typical term then follows from noting that the product of bilinear terms remains bilinear,

that is, if $(b^1_{m,n,q})$ and $(b^2_{m,n,q})$ are two sequences satisfying (3.11) then so does the sequence $(b^1_{m,n,q}b^2_{m,n,q})$.

3.3.3 The Character Sum

By Lemmas 2.2.4 and 2.2.5 we evaluate the sum $\mathcal{C}(m,n,q)$. We make some comments on the case $2|q$, say $2^k||q$ with $k \geq 1$.

If $k \equiv 1 \bmod 2$ then the sum vanishes unless $k = 1, 3$. If $k = 1$ then we see that this matches the case q odd up to a factor of $e\left(\frac{pm}{2}\right)$ and the additional condition that n be even. If $k = 3$ we see that this matches the case q odd with an additional factor of $S(p+l, 0, 8)$ which we may treat as a constant term. Now assume $k \equiv 0 \bmod 2$. Looking at the term

$$S(l, 0, 2^k) + S(p+l, 0, 4)2^{k-2}\delta(2^{k-2}|l)$$

(recall that $l = 4m - p(n/2)^2$ and p is odd), we see that the second term vanishes if $k \geq 3$. On the other hand, the first term can be absorbed with the $S(l, 0, q_2)$ and by multiplicative properties of the Ramanujan sum, this is in fact $S(l, 0, 2^k q_2)$.

3.3.4 Final Simplifications

We now make the following simplifications -

(1) We assume that we've performed all Mellin inversions to the $G^{(k)}(z_0)$ to obtain a bilinear form in the (m,n) and q variables. This comes at a cost of N^ϵ loss and allows us to replace $G^{(k)}(z_0)$ with $\left(1 + \frac{N\sqrt{m}}{q\sqrt{p}}\right)^{-1/2}$ with a dependence on the u and x variables.

(2) We assume that we're treating the case ω_{2j} with $j = 0$, that is, only the first term in Theorem 2.4.2. The additional terms are provide better bounds by a factor of t and can be handled almost identically to this case.

(3) We assume that q is either odd or $q = 2^{2k+1}q_0$ with q_0 odd since the other cases differ from this by a bounded constant term as explained in the previous subsection.

(4) Finally, we observe that u depends on the size of q, which we may handle by applying a dyadic partition on the q-variable.

Applying our dualizations we get

$$S(N) = \sum_{C \text{ dyadic}} S(N; C) + O(N^{-A}) \tag{3.12}$$

where

$$\begin{aligned} S(N, C) =& \frac{N^{3-it}}{Q\sqrt{p}} \sum_{q \sim C} \frac{1}{q^{2+it}} \\ & \times \sum_{n \ll N_0} \sum_{m \ll M_0} \lambda_{\check{f}}(m) \mathcal{C}(m, n, q) I(m, n, q). \end{aligned}$$

Note that we have

$$S(N) \ll N^{\epsilon} \sup_{C \ll Q} S(N, C).$$

With a dyadic partition of C fixed, we now further apply a dyadic partition on the m-variable. Thus we have

$$S(N, C) = \sum_{\substack{M_1 \ll M_0 \\ M_1 \text{ dyadic}}} S(N, C, M_1)$$

where $S(N, C, M_1)$ is defined identically to $S(N, C)$ with the condition $m \ll M_0$ replaced by $m \sim M_1$. Note that

$$S(N, C) \ll N^{\epsilon} \sup_{M_1 \ll M_0} S(N, C, M_1).$$

Therefore the bounds for $S(N)$ follows from bounds on $S(N, C, M_1)$, which we now seek to find.

3.4 Conclusion of the Proof

Using Lemma 2.2.5 and simplifications in the character sum and the integral, we see that (with q decomposing as in the notation of Lemma 2.2.5)

$$S(N,C,M_1) = \frac{N^3}{Q\sqrt{pt}} \int\limits_{|u|\ll C/QK} \int\limits_{x\in I(M_1)} \sum_{q_0^2 q_1 q^* = q \sim C} \frac{g(q,x)}{(q_0^2 q_1 q^*)^{3/2+it}} \times \sum_{n\sim N_0} \sum_{m\sim M_1} \lambda_{\check{f}}(m) W_{k-1}\left(\frac{2N\sqrt{m}}{q\sqrt{p}}\right) \sqrt{q^*} S(4m-pn^2, 0, q_0^2) \left(\frac{\frac{4m-pn^2}{q_1}}{q^*}\right) q_1.$$

Writing the Ramanujan sum as

$$S(r,0;q) = \sum_{d|(r,q)} d\mu\left(\frac{q}{d}\right)$$

and interchanging sums gives

$$\begin{aligned} &\frac{N^3}{Q\sqrt{p}\sqrt{t}} \int\limits_{|u|\ll C/QK} \int\limits_{x\in I(M_1)} \sum_{l\ll C} \frac{\mu(l)}{l^{5/2}} \sum_{dq_1q^*=q\sim C/l} \frac{g(ql,x)}{q^{5/2}} \\ &\times \sum_{n\ll N_0} \sum_{m\ll M_0} \lambda_f(m) \left(\frac{d}{q^*}\right) \left(\frac{\frac{4m-pn^2}{q_1 d}}{q^*}\right) q_1 d\sqrt{q^*}. \end{aligned} \tag{3.13}$$

Applying Cauchy-Schwartz we obtain

$$\frac{\mu(I(M_1))N^3}{Q^2K\sqrt{t}} \left(1+\frac{N\sqrt{M_1}}{C\sqrt{p}}\right)^{-1/2} \sum_{l\ll C} \frac{1}{l} \left(\sum_{dq_1q^*\sim C/l} \left| \sum_{n\ll N_0} \sum_{m\ll M_1} \lambda_{\check{f}}(m) \left(\frac{\frac{4m-pn^2}{q_1 d}}{q^*}\right) \sqrt{q_1 d} \right|^2 \right)^{1/2}.$$

We observe that

$$\mu(I(M_1)) \left(1+\frac{N\sqrt{M}}{C\sqrt{p}}\right)^{-1/2} \ll \begin{cases} CQ/N^{2-\epsilon} & \text{if } M_1 \ll C^2p/N^{2-\epsilon} \\ \frac{M_1^{1/4} C^{1/2} p^{1/4}}{N^{1/2}\sqrt{pK}} & \text{if } M_1 \gg C^2p/N^{2-\epsilon}. \end{cases} \tag{3.14}$$

We now apply Theorem 2.3.1 for each fixed d and q_1 getting

$$\frac{\mu(M_1)}{QK\sqrt{t}M_1^{1/4}}\sum_{l\ll C}\frac{1}{l}\Bigg(\sum_{dq_1\ll C/l}\Bigg(\Big(C+L\Big)||\alpha(q_1,d)||_2^2\Bigg)\Bigg)^{1/2}$$

where

$$L = 2025\sup\Big\{pN_0^2, M_1\Big\} \tag{3.15}$$

and

$$||\alpha(q_1,d)||_2^2 = \sum_{\substack{(pn_1^2-4m_1)(pn_2^2-4m_2)=\square\\ q_1d|(pn_i^2-4m_i)}}\sum\sum\sum\sum|\lambda_f(m_1)\lambda_f(m_2)|. \tag{3.16}$$

We now apply the Deligne/Ramanujan bound (2.11) to trivially bound $|\lambda_{\check{f}}(m_i)| \ll (pt)^{\epsilon}$.

Next we count the number of points in the set

$$\{(n_1,n_2,m_1,m_2) : n_i \sim N_0, m_i \sim M_1, (pn_1^2-4m_1)(pn_2^2-4m_2) = \square, q_1d|(pn_i^2-4m_i)\}.$$

Note that this is bounded by counting the set

$$\{(n_1,n_2,m_1,m_2,\theta,r_1,r_2) : n_i \sim N_0, m_i \sim M_1, pn_i^2-4m_i = q_1d\theta r_i^2\}.$$

We observe that the total number of points (θ,r_1,r_2) is bounded by $\frac{L}{q_1d}$. Now we split into two cases - $C \ll \frac{N^{1+\epsilon}K^{1/2}}{t}$ and $C \gg \frac{N^{1+\epsilon}K^{1/2}}{t}$.

3.4.1 Case I - Small C

Assume

$$C \ll \frac{N^{1+\epsilon}K^{1/2}}{t}. \tag{3.17}$$

Therefore $N_0 \ll K^{1/2}$. Trivially counting $n_1, n_2, \theta, r_2, r_2$ we obtain

$$S(N,C,M_1) \ll \frac{\mu(I(M_1))}{QKt^{1/2}}\left(1+\frac{N\sqrt{M_1}}{C\sqrt{p}}\right)^{-1/2}(C+pK)^{1/2}p^{1/2}K.$$

Bounding C by $NK^{1/2}/t$ gives a better bound than in Case II.

3.4.2 Case II - Large C

We now assume $C \gg \frac{NK^{1/2}}{t}$. This implies $N_0 \sim Ct/N$ and that $n \sim N_0$ by stationary phase analysis. It also implies $pN_0^2 \gg M_0$.

Here we use the crucial fact that $n \sim N_0$. We observe that

$$|pn_i^2 - \theta r_i^2| \ll M_1 \implies \left|n_i - \frac{\sqrt{\theta}r_i}{\sqrt{p}}\right| \ll \left(\frac{M_1}{pN_0}+1\right).$$

This gives a bound on the number of n_i if we fix r_1, r_2 and θ. So summing over r_1, r_2 and θ and q_1 and d gives the point count

$$\left(C+L\right)L\left(\frac{M_1}{pN_0}+1\right)^2.$$

If $M_1 > pN_0$ then we obtain the bound

$$S(N,C,M_1) \ll \frac{pN_0^2M^{3/4}}{pN_0QK\sqrt{t}} \ll \frac{N_0M^{3/4}}{QK\sqrt{t}}. \tag{3.18}$$

Using the trivial bounds $M_1 \ll pK$ and $N_0 \ll t/\sqrt{K}$ gives

$$S(N,C,M_1) \ll \frac{1}{QK}p^{3/4}\sqrt{t}K^{1/4}. \tag{3.19}$$

If $M_1 \ll pN_0$ we have bound

$$S(N,C,M_1) \ll \frac{1}{QK\sqrt{t}}\mu(I(M_1))\left(1+\frac{N\sqrt{M}}{C\sqrt{p}}\right)^{-1/2}pN_0^2. \tag{3.20}$$

Trivially bounding $N_0 \ll t/\sqrt{K}$, $M_1 \ll pK$ and $C \ll Q$ gives

$$S(N, M_1, C) \ll \frac{1}{QK} \frac{p^{3/4} t^{3/2}}{K^{5/4}}. \tag{3.21}$$

From (3.19) and (3.21) we get

$$S(N, C) \ll N^{1/2} p^{1/2} t^{1/2} \left(K^{1/4} + \frac{t}{K^{5/4}} \right). \tag{3.22}$$

Inserting (3.22) into (3.12) we obtain

$$S(N) \ll N^{1/2} p^{1/2+\epsilon} t^{1/2+\epsilon} \left(K^{1/4} + \frac{t}{K^{5/4}} \right). \tag{3.23}$$

This proves Theorem 3.0.1. The bound is optimal with the choice $K = t^{2/3}$ which proves Theorem 1.

Chapter 4

Proof of Theorem 2

Assume that $L_p(\pi_1 \otimes \pi_2, s)$ is trivial. This turns out to be true *ex poste*. By the assumptions on π_1 and π_2, the conductor exponent on $\pi_1 \otimes \pi_2$ is $2a(\pi_1)$. By the approximate functional equation and an application of a smooth partition of unity we have

$$L(\frac{1}{2}, \pi_1 \otimes \pi_2) \ll \sup_{N \ll p^{\frac{1}{2}a(\pi_1 \otimes \pi_2)}} \frac{S(N)}{\sqrt{N}}$$

where

$$S(N) = \sum_{(n,p)=1} \lambda_f(n)\lambda_g(n) V\left(\frac{n}{N}\right).$$

Here we've set f to be an automorphic form attached to the automorphic representation π_1. Similarly, g is an automorphic form attached to the automorphic representation π_2. We rewrite this as

$$S(N) = \sum_{(n,p)=1} \sum_{(m,p)=1} \lambda_f(n) V\left(\frac{n}{N}\right) \lambda_g(m) U\left(\frac{m}{M}\right) \delta(p^\sigma | m-n) \delta\left(\frac{m-n}{p^\sigma}\right)$$

with $\sigma \leq a(\pi_1)/2$. We now assume that $a(\omega_{\pi_2}) \leq \sigma$. We now modify the sum further. Assume $p \neq 2$. Given any character $\chi \bmod p^j$ with $j \geq 2$, there exists a unique character ψ such that $\chi = \left(\frac{\cdot}{p}\right)\psi^2$ or $\chi = \psi^2$ where $\left(\frac{\cdot}{p}\right)$ is the quadratic character mod p. Note that ψ has the same level as χ. Let $\omega_{1/2}$ be said character for ω_{π_1}. We then modify the

sum to

$$S(N) = \sum_{(n,p)=1} \sum_{(m,p)=1} \lambda_f(n)\omega_{1/2}(n)V\left(\frac{n}{N}\right)\lambda_g(m)\omega_{1/2}^{-1}(n)U\left(\frac{m}{M}\right)\delta(p^\sigma|m-n)\delta\left(\frac{m-n}{p^\sigma}\right).$$

We now observe that $g \otimes \omega_{\pi_2}^{-1}$ now has at-most quadratic nebentypus. Furthermore, since $a(\omega_{\pi_2}) \leq \sigma$, we see that $a(\pi_2 \otimes \omega_{1/2}^{-1}) \leq 2\sigma$ and for any twist $\chi \in \mathfrak{X}$, we have $a(\pi_2 \otimes \omega_{1/2}^{-1} \otimes \chi) \leq \sup\{a(\pi_2 \otimes \omega_{1/2}^{-1}, 2a(\chi)\}$. On the other hand, we imposed the condition $a(\omega_{\pi_1}) \leq \frac{a(\pi_1)}{2} - 1$ which remains unchanged for $\pi_1 \otimes \omega_{1/2}$.

Since p is a fixed odd prime, by an abuse of notation, we shall say π_i instead of $\pi_{i,p}$ when referring to the local representation of π_i at p. For example, when we say that π_i is a principal series representation, we mean that $\pi_{i,p}$ is a principal series representation, etc.

Note that if π_1 is supercuspidal, then the assumptions on π_1 guarantee that it is twist-minimal. By our transformations on π_2, we draw the same conclusion when π_2 is also supercuspidal.

In the course in the rest of the proof we assume that $a(\pi_2) \leq 2\sigma$ and π_2 has at most quadratic nebentypus.

Let (θ_i, E_i) be the pair associated to π_i via the Langlands correspondence, ω_i is a uniformizer for E_i, $\mathfrak{o}_{E_i}$ be the ring of integers of E_i and ψ_{E_i} the standard additive character on E_i. If $e_i = 1$ then we take $\omega_i = p$. If $e_i = 2$ then we are only guaranteed that $\omega_i = pu_i$ for a unit $u_i \in \mathbb{Z}_p^\times$.

For any $r \geq e_i$ there exists a constant c_i a unit, such that we have the formula

$$\theta_i(\exp(\omega_i^r u)) = \psi_{E_i}(\omega_i^{-a(\theta_i)+r} c_i u)$$

for any $u \in \mathfrak{o}_{E_i}$.

The assumption on the nebentypus of π_2 guarantees that $\mathrm{tr}(c_2) = 0$ while the assumption on π_1 guarantees that $\mathrm{tr}(c_1) \equiv 0 \bmod p$. To see this, note that $\theta_i|_{\mathbb{Q}_p^\times} = \omega_{\pi_i}$.

The assumption on ω_{π_2} falls out more naturally to simplify components of the proof, while the assumption on ω_{π_1} is made due to technical complications that arise in the proof, namely in the proof of Theorem 4.4.5. It is also used in Section 2.1 but it's use there can be possibly avoided with extra effort.

We now expand the first delta symbol with additive frequencies and the second using the delta symbol of Duke, Friedlander and Iwaniec 2.7

$$\begin{aligned} S(N) =& \frac{1}{p^\sigma Q} \sum_{q \leq Q} \int_{\mathbb{R}} \frac{g(q,x)}{q} \sideset{}{^*}\sum_{a \bmod q} \sum_{u \bmod p^\sigma} \\ & \times \sum_{(m,p)=1} \lambda_g(m) e\left(-\frac{am}{p^\sigma q} - \frac{um}{p^\sigma} - \frac{mx}{p^\sigma qQ} \right) \\ & \times \sum_{(n,p)=1} \lambda_g(n) e\left(\frac{an}{p^\sigma q} + \frac{un}{p^\sigma} + \frac{nx}{p^\sigma qQ} \right) dx. \end{aligned} \tag{4.1}$$

We split this further into the divisibility of q by p and obtain

$$\begin{aligned} S(N) =& \frac{1}{p^\sigma Q} \sum_{t \geq 0} \sum_{\substack{q \leq Q/p^t \\ (q,p)=1}} \int_{\mathbb{R}} \frac{g(p^t q,x)}{p^t q} \sideset{}{^*}\sum_{a \bmod p^t q} \sum_{u \bmod p^\sigma} \\ & \times \sum_{(m,p)=1} \lambda_g(m) e\left(-\frac{am}{p^{\sigma+t} q} - \frac{um}{p^\sigma} - \frac{mx}{p^{\sigma+t} qQ} \right) \\ & \times \sum_{(n,p)=1} \lambda_f(n) e\left(\frac{an}{p^{\sigma+t} q} + \frac{un}{p^\sigma} + \frac{nx}{p^{\sigma+t} qQ} \right) dx. \end{aligned} \tag{4.2}$$

We split $S(N) = S_{t=0}(N) + S_{t>0}(N)$ with the t-parameter as given above.

4.1 Proof Sketch

Recall that $a(\pi_1) = l_1$ and $a(\pi_2) = l_2$. We've assumed that $2\sigma \geq l_2$ and $a(\omega_{\pi_2}) \leq \sigma$. Let's assume that we're in the generic case $N = p^{l_1}$ and $q \sim Q = \sqrt{N/p^\sigma}$. Ignoring any analytic

oscillations, then $S(N)$ is roughly of the shape

$$\sum_{q\sim Q}\sideset{}{^*}\sum_{a \bmod p^\sigma q}\sum_n \lambda_f(n)e\left(\frac{an}{p^\sigma q}\right) \times \sum_m \lambda_g(m)e\left(-\frac{am}{p^\sigma q}\right).$$

In applying the delta symbol we have gambited N. So we need to save N and more.

Applying Voronoi Summation in the n sum gives

$$\sum_{n^*\sim p^{a(\pi_1)}Q^2/N=p^{l_1-\sigma}} \lambda_f(n^*)e\left(-\frac{\overline{ap^{l_1-\sigma}}n^*}{q}\right)\mathcal{C}_{\pi_1}(n^*q,p^\sigma)I(...)$$

and saves $\sqrt{N/p^{l_1-\sigma}}$.

Applying Voronoi Summation in the m-sum gives

$$\sum_{m^*\sim p^{2\sigma}Q^2/N=p^\sigma} \lambda_g(m^*)e\left(\frac{\overline{ap^\sigma}m^*}{q}\right)\mathcal{C}_{\pi_2}(m^*q,p^\sigma)I(...)$$

and saves $\sqrt{N/p^\sigma}$.

The character sum mod p^σ saves $\sqrt{p^\sigma}$. The a mod q sum is a Ramanujan sum and saves $q\sim Q$. So far we have saved

$$\frac{N}{\sqrt{p^{l_1}}}\times\sqrt{p^\sigma}\times Q = N.$$

At this stage we require any additional saving. The sum is now of the shape

$$\sum_{q\sim Q}\sum_{\substack{n\sim p^{l_1-\sigma}\\ mp^{l_1-\sigma}\equiv n \bmod q}}\sum_{m\sim p^\sigma}\lambda_f(n)\lambda_g(m)\mathcal{C}_0(m,n,p^\sigma)I(...).$$

We apply the triangle inequality in the q-variable and Cauchy-Schwartz in the n-variable to obtain a sum of the form

$$\sum_{n\sim p^{l_1-\sigma}}\left|\sum_{\substack{m\sim p^{\sigma}\\ mp^{l_1-\sigma}\equiv n \bmod q}}\lambda_g(m)\mathcal{C}_0(m,n,p^{\sigma})I(...)\right|^2.$$

We majorize the n-sum and open the square. We then apply Poisson summation in the n-variable. Note that the n-sum is essentially of length $p^{l_1-\sigma}/Q$. The diagonal saves

$$\frac{p^{\sigma}}{Q}$$

while the off-diagonal saves

$$\frac{p^{l_1-\sigma}}{p^{\sigma}Q}\times\sqrt{p^{\sigma}}.$$

Equating the two savings gives the optimal choice $\sigma\sim 2l_1/5$ with the optimal saving being $p^{l_1/20}$. Note that by choosing $\sigma\sim 2l_1/5$, by the assumptions on π_1 and π_2, we guarantee that $L_p(\pi_1\otimes\pi_2,s)$ is trivial and the conductor exponent of $\pi_1\otimes\pi_2$ at p is $2a(\pi_1)$.

Note that the above sketch relies on "square-root cancellation" in all p-adic character sums that appear. Not only is this not uniformly true in all ranges, but it also very complicated to prove at the "transition range" post Cauchy-Schwartz.

4.2 Application of Summation Formulae

4.2.1 Case I - $S_{t>0}(N)$

We first treat the n-sum. The treatment of the m-sum is almost identical. The n-sum, for fixed p^tq is given by

$$\sum_{(n,p)=1}\lambda_f(n)e\left(\frac{an}{p^{\sigma+t}q}+\frac{un}{p^{\sigma}}+\frac{nx}{p^{\sigma+t}qQ}\right)V\left(\frac{n}{N}\right).$$

Then in the notation of Theorem 2.6.4, we set $\phi_p = \mathbb{I}_{\mathbb{Z}_p^\times}$ and $\phi_\infty(m) = m^{1/2} e\left(\frac{nx}{p^{\sigma+t}qQ}\right) V\left(\frac{n}{N}\right)$. Set $h = a + qup^t$. Clearly $(h, q) = 1$. We also have $(h, p) = 1$ if $t > 0$. Note that $h \equiv a \bmod q$. Applying Theorem 2.6.4 gives

$$\sum_{(n,p)=1} \frac{\lambda_f(n)}{n^{1/2}} e\left(-\frac{\overline{a[p^{l_1}, p^{2(\sigma+t)}]} np^{\sigma+t}}{q}\right)$$
$$\times \chi_f(\overline{n}q)^{-1} \mathcal{B}_{\pi_{\infty,f},\phi_\infty}\left(\frac{n}{[p^{l_1}, p^{2(\sigma+t)}]q^2}\right) \mathcal{B}_{\pi_{p,f},\Phi_p^{h/p^{\sigma+t}q}}\left(\frac{n}{[p^{l_1}, p^{2(\sigma+t)}]q^2}\right)$$

with the Archimedean Bessel transform given by

$$\mathcal{B}_{\pi_{\infty,f},\phi_\infty}\left(\frac{n}{[p^{l_1}, p^{2(\sigma+t)}]q^2}\right) = \left(\frac{n}{[p^{l_1}, p^{2(\sigma+t)}]q^2}\right)^{1/2} \int_{\mathbb{R}} V\left(\frac{y_1}{N}\right)$$
$$\times e\left(\frac{y_1 x}{p^{\sigma+t}qQ}\right) J_{k_1}\left(\frac{4\pi\sqrt{ny_1}}{[p^l, p^{2(\sigma+t)}]^{1/2}q}\right) dy_1.$$

Applying a change of variable and using (2.18) and (2.19) we see that this is given by a linear combination of the form

$$\frac{N^{3/4}}{q^{1/2}[p_1^l, p^{2(\sigma+t)}]^{1/4}} \sum_{(n,p)=1} \frac{\lambda_f(n)}{n^{1/4}} e\left(-\frac{\overline{a[p^l, p^{2(\sigma+t)}]} np^{\sigma+t}}{q}\right)$$
$$\times \mathcal{B}_{\pi_{p,f},\Phi_p^{h/p^{\sigma+t}q}}\left(\frac{n}{[p^{l_1}, p^{2(\sigma+t)}]q^2}\right) \tag{4.3}$$
$$\times \int_{\mathbb{R}} V(y_1) e\left(\frac{y_1 x}{p^{\sigma+t}qQ} \pm \frac{\sqrt{Nny_1}}{[p^{l_1}, p^{2(\sigma+t)}]^{1/2}q}\right) dy_1.$$

Similarly, if we look at the m-sum then it is given by

$$\sum_{(m,p)=1} \lambda_g(m) e\left(-\frac{hm}{p^{\sigma+t}q} - \frac{mx}{p^{\sigma+t}qQ}\right) V\left(\frac{m}{N}\right).$$

Then in the notation of Theorem 2.6.4, we set $\phi_p = \mathbb{I}_{\mathbb{Z}_p^\times}$ and $\phi_\infty(m) = e\left(-\frac{mx}{p^{\sigma+t}qQ}\right)V\left(\frac{m}{N}\right)$. Applying the theorem gives

$$\sum_{(m,p)=1}\frac{\lambda_g(n)}{m^{1/2}}e\left(\frac{\overline{p^{2(\sigma+t)}}mp^{\sigma+t}}{q}\right)$$
$$\times\,\chi_g(\overline{m}q)^{-1}\mathcal{B}_{\pi_{\infty,g},\phi_\infty}\left(\frac{m}{p^{2(\sigma+t)}q^2}\right)\mathcal{B}_{\pi_{p,g},\Phi_p^{h/p^{\sigma+t}q}}\left(\frac{m}{p^{2(\sigma+t)}q^2}\right).$$

Applying a change of variable and using (2.18) and (2.19) we see that this is given by a linear combination of the form

$$\begin{aligned}&\frac{N^{3/4}}{q^{1/2}p^{(\sigma+t)/2}}\sum_{(m,p)=1}\frac{\lambda_g(m)}{m^{1/4}}e\left(\frac{\overline{ap^{\sigma+t}}m}{q}\right)\\&\times\mathcal{B}_{\pi_{p,f},\Phi_p^{h/p^{\sigma+t}q}}\left(\frac{m}{p^{2(\sigma+t)}q^2}\right)\\&\times\int_{\mathbb{R}}V(y_1)e\left(\frac{y_1x}{p^{\sigma+t}qQ}\pm\frac{\sqrt{Nmy_1}}{p^{\sigma+t}q}\right)dy_1.\end{aligned}\tag{4.4}$$

4.2.2 Case II - $S_{t=0}(N)$

When $t=0$, it is possible that $a\equiv -qu \bmod p$. To handle this we rewrite the u-sum as

$$\sum_{u \bmod p^\sigma}e\left(\frac{u(n-m)}{p^\sigma}\right)=\sum_{0\le r\le\sigma}\sum_{\substack{k=1\\(k,p)=1}}^{p^{\sigma-r}}e\left(\frac{(-a\overline{q}+k\overline{q}p^r)(n-m)}{p^\sigma}\right).$$

Essentially, we are rearranging $u \bmod p^\sigma$ so that $0 \bmod p^\sigma$ is at $-a \bmod p^\sigma$ and "twist" all the harmonics by $\overline{q} \bmod p^\sigma$. Now consider the n-sum given by

$$\sum_n\lambda_f(n)e\left(\frac{\{a+(-a\overline{q}+kp^r)q\}n}{p^\sigma q}+\frac{nx}{p^\sigma qQ}\right)V\left(\frac{n}{N}\right).$$

Consider $a+(-a\overline{q}+kp^r)q$. Since $(a,q)=1$ this is co-prime to q. It is easy to see that $p^r||a+(-a\overline{q}+kp^r)q$ since $(kq,p)=1$. So let h satisfy $p^rh=a+(a\overline{q}+kp^r)q$. Then $h\equiv a\overline{p^r} \bmod q$ and $h\equiv k \bmod p^{\sigma-r}$. Therefore we apply Theorem 2.6.4 with $\phi_p=\mathbb{I}_{\mathbb{Z}_p^\times}$

and $\phi_\infty(m) = e\left(\frac{nx}{p^\sigma qQ}\right)V\left(\frac{m}{N}\right)$ to get

$$\sum_n \frac{\lambda_f(n)}{n^{1/2}} e\left(-\frac{\overline{ap^{l_1}}p^\sigma}{q}\right)$$
$$\times \chi_f(\overline{n}q)^{-1}\mathcal{B}_{\pi_{\infty,f},\phi_\infty}\left(\frac{n}{p^l q^2}\right)\mathcal{B}_{\pi_{p,f},\Phi_p^{h/p^{\sigma-r}q}}\left(\frac{n}{p^l q^2}\right)$$

with the Archimedean Bessel transform given by

$$\mathcal{B}_{\pi_{\infty,f},\phi_\infty}\left(\frac{n}{p^{l_1}q^2}\right) = \left(\frac{n}{p^{l_1}q^2}\right)^{1/2}\int_{\mathbb{R}} V\left(\frac{y_1}{N}\right)e\left(\frac{y_1 x}{p^\sigma qQ}\right)J_{k_1}\left(\frac{4\pi\sqrt{ny_1}}{p^{l_1/2}q}\right)dy_1.$$

We substitute this into the original equation and obtain

$$\frac{N^{3/4}}{p^{l_1/4}q^{1/2}}\sum_{(n,p)=1}\frac{\lambda_f(n)}{n^{1/4}}\mathcal{B}_{\pi_{p,f},\Phi_p^{h/p^{\sigma-r}q}}\left(\frac{n}{q^2p^{l_1}}\right)I(n,q) \tag{4.5}$$

where

$$I_1(n,q) = \sum_{\pm}\int V(y_1)e\left(\frac{Nxy_1}{p^\sigma qQ} \pm \frac{2\sqrt{Nmy_1}}{p^{l_1/2}q}\right)dy_1.$$

Similarly, looking at the m-sum we may apply Theorem (2.6.4) to get

$$\frac{N^{3/4}}{[p^{l_2},p^{2(\sigma-r)}]^{1/4}q^{1/2}}\sum_m \frac{\lambda_g(m)}{m^{1/4}}e\left(\frac{\overline{a[p^{l_2},p^{2(\sigma-r)}]}p^{\sigma-r}}{q}\right)$$
$$\times \chi_g(\overline{m}q)^{-1}\mathcal{B}_{\pi_{p,g},\Phi_p^{-h/p^{\sigma-r}q}}\left(\frac{m}{[p^{l_2},p^{2(\sigma-r)}]q^2}\right)I_2(m,q)$$

where

$$I_2(m,q) = \sum_{\pm}\int U(y_2)e\left(-\frac{Nxy_2}{p^\sigma qQ} \pm \frac{2\sqrt{Nmy_2}}{[p^{l_2},p^{2(\sigma-r)}]^{1/2}q}\right)dy_2.$$

4.3 Intermediate Calculations

4.3.1 Simplifying Character Sums

Case I - $S_{t>0}(N)$

Now let us simplify the character sums. First we look at the case $S_{t>0}(N)$. The character sum is given by

$$\sum_{u \bmod p^{\sigma} a} \sideset{}{^*}\sum_{\bmod p^t q} e\left(\frac{\overline{a} p^{\sigma+t}([p^{l_1}, p^{2(\sigma+t)}]m - [p^{l_2}, p^{2(\sigma+t)}]n)}{q}\right)$$
$$\times \mathcal{B}_{\pi_{p,f}, \Phi_p^{h/p^{\sigma+t}q}}\left(\frac{n}{[p^{l_1}, p^{2(\sigma+t)}]}\right)\mathcal{B}_{\pi_{p,g}, \Phi_p^{h/p^r q}}\left(\frac{m}{[p^{l_2}, p^{2(\sigma+t)}]}\right)$$

First we apply a preliminary Chinese Remainder Theorem to the a sum mod $p^t q$. Then we see that the q-sum is now given by

$$\sum_{a \bmod q}^{*} e\left(\frac{\overline{a} p^{\sigma+t}([p^{l_1}, p^{2(\sigma+t)}]m - [p^{l_2}, p^{2(\sigma+t)}]n)}{q}\right) = S([p^{l_1}, p^{2(\sigma+t)}]m - [p^{l_2}, p^{2(\sigma+t)}]n, 0; q)$$

while the sum mod $p^{\sigma+t}$ becomes, upon substituting the definition of the local Bessel functions in

$$\frac{(1-1/p)^{-2}}{p^{\sigma+t}} \sideset{}{^*}\sum_{a \bmod p^t} \sum_{u \bmod p^{\sigma}} \sum_{\chi_1, \chi_2 \bmod p^{\sigma+t}} \epsilon(\pi_{f,p}\overline{\chi_1})\epsilon(\pi_{g,p}\chi_2)\epsilon(\chi_1)\epsilon(\chi_2)^{-1}$$
$$\times \overline{\chi_1(n)}\chi_2(m)\chi_1\overline{\chi_2}(q(a + p^t q u)).$$

Applying the orthogonality relations of Dirichlet characters this becomes

$$(1-1/p)^{-1} \sideset{}{^*}\sum_{\chi \bmod p^{\sigma+t}} \epsilon(\pi_{f,p}\overline{\chi})\epsilon(\pi_{g,p}\chi)\chi(n\overline{m})$$

which upon applying the Local Langlands Correspondence we may rewrite as

$$(1-1/p)^{-1} \sideset{}{^*}\sum_{\chi \bmod p^{\sigma+t}} \epsilon(\theta_1\overline{\chi})\epsilon(\theta_2\chi)\chi(n\overline{m})$$

Case II - $S_{t=0}(N)$

Next we look at the case $S_{t=0}(N)$. For a fixed $0 \leq r \leq \sigma$ consider the character sum given by

$$\mathcal{C}_0(m,n,p^r,p^\sigma,q) = \sum_{k \bmod p^{\sigma-r}a} \sideset{}{^*}\sum_{\bmod q} e\left(\frac{\overline{a}p^{\sigma-r}(p^{l_1}m - [p^{l_2},p^{2(\sigma-r)}]n)}{q}\right) \times \mathcal{B}_{\pi_{p,f},\Phi_p^{h/p^{\sigma-r}q}}\left(\frac{n}{p^{l_1}}\right)\mathcal{B}_{\pi_{p,g},\Phi_p^{h/p^rq}}\left(\frac{m}{[p^{l_2},p^{2(\sigma-r)}]}\right)$$

unless π_2 is a principal series representation and $2(\sigma-r) = a(\pi_2)$ which we will handle separately.

As before, the sum mod q is a Ramanujan sum given by

$$S(p^{l_1}m - [p^{l_2},p^{2(\sigma-r)}]n, 0; q).$$

If $\sigma - r > 1$ then the sum mod $p^{\sigma-r}$ is given by

$$\frac{(1-1/p)^{-2}}{p^{(\sigma-r)}} \sideset{}{^*}\sum_{k \bmod p^{\sigma-r}} \sum_{\chi_1,\chi_2 \bmod p^{\sigma-r}} \epsilon(\pi_{f,p}\overline{\chi_1})\epsilon(\pi_{g,p}\chi_2)\epsilon(\chi_1)\epsilon(\chi_2)^{-1}\overline{\chi_1(n)}\chi_2(m)\chi_1\overline{\chi_2}(qk))$$

If $\sigma - r = 1$ then it is given by

$$\frac{\epsilon(\pi_{f,p})\epsilon(\pi_{g,p})}{(1-p)^2} + \frac{(1-1/p)^{-1}}{p^{1/2}(1-p)} \sum_{\chi \bmod p}^* \epsilon(\chi)^{-1} \sideset{}{^*}\sum_{k \bmod p} \left[\epsilon(\pi_{g,p})\epsilon(\pi_{f,p}\chi)\chi(n\overline{qk}) + \epsilon(\pi_{g,p}\chi)\epsilon(\pi_{f,p})\chi(m\overline{qk})\right]$$
$$+ \frac{(1-1/p)^{-2}}{p} \sideset{}{^*}\sum_{k \bmod p} \sideset{}{^*}\sum_{\chi_1,\chi_2 \bmod p} \epsilon(\pi_{f,p}\overline{\chi_1})\epsilon(\pi_{g,p}\chi_2)\epsilon(\chi_1)\epsilon(\chi_2)^{-1}\overline{\chi_1(n)}\chi_2(m)\chi_1\overline{\chi_2}(qk)$$

and if $\sigma = r$ then it is given by

$$\epsilon(\pi_{f,p})\epsilon(\pi_{g,p}).$$

As before, we apply the orthogonality relations of Dirichlet characters to deduce for $\sigma - r > 1$,

$$(1-1/p)^{-1} \sideset{}{^*}\sum_{\chi \bmod p^{\sigma-r}} \epsilon(\theta_1\overline{\chi})\epsilon(\theta_2\chi)\chi(n\overline{m})$$

and if $\sigma - r = 1$ then

$$\frac{\epsilon(\theta_1)\epsilon(\theta_2)}{(1-p)^2} + (1-1/p)^{-1} \sum_{\chi \bmod p}^{*} \epsilon(\theta_1\overline{\chi})\epsilon(\theta_2\chi)\chi(n\overline{m}).$$

4.3.2 Bounds for the Character Sum

Let

$$\mathcal{C}_0(m,n,p^j) = \sum_{\chi \in \mathfrak{X}_j} \epsilon(\theta_1\overline{\chi})\epsilon(\theta_2\chi)\chi(\overline{m}n).$$

Recall that we've assumed that E_i is the field associated to θ_i, ω_i and ψ_{E_i} be a uniformizer and the standard additive character of E_i respectively. Let $\mathfrak{o}_i$ and $\mathfrak{p}_i$ be the ring of integers of E_i and it's maximal ideal respectively. Let e_i be the ramification degree of E_i and d_i the order of the different respectively. Let $a_i(j) = e_i j$ and define the function $a_i(j,\theta_i) = \sup\{a(\theta_i), a_i(j)\}$. Let c_i be the constant given by

$$\theta_i(\exp(\omega_i^{e_i}u)) = \psi_{E_i}(\omega_i^{-a(\theta_i)+e_i}u)$$

for $u \in \mathfrak{o}_{E_i}$.

The estimates on the character sum change depending on the size of j relative to the conductor exponents of π_1 and π_2.

Lemma 4.3.1. *Assume that* $a(\pi_2) < 2j < a(\pi_1) - 1$. *Then*

$$\mathcal{C}_0(m,n,p^j) \ll p^{O(1)}p^{j/2}$$

Proof: Opening the root numbers gives

$$\begin{aligned}
&\frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a(\theta_1)}]^{1/2}[\mathfrak{o}_2 : \mathfrak{p}_2^{a_2(j)}]^{1/2}} \sum_{x_1 \bmod \mathfrak{p}^{a(\theta_1)}} \psi_1(\omega_1^{-a(\theta_1)}x_1)\theta_1(x_1) \\
&\times \sum_{x_2 \bmod \mathfrak{p}^{a_2(j)}} \psi_2(\omega_2^{-a_2(j)}x_2)\theta_2(x_2) \\
&\times \sum_{\chi \in \mathfrak{X}_j} \chi_{E_1}^{-1}(x_1)\chi_{E_2}(x_2)\chi(\overline{m}n)
\end{aligned}$$

and set $x_i = x_{i,0}(1 + \omega_i^{\lceil a(\theta,j)/2 \rceil} x_{i,1})$ and decompose $\chi = \chi_1\chi_0$ with $\chi_0 \bmod p^{\lceil j/2 \rceil}$. Then evaluating all the 1-index terms gives the equations, dropping the 0 in the indices

$$Nm_{E_1}(x_1)m \equiv Nm_{E_2}(x_2)n \bmod p^{j/2}$$

$$x_1 \equiv c_1 - \alpha\omega_2^{a(\theta_2)-a_1(j)} \bmod \mathfrak{p}_1^{a(\theta_2)/2}$$

$$x_2 \equiv c_2\omega_2^{a_2(j)-a(\theta_2)} - \alpha \bmod \mathfrak{p}_2^{a_2(j)/2}.$$

At this stage we would like to replace x_1 and x_2 everywhere with α and get a single equation α in the supercuspidal case. This comes with two complications This requires the norm (and trace) maps to preserve the congruence, a proof of which can be deduced from Proposition 5 of Chapter V in [31]. Second has to do with E_i being purely ramified. We observe that $Nm(c_i)$ is divisible by p. Therefore, when $c_i(j) < a(\theta_i)$ we require dividing the congruence by a factor of ω. Equivalently, we substitute the congruence mod $\mathfrak{p}^r$ into the congruence p^j and divide it by p. We shall by an abuse of notation, still call the corresponding term Nm (and tr). We shall henceforth apply these substitutions throughout the rest of this chapter without additional comments.

For notational purposes, we set $z = n\overline{m}$. The bound on the character sum is equivalent to proving that the number of α that satisfies this equation is $O(1)$. Note that the exponents of ω_i are always divisible by e_i. As a result, they always lie in $\mathbb{Z}_p$ and if the exponent is non-zero, it lies in $p\mathbb{Z}_p$.

Since $a(\pi_1) > 2j$,

$$Nm_{E_1}(x_1) \equiv Nm(c_1) \bmod p.$$

Since $2j > a(\pi_2)$, mod p the equation becomes

$$f(\alpha) := \alpha^2 n - Nm(c_1)m \equiv 0 \bmod p.$$

Clearly for α a unit, $f'(\alpha) \not\equiv 0 \bmod p$. By Hensel's Lemma 2.1.5, we conclude that there are $O(1)$ many solutions (in fact at most 2). □

Lemma 4.3.2. *Assume π_1 is supercuspidal and $2j = a(\pi_1)$ or $2j = a(\pi_1) - 1$. Then*

$$\mathcal{C}_0(m, n, p^j) \ll p^{j/2}.$$

Proof: As before we get the simultaneous equation

$$Nm_{E_1}(x_1)m \equiv Nm_{E_2}(x_2)n \bmod p^{j/2}$$

$$x_1 \equiv c_1 - \alpha \bmod \mathfrak{p}_1^{a_1(j)/2}$$

$$x_2 \equiv c_2\omega_2^{a_2(j)-a(\theta_2)} - \alpha \bmod \mathfrak{p}_2^{a_2(j)/2}.$$

Observe that $a(\pi_1) = 2j$ is only possible if π_1 is associated to an unramified field E_1 while $a(\pi_1) = 2j + 1$ when π_1 is associated to a ramified field E_1. Observe in both cases we have $a(\theta_1) = a_1(j)$ and in both cases we have

$$Nm(c_1 - \alpha) = Nm(c_1) - \mathrm{tr}(c_1)\alpha + \alpha^2$$

for $\alpha \in \mathbb{Q}_p$. Using that $tr(c_1) \equiv 0 \bmod p$ we get the equation

$$(Nm(c_1) + \alpha^2)m \equiv \alpha^2 n \bmod p.$$

We now make the change of variable $\alpha \mapsto 1/\alpha$ and get the equation

$$\alpha^2 Nm(c_1) \equiv \left(1 - \frac{m}{n}\right) \bmod p.$$

Clearly, for any solution to the above, the derivative is non-zero, so by Hensel's Lemma we are done. □

Lemma 4.3.3. *Assume π_2 is supercuspidal and $2j = a(\pi_2)$ or $2j + 1 = a(\pi_2)$. Then*

$$\mathcal{C}_0(m, n, p^j) \ll p^{j/2}.$$

Proof: The simultaneous equations are now

$$Nm_{E_1}(x_1)m \equiv Nm_{E_2}(x_2)n \bmod p^{j/2}$$

$$x_1 \equiv c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha \bmod \mathfrak{p}_1^{a(\theta_1)/2}$$

$$x_2 \equiv c_2 - \alpha \bmod \mathfrak{p}_2^{a_2(j)/2}.$$

Again, if $2j = a(\pi_2)$ then π_2 must be associated to an unramified field E_2 and if $2j+1 = a(\pi_2)$ then then π_2 must be associated to a ramified field E_2. In both cases, $a(\theta_2) = a_2(j)$. Applying the same arguments as in the previous lemma, using the fact that $\mathrm{tr}(c_2) = 0$ we get the single equation

$$(Nm(c_2) + \alpha^2)n \equiv mNm(c_1) \bmod p.$$

Again, it is easy to see any solution above has non-zero derivative mod p, so by Hensel's Lemma we are done. □

Now suppose $2j = a(\pi_2)$ when $\pi_2 = \pi(\chi_1, \chi_1^{-1}\psi)$ is a principal series representation. Then we encounter sums of the form

$$\sum_{\substack{\chi \in \mathfrak{X}_j \\ a(\chi\chi_1)=r}} \epsilon(\chi_1\chi)\epsilon(\chi_2^{-1}\psi\chi)\epsilon(\theta_2\overline{\chi})\chi(\overline{n}m)$$

which can be rewritten as

$$\chi_1^{-1}(\overline{n}m) \sum_{\chi \in \mathfrak{X}_r} \epsilon(\theta_1\chi_1\overline{\chi})\epsilon(\chi)\epsilon(\chi_1^{-2}\psi\chi)\chi(\overline{n}m).$$

Note that if $a(\chi_2\chi) < j$ then $a(\overline{\chi_1}\psi\chi) = j$ since $a(\psi) < j$. Similarly, if $a(\overline{\chi_1}\psi\chi) < j$ then $a(\chi_1\chi) = j$. Therefore, the cases $a(\chi_1\chi) < j$ and $a(\overline{\chi_1}\psi\chi) < j$ are mutually independent.

If $r = 1$ we trivially bound it by p. So we assume that $r \geq 2$. We open out the root numbers and obtain

$$\begin{aligned}&\frac{1}{[\mathfrak{o}_{E_1} : \mathfrak{p}_{E_1}^{a(\theta_1)}]^{1/2} p^{r/2} p^{j/2}} \sum_{\chi \in \mathfrak{X}_r} \chi(\overline{n}m) \sum_{y_1 \bmod p^r} \chi(y_1) e\left(\frac{y_1}{p^r}\right) \\ &\times \sum_{y_2 \bmod p^j} \chi_1^{-2} \chi(y_2) e\left(\frac{y_2}{p^j}\right) \\ &\times \sum_{x \bmod \mathfrak{p}_{E_1}^{a(\theta_1)}} \chi_{1,E_1} \chi_{E_1}^{-1} \theta_1(x) \psi_{E_1}(\omega_1^{-a(\theta_1)} x).\end{aligned}$$

We now set $y_1 = y_{2,0}(1 + p^{r/2} y_{1,1})$, $y_2 = y_{2,0}(1 + p^{j/2} y_{2,1})$, $x = x_0(1 + p^{a_1(j)/2} x_1)$ and $\chi = \psi_0 \psi_1$ with $\psi_1 \in G_{r/2}$. We observe that all the indices with 1 and independent of each other and we execute them. We get the following set of congruences

$$y_1 \equiv \alpha \bmod p^{r/2}$$

$$y_2 \equiv -2\alpha_1 + p^{j-r}\alpha \bmod p^{j/2},$$

$$x \equiv c_1 - \alpha \omega_1^{a(\theta_1) - a_1(j)} + \alpha_1 \omega_1^{a(\theta_1) - a_1(j)} \bmod \mathfrak{p}^{a_2(j)/2},$$

$$y_1 y_2 m \equiv Nm(x) n \bmod p^{r/2}.$$

where α_1 is the constant associated to χ_1. We observe that all free variables are determined by α, so it remains to prove that there are $p^{O(1)}$ many solutions in α. Fist assume $r < j$. Substituting all variables into the last equation and reducing mod p gives

$$-2\alpha_1 \alpha m \equiv c_1 n \bmod p.$$

By Hensel's Lemma, we conclude that there is at-most one solution to the equation in α. So assume that $r = j$. We also assume that $a(\chi_1^{-2}\chi) = j$ otherwise we apply an identical argument as in the case $r < j$ but with α_1 replaced with $-\alpha_1$ to deduce stronger bounds. Once again substituting and reducing mod p gives

$$\alpha(\alpha - 2\alpha_1) m \equiv c_1 n \bmod p,$$

which we factorize as

$$(\alpha-\alpha_1)^2 \equiv \alpha_1^2 + \frac{c_1 n}{m} \bmod p.$$

Note that by assumption we have $\alpha \not\equiv \pm\alpha_1 \bmod p$. Thus, any solution to the above equation has non-zero derivative in α. By Hensel's Lemma there are at-most 2 solutions. We record these observations in the following lemma.

Lemma 4.3.4. *Let $1 \leq r < j$. Assume π_2 is a principal series and $2j = a(\pi_2)$. We have the bound*

$$\sum_{\substack{\chi\in\mathfrak{X}_j \\ a(\chi\chi_1)=r}} \epsilon(\chi_1\chi)\epsilon(\chi_1^{-1}\psi\chi)\epsilon(\theta_1\overline{\chi})\chi(\overline{n}m) \ll p^{O(1)}p^{r/2},$$

$$\sum_{\substack{\chi\in\mathfrak{X}_j \\ a(\chi\chi_1^{-1})=r}} \epsilon(\chi_1\chi)\epsilon(\chi_1^{-1}\psi\chi)\epsilon(\theta_1\overline{\chi})\chi(\overline{n}m) \ll p^{O(1)}p^{r/2}.$$

$$\sum_{\substack{\chi\in\mathfrak{X}_j \\ a(\chi\chi_1^{-1})=a(\chi\chi_1)=j}} \epsilon(\chi_1\chi)\epsilon(\chi_1^{-1}\psi\chi)\epsilon(\theta_1\overline{\chi})\chi(\overline{n}m) \ll p^{O(1)}p^{j/2}.$$

Next assume that π_1 is a principal series representation and $2j = a(\pi_1)$. Note that we may re-write the character sum as

$$\sum_{\substack{\chi\in\mathfrak{X}_j \\ a(\chi_1\chi)+a(\overline{\chi_1}\chi_2\chi)=r}} \epsilon(\chi_1\chi)\epsilon(\overline{\chi_1}\chi_2\chi)\epsilon(\theta_2\overline{\chi})\chi(\overline{n}m),$$

with $a(\chi_1) = j$ and $a(\chi_2) < a(\chi_1)$. Note that if $a(\chi_1\chi) < j$ then $a(\overline{\chi_1}\chi_2\chi) = j$ since $a(\chi_2) < j$. Similarly, if $a(\overline{\chi_1}\chi_2\chi) < j$ then $a(\chi_1\chi) = j$. Therefore, the cases $a(\chi_1\chi) < j$ and $a(\overline{\chi_1}\chi_2\chi) < j$ are mutually independent.

Let's treat the case $a(\chi_1\chi) = r < j$ first. The character sum can be rewritten as

$$\chi_1(n\overline{m}) \sum_{\chi\in\mathfrak{X}_r} \epsilon(\chi)\epsilon(\overline{\chi_1^2}\chi_2\chi)\epsilon(\theta_2\chi_1\overline{\chi})\chi(\overline{n}m).$$

If $r = 0$ or 1 we trivially bound it. So we assume $r \geq 2$. As before, we get an equation

$$\alpha(-2\alpha_1 + p^{j-r}\alpha_2 + p^{j-a(\chi_2)}\alpha_2)m \equiv Nm_{E_2}(\alpha_1 - \alpha - c_2\omega_2^{a_2(j)-a(\theta_2)})n \bmod p^{r/2},$$

which reduces mod p to

$$-2\alpha_1\alpha m \equiv \alpha_1^2 n \bmod p,$$

which further reduces (since α_1 is a unit) to

$$-2\alpha_1 m \equiv n\alpha \bmod p.$$

By Hensel's Lemma, this equation has non-zero derivative mod p in α, so the lift is unique.

Next assume that $a(\overline{\chi_1}\chi_2\chi) = r < j$. The character sum can be rewritten as

$$\overline{\chi_1}\chi_2(n\overline{m}) \sum_{\chi \in \mathfrak{X}_r} \epsilon(\chi)\epsilon(\chi_1^2\overline{\chi_2}\chi)\epsilon(\theta_2\overline{\chi_1}\chi_2\overline{\chi})\chi(\overline{n}m).$$

If $r = 1$ we trivially bound it. So we assume $r \geq 2$. As before, we get an equation

$$\begin{aligned}&\alpha(2\alpha_1 + p^{j-r}\alpha - p^{j-a(\chi_2)}\alpha_2)m \\ \equiv\ & Nm_{E_2}(-\alpha_1 - \alpha\omega_2^{a_2(j)-a_2(r)} + \alpha_2\omega_2^{a_2(j)-c_2(a(\chi_2))} + c_2\omega_2^{a_2(j)-a(\theta_2)})n \bmod p^{r/2},\end{aligned}$$

which reduces mod p to

$$2\alpha_1\alpha m \equiv \alpha_1^2 n \bmod p,$$

which further reduces (since α_1 is a unit) to

$$2\alpha m \equiv n \bmod p.$$

This equation has non-zero derivative mod p in α, so by Hensel's Lemma the lift is unique.

Now suppose that both $a(\chi_1\chi) = a(\overline{\chi_1}\chi_2\chi) = j$. We now get an equation of the shape

$$\alpha(2\alpha_1 + \alpha - p^{j-a(\chi_2)}\alpha_2)m \equiv Nm_{E_2}(-\alpha_1 - \alpha + \alpha_2\omega_2^{a_2(j)-c(a(\chi_2)} - c_2\omega_2^{a_2(j)-a(\theta_2)})n \bmod p^{j/2},$$

which reduces mod p to

$$\alpha(2\alpha_1 + \alpha)m \equiv (\alpha_1 + \alpha)^2 n \bmod p.$$

We rewrite this equation as

$$(m-n)\alpha^2 + 2\alpha_1(m-n)\alpha + n\alpha_1^2 \equiv 0 \bmod p.$$

Observe that there are no solutions to this equation mod p if $m \equiv n \bmod p$. So we assume that $m \not\equiv n \bmod p$. Then this equation factorizes as

$$(\alpha_1 + \alpha)^2 \equiv \alpha_1^2 \frac{m}{m-n} \bmod p.$$

By assumption, $\alpha \not\equiv \pm\alpha_1 \bmod p$. Therefore, the derivative is non-zero for any admissible solution mod p, so by Hensel's Lemma, the lift is unique. We record these observations in the following lemma.

Lemma 4.3.5. *Let $1 \leq r < j$. Assume π_2 is a principal series and $2j = a(\pi_1)$. We have the bounds*

$$\sum_{\substack{\chi \in \mathfrak{X}_j \\ a(\chi\chi_1)=r}} \epsilon(\chi_1\chi)\epsilon(\chi_1^{-1}\chi_2\chi)\epsilon(\theta_2\overline{\chi})\chi(\overline{n}m) \ll p^{O(1)}p^{r/2},$$

$$\sum_{\substack{\chi \in \mathfrak{X}_j \\ a(\chi\chi_1^{-1}\chi_2)=r}} \epsilon(\chi_1\chi)\epsilon(\chi_1^{-1}\chi_2\chi)\epsilon(\theta_2\overline{\chi})\chi(\overline{n}m) \ll p^{O(1)}p^{r/2},$$

$$\sum_{\substack{\chi \in \mathfrak{X}_j \\ a(\chi\chi_1)=a(\chi\chi_1^{-1}\chi_2)=j}} \epsilon(\chi_1\chi)\epsilon(\chi_1^{-1}\chi_2\chi)\epsilon(\theta_2\overline{\chi})\chi(\overline{n}m) \ll p^{O(1)}p^{j/2}.$$

Now we treat the case $2j < a(\pi_2) - 1$.

Lemma 4.3.6. *Assume that $2j < a(\pi_2)-1$. Let ν be the largest integer such that $2c_2(\nu) \leq a(\theta_2)$. Then*

$$\mathcal{C}_0(m, n, p^j) = \varphi(p^\nu) \sum_{\chi \in \mathfrak{X}_j/\sim_\nu} \mathbb{I}(mc_{\theta_1,\theta_2,\chi} \equiv n \bmod p^\nu).$$

Proof: This follows by breaking the χ sum into G_ν orbits and applying Deligne's Stabilization Lemma [7]. □

Note $\nu \sim \frac{a(\pi_2)}{4}$. As a result we see that the character sum is in fact a Ramanujan sum when $j \leq \frac{a(\pi_2)}{4}$.

Now we treat the case $2j > a(\pi_1)$.

Lemma 4.3.7. *Assume that* $2j > a(\pi_1)$. *Then*

$$\mathcal{C}_0(m,n,p^j) = \varphi(p^{\lfloor \frac{j}{2} \rfloor}) \sum_{\chi \in \mathfrak{X}_j/\sim \lfloor j/2 \rfloor} \mathbb{I}(c_{\chi,\theta_1,\theta_2} m \equiv n \bmod p^{j/2}).$$

Proof: This follows by breaking the χ sum into $G_{\lfloor j/2 \rfloor}$ orbits and applying Deligne's Stabilization Lemma 2.1.4. □

4.3.3 Bounds for the Integral I

Let m, n be positive real numbers, not necessarily integers. Let t be a non-negative real number. Let $q \geq 1$. For $\epsilon_1, \epsilon = \pm 1$ we define

$$\begin{aligned} I^{\epsilon_1,\epsilon_2}(m,n,p^t,q) =& 4 \int g(p^t q, x) \int V(y_1^2) y_1 U(y_2^2) y_2 \\ & \times e\left(\frac{Nx(y_1^2 - y_2^2)}{p^{\sigma+t} qQ} + \epsilon_1 \frac{\sqrt{Nn} y_1}{q([p^{l_1}, p^{2(\sigma+t)}])^{1/2}} + \epsilon_2 \frac{\sqrt{Nm} y_2}{q[p^{l_2}, p^{2(\sigma+t)}]^{1/2}} \right) dx dy_1 dy_2, \end{aligned}$$

except possibly when π_1 is a principal series and $2(\sigma + t) = a(\pi_1)$ in which case we get a collection of integrals

$$\begin{aligned} I^{\epsilon_1,\epsilon_2}(m,n,p^t,p^r,q) =& 4 \int g(p^t q, x) \int V(y_1^2) y_1 U(y_2^2) y_2 \\ & \times e\left(\frac{Nx(y_1^2 - y_2^2)}{p^{\sigma+t} qQ} + \epsilon_1 \frac{\sqrt{Nn} y_1}{q p^{(\sigma+t+r)/2}} + \epsilon_2 \frac{\sqrt{Nm} y_2}{q p^{\sigma+t}} \right) dx dy_1 dy_2, \end{aligned}$$

for $0 \leq r \leq \sigma + t$.

When $t = 0$ we get the integrals

$$\begin{aligned} I^{\epsilon_1,\epsilon_2}(m,n,q,p^j) =& 4 \int g(q, x) \int V(y_1^2) y_1 U(y_2^2) y_2 \\ & \times e\left(\frac{Nx(y_1^2 - y_2^2)}{p^{\sigma} qQ} + \epsilon_1 \frac{\sqrt{Nn} y_1}{q p^{l_1/2}} + \epsilon_2 \frac{\sqrt{Nm} y_2}{q[p^{l_2}, p^{2j}]^{1/2}} \right) dx dy_1 dy_2 \end{aligned}$$

except possibly when π_2 is a principal series and $2j = a(\pi_2)$ in which case we get a collection of integrals

$$I^{\epsilon_1,\epsilon_2}(m,n,q,p^j,p^r) = 4\int g(q,x)\int V(y_1^2)y_1U(y_2^2)y_2 \times e\left(\frac{Nx(y_1^2-y_2^2)}{p^\sigma qQ}+\epsilon_1\frac{\sqrt{Nn}y_1}{qp^{l_1/2}}+\epsilon_2\frac{\sqrt{Nm}y_2}{qp^{(j+r)/2}}\right)dxdy_1dy_2,$$

for $0 \le r \le j$. Let $C \ll Q/p^t$ be a parameter and let $q \sim C$. We now partition the x-integral into two parts, namely

$$|x| \ll p^\epsilon \frac{p^{\sigma+t}CQ}{N}, \quad |x| \gg p^\epsilon \frac{p^{\sigma+t}CQ}{N}.$$

Set $X_C = p^\epsilon \frac{p^{\sigma+t}CQ}{N}$. In the range $|x| \gg X_C$, we apply a smooth dyadic partition to x, say into $|x| \sim X$ with $X_C \ll X \ll p^\epsilon$. We thus have

$$I = I_{X_C} + \sum_{\substack{X_C \ll X \ll p^\epsilon \\ X \text{ dyadic}}} I_X$$

Assume $t > 0$. If $|x| \ll X_C$ then we observe that by repeated integration by parts that the integral is negligibly small unless

$$|n| \ll p^\epsilon \frac{[p^{l_1}, p^{2(\sigma+t)}]C^2}{N} =: N_C,$$

$$|m| \ll p^\epsilon \frac{p^{2(\sigma+t)}C^2}{N} =: M_C,$$

except possibly when π_1 is a principal series representation and $2(\sigma+t) = a(\pi_1)$. In this case, the bound for m remains the same while the bound for n becomes

$$|n| \ll \frac{p^{\sigma+t+r}C^2}{N} =: N_C(r).$$

Next assume that $|x| \gg X_C$, say $|x| \sim X$. Then we observe by repeated integration by parts that the integral is negligibly small unless

$$p^{-\epsilon}\frac{NX^2[p_1^l, p^{2(\sigma+t)}]}{Q^2 p^{2(\sigma+t)}} \ll |n| \ll p^{\epsilon}\frac{NX^2[p_1^l, p^{2(\sigma+t)}]}{Q^2 p^{2(\sigma+t)}} := N(X),$$

$$p^{-\epsilon}\frac{NX^2}{Q^2} \ll |m| \ll p^{\epsilon}\frac{NX^2}{Q^2} := M(X),$$

except possibly when π_1 is a principal series representation and $2(\sigma + t) = a(\pi_1)$. In this case, the bound for m remains the same while the bound for n becomes

$$p^{-\epsilon}\frac{NX^2 p^{\sigma+t+r}}{Q^2 p^{2(\sigma+t)}} \ll n \ll p^{\epsilon}\frac{NX^2 p^{\sigma+t+r}}{Q^2 p^{2(\sigma+t)}} =: N(X, r).$$

for $0 \leq r \leq \sigma + t$. Note that we always have

$$N(X) \ll \frac{N[p^{l_1}, p^{2(\sigma+t)}]}{Q^2 p^{2(\sigma+t)}},$$

$$M(X) \ll p^{\epsilon}\frac{N}{Q^2}$$

and

$$N(X, r) \ll \frac{N p^{\sigma+t+r}}{Q^2 p^{2(\sigma+t)}}$$

Now assume $t = 0$. Let $j \leq \sigma$. If $|x| \ll X_C$ then by repeated integration by parts

$$|n| \ll \frac{p^{l_1} C^2}{N} =: N_C$$

$$|m| \ll p^{\epsilon}\frac{[p^{l_2}, p^{2j}]C^2}{N} =: M_C,$$

except possibly when π_2 is a principal series representation and $2r = a(\pi_2)$. In this case the bound for n remains the same while the bound for m becomes

$$|m| \ll p^{\epsilon}\frac{p^{j+r} C^2}{N} =: M_C(r).$$

Next assume that $|x| \gg X_C$, say $|x| \sim X$. Then we observe by repeated integration by parts that the integral is negligibly small unless

$$p^{-\epsilon}\frac{NX^2p^{l_1}}{Q^2p^{2\sigma}} \ll |n| \ll p^{\epsilon}\frac{NX^2p^{l_1}}{Q^2p^{2\sigma}} := N(X),$$

$$p^{-\epsilon}\frac{NX^2[p^{l_2},p^{2j}]}{p^{2\sigma}Q^2} \ll |m| \ll p^{\epsilon}\frac{NX^2[p^{l_2},p^{2j}]}{p^{2\sigma}Q^2} := M(X),$$

except possibly when π_2 is a principal series representation and $2j = a(\pi_2)$. In this case the bound for n remains the same while the bound for m becomes

$$p^{-\epsilon}\frac{NX^2p^{j+r}}{p^{2\sigma}Q^2} \ll |m| \ll p^{\epsilon}\frac{NX^2p^{j+r}}{p^{2\sigma}Q^2} := M(X,r).$$

Note that we always have

$$N(X) \ll \frac{Np^{l_1}}{p^{2\sigma}Q^2},$$

$$M(X) \ll p^{\epsilon}\frac{N[p^{l_2},p^{2r}]}{p^{2\sigma}Q^2}.$$

and

$$M(X,r) \ll p^{\epsilon}\frac{Np^{j+r}}{p^{2\sigma}Q^2}.$$

Now assume $t \geq 0$. Assume $|x| \leq X_C$. The x-integral is now given by

$$X_C \int g(p^tq, X_Cx)W_{X_C}(x)e\left(\frac{NX_Cx(y_1^2-y_2^2)}{p^{\sigma+t}qQ}\right)dx,$$

where W_{X_C} is a smooth bump function supported on $[-1,1]$ with derivatives satisfying $W^{(j)} \ll 1$ uniformly.

Next assume that $|x| \sim X$ with $X_C \ll X \ll p^{\epsilon}$. Then the x-integral is given by

$$X \int g(p^tq, Xx)W(x)e\left(\frac{NXx(y_1^2-y_2^2)}{p^{\sigma+t}qQ}\right)dx,$$

where W is a smooth bump function supported on $[-2,-1] \cup [1,2]$ with derivatives satisfying $W^{(j)} \ll 1$. In both cases $X \ll X_C$ and $X \gg X_C$, we observe the following - when

$p^tC \ll Q^{1-\epsilon}$ we use (2.22) to replace $q(p^tq, x)$ by 1. Then using repeated integration parts in the x-variable, we deduce that $y_1 - y_2 \ll Q^\epsilon \frac{p^{\sigma+t}CQ}{N}$. When $p^tC \gg Q^{1-\epsilon}$ then we use (2.24) and repeated integration by parts in the x-variable to deduce $y_1 - y_2 \ll Q^\epsilon \frac{p^{\sigma+t}CQ}{N}$ again.

We set $y_2 = y_1 + u$ with $u \ll Q^\epsilon \frac{p^{\sigma+t}qQ}{N}$. Assume $t > 0$. For $|x| \leq X_C$ the integral now becomes

$$X_C \int g(p^tq, X_Cx)W_{X_C}(x) \int e\left(\frac{NxX_Cu^2}{p^{\sigma+t}qQ}\right) \int V(y^2)U((y+u)^2)y(y+u)$$
$$\times e\left(\epsilon_1 \frac{2\sqrt{Nn}y}{q([p^{l_1}, p^{2(\sigma+t)}])^{1/2}} + \epsilon_2 \frac{2\sqrt{Nm}(y+u)}{q[p^{l_2}, p^{2(\sigma+t)}]^{1/2}}\right) dydudx,$$

except possibly when π_1 is a principal series and $2(\sigma + t) = a(\pi_1)$ in which case we get a collection of integrals

$$X_C \int g(p^tq, X_Cx)W_{X_C}(x) \int e\left(\frac{NxX_Cu^2}{p^{\sigma+t}qQ}\right) \int V(y^2)U((y+u)^2)y(y+u)$$
$$\times e\left(\epsilon_1 \frac{2\sqrt{Nn}y}{qp^{(\sigma+t+r)/2}} + \epsilon_2 \frac{2\sqrt{Nm}(y+u)}{qp^{(\sigma+t)}}\right) dydudx,$$

where $0 \leq r \leq \sigma + t$. Similarly,

$$X \int g(p^tq, Xx)W(x) \int e\left(\frac{NxXu^2}{p^{\sigma+t}qQ}\right) \int V(y^2)U((y+u)^2)y(y+u)$$
$$\times e\left(\frac{Nyu}{p^{\sigma+t}qQ} + \epsilon_1 \frac{2\sqrt{Nn}y}{q([p^{l_1}, p^{2(\sigma+t)}])^{1/2}} + \epsilon_2 \frac{2\sqrt{Nm}(y+u)}{q[p^{l_2}, p^{2(\sigma+t)}]^{1/2}}\right) dydudx,$$

except possibly when π_1 is a principal series and $2(\sigma + t) = a(\pi_1)$ in which case we get a collection of integrals

$$X \int g(p^tq, Xx)W(x) \int e\left(\frac{NxXu^2}{p^{\sigma+t}qQ}\right) \int V(y^2)U((y+u)^2)y(y+u)$$
$$\times e\left(\epsilon_1 \frac{2\sqrt{Nn}y}{qp^{(\sigma+t+r)/2}} + \epsilon_2 \frac{2\sqrt{Nm}(y+u)}{qp^{(\sigma+t)}}\right) dydudx.$$

Next we study the y-integral. Observe that any term which appears with u is non-oscillatory and therefore we can absorb it into the weight function. We then see that the

phase function in y is given by

$$P(y) = \epsilon_1 \frac{2\sqrt{Nn}y}{q([p^{l_1}, p^{2(\sigma+t)}])^{1/2}} + \epsilon_2 \frac{2\sqrt{Nm}y}{q[p^{l_2}, p^{2(\sigma+t)}]^{1/2}}.$$

Applying the first derivative bound gives that for a fixed u, the y-integral can be bounded by

$$\frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{q([p^{l_1}, p^{2(\sigma+t)}])^{1/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{q[p^{l_2}, p^{2(\sigma+t)}]^{1/2}}\right|}.$$

Alternatively, it can also be trivially bounded. We then bound the u-integral by it's length. An identical argument for the case $t = 0$ can also be done and we get the following lemma.

Lemma 4.3.8. *Assume $t > 0$ and $C \ll Q/p^t$. We have the bound*

$$I_{X_C}^{\epsilon_1,\epsilon_2}(m, n, p^t, q) \ll Q^\epsilon \times X_C \times \frac{p^{\sigma+t}CQ}{N} \times \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{q([p^{l_1}, p^{2(\sigma+t)}])^{1/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{qp^{\sigma+t}}\right|}\right\},$$

$$I_{X}^{\epsilon_1,\epsilon_2}(m, n, p^t, q) \ll Q^\epsilon \times X \times \frac{p^{\sigma+t}CQ}{N} \times \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{q([p^{l_1}, p^{2(\sigma+t)}])^{1/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{qp^{\sigma+t}}\right|}\right\}.$$

except possibly when $2(\sigma + t) = a(\pi_1)$ in which case we get the bound

$$I_{X_C}^{\epsilon_1,\epsilon_2}(m, n, p^t, p^r, q) \ll Q^\epsilon \times X_C \times \frac{p^{\sigma+t}CQ}{N} \times \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{q^{(\sigma+t+r)/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{qp^{\sigma+t}}\right|}\right\},$$

$$I_{X_C}^{\epsilon_1,\epsilon_2}(m, n, p^t, p^r, q) \ll Q^\epsilon \times X \times \frac{p^{\sigma+t}CQ}{N} \times \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{q^{(\sigma+t+r)/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{qp^{\sigma+t}}\right|}\right\},$$

for $0 \leq r \leq \sigma + t$.

Lemma 4.3.9. *Assume* $C \ll Q$ *and* $j \leq \sigma$. *We have the bound*

$$I_{X_C}^{\epsilon_1,\epsilon_2}(m,n,q,p^j) \ll Q^{\epsilon} \times X_C \times \frac{p^{\sigma} CQ}{N} \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{qp^{l_1/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{q[p^{l_2}, p^{2j}]^{1/2}}\right|}\right\},$$

$$I_{X}^{\epsilon_1,\epsilon_2}(m,n,q,p^j) \ll Q^{\epsilon} \times X \times \frac{p^{\sigma} CQ}{N} \times \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{qp^{l_1/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{q[p^{l_2}, p^{2j}]^{1/2}}\right|}\right\},$$

except possibly when π_2 *is a principal series representation and* $2j = a(\pi_2)$ *in which case we get the bound*

$$I_{X_C}^{\epsilon_1,\epsilon_2}(m,n,q,p^j,p^r) \ll Q^{\epsilon} \times X_C \times \frac{p^{\sigma} CQ}{N} \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{qp^{l_1/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{qp^{(j+r)/2}}\right|}\right\},$$

$$I_{X}^{\epsilon_1,\epsilon_2}(m,n,q,p^j,p^r) \ll Q^{\epsilon} \times X \times \frac{p^{\sigma} CQ}{N} \times \inf\left\{1, \frac{1}{\left|\epsilon_1 \frac{2\sqrt{Nn}}{qp^{l_1/2}} + \epsilon_2 \frac{2\sqrt{Nm}}{qp^{(j+r)/2}}\right|}\right\},$$

for $0 \leq r \leq j$.

4.3.4 Combining Dualizations

Combining our dualizations we get

$$S_{t=0}(N) = \sum_{j \leq \sigma} S(N; p^j),$$

$$S(N;p^j)=\frac{N^{3/2}}{Qp^\sigma}\sum_{\pm}\sum_{\pm}\sum_{\substack{(de,p)=1\\ de\le Q}}\frac{\mu(e)}{e^2dp^{l_1/4}[p^{l_2},p^{2j}]^{1/4}}$$
$$\times\sum_{p^{l_1}m\equiv[p^{l_2},p^{2j}]n \bmod d}\sum_n\sum_m\frac{\lambda_f(n)\lambda_g(m)}{(nm)^{1/4}}\mathcal{C}_0(m,n,p^j)I^{\pm,\pm}(m,n,de),$$

provided that if π_2 is a principal series then $2j\neq a(\pi_2)$. In case π_2 is a principal series and $2j=a(\pi_2)$ we get

$$S_{\pi_2,pr}(N;p^j)=\frac{N^{3/2}}{Qp^\sigma}\sum_{\pm}\sum_{\pm}\sum_{0\le v\le j}\sum_{\substack{(de,p)=1\\ de\le Q}}\frac{\mu(e)}{e^2dp^{l_1/4}p^{(j+r)/4}}$$
$$\times\sum_{p^{l_1}m\equiv p^{j+r}n \bmod d}\sum_n\sum_m\frac{\lambda_f(n)\lambda_g(m)}{(nm)^{1/4}}\mathcal{C}_0(m,n,p^j,p^v)I^{\pm,\pm}(m,n,p^j,p^v,de).$$

Similarly,

$$S_{t>0}(N)=\sum_{t>0}S(N;p^{\sigma+t}),$$

$$S(N;p^{\sigma+t})=\frac{N^{3/2}}{Qp^{\sigma+t}}\sum_{\pm}\sum_{\pm}\sum\sum_{\substack{(de,p)=1\\ de\le Q/p^t}}\frac{\mu(e)}{e^2d([p^{l_1},p^{2(\sigma+t)}])^{1/4}p^{(\sigma+t)/2}}$$
$$\sum_{[p^{l_1},p^{2(\sigma+t)}]m\equiv p^{2(\sigma+t)}n \bmod d}\sum_n\sum_m\frac{\lambda_f(n)\lambda_g(m)}{(nm)^{1/4}}\mathcal{C}(m,n,p^{\sigma+t},q)I^{\pm,\pm}(m,n,p^t,de).$$

provided that if π_1 is a principal series then $2(\sigma+t)\neq a(\pi_2)$. In case π_1 is a principal series and $2(\sigma+t)=a(\pi_2)$ we get

$$S_{\pi_1,pr}(N;p^{\sigma+t})=\frac{N^{3/2}}{Qp^{\sigma+t}}\sum_{\pm}\sum_{\substack{(de,p)=1\\ de\le Q/p^t}}\sum_{r\le\sigma+t}\frac{\mu(e)}{e^2d(p^{(\sigma+t+r)/4}p^{(\sigma+t)/2}}$$
$$\sum_{p^{\sigma+t+r}m\equiv p^{2(\sigma+t)}n \bmod d}\sum_n\sum_m\frac{\lambda_f(n)\lambda_g(m)}{(nm)^{1/4}}\mathcal{C}_0(m,n,p^{\sigma+t},p^r,q)I^{\pm,\pm}(m,n,p^t,p^r,de).$$

We apply a dyadic partition to the q-sum. If $j\le\sigma$ we get

$$S(N;p^j)=\sum_{\substack{C\ll Q\\ C \text{ dyadic}}}S(N;p^j,C),$$

where $S(N;p^j,C)$ is defined identically to $S(N;p^j)$ with the additional condition that $de \sim C$. Similarly, if $t > 0$ we get

$$S(N;p^{\sigma+t}) = \sum_{\substack{C \ll Q/p^t \\ C \text{ dyadic}}} S(N;p^{\sigma+t},C).$$

We now apply a dyadic partition to the n-sum, say $n \sim N_0$. If $N_0 \ll N_C$ then we must also have $m \ll M_C$ and only I_C is potentially non-negligible. If $N_0 \gg N_C$ then we that the x-variable in the integral lies in the range

$$p^{-\epsilon}\frac{Qp^{\sigma+t}\sqrt{N_0}}{N[p^{l_1},p^{2(\sigma+t)}]^{1/2}} \ll |x| \ll p^{\epsilon}\frac{Qp^{\sigma+t}\sqrt{N_0}}{N[p^{l_1},p^{2(\sigma+t)}]^{1/2}} =: X(N_0),$$

and I_X is potentially non-negligible for those X satisfying

$$X(N_0)p^{-\epsilon} \ll |X| \ll X(N_0)p^{\epsilon}.$$

Note that as a result we must also have $|m|$ lying in the range

$$p^{-\epsilon}M(X(N_0)) \ll |m| \ll p^{\epsilon}M(X(N_0)).$$

If $j \leq \sigma$ we thus split $S_0(N;p^j,C)$ as

$$S(N;p^j,C) = \sum_{\substack{N_0 \ll N_C \\ N_0 \text{ dyadic}}} S(N;p^j,C,N_0) + \sum_{\substack{N_0 \gg N_C \\ N_0 \text{ dyadic}}} S(N;p^j,C,N_0).$$

Similarly, if $t > 0$ we get

$$S(N;p^{\sigma+t},C) = \sum_{\substack{N_0 \ll N_C \\ N_0 \text{ dyadic}}} S(N;p^{\sigma+t},C,N_0) + \sum_{\substack{N_0 \gg N_C \\ N_0 \text{ dyadic}}} S(N;p^{\sigma+t},C,N_0).$$

We observe that the bounds from the range $N_0 \ll N_C$ are stronger than those coming from $N_0 \gg N_C$. We therefore only consider the case $N_0 \gg N_C$.

When $\sigma+t > a(\pi_1)$, then $M(X(N_0)) \sim N_0$. Applying Lemma 4.3.7, Lemma 4.3.8 and the Rankin-Selberg bound (2.13) we obtain the bound

$$\begin{aligned}
S(N;p^{\sigma+t},C,N_0) \ll & \frac{N^{3/2}}{Qp^{2(\sigma+t)}}\sum_{e\leq C}\frac{1}{e^2}\sum_{d\sim C/e}\frac{1}{d}\times X(N_0)\times\frac{p^{\sigma+t}CQ}{N} \\
& \times \sum_{\chi\in\mathfrak{X}_{\sigma+t}/\sim(\sigma+t)/2}\sum_{\substack{m\sim N_0}}\sum_{\substack{n\sim N_0 \\ m\equiv n \bmod d \\ c_{\theta_1,\theta_2,\chi}m\equiv n \bmod p^{(\sigma+t)/2}}}\left(\frac{|\lambda_f(m)|^2}{m^{1/2}}+\frac{|\lambda_f(n)|^2}{n^{1/2}}\right) \\
\ll & \frac{N^{1/2}CX(N_0)p^{(\sigma+t)/2}}{p^{\sigma+t}}\sum_{e\leq C}\frac{1}{e^2}\sum_{d\sim C/e}\frac{N_0^{1/2}}{d}\left(\frac{N_0}{p^{(\sigma+t)/2}d}+1\right) \\
\ll & \frac{N^{1/2}CX(N_0)}{p^{\sigma+t}}\left(\frac{N_0^{3/2}}{C}+p^{(\sigma+t)/2}N_0^{1/2}\right).
\end{aligned} \tag{4.6}$$

When $2(\sigma+t)\leq a(\pi_1)$ we have bound

$$\begin{aligned}
S(N;p^{\sigma+t},C,N_0) \ll & \frac{N^{3/2}}{Qp^{3(\sigma+t)/2+l_1/4}}\sum_{e\leq C}\frac{1}{e^2}\sum_{d\sim C/e}\frac{1}{d}\times X(N_0)\times\frac{p^{\sigma+t}CQ}{N}\times p^{(\sigma+t)/2} \\
& \times \sum_{\substack{m\sim M(X(N_0)) \\ mp^{l_1}\equiv np^{2(\sigma+t)} \bmod d}}\sum_{n\sim N_0}\left(\frac{|\lambda_f(m)|^2}{(mn)^{1/4}}+\frac{|\lambda_f(n)|^2}{(mn)^{1/4}}\right) \\
\ll & \frac{N^{1/2}CX(N_0)}{p^{l_1/4}}\sum_{e\leq C}\frac{1}{e^2}\sum_{d\sim C/e}\frac{1}{d}\Bigg\{N_0^{3/4}\left(\frac{M_0(X(N_0))^{3/4}}{d}+1\right) \\
& + M(X(N_0))^{3/4}\left(\frac{N_0^{3/4}}{d}+1\right)\Bigg\}.
\end{aligned}$$

When $a(\pi_2)-1\leq 2j\leq 2\sigma$ we obtain the bound

$$\begin{aligned}
& \frac{N^{3/2}}{Qp^{\sigma+j/2+l_1/4}}\sum_{e\leq C}\frac{1}{e^2}\sum_{d\sim C/e}\frac{1}{d}\times X(N_0)\times\frac{p^{\sigma}CQ}{N}\times p^{j/2} \\
& \times \sum_{\substack{m\sim M(X(N_0)) \\ mp^{l_1}\equiv np^{2j} \bmod d}}\sum_{n\sim N_0}\left(\frac{|\lambda_f(m)|^2}{m^{1/2}n^{2a}}+\frac{|\lambda_f(n)|^2}{n^{1/2-2a}}\right) \\
\ll & \frac{N^{1/2}CX(N_0)}{p^{l_1/4}}\sum_{e\leq C}\frac{1}{e^2}\sum_{d\sim C/e}\frac{1}{d}\Bigg\{N_0^{3/2-2a}\left(\frac{M_0(X(N_0))^{1/2}}{d}+1\right) \\
& + M(X(N_0))\left(\frac{N_0^{1-2a}}{d}+1\right)\Bigg\}.
\end{aligned} \tag{4.7}$$

for some exponent $a \in \mathbb{R}$ that we will choose later.

When $2j < a(\pi_2)$, we apply Lemma 4.3.6 and obtain the bound

$$\begin{aligned}
&\frac{N^{3/2}}{Qp^{\sigma+l_1/4+l_2/4}} \sum_{e\leq C} \frac{1}{e^2} \sum_{d\sim C/e} \frac{1}{d} \times X(N_0) \times \frac{p^\sigma CQ}{N} \\
&\times \sum_{\chi\in\mathfrak{X}_j/\sim a(\pi_2)/4} \sum_{m\sim M(X(N_0))} \sum_{\substack{n\sim N_0 \\ mp^{l_1}\equiv np^{l_2} \bmod d \\ c_{\theta_1,\theta_2,\chi} m\equiv n \bmod p^{a(\pi_2)/4}}} \left(\frac{|\lambda_f(m)|^2}{m^{1/4}n^{2c}} + \frac{|\lambda_f(n)|^2}{m^{1/4}n^{1/2-2c}}\right) \\
&\ll \frac{N^{1/2}CX(N_0)p^{j-\frac{a(\pi_2)}{4}}}{p^{l_1/4+l_2/4}} \sum_{e\leq C} \frac{1}{e^2} \sum_{d\sim C/e} \frac{1}{d} \Bigg\{ N_0^{1/2+2c} \left(\frac{M(X(N_0))^{3/4}}{p^{a(\pi_2)/4}d} + 1\right) \\
&+ M(X(N_0))^{3/4} \left(\frac{N_0^{1-2c}}{p^{a(\pi_2)/4}d} + N_0^{-2c}\right)\Bigg\}
\end{aligned} \tag{4.8}$$

for a real constant $c \in \mathbb{R}$. When π_2 is a principal series and $v \leq 4j/5$ we obtain the bound

$$\begin{aligned}
&\frac{N^{3/2}}{Qp^{\sigma+l_1/4}} \sum_{e\leq C} \frac{1}{e^2} \sum_{d\sim C/e} \frac{1}{d} \times X(N_0) \times \frac{p^\sigma CQ}{N} \\
&\times \sum_{v\leq 4j/5} \frac{p^{v/2}}{p^{(j+v)/4}} \sum_{m\sim M(v,X(N_0))} \sum_{\substack{n\sim N_0 \\ mp^{l_1}\equiv np^{l_2} \bmod d}} \left(\frac{|\lambda_f(m)|^2}{m^{1/2}n^{2b}} + \frac{|\lambda_f(n)|^2}{n^{1/2-2b}}\right) \\
&\ll \frac{N^{1/2}CX(N_0)}{p^{l_1/4}} \sum_{v\leq 4j/5} \frac{p^{v/2}}{p^{(j+v)/4}} \sum_{e\leq C} \frac{1}{e^2} \sum_{d\sim C/e} \frac{1}{d} \Bigg\{ N_0^{1/2+2b} \left(\frac{M(v,X(N_0))}{d} + 1\right) \\
&+ M(v,X(N_0))^{1/2} \left(\frac{N_0^{1-2b}}{d} + 1\right)\Bigg\}.
\end{aligned} \tag{4.9}$$

for a constant $b \in \mathbb{R}$ that we choose later.

4.4 Cauchy-Schwartz and Poisson Summation

Applying Cauchy-Schwartz we see that

$$\begin{aligned}
S(N;p^j,C,N_0) \ll &\frac{N^{3/2}}{Qp^\sigma p^{l_1/4}} \sum_{0\leq j\leq\sigma} \frac{1}{[p^{l_2},p^{2j}]^{1/4}} \sum_{e\leq C} \frac{|\mu(e)|}{e^2} \\
&\times \sum_{d\sim C/e} \frac{1}{d} \left(\sum_{n\sim N_0} \frac{|\lambda_f(n)|^2}{n^{1/2}}\right)^{1/2} \Sigma(d,e,p^j),
\end{aligned}$$

$$\Sigma := \Sigma(d,e,p^j) := \left(\sum_{n\sim N_0} \left| \sum_{\substack{m \\ mp^{l_1}\equiv n[p^{l_2},p^{2j}] \bmod d}} \frac{\lambda_g(m)}{m^{1/4}} \mathcal{C}(m,n,p^j) I(m,n,de,p^j) \right|^2 \right)^{1/2}. \tag{4.10}$$

In the special case that π_2 is a principal series and $2j = a(\pi_2)$ we get

$$\begin{aligned} S_{\pi_2,pr}(N;p^j,C,N_0) \ll & \frac{N^{3/2}}{Qp^{\sigma}p^{l_1/4}} \sum_{0\le v\le j} \frac{1}{p^{(j+v)/4}} \sum_{e\le C} \frac{|\mu(e)|}{e^2} \\ & \times \sum_{d\sim C/e} \frac{1}{d} \left(\sum_{n\sim N_0} \frac{|\lambda_f(n)|^2}{n^{1/2}} \right)^{1/2} \Sigma(d,e,p^j,p^v), \end{aligned}$$

$$\Sigma := \Sigma(d,e,p^j,p^v) := \left(\sum_{n\sim N_0} \left| \sum_{\substack{m \\ np^{j+r}\equiv mp^{l_1} \bmod d}} \frac{\lambda_g(m)}{m^{1/4}} \mathcal{C}_0(m,n,p^j,p^v) \chi(m) I(m,n,de,p^j,p^v) \right|^2 \right)^{1/2}. \tag{4.11}$$

We shall restrict the range to $\frac{3\sigma}{4} \le j$ and when π_2 is a principal series and $2j = a(\pi_2)$, then we restrict to $v \ge 4j/5$. Similarly, for $t > 0$

$$\begin{aligned} S(N;p^{\sigma+t},C,N_0) \ll & \frac{N^{3/2}}{Qp^{3(\sigma+t)/2}[p^{l_1},p^{2(\sigma+t)}]^{1/4}} \sum_{e\le C} \frac{|\mu(e)|}{e^2} \\ & \times \sum_{d\sim C/e} \frac{1}{d} \left(\sum_{n\sim N_0} \frac{|\lambda_f(n)|^2}{n^{1/2}} \right)^{1/2} \Sigma(d,e,p^{\sigma+t}). \end{aligned}$$

We shall restrict the range to $\sigma+t < a(\pi_1)-1$. For notational purposes, we write $I(m,n,q)$ instead of $I(m,n,de)$. Applying the l^2-Rankin-Selberg bound (2.13), we have

$$\sum_{n\sim N_0} \frac{|\lambda_f(n)|^2}{n^{1/2}} \ll C(\pi_1)^{\epsilon} N_0^{1/2}. \tag{4.12}$$

First assume that either $2j > a(\pi_2)$ or if $2j = a(\pi_2)$ then π_2 is not a principal series representation. We smoothen out the n-sum and open the square obtaining

$$\sum_{\substack{m_1,m_2 \\ m_1\equiv m_2 \bmod d}} \frac{\lambda_g(m_1)\overline{\lambda_g(m_2)}}{(m_1m_2)^{1/4}}$$
$$\times \sum_{\substack{n \\ n[p^{l_1},p^{2j}]\equiv m_1[p^{l_2},p^{2j}] \bmod d}} V\left(\frac{n}{N_0}\right) I(m_1,n,p^j,q)\overline{I(m_2,n,p^j,q)}\mathcal{C}(m_1,n,p^j)\overline{\mathcal{C}(m_2,n,p^j)}.$$

We may rewrite the character sum now as

$$\mathcal{C}_0(...)\overline{\mathcal{C}_0(...)} = \sum_{\chi_1,\chi_2\in\mathfrak{X}_j} \epsilon(\theta_1\overline{\chi_1})\epsilon(\theta_2,\chi_1)\overline{\epsilon(\theta_1\overline{\chi_1})\epsilon(\theta_2\chi_2)}\chi_1(\overline{m_1})\chi_2(m_2)\chi_1\overline{\chi_2}(n)$$

We now decompose the sum based on the conductor of $\chi_1\overline{\chi_2}$, say $a(\chi_1\overline{\chi_2}) = r$ with $r \le j$. If $r = 0$ then $\chi_1 = \chi_2$. We see all the root numbers cancel and the sum becomes

$$\sum_{\chi\in\mathfrak{X}_j} \chi(\overline{m_1}m_2) = S(m_1 - m_2, 0, p^j).$$

We then apply Poisson summation in the n-sum getting

$$\frac{N_0}{d}\sum_{m_1\sim M_0}\sum_{m_2\sim M_0} \frac{\lambda_g(m_1)\overline{\lambda_g(m_2)}}{(m_1m_2)^{1/4}}$$
$$\times \sum_{n\in\mathbb{Z}} e\left(-\frac{m_1np^\eta}{d}\right)\mathcal{I}(m_1,m_2,n,q)S(m_1-m_2,0,p^j),$$

$$\mathcal{I}(m_1,m_2,n,q) = \int V(y)I(m_1,N_0y,q)\overline{I(m_2,N_0y,q)}e\left(-\frac{N_0ny}{d}\right)dy.$$

Here η is a constant integer that depends on j, l_1 and l_2. It can be explicitly calculated but we will not need to know it's exact value. By repeated integration by parts we see that

$$|n| \ll p^\epsilon \frac{d}{N_0}\left(1 + \frac{\sqrt{NN_0}}{[p^l, p^{2(\sigma+t)}]^{1/2}C}\right) + 1 =: N_1(0).$$

We make the change of variable $\chi_2' = \overline{\chi_1}\chi_2$ so that $a(\chi_1\overline{\chi_2}) = a(\chi_2') = r$.

Now suppose that $0 < r < j$. We rewrite the character sum as

$$\sum_{\chi_1 \in \mathfrak{X}_j} \sum_{\chi_2 \in \mathfrak{X}_r} \epsilon(\theta_1 \overline{\chi_1}) \epsilon(\theta_2 \chi_1) \overline{\epsilon(\theta_1 \overline{\chi_1 \chi_2}) \epsilon(\theta_2 \chi_1 \chi_2)} \chi_1(\overline{m_1}) \chi_2(m_2) \overline{\chi_2}(n).$$

We will now open the root numbers. Recall that E_i is the field associated to θ_i, ω_i and ψ_{E_i} the uniformizer and the standard additive character of E_i respectively. Recall that the function $a_i(j, \theta_i)$ depends on E_i. Suppress the i in the subscript of the function c_i. For most cases it should not cause confusion. Where it does cause confusion, we've inserted it back. We open the root numbers getting

$$\begin{aligned}
&\frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a(\theta_1,j)}][\mathfrak{o}_2 : \mathfrak{p}_2^{a(\theta_2,j)}]} \sum_{\chi_1 \in \mathfrak{X}_j} \sum_{\chi_2 \in \mathfrak{X}_r} \sum_{x_1, x_3 \bmod \mathfrak{p}_1^{a(\theta_1,j)}} \sum_{x_2, x_4 \bmod \mathfrak{p}_2^{a(\theta_2,j)}} \theta_1(x_1 x_3^{-1}) \theta_2(x_2 x_4^{-1}) \\
&\times \chi_{1,E_1}^{-1}(x_1 x_3^{-1}) \chi_{1,E_1}(x_2 x_4^{-1}) \chi_{2,E_1}(x_3) \chi_{2,E_2}(x_4^{-1}) \\
&\times \psi_{E_1}(\omega_1^{-a(\theta_1,j)}(x_1 - x_3)) \psi_{E_2}(\omega_2^{-a(\theta_2,j)}(x_2 - x_4)) \chi_1(\overline{m_1} m_2) \overline{\chi_2}(\overline{m_2} n).
\end{aligned}$$

We make the change of variable $x_1 \mapsto x_1 x_3^{-1}$ and $x_2 \mapsto x_2 x_4^{-1}$ which transforms the sum to

$$\begin{aligned}
&\frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a(\theta_1,j)}][\mathfrak{o}_2 : \mathfrak{p}_2^{a(\theta_2,j)}]} \sum_{\chi_1 \in \mathfrak{X}_j} \sum_{\chi_2 \in \mathfrak{X}_r} \sum_{x_1, x_3 \bmod \mathfrak{p}_1^{a(\theta_1,j)}} \sum_{x_2, x_4 \bmod \mathfrak{p}_2^{a(\theta_2,j)}} \theta_1(x_1) \theta_2(x_2) \\
&\times \chi_{1,E_1}^{-1}(x_1) \chi_{1,E_1}(x_2) \chi_{2,E_1}(x_3) \chi_{2,E_2}(x_4^{-1}) \\
&\times \psi_{E_1}(\omega_1^{-a(\theta_1,j)} x_3(x_1 - 1)) \psi_{E_2}(\omega_2^{-a(\theta_2,j)} x_4(x_2 - 1)) \chi_1(\overline{m_1} m_2) \overline{\chi_2}(\overline{m_2} n).
\end{aligned}$$

Evaluating the x_3 and x_4 sums we deduce that $x_1 - 1 \in \mathfrak{p}^{a(\theta_1,j)-a_1(r)}$ and $x_2 - 1 \in \mathfrak{p}_2^{a(\theta_2,j)-a_2(r)}$ respectively. This transforms the sum into

$$\begin{aligned}
&\frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a_1(r)}]^{1/2}[\mathfrak{o}_2 : \mathfrak{p}_2^{a_2(r)}]^{1/2} p^{r/2}} \sum_{\chi_1 \in \mathfrak{X}_j} \sum_{\chi_2 \in \mathfrak{X}_r} \\
&\times \sum_{x_1 \bmod \mathfrak{p}_1^{a_1(r)}} \theta_1 \chi_{1,E_1}^{-1}(1 - \omega_1^{a(\theta_1,j)-a_1(r)} x_1) \chi_{2,E_2}^{-1}(x_1) \\
&\times \sum_{x_2 \bmod \mathfrak{p}_2^{a_2(r)}} \theta_2 \chi_{1,E_2}(1 - \omega_2^{a(\theta_2,j)-a_2(r)} x_2) \chi_{2,E_2}(x_2) \\
&\times \chi_1(\overline{m_1} m_2) \sum_{z \bmod p^r} \overline{\chi_2}(z\overline{nm_2}) e\left(\frac{z}{p^r}\right).
\end{aligned}$$

Set $\nu(r) = \frac{a(\theta_2,j)-a_2(r)}{e_2}$. We decompose the χ_1 sum into G_ν orbits. We then evaluate the sum over G_ν. We observe that the sum vanishes unless $m_1 \equiv m_2 \bmod p^{\nu(r)}$. Set $k(m_1, m_2)$ to be the element in $\mathbb{Z}_p$ that satisfies

$$m_1 = \exp(p^{\nu(r)} k(m_1, m_2)) m_2.$$

If $\nu(r) < j$ we see that the sum has now become

$$\begin{aligned}
&\frac{\varphi(p^{\nu(r)})}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a_1(r)}]^{1/2}[\mathfrak{o}_2 : \mathfrak{p}_2^{a_2(r)}]^{1/2}} \sum_{\chi_1 \in \mathfrak{X}_j/\sim_{\nu(r)}} \sum_{\chi_2 \in \mathfrak{X}_r} \\
&\times \sum_{x_1 \bmod \mathfrak{p}_1^{a_1(r)}} \theta_1 \chi_{1,E_1}^{-1}(1 - \omega_1^{a(\theta_1,j)-a_1(r)} x_1) \chi_{2,E_2}^{-1}(x_1) \\
&\times \sum_{x_2 \bmod \mathfrak{p}_2^{a_2(r)}} \chi_{1,E_2} \theta_2(1 - \omega_2^{a(\theta_2,j)-a_2(r)} x_2) \chi_{2,E_2}(x_2) \\
&\times \chi_1(\exp(k(m_1, m_2) p^\nu) \chi_2(\overline{m_2} n).
\end{aligned}$$

If $\nu(r) \geq j$ then we see that the χ_1-sum condition is empty. Necessarily, if $r \geq 1$ then $a_i(j) < a(\theta_i)$. The sum becomes

$$\begin{aligned}&\frac{\varphi(p^j)}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a_1(r)}]^{1/2}[\mathfrak{o}_2 : \mathfrak{p}_2^{a_2(r)}]^{1/2}}\\ &\times \sum_{x_1 \bmod \mathfrak{p}_1^{a_1(r)}} \theta_1(1-\omega_1^{a(\theta_1)-a_1(r)}x_1)\\ &\times \sum_{x_2 \bmod \mathfrak{p}_2^{a_2(r)}} \theta_2(1-\omega_2^{a(\theta_2)-a_2(r)}x_2)\\ &\times \sum_{\chi_2 \in \mathfrak{X}_r} \chi_{2,E_1}^{-1}(x_1)\chi_{2,E_2}(x_2)\chi_2(\overline{m_2}n).\end{aligned}$$

We now apply Poisson summation in the n-variable getting

$$\begin{aligned}&\frac{N_0}{d}\sum_{0<r\leq j}\frac{\varphi(p^{\nu(r)})}{p^{r/2}}\sum_{m_1\equiv m_2 \bmod p^{j-\nu(r)}d}\sum_{m_2}\frac{\lambda_g(m_1)\overline{\lambda_g(m_2)}}{(m_1m_2)^{1/4}}\\ &\times \sum_{\substack{n\in\mathbb{Z}\\(n,p)=1}} e\left(-\frac{p^\eta m_1 n}{d}\right)\mathcal{I}(m_1,m_2,n,p^r,q)\mathcal{C}(m_1,m_2,n,p^r,d),\end{aligned}$$

$$\mathcal{I}(m_1,m_2,n,p^j,p^r,q) = \int V(y)I(m_1,N_0y,q)\overline{I(m_2,N_0y,q)}e\left(-\frac{N_0ny}{p^rd}\right)dy.$$

$$\begin{aligned}\mathcal{C}(...) =& \frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a_1(j)}]^{1/2}[\mathfrak{o}_2 : \mathfrak{p}_2^{a_2(r)}]^{1/2}}\sum_{\chi_1\in\mathfrak{X}_j/\sim\nu(r)}\sum_{\chi_2\in\mathfrak{X}_r}\\ &\times \sum_{x_1 \bmod \mathfrak{p}_1^{a_1(r)}} \theta_1\chi_{1,E_1}^{-1}(1-\omega_1^{a(\theta_1,j)-a_1(r)}x_1)\chi_{2,E_2}^{-1}(x_1)\\ &\times \sum_{x_2 \bmod \mathfrak{p}_2^{a_2(r)}} \chi_{1,E_2}\theta_2(1-\omega_2^{a(\theta_2,j)-a_2(r)}x_2)\chi_{2,E_2}(x_2)\\ &\times \chi_1(\exp(k(m_1,m_2)p^{\nu(r)})\epsilon(\chi_2)\overline{\chi_2}(nm_2).\end{aligned}$$

Here, η is a constant that depends on l_1, l_2, j and r but we will not need to know the value of this constant explicitly.

By repeated integration by parts, we see that the integral $\mathcal{I}$ is negligibly small unless

$$|n| \ll N^\epsilon \frac{p^r d}{N_0}\left(1+\frac{\sqrt{NN_0}}{[p^{l_1},p^{2(\sigma+t)}]^{1/2}C}\right) =: N_1(r).$$

We write

$$\Sigma = \Sigma^{=} + \Sigma^{\neq}$$

where

$$\begin{aligned}\Sigma^{=} =& \frac{N_0}{d} \sum_{\substack{0\le r\le j\\ j\le \nu(r)}} \frac{1}{p^{r/2}} \sum_{m_1} \sum_{\substack{m_2\\ m_1\equiv m_2 \bmod d}} \frac{\lambda_g(m_1)\overline{\lambda_g(m_2)}}{(m_1m_2)^{1/4}} S(m_1-m_2,0,p^j)\\ &\times \sum_{n\in\mathbb{Z}} e\bigg(-\frac{m_1np^{\eta}}{d}\bigg)\mathcal{I}(m_1,m_2,n,p^j,p^r,q)\mathcal{C}(m_1,m_2,n,p^r,p^j),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\Sigma^{\neq} =& \frac{N_0}{d} \sum_{\substack{0\le r\le j\\ \nu(r)<j}} \frac{\varphi(p^{\nu(r)})}{p^{r/2}} \sum_{m_1} \sum_{\substack{m_2\\ m_1\equiv m_2 \bmod p^{\nu(r)}d}} \frac{\lambda_g(m_1)\overline{\lambda_g(m_2)}}{(m_1m_2)^{1/4}}\\ &\times \sum_{\substack{n\in\mathbb{Z}\\ (n,p)=1}} e\bigg(-\frac{p^{\eta}m_1n}{d}\bigg)\mathcal{I}(m_1,m_2,n,p^j,p^r,q)\mathcal{C}(m_1,m_2,n,p^j,p^r),\end{aligned} \tag{4.14}$$

where for $r<j$

$$\begin{aligned}\mathcal{C}(m_1,m_2,n,p^r,p^j) =& \frac{1}{[\mathfrak{o}_1:\mathfrak{p}_1^{a_1(r)}]^{1/2}[\mathfrak{o}_2:\mathfrak{p}_2^{a_2(r)}]^{1/2}} \sum_{\chi_1\in\mathfrak{X}_j/\sim\nu(r)} \sum_{\chi_2\in\mathfrak{X}_r}\\ &\times \sum_{x_1 \bmod \mathfrak{p}_1^{a_1(r)}} \theta_1\chi_{1,E_1}^{-1}(1-\omega_1^{a(\theta_1,j)-a_1(r)}x_1)\chi_{2,E_2}^{-1}(x_1)\\ &\times \sum_{x_2 \bmod \mathfrak{p}_2^{a_2(r)}} \chi_{1,E_2}\theta_2(1-\omega_2^{a(\theta_2,j)-a_2(r)}x_2)\chi_{2,E_2}(x_2)\\ &\times \chi_1(\exp(k(m_1,m_2)p^{\nu})\epsilon(\chi_2)\overline{\chi_2}(nm_2).\end{aligned} \tag{4.15}$$

In case $2j < a(\pi_2) - 1$, so that $\nu \geq 1$, we use the form in (4.15) for the case $r = j$ as well. In case $2j \geq a(\pi_2) - 1$ then we write the case $r = j$ as

$$\begin{aligned}\mathcal{C}(m_1, m_2, n, p^j, p^j) =& \frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a(\theta_1,j)}][\mathfrak{o}_2 : \mathfrak{p}_2^{a(\theta_2,j)}]p^{j/2}} \sum_{\chi_1 \in \mathfrak{X}_j} \sum_{\chi_2 \in \mathfrak{X}_j} \\ & \times \sum_{x_1, x_3 \bmod \mathfrak{p}_1^{a(\theta_1,j)}} \sum_{x_2, x_4 \bmod \mathfrak{p}_2^{a(\theta_2,j)}} \theta_1(x_1 x_3^{-1})\theta_2(x_2 x_4^{-1}) \\ & \times \chi_{1,E_1}^{-1}(x_1 x_3^{-1})\chi_{1,E_1}(x_2 x_4^{-1})\chi_{2,E_1}(x_3)\chi_{2,E_2}(x_4^{-1}) \\ & \times \psi_{E_1}(\omega_1^{-a(\theta_1,j)}(x_1 - x_3))\psi_{E_2}(\omega_2^{-a(\theta_2,j)}(x_2 - x_4)) \\ & \times \chi_1(\overline{m_1} m_2)\chi_2(m_2) \sum_{z \bmod p^j} \overline{\chi_2}(z) e\left(\frac{zn}{p^j}\right).\end{aligned} \tag{4.16}$$

Now assume that $\pi_2 = \pi(\chi, \chi^{-1}\psi)$ is a principal series representation and $2j = a(\pi_2)$. Fix $r \leq v$ so that when we open the square the sum becomes

$$\sum_{\substack{m_1, m_2 \\ m_1 \equiv m_2 \bmod d}} \frac{\lambda_g(m_1)\chi(m_1)\overline{\lambda_g(m_2)\chi(m_2)}}{(m_1 m_2)^{1/4}}$$

$$\times \sum_{\substack{n \\ np^{l_1} \equiv m_1 p^{j+v} \bmod d}} V\left(\frac{n}{N_0}\right) I(m_1, n, p^j, , p^v q)\overline{I(m_2, n, p^j, p^v, q)}\mathcal{C}(m_1, n, p^j, p^v)\overline{\mathcal{C}(m_2, n, p^j, p^v)}.$$

We see that the character sum becomes

$$\mathcal{C}_0(...)\overline{\mathcal{C}_0(...)} = \sum_{\chi_1, \chi_2 \in \mathfrak{X}_v} \epsilon(\theta_1\chi\chi_1)\epsilon(\chi^2\psi\overline{\chi_1})\epsilon(\overline{\chi_1})\overline{\epsilon(\theta_1\chi\chi_2)\epsilon(\chi^2\psi\overline{\chi_2})\epsilon(\overline{\chi_2})}\chi_1(\overline{m_1})\chi_2(m_2)\chi_1\overline{\chi_2}(n).$$

As before, we decompose the sum based on the conductor of $\chi_1\overline{\chi_2}$ with $a(\chi_1\overline{\chi_2}) = r$ with $r \leq v$. If $r = 0$ then $\chi_1 = \chi_2$, all root numbers cancel and the sum becomes $S(m_1 - m_2, 0, p^v)$.

Now suppose $0 < r < v$. We rewrite the character sum as

$$\sum_{\chi_1 \mathfrak{X}_v} \sum_{\chi_2 \in \mathfrak{X}_r} \epsilon(\theta_1\chi\chi_1)\epsilon(\chi^2\psi\overline{\chi_1})\epsilon(\overline{\chi_1})\overline{\epsilon(\theta_1\chi\chi_1\chi_2)\epsilon(\overline{\chi_1\chi_2})\epsilon(\chi^2\psi\overline{\chi_1\chi_2})}\chi_1(\overline{m_1}m_2)\chi_2(m_2\overline{n}).$$

Instead of writing $\theta_1\chi$, we simple write θ_1 with the understanding that we've absorbed the twist χ into θ_1. This turns out not to affect any arguments of the proof and declutters notation. Retaining previous notation, we open the root numbers getting

$$\begin{aligned}\frac{1}{[\mathfrak{o}_{E_1}:\mathfrak{p}_1^{a(\theta_1)}]p^{j+v}}\sum_{\chi_1\in\mathfrak{X}_v}\sum_{\chi_2\in\mathfrak{X}_r}\sum_{x_1,x_4 \bmod \mathfrak{p}^{a(\theta_1)}}&\theta_1\chi_{1,E_1}(x_1x_4^{-1})\chi_{2,E_1}(x_4^{-1})\psi_{E_1}(\omega^{-a(\theta_1)}(x_1-x_4))\\ &\times\sum_{x_2,x_5 \bmod p^j}\chi^2\psi\chi_1^{-1}(x_2x_5^{-1})\chi_2(x_5)e\left(\frac{x_2-x_5}{p^j}\right)\\ &\times\sum_{x_3,x_6 \bmod p^v}\chi_1^{-1}(x_3x_6^{-1})\chi_2(x_6)e\left(\frac{x_3-x_6}{p^v}\right)\\ &\times\chi_1(m_1\overline{m_2})\chi_2(m_2\overline{n}).\end{aligned}$$

We make the change of variable $x_1x_4^{-1}\mapsto x_1$, $x_2x_5^{-1}\mapsto x_2$ and $x_3x_6^{-1}\mapsto x_3$. This converts the character sum into

$$\begin{aligned}\frac{1}{[\mathfrak{o}_{E_1}:\mathfrak{p}_1^{a(\theta_1)}]p^{j+v}}\sum_{\chi_1\in\mathfrak{X}_v}\sum_{\chi_2\in\mathfrak{X}_r}\sum_{x_1,x_4 \bmod \mathfrak{p}^{a(\theta_1)}}&\theta_1\chi_{1,E_1}(x_1)\chi_{2,E_1}(x_4^{-1})\psi_{E_1}(\omega^{-a(\theta_1)}(x_4(x_1-1))\\ &\times\sum_{x_2,x_5 \bmod p^j}\chi^2\psi\chi_1^{-1}(x_2)\chi_2(x_5)e\left(\frac{x_5(x_2-1)}{p^j}\right)\\ &\times\sum_{x_3,x_6 \bmod p^v}\chi_1(x_3)\chi_2(x_6)e\left(\frac{x_6(x_3-1)}{p^v}\right)\\ &\times\chi_1(m_1\overline{m_2})\chi_2(m_2\overline{n}).\end{aligned}$$

Evaluating the x_4 sum we see that it vanishes unless $x_1-1\in\mathfrak{p}_1^{a(\theta_1)-a_1(r)}$. Evaluating the x_5 sum we see that it vanishes unless $x_2-1\in(p^{j-r})$. Evaluating the x_6 sum we see that it vanishes unless $x_3-1\in(p^{v-r})$. The sum now transforms into

$$\begin{aligned}\frac{1}{[\mathfrak{o}_{E_1}:\mathfrak{p}_1^{a_1(r)}]^{1/2}p^r}\sum_{x_1 \bmod \mathfrak{p}^{a_1(r)}}&\theta_1\chi_{1,E_1}(1-\omega_1^{a(\theta_1)-a_1(r)}x_1)\chi_{2,E_1}(x_1)\\ &\times\sum_{x_2 \bmod p^r}\chi^2\psi\chi_1^{-1}(1-p^{j-r}x_2)\chi_2^{-1}(x_2)\\ &\times\sum_{x_3 \bmod p^r}\chi_1^{-1}(1-p^{v-r}x_3)\chi_2^{-1}(x_3)\\ &\times\chi_1(m_1\overline{m_2})\chi_2(m_2\overline{n}).\end{aligned}$$

Decomposing the χ_1-sum into G_{v-r} orbits, we see that the G_{v-r} orbit vanishes unless $m_1 \equiv m_2 \bmod p^{v-r}$. Set $k(m_1, m_2)$ to satisfying $m_1 = \exp(p^{v-r}k(m_1, m_2))$. This transforms the character sum into

$$\begin{aligned}\frac{\varphi(p^{v-r})}{[\mathfrak{o}_{E_1} : \mathfrak{p}_1^{a_1(r)}]^{1/2}p^r} \sum_{\chi_1 \in \mathfrak{X}_v \sim_{v-r}} \sum_{\chi_2 \in \mathfrak{X}_r} \sum_{x_1 \bmod \mathfrak{p}^{a_1(r)}} \theta_1\chi_{1,E_1}(1-\omega_1^{a(\theta_1)-a_1(r)}x_1)\chi_{2,E_1}(x_1)\\ \times \sum_{x_2 \bmod p^r} \chi^2\psi\chi_1^{-1}(1-p^{j-r}x_2)\chi_2^{-1}(x_2)\\ \times \sum_{x_3 \bmod p^r} \chi_1^{-1}(1-p^{v-r}x_3)\chi_2^{-1}(x_3)\\ \times \chi_1(\exp(k(m_1,m_2)p^{v-r})\chi_2(\overline{m_2 n}).\end{aligned}$$

When $r = v$ we simply open the root numbers obtaining

$$\begin{aligned}\frac{1}{[\mathfrak{o}_{E_1} : \mathfrak{p}_1^{a(\theta_1)}]p^{j+v}} \sum_{\chi_1 \in \mathfrak{X}_v} \sum_{\chi_2 \in \mathfrak{X}_v} \sum_{x_1,x_4 \bmod \mathfrak{p}^{a(\theta_1)}} \theta_1\chi_{1,E_1}(x_1)\theta_1\chi_{2,E_1}(x_4^{-1})\psi_{E_1}(\omega^{-a(\theta_1)}(x_1-x_4))\\ \times \sum_{x_2,x_5 \bmod p^j} \chi^2\psi\chi_1^{-1}(x_2)\chi^2\psi\chi_2^{-1}(x_5^{-1})e\left(\frac{x_2-x_5}{p^j}\right)\\ \times \sum_{x_3,x_6 \bmod p^v} \chi_1^{-1}(x_3)\chi_2^{-1}(x_6^{-1})e\left(\frac{x_3-x_6}{p^v}\right)\\ \times \chi_1(\overline{m_1})\chi_2(m_2)\chi_1\overline{\chi_2}(n).\end{aligned}$$

. We then apply Poisson summation in the n-sum to get

$$\Sigma(p^j, p^v) = \Sigma_{pr}^{=} + \Sigma_{pr}^{\neq},$$

$$\begin{aligned}\Sigma_{pr}^{=} =& \frac{N_0}{d}\sum_{m_1}\sum_{m_2}\frac{\lambda_g(m_1)\overline{\lambda_g(m_2)}}{(m_1m_2)^{1/4}}\\ &\times \sum_{n\in\mathbb{Z}} e\left(-\frac{m_1np^\eta}{d}\right)\mathcal{I}_{pr}(m_1,m_2,n,p^j,p^v,q)S(m_1-m_2,0,p^v),\end{aligned} \tag{4.17}$$

$$\Sigma^{\neq} = \frac{N_0}{d} \sum_{0<r\leq v} \frac{\varphi(p^{v-r})}{p^{r/2}} \sum_{m_1 \equiv m_2 \bmod p^{v-r}d} \sum_{m_2} \frac{\lambda_g(m_1)\overline{\lambda_g(m_2)}}{(m_1 m_2)^{1/4}} \times \sum_{\substack{n\in\mathbb{Z} \\ (n,p)=1}} e\left(-\frac{p^n m_1 n}{d}\right) \mathcal{I}_{pr}(m_1, m_2, n, p^j, p^v, p^r, q) \mathcal{C}_{pr}(m_1, m_2, n, p^j, p^v, p^r, d), \tag{4.18}$$

$$\begin{aligned} \mathcal{C}_{pr}(m_1, m_2, n, p^j, p^v, p^r) = & \frac{\varphi(p^{v-r})}{[\mathfrak{o}_{E_1} : \mathfrak{p}_1^{a_1(r)}]^{1/2} p^r} \sum_{\chi_1 \in \mathfrak{X}_v \sim_{v-r}} \sum_{\chi_2 \in \mathfrak{X}_r} \\ & \times \sum_{x_1 \bmod \mathfrak{p}^{a_1(r)}} \theta_1 \chi_{1,E_1}(1 - \omega_1^{a(\theta_1)-a_1(r)} x_1) \chi_{2,E_1}(x_1) \\ & \times \sum_{x_2 \bmod p^r} \chi^2 \psi \chi_1^{-1}(1 - p^{j-r} x_2) \chi_2^{-1}(x_2) \\ & \times \sum_{x_3 \bmod p^r} \chi_1^{-1}(1 - p^{v-r} x_3) \chi_2^{-1}(x_3) \\ & \times \sum_{z \bmod p^r} \chi_2(\overline{m_2 z}) e\left(\frac{nz}{p^r}\right) \\ & \times \chi_1(\exp(k(m_1, m_2) p^{v-r}). \end{aligned} \tag{4.19}$$

if $r < v$ and

$$\begin{aligned} \mathcal{C}_{pr}(m_1, m_2, n, p^j, p^v, p^v) = & \frac{1}{[\mathfrak{o}_{E_1} : \mathfrak{p}_1^{a(\theta_1)}] p^{j+v}} \sum_{\chi_1 \in \mathfrak{X}_v} \sum_{\chi_2 \in \mathfrak{X}_v} \\ & \sum_{x_1, x_4 \bmod \mathfrak{p}^{a(\theta_1)}} \theta_1 \chi_{1,E_1}(x_1) \theta_1 \chi_{2,E_1}(x_4^{-1}) \psi_{E_1}(\omega^{-a(\theta_1)}(x_1 - x_4)) \\ & \times \sum_{x_2, x_5 \bmod p^j} \chi^2 \psi \chi_1^{-1}(x_2) \chi^2 \psi \chi_2^{-1}(x_5^{-1}) e\left(\frac{x_2 - x_5}{p^j}\right) \\ & \times \sum_{x_3, x_6 \bmod p^v} \chi_1^{-1}(x_3) \chi_2^{-1}(x_6^{-1}) e\left(\frac{x_3 - x_6}{p^v}\right) \\ & \times \sum_{z \bmod p^v} \chi_1 \overline{\chi_2}(z) e\left(\frac{nz}{p^v}\right) \\ & \times \chi_1(\overline{m_1}) \chi_2(m_2) \end{aligned} \tag{4.20}$$

if $r = v$. By repeated integration by parts, we see that the integral is negibly small unless

$$|n| \ll \frac{p^r d}{N_0} \left(\frac{\sqrt{N N_0}}{p^{l_1/2} q} + 1\right) =: N_1(r)$$

4.4.1 Bounds for the Integral II

We estimate the size of the integral.

Lemma 4.4.1. *Assume* $2(\sigma+t) \leq a(\pi_1)$. *If* $N_0 \ll N_C$ *then*

$$\mathcal{I}(m_1, m_2, n, p^t, p^r, q) \ll \left(\frac{p^{\sigma+t}CQ}{N}\right)^2 \times X_C^2 \times \inf\left\{1, \frac{\frac{2\sqrt{NN_0}}{p^{l_1/2}q}}{\left|\frac{2\sqrt{Nm_1}}{p^{\sigma+t}q} - \frac{2\sqrt{Nm_2}}{p^{\sigma+t}q}\right|}\right\},$$

else if $N_0 \gg N_C$ *then*

$$\mathcal{I}(m_1, m_2, n, p^t, q) \ll \left(\frac{p^{\sigma+t}CQ}{N}\right)^2 \times X(N_0)^2 \times \inf\left\{1, \frac{\frac{2\sqrt{NN_0}}{q[p^{l_1}, p^{2(\sigma+t)}]^{1/2}}}{\left|\frac{2\sqrt{Nm_1}}{p^{\sigma+t}q} - \frac{2\sqrt{Nm_2}}{p^{\sigma+t}q}\right|}\right\}.$$

Proof: We use the inequality

$$\mathcal{I}(m_1, m_2, n, p^t, p^r, q) \leq \int |V(y)||I(m_1, N_0 y, p^t, q)I(m_2, N_0 y, p^t, q)|dy,$$

and estimate each I with Lemma 4.3.8. The lemma then follows from the elementary estimate for real constants a, b and c with $c \neq 0$,

$$\int_{\mathbb{R}} \inf\left\{1, \frac{1}{|a-cx|}\right\} \inf\left\{1, \frac{1}{|b-cx|}\right\} dx \ll \inf\left\{1, \frac{|c|}{|a-b|}\right\}.$$

□

Lemma 4.4.2. *Assume that $j \leq \sigma$ and if $2j = a(\pi_2)$ then π_2 is not a principal series. If $N_0 \ll N_C$ then*

$$\mathcal{I}(m_1, m_2, n, p^j, p^r, q) \ll \left(\frac{p^\sigma CQ}{N}\right)^2 \times X_C^2 \times \inf\left\{1, \frac{\frac{2\sqrt{NN_0}}{p^{l_1/2}q}}{\left|\frac{2\sqrt{Nm_1}}{[p^{l_2}, p^{2j}]q} - \frac{2\sqrt{Nm_2}}{[p^{l_2}, p^{2j}]q}\right|}\right\},$$

else if $N_0 \gg N_C$ then

$$\mathcal{I}(m_1, m_2, n, p^j, q) \ll \left(\frac{p^\sigma CQ}{N}\right)^2 \times X(N_0)^2 \times \inf\left\{1, \frac{\frac{2\sqrt{NN_0}}{p^{l_1/2}q}}{\left|\frac{2\sqrt{Nm_1}}{[p^{l_2}, p^{2j}]q} - \frac{2\sqrt{Nm_2}}{[p^{l_2}, p^{2j}]q}\right|}\right\},$$

Lemma 4.4.3. *Assume that π_2 is a principal series. If $N_0 \ll N_C$ then*

$$\mathcal{I}_{pr}(m_1, m_2, n, p^j, p^v, p^r, q) \ll \left(\frac{p^\sigma CQ}{N}\right)^2 \times X_C^2 \times \inf\left\{1, \frac{\frac{2\sqrt{NN_0}}{p^{l_1/2}q}}{\left|\frac{2\sqrt{Nm_1}}{p^{j+v}q} - \frac{2\sqrt{Nm_2}}{p^{j+v}q}\right|}\right\},$$

else if $N_0 \gg N_C$ then

$$\mathcal{I}_{pr}(m_1, m_2, n, p^j, p^v, p^r, q) \ll \left(\frac{p^\sigma CQ}{N}\right)^2 \times X(N_0)^2 \times \inf\left\{1, \frac{\frac{2\sqrt{NN_0}}{p^{l_1/2}q}}{\left|\frac{2\sqrt{Nm_1}}{p^{j+v}q} - \frac{2\sqrt{Nm_2}}{p^{j+v}q}\right|}\right\},$$

The proofs of both Lemma 4.4.2 and Lemma 4.4.3 are nearly identical to the proof for Lemma 4.4.1 so we omit it.

4.4.2 Bounds for the Character Sum II

In this sub-section, we take on proving the most technically complicated part of this chapter.

Lemma 4.4.4. *Assume $r \geq 1$ and $\nu(r) \geq j$. Then*

$$\mathcal{C}(m_1, m_2, n, p^j, p^r) \ll p^{r/2}.$$

Proof: Since $\nu(r) \geq j$ we see that the χ_1-sum is degenerate. We thus see the character sum is

$$\begin{aligned}
&\frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a_1(r)}]^{1/2}[\mathfrak{o}_2 : \mathfrak{p}_2^{a_2(r)}]^{1/2}p^{r/2}} \\
&\times \sum_{x_1 \bmod \mathfrak{p}_1^{a_1(r)}} \theta_1(1 - \omega_1^{a(\theta_1)-a_1(r)}x_1) \\
&\times \sum_{x_2 \bmod \mathfrak{p}_2^{a_2(r)}} \theta_2(1 - \omega_2^{a(\theta_2)-a_2(r)}x_2) \\
&\times \sum_{z \bmod p^r} e\left(\frac{nz}{p^r}\right) \\
&\times \sum_{\chi_2 \in \mathfrak{X}_r} \chi_{2,E_1}^{-1}(x_1)\chi_{2,E_2}(x_2)\chi_2(\overline{m_2 z}).
\end{aligned}$$

We We then write $\chi_2 = \chi_0\chi$ with $\chi_0 \bmod p^{r/2}$, $x_i = x_{i,0}(1 + x_{i,1}\omega_i^{a_i(r)/2})$ and $z = z_0(1 + p^{r/2}z_1)$. We observe that the χ_0, $x_{i,1}$ and z_1 sums are independent and we evaluate them. Let α be the constant associated to χ. We then get the collection of equations, dropping the 0 in the subscript

$$c_1 \frac{x_1}{1 - \omega_1^{a(\theta_1)-a_1(r)}x_1} \equiv \alpha \bmod \mathfrak{p}_1^{a_1(r)/2},$$

$$c_2 \frac{x_2}{1 - \omega_2^{a(\theta_2)-a_2(r)}x_1} \equiv \alpha \bmod \mathfrak{p}_2^{a_2(r)/2}$$

$$Nm(x_1)m_2 z \equiv Nm(x_2) \bmod p^{r/2},$$

$$\alpha \equiv nz \bmod p^{r/2}.$$

To conclude the lemma, we need to show there are $O(1)$ many solutions in all free variables. First we observe that x_1, x_2 and z are determined as functions of α. Observe that since $\nu(r) \geq j$, necessarily, $a(\theta_i) - a_i(r) \geq \frac{a_i(r)}{2}$. Therefore we get the equations

$$c_1 x_1 \equiv \alpha \bmod \mathfrak{p}_1^{a_1(r)/2},$$

$$c_2 x_2 \equiv \alpha \bmod \mathfrak{p}_2^{a_2(r)/2},$$

$$Nm(x_1) m_2 n\alpha \equiv Nm(x_2) \bmod p^{r/2}.$$

Substituting x_1 and x_2 into the last equation we get the single equation

$$Nm(c_2) n m_2 \alpha \equiv Nm(c_1) \bmod p^{r/2}.$$

By Hensel's lemma, we then conclude that the choice of α is unique. □

Lemma 4.4.5. *Assume* $a(\pi_2) - 1 \leq 2j < a(\pi_1) - 1$. *Assume* $0 < r < j$. *Then*

$$\mathcal{C}(m_1, m_2, n, p^j, p^r) \ll p^{O(1)} p^r.$$

Proof: Note that trivially we have the bound

$$\mathcal{C}(m_1, m_2, n, p^r, p^j) \ll p^{Cr} \mathbb{I}(m_1 \equiv m_2 \bmod p^{j-r})$$

for an absolute constant C. We use this bound when $r = 1$ and the statement of the theorem follows in this case. So from now on we assume that $r \geq 2$.

We open the definition of $\epsilon(\chi_2)$ in the character sum and it becomes

$$\frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a_1(j)}]^{1/2}[\mathfrak{o}_2 : \mathfrak{p}_2^{a_2(r)}]^{1/2}p^{r/2}} \sum_{\chi_1 \in \mathfrak{X}_j/\sim j-r} \sum_{\chi_2 \in \mathfrak{X}_r}$$
$$\times \sum_{x_1 \bmod \mathfrak{p}_1^{a_1(r)}} \theta_1\chi_{1,E_1}^{-1}(1-\omega_1^{a(\theta_1,j)-a_1(r)}x_1)\chi_{2,E_2}^{-1}(x_1)$$
$$\times \sum_{x_2 \bmod \mathfrak{p}_2^{a_2(r)}} \chi_{1,E_2}\theta_2(1-\omega_2^{a(\theta_2,j)-a_2(r)}x_2)\chi_{2,E_2}(x_2)$$
$$\times \chi_1(\exp(k(m_1,m_2)p^{j-r}) \sum_{z \bmod p^r} \overline{\chi_2}(z\overline{nm_2})e\left(\frac{z}{p^r}\right).$$

We then write $\chi_2 = \chi_0\chi$ with $\chi_0 \bmod p^{r/2}$. Similarly, we note that there's a bijection between $\chi_1 \in \mathfrak{X}_j/\sim j-r$ with elements $\alpha_1 \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ given by

$$\chi_1(\exp(kp^{j-r})) = e\left(\alpha_1\frac{k}{p^r}\right).$$

If $t \geq 1$ then there exists a unique element $\alpha_2 \in (\mathbb{Z}/p^{r-t}\mathbb{Z})^\times$ such that

$$\chi(\exp(kp^t)) = e\left(\alpha_2\frac{k}{p^{r-t}}\right).$$

Let Nm_{E_i} and tr_{E_i} be the standard norm and trace associated to E_i respectively. Let c_i be the constants given by

$$\theta_i(\exp(\omega_i^r u)) = \psi_{E_i}(\omega_i^{-a(\theta_i)+r}u)$$

for any $r \geq e_i$ and $u \in \mathfrak{o}_{E+i}$.

We write $\alpha_1 = \alpha_{0,1} + \alpha_{1,1}p^{r/2}$ and do the same for α_2 taking $t = r/2$. We then write $x_i = x_{i,0}(1+\omega_i^{a_i(r)/2}x_{i,1})$ and $z = z_0(1+p^{r/2}z_1)$. One observes that the $x_{i,1}, z_i, \chi$ and $\alpha_{i,1}$ sums are all independent of each other and we evaluate them. In doing so we get the following congruences in terms of the same variables with 0 in their indices. We drop the 0 in the indices for notational purposes and get the following system of congruences.

$$(c_1\omega_1^{a(\theta_1,j)-a(\theta_1)} - \alpha_1\omega_1^{a(\theta_1,j)-a_1(j)})\frac{x_1}{1-\omega_1^{a(\theta_1,j)-a_1(r)}x_1} \equiv \alpha_2 \bmod \mathfrak{p}_1^{a_1(r)/2}$$

$$(c_2\omega_2^{a(\theta_2,j)-a(\theta_2)} + \alpha_1\omega_2^{a(\theta_2,j)-a_2(j)})\frac{x_2}{1-\omega_2^{a(\theta_2,j)-a_2(r)}x_2} \equiv -\alpha_2 \bmod \mathfrak{p}_2^{a_2(r)/2}$$

$$\alpha_2 \equiv nm_2 z \bmod p^{r/2}$$

$$Nm_{E_1}(x_1) \equiv Nm_{E_2}(x_2)z \bmod p^{r/2}$$

$$\mathrm{tr}_{E_1}\frac{\omega_1^{a(\theta_1,j)-a_1(j)}x_1}{1-\omega_1^{a(\theta_1,j)-a_1(r)}x_1} - \mathrm{tr}_{E_2}\frac{\omega_2^{a(\theta_2,j)-a_2(j)}x_2}{1-\omega_2^{a(\theta_2,j)-a_2(r)}x_2} \equiv k(m_1,m_2) \bmod p^{r/2}$$

The bound on the character sum then amounts to showing that the above set of equations has at-most $p^{O(1)}$ many solutions in all the free variables $x_1, x_2, \alpha_1, \alpha_2$ and z. By our assumptions, $a_2(j) \geq a(\theta_2)$ and $a_1(j) < a(\theta_1)$ so that the equations become (rearranging some equations)

$$\frac{x_1}{1-\omega_1^{a(\theta_1,j)-a_1(j)}x_1} \equiv \frac{\alpha_2}{c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)}} \bmod \mathfrak{p}_1^{a_1(r)/2},$$

$$\frac{x_2}{1-\omega_2^{a(\theta_2,j)-a_2(r)}x_2} \equiv -\frac{\alpha_2}{c_2\omega_2^{a(\theta_2,j)-a(\theta_2)} + \alpha_1} \bmod \mathfrak{p}_2^{a_2(r)/2},$$

$$\alpha_2 \equiv nm_2 z \bmod p^{r/2},$$

$$Nm_{E_1}(x_1) \equiv Nm_{E_2}(x_2)z \bmod p^{r/2},$$

$$\mathrm{tr}_{E_1}\frac{\omega_1^{a(\theta_1,j)-a_1(j)}x_1}{1-\omega_1^{a(\theta_1,j)-a_1(r)}x_1} - \mathrm{tr}_{E_2}\frac{\omega_2^{a(\theta_2,j)-a_2(j)}x_2}{1-\omega_2^{a(\theta_2,j)-a_2(r)}x_2} \equiv k(m_1,m_2) \bmod p^{r/2}$$

Note that the rearrangement of the first two equations is valid because both $c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)}$ and $c_2\omega_2^{a(\theta_1,j)-a_2(j)} + \alpha_1$ are units. We observe that all other free variables are determined as a function of α_1 and α_2. Next we observe that

$$\begin{aligned}
\mathrm{tr}_{E_1}\frac{x_1}{1-\omega_1^{a(\theta_1,j)-a_1(r)}x_1} &\equiv \mathrm{tr}_{E_1}\frac{\alpha_2}{c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)}} \\
&\equiv \alpha_2 \mathrm{tr}_{E_1}\frac{1}{c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)}} \\
&\equiv \alpha_2\frac{\mathrm{tr}_{E_1}(c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)})}{Nm(c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)})} \bmod p^{r/2}.
\end{aligned}$$

A similar calculation gives

$$\mathrm{tr}_{E_2}\frac{x_2}{1-\omega_2^{a_2(j)-a_2(r)}x_2} \equiv -\alpha_2\frac{\mathrm{tr}_{E_2}(c_2\omega_2^{a(\theta_2,j)-a(\theta_2)}+\alpha_1)}{Nm(c_2\omega_2^{a(\theta_2,j)-a(\theta_2)}+\alpha_1)} \bmod p^{r/2}$$

With these calculations, we can rewrite the trace equation as

$$\alpha_2\left(\frac{\mathrm{tr}_{E_1}\omega_1^{a(\theta_1)-a_1(j)}(c_1-\alpha_1\omega_1^{a(\theta_1)-a_1(j)})}{Nm(c_1-\alpha_1\omega_1^{a(\theta_1)-a_1(j)})}+\frac{\mathrm{tr}_{E_2}(c_2\omega_2^{a(\theta_2,j)-a(\theta_2)}+\alpha_1)}{Nm(c_2\omega_2^{a(\theta_2,j)-a(\theta_2)}+\alpha_1)}\right) \equiv k(m_1,m_2) \bmod p^{r/2}.$$

Next we observe that

$$x_1 \equiv \frac{\alpha_2}{(c_1-\alpha_1\omega_1^{a(\theta_1)-a_1(j)})+\omega_1^{a(\theta_1)-a_1(r)}\alpha_2} \bmod \mathfrak{p}_1^{a_1(r)/2}.$$

Therefore

$$Nm(x_1) \equiv \alpha_2^2 Nm_{E_1}^{-1}(c_1-\alpha_1\omega_1^{a(\theta_1)-a_1(j)}+\omega_1^{a(\theta_1)-a_1(r)}\alpha_2) \bmod p^{r/2}.$$

Similarly,

$$x_2 \equiv -\frac{\alpha_2}{c_2\omega_2^{a_2(j)-a(\theta_2)}+\alpha_1-\omega_2^{a_2(j)-a_2(r)}\alpha_2} \bmod \mathfrak{p}_2^{a_2(r)/2},$$

$$Nm(x_2) \equiv \alpha_2^2 Nm_{E_2}^{-1}((c_2\omega_2^{a_2(j)-a(\theta_2)}+\alpha_1-\omega_2^{a_2(j)-a_2(r)}\alpha_2) \bmod p^{r/2}.$$

So we obtain the following simultaneous equations in α_1 and α_2 (using the fact that trace is a linear function of α_i and $tr(c_2)=0$)

$$\alpha_2\left(\frac{\mathrm{tr}_{E_1}\omega_1^{a(\theta_1)-a_1(j)}(c_1-\alpha_1\omega_1^{a(\theta_1)-a_1(j)})}{Nm(c_1-\alpha_1\omega_1^{a(\theta_1)-a_1(j)})}+\frac{2\alpha_1}{Nm(c_2\omega_2^{a(\theta_2,j)-a(\theta_2)}+\alpha_1)}\right) \equiv k(m_1,m_2) \bmod p^{r/2},$$

$$\begin{aligned} &Nm_{E_2}((c_2\omega_2^{a_2(j)-a(\theta_2)}+\alpha_1-\omega_2^{a_2(j)-a_2(r)}\alpha_2)nm_2 \\ &\equiv \alpha_2 Nm_{E_1}(c_1-\alpha_1\omega_1^{a(\theta_1)-a_1(j)}+\omega_1^{a(\theta_1)-a_1(j)}\alpha_2) \bmod p^{r/2}. \end{aligned}$$

Observe that from the first equation we may write α_2 as a function of α_1. Using the fact that $tr(c_1) \equiv 0 \bmod p$, that $j - r > 0$, that $2j < a(\pi_1)$ we see that the system mod p is

$$2\alpha_1\alpha_2 \equiv Nm(c_2\omega_2^{a_2(j)-a(\theta_2)} - \alpha_1)k(m_1, m_2) \bmod p,$$

$$Nm_{E_2}(c_2\omega_2^{a_2(j)-a(\theta_2)} - \alpha_1)n \equiv \alpha_2 Nm_{E_1}(c_1) \bmod p.$$

We use the second equation we rewrite α_2 as a function of α_1 and substitute this into the first equation. This gives a single equation in α_1 -

$$Nm_{E_2}(\alpha_1 - c_2\omega_2^{a_2(j)-a(\theta_2)})(2n\alpha_1 - Nm(c_1)k(m_1, m_2)) \equiv 0 \bmod p.$$

We observe that since $Nm_{E_2}(\alpha_1 - c_2\omega_2^{a_2(j)-a(\theta_2)})$ is always a unit, the only possible solution is

$$\alpha_1 \equiv \frac{Nm(c_1)k(m_1, m_2)}{2n} \bmod p.$$

If $k(m_1, m_2)$ is not a unit then there are no solutions in α_1. So we assume that $k(m_1, m_2)$ is a unit. Set $P(\alpha) = Nm_{E_2}(\alpha_1 - c_2\omega_2^{a_2(j)-a(\theta_2)})(2n\alpha_1 - Nm(c_1)k(m_1, m_2))$. Note that $Nm_{E_2}(\alpha_1 - c_2\omega_2^{a_2(j)-a(\theta_2)}) = \alpha_1^2 + Nm(c_2\omega_2^{a_2(j)-a(\theta_2)})$. Then we observe that for this particular choice of α_1 we have that

$$P'(\alpha_1) \equiv 2nNm_{E_2}(\alpha_1 - c_2\omega_2^{a_2(j)-a(\theta_2)}) \not\equiv 0 \bmod p,$$

assuming $p \neq 2$ and since n and m_2 are units mod p. Therefore, by Hensel's Lemma, we conclude there are only finitely many solutions. □

Note that the proof works equally well for $\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^j, p^r)$ so we record this in the next lemma.

Lemma 4.4.6. *Assume that π_2 is a principal series representation and $2j = a(\pi_2)$. Assume $0 < r < j$. Then*

$$\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^j, p^r) \ll p^{O(1)}p^r.$$

Lemma 4.4.7. *Assume* $2j < a(\pi_2) - 1$, $0 < r \leq j$ *and* $j > \nu(r)$. *Then*

$$\mathcal{C}(m_1, m_2, n, p^j, p^r) \ll p^{r/2+\nu(r)/2+\inf\{\nu(r)/2, a(\pi_2)/2-j\}} \mathbb{I}(k(m_1, m_2) \equiv 0 \bmod p^{\inf\{\nu(r)/2, a(\pi_2)/2-j\}}).$$

Proof: Set $\lambda_2 = (a(\theta_2) - a_2(j))/e_2$. Following the calculations in Lemma 4.4.5 our simultaneous equations are now

$$\alpha_2 \left(\frac{\mathrm{tr}_{E_1}(\omega_1^{a(\theta_1)-a_1(\pi_2)/2}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha_1))}{Nm(c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)})} + \frac{2(pu_2)^{\lambda_2}\alpha_1}{Nm(c_2 + \alpha_1\omega_2^{a(\theta_2)-a_2(j)})} \right) \equiv k(m_1, m_2) \bmod p^{\nu(r)/2}, \tag{4.21}$$

$$\begin{aligned} &Nm_{E_2}((c_2 + \alpha_1\omega_2^{a(\theta_2)-a_2(j)} - \omega_2^{a(\theta_2)-a_2(r)}\alpha_2)nm_2 \\ &\equiv \alpha_2 Nm_{E_1}(c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)} + \omega_1^{a(\theta_1)-a_1(r)}\alpha_2) \bmod p^{r/2}. \end{aligned} \tag{4.22}$$

Note that we've assumed that $j \geq 4\sigma/5$. Therefore $\lambda_2 < a(\theta_1) - a(\pi_2)/2$. Now observe that a solution exists if and only if $v_p(k(m_1, m_2)) = \inf\{\lambda_2, \nu(r)/2\}$. If $\lambda_2 \geq \nu(r)/2$ then we see that the congruence is degenerate and we have the bound

$$\mathcal{C}(m_1, m_2, n, p^j, p^r) \ll p^{r/2+\nu(r)} \mathbb{I}(k(m_1, m_2) \equiv 0 \bmod p^{\nu(r)/2}).$$

So we assume that $\lambda_2 < \nu(r)/2$. Set K to satisfy $p_2^{\lambda} K = k(m_1, m_2)$. Dividing (4.21) through by p^{λ_2} gives

$$\alpha_2 \left(\frac{\mathrm{tr}_{E_1}(\omega_1^{a(\theta_1)-a(\pi_2)/2-a_1(\lambda_2)}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha_1))}{Nm(c_1 - \alpha_1\omega_1^{a(\theta_1)-a_1(j)})} + \frac{2u_2^{\lambda_2}\alpha_1}{Nm(c_2 + \alpha_1\omega_2^{a(\theta_2)-a_2(j)})} \right) \equiv K \bmod p^{\nu(r)/2-\lambda_2}. \tag{4.23}$$

Reducing both equations mod p gives

$$\alpha_2\alpha_1 u_2^{\lambda_2} \equiv k(m_1, m_2)Nm(c_2) \bmod p,$$

$$Nm(c_2)\alpha_2 nm_2 \equiv Nm(c_1) \bmod p.$$

Substituting α_2 as a function of α_1 and clearing denominators, we see that this is an equation in α_1 with unit coefficients. By Hensel's lemma, the choice of α_1 is therefore

unique upto $p^{\nu(r)/2-\lambda_2}$ and we already know that once α_1 is fixed, α_2 is determined mod $p^{r/2}$. □

Lemma 4.4.8. *Assume $a(\pi_2) < 2j < a(\pi_1) - 1$. Then*

$$\mathcal{C}(m_1, m_2, n, p^j, p^j) \ll p^{O(1)} p^j$$

Proof: Recall that in the case $r = j$ the sum is given by

$$\frac{1}{[\mathfrak{o}_1 : \mathfrak{p}_1^{a(\theta_1,j)}][\mathfrak{o}_2 : \mathfrak{p}_2^{a(\theta_2,j)}] p^{j/2}} \sum_{\chi_1 \in \mathfrak{X}_j} \sum_{\chi_2 \in \mathfrak{X}_j} \sum_{x_1,x_3 \bmod \mathfrak{p}_1^{a(\theta_1,j)}} \sum_{x_2,x_4 \bmod \mathfrak{p}_2^{a(\theta_2,j)}} \theta_1(x_1 x_3^{-1}) \theta_2(x_2 x_4^{-1})$$
$$\times \chi_{1,E_1}^{-1}(x_1 x_3^{-1}) \chi_{1,E_1}(x_2 x_4^{-1}) \chi_{2,E_1}(x_3) \chi_{2,E_2}(x_4^{-1})$$
$$\times \psi_{E_1}(\omega_1^{-a(\theta_1,j)}(x_1 - x_3)) \psi_{E_2}(\omega_2^{-a(\theta_2,j)}(x_2 - x_4)) \chi_1(\overline{m_1} m_2) \chi_2(m_2)$$
$$\times \sum_{z \bmod p^j} \overline{\chi_2}(z) e\left(\frac{zn}{p^j}\right).$$

For $j = 1, 3$ we write $x_j = x_{j,0}(1 + \omega_1^{a(\theta_1,j)/2} x_{j,1})$. For $j = 2, 4$ we write $x_j = x_{j,0}(1 + \omega_2^{a(\theta_2,j)/2} x_{j,1})$. We write $z = z_0(1 + p^{j/2} z_1)$. We write $\chi_i = \chi_{i,0}\chi_{i,1}$ where $\chi_{i,0} \in G_{j/2}$ and $\chi_{i,1} \in \mathfrak{X}_j/\sim j/2$. As before we associate elements in $\chi_i \in \mathfrak{X}_j/\sim j/2$ to elements in $\alpha_i \in (\mathbb{Z}/p^{j/2}\mathbb{Z})^\times$. We once again observe that all the elements $x_{i,1}, \chi_{i,1}$ and z_1 are independent of each other and we may evaluate them. In doing so, we get the following congruences

$$c_1 + x_1 \equiv \alpha_1 \omega_1^{a(\theta_1) - a_1(j)} \bmod \mathfrak{p}_1^{a(\theta_1)/2},$$

$$c_2 \omega_2^{a_2(j) - a(\theta_2)} + x_2 \equiv \alpha_1 \bmod \mathfrak{p}_2^{a_2(j)/2},$$

$$c_1 + x_3 \equiv (\alpha_2 - \alpha_1) \omega_1^{a(\theta_1) - a_1(j)} \bmod \mathfrak{p}_1^{a(\theta_1)/2},$$

$$c_2 \omega_2^{a_2(j) - a(\theta_2)} + x_4 \equiv \alpha_2 - \alpha_1 \bmod \mathfrak{p}_2^{a_2(j)/2},$$

$$Nm_{E_2}(x_2 x_4^{-1}) m_2 \equiv Nm_{E_1}(x_1 x_3^{-1}) m_1 \bmod p^{j/2},$$

$$Nm_{E_1}(x_3) m_2 \equiv Nm_{E_2}(x_4) z \bmod p^{j/2},$$

$$\alpha_2 \equiv nz \bmod p^{j/2}.$$

We now observe that if we fix (α_1, α_2) then we determine all other free variables. We make the change of variable $\alpha_2 \mapsto \alpha_1 - \alpha_2$. This imposes the additional condition that $\alpha_1 - \alpha_2$ is a unit. Note that α_2 is a unit because we required that $\chi_1\overline{\chi_2}$ also be primitive mod p^j. The equations can be rewritten as

$$c_1 + x_1 \equiv \alpha_1 \omega_1^{a(\theta_1)-a_1(j)} \bmod \mathfrak{p}_1^{a(\theta_1)/2},$$

$$c_2\omega_2^{a_2(j)-a(\theta_2)} + x_2 \equiv \alpha_1 \bmod \mathfrak{p}_2^{a_2(j)/2},$$

$$c_1 + x_3 \equiv -\alpha_2\omega_1^{a(\theta_1)-a_1(j)} \bmod \mathfrak{p}_1^{a(\theta_1)/2},$$

$$c_2\omega_2^{a_2(j)-a(\theta_2)} + x_4 \equiv -\alpha_2 \bmod \mathfrak{p}_2^{a_2(j)/2},$$

$$Nm_{E_2}(x_2x_4^{-1})m_2 \equiv Nm_{E_1}(x_1x_3^{-1})m_1 \bmod p^{j/2},$$

$$Nm_{E_1}(x_3)m_2 \equiv Nm_{E_2}(x_4)z \bmod p^{j/2},$$

$$\alpha_1 - \alpha_2 \equiv nz \bmod p^{j/2}.$$

Substituting the values of z, $x_1, ..., x_4$ we get the simultaneous equation

$$f(\alpha_1)nm_1 \equiv g(\alpha_1)(\alpha_1 - \alpha_2) \bmod p^{j/2}.$$

$$f(\alpha_2)nm_2 \equiv g(\alpha_2)(\alpha_1 - \alpha_2) \bmod p^{j/2}$$

where

$$f(\beta) = Nm_{E_1}(\beta\omega_1^{a(\theta_1)-a_1(j)} - c_1),$$

$$g(\beta) = Nm_{E_2}(\beta - c_2\omega_2^{a_2(j)-a(\theta_2)}).$$

we thus have a simultaneous equation in two variables. If the Jacobian of these equations doesn't vanish mod p then by Hensel's Lemma 2.1.5 we conclude the lift is unique and therefore there are only $O(1)$ many solutions.

Set

$$F_1(\beta_1, \beta_2) := f(\beta_1)nm_1 - g(\beta_1)(\beta_1 - \beta_2),$$

$$F_2(\beta_1, \beta_2) := f(\beta_2)nm_2 - g(\beta_2)(\beta_1 - \beta_2).$$

We then have

$$\frac{\partial F_1}{\partial \beta_1} = f'(\beta_1)nm_1 - g'(\beta_1)(\beta_1 - \beta_2) - g(\beta_1),$$

$$\frac{\partial F_2}{\partial \beta_2} = f'(\beta_2)nm_2 - g'(\beta_2)(\beta_1 - \beta_2) + g(\beta_2),$$

$$\frac{\partial F_1}{\partial \beta_2} = g(\beta_1),$$

$$\frac{\partial F_2}{\partial \beta_1} = -g(\beta_2).$$

The Jacobian J is now given by

$$\begin{aligned}
&\det \begin{pmatrix} \frac{\partial F_1}{\partial \beta_1} & \frac{\partial F_2}{\partial \beta_1} \\ \frac{\partial F_1}{\partial \beta_2} & \frac{\partial F_2}{\partial \beta_2} \end{pmatrix} \\
=&n^2 m_1 m_2 f'(\beta_1) f'(\beta_2) + g'(\beta_1) g'(\beta_2)(\beta_1 - \beta_2)^2 \\
&- (\beta_1 - \beta_2)\Big(nm_1 f'(\beta_1) g'(\beta_2) + nm_2 f'(\beta_2) g'(\beta_1) \\
&+ g'(\beta_1) g(\beta_2) - g(\beta_1) g'(\beta_2) \Big) \\
&+ nm_1 f'(\beta_1) g(\beta_2) - nm_2 f'(\beta_2) g(\beta_1).
\end{aligned}$$

Notice the cancellation of the term $g(\beta_1)g(\beta_2)$. Crucially, since $a_1(j) < a(\pi_1) - 1$ we observe that mod p, we see that

$$F_1 \equiv c_1' nm_1 - g(\beta_1)(\beta_1 - \beta_2) \bmod p,$$

$$F_1 \equiv c_1' nm_2 - g(\beta_2)(\beta_1 - \beta_2) \bmod p,$$

so that

$$f'(\beta_1) \equiv f'(\beta_2) \equiv 0 \bmod p.$$

If in addition $J(\alpha_1, \alpha_2) \equiv 0 \bmod p^r$ for some $1 \leq r \leq j/2$ then we must have

$$g'(\alpha_1)g'(\alpha_2) + R(\alpha_1, \alpha_2) \equiv 0 \bmod p,$$

where $R(\beta_1, \beta_2)$ satisfies

$$g(\beta_1)g'(\beta_2) - g'(\beta_1)g(\beta_2) = (\beta_1 - \beta_2)R(\beta_1, \beta_2).$$

We now use the fact that $g(\alpha)$ is a quadratic polynomial. Using the fact that $tr(c_2) = 0$ we have

$$g(\beta) = Nm_{E_2}(\omega_2^{a(\theta_2,j)-a_2(j)}\beta) + Nm_{E_2}(c_2\omega_2^{a(\theta_2,j)-a(\theta_2)}).$$

Therefore, when $a_2(j) \geq a(\theta_2)$ we get that

$$g(\beta) = \beta^2 + Nm(c_2\omega_2^{a_2(j)-a(\theta_2)}),$$

$$g'(\beta) = 2\beta.$$

Note that $Nm(c_2\omega_2^{a_2(j)-a(\theta_2)})$ depends on $a_2(j) - a(\theta_2)$. Substituting this into the Jacobian equation, we get

$$-2\frac{g(\alpha_1)\alpha_2 - g(\alpha_2)\alpha_1}{\alpha_1 - \alpha_2} \equiv 4\alpha_1\alpha_2 \bmod p.$$

An algebraic manipulation shows

$$g(\alpha_1)\alpha_2 - g(\alpha_2)\alpha_1 = (\alpha_1 - \alpha_2)(\alpha_1\alpha_2 - Nm(c_2\omega_2^{a_2(j)-a(\theta_2)}))$$

and therefore we require that

$$-\alpha_1\alpha_2 + Nm(c_2\omega_2^{a_2(j)-a(\theta_2)}) \equiv 2\alpha_1\alpha_2 \bmod p$$

which can be rewritten as

$$\alpha_1\alpha_2 \equiv \frac{Nm(c_2\omega_2^{a_2(j)-a(\theta_2)})}{3} \bmod p.$$

Since $a_2(j) > a(\theta_2)$, $Nm(c_2\omega_2^{a_2(j)-a(\theta_2)}) \equiv 0 \bmod p$. Then $J \equiv 0 \bmod p$ is equivalent to $\alpha_1\alpha_2 \equiv 0 \bmod p$. Since both α_1 and α_2 are units, $J \equiv 0 \bmod p$ cannot occur. This concludes the proof. □

Lemma 4.4.9. *Assume π_2 is supercuspidal and $2j = a(\pi_2)$ or $2j+1 = a(\pi_2)$. Let $0 \le t \le j/2$ be the largest integer satisfying the relation*

$$\begin{aligned} &m_1m_2n(Nm(c_1)^2 - m_1m_2Nm(c_2)n^2) \\ &+\mathrm{tr}(c_1)(pu_1)^{(a(\theta_1)-a_1(j))/e_1}(m_2-m_1)(Nm(c_1)^2 - 2m_1m_2Nm(c_2)n^2) \equiv 0 \bmod p^t. \end{aligned} \tag{4.24}$$

Then

$$\mathcal{C}(m_1, m_2, n, p^j, p^j) \ll p^{j+t/2}.$$

Proof: Recall that the simultaneous equations are now given by

$$F_1 : f(\alpha_1)nm_1 \equiv g(\alpha_1)(\alpha_1 - \alpha_2) \bmod p^{j/2}.$$

$$F_2 : f(\alpha_2)nm_2 \equiv g(\alpha_2)(\alpha_1 - \alpha_2) \bmod p^{j/2}$$

with

$$f(\alpha) = Nm_{E_1}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha),$$

$$g(\alpha) = \alpha^2 + Nm(c_2).$$

Now we use the fact that $j \le \frac{2l_1}{5}$. This implies that

$$Nm_{E_1}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha_1) \equiv Nm(c_1) - (pu_1)^r tr(c_1)\alpha_1 \bmod p^{j/2},$$

where $r = \frac{a(\theta_1)-a_1(j)}{e} \geq j/4$. For notational simplicity, we set $Nm(c_1) = N_1$ and $Nm(c_2) = N_2$ and $\mathrm{tr}(c_1) = B$. Then we observe that from F_1 and F_2, we have the relation

$$\frac{\alpha_1^2 + N_2}{N_1 - (pu_1)^r B\alpha_1} m_1 \equiv \frac{\alpha_2^2 + N_2}{N_1 - (pu_1)^r B\alpha_2} \bmod p^{j/2}. \tag{4.25}$$

From F_1 we rewrite α_2 as a function of α_1

$$\alpha_2 \equiv \alpha_1 - \frac{\alpha_1^2 + N_2}{N_1 - (pu_1)^r B\alpha_1} n m_1 \bmod p^{j/2}.$$

Then substituting into (4.25) we get

$$m_1 \frac{g(\alpha_1)}{f(\alpha_1)} \equiv m_2 \frac{g}{f}\left(\alpha_1 - n m_1 \frac{g(\alpha_1)}{f(\alpha_1)}\right) \bmod p^{j/2}.$$

In terms of polynomials in f and g, the right-hand side becomes

$$m_2 \frac{g(\alpha_1) - 2\alpha_1 n m_1 \frac{g}{f}(\alpha_1) + n^2 m_1^2 \frac{g^2}{f^2}(\alpha_1)}{f(\alpha_1) + (pu_1)^r B n m_1 \frac{g}{f}(\alpha_1)}.$$

Note that this uses explicitly the shape of $g(\alpha) = \alpha^2 + c$. For different $g(\alpha)$ we get different relations.

Observe that both $g(\alpha)$ and $f(\alpha)$ are units. We then take out a factor of $\frac{g}{f}(\alpha_1)$ and cancel it against the left-hand side getting the equation

$$m_1 \equiv m_2 \frac{1 - 2\alpha_1 n m_1 \frac{1}{f}(\alpha_1) + n^2 m_1^2 \frac{g}{f^2}(\alpha_1)}{1 + (pu_1)^r B n m_1 \frac{g}{f^2}(\alpha_1)} \bmod p^{j/2}.$$

We now cross-multiply to get

$$m_1\left(1 + (pu_1)^r B n m_1 \frac{g}{f^2}(\alpha_1)\right) \equiv m_2\left(1 - 2\alpha_1 n m_1 \frac{1}{f}(\alpha_1) + n^2 m_1^2 \frac{g}{f^2}(\alpha_1)\right) \bmod p^{j/2}.$$

Moving all terms to one side and clearing f^2 from the denominators gives

$$(m_2 - m_1)f(\alpha_1)^2 - 2nm_1m_2\alpha_1 f(\alpha_1) + n^2m_1^2m_2g(\alpha_1) - (pu_1)^r Bnm_1^2 g(\alpha_1) \equiv 0 \bmod p^{j/2}.$$

We now substitute in the explicit polynomial for f and g. Using that $r \geq j/4$ so that $2r \geq j/2$ we see that we get a quadratic polynomial in α_1 -

$$A_2\alpha_1^2 + A_1\alpha_1 + A_0 \equiv 0 \bmod p^{j/2},$$

$$A_2 = m_1^2(m_2n^2 + B(pu_1)^r n) + 2m_1m_2B(pu_1)^r n,$$

$$A_1 = -2N_1(m_1m_2n + (m_2 - m_1)B(pu_1)^r),$$

$$A_0 = m_1^2N_2(m_2n^2 - B(pu_1)^r n) + (m_2 - m_1)N_1^2.$$

We then have that the discriminant of the quadratic is given by

$$\Delta := A_1^2 - 4A_2A_0$$

which can be explicitly written out as

$$\begin{aligned}\Delta =& 4m_1^2 n\Big\{m_1m_2n(N_1^2 - m_1m_2N_2n^2) \\ &+ B(pu_1)^r(m_2 - m_1)(N_1^2 - 2m_1m_2N_2n^2)\}\end{aligned}$$

Now it's well-known that if $p^t||\Delta$ then the number of solutions to α_1 is bounded by

$$\begin{cases} 2p^{t/2} & \text{if } t \equiv 0 \bmod 2 \\ 0 & \text{otherwise} \end{cases}.$$

Since p is odd and m_1 and n are units, we have that if $\Delta \equiv 0 \bmod p^t$ then the congruence

$$m_1m_2n(N_1^2 - m_1m_2N_2n^2) + B(pu_1)^r(m_2 - m_1)(N_1^2 - 2m_1m_2N_2n^2) \equiv 0 \bmod p^t$$

must be satisfied. □

Note that the proof of Lemma 4.4.9 works equally well for $\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^j, p^j)$. So we record this statement in the next lemma.

Lemma 4.4.10. *Assume π_2 is a principal series representation with $2j = a(\pi_2)$. Let $0 \leq t \leq j/2$ be the largest integer satisfying the relation*

$$\begin{aligned} &m_1m_2n(Nm(c_1)^2 - m_1m_2Nm(c_2)n^2) \\ &+\mathrm{tr}(c_1)(pu_1)^{(a(\theta_1)-a_1(j))/e_1}(m_2 - m_1)(Nm(c_1)^2 - 2m_1m_2Nm(c_2)n^2) \equiv 0 \bmod p^t. \end{aligned} \tag{4.26}$$

Then

$$\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^j, p^j) \ll p^{j+t/2}.$$

Let $h(m_1, m_2, n)$ be the polynomial in m_1, m_2 and n in (4.26). We now show that if we fix m_1 and m_2, the number of solutions in $n \bmod p^t$ is at-most $O(1)$.

Proposition 4.4.11. *Let $(m_1m_2, p) = 1$. Let $t \geq 0$ be an integer. The number of solutions $\beta \in (\mathbb{Z}/p^t\mathbb{Z})^\times$ to the equation $h(m_1, m_2, \beta) \equiv 0 \bmod p^t$ is $O(1)$.*

Proof: If $t = 0$ there is nothing to prove so assume that $t > 0$. Set $\mathrm{tr}(c_1) = p^sB$ with $s \leq j/2$ and B a unit mod p. If $t \leq r + s$ then we have

$$m_1m_2\beta(N_1^2 - m_1m_2N_2\beta^2) \equiv 0 \bmod p^t.$$

Since m_1, m_2 and β are units, we require that

$$N_1^2 - m_1m_2N_2\beta^2 \equiv 0 \bmod p^t. \tag{4.27}$$

By Hensel's Lemma, there are at-most 2 solutions in β. Now assume $t > r + s$. Necessarily, β must satisfy (4.27) mod p^{r+s}. We then observe that this is a cubic in β. Reducing mod p, it can be rewritten as

$$h(m_1, m_2, \beta) \equiv m_1m_2N_1^2\beta - (m_1m_2)^2N_2\beta^3 \bmod p.$$

We thus have

$$h'(m_1, m_2, \beta) \equiv m_1 m_2 (N_1^2 - 3m_1 m_2 N_2 \beta^2) \bmod p.$$

We use (4.27) to deduce that

$$N_1^2 - 3m_1 m_2 N_2 \beta^2 \equiv -2N_1^2 \bmod p.$$

Therefore, $h' \not\equiv 0 \bmod p$ so by Hensel's Lemma, there is at-most $O(1)$ (in fact at-most 3) solutions. □

Equally well, we could have shown that the same is true for m_1 and m_2 assuming all the other variables were fixed, but we do not pursue this.

Now we treat the cases $\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^v, p^v)$ with $v < j$. Recall that $\pi_2 = \pi(\chi, \chi^{-1}\psi)$ where $a(\chi) = j$ and $\psi^2 = 1$. Let β be the constant associated to χ. Following the proof of Lemma 4.4.9 the simultaneous equations become

$$Nm_{E_1}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha_1)m_1 n \equiv -\alpha_1(\beta - p^{j-v}\alpha_1)(\alpha_1 - \alpha_2) \bmod p^{v/2},$$

$$Nm_{E_1}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha_1)m_1 n \equiv -\alpha_2(\beta - p^{j-v}\alpha_2)(\alpha_1 - \alpha_2) \bmod p^{v/2}.$$

Note that here c_1 also depends on χ coming from π_2 but this fact won't be important. We thus get

$$\frac{Nm_{E_1}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha_1)}{\alpha_1(\beta - p^{j-v}\alpha_1)} m_1 \equiv \frac{Nm_{E_1}(c_1 - \omega_1^{a(\theta_1)-a_1(j)}\alpha_2)}{\alpha_2(\beta - p^{j-v}\alpha_2)} m_2 \bmod p^{v/2}. \tag{4.28}$$

Setting $t = \frac{a(\theta_1)-a_1(j)}{e_1} \geq j/4$ and

$$f(\alpha) = Nm_{E_1}(c - \omega_1^{a(\theta_1)-a_1(j)}\alpha),$$

$$g(\alpha) = -\alpha(\beta - p^{j-v}\alpha),$$

$$B = \mathrm{tr}(c_1),$$

then following calculations in Lemma 4.4.9 we see that the right-hand side of the 4.28 becomes

$$m_2 \frac{g(\alpha_1) + nm_1(\beta - 2p^{j-v}\alpha_1)\frac{g}{f}(\alpha_1) + p^{j-v}n^2m_1^2\frac{g^2}{f^2}(\alpha_1)}{f(\alpha_1) + (pu_1)^t Bnm_1\frac{g}{f}(\alpha_1)}.$$

Since $f(\alpha)$ and $g(\alpha)$ are both units, we clear it from both sides and get the equation

$$m_1 \equiv m_2 \frac{1 + nm_1(\beta - 2p^{j-v}\alpha_1)\frac{1}{f}(\alpha_1) + p^{j-v}n^2m_1^2\frac{g}{f^2}(\alpha_1)}{1 + (pu_1)^t Bnm_1\frac{g}{f^2}(\alpha_1)} \bmod p^{v/2}.$$

Clearing denominators and rearranging we get the equation

$$(m_1-m_2)f(\alpha_1)^2 - nm_1m_2(\beta-2p^{j-v}\alpha_1)f(\alpha_1) + ((pu_1)^t Bnm_1^2 - p^{j-v}n^2m_1^2m_2)g(\alpha_1) \equiv 0 \bmod p^{v/2}.$$

This is a quadratic in α_1 which we write as

$$A_2\alpha_1^2 + A_1\alpha_1 + A_0 \equiv 0 \bmod p^{v/2},$$

$$A_2 = nm_1p^{j-v}\Big\{(pu_1)^t B(m_1 - 2m_2) - nm_1m_2p^{j-v}\Big\},$$

$$A_1 = (2Nm(c_1) + \beta nm_1)\Big\{p^{j-v}nm_1m_2 - (m_1 - m_2)(pu_1)^t B\Big\},$$

$$A_0 = Nm(c_1)\{(m_1 - m_2)Nm(c_1) - nm_1m_2\beta\}.$$

Recall that $\mathrm{tr}(c_1) \equiv 0 \bmod p$. Let $s \leq v/2$ be the largest integer such that $\mathrm{tr}(c_1) \equiv 0 \bmod p^s$. Set $B = B_0p^s$ with B_0 a unit mod p. We observe that the congruence has no solutions unless

$$A_0 \equiv 0 \bmod p^{\inf\{t+s, j-v\}}.$$

This implies that

$$nm_1m_2\beta \equiv (m_1 - m_2)Nm(c_1) \bmod p^{\inf\{t+s, j-v\}},$$

which we assume from now on. We require the p-adic divisibility of A_i. Set p_i^δ to satisfy

$p^{\delta_i}||A_i$. By assumption, $u = \inf\{t-s, j-v\} \leq \delta_i$. Let $A_{i,red}$ be units such that $A_i = p^{\delta_i}A_{i,red}$. Then our equations in α becomes

$$A_{0,red}p^{\delta_0} + A_{1,red}p^{\delta_1}\alpha + A_{2,red}p^{\delta_2}\alpha^2 \equiv 0 \bmod p^{v/2}.$$

Let $\delta = \inf\{\delta_i\}_{i=0,1,2}$. Then we reduce the equation mod p^δ and get

$$A_0'p^{\delta_0-\delta} + A_{1,red}p^{\delta_1-\delta}\alpha + A_{2,red}p^{\delta_2-\delta}\alpha^2 \equiv 0 \bmod p^{v/2-\delta}.$$

If $\delta \geq v/2$ then the equation is degenerate in which case we trivially count the number of α. So assume $\delta < v/2$. We now split into various cases depending on $\delta = \delta_i$. First suppose $\delta = \delta_0$. If $\delta_1 > \delta_0$ then we observe that mod p this becomes

$$A_0' + A_{2,red}p^{\delta_2-\delta_0}\alpha^2 \equiv 0 \bmod p.$$

If $\delta_2 > \delta$ then there are no solutions. If $\delta_2 = \delta_0$ there are at-most 2 solutions mod $p^{v/2-\delta}$ by Hensel's Lemma. So now assume that $\delta_1 = \delta_0$. If $\delta_2 > \delta_0$ there is at-most one solution mod $p^{v/2-\delta}$ by Hensel's Lemma. If $\delta_2 = \delta$ then this is a quadratic in α with all coefficients units. The number of solutions is controlled by $p^t||\Delta_{red}$ for $0 \leq t \leq v/2-\delta$ where

$$\Delta_{red} = (A_{1,red})^2 - 4A_{0,red}A_{2,red},$$

and is given by

$$\begin{cases} 2p^{s/2} & \text{if } s \equiv 0 \bmod 2 \\ 0 & \text{otherwise} \end{cases}.$$

Now assume that $\delta = \delta_1$. We've already covered the case $\delta = \delta_1$ so we assume that $\delta_0 > \delta$. Then we observe that mod p the equation becomes

$$\alpha(A_{1,red} + A_{2,red}p^{\delta_2-\delta}\alpha) \equiv 0 \bmod p,$$

and since α is a unit this becomes

$$A_{1,red} + A_{2,red} p^{\delta_2 - \delta} \alpha \equiv 0 \bmod p.$$

If $\delta_2 > \delta$ there are no solutions. If $\delta_2 = \delta$ then by Hensel's Lemma there is at-most one solution mod $p^{v/2-\delta}$.

Finally, assume that $\delta = \delta_2$. We've already calculated the cases when $\delta_0 > \delta, \delta_1 = \delta$ and $\delta_0 = \delta, \delta_1 > \delta$ and $\delta_1 = \delta_2 = \delta$. So assume that $\delta_0 > \delta, \delta_1 > \delta$. Then reducing mod p we see that

$$A_{2,red} \alpha^2 \equiv 0 \bmod p$$

which has no solutions.

Now these equations appear extremely complicated to solve in general, but we'll only require the case when $j - v < (a(\theta_1) - a_1(j))/e_1$.

Looking at the equation for A_2, we observe that $\delta_2 = 2(j-v)$ and hence $j - v \leq \delta \leq 2(j-v)$. If $\delta < 2(j-v)$ then a solution exists if and only if $\delta = \delta_0 = \delta_1$ and there is at-most one solution. If $j - v < \delta_0 < 2(j-v)$ then from the equation defining A_2 we require that $p^{\delta-(j-v)} || (2Nm(c_1) + \beta n m_1)$ for a solution to exist. Note that from $A_0 \equiv 0 \bmod p^{\delta_0}$ this translates to $p^{\delta-(j-v)} || (m_1 + m_2)$.

Now assume that $\delta = 2(j-v)$. Again, a solution exists if and only if at-least one of $\delta_0 = \delta$ or $\delta_1 = \delta$ and there is at-most one solution if one $\delta_0 > \delta, \delta_1 = \delta$ or $\delta = \delta_0, \delta_1 > \delta$. We record these observations in the following lemma.

Lemma 4.4.12. *Assume π_2 is a principal series representation. Assume $2j = a(\pi_2)$ and $v < j$ with $v - j < \frac{a(\theta_1) - a_1(j)}{e_1}$. Let δ_0 be the largest integer such that $p^{\delta_0} | (nm_1m_2\beta - (m_1 - m_2)Nm(c_1))$. The character sum vanishes unless $\delta_0 \geq j - v$.*

(1) If $j - v < \delta_0 < 2(j-v)$ then the character sum vanishes unless $p^{\delta_0-(j-v)} || (m_1 + m_2)$. In that case we have the bound

$$\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^v, p^v) \ll p^{v + \inf\{v/2, \delta_0\}} \mathbb{I}(p^{\delta_0} | (nm_1m_2\beta - (m_1 - m_2)Nm(c_1))) \times \mathbb{I}(p^{\delta_0-(j-v)} | (m_1 + m_2)).$$

(2) Assume $\delta_0 > 2(j-v)$. Then the character sum vanishes unless $p^{j-v}||(m_1+m_2)$. In that case we have the bound

$$\mathcal{C}_{pr}(m_1,m_2,n,p^j,p^v,p^v) \ll p^{v+\inf\{v/2,2(j-v)\}}\mathbb{I}(p^{\inf\{v/2,\delta_0\}}|(nm_1m_2\beta-(m_1-m_2)Nm(c_1)))$$
$$\times \mathbb{I}(p^{\inf\{v/2,j-v\}}|(m_1+m_2)).$$

(3) Assume $\delta_0 = 2(j-v)$. Set δ_1 to be the largest integer satisfying $p^{\delta_1}|(2Nm(c_1)+\beta nm_1)$. Then the character sum vanishes unless $\delta_1 \geq j-v$.

(i) Assume $\delta_1 > j-v$. We have bound

$$\mathcal{C}_{pr}(m_1,m_2,n,p^j,p^v,p^v) \ll p^{v+\inf\{v/2,\delta_1\}}\mathbb{I}(p^{\inf\{v/2,2(j-v)\}}|(nm_1m_2\beta-(m_1-m_2)Nm(c_1))$$
$$\times \mathbb{I}(p^{\inf\{v/2,\delta_1\}}|(2Nm(c_1)+\beta nm_1)).$$

(ii) Assume $\delta_1 = j-v$. Let $0 \leq s \leq v-2(j-v)$ be the largest integer satisfying

$$\Delta_{red} \equiv 0 \bmod p^s.$$

We have bound

$$\mathcal{C}_{pr}(m_1,m_2,n,p^j,p^v,p^v) \ll p^{v+\inf\{v/2,2(j-v)\}}\mathbb{I}(p^{\inf\{v/2,2(j-v)\}}|(nm_1m_2\beta-(m_1-m_2)Nm(c_1))$$
$$\times \mathbb{I}(p^{\inf\{v/2,j-v\}}|(m_1+m_2)$$
$$\times p^{s/2}\mathbb{I}(p^s|\Delta_{red}).$$

Lemma 4.4.13. *Assume $\delta_0=\delta_1=\delta_2=2(j-v)$. Assume $0 \leq s \leq v/2-\delta$. Assume m_1 and m_2 are integers with $(m_1m_2,p)=1$. The number of $n \in (\mathbb{Z}/p^{\delta+s}\mathbb{Z})^\times$ that satisfy the congruences $A_0, A_1 \equiv 0 \bmod p^{2\delta}$ and $\Delta_{red} \equiv 0 \bmod p^s$ is $O(1)$.*

Proof: We want to apply Hensel's Lemma but to do so we'll first have to separate n from the congruences coming from $A_0, A_1 \equiv 0 \bmod p^{2(j-v)}$. Set $n = n_0 + p^{2(j-v)}n_1$. Then it is easy to see that n_1 is independent of the congruences defined by A_0 and A_1. Note that n_1 may not be a unit mod p any more. Clearly, the choice of n_0 is unique. Let $\Delta(n) = A_1(n)^2 - 4A_0(n)A_2(n)$ be the un-reduced discriminant and $\Delta_{red}(n) = A_{1,red}^2(n) -$

$4A_{0,red}(n)A_{2,red}(n)$ the reduced discriminant, where we now treat them as functions of n assuming m_1 and m_2 fixed. Then we have

$$\Delta_{red}(n) = \frac{\Delta(n)}{p^{4(j-v)}}.$$

We have the formal p-adic expansion

$$\Delta(n) = \Delta(n_0 + p^{2(j-v)}n_1) = \Delta(n_0) + \sum_{k=1}^{\infty} \frac{1}{k!}\Delta^{(k)}(n_0)(p^{2(j-v)}n_1)^k.$$

Note that mod p^s for any $s \in \mathbb{N}$, this is a finite expansion. To study $\Delta_{red}(n)$, it is enough to study $\frac{\Delta(n)}{p^{4(j-v)}}$. We observe that for $k \geq 3$, the coefficient of n_1^k is divisible by $p^{5(j-v)}$. We next study the p-adic divisibility of $\Delta'(n)$ and $\Delta''(n)$. By definition

$$\Delta'(n) = 2A_1'(n)A_1(n) - 4(A_0(n)A_2'(n) + A_0'(n)A_2(n)),$$

$$\Delta''(n) = 2A_1''(n)A_1(n) + 2(A_1'(n))^2 - 4(A_0(n)A_2''(n) + 2A_0'(n)A_2'(n) + A_0''(n)A_2(n)).$$

One can explicitly calculate all the derivatives of A_i as functions of n. These are

$$A_0'(n) = m_1m_2\beta, \quad A_0''(n) = 0,$$

$$A_1'(n) = \beta m_1\left\{p^{j-v}nm_1m_2 - (m_1 - m_2)(pu_1)^tB\right\} + p^{j-v}m_1m_2(2Nm(c_1) + \beta nm_1),$$

$$A_1''(n) = 2\beta p^{j-v}m_1^2m_2,$$

$$A_2'(n) = m_1p^{j-v}\left\{(pu_1)^tB(m_1 - 2m_2) - nm_1m_2p^{j-v}\right\} - nm_1^2m_2p^{2(j-v)},$$

$$A_2''(n) = -2m_1^2m_2p^{2(j-v)}.$$

With these calculations, we observe that $p^{2(j-v)}|\Delta''(n)$ for any n. On the other hand,

$$v_p(A_1'(n_0)A_1(n_0)), v_p(A_0(n_0)A_2'(n_0)) \geq 3(j-v)$$

while

$$v_p(A_0'(n_0)A_2(n_0)) = 2(j-v),$$

so

$$v_p(\Delta'(n_0)) = 2(j-v).$$

Therefore, when we divide $\Delta(n)$ by $p^{4(j-v)}$, we see that

$$\frac{\Delta(n)}{p^{4(j-v)}} \equiv U(n_0) + V(n_0)n_1 + H(n_0, n_1) \bmod p^s$$

where H is a polynomial in n with coefficients that are functions of n_0, all divisible by p^{j-v}, $U(n_0)$ is a function of n_0 only, and $V(n_0)$ is a unit. Reducing mod p, the equation then becomes

$$U(n_0) + V(n_0)n_1 \bmod p.$$

Differentiating with respect to n_1, we see that since $V(n_0$ is a unit mod p, the derivative is a unit mod p. By Hensel's Lemma, we thus see that the lift of n_1 is therefore unique mod p^s. □

Now we treat the case $\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^r, p^v)$ with $0 < v < r < j$.

Lemma 4.4.14. *Assume that $0 < r < v < j$. We have bound*

$$\begin{aligned}\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^v, p^r) \ll & p^{r+\inf\{r/2, j-v\}} \\ & \times \mathbb{I}(k(m_1, m_2)Nm(c_1) - nm_2\beta(1 - p^{v-r}k(m_1, m_2)) \\ & \equiv 0 \bmod p^{\inf\{r/2, j-v\}})\end{aligned}$$

.

Proof: The simultaneous equations become

$$\alpha_2\left(\frac{\mathrm{tr}\omega^{a(\theta_1)-a_1(v)}(c_1 - \omega_1^{a(\theta_1)-a_1(r)}\alpha_1)}{Nm(c_1 - \omega_1^{a(\theta_1)-a_1(r)}\alpha_1)} + \frac{p^{j-v}}{\beta - p^{j-r}\alpha_1} + \frac{1}{\alpha_1}\right) \equiv k(m_1, m_2) \bmod p^{r/2}$$

$$\alpha_2 Nm((c_1 - \omega^{a(\theta_1)-a_1(v)}(\alpha_1 - p^{v-r}\alpha_2)) \equiv nm_2(\beta + p^{j-v}(\alpha_1 - p^{v-r}\alpha_2))(\alpha_1 - p^{v-r}\alpha_2) \bmod p^{r/2}.$$

Note that there are no solutions unless $k(m_1, m_2)$ is a unit mod p which we now assume. Set

$$\alpha_2 = \frac{k(m_1, m_2)}{g(\alpha_1)} \bmod p^{r/2}.$$

Set $\lambda_1 = (a(\theta_1) - a_1(v))/e_1$. Set $q = \sup\{j - v, v - r\}$. For the moment assume that $q < r/2$. Then $\lambda_1 > q$. We reduce both equations mod p^{q+1}. Setting $K = k(m_1, m_2)$, the first equation becomes

$$\alpha_2 \frac{\beta - p^{j-v}\alpha_1(1 - p^{v-r})}{\alpha_1(\beta - p^{j-v}\alpha_1} \equiv K \bmod p^{q+1} \tag{4.29}$$

while the second equation becomes

$$\alpha_2 Nm(c_1) \equiv nm_2(\alpha_1 - p^{v-r}\alpha_2)(\beta - p^{j-v}(\alpha_1 - p^{v-r}\alpha_2) \bmod p^{q+1} \tag{4.30}$$

Substituting (4.29) into (4.30) we get a single polynomial in α_1 -

$$nm_2 p^{j-v}\alpha_1^2 + \left(KNm(c_1) - nm_2\beta(1 - p^{v-r}K\right)\alpha_1 \equiv 0 \bmod p^{q+1} \tag{4.31}$$

Since α_1 is a unit, dividing through by α_1 gives the condition

$$nm_2 p^{j-v}\alpha_1 + \left(KNm(c_1) - nm_2\beta(1 - p^{v-r}K\right) \equiv 0 \bmod p^{q+1}$$

Reducing mod $j - v$ gives the condition

$$KNm(c_1) - nm_2\beta(1 - p^{v-r}K) \equiv 0 \bmod p^{j-v}.$$

Set $KNm(c_1) - nm_2\beta(1-p^{v-r}K) = p^{j-v}C$ for some constant C. Dividing the congruence through by p^{j-v} gives the residual congruence

$$nm_2\alpha_1 + C \equiv 0 \bmod p^{q+1-(j-v)} \tag{4.32}$$

which α_1 must satisfy. Now substituting $KNm(c_1) - \beta(1-p^{v-r}K) = p^{j-v}C$ into (4.31) we get the congruence

$$nm_2p^{j-v}\alpha_1^2 + p^{j-v}C\alpha_1 \equiv 0 \bmod p^{q+1}$$

Dividing through by p^{j-v} gives

$$nm_2\alpha_1^2 + C\alpha_1 \equiv 0 \bmod p^{q+1-(j-v)}$$

We then differentiate in α_1 which gives

$$2nm_2\alpha_1 + C \bmod p$$

which at a solution is a unit mod p. Hence by Hensel's lemma, the lift of α_1 to $p^{r/2-(j-v)}$ is unique. We thus get the bound

$$\begin{aligned}\mathcal{C}_{pr}(m_1, m_2, n, p^j, p^v, p^r) \ll & p^{r+\inf\{r/2, j-v\}} \times \mathbb{I}(KNm(c_1) - \beta(1-p^{v-r}K) \\ & \equiv 0 \bmod p^{\inf\{r/2, j-v\}})\end{aligned}$$

as desired. □

4.4.3 The Zero Frequency - $\Sigma^{=}$

First assume that $2j \geq a(\pi_2) - 1$ and $j > \sigma$. Using Lemma 4.4.1 and the Rankin-Selberg bound 2.13, if $N_0 \ll N_C$ then

$$\begin{aligned}\Sigma^{=} &\ll \sum_{\mu=0,1} \frac{p^{j-\mu}N_0}{d} \sum_{\substack{m_1,m_2 \ll M_C \\ m_1 \equiv m_2 \bmod p^{j-\mu}d}} \frac{|\lambda_g(m_1)\lambda_g(m_2)|}{|m_1 m_2|^{1/4}} \sum_{|n| \ll N_1(0)} |\mathcal{I}(m_1, m_2, n, p^j, q)| \\ &\ll \sum_{\mu=0,1} \frac{p^{j-\mu}N_0}{d} \sum_{|n| \ll N_1(0)} \sum_{\substack{m_1,m_2 \ll M_C \\ m_1 \equiv m_2 \bmod p^{j-\mu}d}} |\mathcal{I}(m_1, m_2, n, p^j, q)| \left(\frac{|\lambda_g(m_1)|^2}{m_1^{1/2}} + \frac{|\lambda_g(m_2)|^2}{m_2^{1/2}} \right) \\ &\ll \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma+t}CQ}{N} \right)^2 \times X_C^2 \\ &\times \left(\frac{M_C^{3/2}}{p^j d} + M_C^{1/2} \right) \times N_1(0),\end{aligned} \tag{4.33}$$

and if $N_0 \gg N_C$ then

$$\begin{aligned}\Sigma^{=} &\ll \sum_{\mu=0,1} \frac{p^{j-\mu}N_0}{d} \sum_{\substack{m_1,m_2 \sim M(X(N_0)) \\ m_1 \equiv m_2 \bmod p^{j-\mu}d}} \frac{|\lambda_g(m_1)\lambda_g(m_2)|}{|m_1 m_2|^{1/4}} \sum_{|n| \ll N_1(0)} |\mathcal{I}(m_1, m_2, n, p^j, q)| \\ &\ll \sum_{\mu=0,1} \frac{p^{j-\mu}N_0}{d} \sum_{|n| \ll N_1(0)} \sum_{\substack{m_1,m_2 \sim M(X(N_0)) \\ m_1 \equiv m_2 \bmod p^{j-\mu}d}} |\mathcal{I}(m_1, m_2, n, p^j, q)| \left(\frac{|\lambda_g(m_1)|^2}{m_1^{1/2}} + \frac{|\lambda_g(m_2)|^2}{m_2^{1/2}} \right) \\ &\ll \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma+t}CQ}{N} \right)^2 \times X(N_0)^2 \\ &\times \left(\frac{M(X(N_0))^{3/2}}{p^j d} + M(X(N_0))^{1/2} \right) \times N_1(0).\end{aligned} \tag{4.34}$$

Next assume that $2j \geq a(\pi_2) - 1$ and $j > \sigma$. Using Lemma 4.4.1 and the Rankin-Selberg bound 2.13, if $N_0 \ll N_C$ then

$$\begin{aligned}\Sigma^{=} &\ll \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma}CQ}{N} \right)^2 \times X_C^2 \\ &\times \left(\frac{M_C^{3/2}}{p^j d} + M_C^{1/2} \right) \times N_1(0),\end{aligned} \tag{4.35}$$

and if $N_0 \gg N_C$ then

$$\begin{aligned}\Sigma^{=} \ll & \frac{p^j N_0}{d} \times \left(\frac{p^\sigma CQ}{N}\right)^2 \times X(N_0)^2 \\ & \times \left(\frac{M(X(N_0))^{3/2}}{p^j d} + M(X(N_0))^{1/2}\right) \times N_1(0).\end{aligned} \tag{4.36}$$

Next assume that $2j < a(\pi_2) - 1$. Then we get the bound

$$\begin{aligned}\Sigma^{=} \ll & \frac{p^j N_0}{d} \times \left(\frac{p^\sigma CQ}{N}\right)^2 \times X(N_0)^2 \\ & \times \sum_{\nu(r)\geq j} \left(\frac{M(X(N_0))^{3/2}}{p^j d} + M(X(N_0))^{1/2}\right) \times N_1(r),\end{aligned} \tag{4.37}$$

4.4.4 The Non-zero Frequency $\Sigma^{\neq}$

Assume $2j > a(\pi_2)$. If $j > \sigma$, applying Lemma 4.4.5, Lemma 4.4.1 and the Rankin-Selberg bound (2.13) we see that if $N_0 \ll N_C$ then

$$\begin{aligned}\Sigma^{\neq} \ll & \frac{N_0}{d} \sum_{r\leq j} \frac{p^j}{p^{r/2}} \sum_{\substack{m_1 \ll M_C \\ m_1 \equiv m_2 \bmod p^{j-r}d}} \sum_{m_2 \ll M_C} \frac{|\lambda_g(m_1)\lambda_g(m_2)|}{(m_1 m_2)^{1/4}} \\ & \times \sum_{0<|n|\ll N_1(r)} |\mathcal{I}(m_1, m_2, n, p^j, p^r, d)| \\ \ll & \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma+t} qQ}{N}\right)^2 \times X_C^2 \\ & \times \sum_{r\leq j} \frac{N_1(r)}{p^{r/2}} \sum_{\substack{m_1 \ll M_C \\ m_1 \equiv m_2 \bmod p^{j-r}d}} \sum_{m_2 \ll M_C} \left(\frac{|\lambda_g(m_1)|^2}{m_1^{1/2}} + \frac{|\lambda_g(m_2)|^2}{m_2^{1/2}}\right) \\ \ll & \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma+t} qQ}{N}\right)^2 \times X_C^2 \\ & \times \sum_{r\leq j} \frac{1}{p^{r/2}} \left(\frac{M_C^{3/2}}{p^{j-r} d} + M_C^{1/2}\right) \times N_1(r),\end{aligned} \tag{4.38}$$

and if $N_0 \gg N_C$ then

$$\begin{aligned}\Sigma^{\neq} \ll & \frac{N_0}{d}\sum_{r\leq j}\frac{p^j}{p^{r/2}}\sum_{\substack{m_1\sim M(X(N_0))\\ m_1\equiv m_2 \bmod p^{j-r}d}}\sum_{m_2\sim M(X(N_0))}\frac{|\lambda_g(m_1)\lambda_g(m_2)|}{(m_1m_2)^{1/4}}\\ & \times\sum_{0<|n|\ll N_1(r)}|\mathcal{I}(m_1,m_2,n,p^j,p^r,d)|\\ \ll & \frac{p^jN_0}{d}\times\left(\frac{p^{\sigma+t}qQ}{N}\right)^2\times X(N_0)^2\\ & \times\sum_{r\leq j}\frac{N_1(r)}{p^{r/2}}\sum_{\substack{m_1\sim M(X(N_0))\\ m_1\equiv m_2 \bmod p^{j-r}d}}\sum_{m_2\sim M(X(N_0))}\left(\frac{|\lambda_g(m_1)|^2}{m_1^{1/2}}+\frac{|\lambda_g(m_2)|^2}{m_2^{1/2}}\right)\\ \ll & \frac{p^jN_0}{d}\times\left(\frac{p^{\sigma+t}qQ}{N}\right)^2\times X(N_0)^2\\ & \times\sum_{r\leq j}\frac{1}{p^{r/2}}\left(\frac{M(X(N_0))^{3/2}}{p^{j-r}d}+M(X(N_0))^{1/2}\right)\times N_1(r).\end{aligned} \tag{4.39}$$

Next, assume $2j > a(\pi_2)$ and $j \leq \sigma$, applying Lemma 4.4.5, Lemma 4.4.2 and the Rankin-Selberg bound (2.13) we see that if $N_0 \ll N_C$ then

$$\begin{aligned}\Sigma^{\neq} \ll & \frac{p^jN_0}{d}\times\left(\frac{p^{\sigma}qQ}{N}\right)^2\times X_C^2\\ & \times\sum_{r\leq j}\frac{1}{p^{r/2}}\left(\frac{M_C^{3/2}}{p^{j-r}d}+M_C^{1/2}\right)\times N_1(r),\end{aligned} \tag{4.40}$$

and if $N_0 \gg N_C$ then

$$\begin{aligned}\Sigma^{\neq} \ll & \frac{p^jN_0}{d}\times\left(\frac{p^{\sigma}qQ}{N}\right)^2\times X(N_0)^2\\ & \times\sum_{r\leq j}\frac{1}{p^{r/2}}\left(\frac{M(X(N_0))^{3/2}}{p^{j-r}d}+M(X(N_0))^{1/2}\right)\times N_1(r).\end{aligned} \tag{4.41}$$

Assume that π_2 is supercuspidal and $2j = a(\pi_2)$ or $2j+1 = a(\pi_2)$. We use Lemma 4.4.9 with Proposition 4.4.11 to get that

$$\begin{aligned}
&\frac{p^j N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \times X_C \\
&\times \sum_{\substack{m_1 \ll M_C \\ m_1 \equiv m_2 \bmod d}} \sum_{m_2 \ll M_C} \left(\frac{|\lambda_g(m_1)|^2}{m_1^{1/2}} + \frac{|\lambda_g(m_2)|^2}{m_2^{1/2}}\right) \\
&\times \frac{1}{p^{j/2}} \sum_{0 \le t \le j/2} p^{t/2} \sum_{\substack{n \ll N_1(j) \\ f(n) \equiv 0 \bmod p^t}} 1 \qquad (4.42) \\
\ll &\frac{p^j N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \times X_C^2 \times \left(\frac{M_C^{3/2}}{d} + M_C^{1/2}\right) \\
&\times \frac{1}{p^{j/2}} \times \left(N_1(j) + p^{j/4}\right)
\end{aligned}$$

Therefore if $N_0 \ll N_C$ then

$$\begin{aligned}
\Sigma^{\not=} \ll &\frac{p^j N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \times X(N_0)^2 \\
&\left\{ \times \sum_{0<r<j} \frac{1}{p^{r/2}} \left(\frac{M_C^{3/2}}{p^{j-r}d} + M_C^{1/2}\right) \times N_1(r) \right. \qquad (4.43) \\
&\left. \left(\frac{M_C^{3/2}}{d} + M_C^{1/2}\right) \times \frac{1}{p^{j/2}} \times \left(N_1(j) + p^{j/4}\right) \right\}
\end{aligned}$$

and if $N_0 \gg N_C$ then

$$\begin{aligned}
\Sigma^{\not=} \ll &\frac{p^j N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \times X(N_0)^2 \\
&\left\{ \times \sum_{0<r<j} \frac{1}{p^{r/2}} \left(\frac{M(X(N_0))^{3/2}}{p^{j-r}d} + M(X(N_0))^{1/2}\right) \times N_1(r) \right. \qquad (4.44) \\
&\left. \left(\frac{M(X(N_0))^{3/2}}{d} + M(X(N_0))^{1/2}\right) \times \frac{1}{p^{j/2}} \times \left(N_1(j) + p^{j/4}\right) \right\}
\end{aligned}$$

Next assume that π_2 is a principal series representation and $2j = a(\pi_2)$. Let $v \leq j$. If $v = j$ then we obtain

$$\begin{aligned}\Sigma_{pr}(p^j, p^j) \ll & \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma+t} qQ}{N}\right)^2 \times X_C^2 \\ & \times \left\{ \left(\frac{M_0^{3/2}}{p^j d} + M_C^{1/2}\right) \times N_1(0) \right. \\ & \left. + \frac{1}{p^{j/2}} \times \left(\frac{M_C^{3/2}}{d} + M_C^{1/2}\right) \times (N_1(j) + p^{j/4}) \right\},\end{aligned} \tag{4.45}$$

and if $N_0 \gg N_C$ then

$$\begin{aligned}\Sigma_{pr}(p^j, p^j) \ll & \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma} qQ}{N}\right)^2 \times X(N_0)^2 \\ & \times \left\{ \left(\frac{M(N(X_0))^{3/2}}{p^j d} + M(X(N_0))^{1/2}\right) \times N_1(0) \right. \\ & \left. + \frac{1}{p^{j/2}} \times \left(\frac{M(N(X_0))^{3/2}}{d} + M(N(X_0))^{1/2}\right) \times (N_1(j) + p^{j/4}) \right\}.\end{aligned} \tag{4.46}$$

If $v < j$ and $N_0 \ll N_C$ then we obtain the bound (since $v \geq 4j/5$ so $2(j-v) \leq v/2$)

$$\begin{aligned}\Sigma^{\not=}_{pr} \ll & \frac{p^v N_0}{d} \times \left(\frac{p^{\sigma} qQ}{N}\right)^2 \times X_C^2 \\ & \times \left\{ \sum_{0<r<v} \frac{1}{p^{r/2}} \left(\frac{M_C^{3/2}}{p^{v-r} d} + M_C^{1/2}\right) \times \left(N(r) + p^{\inf\{r/2, j-v\}}\right) \right. \\ & + \sum_{s=j-v}^{2(j-v)-1} \frac{p^s}{p^{v/2}} \times \left(\frac{M_C^{3/2}}{p^{s-(j-v)} d} + M_C^{1/2}\right) \times \left(\frac{N(v)}{p^s} + 1\right) \\ & \left. + \sum_{t=0}^{v/2-2(j-v)} \frac{p^{2(j-v)+t/2}}{p^{v/2}} \times \left(\frac{M_C^{3/2}}{p^{j-v} d} + M_C^{1/2}\right) \times \left(\frac{N(v)}{p^{2(j-v)+t}} + 1\right) \right\}\end{aligned} \tag{4.47}$$

and if $N_0 \gg N_C$ then

$$\begin{aligned}\Sigma^{\not=_{pr}} \ll & \frac{p^v N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \times X_0^2 \\ & \times \left\{ \sum_{0<r<v} \frac{1}{p^{r/2}} \left(\frac{M(X(N_0))^{3/2}}{p^{v-r}d} + M(X(N_0)) \right) \times \left(N(r) + p^{\inf\{r/2, j-v\}} \right) \right. \\ & + \sum_{s=j-v}^{2(j-v)-1} \frac{p^s}{p^{v/2}} \times \left(\frac{M(X(N_0))^{3/2}}{p^{s-(j-v)}d} + M(X(N_0)^{1/2}) \right) \times \left(\frac{N(v)}{p^s} + 1 \right) \\ & \left. + \sum_{t=0}^{v/2-2(j-v)} \frac{p^{2(j-v)+t/2}}{p^{v/2}} \times \left(\frac{M(X(N_0))^{3/2}}{p^{j-v}d} + M(X(N_0))^{1/2} \right) \times \left(\frac{N(v)}{p^{2(j-v)+t}} + 1 \right) \right\}\end{aligned} \tag{4.48}$$

Assume that $2j < a(\pi_2) - 1$ with $4\sigma/5 \leq j$. If $N_0 \ll N_C$ then we obtain the bound

$$\begin{aligned}\Sigma^{\not=} \ll & \frac{N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \times X_C^2 \\ & \times \sum_{\substack{0<r\leq j \\ \nu(r)<j}} \frac{p^{r/2+\nu(r)/2+\inf\{\nu(r)/2, a(\pi_2)/2-j\}}}{p^{r/2}} \times \left(\frac{M_C^{3/2}}{p^{j-\nu(r)-\inf\{\nu(r)/2, a(\pi_2)/2-j\}}d} + M_C^{1/2} \right) \\ & \times N_1(r)\end{aligned} \tag{4.49}$$

and if $N_0 \gg N_C$ then we obtain the bound

$$\begin{aligned}\Sigma^{\not=} \ll & \frac{N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \times X(N_0)^2 \\ & \times \sum_{\substack{0<r\leq j \\ \nu(r)<j}} \frac{p^{r/2+\nu(r)/2+\inf\{\nu(r)/2, a(\pi_2)/2-j\}}}{p^{r/2}} \\ & \times \left(\frac{M(X(N_0))^{3/2}}{p^{j-\nu(r)-\inf\{\nu(r)/2, a(\pi_2)/2-j\}}d} + M(X(N_0))^{1/2} \right) \times N_1(r)\end{aligned} \tag{4.50}$$

4.5 Final Estimates

Note that all estimates when $N_0 \ll N_C$ are dominated by the estimates from $N_0 \gg N_C$. So we assume that $N_0 \gg N_C$ from now on. Next we observe that estimates for N_0 are an increasing function of N_0, so we take N_0 as large as possible. This means that if

$2(\sigma+t) < a(\pi_1)$ or π_1 is supercuspidal and $2(\sigma+t) = a(\pi_1)$ we have that

$$N_0 = \frac{Np^{l_1}}{p^{2(\sigma+t)}Q^2} = p^{l_1-\sigma-2t} \tag{4.51}$$

If $2(\sigma+t) > a(\pi_1)$ then

$$N_0 = \frac{N}{Q^2} = p^{\sigma}. \tag{4.52}$$

If π_1 is a principal series and $2(\sigma+t) = a(\pi_1)$ then

$$N_0(v) = \frac{Np^{\frac{l_1}{2}+v}}{p^{2(\sigma+t)}Q^2} = p^{\frac{l_1}{2}+v-\sigma-2t} \tag{4.53}$$

for $v \leq \sigma + t$.

This implies that $X(N_0) \sim 1$ and therefore $M(X(N_0)) \sim M(1)$. If $t > 0$ then

$$M(1) \sim \frac{N}{Q^2} = p^{\sigma} \tag{4.54}$$

We have that if $j \leq \sigma$ and $2j < a(\pi_2)$ or if π_2 is supercuspidal and $2j = a(\pi_2)$ then

$$M(1) \sim \frac{N[p^{l_2}, p^{2j}]}{p^{2\sigma}Q^2} = p^{\sup\{l_2,2j\}-\sigma}. \tag{4.55}$$

If π_2 is a principal series and $2j = a(\pi_2)$ then

$$M(1, v) \sim \frac{Np^{\frac{l_2}{2}+v}}{p^{2\sigma}Q^2} = p^{\frac{l_2}{2}+v-\sigma} \tag{4.56}$$

for $v \leq \frac{l_2}{2}$. We write M instead of $M(1)$ from now on.

So thus assume that from now on N_0 and M are as given. Note that therefore

$$N_1(0) = 1 + \frac{d}{N_0},$$

$$N_1(r) = \frac{p^r d}{N_0},$$

for any $r \geq 1$. Assume $2j > a(\pi_2)$. Observing that Σ is then dominated by $\Sigma^{=}$ and the term with $r = j$ in $\Sigma^{\neq}$, we see that

$$\begin{aligned}\Sigma \ll & \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma+t} qQ}{N}\right)^2 \\ & \times \left\{ \left(\frac{M^{3/2}}{p^j d} + M^{1/2}\right) \times N_1(0) \right. \\ & \left. + \frac{1}{p^{j/2}} \times \left(\frac{M^{3/2}}{d} + M^{1/2}\right) \times N_1(j) \right\}.\end{aligned} \tag{4.57}$$

In the case π_2 is supercuspidal and $2j = a(\pi_2)$ or $2j + 1 = a(\pi_2)$ or $2j = a(\pi_2)$ and $v = j$ we have that

$$\begin{aligned}\Sigma \ll & \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma} qQ}{N}\right)^2 \\ & \times \left\{ \left(\frac{M^{3/2}}{p^j d} + M^{1/2}\right) \times N_1(0) \right. \\ & \left. + \frac{1}{p^{j/2}} \times \left(\frac{M^{3/2}}{d} + M^{1/2}\right) \times (N_1(j) + p^{j/4}) \right\}.\end{aligned} \tag{4.58}$$

$$\begin{aligned}\Sigma(p^j, p^j) \ll & \frac{p^j N_0}{d} \times \left(\frac{p^{\sigma} qQ}{N}\right)^2 \\ & \times \left\{ \left(\frac{M^{3/2}}{p^j d} + M^{1/2}\right) \times N_1(0) \right. \\ & \left. + \frac{1}{p^{j/2}} \times \left(\frac{M^{3/2}}{d} + M^{1/2}\right) \times (N_1(j) + p^{j/4}) \right\}.\end{aligned} \tag{4.59}$$

If π_2 is a principal series and $\frac{4j}{5} \leq v < 4j/5$ then

$$\begin{aligned}\Sigma(p^j, p^v) \ll & \frac{p^v N_0}{d} \times \left(\frac{p^{\sigma} qQ}{N}\right)^2 \\ & \times \left\{ \left(\frac{M^{3/2}}{p^v d} + M^{1/2}\right) \times N_1(0) \right. \\ & + \frac{1}{p^{v/2}} \times \left(\frac{M^{3/2}}{d} + M^{1/2}\right) \times \left(N(v) + p^{j-v}\right) \\ & \left. + \frac{1}{p^{v/2}} \times \left(\frac{M^{3/2}}{p^{j-v} d} + M^{1/2}\right) \times \left(N(v) + p^{v/4+(j-v)}\right) \right\}.\end{aligned} \tag{4.60}$$

Assume that $2j < a(\pi_2) - 1$ and $4\sigma/5 \leq j$. Replacing $\nu(r) = \frac{a(\theta_2) - a_2(r)}{e_2}$ by $\frac{a(\pi_2)}{2} - r$,

we see that $\nu(r) \geq j$ is equivalent to $r \leq \frac{a(\pi_2)}{2} - j$. Similarly, $\nu(r) > j$ is equivalent to $r > \frac{a(\pi_2)}{2} - j$. We have

$$\begin{aligned}\Sigma \ll & \frac{p^j N_0}{d} \times \left(\frac{p^\sigma qQ}{N}\right)^2 \\ & \times \left\{\left(\frac{M^{3/2}}{p^j d} + M^{1/2}\right) \times \left(N_1(0) + \frac{N_1(\frac{a(\pi_2)}{2} - j)}{p^{(\frac{a(\pi_2)}{2} - j)/2}}\right)\right. \\ & \left. + \frac{p^{\frac{a(\pi_2)}{2} - j}}{p^{j/2}} \times \left(\frac{M^{3/2}}{p^{j/2 - \frac{a(\pi_2)}{4}} d} + M^{1/2}\right) \times N_1(j)\right\}.\end{aligned} \tag{4.61}$$

Next we observe that the bounds are increasing functions of C where $q \sim C$. When $t = 0$ we consider only the case $C = Q$ and if $t > 0$ (that is $p^{\sigma+t}$ with $t > 0$) then $C = Q/p^t$.

In the range $2(\sigma + t) > a(\pi_1)$ we use the bound (4.6) with (4.52) and (4.54) to obtain the bound

$$S(N, p^{\sigma+t}, Q, N_0) \ll \frac{N}{p^{\sigma+3t/2}}\left(\frac{p^{3\sigma/2+t/2}}{N^{1/2}} + p^{\sigma/2}\right) \tag{4.62}$$

For π_1 supercuspidal and $2(\sigma + t) = a(\pi_1)$ or $2(\sigma + t) = a(\pi_1) + 1$ we obtain the bound

$$S(N, p^{\sigma+t}, Q, N_0) \ll \frac{N}{p^{\sigma+3t/2}}\left(\frac{p^{3\sigma/2+t/2}}{N^{1/2}} + p^{\sigma/2}\right) \tag{4.63}$$

Similarly, for π_1 a principal series and $2(\sigma + t) = a(\pi_1)$ we obtain the bound

$$S_{\pi_1,pr}(N, p^{\sigma+t}, Q, N_0) \ll \frac{N}{p^{\sigma+3t/2}}\left(\frac{p^{3\sigma/2+t/2}}{N^{1/2}} + p^{\sigma/2}\right) \tag{4.64}$$

When $2(\sigma+t) < a(\pi_1)$ using (4.57), (4.51) and (4.54) we obtain the bound

$$\begin{aligned}
S(N,p^{\sigma+t},Q,N_0) \ll & \frac{N^{3/2}}{Qp^{l_1/4+3(\sigma+t)/2}} \times N_0^{1/4} \times \sum_{e\leq Q/p^t} \frac{1}{e^2} \sum_{d\sim Q/p^t e} \frac{1}{d} \\
& \times \left[\frac{p^{\sigma+t}N_0}{d} \times \left\{ \left(\frac{M^{3/2}}{p^{\sigma+t}d} + M^{1/2} \right) \times \left(\frac{d}{N_0} + 1 \right) \right.\right. \\
& \left.\left. + \frac{1}{p^{(\sigma+t)/2}} \times \left(\frac{M^{3/2}}{d} + M^{1/2} \right) \times \frac{p^{\sigma+t}d}{N_0} \right\} \right]^{1/2} \\
\ll & \frac{N^{3/2}N_0^{1/4}}{p^{l_1/4}} \left(\frac{M^{3/4}N_0^{1/2}}{Q^2p^{3\sigma/2+t/2}} \right. \\
& \left. + \frac{N_0^{1/2}M^{1/4}}{Q^{3/2}p^{\sigma+t/2}} + \frac{M^{3/4}}{Q^{3/2}p^{3\sigma/4+t/4}} + \frac{M^{1/4}}{Qp^{3(\sigma+t)/4}} \right) \\
\ll & N^{1/2}p^{\frac{l_1}{2}-\frac{\sigma}{2}-\frac{t}{2}} + N^{3/4}p^{\frac{l_1}{2}-\frac{3\sigma}{4}-\frac{t}{2}} \\
& + N^{3/4}p^{\frac{\sigma}{2}-\frac{t}{4}} + Np^{-\frac{\sigma}{4}-\frac{3t}{4}}
\end{aligned} \tag{4.65}$$

When $4\sigma/5 \leq j \leq \sigma$ and $a(\pi_2) < 2j \leq \sigma$ we use (4.57) to obtain

$$\begin{aligned}
S(N,p^{j},Q,N_0) \ll & \frac{N^{3/2}}{Qp^{\sigma+l_1/4+j/2}} \times N_0^{1/4} \times \sum_{e\leq Q} \frac{1}{e^2} \sum_{d\sim Q/e} \frac{1}{d} \\
& \times \left[\frac{p^{j}N_0}{d} \times \left\{ \left(\frac{M^{3/2}}{p^{j}d} + M^{1/2} \right) \times \left(\frac{d}{N_0} + 1 \right) \right.\right. \\
& \left.\left. + \frac{1}{p^{j/2}} \times \left(\frac{M^{3/2}}{d} + M^{1/2} \right) \times \frac{p^{j}d}{N_0} \right\} \right]^{1/2} \\
\ll & \frac{N^{3/2}N_0^{1/4}}{Qp^{\sigma+l_1/4+j/2}} \left(\frac{M^{3/4}N_0^{1/2}}{Q} + \frac{p^{j/2}N_0^{1/2}M^{1/4}}{Q^{1/2}} \right. \\
& \left. + \frac{p^{3j/4}M^{3/4}}{Q^{1/2}} + p^{3j/4}M^{1/4} \right) \\
\ll & N^{1/2}p^{\frac{l_1}{2}-\frac{3\sigma}{8}} + N^{3/4}p^{\frac{l_1}{2}-\frac{3\sigma}{4}} + N^{3/4}p^{\frac{\sigma}{2}} + Np^{-\frac{\sigma}{4}}
\end{aligned} \tag{4.66}$$

When $j \leq 4\sigma/5$ and $a(\pi_2) < 2j$ we use (4.7) and (4.55) to obtain

$$\begin{aligned} S(N,p^j,Q,N_0) \ll & \frac{N^{3/2}}{Qp^{\sigma+l_1/4}} \sum_{e\leq Q} \frac{1}{e^2} \sum_{d\sim Q/e} \frac{1}{d} \times \left\{ N_0^{1/2+2a}\left(\frac{M}{d}+1\right) \right. \\ & \left. + M^{1/2}\left(\frac{N_0^{1-2a}}{d} + N_0^{-2a}\right)\right\} \\ \ll & N^{1/2}p^{(1/4+2a)l_1+(1/10-2a)\sigma} + Np^{(1/4+2a)l_1-(1+2a)\sigma} \\ & + N^{1/2}p^{(3/4-a)l_1+(a-7/10)\sigma} + Np^{-l_1/4-\sigma/5} \end{aligned} \tag{4.67}$$

When $j < 4\sigma/5$ and $a(\pi_2)/2 < 2j < a(\pi_2) - 1$ we use (4.8) to obtain

$$\begin{aligned} S(N,p^j,Q,N_0) \ll & \frac{N^{1/2}Qp^{j-\frac{a(\pi_2)}{4}}}{p^{l_1/4+l_2/4}} \sum_{e\leq Q} \frac{1}{e^2} \sum_{d\sim Q/e} \frac{1}{d} \left\{ N_0^{1/2+2a}\left(\frac{M}{p^{a(\pi_2)/4}d}+1\right) + M^{1/2}\left(\frac{N_0^{1-2a}}{p^{a(\pi_2)/4}d}+1\right)\right\} \\ \ll & \frac{N^{1/2}}{p^{l_1/4+l_2/4}} \left\{ p^{j-l_2/2}\left(N_0^{1/2+2a}M + M^{1/2}N_0^{1-2a}\right) + Qp^{j-l_2/4}\left(N_0^{1/2+2a}+M^{1/2}\right)\right\} \\ \ll & N^{1/2}p^{(1/4+2a)l_1+3l_2/4-(3/2+2a)\sigma} + N^{1/2}p^{(3/4-a)l_1+l_2/4+(a-3/2)\sigma} \\ & + Np^{(1/4+2a)l_1-(1+2a)\sigma} + Np^{l_2/2-l_1/4-\sigma}. \end{aligned} \tag{4.68}$$

When π_2 is a principal series, $a(\pi_2) = 2j$ and $v \leq 4\sigma/5$ with $j \geq 4\sigma/5$, the contribution to $S_{\pi_2,pr}(N,p^j,Q,N_0)$ using (4.9) to get

$$\begin{aligned} & \frac{N^{3/2}}{Qp^{\sigma+l_1/4}} \sum_{0\leq v\leq 4j/5} \frac{p^{v/2}}{p^{(j+v)/4}} \sum_{e\leq Q} \frac{1}{e^2} \sum_{d\sim Q/e} \frac{1}{d} \times \left\{ N_0^{1/2+2b}\left(\frac{M(v)^{3/4}}{d}+1\right) \right. \\ & \left. + M(v)^{3/4}\left(\frac{N_0^{1-2b}}{d}+N_0^{-2b}\right)\right\} \\ \ll & N^{1/2}p^{(1/4+2b)l_1+l_2/4-(9/20+2b)\sigma} + Np^{(1/4+2b)l_1-l_2/8-(4/5+2b)\sigma} \\ & + N^{1/2}p^{(3/4-2b)l_1+l_2/4+(2b-19/20)\sigma} + Np^{-(1/4+2b)l_1+l_2/4+(2b-9/20)\sigma}. \end{aligned} \tag{4.69}$$

When $4\sigma/5 \le v < j$ we use (4.60) to obtain

$$\begin{aligned}
&\frac{N^{3/2}}{Qp^{\sigma+l_1/4}} \sum_{4j/5\le v<j} \frac{1}{p^{(j+v)/4}} \times N_0^{1/4} \times \sum_{e\le Q} \frac{1}{e^2} \sum_{d\sim Q/e} \frac{1}{d} \\
&\times \left[\frac{p^v N_0}{d} \left\{ \left(\frac{M(v)^{3/2}}{p^v d} + M(v)^{1/2} \right) \times \left(\frac{d}{N_0} + 1 \right) \right.\right. \\
&+ \frac{1}{p^{v/2}} \times \left(\frac{M(v)^{3/2}}{d} + M^{1/2} \right) \times \left(\frac{p^v d}{N_0} + p^{j-v} \right) \\
&\left.\left. + \frac{1}{p^{v/2}} \times \left(\frac{M^{3/2}}{p^{j-v} d} + M^{1/2} \right) \times \left(\frac{p^v d}{N_0} + p^{v/4+(j-v)} \right) \right\} \right]^{1/2} \\
\ll &\frac{N^{3/2}}{Qp^{\sigma+l_1/4}} \times N_0^{1/4} \times \sum_{4j/5\le v<j} \frac{1}{p^{(j+v)/4}} \\
&\times \left(\frac{M(v)^{3/4}}{Q^{1/2}} + \frac{N_0^{1/2} M(v)^{3/4}}{Q} + p^{v/2} M(v)^{1/4} + \frac{p^{v/2} N_0^{1/2} M(v)^{1/4}}{Q^{1/2}} \right. \\
&+ \frac{p^{3v/4} M(v)^{3/4}}{Q^{1/2}} + \frac{N_0^{1/2} p^{j/2-v/4} M(v)^{3/4}}{Q} + p^{3v/4} M(v)^{1/4} \\
&+ \frac{N_0^{1/2} p^{j/2-v/4} M(v)^{1/4}}{Q^{1/2}} + \frac{p^{-j/2+5v/4} M(v)^{3/4}}{Q^{1/2}} \\
&\left. + \frac{N_0^{1/2} p^{3v/8} M(v)^{3/4}}{Q} + \frac{N_0^{1/2} p^{j/2-v/8} M(v)^{1/4}}{Q^{1/2}} \right) \\
\ll &N^{3/4} p^{7l_2/8-5\sigma/4} + N^{3/4} p^{l_2/4+l_1/2-5\sigma/4} + N^{1/2} p^{11l_2/16+l_1/2-3\sigma/2} + N p^{3l_2/8-\sigma}.
\end{aligned} \tag{4.70}$$

When $v = j$ we use (4.59) to obtain

$$\begin{aligned}
&\frac{N^{3/2}}{Qp^{\sigma+l_1/4+j/2}} \times N_0^{1/4} \times \sum_{e\le Q} \frac{1}{e^2} \sum_{d\sim Q/e} \frac{1}{d} \\
&\times \left[\frac{p^j N_0}{d} \times \left\{ \left(\frac{M^{3/2}}{p^j d} + M^{1/2} \right) \times \left(\frac{d}{N_0} + 1 \right) \right.\right. \\
&\left.\left. + \frac{1}{p^{j/2}} \times \left(\frac{M^{3/2}}{d} + M^{1/2} \right) \times \left(\frac{p^j d}{N_0} + p^{j/4} \right) \right\} \right]^{1/2} \\
\ll &\frac{N^{3/2} N_0^{1/4}}{Qp^{\sigma+l_1/4+j/2}} \left(\frac{M^{3/4}}{Q^{1/2}} + \frac{N_0^{1/2} M^{3/4}}{Q} + p^{j/2} M^{1/4} + \frac{p^{j/2} N_0^{1/2} M^{1/4}}{Q^{1/2}} \right. \\
&\left. + \frac{p^{3j/4} M^{3/4}}{Q^{1/2}} + \frac{p^{3j/8} N_0^{1/2} M^{3/4}}{Q} + p^{3j/4} M^{1/4} + \frac{p^{3j/8} N_0^{1/2} M^{1/4}}{Q^{1/2}} \right) \\
\ll &N^{3/4} p^{7l_2/8-5\sigma/4} + N^{1/2} p^{l_1/2+11l_2/16-3\sigma/2} + N p^{3l_2/8-\sigma} \\
&+ N^{3/4} p^{l_1/2+l_2/4-5\sigma/4}.
\end{aligned} \tag{4.71}$$

Combining (4.71), (4.70) and (4.69) we get the bound

$$\begin{aligned}S_{pr,\pi_2}(N,p^j,Q,N_0) \ll & N^{3/4}p^{7l_2/8-5\sigma/4}+N^{3/4}p^{l_1/2+l_2/4-5\sigma/4}\\ &+N^{1/2}p^{l_1/2+11l_2/16-3\sigma/2}+Np^{3l_2/8-\sigma}\\ &+N^{1/2}p^{(1/4+2b)l_1+l_2/4-(9/20+2b)\sigma}+Np^{(1/4+2b)l_1-l_2/8-(4/5+2b)\sigma}\\ &+N^{1/2}p^{(3/4-2b)l_1+l_2/4+(2b-19/20)\sigma}+Np^{-(1/4+2b)l_1+l_2/4+(2b-9/20)\sigma}.\end{aligned} \tag{4.72}$$

When $a(\pi_2)$ is supercuspidal and $2j=a(\pi_2)$ or $2j+1=a(\pi_2)$ we use (4.58) to obtain

$$\begin{aligned}S(N,p^j,Q,N_0) \ll & \frac{N^{3/2}}{Qp^{\sigma+l_1/4+j/2}}\times N_0^{1/4}\times\sum_{e\leq Q}\frac{1}{e^2}\sum_{d\sim Q/e}\frac{1}{d}\\ &\times\left[\frac{p^jN_0}{d}\times\left\{\left(\frac{M^{3/2}}{p^jd}+M^{1/2}\right)\times\left(\frac{d}{N_0}+1\right)\right.\right.\\ &\left.\left.+\frac{1}{p^{j/2}}\times\left(\frac{M^{3/2}}{d}+M^{1/2}\right)\times\left(\frac{p^jd}{N_0}+p^{j/4}\right)\right\}\right]^{1/2}\\ \ll & \frac{N^{3/2}N_0^{1/4}}{Qp^{\sigma+l_1/4+j/2}}\left(\frac{M^{3/4}}{Q^{1/2}}+\frac{N_0^{1/2}M^{3/4}}{Q}+p^{j/2}M^{1/4}+\frac{p^{j/2}N_0^{1/2}M^{1/4}}{Q^{1/2}}\right.\\ &\left.+\frac{p^{3j/4}M^{3/4}}{Q^{1/2}}+\frac{p^{3j/8}N_0^{1/2}M^{3/4}}{Q}+p^{3j/4}M^{1/4}+\frac{p^{3j/8}N_0^{1/2}M^{1/4}}{Q^{1/2}}\right)\\ \ll & N^{3/4}p^{7l_2/8-5\sigma/4}+N^{1/2}p^{l_1/2+11l_2/16-3\sigma/2}+Np^{3l_2/8-\sigma}\\ &+N^{3/4}p^{l_1/2+l_2/4-5\sigma/4}\end{aligned} \tag{4.73}$$

When $4\sigma/5 \leq j \leq \sigma$ and $2j < a(\pi_2) - 1$ (and this automatically guarantees that $j \geq a(\pi_2)/4$ since $a(\pi_2) \leq 2\sigma$) we use (4.61) to obtain

$$\begin{aligned} S(N, p^j, Q, N_0) \ll & \frac{N^{3/2}}{Qp^{\sigma+l_1/4+l_2/4}} \times N_0^{1/4} \times \sum_{e \leq Q} \frac{1}{e^2} \sum_{d \sim Q/e} \frac{1}{d} \\ & \times \left[\frac{p^j N_0}{d} \times \left\{ \left(\frac{M^{3/2}}{p^j d} + M^{1/2} \right) \times \left(1 + \frac{p^{a(\pi_2)/4 - j/2} d}{N_0} \right) \right. \right. \\ & \left. \left. + \frac{p^{a(\pi_2)/2 - j}}{p^{j/2}} \times \left(\frac{M^{3/2}}{p^{j/2 - a(\pi_2)/4} d} + M^{1/2} \right) \times \frac{p^j d}{N_0} \right\} \right]^{1/2} \\ \ll & \frac{N^{3/2} N_0^{1/4}}{Qp^{\sigma+l_1/4+l_2/4}} \left(\frac{M^{3/4} N_0^{1/2}}{Q} + \frac{p^{j/2} N_0^{1/2} M^{1/4}}{Q^{1/2}} \right. \\ & \left. + \frac{p^{3l_2/8} M^{3/4}}{Q^{1/2}} + p^{l_2/4 + j/4} M^{1/4} \right) \\ \ll & N^{1/2} p^{\frac{l_1}{2} - \frac{3\sigma}{8}} + N^{3/4} p^{\frac{l_1}{2} \frac{3\sigma}{4}} \\ & + N^{3/4} p^{\frac{l_2}{8} + \frac{\sigma}{4}} + N p^{\frac{l_2}{8} - \frac{\sigma}{2}}. \end{aligned} \tag{4.74}$$

When $j < a(\pi_2)/4$ we use (4.8)

$$\begin{aligned} S(N, p^j, Q, N_0) \ll & \frac{N^{1/2} Q p^j}{p^{l_1/4 + l_2/4}} \sum_{e \leq Q} \frac{1}{e^2} \sum_{d \sim Q/e} \frac{1}{d} \left\{ N_0^{1/2 + 2c} \left(\frac{M^{3/4}}{p^j d} + 1 \right) \right. \\ & \left. + M^{3/4} \left(\frac{N_0^{1-2c}}{p^j d} + N_0^{-2c} \right) \right\} \\ \ll & N^{1/2} p^{(1/4+2c) l_1 + l_2/2 - (5/4+2c)\sigma} + N p^{(1/4+2c) l_1 - (1+2c)\sigma} \\ & + N^{1/2} p^{(3/4-2c) l_1 + l_2/2 + (2c - 7/4)\sigma} + N p^{-(1/4+2c) l_1 + 3l_2/4 + (2c - 5/4)\sigma}. \end{aligned} \tag{4.75}$$

First we observe that comparing (4.63), (4.65) and (4.62) to the other bounds, we see that the case $t > 0$ is dominated by those for $t = 0$. Using $l_2 \leq 2\sigma$, choosing $a = c = \frac{1}{12}$ and $b = \frac{l_2/2\sigma + 1/5}{12}$ (note that the terms with b only appear when $l_2 \geq 8\sigma/5$), adding (4.72), (4.73),(4.66), (4.74), (4.68) and (4.75) and taking $\sigma \sim 2l_1/5$ we see that

$$S(N) \ll N^{1/2} p^{9l_1/20} + N^{3/4} p^{l_1/5} + N p^{-l_1/20}. \tag{4.76}$$

We thus observe that since $N \ll p^{l_1}$ ignoring factors of p^{ϵ} and $p^{O(1)}$

$$L\left(\frac{1}{2}, \pi_1 \otimes \pi_2\right) \ll p^{9l_1/20} = p^{\frac{a(\pi_1 \otimes \pi_2)}{4}\left(1-\frac{1}{10}\right)} \tag{4.77}$$

as desired.

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