Multiattribute acceptance sampling plans

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Preface

Irrespective of the type of product, evaluation of conformity to specified requirements of its quality characteristics is an integral part of quality assurance. Although they form a set of necessary verification activities almost at all stages of production, these activities, known as inspection do not add value to the product on their own and are to be kept at their minimum. The sampling inspection where a portion of a collection of product units is inspected on a set of characteristics with a view to making decision about acceptance or otherwise becomes relevant in this context.

The number of elements of the set of characteristics to be verified for this purpose is unlikely to be just one in most of the practical situations. We may call this as multiattribute inspection as employed for verification of materials procured from outside and further at all stages of production, through semi finished and finished or assembly stages to final despatch to the customers. At all such stages consecutive collections of products called lots, are submitted for acceptance or alternative disposition.

In this context one observes that the defectives with respect to different characteristics may in some situations be considered as jointly independent, whereas in some situations, occurrences of one type of defect may preclude occurrence of any other type. In some other situations, it is more natural to count number of defects for each unit of product so that the quality of a lot is expressed as an ordered set of average number of defects per unit for each type of defect instead of proportion defectives in an aggregate.

The present exercise attends to the designing of the procedure of sampling inspection by attribute with the above scope and purpose in mind. One may call them as multiattribute single sampling plans(MASSP).

The first objective is to establish a sampling scheme in line with available international standards tabulated on the basis of Acceptable Quality Level(AQL) in such multiattribute situations. These Standards in general prescribe that separate plans are to be chosen for the different classes of attributes. We examine the consequences of constructing a sampling plan by this prescribed method in a multiattribute situation. We first consider the effective producer's risks. Secondly, it has been thought as reasonable to expect that the OC function should be more sensitive to the changes in the defect level of more important attributes, particularly, in a situation, where unsatisfactory defect level occurs due to more serious type of defects. A measure of sensitivity has been defined, for this purpose. Plans constructed with all possible practically useful combinations of AQL for three attributes have been examined for both the as dicussed. It has been found that the features observed depict a picture far from the ideal on both the counts for most of the plans.

Further, it has been discovered that these two desirable properties remain generally dissatisfied inherently absent for the class of plans (we call these as MASSP of C kind), where we accept a lot if the number of defectives observed in the sample is less than the respective acceptance numbers stipulated separately for each attribute depending upon the AQL.

We have, therefore, introduced a sampling scheme consisting of plans with same sample size as used by the International standards for a given lot size, but with a different acceptance criterion. (we call this as MASSP of A kind). For these plans the OC function is more sensitive to the changes in the defect level of more important attributes characterized by low AQL values. Further, they can be deigned to ensure a reasonable producer's risk. These plans are tabulated for ready use by the industries, encountering the need for multiattribute inspection schemes.

Next we work out methods of formulating multiattribute sampling plans of given strength with different acceptance criteria and develop step by step algorithm for obtaining these plans. The sample sizes of the A kind plans are likely to be much less than the comparable plans of C kind.

Next we draw our attention to formulating economic plans. We have restricted ourselves to those situations, where a) the inspection is nondestructive b) we take a single sample of size n and inspect for all attributes c) the occurrences of defects of different types are jointly independent or d) mutually exclusive

e) the cost functions are linear and f) the distributions of process averages are either all discrete and when continuous the prior distributions are assumed to be independent for the different characteristics.

We have used data collected from industries to understand the nature of quality variations. For example, in one such industry producing metal containers, one observes that most of the time the process is stable at a certain process average and shifts to an unsatisfactory level for a small duration of time. The process average during the latter phase, although not quite stable hovers around a higher level. It has been verified from the data collected by us that we can approximate this process by a discrete two point prior distribution. In another situation, in a pharmaceutical company we find the quality variations with respect to different type of defects are independent and justify the use of gamma distribution as the prior for each attribute.

For the above cases we develop general cost models under the assumptions mentioned; study the cost functions and the optimality properties of different sampling schemes as proposed and compare them in terms of costs. Further, it has been observed that the standard distributions like beta and gamma as models of quality variation may not fit for the data collected by us from shop floor observations during production of ceiling fans, garments, cigarettes etc. In such cases we have to take recourse to direct computation of the cost function derived from the empirical distributions obtained from past data. We could demonstrate that in general it should be possible to obtain an optimal MASSP of the kind defined by us by the methods developed. Moreover, the choice of acceptance criterion does affect the costs of optimal MASSP's.

While providing professional help to industries I have noted that the available literature and published plans are not adequate to address the industry requirements for verification of product quality as a producer and also as a consumer in a multiattribute situation. I submit that the theory of multiattribute plans can not and should not be taken as a mere extension of the theory of sampling plans developed for single attribute. The present exercise may be viewed as an attempt to make a beginning towards development of a comprehensive theory of multiattribute acceptance sampling focusing on the real life problems.

In this endeavour I am most indebted to those industries who shared with me the knowledge of domain, valuable data and all other relevant information and offered liberally the opportunities required for my applications. I shall feel most gratified if any of them find the outcome of this exercise at all useful.

The founder of the theory of sampling inspection by attribute is Harold F. Dodge. However, the statistical theory of sampling inspection by attribute has been most comprehensively established, elaborated and explained by Anders Hald. Without their work the present dissertation could not have conceived of.

Professor Kalyan Bidhan Sinha, the ex-director of the Indian Statistical Institute always urged us to innovate new theory based on practice. I am indebted to him for providing the motivation of writing this thesis.

My supervisor Professor A. C. Mukhopadhyaya has always been my inspiration for undertaking the present exercise and bringing it to a presentable shape. He is also my teacher. Whatever little theory I could assimilate is perhaps because of him and other senior teachers of the Indian Statistical Institute. Among them I must mention Professor C. R. Prasad, Professor R. J. Pandey and the late Professor K. G. Ramamurthy.

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Part 0 : Introduction

0.1 Purpose of sampling inspection

The very purpose of quality management comprising of quality planning, quality control, quality assurance and quality improvement is to achieve simultaneously, improvement in customer satisfaction, improvement in productivity and reduction in costs of product at all stages. From the very beginning statistical methods have been developed, established and implemented as support activities for the above purpose. While Shewhart (1931) focused his attention on economic control of quality of manufactured product, Dodge and Romig (1929) attended to quality assurance activities of an organization for protection of the interest of the internal and external customers at reduced cost of inspection and cost of production, and also for attainment of uniform quality. The latter established methodologies of sampling inspection for separating lots or batches of mass produced items into lots of satisfactory and unsatisfactory quality.

0.2 Theory and application of sampling inspection - a short review

In the following sections we try to underline the nature of major developments of techniques of sampling inspection for industrial quality assurance since the pre-war time. In this context we limit our discussions at the beginning to single sampling plans as employed for lot-by-lot inspection by attribute. The developments relating to double and multiple sampling plans for attributes, continuous sampling and chain sampling plans, sequential sampling plans and their modification and the whole subject matter of sampling plans by variables and all methods pertaining to bulk materials are scrupulously excluded from our discussions as they are not relevant as far as the scope of the present thesis is concerned. The historical perspective discussed focuses attention only to that part of the development which is conceptually and theoretically related to the problems studied in the present thesis. A single sampling plan is defined by means of three parameters c, n, and N and the following rule : From each lot of size N take a random sample of size n. If the number of defectives in the sample is less than or equal to the acceptance number c, accept the lot, otherwise reject the lot.

0.2.1 Early development

Prior to 1929, the research in the area of sampling inspection was insignificant and the history, however, scanty, can be obtained from the issues of the Bell system Technical journal . Paul Peach (1950) in a review of developments prior to 1941 reported the work done by Molina and Crowell on sampling techniques and use of Bayes' theorem for this purpose. All the work reached a climax in October 1929, when The Bell System Technical journal printed the historic paper "A method of sampling Inspection" by H.F Dodge and H.G.Romig. The paper made the following points:
Sampling inspection problems have two aspects, the economic and the statistical. Before an intelligent inspection plan can be designed, the economic objectives must be recognized; otherwise the problem cannot be stated accurately. Any sampling procedure involves a chance that a bad lot will be passed; this probability is called the consumer's risk. In order to state the consumer's risk unconditionally we need to know the probability that a bad lot will be submitted. Since this can hardly be known in practice, we must be content with a conditional statement. Without any a priori knowledge of the product quality, and assuming only random sampling, we can design a scheme such that the probability of accepting a lot of quality X or worse shall not exceed Y, provided such a lot is submitted. This requirement does not uniquely specify an inspection plan, and we are therefore at liberty to impose an additional requirement, such as e.g. that the average number of pieces inspected under some assumed set of conditions should be a minimum.

0.2.2 The Dodge and Romig's system of LTPD plans

Lot Tolerance Percent Defective (LTPD) is the poorest quality in an individual lot, expressed in per cent defective, that should be accepted. The Dodge Romig's (1929) system of LTPD plans, with total inspection of rejected lots, is established by setting the consumer's risk (the chance of acceptance of lots of LTPD quality) equal to 10 percent and by minimizing the average total inspection for product of producer's process average quality.

0.2.3 Dodge and Romig's system of AOQL plans

Dodge and Romig (1941), first developed in 1927, the concept of average outgoing quality (AOQ) and introduced tables based on it. AOQ is defined as "The expected average quality of outgoing product including all accepted lots plus all rejected lots after the latter have been inspected 100 percent and defective units replaced by good units." The AOQ as function of product quality (p), is initially increasing in p, attains a maximum and then decreases as p increases. It takes on the value zero for $p = 0$ and $p = 1$ and reaches a maximum at some point in the range $0 < p < 1$. The average outgoing quality limit (AOQL) is defined as the maximum of AOQ. The Dodge and Romig (1941) system of AOQL plans for acceptance /

rectification is defined by setting the AOQL and by minimizing the average amount of total inspection (ATI) for product of producer's process average quality .

The LTPD and the AOQL systems use the same optimality criterion, namely the minimization of average total inspection costs for the product of process average quality. The constraints are meant to ensure the quality of the accepted lot or the quality of the outgoing product in general.

0.2.4 The concept of OC curve

The concepts of Operating Characteristic (OC) curve (earlier known as probability of acceptance curve), producer's risk and consumer's risk were introduced by the engineers of Western Electric Company in 1923. Dodge and Romig (1959) formalized this concept by differentiating between what they called type A and type B OC curves. The type A OC curve of a sampling plan is the probability of acceptance, as a function of a given lot quality, where lot quality is expressed as fraction defective in the lot and samples are drawn without replacement from such a lot. Further, the conceptually infinite output of units from a process for which the probability of producing a defective unit is p is called product of quality p. The type B OC curve of a sampling plan is defined as the curve showing probability of accepting a lot as a function of product quality, where product quality is expressed as fraction defective p. This probability of acceptance is obtained by assuming a binomial distribution of lot quality values, since products of quality specified by the population fraction defective are randomly subdivided into a collection of lots. It can be shown rigorously [See Hald (1981)] that, random samples taken without replacement from such a collection of lots having such a binomial distribution may be treated mathematically, as if they were actually samples from an infinite universe having a fraction defective p. The LTPD value refers to a specified lot fraction defective and the corresponding probability of acceptance, the consumer's risk, refers to the type A OC curve and is more specifically the consumer's A risk. The producer's risk, the probability of rejection of a product of acceptable quality refers to the corresponding type B probability of acceptance subtracted from 1. More specifically, this is producer's B risk. [See Dodge and Romig (1959)]

0.2.5 Sampling plans based on OC function

0.2.5.1 Sampling plans with given producer's and consumer's risks

We note that for large lots, the type B OC and type A OC curves of any sampling plan can be approximated by the corresponding Poisson probabilities, for lots containing small proportion defective. Given two quality levels, satisfactory (p_1) and unsatisfactory (p_2), one can find the acceptance number and the sample size of a single sampling plan such that the risks of rejection at p_1, i.e. producer's risk and risk of acceptance at p_2, i.e. consumer's risk are less than the respective stipulated values. Peach (1947) lists sampling plans for which the producer's risk and consumer's risk are both set at 0.05. Grubbs (1949) developed tables for producer's risk = 0.05 and consumer's risks = 0.10, Horsnell (1954) and Bowker and Lieberman(1955) produced tables for various values of producer's and consumer's risks. All these plans assume large lot sizes. Hald and Kousgaard (1966) provided a very comprehensive set of tables to obtain a single sampling plan with given producer's risk and consumer's risk. Hamaker (1950) prepared tables of single sampling plans using eight values of the indifference quality level (IQL) i.e. the quality at which both the consumer's and the producer's risk are equal to 0.50.

0.2.5.2 System of sampling plans with sample size as given function of lot size

If the sample size is a given function of lot size, the only unknown is the acceptance number, which may be determined by specifying a requirement to the OC function. In this case one may decide to use the relation like $n = a\sqrt{N} + b$ and fix the value of IQL (say). Then we may compute n for consecutive values of c, and find N from the assumed relationship of n and N. Hald(1981) observed that a system of this type has the same structure as that Bayesian sampling plans [discussed in subsequent sections] with continuous prior. The Military Standard 105D (discussed in the next section) is the most notable example of this.

Golub (1953) proposed plans when sample size was fixed by economic, administrative or other practical reasons and acceptance number was determined such that sum of producer's and the consumer's risks was minimized. Much later, Majumdar (1985) extended this system to double sampling plans.

0.2.6 Wartime development

0.2.6.1

Olds (1950) observed that at the time of the beginning of the war, companies used to take pride in the fact that no item of product was released to the consumer without 100% inspection. Any hint that bad material escaped this screening process was very distasteful and the weight of tradition blocked the sanction of any procedure based on the assumption that less than perfection could be tolerated. During war time the manufacturing industry was faced with a shortage of experienced inspectors. Some of them found that rational sampling procedure had resulted in higher level of quality. These two factors contributed to the decision that, in so far as possible, the acceptance inspection of ordnance materials (excluding critical characteristics) would be done on a sampling basis. Thereafter, work on standard sampling procedures and tables were undertaken, and educational training programs for the inspection personnel started in 1942. By 1944 twelve separate tables were constructed and put into operations by the office of US army ordnance. Around the same time, the office of procurement of materials of the US Navy department prepared a manual called "Standard Sampling Procedure".

0.2.6.2 Military Standards

The army ordnance tables were constructed under immediate assistance of Dodge and Gause. The most significant feature of these tables is the introduction of the concept of an acceptable quality level defined as the maximum percent defective which can be considered satisfactory as process average. Over the last 50 years, this system has developed into International Standards ISO 2859/Miliary standard 105 D/Military standard 105E/ ANSI/ASQC Z1.4. All these standards are either similar, or identical. The most widely used standard is Military Standard 105 D (1963) (abbreviated as MIL-STD-105D) adopted as an international standard ISO 2859 (1974).

This system is not based on an explicit mathematical model. For any lot size the table gives the corresponding sample size. The relationship was based on what was considered reasonable in practice. There is no explicitly theoretical foundation for the relation between sample size (n) and lot size (N). As the lot size increases, the sample size increases but at a lesser rate such that $n/N \to 0$ as $N \to \infty$.

The user of the table may choose between various "levels" of this relationship, called inspection levels. Next, one has to choose an AQL (given in percent defectives or defects per hundred) and find from the table the acceptance number. For given AQL values, the acceptance numbers are determined so that the producer's risk is small and decreasing with lot size.

As an important special feature of this system, it should be noted that the table contains effectively a set of three plans for each AQL, designated as normal, tightened and reduced inspection. If inspection results indicate that the producer's process average is possibly higher than the AQL agreed upon, then tightened inspection is introduced, which means that the consumer gets a higher protection against receiving unsatisfactory product and the producer's risk at the AQL increases. Moreover, if the inspection records show that the producer consistently submits products of better quality than the AQL it is *permissible to switch from normal to reduced inspection.* In this case the sample sizes are reduced but the acceptance numbers are adjusted, such that consumer's risk remains small.

0.2.6.3 Other sampling schemes

The other lot by lot attribute plans such as double and multiple sampling plans by Dodge and Romig (1941) and sequential sampling plans have been left out of the scope of our discussions to do justice to the focus of our present submission. However we must mention here that the theory of sequential probability ratio test was developed by Wald in 1943 and later by Barnard in the same year. The results were published by Wald (1945, 1947) and Barnard (1946) later. Detailed instruction for application, tables etc have been given by the Statistical Research Group, Columbia University (1945). These plans were developed taking into considerations the various economic aspects of production-inspection processes.

0.2.7 Bayesian sampling plans

Dodge (1950) remarked that "A product with a history of consistently good quality requires less inspection than one with no history or a history of erratic quality." Selection of a plan, therefore should depend upon the purpose, the quality history and the extent of knowledge of the process. This approach has resulted in formulation of what is known as Bayesian sampling plans. In this approach we attempt to obtain sampling plans minimizing overall average costs (consisting of inspection, acceptance and rejection cost) with respect to a given prior distribution of the process average. Dodge-Romig's model may be considered as a Bayesian plan, where cost function is given by the ATI and the prior distribution is a one point prior with outliers.

Hamaker (1951) and Anscombe (1951) are the early contributors to such economic theory of sampling inspection. The linear cost model is most common and there are a number of different ways to formulate it. The most detailed one is proposed by Gutheri and Johns(1959) and further simplified by Hald (1960). An essential feature of the model established by Hald (1965) is the critical or (economic) break-even-quality (p_r), at which the cost of acceptance and cost of rejection of a lot are equal. In a suitable common unit, the expected loss per batch is expressed as, cost of sampling + loss due to wrong decision. This loss is a function of lot size (N), sample size (n), acceptance number (c), p_r and the prior distribution of the process average.

In the absence of complete knowledge of the prior distribution a theoretical model is usually proposed to approximate the real situation. Sittig (1951) and later Champernowene (1953) used different forms of prior distributions (including beta) to arrive at the optimal sampling plans. Barnard (1954) proposed double binomial prior distribution. Taylor (1957), Horsnell (1957), and Wetherill (1960) proposed different forms of prior distribution. Hald provided tables for double binomial prior and subsequently, tables based on gamma prior [Hald (1981)], which can also be used as good approximation for beta prior distribution. Hald (1964, 1967, 1968) studied the asymptotic properties of Bayesian single sampling plans for an arbitrary process average curve differentiable in the neighborhood of breakeven quality value and a general loss function and investigated the efficiency of non-optimal plans. Chiu (1974) proposed a new prior based on a production situation where non-conformities arise, when a continuous variable product characteristic violates the specification limit. Under the assumption that this variable follows a normal distribution within each batch and the mean of the variable has a normal prior distribution he defines a prior distribution of the process average defective from such a process. This distribution may have different shapes from those of beta distributions, even when they have the same mean and variance, and Chiu (1974) refutes the contention of Weiler (1965) that beta distribution can generally approximate other prior distribution. He uses published data in support of his claim.

There have been many other studies devoted to economic design of acceptance sampling plans. For example Pandey (1974, 1984) has derived the Bayesian solutions for a two point prior for sampling plans with three decision criteria where lots are classified in three grades. For $p < p_a$ the product is accepted, for $p_a \leq p < p_b$ the product is 100% inspected and rectified, and for $p > p_b$, the product should be discarded. A survey paper by Wetherill and Chiu (1975) cites 253 references in acceptance sampling plans. Later works in this areas have been reviewed by Montgomery (1990).

0.3 The relevance of sampling inspection in today's economic scenario:

0.3.1: Sampling inspection as a quality practice

Many leaders in the quality field had argued, since 1950, against sampling inspection as a good quality practice and had emphasized instead on prevention of defects. Deming (1950) proposed a decision rule as an alternative to formal sampling inspection plan. If k_1 and k_2 are the inspection costs of one unit and the cost of accepting a defective unit respectively then the rule is to accept without inspection if the process average quality $p < k_1/k_2$, else do 100% inspection of the lot. Hald (1981), however, has taken this point into account when he defines the minimum unavoidable cost as the cost corresponding to the situation where perfect knowledge of quality exists without costs and all lots are classified correctly. If the process average remains stable at a given level one can perhaps take recourse to such rules.

Mood (1943) proved that the number of non-conformance in a random sample from the lot is independent of the number of non-conformance in the remainder of the lots if the process is under binomial control. Thus, in this situation, sample provides no information about the quality. These results are used to suggest not to use sampling inspection [Gitlow et. al (1991)]. One should however note that sampling inspection is not an alternative to process control. The purpose of the process control is to attain a state such that products are made right first time and every time. It is, however, a common experience that even for the most well established processes the process average shifts and the outputs from this process might go into next process or might be delivered to the consumers before such shifts are detected, corrective actions are initiated and effectively implemented. As pointed out by Milligan (1991) that well run processes go out of control occasionally, reliable and conscientious suppliers have problems from time to time, transportation networks sometime expose products and materials to difficult and inhospitable conditions often without warning.

Sampling inspection is often more economical and efficient than 100 per cent inspection for most of the processes in reality and may prove to be effective in protecting the internal and external customers. Moreover, a careful analysis of the results of sampling inspection can be and should be used for the purpose of continual improvement. All process models, starting from Dodge's, assume that the process average varies and the purpose of sampling inspection is to safeguard against accepting poor quality products.

0.3.2: Technological feasibility of 100% inspection

One notices that hundred percent inspection is becoming more and more feasible because of technological changes in certain manufacturing sectors. The use of laser, machine vision, pattern recognition and other techniques provide accurate and consistent results. With emphasis on defect level in the parts-per-million range, many industries are increasingly accepting on-machine automated 100 percent inspection and testing. Juran et. al (1996) mention seven categories of potential application of automated inspection. This result in reduction of amount of inspection at the end of the line and at times they are considerably more cost effective. A case in point is the use of a self inspecting and self correcting modern technology such as computerized numerical control system (CNC). [Tang and Tang (1989)]

Even in these situation, taking such technological advantages may not be feasible for all processes providing inputs to a major process of an organization. Some of them are likely to be located elsewhere, using technology with lower degree of sophistication. To take recourse to 100% inspection at each receiving /delivery end is not feasible and may prove to be disastrous.

Moreover, inspection whether 100% or sampling does not correct the causes of quality problems. It therefore follows that as a good management practice an organization does not rely on mass inspection. On the other hand one makes attempt to bring uniformity in quality assurance system through standardization so that improvement becomes meaningful.

Any modern organization is likely to employ different types of verification alternatives such as no inspection, on line self-inspection, lot-by-lot hundred percent inspection and lot-by-lot sampling inspection depending upon the requirements of internal and external customers at various stages of the product realization processes. It is important to establish and implement a quality management system so that verification becomes useful for the dual purpose of a) pinpointing areas of corrective action and b) providing necessary data for identification and prioritization of activities of preventive and improvement action.

0.4 Scope of the present inquiry

0.4.1: The multiattribute sampling inspection

At the outset, we take into account the fact that the number of elements of the set of characteristics to be verified, for the purpose of acceptance/rejection of lots or batches of mass produced items, is unlikely to be just one in most of the practical situations. For example, a sample of finished garment is verified for 15 characteristics (all attribute type) before delivery. Hansen (1957) has listed 130 attributes for automotive final inspection and road test. We attend to the problems specific to formulating lot-by-lot multiattribute sampling inspection schemes in such situations.

0.4.2 The existing method of choosing sampling inspection plans based on seriousness of characteristics.

The quality characteristics are decidedly unequal in their effect on fitness for use. A relatively few are serious i.e. of critical importance; many are minor. Clearly, the more important the characteristic, the greater the attention it should receive in such matters as: extent of quality planning, precision of processes, tooling and instrumentation, sizes of samples, strictness of conformance etc. To this end many companies utilize formal systems of seriousness classification. Juran and Gryna(1996) tabulate one such classification scheme in food industry.

The general practice of the industry has been to assign different AQL values and employ effectively a parallel system of sampling inspections. For example, Hansen (1957) reported adaptation of MIL-STD-105A to the acceptance inspection of the M38A1 truck commonly called 'the jeep' manufactured by Wiley's Motors Inc, in Toledo, Ohio. Two hundred and four characteristics were classified in four classes, namely: special defects (hundred per cent inspection) comprising of 11 attributes, major defects (AQL 15 per one hundred vehicle) comprising of 14 attributes, minor defects (AQL 150 defects per 100 vehicle) comprising of 69 attributes, incidental defects (400 defects per 100 vehicles) comprising of 110 attributes.

The MIL-STD-105D recommends classification of defects and designate different AQLs for groups of defects or for individual defects as the case may be. However, since sample size is taken as a function of lot size (for most of the time), the acceptance number get affected by variable AQL's. For lot sizes around 50, for the vehicle example, we take a sample of size 8 (inspection level II, normal inspection) and inspect for all three classes of attributes and accept the 'jeep', if the number of major defects is less than or equal to 3, the number of minor defects is less than 15 and the number of incidental defects is less than 44.

0.4.3 The producer's risk, consumer's risk and the sensitivity of the OC function for a multiattribute sampling plan

For a multiattribute sampling plan we might consider two aspects of the OC function a) the effective producer's and the consumer's risks and b) sensitivity of the OC function with changes in defect levels of different classes/types of attributes. Secondly, we thought it reasonable to expect that the OC function should be more sensitive to the changes in the defect level of more important attributes, particularly in a situation where unsatisfactory defect level occurs due to more serious type of attributes. It is, therefore, necessary to develop a measure for the sensitivity. We may, thereafter, investigate the consequences of using the sampling plans of the type being used in practice.

0.4.4 The nature of defect occurrence

We would of course keep in mind that, in all situations, attributes need not be classified in mutually exclusive category as given in the above example. In some situations, defect occurrence may be mutually exclusive i.e. occurrence of one type of defect preclude the occurrence of another. For example the defect 'Omission of tags' preclude the defect 'Wrong size tag' of garment verification. On the other hand, defect of different types may occur as jointly independently. For example, the functional defect and surface defect for a metal closure might occur as independent of each other. We should therefore try to arrive at a general expression for the type B OC function in both the situations where defects occurrences are either jointly independent or they occur mutually exclusively.

0.4.5 Alternative acceptance criteria

Our next query is whether it is possible to improve the discrimination power in the above sense by taking recourse to alternative acceptance criteria. For the case of three attributes we might take a sample of size n and observe x_1, x_2, x_3 the number of defectives for the first, second, and third attributes respectively, arranged in the order of importance. Instead of the acceptance criterion: accept if $x_1 \leq c_1, x_2 \leq c_2, x_3 \leq c_3$, we may consider the acceptance criterion: accept if $x_1 \leq a_1, x_1 + x_2 \leq a_2; x_1 + x_2 + x_3 \leq a_3$. Bray, Lyon and Burr (1973) have proposed similar acceptance criterion in the context of a situation where an item is verified for a single attribute but can be classified as good, marginal and bad. The alternative mentioned in the context of multiattribute plans was first proposed by Majumdar (1979) and successfully tried by industries.(e.g. Metal box,India) thereafter.

0.4.6 A generalized acceptance sampling scheme

We would hereafter attempt to develop a sampling inspection scheme using the sample size - lot size relation of the MIL- STD-105D standard, using the above mentioned alternative acceptance criterion. In this exercise we would try to induct the features: a) producer's risk (i.e. the effective producer's risk for a multiattribute inspection plan) is reasonable from the point of view of the industry, and b) the scheme possesses the property that the OC function for a given plan is more sensitive to the changes in the defect level of more important attribute (s) .

0.4.7 Sampling plans with given producer's and consumer's risks

Having established a general expression for the type B OC function for a multiattribute single sampling plan (MASSP) we draw our attention to the construction of a MASSP, given two quality levels in the form of vectors, satisfactory ($\boldsymbol{p}$) and unsatisfactory ($\boldsymbol{p'}$), such that the risks of rejection at $\boldsymbol{p}$ (producer's risk) and risk of acceptance at $\boldsymbol{p'}$ (consumer's risk) are nearly same as, but not more than those stipulated. Note that the $\boldsymbol{p}$ and $\boldsymbol{p'}$ denote the process average vectors.

It is obvious that the acceptance parameters of the sampling plans which will be arrived at will be specific to the acceptance criteria. We shall, therefore, attempt to develop sampling inspection schemes accordingly. We may choose different such schemes based on different criteria compare them with respect to the sample size(say).

0.4.8 Bayesian Plans

0.4.8.1 The existing models

The economic design of multiattribute sampling schemes taking account of Bayesian principles based on appropriate prior distribution was considered by Schmidt and Bennett (1972), and further by Case, Schmidt and Bennett (1975), Ailor, Schimdt and Bennet (1975), Majumdar (1980, 1990, 1997), Moskowitz, Plante and Tang (1986), Moskowitz, Plante and Tang and Ravindran (1984) and Tang, Plante and Moskowitz (1986).

Schimdt et. al (1972) assumed 1) destructive inspection and 2) scrapping of rejected lots. In Ailor, Schimdt and Bennett (1975) 's model the characteristics of interest could be a mixture of attributes and variables and the corrective action on the rejected lot would be either scrapping or screening.

Case et. al (1975) considered no screening/sorting on the rejected lots; all rejected lots are scrapped. Tang et. al (1986) classified attributes in two classes, scrappable and screenable . For the rejection due to scrappable defects, cost of rejection is proportional to the lot size, irrespective of number of defects present in the remainder of the lot. For rejection due to screenable attributes, the cost of rejection is proportional to the number of items screened (inspected) and no fixed cost is incurred. Whenever a lot is rejected due to scrappable defect, the remainder of the lot is not at all inspected for any other attributes (scrappable or screenable). In the event the lot is not rejected on the basis of scrappable attribute (s) but on the basis of screenable attribute(s), the lot is not tested for scrappable attributes at all.

Moskowwitz, Plante,Tang, Ravindran (1984) introduced the concept of multiattribute utility theory into quality control acceptance schemes and developed a general multiattribute sampling scheme, using the criterion of maximizing total utility. The process averages $(p_i; i = 1, 2, ..., r)$ are considered as independently distributed beta variables. The cost model considered is quite general.

The above models assume that the process averages corresponding to the attributes included are jointly independent and each has a continuous prior distribution for the process average measuring the lot quality. Optimal solutions for specific problems using direct search or pattern search methods are obtained. Moreover each attribute has its own sample size and acceptance number.

0.4.8.2 The situations considered by us

In the present context of the economy we note the following:

a) Screening is becoming more and more feasible due to rapid growth in computerized testing and inspection system. In many fields today the sampling inspection is relevant only for deciding whether to accept or screen. The basis for discerning the attributes in these situations depends on their contribution to cost components. In any case even for a scrappable attribute, one may not be willing to reject the whole lot at a fixed cost as assumed by Case et. al (1975) or at a cost proportional to the lot size [Tang et. al (1986)], more so in the situation of nondestructive testing.

Moreover, the assumption that a lot rejected for a single scrappable attribute is not screened at all for other attributes (scrappable or screenable) may not hold in many situations. For example, a lot rejected for a scrappable attribute like undersize diameter may be screened for defect like oversize diameter for which one can rework an item. However, this is not to say that the assumptions made by the authors in the earlier models do not hold in all situations.

b) If we use different sample sizes for different attributes we save (in most situations) only testing cost and not on sampling cost, because we may (in many situations) have to draw a sample of size max (n_i), n_i denoting the sample size for the i th attribute, $i > 1$; to enable testing for all attributes. There are many practical situations where testing is done simultaneously on all attributes. Examples can be cited for finished garment checking, visual inspection of plastic containers, regulatory testing for packaged commodities like biscuits, nondestructive testing for foundry and forged items, etc. For testing of components of assembled units the general practice is to test all components in respect of all the sampled items. Similar practices are considered as practical for screening/sorting of the rejected lot.

c) The defect occurrences for different attributes in the lot or in the sample are considered as jointly independent. There are many situations where defect occurrences in the lot/sample may be mutually exclusive. This may happen due to the very nature of defect occurrences e.g. a shirt with a button missing and a shirt with a wrong button; undersize and over size dimensions verified by go-no go gauge or when defects are classified in mutually exclusive classes e.g. critical, major and minor. It would therefore be necessary to take care of both the situations.

We have, however, restricted ourselves to those situation where a) the inspection is nondestructive b) we take a single sample of size n and inspect for all attributes c) the occurrences of defects of the different types are jointly independent or d) mutually exclusive e) for each attribute there is a cost component for rejection, proportional to the number of defective items inspected in the sample or in the rejected lots f) for each attribute there is a cost of acceptance of a defective unit g) the prior distribution of process averages are either discrete or continuous.

The cost models discussed in the thesis were developed in Majumdar (1980) and Majumdar (1990), Majumdar (1997) which may be considered as extension of the cost model proposed by Hald (1965) for the single attribute case to the multiattribute situation.

0.4.8.3 Prior distributions

In a multiattribute product situation, the prior distribution under continuity assumptions in respect of the process average levels corresponding to the distinct attributes have always been assumed to be independent so far. To find out what distribution is to be used as prior for a process average level corresponding to each of the attributes, we need to analyze inspection data on multiattribute product inspection. As noted by Hald (1981) and also by Chiu (1974) that published data on quality variation are very scarce. We have, therefore collected data from industry and used them to understand the appropriateness of any assumption regarding the distribution of process average levels. These data have been obtained while providing professional assistance to these organizations. The data are analyzed to examine whether the prior distribution can be considered as discrete e.g. two point prior depicting dependence of the attributes in a sort of extreme. In case where two point discrete prior is inappropriate, a continuous prior is assumed with usual independence assumptions and the parameters of the theoretical distribution are estimated from the empirical frequency distribution obtained.

0.4.8.4 Economic sampling scheme

We focus primarily on the relative economy of different alternative acceptance criteria in the above situations. For this we try to obtain general results as for as possible based on general prior distributions. Further we do necessary computations mostly to verify these results. We write simple computer programs as Excel Macros, using Visual Basic Application (VBA). We have extensively used the built in Excel worksheet functions of beta, binomial, Poisson, gamma, χ^2, negative binomial etc. in our programs.

0.4.8.5 Restricted Bayesian Plans

As pointed out by Hald (1981), the Bayesian solution is theoretically interesting, but not very useful in practice, because it rests on the assumption that the prior distribution is stable and that no outliers occur so that for small and medium lot sizes the Bayesian plan will often be to accept without inspection or to use a sampling plan with poor discriminating power. To remedy this effect one imposes a restriction on the OC function. These plans are known as restricted Bayesian plans. The Dodge-Romig's LTPD plans may be considered as restricted Bayesian Plans. We have not been able to locate any published work on restricted Bayesian plans in the multiattribute context. We wish to extend our inquiry in this field in future and the present thesis does not include any inquiry into this topic.

0.5 Summary of the dissertation : Sampling inspection schemes by attribute for multiple quality characteristics

The dissertation has three parts, Part 1 is divided into 3 chapters, part 2 into 5 chapters and part 3 has only one chapter. The partwise and chapterwise summaries of the dissertation, highlighting the major theoretical findings are enumerated in the sections which follow.

0.5.1 Part1 : Multiattribute sampling inspection Plans based on OC function

0.5.1.1 Chapter 1: Introductory concepts

Irrespective of the type of product, evaluation of conformity to specified requirements of its quality characteristics is an integral part of quality assurance. Although they form a set of necessary verification activities almost at all stages of production these activities, known as inspection do not add value to the product on their own and are to be kept at their minimum. The sampling inspection where a portion of a collection of product units is inspected on a set of characteristics with a view to making decision about acceptance or otherwise becomes relevant in this context. We attend to the designing of the procedure of sampling inspection with the above purpose in mind. The scope of the present exercise however is limited to only attribute type verification of a recognizable collection of countable discrete pieces, called a lot, submitted for acceptance on a more or less continuous basis.

At the outset we take into account the fact that number of elements of the set of characteristics to be verified is unlikely to be just one in most of the practical situations. Multiattribute sampling inspections are employed at all stages of production, starting from raw materials through semi finished to finished or assembly stages, where consecutive collections of products called lots, are submitted for acceptance or otherwise. The inspection lot is either a suitable production batch or a shift's production, or a customer's order specific collection.

In this context we consider those situations where it is possible to take a sample of size n and inspect for all the attributes in any order. For most of the cases the defect occurrences for different types of attributes are independent. For some attributes, however, the defect occurrences are such that occurrence of one type of defect precludes the occurrence of any other type. We suppose that there are r attribute characteristics inspectable separately for any product unit and if we take a sample of size n from a lot of size N, all the r characteristics are observable in any order. The sampling scheme we are interested in should be applicable when lots are submitted consecutively in a more or less continuous manner.

For a single attribute, Dodge and Romig (1959) defined product of quality p as the the conceptually infinite output of units from a stable process for which the probability of producing a defective unit is p, which is also called the long term fraction defective in the process or the process average. Also, it is assumed that in a stable process, the probability of a unit being defective does not depend on whether any other unit in the process turns out to be defective or not. The concept of type B OC curve of a sampling plan as formally defined by Dodge and Romig (1959) is the curve showing probability of accepting a lot as a function of product quality, where product quality is expressed as fraction defective p. So lots are outcomes of a production process, which constitutes as described a Bernoullian sequence with parameter p i.e. in a lot of size N, the number of defectives in the lot, X follows binomial distribution with parameter N, p and if a sample of a fixed size n is drawn at random without replacement from the lot, then the number of defectives observable in such a sample, x follows binomial distribution with parameters n, p. Thus,

$$Pr(x|p) = b(x, n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

For a single sampling plan the type B OC function in this case is given as,

$$B(c, n, p) = \sum_{x=0}^{c} b(x, n, p).$$

In case of $r(\geq 2)$ attributes, $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ denotes the process average vector, where p_i is the long term fraction defective in the process in respect of the i th attribute, $i = 1, 2, ..., r$.

For a lot of size N, the quality is measured by the vector $\boldsymbol{X} = (X_1, X_2, ..., X_r)$, where $X_i =$ number of defectives in the lot in respect of the i th attribute, $i = 1, 2, ..., r$.

A random sample of size n is drawn from a lot and the verification of non-conformity with respect to (w.r.t.) r specified attributes can be carried out in the sample in any order. The number of defectives vector for the sample is denoted as $\boldsymbol{x} = (x_1, x_2, ..., x_r)$.

Two situations are considered viz., (i) Defective units w.r.t. the different attributes occur independently i.e. $X_1, X_2, ..., X_r$ are jointly independently distributed and $x_1, x_2, ..., x_r$ are consequently jointly independently distributed. (ii) Defective units w.r.t. the different attributes occur mutually exclusively i.e. the occurrence of one kind of defect precludes the occurrence of another kind of defect. In this case $(x_1, x_2, ..., x_r)$ follows multinomial distribution with parameters $n, p_1, p_2, ..., p_r$

In both the situations mentioned, under Poisson conditions [Hald (1981)], which means $p_i \to 0, np_i$ finite, for all i and in addition $\sum_{i=1}^{r} p_i \to 0$ in situation (ii) in the production setup assumed already,

the type B probability for a realization of $x_1, x_2, ..., x_r$ in a sample of size n can be approximated as

$$\prod_{i=1}^{r} g(x_i, np_i),$$

where $g(y, m)$ represents an individual term of Poisson distribution denoted by $e^{-m}m^y/y!$, $y = 0, 1, 2...$

...(0.5.1)

Different sampling schemes and related acceptance criteria are studied. In all the schemes a single sample of size n is drawn from the lots. assuming the acceptance criterion to be as follows : if $\boldsymbol{x} \in A$, accept the lot and otherwise reject it.

Then, the type B probability of acceptance of the lot at $\boldsymbol{p}$ under Poisson conditions can be approximated as,

$$P(\boldsymbol{p}) = \sum_{\boldsymbol{x} \in A} \prod_{i=1}^{r} g(x_i, np_i)$$

...(0.5.2)

The inspection scenario may change from identification of a defective unit to the counting of defects in a unit. In case we count the number of defects per item for each characteristic we may construct a model for which the number of defects for a unit for i th the characteristic equals to p_i in the long run. We assume the number of defects for r distinct characteristics in a unit are indepedently distributed. The output of such a process is called a product of quality $(p_1, p_2, .., p_r)$.

In this case x_i = number of defects w.r.t. attribute i in a random sample of size $n, i = 1, 2, ..., r$. The expression (0.5.1) holds here exactly for a realization of $\boldsymbol{x} = (x_1, x_2, ..., x_r)$ in a random sample size n and the probability of acceptance, $P(\boldsymbol{p})$ given by (0.5.2) holds exactly too.

Along with the two situations under appropriate assumptions mentioned before, where we count the number defective units w.r.t. different characteristics in the sample, we include under Poisson conditions the present situation of defect verifications also w.r.t. a number of characteristics under considerations. The type B OC function given in (0.5.2) for the appropriate acceptance criterion stated covers all these situations of defectives / defects under given assumptions discussed.

0.5.1.2 Chapter 2: Multiattribute sampling scheme based on AQL

This chapter is devoted to the objective of establishing, for a multiattribute situation, a sampling scheme in line with available international standards tabulated on the basis of Acceptable Quality Level (AQL). The published plans most widely used is the US Military standard 105D (1963) [abbreviated as MIL-STD-105D], which has been adopted as an international standard (ISO 2859). The standard is highly suitable for the consumers.

The sample size used in MIL-STD-105D is determined from the given lot size and by the choice of inspection level. The acceptance number (c) is arrived at, such that it remains more or less same for the same value of the product of sample size and the AQL to ensure that the Poisson probability of acceptance (rejection) remains same for the plans. This ensures a desired producer's risk. There are 13 sets of n.AQL values and 11 acceptance numbers chosen in such a way that the producer's risk (excepting for $c = 0$) varies from maximum of 9.02% to a minimum of 1.44%. With increase in the n.AQL values, c increases, so that producer's risk decreases up to a level of 2% and thereafter it is kept at less than 2%. There is no theoretical foundation for the relation between sample size and lot size. As the lot size increases, the sample size increases but at a lesser rate such that $n/N \to 0$ as $N \to \infty$.

The Standard prescribes that separate plans are to be chosen for the different classes of attributes. For example, the plan for a critical defect will have generally a lower AQL value than the plan for a major defect and the plan for a major defect will have an AQL value lower than the plan for a minor defect.

To examine the consequences of constructing a sampling plan by this method for our multiattribute situation we first consider the effective producer's risks.

Secondly, we consider it reasonable to expect that the OC function should be more sensitive to the changes in the defect level of more important attributes, particularly, in a situation, where unsatisfactory defect level occurs due to more serious type of defects. Suppose that the defect/ defective on i th attribute is more serious than that on the j th attribute then, as the total defect level changes from low to high, the absolute value of the slope of the OC w.r.t the i th type of defect / defective as a function of the total defect / defective level should have always higher value than the absolute slope of the OC w.r.t the j th type of defect / defective, assuming that the relative contribution of a characteristic to the total defect / defective remains constant.

We examine the cases of two manufactures: one who needs to verify the quality of plastic components/washer procured from vendors for packaging cosmetics and, another, a garment manufacturer who wants to make sure the quality of his product before shipment. Both of them categorize their defect types as critical, major and minor and choose the AQL's accordingly. For a given lot size they find the sample size using the MIL-STD-105-D and the value of three acceptance numbers. A lot is accepted only when the acceptance criterion is satisfied for each one of the three types of defects. In these examples we obtain the producer's risk as more than 10.5% for the first set of plans (plastic container) and around 22% for the second set (garment) of plans. It would appear that we are asking for the moon from the producer.

We have thereafter computed the slope of the OC in the manner as explained, at different process averages keeping the defect contribution of each type of attribute as proportional to its AQL value. Using this as a measure of sensitivity, the OC should appear to be more sensitive to the changes of major defect than to the changes of minor defect, when the overall quality level becomes unsatisfactory. But unfortunately the features observed depict a picture far from this ideal.

We have further considered the class of plans where we take a sample of size n, accept the lot if and only if $x_i \leq c_i; \forall i$. We call these plans as multiattribute single sampling plans (MASSP) of C kind. Note that all MASSP's constructed from MIL-STD-105D as suggested above are C type plans.

The probability of acceptance for this plan at $(p_1, p_2, ..., p_r)$ under Poisson conditions :

$$\sum_{x_1=0}^{c_1} \sum_{x_2=0}^{c_2} ... \sum_{x_r=0}^{c_r} \prod_{i=1}^{r} g(x_i, m_i) = \prod_{i=1}^{r} G(c_i, m_i)$$

. where $G(c_i, m_i) = \sum_{x_i=0}^{c_i} g(x_i, m_i); \quad m_i = np_i.$

...(0.5.3)

We obtain, writing $PC'j$ as the partial differential coefficient of the OC function w.r.t

m_j,

$$-PC'_j = g(x_j, m_j) \prod_{i=1:i\neq j}^{r} G(c_i, m_i)$$

...(0.5.4)

Defining
$m = m_1 + m_2 + ... + m_r$; $\rho_i = m_i/m$, we write,
$$Slope_j(m) = g(x_j, m\rho_j) \prod_{i=1:i\neq j}^{r} G(c_i, m\rho_i)$$

...(0.5.5)

We have shown that for a given set of ρ_i, the function $H(m) = Slope_i(m) - Slope_{i+1}(m)$ $i = 1, 2, ..., r-1$ undergoes atmost one change of sign from positive to negative for $m \geq 0; i = 1, 2, ..., r-1$. Further, for $\rho_{i+1}/\rho_i > c_{i+1}/c_i$, there is exactly one real positive root for $H(m) = 0$.

If we now set $\rho_{i+1}/\rho_i = AQL_{i+1}/AQL_i$ and try to ensure a reasonable producer's risk, the inequality, $\rho_{i+1}/\rho_i > AQL_{i+1}/AQL_i$ is found to be a reasonable assumption to be usually satisfied. It, therefore, follows that in such a case, the plan will fail to satisfy the condition $Slope_i(m) \geq Slope_{i+1}(m)$ for all $m > 0$. We conclude that in general there is no good C kind plan in the sense defined as above.

For $r = 3$, we have further constructed the sampling plans taking all 286 possible ordered triplets of n.AQL values and possible n and c values as tabulated in the MIL-STD-105D table. We find that the producer's risk varies from 3.6% to 34.2%. There are 103 plans with producer's risk more than 16 % and there are only 34 plans with producer's risk less than or equal to 6%.
Further, there are 183 plans which do not always satisfy the condition of higher absolute slope of the OC for more serious type of attribute defect associated with lower value of AQL, as compared to the slope of OC for the attribute associated with higher value of AQL, in the range of p ($p = p_1 + p_2 + p_3$, $p_1 : p_2 : p_3 = AQL_1 : AQL_2 : AQL_3$), from $p = 0$ to the limiting quality, defined as the value of process average for which the probability of acceptance is around 0.10.

We have, therefore, introduced in this chapter a sampling scheme consisting of plans with alternative acceptance criteria : accept if $x_1 \leq a_1; x_1 + x_2 \leq a_2; ...; x_1 + x_2 + .. + x_r \leq a_r$; reject otherwise. We call this plan an MASSP of A kind. We prove that $-PA'_i \geq -PA'_{i+1}$ for $i = 1, 2, .., r-1$ where PA denotes the probability of acceptance for the plan at a given process average and $-PA'_i$ is the absolute value of the slope of the OC function w.r.t m_i at a given $m = m_1 + m_2 + ... + m_r$, assuming m_i/m fixed, $i = 1, 2, ..., r$.

The above property of the A kind MASSP's, therefore, allows us to order the attributes in order of their relative discriminating power. It also follows that if the attributes are ordered in the ascending order of AQL values, then it is possible to construct a sampling scheme ensuring an acceptable producer's risk and also satisfying the condition of higher absolute slope for the lower AQL attribute. Using the set of n.AQL values chosen from MIL-STD-105D we establish a MASSP scheme consisting of A kind MASSP's. These plans will have same sample size as used by the MIL-STD-105D standard for a given lot size, but will have different acceptance criteria. We have done this exercise for $r = 3$, using the following procedure.

There are 13 n.AQL values in the MIL-STD-105D. We get 286 triplets such that $n.AQL_1 < n.AQL_2 < n.AQL_3$. We further order these 286 plans in the usual lexicographic fashion. We start with the first combination of (0.1256, 0.1991, 0.3156) and choose $a_1 = 1, a_2 = 1$, and $a_3 = 2$. This gives producer's risk of around 5.5%. All other sets of acceptance numbers are worked out such that a plan positioned later will have a lesser producer's risk, so that producer's risk decreases up to a level of 5%, thereafter it is kept at less than 5%. It is heartening to note that the largest value of ratio of the limiting quality level (as defined) to the total AQL has ranged from 1.5 to 7.4, so that the OC appears to be quite steep. For the MIL-STD-105D single sampling plans the comparable ratio varies from 1.7 to 18.3. [Hald (1981)]

These plans are tabulated for ready use by the industries, encountering the need for multiattribute inspection schemes as described already. Similar exercises can be done for other values of r.

0.5.1.3 Chapter 3: Sampling Plans with given producer's and cosumer's risk

We have considered in the last chapter two schemes: one, consisting of plans of A kind and, the other consisting of plans of C kind, which differ in terms of acceptance criteria. In this chapter, we try to construct MASSP's defined by two equations $Q(\boldsymbol{p}) = \alpha$ and $P(\boldsymbol{p\prime}) = \beta$. where $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ is a satisfactory quality level and $\boldsymbol{p\prime} = (p'_1, p'_2, ..., p'_r)$ is an unsatisfactory quality level; $Q(\boldsymbol{p})$ is the probability of rejection at $\boldsymbol{p}$ and $P(\boldsymbol{p\prime})$ is the probability of acceptance at $\boldsymbol{p\prime}$. Further α and β are the stipulated producer's and consumer's risks respectively.

For r = 2, we have been able to formulate procedures to obtain both C kind and A kind plans given the above mentioned risks.

For the C type plan we solve : $G(c_1, np_1)G(c_2, np_2) = 1-\alpha$ and $G(c_1, np'_1).G(c_2, np'_2) = \beta$. We restrict to the situation where $p_1/(p_1 + p_2) = p'_1/(p'_1 + p'_2) = \rho$. For a given ρ we first define $m_P(c_1, c_2, \rho)$ as the value of m satisfying the equation $G(c_1, m\rho)G(c_2, m(1 - \rho)) = P$ and note that we must have n, c_1, c_2 such that $m_{1-\alpha}(c_1, c_2, \rho) = m; \quad m_\beta(c_1, c_2, \rho) = m'$; $m = np_1 + np_2; m' = np'_1 + np'_2$.

We introduce the auxiliary function $R(c_1, c_2, \rho, \alpha, \beta) = m_\beta(c_1, c_2, \rho)/m_{1-\alpha}(c_1, c_2, \rho)$. We show that if we increase c_2 keeping c_1 as fixed then as $c_2 \to \infty$, $R(c_1, c_2, \rho, \alpha, \beta)$ first decreases to a minimum and then increases and go on increasing so that the function has a unique minimum in the range $m > 0$. We call this as $c_2^{(c_1)}$. We have further noted that $c_2^{(c_1)} > c_2^{(c_1+1)}$.

The task is now to find uniquely the value of c_1, and c_2 such that: $R(c_1, c_2 - 1, \rho, \alpha, \beta) > p'/p \geq R(c_1, c_2, \rho, \alpha, \beta)$ for the smallest value of c_1. Since c_1, c_2 are all integers we must consider the set of c_1, c_2 values for which $R(c_1, c_2, \alpha, \beta)$ is less than p'/p and choose the c_1 and c_2 for which $m_\beta(c_1, c_2, \rho)$ is minimum. This will ensure us the minimum sample size obtained as $n = m_\beta(c_1, c_2, \rho)/p'$, where $p' = p'_1 + p'_2$, satisfying the stipulations of the producer's and consumer's risks at specified points.

We could, therefore develop a step by step procedure to obtain the desired sample plan. To facilitate the above task we may construct a table containing $c_1, c_2, m_\beta(c_1, c_2, \rho), m_{1-\alpha}(c_1, c_2, \rho)$ arranged in descending order of $R(c_1, c_2, \rho, \alpha, \beta)$ for a given α, β, and ρ. We present the table for $\alpha = 0.05$, $\beta = 0.10$ and $\rho = 0.1$.

For the A kind plans of given strength we want to satisfy the equations : $PA(a_1, a_2; np_1, np_2) = 1 - \alpha$; $PA(a_1, a_2; np'_1, np'_2) = \beta$; $PA(a_1, a_2; np_1, np_2)$ denotes the probability of acceptance at (p_1, p_2). We restrict to the situation where $p_1/(p_1 + p_2) = p'_1/(p'_1 + p'_2) = \rho$. For a given ρ we define $ma_P(a_1, a_2, \rho)$ as the value of m satisfying the equation : $\sum_{x_1=0}^{x_1=a_1} g(x_1, m\rho)G(a_2 - x_1, m(1-\rho)) = P$. We have to obtain n, a_1, a_2 for a given ρ, α and β such that $ma_{1-\alpha}(a_1, a_2.\rho) = m$ and $ma_\beta(a_1, a_2, \rho) = m'$.

As before we introduce the auxiliary function $Ra(a_1, a_2, \rho, \alpha, \beta) = ma_\beta(a_1, a_2, \rho)/ma_{(1-\alpha)}(a_1, a_2, \rho)$. We find that $ma_P(a_1, a_2, \rho)$ is an increasing function of a_1, a_2 ; $Ra(a_1, a_2, \rho, \alpha, \beta)$ is a decreasing function of a_1 and a_2, keeping the other parameters fixed.
We obtain the smallest $a_2 = a_2^*$ for which $Ra(0, a_2^*, \rho, \alpha, \beta) > p'/p \geq Ra(a_2^*, a_2^*, \rho, \alpha, \beta)$ and then find a_1^* such that, $Ra(a_1^* - 1, a_2^*, \rho, \alpha, \beta) > p'/p \geq Ra(a_1^*, a_2^*, \rho, \alpha, \beta)$;. and the desired value of n is obtained by dividing the value of $ma_\beta(a_1^*, a_2^*, \rho)$ by p'.

To facilitate this task we construct a table containing $a_1, a_2, ma_\beta(a_1, a_2, \rho), ma_{1-\alpha}(a_1, a_2, \rho)$ arranged in descending order of $Ra(a_1, a_2, \alpha, \beta, \rho)$. For $\alpha = 0.05, \beta = 0.10$ and $\rho = 0.1$ the result has been presented in this chapter.

We now introduce a MASSP of D kind as the one with the following rule : from each lot of size N, take a sample of size n, accept if total number of defects of all types put together is less than or equal to k, otherwise reject the lot.

For obtaining the D type MASSP of given strength, we use the fact that under Poisson conditions the OC at $(p_1, p_2, ..., p_r)$ is identical with that of single sampling plan for single attribute with sample size n and with acceptance number k, at $p = p_1 + p_2 + .. + p_r$. Thus, the fixed risk D type plan satisfies $Rd(k-1) > p'/p \geq Rd(k)$ where $Rd(k) = md_\beta(k)/md_{1-\alpha}(k)$; $G(k, md_P) = P$. We obtain n from $n = md_\beta(k)/p'$; $p' = p'_1 + p'_2 + ... + p'_r$. $p = p_1 + p_2 + ... + p_r$. These plans have been extensively tabulated by Hald(1981) for different values α, β, and p'/p

for integer as well as noninteger values of k.

We are now in a position to compare the sample size of a plan of A kind with that of plan of C kind of the same strength. We do this by taking the sample size of a D kind plan of the same strength as the datum. As an example, we have taken $\alpha = 0.05$, $\beta = 0.10$ and $\rho = 0.1$, plotted the sample sizes of the A kind plan, C kind plan and the D kind plan against different values of p'/p. In this case the sample size for the plan A is marginally higher than that of Plan D. The sample size for the Plan C is significantly more than the equivalent Plan D as well as that for plan A in this case. Similar results hold for other values of ρ as well.

To summarise the results of this chapter we note that for $r = 2$ it is possible to uniquely determine the fixed risk minimum sample size MASSP's for a given value of ρ. Taking note of the results of chapter 2.2 that the A kind plans also satisfy the condition of higher absolute slope of the OC for the lower AQL attribute and the plans of C kind and D kind do not ensure the same, and the fact the sample size of the A kind plans are likely to be much less than the comparable C kind plans, we might choose an A kind plan as a better alternative in general.

0.5.2 Part 2: Bayesian multiattribute sampling inspection plans - general cost models and discrete prior distributions

0.5.2.1 Chapter 1: General cost models

To start with we look at the models used by various authors proposed in the area of Bayesian MASSP's. Among them are Case et.al (1975), Moskowitz et.al (1984), and Tang et.al (1986). These models are reviewed and a general cost model which was developed by the author Majumdar (1980, 1990) and extended later by Majumdar (1997) for nondestructive testing under assumption of independence and also for mutually exclusive categories of defects/defectives using n_i (sample size for the i th attribute) $= n$. for all i, in line with the one presented by Hald (1965) for a single attribute, is included in this chapter. The details with the appropriate distributions are worked out in the later chapters.

We have restricted ourselves to those situations where a) the inspection is nondestructive b) we take a single sample of size n and inspect for all attributes c) the occurrences of defects of different types are jointly independent or d) mutually exclusive e) for each attribute there is a cost component for rejection proportional to the number of units inspected in the sample or in the rejected lots f) for each attribute there is a per unit cost of acceptance and rejection of defective unit g) the distribution of process averages are either all discrete and when continuous the prior distributions are assumed as independent.

It has been shown that under Poisson consitions the same expression for probability of acceptance for any specified acceptance criteria can be used in both the situations of independence and mutually exclusive occurrences of defect types. And therefore, the same expression for cost functions can be used in both the situations under Poisson conditions. Moreover this expression can also be used as an exact form, in case we count the number of defects per item for each characteristic and assume that the number of defects for r distinct characteristics in a unit are indepedently distributed. Our purpose is to focus on relative merits of alternative acceptance criteria, we therefore choose the general models developed in Majumdar (1980, 1990, 1997).

0.5.2.2 Chapter 2: The expected cost model for discrete prior distributions

By Bayesian plans we understand the plans obtained by minimizing average cost which has three identifiable components viz. inspection cost, acceptance cost and rejection cost. For these plans the process average vector is taken to be a random variable. In our present context the prior distribution (i.e. the distribution of process average) is the expected distribution of lot quality vector on which the sampling plan is going to operate. For the multiattribute Bayesian plans considered by others the process average for each attribute is assumed to follow a beta distribution so that the lot quality distribution for each attribute follows a beta binomial. Thus, in a situation when defect occurrences are jointly independent the product of individual beta distributions is chosen as an apprpriate prior. Even when the prior seems to be quite appropriate the process will occasionally go out of control and some lots of poorer quality will be produced before the process gets corrected. Such a situation can be modelled satisfactorily by a beta prior with outliers. On the other hand the Dodge-Romig's models which have been based on a one point prior with outliers can also be extended to the multiattribute case.

As an alternative to these models, consider that the process average is a random variable which may take on two values, a satisfactory and an unsatisfactory quality level with given probabilities. This two point prior may also be considered as a simplification of the models discussed before, since the model accommodates in a way the outliers which might also be present. Hald (1981) emphasized that the distribution of lot quality derived from the past inspection records, should be used taking into account the information of normal quality variations. He pointed out that *"It is a peculiar fact that published data regarding quality variations are very scarce even if enormous amounts must exist in inspection records."* For the multiattribute case we observed in a number of real life situations that when the process performs at an unsatisfactory level it generally happens (e.g.for a manufacturing operation) that the quality level is poor for all the attributes.
In this chapter we have presented one such set of data for a manufacturing process viz. RS closures in a factory engaged in manufacturing of containers. On the spot observations are made using a p chart as data format on two attributes. In a typical scenario, one observes

that most of the time the process is stable at a certain process average. However, the process works unsatisfactorily for about 1.5 hours on an average in a day (24 hours). During this time both the type of defects occur more frequently. The process average during the phase, although not quite stable hovers around a higher level. It has been verified in this case by the data collected by us, we can approximate the process by a discrete two point prior distribution.

Extending the result to the case of more than two attributes ($r > 2$), we assume that the process average vector is a random variable which may take on two values, $\boldsymbol{p} = (p_1, p_2, ...p_r)$ and $\boldsymbol{p'} = (p'_1, p'_2, ...p'_r)$ a satisfactory and an satisfactory quality level with given probabilities. w_1 and w_2 respectively.

Using the notation $P(\boldsymbol{p})$ for the probability of acceptance at $\boldsymbol{p}$ and $Q(\boldsymbol{p}) = 1 - P(p)$ we have obtained the average costs per lot, $K(N, n)$ as a function of prior distribution parameters and cost elements. This is a general cost model for r characteristics.
Denoting the expected cost of sampling per unit, expected cost of acceptance per unit when all lots are accepted, expected cost of rejection per unit when all lots are rejected by k_s, k_a, k_r respectively and the minimum avoidable cost per lot when all lots are classified correctly by K_m and per unit by k_m we have noted that sampling inspection should only be taken recourse to, if the average cost per unit k_{Avg} is such that $k_{Avg} - k_m < min[k_a - k_m, k_r - k_m]$. We then can write the regret function, the multiattribute analogue of the regret function defined by Hald (1965) $R(N, n)$ as:

$$R(N, n) = [K(N, n) - K_m(N, n)]/(k_s - k_m)$$

.

...(0.5.6)

It has been shown that:
$R(N, n) = n + (N - n)[\gamma_1 Q(\boldsymbol{p}) + \gamma_2 P(\boldsymbol{p'})]$ where $\gamma'_j s$ are function of costs and the prior distribution parameters.

0.5.2.3 Chapter 3: Cost of MASSP's of A kind

In this chapter we attempt to evaluate the cost implication of the A kind MASSP's for a given ordering of the attributes. The results obtained can be used to order the attributes to achieve a lesser value of the regret as defined in the previous chapter. We assume as in the previous chapter that the Poisson conditions hold, so that the type B OC function can be expressed in terms of the sum of products of the individual Poisson probabilities; the sum being taken over $(x_1, x_2, .., x_r) \in A$ and A is the set of $(x_1, x_2, .., x_r)$ for which we decide to accept a lot. We recall that given two levels of process average $\boldsymbol{p}$ and $\boldsymbol{p'}$ and the values of γ_1, γ_2, the regret is a function of the lot size N, sample size n, and the acceptance numbers as applicable to a specific acceptance criterion. For a given N, we consider the optimality of different sampling schemes in terms of n and the acceptance parameters. We use the following additional notations for our discussions. $p_{(j)} = p_1 + p_2 + ... + p_j$; $p = p_1 + p_2 + ... + p_r$; $p' = p'_1 + p'_2 + ... + p'_r$.

We first notice that a D kind plan with parameter (k, n) has the same probability of acceptance, under Poisson conditions, at each value of $\boldsymbol{p}$, as that of an A kind plan with sample size n with acceptance numbers $a_1 = a_2 = ... = a_r = k$. Hence both the plans will have the same regret, for given values of $N, \boldsymbol{p}$, $\boldsymbol{p'}$,γ_1 and γ_2. It makes sense to opt for an A kind plan for which all the acceptance numbers $(a_1, a_2, ..., a_r)$ are not all equal. However, for $r > 2$ there may be situations where for the optimal plan we may obtain $a_i = a_j$ for some $i \neq j$. As shown in this chapter this will depend upon the ordering of the attributes and the ratio $p'_{(i)}/p_{(i)}$. We briefly present the main results proved in this chapter.

Situation 1:

$p'_{(r-1)}/p_{(r-1)} > (p'/p)$

In this case if the optimal D plan has the parameters (k, n), the regret function value of the plan A with acceptance criterion: $x_{(i)} \leq k - 1$; for $i = 1, 2, ..., r - 1; x_{(r)} \leq k$; is less than the regret function value of the optimal plan D (k, n) if,
$p'_{(r-1)}/p_{(r-1)})^k > (p'/p)^{k+1}$.

Situation 2:

$[p'_{(r-2)}/p_{(r-2)}] > [p'_{(r-1)}/p_{(r-1)}]$

Let us consider the set of A plans with acceptance criteria : $x_{(i)} \leq a_{r-1}$ for $i = 1, 2, ..., r-1$; $x_{(r)} \leq a_r$. In this condition we suppose n, a_{r-1}, a_r are the optimal parameters with acceptance criteria as specified above, for a given $N, \boldsymbol{p}, \boldsymbol{p'}, \gamma_1, \gamma_2$, etc and we construct another plan with the same sample size n but with acceptance criterion changed to : $x_{(r-2)} \leq a_{r-1} - 1; x_{(r-1)} \leq a_{r-1}, x_{(r)} \leq a_r$. We show that the second plan will have lesser regret value than the so called optimal plan in the specified set if,
$\left[p'_{(r-2)}/p_{(r-2)}\right]^{a_{r-1}} > \left[p'_{(r-1)}/p_{(r-1)}\right]^{a_{r-1}+1}$

Situation 3 :

$\left[p'_{(j-1)}/p_{(j-1)}\right] > [p'_{(j)}/p_{(j)}]$

We consider a set of 'A' plans such that for some $j(i < j \leq r), a_i = a_j$ i.e. to say that all acceptance numbers are equal for $i \leq j$. Let the optimal plan from the above set has the parameter $n, a_1 = a_2 = ... = a_j, a_{j+1}, ..., a_r$ for given $N, \boldsymbol{p}$ and $\boldsymbol{p'}$ γ_1, γ_2. If we now construct a plan with acceptance numbers b_i such that $b_1 = b_2 ... = b_{j-1} = a_j - 1$, and $b_i = a_i$ for

$i = j+1, ..., r$, then this plan will have a lesser regret value if,
$[p'_{(j-1)}/p_{(j-1)}]^{a_j} > [p'_{(j)}/p_{(j)}]^{a_j+1}$.

Situation 4 :

$[p'_{j-1}/p_{j-1}] \leq [p'_j/p_j]$
In this case there exists a minimum regret A plan with ; $a_{j-1} = a_j$.
In the chapter 2.5 we verify these results by sample computations.

0.5.2.4 Chapter 4 : Cost of MASSP's of C kind

In this chapter we introduce the concept of the OC distribution and the OC moments of the C kind plans. The approach of moment equivalent plans was originally used by Hald (1981) for finding the approximate OC curve for a double sampling plan (in a single attribute situation) from the OC of the equivalent single sampling plan. We use this approach in the case of MASSP of C kind to investigate some of its optimality properties.

We first note that if we treat all $\rho_i = p_i/p$ as fixed for $i = 1, 2, ..., r$; $p = p_1 + p_2 + ... + p_r$, the Poisson OC of a given C kind plan with acceptance numbers $c_1, c_2, ..., c_r$ and sample size n can be considered as a function of $m = np$ denoted by $P(m)$. Since the $P(m)$ is a monotonically decreasing function of m with the property: $P(m) = 1$ for $m = 0$ and $P(m) = 0$ for $m \to \infty$, the function $Q(m) = 1 - P(m)$ satisfies all the required properties of a distribution function, even though we do not regard m as a random variable in the present context. Thus, for a given $(\rho_1, \rho_2, ..., \rho_r)$, the OC distribution for a C Plan is given by the distribution function $1 - P(m), 0 < m < \infty$. For the OC distribution so defined we have obtained the expressions of moments of different orders. We may call them as OC moments.

We may now construct a moment equivalent single sampling plan with parameters (a_0, n_0) for a given value of $(\rho_1, \rho_2, ..., \rho_r)$ such that the OC distribution of the single sampling plan has the same mean and variance as those of the OC distribution of the given C kind plan. For the moment equivalent single sampling plan we get $a_0 = E^2(m)/V(m); n_0 = nE(m)/V(m)$. It has been proved in this chapter, that the moment equivalent single sampling plan so obtained for any specified value of $(\rho_1, \rho_2, ..., \rho_r)$ has sample size $n_0 < n$.

We note that corresponding to the two process average points, $\boldsymbol{p}$ and $\boldsymbol{p'}$ of the two point prior distribution, we have two ρ vectors $(\rho_1, \rho_2, ..., \rho_r)$ and $(\rho'_1, \rho'_2, ..., \rho'_r)$ and therefore we may construct two moment equivalent SSP's (a_0, n_0) and $(a'_0, n'_0\ )$ corresponding to these two ρ vectors. We have proved that both n_0 and n'_0 are $< n$.

Hald(1981) assumes near identity of the OC curves of the moment equivalent single sampling plan for estimating the probability of acceptance in case of a double sampling plan for different process average values. Hald's assumptions has been corroborated by numerical computations. Following Hald's argument we may also assume the near identity of the OC curves i.e. the OC curve as a function of p for a given value of $(\rho_1, \rho_2, ..., \rho_r)$ and that of the OC curve of the moment equivalent single sampling plan.We use the above logic (although mathematically non-rigorous) and numerically justify the near identity of the OC curve for a given plan of a MASSP of C kind and that of the equivalent SSP (under the restriction of a specified ρ vector). Note that we require effectively this near identity only for the tail probabilities.

We thereafter consider three possible scenario (a) $a_0 > a'_0, n_0 < n'_0$ (b) $a_0 < a'_0, n_0 > n_{0'}$ and (c) $a_0 > a'_0, n_0 > n'_0$.
We show that corresponding to the MASSP of C kind with sample size n and acceptance numbers $c_1, c_2, ..., c_r$ the SSP (a_0, n_0) and the SSP (a'_0, n'_0) will have lesser regret value in the situation (a) and the situation (b) respectively. In the situation (c) we notice (using Hald's results) that the OC's of two SSP's (a_0, n_0) and (a'_0, n'_0) intersect at some point p_0(say). If $p < p_0 < p'$ or if $p_0 < p$ then, the SSP (a_0, n_0) will have lesser regret value than the regret value of the corresponding MASSP. In case, $p_0 > p'$ neither of the SSP (a_0, n_0) and the SSP (a'_0, n'_0) may have lesser regret than the optimal C plan. In this case the plan $(a''_0, n''_0); a'' > a_0, n''_0 > n_0$ will have lesser regret if $n > n''_0$.

Further the Poisson OC of the MASSP D kind with (k, n) at any process average $\boldsymbol{p}$ is identical with the Poisson OC of the SSP (k, n) at $p = p_1 + p_2 + ... + p_r$.
It therefore follows that given an optimal MASSP of C kind it should be possible to construct a D kind plan with lesser regret values in first two situations as above. For the third situation we may have to satisfy the additional condition mentioned as above to obtain such a D kind plan.

In the next chapter we undertake some numerical exercises to demonstrate these results.

0.5.2.5 Chapter 5: Results of numerical verification

This chapter presents examples of comparison of the regret value of A kind, C kind and D kind plans for different situations using the results obtained in earlier chapters and provides a visual basic programme written as Excel Macro for obtaining optimal A kind and C kind plans. For the purpose of verification we consider the case where the number of characteristics $r = 3$. In chapter 2.2.2 we have noted that for a given two point discrete prior distribution of $\boldsymbol{p}$ the regret function is

$$R(N, n) = n + (N - n)[\gamma_1 Q(\boldsymbol{p}) + \gamma_2 P(\boldsymbol{p'})]$$

Where γ_j's are functions of cost parameters and the parameters of the two point prior distribution. Moreover, the $P(\boldsymbol{p})$ denotes the type B probability of acceptance at $\boldsymbol{p}$ and $Q(\boldsymbol{p}) = 1 - P(\boldsymbol{p})$. We consider optimality properties for different acceptance criterion given the values of γ_1, γ_2, $\boldsymbol{p}$ and $\boldsymbol{p'}$.

1. $p'_1/p_1 = p'_2/p_2 = p'_3/p_3$

From the results of chapter 2.2.4 it follows that for the optimal C plan with sample size n, acceptance parameters $c_1, c_2, ..., c_r$ and given values of p_i/p; p'_i/p', we can construct a moment equivalent D plan with sample size n_0 and acceptance number k_0 which will have approximately same probability of acceptance as the plan C at $\boldsymbol{p}$ and at $\boldsymbol{p'}$. Further, this plan has lesser regret than the optimal C plan. For a typical value of $\gamma_1 = 1, \gamma_2 = 0.7, (p_1, p_2, p_3) = (0.002, 0.008, 0.02)$ and $p'_1/p_1 = p'_2/p_2 = p'_3/p_3 = 5$ we compute the optimal parameters of the C plan and those of the equivalent D type plan. We find that regret value of the moment equivalent SSP is less than the optimal C plan. Moreover the regret value of the optimal D plan is still lesser than that of each of these two plans.

2. p'_i/p_i **are not same for all** i

a) For r = 3, let $p'_1 = 0.01, p'_2 = 0.04, p'_3 = 0.10, p'_1/p_1 = 5, p'_2/p_2 = 5, p'_3/p_3 = 3$ and we take $\gamma_1 = 1, \gamma_2 = 0.7$. Since $p'_1/p_1 = p'_2/p_2 = 5$ we notice from the results of the chapter 2.3 (situation 4) that $a_1 = a_2$ for the optimal A plan. Also in this case $p'_{(2)}/p_{(2)} = 5$ and $p'/p = 3.46$. Thus for $k \geq 4$ we satisfy the inequality $(p'_{(2)}/p_{(2)})^k > (p'/p)^{(k+1)}$. This means there exists an A plan cheaper than the optimal D plan for which the k value is greater than or equal to 4. (Situation 1 in chapter 2.4).

We have computed the parameters of the optimal D plan, optimal A plan and the corresponding regret value for lot sizes 1000 (1000) 10000. We find that for all the optimal A plans $a_1 = a_2$ and $a_3 > a_2$. Further we compute the corresponding regret value of the optimal C kind which is higher than that of the optimal D plan and the optimal A plan.

b) For r = 3, let $p'_1 = 0.01, p'_2 = 0.04, p'_3 = 0.10, p'_1/p_1 = 8, p'_2/p_2 = 5, p'_3/p_3 = 3, \gamma_1 = 1, \gamma_2 = 0.7$. For $k > 2$ we satisfy the inequality $(p'_{(2)}/p_{(2)})^k > (p'/p)^{(k+1)}$. This means there exists an A plan cheaper than the optimal D plan for which the k value is greater than 2. (Situation 1 of chapter 2.3). Also by results of the chapter (situation 3) $a_3 > a_2$ for $a_3 > 2$. Further $(p'_{(1)}/p_{(1)})^{a_2} > (p'_{(2)}/p_{(2)})^{a_2+1}$ for $a_2 > 4$ and hence $a_2 > a_1$ for $a_2 > 4$ (situation 2).

We present the parameters of the optimal D plan, optimal A plan and the corresponding regret values for lot sizes 1000 (1000) 10000.For all the optimal A plan $a_2 > a_1$ and $a_3 > a_2$. We have also presented the parameters of the optimal C plan and the corresponding regret value which is higher than that of the optimal D plan and the optimal A plan.

0.5.3 Part3 Bayesian multiattribute sampling inspection plans for continuous prior distribution

0.5.3.1 Chapter 1: Bayesian sampling inspection plans for continuous prior distribution

In this chapter it will be assumed that the process average defective (defects) for each attribute has a continuous prior distribution. We examine in particular, the problems of choice of a theoretical distribution as relevant to a multiattribute situation. The most widely used continuous prior distribution for the process average quality p_i is the beta distribution.

$$\beta(p_i, s_i, t_i) = {p_i}^{s_i - 1}(1 - p_i)^{t_i - 1}/\beta(s_i, t_i), s_i > 0, t_i > 0.$$

We express the above using the parameters ; $\bar{p_i}$ and s_i where $\bar{p_i} = s_i/(s_i + t_i)$.
When $\bar{p_i}$ and $\bar{p_i}/s_i$ are small ; more precisely, if $\bar{p_i} < 0.1$ and $\bar{p_i}/s_i < 0.2$, this distribution can be approximated by a gamma distribution:

$$f(p_i, \bar{p_i}, s_i)dp_i = e^{(-v_i)}(v_i)^{s-1}dv_i/\Gamma(s_i); \quad v_i = s_i p_i/\bar{p_i}$$

with mean $E(p_i) = \bar{p_i}$ and the shape parameter, s_i. Hald (1981) used this approximation to tabulate the optimal sampling single sampling plans. Most of his results are based on assuming gamma as the right distribution in the effective range. Corresponding to a beta distribution of the single attribute process average of quality p_i, the distribution of the lot quality denoted by X_i as well as sample quality x_i become a beta-binomial distribution which can similarly be approximated as a gamma-Poisson distribution.

The gamma Poisson distribution assumed appears to be the right distribution when we are counting number of defects instead of defectives. Thus, in all the situations where we are counting number of defectives / defects we write the joint prior distribution of $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ under the assumption of independence. The joint distribution term of $\boldsymbol{x} = (x_1, x_2, ..., x_r)$ is thus assumed to be the product of r gamma-Poisson distribution terms. The expression for the average costs connected with a sampling scheme is to be worked out from the general cost model expression, obtained under these assumptions.

To verify how far the assumed probability distributional forms fits into a real life situation, we collected the inspection data for 86 lots containing about 25500 pieces each of filled vials of an eye drop produced by an established pharmaceutical company based at Kolkata. Each vial is inspected for six characteristics. From the criticality point of view, however, the defects can be grouped in two categories. We have been able to justify, by using the χ^2 goodness of fit analysis, the assumption that the distributions of lot quality follow the assumed theoretical gamma-Poisson distributions.

Further, the scatter plot of the observed numbers of defects of the second category against those of the first category exhibits no specific pattern. It would be therefore reasonable to assume that the p_i's are independently distributed in the present context.

The beta and gamma distributions are, however, not always appropriate. For example, we could not fit the gamma or beta distribution in case of quality variation of ceiling fans, garments and cigarettes. In such cases we will have to take recourse to direct computation of the average cost function derived from the empirical distributions observed. Nevertheless, since the gamma (or beta) distribution is likely to be appropriate at least in some situations and neat theoretical expressions can be obtained in such cases, we study in details the expected cost functions under such assumptions.

In chapter 2.1 we have developed the expression for the generalized cost function at a given process average. We make use of this to obtain the expression for the average cost when the process averages follow independent continuous distributions in general. We use the notations of cost parameter A_i, R_i, S_i as given in chapter 2.1, and prove that, the optimal plan in this situation for a specified acceptance criteria $(x_1, x_2, ., ., .x_r) \in A$ is obtained by minimizing the function :

$$K(N,n)/(A_1-R_1) = nk'_s + (N-n)k'_r + (N-n)[\sum_{x_1,x_2,...x_r\in A}$$

$$\{\sum_{i=1}^{r} d_i.\bar{p}_i.(s_i+x_i)/(s_i+n\bar{p}_i) - d_0\}\prod_{i=1}^{r} g(x_i,n.\bar{p}_i,s_i)]$$

Where $g(x_i, n\, s_i)$ is a gamma-Poisson distribution term as

$$g(x_i, n\, s_i) = \frac{\Gamma(s_i+x_i)}{x!\Gamma s}\theta_i^{s_i}.(1-\theta_i)^{x_i}; \quad \theta_i = \frac{s_i}{(s_i+n.\bar{p}_i)}$$

$k'_s = k_s/(A_1-R_1), k'_r = k_r/(A_1-R_1)$
$d_0 = (R_0-A_0)/(A_1-R_1);\ d_i = (A_i-R_i)/(A_1-R_1)$ for $i = 1, 2, ..., r$.

...(0.5.7)

Using the above expression we may now construct optimal A kind and C kind plans for a given lot size N, cost parameters and the parameters of the prior distributions $\bar{p}_i, s_i$ and compare their relative merits in a given situation. We demonstrate this with one example obtained in respect of plastic containers used for cosmetics.The outcome of this exercise has been tabulated as the obtained optimal A, C and D plans for lot sizes from 30000(10000)50000 and the corresponding costs.

We find that the optimal A kind plans are cheaper than the optimal C kind plans and optimal D kind plans. The optimal C kind plans are cheaper than the optimal D kind plans. The behaviour of cost function near the neighbourhood of the optimum sample size and the acceptance parameters for the Plan A kind, C kind and D kind have been presented. We thus demonstrate that it should be possible to obtain an optimal MASSP by the above methods and note that the choice of acceptance criteria does affect the costs of optimal MASSP's. However, we have not attempted to obtain general theoretical results on the choice of acceptance criteria as in the case of two point discrete prior distributions.

Part 1 : Multiattribute sampling plans based on OC function

1.1 Introductory concepts

1.1.1 Relevance of sampling inspection

The term quality as applicable to an industrial product is the ability of a set of characteristics to fulfill the requirements of customers and users. An intended product is the result of a value addition process of transforming inputs into outputs. A product may be a service, a software, an intellectual product, a piece of information as well as materials offered either in discrete units of number of pieces, pairs, cartons, boxes etc. or as bulk items.
Irrespective of the type of product, evaluation of conformity to specified requirements of the quality characteristics by observation and judgment accompanied by measurement, testing and/or gauging is an integral part of quality assurance. We call this inspection. Since the focus on today's quality control is more on making things right at first time, every time, it is important to establish a system for achieving economy, productivity and customer satisfaction simultaneously. We notice that inspection on its own does not add value to the product but is a set of necessary verification activities at different stages of production process. So it is to be kept at its minimum. The sampling inspection where a portion of a collection of product units is inspected on a set of characteristics with a view to making decision about acceptance or otherwise of the product becomes relevant in this context. We attend to the designing of the procedure of sampling inspection with the above purpose in mind. The scope of the present exercises, presented in the dissertation, however, is limited to only attribute type verification of collections of countable discrete pieces, called lots, received consecutively on a more or less continuous basis.

1.1.2 Multiattribute sampling inspection

At the outset we take into account the fact that the number of elements of the set of characteristics to be verified is unlikely to be just one in most of the practical situations. For example, a sample of finished garment is verified for 15 characteristics (all attribute type) before delivery.[Table 1.1.1]

Table 1.1.1 Checklist for verification of presentation defects in finished and packed garments

Sl.No.	Defect type
1.	Omission of tags
2.	Torn / crushed poly bag
3.	Main label center out
4.	Puckering in the folding of the garment
5.	Bending of the garment
6.	Scratch marks on the box
7.	Deformation of the box
8.	Twisted collar
9.	Button down up down
10.	Bubbling
11.	Incomplete button stitching
12.	Visible fusing marks
13.	Wrong size tag
14.	Wrong bar code
15.	Wrong hang tag

Multiattribute sampling inspection is employed in all stages of production from raw materials through semi-finished, finished to final assembly stages. The inspection lot is either the normal production batch or a shift's production, or customer ordered specific collection. We consider those situations where it is possible to take a sample of size n from a lot and inspect for all the attributes in any order in the sample. For most of the cases, the defect occurrence pertaining to a particular attribute in a lot is independent of occurrence of defects with respect to others attributes. For some attributes, however, the defect occurrences are such that occurrence of one type of defect, precludes the occurrence of any other type. For example, the defect 'Omission of tags' precludes the defect 'Wrong size tag' in the above example of garment verification.

1.1.3 The type B OC function for a generalized multiattribute single sampling plan

We suppose that there are r attribute characteristics inspectable separately for any product and if we take a sample of size n from a lot of size N, all the r characteristics are observable in any order. The sampling scheme we are interested in should be applicable when lots are submitted one by one in a consecutive manner, selected from a continuous production process.

We recall that the type B OC curve of a sampling plan for a single attribute is defined by Dodge and Romig (1959) as the curve showing probability of accepting a lot as a function of product quality, where product quality is expressed as the long term fraction defective p or the probability of a unit or item being defective in the production process under stable conditions. This probability of acceptance is obtained by assuming a binomial distribution of lot quality values, since product of quality p has been, it is assumed, randomly subdivided into a collection of lots of size N each. It can be shown rigorously [See Hald (1981)] that random samples taken without replacement from such a collection of lots, may be treated mathematically as if they were actually samples from an infinite universe having a fraction defective p. In other words, the situation resembles a Bernoullian sequence of trials with a probability of success given by the process average p. The probability of obtaining x defectives in a sample of size n, taken without replacement from a lot is thus independent of lot size and follows binomial distribution with parameter n, p.

For more than one characteristic we consider two situations: (a) the defect occurrences are independent (b) the defect occurrences are mutually exclusive i.e. a product item can contain only one type of defect, existence of one precluding the existence of all other types.

a) When defect occurences are independent

When the defect occurrences of different types are independent the probability of any $(X_1, X_2, ..., X_r)$, X_i being the number of defectives in a lot of type $i, i = 1, 2, ..., r$ is

$$Pr(X_1, X_2, ..., X_r) = \prod_{i=1}^{r} Pr(X_i).$$

...(1.1.1)

The probability of obtaining x_i defectives corresponding to a characteristic i from a sample of size n drawn from a lot of size N is

$$Pr(x_i \mid X_i) = \binom{n}{x_i}\binom{N-n}{X_i - x_i} \Big/ \binom{N}{X_i}.$$

...(1.1.2)

If now X_i is assumed to follow binomial with parameters N and p_i and they are independent as given by (1.1.1), the unconditional probability of obtaining $(x_1, x_2, ..., x_r)$ defectives in sample of size n is

$$\sum_{X_1}\sum_{X_2}\cdots\sum_{X_r}\prod_{i=1}^{r}\left[\binom{n}{x_i}\binom{N-n}{X_i-x_i}/\binom{N}{X_i}\right]\left[\binom{N}{X_i}p_i^{X_i}(1-p_i)^{N-X_i}\right]$$
$$=\prod_{i=1}^{r}\binom{n}{x_i}p_i^{x_i}(1-p_i)^{n-x_i}.$$

If b(., ., .) denotes an individual term of the binomial distribution, then

$$Pr(x_1, x_2..., x_r \mid p_1, p_2...p_r) = \prod_{i=1}^{r} b(x_i, n_i, p_i), x_i = 0, 1, 2, ..., n; \quad 0 < p_i < 1; i = 1, 2, ..., r.$$

...(1.1.3)

b) When the defect occurrences are mutually exclusive

When the defect occurrences are mutually exclusive, the expression for the probability of observing $(x_1, x_2, ..., x_r)$ defective in a sample of size n from a lot of size N containing $(X_1, X_2, ..., X_r)$ defectives will be multivariate hypergeometric as

$$Pr(x_1, x_2, ..., x_r \mid X_1, X_2, ..., X_r) = \binom{X_1}{x_1}\binom{X_2}{x_2}\cdots\binom{X_r}{x_r}\binom{N-X_1-X_2...-X_r}{n-x_1-x_2...-x_r}/\binom{N}{n}$$

...(1.1.4)

At any process average vector $(p_1, p_2, ..., p_r)$ the joint probability distribution of $(X_1, X_2, ..., X_r)$ can be assumed to be multinomial $(N, p_1, p_2, ..., p_r)$ such that

$$Pr(X_1, X_2, ..., X_r \mid p_1, p_2, ..., p_r) = \binom{N}{X_1}\binom{N-X_{(1)}}{X_2}\cdots\binom{N-X_{(r-1)}}{X_r}p_1^{X_1}...p_r^{X_r}(1-p_{(r)})^{(N-X_{(r)})}$$

where

$$X_{(i)} = X_1 + X_2 + ... + X_i \quad and \quad p_{(i)} = p_1 + p_2 + ... + p_i; \quad i = 1, 2, ..., r.$$

...(1.1.5)

From (1.1.4) and (1.1.5) it follows that the unconditional probability is

$$Pr(x_1, x_2, ..., x_r \mid p_1, p_2, ..., p_r) = \binom{n}{x_1}\binom{n - x_{(1)}}{x_2}...\binom{n - x_{(r-1)}}{x_r} p_1^{x_1}...p_r^{x_r}(1 - p_{(r)})^{(n - x_{(r)})},$$

. where

$$x_{(i)} = x_1 + x_2 + ... + x_i, i = 1, 2, ..., r \quad 0 \leq x_{(1)} \leq x_{(2)} ... \leq x_{(r)} \leq n; \quad 0 < p_i < 1, \quad i = 1, 2, ..., r; p_{(r)} < 1.$$

...(1.1.6)

Thus the average probability follows multinomial distribution with parameter $(n, p_1, p_2, ..., p_r)$.

1.1.4 Poisson conditions

Hald (1981) has used the phrase 'under Poisson conditions' when Poisson probability can be used as an approximation to the binomial in the expressions of type B OC function. We use the phrase to cover the situations as described below.

(i) Poisson as approximation to binomial and multinomial

If $p_i \to 0$, $n \to \infty$ and $np_i \to m_i$ then the binomial probability $b(x_i, n, p_i)$ tends to Poisson probability $g(x_i, np_i)$ where

$$b(x_i, n, p_i) = \binom{n}{x_i} p_i^{x_i} (1 - p_i)^{n - x_i}$$

and

$$g(x_i, np_i) = e^{-np_i} (np_i)^{x_i} / (x_i)!.$$

Under this conditions for $i = 1, 2, ..., r$, the expression given as equation (1.1.3) can be modified as

$$Pr(x_1, x_2, ..., x_r \mid p_1, p_2, ..., p_r) = \prod_{i=1}^{r} g(x_i, np_i); \quad x_i \geq 0, i = 1, 2, ..., r.$$

...(1.1.7)

If we also make an additional assumption that $\sum_{i=1}^{r} p_i \to 0$ then the equation (1.1.6) can also be modified as (1.1.7).

(ii) Poisson as an exact distribution and occurrences of defect types independent

In case we count the number of defects per item for each characteristic, we may construct a model for which the expected number of defects for a item for i th the characteristic equals to p_i in the long run. We assume the number of defects for r distinct characteristics in a item are independently distributed. The output of such a process is called a product of quality $(p_1, p_2, ..., p_r)$, the parameter vector representing the mean occurrence rates (of defects) per observational unit. The total number of defects for any characteristic in a lot of size N from such a process will vary at random according to a Poisson law with parameter Np_i for the i th characteristic under usual circumstances.

Similarly, the distribution of number of defects on attribute i, in a random sample of size n drawn from a typical lot will be be a Poisson variable with parameter np_i. Independence of the different characteristics will be naturally maintained in the sample, so that the joint probability of occurrence can be expressed as (1.1.7).

Here $p_i, i = 1, 2, ..., r$ denotes the average number of defects per item in respect of characteristics i instead of proportion defectives, since in this situation we are dealing with defects rather than defectives.

1.1.5 Expressions of type B OC function under Poisson conditions

Suppose now we employ a sampling plan given hereunder.
We draw a random sample of size n from a lot of size N; observe the number of defectives/defects for each of r attribute characteristics, $\boldsymbol{x} = (x_1, x_2, ..., x_r)$. giving the vector of defectives/defects w.r.t. the r attribute types.

Let A be the set of $(x_1, x_2, ..., x_r)$ combinations for which we decide to accept the lot i.e. if $(x_1, x_2, ..., x_r) \in A$, accept the lot and reject it otherwise.

Theorem 1.1.1 Irrespective of whether the defectives (defects) occurrences are independent or mutually exclusive, the probability of acceptance under Poisson conditions at a process average $(p_1, p_2, ..., p_r)$ is given by

$$P(\boldsymbol{p}) = \sum_{\boldsymbol{x} \in A} \prod_{i=1}^{r} g(x_i, m_i)$$

...(1.1.8)

Proof: The proof follows from the results obtained as (1.1.7) and the discussions in section 1.1.4(ii)

1.2 Multiattribute sampling schemes based on AQL

1.2.1 Scope

In this chapter we shall discuss the problem of determining a multiattribute single sampling plan (MASSP)and attempt at establishing a sampling scheme satisfying certain requirements of the OC function. The main issues we address are
(a) What should be the basis for designing such a sampling scheme? (b) What should be the acceptance criteria? (c) How will the sample size be related to the the lot size? (d) How will the parameters for a given acceptance criterion be determined for a given sample size?

1.2.2 Using published plans in a multiattribute situation

The MIL-STD-105D and its derivatives

At the outset it is to be made clear that there is no published plan specifically for a multiattribute situation. We, therefore, look at the plans available for single attribute and examine the consequences of using them in a multiattribute situation. The published set of plans most widely used is the US Military standard 105D (1963), which has been adopted as the international standard ISO 2859 (1974). We denote this standard as MIL-STD-105D. The standard is based on what is known as Acceptable Quality Level (AQL) and considered to be most suitable for the consumers.

The Acceptable Quality Level (AQL) is defined as the maximum percent defective (or the number of defects per hundred units) that for purpose of acceptance sampling can be considered satisfactory as a process average. Thus the lots produced at process average of AQL or better level should have a high probability of getting accepted. A producer can always increase his acceptance probability by improving his process average.

This system of selecting an approprite sample size is not based on an explicit mathematical model. For any lot size, the table gives the corresponding sample size. The relationship is based on what is considered reasonable in practice. There is no theoretical foundation for the relation between sample size (n) and lot size (N). As the lot size increases, the sample size increases, but at a lesser rate such that $n/N \to 0$ as $N \to \infty$.

The user of the table may choose between various 'levels' of this relationship, called inspection levels. Next, one has to choose from practical considerations an AQL (given in percent defectives or defects per hundred) and find from the table the acceptance number. For given AQL, the acceptance number is determined so that the producer's risk is reasonably small and is decreasing with lot size. The acceptance number (c) has been arrived at such that it remains same at a given sample size multiplied by the AQL to ensure that the Poisson probability of acceptance (rejection) remains same in such cases. This ensures a desired producer's risk. There are 13 sets of n AQL values and 11 acceptance numbers chosen in such a way that the producer's risk (excepting for c =0) varies from a maximum of 9% to a minimum of 1.44%. With increase in the n. AQL values, c increases, so that producer's risk decreases upto a level of 2% and thereafter it is kept at less than 2%. To attend to utmost simplicity of the standard in numerical and administrative respects, a set of five preferred numbers has been used for each decade as values of AQL and a proportional set has been used for values of the sample size. These preferred numbers constitute a geometric series with ratio equal to the fifth root of 10 i.e. 1.585 [See Hald (1981) for a detailed discussion.]

In case of a multiattribute situation the Standard prescribes that a separate plan is to be chosen for each class of attributes. For example, the plan for critical defects will have generally a lower AQL than the plan for major defects and the plan for major defects will have an AQL lower than the plan for minor defects.

1.2.3 The consequence of using MIL-STD-105D in a multiattribute situation - two live examples

Defect classifications and formulating the Sampling plans

To illustrate this we may consider the case of a consumer who uses a plastic component/washer for packaging cosmetics and procures these items from vendors. He classifies his defects for the plastic component as under:

Table1.2.1 Defect Classification for plastic containers

Critical defects (AQL: 0.15%)	Major defects (AQL: 1.0%)	Minor defects (AQL: 4 %)
1.Internal contamination 2.Extended neck inside the containers 3.BFC not as per standard 4.Missing Recycle logo for export containers	1.Improper neck finishing 2.Prominently visible parting line 3.Weak body/bottom/shoulder 4.External count animation 5. Colour variation	1.Black spot 2.Visible parting line 3.Shrink mark

We take this case and the case of the finished garment verification cited as an example in chapter 1.1, where the inspection is done at the producer's end before shipment of the product. We attempt to adopt the MIL-STD-105D for both the cases. The sampling plans employed using MIL-STD-105D inspection level II for normal inspection for a typical lot size are worked out as given in the following table.

Table 1.2.2 Sampling plans using MIL-STD-105D (normal inspection, inspection level II)

Product	Lot Size	Sample Size	Defect category					
			Critical		Major		Minor	
			AQL	c	AQL	c	AQL	c
Plastic container	15,000	315	0.15%	1	1.0%	7	4%	21
Garment	50,000	500	0.1%	1	1%	10	2.5%	21

The Producer's Risk

As applicable to multiattribute case, the consumer wants to make reasonably sure that the product of satisfactory quality is accepted and he therefore chooses an AQL as $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ say, and a small risk Q($\boldsymbol{p}$) = α, where Q($\boldsymbol{p}$) denotes the probability of rejection at $\boldsymbol{p}$. Since the lot is accepted only when the acceptance criterion is satisfied for all the attributes independently, the producer's risk will always be more than the target producer's risk for the individual characteristics in the MIL-STD-105D. In these examples, we compute the overall producer's risk as more than 10.5% for the first set of plans (plastic container) and around 22% for the second set of plans (garment). It would appear that we are asking for the moon from the producer.

The OC performance

Secondly, we thought it reasonable to expect that the OC function should be more sensitive to the changes in the defect level of more important attributes, particularly, where unsatisfactory defect level occurs due to more serious type of defects. Suppose that the defect on i

th attribute is more serious than that on the j th attribute. Then as the total defect level changes from low to high, the absolute value of the slope of the OC w.r.t. the i th type of defect as a function of the total defect level should have higher value than the absolute slope of the OC w.r.t. the j th type of defect, assuming, that the relative contribution of a characteristic to the total defect remains constant.

To examine this, we may compute the slope of the OC at different total process averages keeping the defect contribution of each attribute as proportional to its AQL value. We, therefore, define $Slope_j$ as absolute value of the partial differential coefficient of the Poisson OC function with respect to m_j; $\quad m_j = np_j$. $Slope_j$ is a function of m and therefore of p, where $m = m_1 + m_2 + ... + m_r$ and $p_1 + p_2 + ... + p_r = p$. Let us recall that $m_i/m = p_i/p$ is held constant for each i, $i = 1, 2, ..., r$.

For r = 3 we find that the Poisson probability of acceptance at (p_1, p_2, p_3) for a MASSP with sample size n and acceptance criterion: accept if $x_1 \leq c_1, x_2 \leq c_2, x_3 \leq c_3$ and reject otherwise, is

$$PC(c_1, c_2, c_3; m_1, m_2, m_3) = G(c_1, m_1)G(c_2, m_2)G(c_3, m_3); \quad m_i = np_i, i = 1, 2, 3$$

$$G(c_i, m_i) = \sum_{x_i=0}^{x_i=c} g(x_i, m_i).$$

...(1.2.1)

We may now write, for $j = 1, 2, 3$

$$Slope_j = -PC'_j = g(x_j, m_j) \prod_{i=1:i\neq j}^{3} G(c_i, m_i); \quad i = 1, 2, 3.$$

...(1.2.2)

We plot these three slopes against the overall proportion defective (defects per hundred) p, keeping the relative contributions of the characteristics to the total defect at $p_1 : p_2 : p_3 = AQL_1 : AQL_2 : AQL_3$, where AQL_j is defined as the AQL value for the j th attribute. We use $Slope_j$ as a function of p to measure the discriminating power of the plan with respect to infininitesimal change in defect level for the j th attribute for different values of p keeping the ratio $p_j/p; j = 1, 2, 3$ separately.

For example, for the first case, if the defect level remains at AQL for all the three attribute the overall defect level is 5.15%. The relative contribution is 0.15/5.15, 1/5.15 and 4/5.15 respectively for the three mutually exclusive classes critical, major and minor. We take it that an item branded as critical defective is not branded as a major defective or as a minor defective. We compute the absolute value of the slope and plot it against the overall proportion defectives as proposed above. We would expect the slope as defined will be highest with respect to critical defects, followed by that with respect to major defects and least with respect to minor defects for all p, when plotted against $p = (p_1 + p_2 + p_3)$.

This desirable feature is not present for the first case (plastic containers) as can be seen from Figure 1.2.1 at the end of this chapter. The OC appears to be more sensitive to the changes of minor defect than to the changes of the major defect when the overall quality level becomes unsatisfactory in general.

1.2.4 The properties of the OC function of the MIL-STD-105D normal inspection when used in a multiattribute situation

We now assess the performance of the MASSP's using the acceptance numbers and the sample sizes of the MIL-STD-105D using normal inspection. We choose the case of three attributes. Note that there are 13 sets of n.AQL values and 11 acceptance numbers for the normal inspection table of the Standard.

When used for the single attribute the properties of the plans are summarized by Hald (1981). For different values of $m = n$.AQL, Hald (1981) tabulated the acceptance number c, the producer's risk i.e. the probability of rejection at AQL (p_1) as $Q(p_1)$, and the value $p_{0.10}$ the process average for which the probability of acceptance is 0.10 and also the ratio, $p_{0.10}/p_1$. These are presented below as table 1.2.3.

Table 1.2.3 Performance of the single sampling plans for single attribute using MIL-STD-105D for normal inspection AQL = p_1.

n.AQL	c	Q(p_1)	$p_{0.1}/p_1$
0.1256	0	0.118	18.3
0.19910	↑	-	-
0.3155	↓	-	-
0.5000	1	0.0902	7.78
0.7924	2	0.0463	6.72
1.256	3	0.0388	5.32
1.991	5	0.0162	4.66
3.155	7	0.0156	3.73
5.000	10	0.0137	3.08
7.924	14	0.016	2.64
12.56	21	0.0099	2.24
19.91	30	0.0127	1.92
31.55	44	0.014	1.70

We now consider the case of r = 3. Let $n.AQL_i$ denote the product of n and the AQL for the i th attribute, $i = 1, 2, 3$. Further, let $n.AQL_1 < n.AQL_2 < n.AQL_3$. There are 286 possible combinations of $n.AQL_1$, $n.AQL_2$ and $n.AQL_3$ chosen from the above set of n. AQL given in the Table 1.2.3.

For all these combinations of $(n.AQL_1, n.AQL_2, n.AQL_3)$ and the corresponding acceptance numbers as (c_1, c_2, c_3) we compute the effective producer's risk as

$$1 - G(c_1, n.AQL_1).G(c_2, n.AQL_2).G(c_3, n.AQL_3).$$

...(1.2.3)

Let $p = p_1 + p_2 + p_3$ and $\rho_i = p_i/p$. Now for $\rho_i = AQL_i/(AQL_1 + AQL_2 + AQL_3); i = 1, 2, 3$ we may define, $p^{(3)}{}_{0.1}$ as the value of p for which,

$$G(c_1, np\rho_1).G(c_2, np\rho_2).G(c_3, np\rho_3) = 0.1$$

...(1.2.4)

Note that, for single attribute Hald (1981) has used the notation $p_{0.1}$ for the process average at which we get a probability of acceptance of 0.10.We further define TotalAQL as $AQL_1 + AQL_2 + AQL_3$.

...(1.2.5)

We use the value $p^{(3)}{}_{0.1}$/TotalAQL as a measure comparable to the ratio $p_{0.1}/p_1$ for the single attribute.

We further compute the absolute value of the slope of the OC w.r.t. j th attribute $j = 1, 2, 3$ for different values of p as

$$Slope_j = g(x_j, np_j) \prod_{i=1:i \neq j}^{3} G(c_i, np_i), \quad p_i/(p_1 + p_2 + p_3) = AQL_i/(AQL_1 + AQL_2 + AQL_3).$$

...(1.2.6)

We examine whether the desirable property of $Slope_1 \geq Slope_2 \geq Slope_3$ holds uniformly in the range TotalAQL $\leq p \leq p^{(3)}{}_{0.1}$. It has been seen that

i) the producer's risk varies from 3.6% to 34.2%

ii) there are 103 plans with producer's risk more than 16 %.

iii) There are only 34 plans with producer's risk less than or equal to 6%

iv) There is no plan for which producer's risk is less than or equal to 2%.

We now compute the slopes in the range $0 \leq p \leq p^{(3)}{}_{0.1}$, keeping $p_1 : p_2 : p_3 = AQL_1 : AQL_2 : AQL_3$ as fixed and examine whether $Slope_1 \geq Slope_2 \geq Slope_3$ for all these p for all the plans. These results are summarised and given below in Table 1.2.4.

Table 1.2.4 Summary table, giving information about slope comparisons for the 286 sampling plans with 3 classes of attributes

Conditions	Number of MASSP's satisfying the given condition in the range: $0 < p \leq$TotalAQL	$0 < p \leq p^{(3)}{}_{0.1}$
Number of plans with $Slope_1 < Slope_2$	9	37
Number of plans with $Slope_2 < Slope_3$	2	161
Number of plans with $Slope_1 < Slope_2$ and $Slope_2 < Slope_3$	0	25
Number of plans with $Slope_1 > Slope_2 > Slope_3$	275	113

It is clear from the above table that, while using different AQL values as appropriate to the respective importance of the attributes and choosing the corresponding acceptance numbers there is no guarantee that we would get a good plan in terms of risks as well as the discriminating power in a multiattribute situation. As regards the ratio $p^{(3)}{}_{0.1}/$ TotalAQL, it has ranged from 1.7 to 12.

We now try to generalize the above results more formally for a class of plans where a sample of given size n is chosen and the acceptance numbers are different for different attributes.

1.2.5 The C kind plan and the properties of its OC Function

We first note all these MASSP's are based on the following acceptance criterion. We take a sample of size n, observe the number of defects for the i th attribute as x_i and apply the following acceptance criterion, accept if: $x_i \leq c_i$ for all $i = 1, 2, ..., r$; reject otherwise.

We will henceforth call these plans as C kind plans. The OC function under Poisson conditions is given by

$$PC(c_1, c_2, ..., c_r; m_1, m_2, ..., m_r) = \prod_{i=1}^{r} G(c_i, m_i); \quad m_i = np_i \, \forall i$$

...(1.2.7)

To compare the relative changes in the OC function for changes in p_i, we compute the PC_i' obtained by differentiating the OC function as given in equation (1.2.7) with respect to m_i. Thus,

$$Slope_j = -PC_j' = g(c_j, m_j) \prod_{i=1:i\neq j}^{r} G(c_i, m_i), j = 1, 2, ...r; c_j \geq 0, j = 1, 2, ...r$$

...(1.2.8)

Let $m = m_1 + m_2 + ... + m_r; 0 \leq m_1 \leq m_2 ... \leq m_r$; $\rho_i = m_i/m$ is fixed for $i = 1, 2, ..., r$ and only m varies. In this case, $0 \leq c_1 \leq c_2 ... \leq c_r$.
Theorem 1.2.1 justifies to some extent the numerical results presented in Table 1.2.4.

Theorem 1.2.1

The function, $H(m) = Slope_i - Slope_{i+1}$ undergoes atmost one change of sign from positive to negative for $m \geq 0$.

Proof We need to show that the above statement is true for

$$F(m) = g(c_i, m\rho_i)G(c_{i+1}, m\rho_{i+1}) - G(c_i, m\rho_i)g(c_{i+1}, m\rho_{i+1}).$$

... (1.2.9)

Without loss of generality we take $i = 1$, so that $F(m)/\left[e^{-m}m^{c_1}\right]=$

$$\sum_{i=0}^{c_1} m^{c_2-i}\left[\rho_2^{c_2-i}\rho_1^{c_1}/((c_2-i)!c_1!)) - \rho_1^{c_1-i}\rho_2^{c_2}/((c_1-i)!.c_2!))\right] + \sum_{x=0}^{c_2-c_1-1}((m\rho_2)^x/x!)(\rho_1^{c_1}/c_1!)$$

$0 \leq c_1 \leq c_2 < \infty$

...(1.2.10)

We first note that all the coefficients of m^x *for* $x \leq c_2 - c_1 - 1$ *are* ≥ 0 *in the second series function of the right hand side in* (1.2.10).

Let us now consider the coefficient of m^{c_2-i} in the first series of the right hand side. For $i=0$ the coefficient of $m^{c_2-i}=0$. Let for some i, say $i=j\leq c_1$, the coefficient of $m^{c_2-j}>0$. Then,

$$(\rho_2/\rho_1)^j < \frac{(c_2!)(c_1-j)!}{(c_1!)(c_2-j)!}$$

...(1.2.11)

This implies

$$(\rho_2/\rho_1)^j < \left[\frac{(c_2-j+1)}{(c_1-j+1)}\right]^j$$

i.e.

$$(\rho_2/\rho_1) < \frac{(c_2-j+1)}{(c_1-j+1)} < \frac{(c_2-j)}{(c_1-j)}.$$

...(1.2.12)

Also note that in this case,

$$\frac{c_2!(c_1-j-1)!}{(c_2-j-1)!c_1!} > \left[\frac{\rho_2}{\rho_1}\right]^j \frac{(c_2-j)}{(c_1-j)} > \left[\frac{\rho_2}{\rho_1}\right]^{j+1}$$

...(1.2.13)

Thus if for $i=j\geq 1; j\leq c_1-1$, the coefficient of $m^{c_2-j}>0$, then the coefficient of m^{c_2-j-1} >0.

Further if for some j of the first series

$$(\rho_2/\rho_1)^j > \frac{c_2!.(c_1-j)!}{c_1!.(c_2-j)!}$$

then the coefficient of $m^{(c_2-j)}<0$ and it follows that

$$(\rho_2/\rho_1)^{j-1} > \frac{c_2!.(c_1-j+1)!}{c_1!.(c_2-j+1)!}$$

and thus the coefficient of $m^{(c_2-j+1)}<0$.

It therefore follows that if for some j, $1\leq j\leq c_1-1$, the coefficient of $m^{(c_2-j)}<0$ then for all i the coefficient of $m^{(c_2-i)}<0, \quad 1\leq i<j\leq c_1$.

From the above three observations we may conclude that the equation $F(m)=0$ can have atmost one real positive root. This is therefore true also for $H(m)$.

Corollary

For $\rho_2/\rho_1 > c_2/c_1$ there is only one real positive root for $F(m) = 0$. Note that this makes the coefficients of $m^i < 0$ for $i = c_2 - 1, c_2 - 2, ..., c_2 - c_1$.

There are no 'good' C kind Plans

If we now set

$$\rho_{i+1}/\rho_i = AQL_{i+1}/AQL_i$$

and also try to ensure a reasonable producer's risk α (say) then for each i, the value of $\mathrm{G}(c_i, n.AQL_i) \geq (1 - \alpha)$.

...(1.2.14)

For example we may calculate c_i from $G(c_i, n.AQL_i) = 0.95$. We get from Cornish-Fisher expansion,

$$n.AQL_i \simeq c_i - 1.6449\sqrt{c_i + 1} + 1.5685 + 0.1962/\sqrt{c_i + 1}.$$

...(1.2.15)

It can be easily verified that for $c_{i+1} \geq c_i$ in this case,

$$\left[\frac{n.AQL_{i+1}}{n.AQL_i}\right] > \left[\frac{c_{i+1}}{c_i}\right]$$

...(1.2.16)

It, therefore, follows that in this case $(\rho_{i+1}/\rho_i) > (c_{i+1}/c_i)$. Surely for some process average this plan will fail to satisfy the condition $Slope_i \geq Slope_{i+1}$ for all positive m as evident from the corollary to theorem 1.2.1.

1.2.6 An alternative sampling scheme

As discussed the OC of C kind Plans do not possess in general the property to become more sensitive (in the sense, defined) to the changes in the process average of the attribute with lower AQL. We now propose the following alternatives.

The A kind Plan

We take a sample of size n, observe the number of defectives or defects in the sample for the i th attribute as x_i for all $i = 1, 2, ..., r$ and apply the following acceptance criterion: accept if,
$x_1 \leq a_1$,

$x_1 + x_2 \leq a_2$,
...,
...,
$x_1 + ... + x_r \leq a_r$; and reject otherwise.

In all the later discussions we use the notation $x_{(j)}$ for $= x_1 + ... + x_j$, $j = 1, 2, ..., r$. Note that $a_i \leq a_j$ for $j \geq i$. The OC function under Poisson conditions is given by,

$$PA(a_1, a_2, .., a_r; m_1, m_2, ..., m_r) = \sum_{x_1=0}^{a_1} ... \sum_{x_r=0}^{a_r - x_{(r-1)}} \prod_{i=1}^{r} g(x_i, m_i)$$

...(1.2.17)

To compare the relative changes in the OC function for changes in p_i, we use the absolute value of the partial differential coefficient of the Poisson OC function as given in equation (1.2.17) with respect to m_i; $m_i = np_i$. This is used as a measure of discriminating power of the OC function w.r.t. j th attribute, treating it as function of $m = m_1 + m_2 + ...m_r$ and keeping m_i/m fixed for $i = 1, 2, ..., r$

Theorem 1.2.3

For $i \leq j$ the discriminating power of the plan (as defined) w.r.t. i th attribute is more than or equal to that for the j th attribute for every $m = m_1 + m_2... + m_r$.

Proof

Denoting the partial differential coefficient of the Poisson OC function as given in equation (1.2.17) with respect to $m_i; m_i = np_i$ by PA'_i, we get

$$-PA'_i = PA(a_1, a_2, .., a_i, ..., a_r; m_1, m_2, ..., m_r) - PA(a_1, a_2, ..., a_i - 1, a_{i+1} - 1, ..a_r - 1; m_1, m_2, ..., m_r)$$

for $a_i > 0$,

...(1.2.18)

and

$$-PA'_i = PA(a_1, a_2, .., a_i, , a_r; m_1, m_2, ..., m_r)$$

for $a_i = 0$.

...(1.2.19)

Thus, the OC function is a decreasing function of m_i or p_i. Moreover, the OC function is increasing in each a_i. Note that $a_j = 0$ implies $a_i = 0$ for all $i \leq j$. It therefore follows from (1.2.18) and (1.2.19) that

$$-PA'_i \geq -PA'_{i+1} \quad \forall m, \quad m = m_1 + m_2 ... + m_r$$

...(1.2.20)

This proves the theorem.

The above property of the plan A therefore allows us to order the attributes in order of relative discriminating power. If the attributes are ordered in the ascending order of AQL value, then it is possible to construct a sampling scheme ensuring an acceptable producer's risk and also satisfy the condition $Slope_i \geq Slope_{i+1}$, for all $m, i = 1, 2, ..., r-1$. For comparison m_i/m is kept fixed, $i = 1, 2, ..., r$.

1.2.7 Construction of A kind MASSP's using the sample size and AQL from MIL-STD-105D

Using the set of n. AQL values chosen from MIL-STD-105D, we shall illustrate this for r = 3. There are 13 n. AQL values. We get 286 combinations such that $n.AQL_1 < n.AQL_2 < n.AQL_3$. We further order the triplets lexicographically, considering the $n.AQL$'s in their natural order.

We start with the first combination of triplet (0.1256, 0.1991, 0.3155) and choose $a_1 = 1, a_2 = 1$,and $a_3 = 2$. This gives producer's risk of around 5.5 %. All other sets of acceptance numbers are worked out such that a plan positioned at a higher level will have a lesser producer's risk, so that producer's risk decreases up to a level of 5% and thereafter it is kept at less than 5%.

The results of this exercise are presented in Table 1.2.4. Using the notation $\rho_i = AQL_i/(AQL_1 + AQL_2 + AQL_3)$ we have defined $p^{(3)}{}_{0.1}$ as the value of the p such that, $PA(a_1, a_2, a_3; np\rho_1, np\rho_2, np\rho_3) = 0.1$.

It is heartening to note that the ratio of the $p^{(3)}{}_{0.1}$ to $TotalAQL$ as defined is also decreasing from 2.3 to 1.4 with increase in $n.TotalAQL$, so that the OC appears to be quite steep.

1.2.8 The D kind plans

Consider the MASSP where we take a sample of size n, observe the number of defectives (defects) for the i th attribute as x_i for all $i = 1, 2, ..., r$ and apply the acceptance rule: accept if $x_1 + x_2 + ... + x_r \leq k$; reject otherwise

The probability of acceptance at $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ under Poisson conditions :

$$PD(k; m_1, m_2, .., m_r) = \sum_{x_1=0}^{k} ... \sum_{x_r=0}^{k-x_{(r-1)}} \prod_{i=1}^{r} g(x_i, m_i)$$

where $m_i = np_i; i = 1, 2, ..., r$.
This plan does not distinguish between the types of defects. This plan is considered as a datum plan to compare the efficiency of other schemes. The probability of acceptance under Poisson conditions at $\boldsymbol{p}$ is equal to that of a single sampling plan at a process average $p = p_1 + p_2 + ... + p_r$ and

$$PD(k; m_1, m_2, ..., m_r) = G(k, m)$$

...(1.2.22)

We, further, note that under Poisson condition and for $a_1 = a_2 = ... = a_r = k$,

$$PA(a_1 = k, a_2 = k, ..., a_r = k; m_1, m_2, ..., m_r) = PD(k; m_1, m_2, .., m_r) = G(k, m)$$

...(1.2.23)

In the next chapters we make use of this plan to compare the costs and efficiencies of the A type and the C type plans.

Figure 1.2.1 Absolute value of Slope of the OC function of the MASSP adopted from MIL-STD-105D Inspection level II - Normal inspection used for incoming inspection of plastic containers. size (N) = 15000. Chosen AQL values are 0.15% , 1% and 4% for critical, major and minor defectives. Sample size = 315, c_1 = 1, c_2 = 7, c_3 =21. The curve is plotted against p, for p_1: p_2: p_3= 0.15 : 1 : 4.0

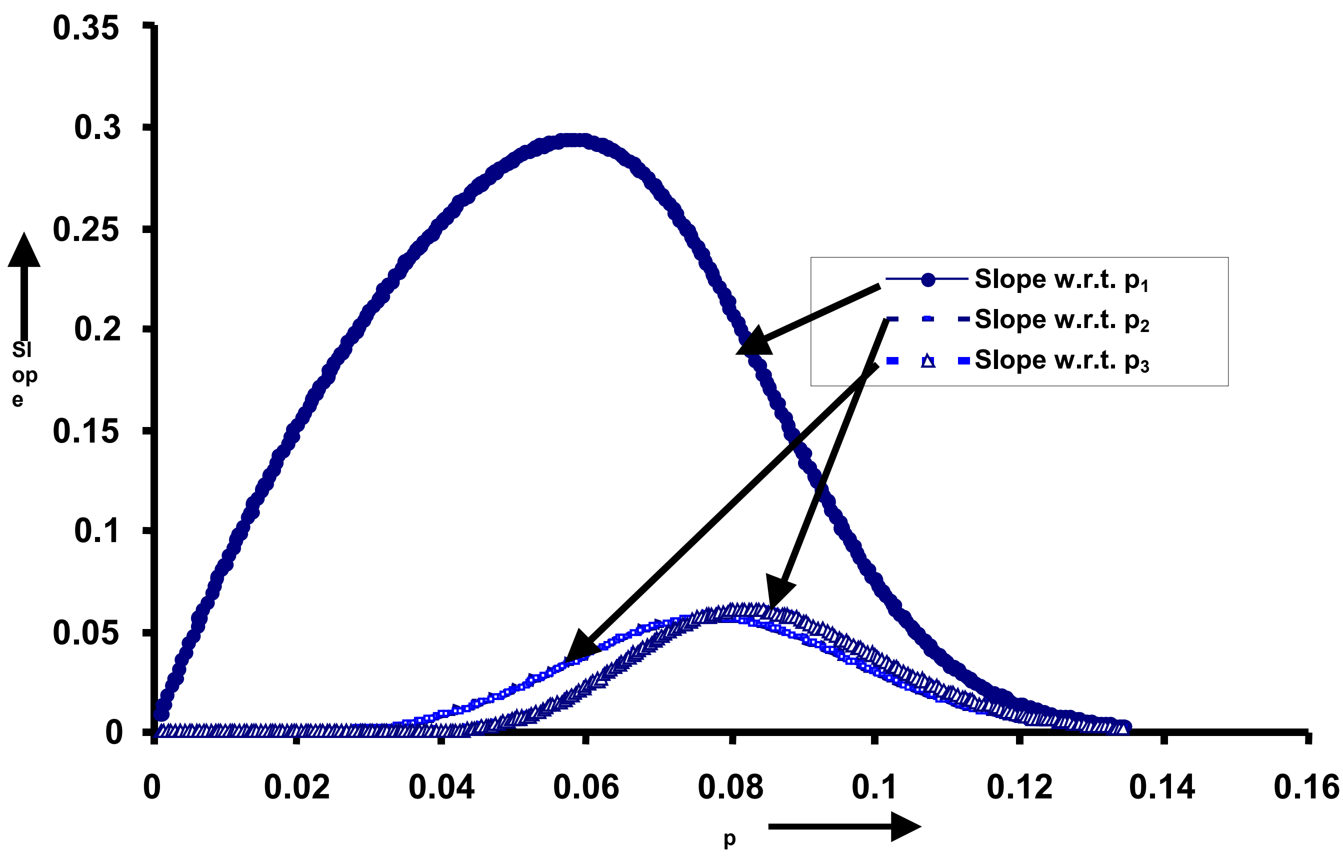


Table 1.2.5 Explanatory notes relating to three attribute A kind single sampling plans for normal inspection

1. Construction setup

This table is constructed using the set of n.AQL values chosen for MIL-STD-105D. There are 13 n.AQL values. We get 286 combinations each of size 3, such that $n.AQL_1 < n.AQL_2 < n.AQL_3$. We order these combinations lexicographically, considering the AQL in their natural order and also keeping the TotalAQL in descending order.

We have started with the first combination as the set (0.12, 0.20. 0.32) and choose $a_1 = 1, a_2 = 1$, and $a_3 = 2$. This gives producer's risk of around 5.5%. All other sets of acceptance numbers are worked out such that a plan positioned at a higher level will have a lesser producer's risk, so that producer's risk decreases up to a level of 5% and thereafter, it is kept at less than 5%.

2. Column identification

Each plan is identified by its serial number given in column 1. The next three columns (column 2, column 3, column 4) contain the $n.AQL_i$ values for $i = 1, 2$, and 3 respectively. From column 5, column 6 and column 7, we obtain the three acceptance numbers of the A kind plan viz. a_1, a_2 and a_3. The next column i.e. the 8 th column gives the producer's risk i.e. the probability of acceptance at the process average (AQL_1, AQL_2, AQL_3). The last column contains the ratio $p^{(3)}{}_{0.10}$ to TotalAQL.

3. Obtaining an A kind sampling plan from the table

We may use the table as follows: Suppose the lot size is 15000 and we use Inspection Level II. From the MIL-STD-105D we get the sample size as 315. Suppose there are three Attributes viz. Attribute 1, Attribute 2 and Attribute 3. We choose their respective AQL's (say), as 0.15%, 1.0% and 4%. This gives $n.AQL_1 = 0.4725, n.AQL_2 = 3.15, n.AQL_3 = 12.6$. We may choose the plan number 145 and get the values of $a_1 = 3, a_2 = 9, a_3 = 23$.

We would, therefore, take a sample of size of 315 from the lot of size 15000, inspect these 315 pieces for all the three attributes and observe the number of defects as x_1, x_2, and x_3 for the first, second and the third attribute respectively. If $x_1 \leq 3, x_1 + x_2 \leq 9, x_1 + x_2 + x_3 \leq 23$, we shall accept the lot, else we shall reject the lot.

For this plan, the producer's risk at AQL = (0.001, 0.01, 0.04) is around 4.5%, as given in column 8. From column 9 we the $p^{(3)}{}_{0.10}$/TotalAQL = 1.8. The probability of acceptance (i.e. the consumer's risk) at this process average (0.0018, 0.018, 0.072) is around 10%.

Table 1.2.5: Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p_{0.10}^{(3)}$ /Total AQL
1	0.1256	0.1991	0.3155	1	1	2	0.055	7.9
2	0.1256	0.1991	0.5	1	2	2	0.054	6.4
3	0.1256	0.3155	0.5	1	2	3	0.026	6.9
4	0.1991	0.3155	0.5	1	2	3	0.039	6.4
5	0.1256	0.1991	0.7924	1	2	3	0.034	5.9
6	0.1256	0.3155	0.7924	1	2	3	0.045	5.4
7	0.1991	0.3155	0.7924	2	2	3	0.050	5.1
8	0.1256	0.5	0.7924	1	2	4	0.038	5.4
9	0.1991	0.5	0.7924	2	2	4	0.043	5.1
10	0.1256	0.1991	1.256	1	2	4	0.030	5
11	0.3155	0.5	0.7924	2	3	4	0.030	4.9
12	0.1256	0.3155	1.256	1	2	4	0.040	4.7
13	0.1991	0.3155	1.256	2	2	4	0.044	4.5
14	0.1256	0.5	1.256	1	3	4	0.049	4.2
15	0.1991	0.5	1.256	2	4	4	0.049	4.1
16	0.3155	0.5	1.256	2	3	5	0.027	4.4
17	0.1256	0.7924	1.256	1	3	5	0.037	4.2
18	0.1991	0.7924	1.256	2	3	5	0.038	4.1
19	0.1256	0.1991	1.991	1	2	5	0.039	4
20	0.3155	0.7924	1.256	2	3	5	0.050	3.8
21	0.1256	0.3155	1.991	1	2	5	0.049	3.8
22	0.1991	0.3155	1.991	2	3	5	0.044	3.7
23	0.5	0.7924	1.256	3	4	5	0.048	3.6
24	0.1256	0.5	1.991	1	2	6	0.043	3.9
25	0.1991	0.5	1.991	2	2	6	0.048	3.8
26	0.3155	0.5	1.991	2	3	6	0.033	3.7
27	0.1256	0.7924	1.991	1	3	6	0.043	3.6
28	0.1991	0.7924	1.991	2	3	6	0.044	3.5
29	0.3155	0.7924	1.991	2	4	6	0.043	3.4
30	0.5	0.7924	1.991	4	6	6	0.050	3.2
31	0.1256	1.256	1.991	1	4	7	0.036	3.4
32	0.1991	1.256	1.991	1	4	7	0.049	3.3
33	0.1256	0.1991	3.155	1	2	7	0.035	3.4
34	0.3155	1.256	1.991	2	4	7	0.045	3.2
35	0.1256	0.3155	3.155	1	2	7	0.043	3.2
36	0.1991	0.3155	3.155	2	2	7	0.046	3.2
37	0.5	1.256	1.991	3	5	7	0.041	3.1
38	0.1256	0.5	3.155	1	3	7	0.047	3.1
39	0.1991	0.5	3.155	2	3	7	0.047	3

Table 1.2.5 (contd.) : Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p_{0.10}^{(3)}$ /Total AQL
40	0.3155	0.5	3.155	4	4	7	0.050	3
41	0.7924	1.256	1.991	3	5	8	0.038	3.2
42	0.1256	0.7924	3.155	1	3	8	0.040	3.2
43	0.1991	0.7924	3.155	2	3	8	0.040	3.1
44	0.3155	0.7924	3.155	3	3	8	0.050	3
45	0.5	0.7924	3.155	3	4	8	0.044	2.9
46	0.1256	1.256	3.155	2	4	8	0.050	2.8
47	0.1991	1.256	3.155	2	5	8	0.048	2.8
48	0.3155	1.256	3.155	2	4	9	0.041	3
49	0.5	1.256	3.155	2	5	9	0.044	2.9
50	0.7924	1.256	3.155	4	5	9	0.050	2.7
51	0.1256	1.991	3.155	2	6	9	0.045	2.7
52	0.1256	0.1991	5	2	2	9	0.049	2.7
53	0.1991	1.991	3.155	2	6	9	0.049	2.7
54	0.1256	0.3155	5	1	2	10	0.037	2.8
55	0.3155	1.991	3.155	2	5	10	0.048	2.8
56	0.1991	0.3155	5	1	2	10	0.050	2.8
57	0.1256	0.5	5	1	3	10	0.038	2.7
58	0.5	1.991	3.155	2	6	10	0.048	2.7
59	0.1991	0.5	5	1	3	10	0.050	2.7
60	0.3155	0.5	5	2	3	10	0.045	2.6
61	0.1256	0.7924	5	2	3	10	0.050	2.6
62	0.7924	1.991	3.155	3	7	10	0.048	2.6
63	0.1991	0.7924	5	2	4	10	0.045	2.6
64	0.3155	0.7924	5	3	5	10	0.048	2.5
65	0.5	0.7924	5	2	4	11	0.045	2.6
66	0.1256	1.256	5	1	4	11	0.046	2.6
67	1.256	1.991	3.155	4	7	11	0.045	2.6
68	0.1991	1.256	5	2	4	11	0.045	2.6
69	0.3155	1.256	5	2	5	11	0.042	2.5
70	0.5	1.256	5	3	5	11	0.049	2.4
71	0.7924	1.256	5	3	5	12	0.046	2.5
72	0.1256	1.991	5	2	5	12	0.045	2.5
73	0.1991	1.991	5	3	5	12	0.050	2.4
74	0.3155	1.991	5	2	6	12	0.044	2.4
75	0.5	1.991	5	5	6	12	0.050	2.4
76	0.7924	1.991	5	3	6	13	0.050	2.4
77	1.256	1.991	5	5	8	13	0.045	2.3
78	0.1256	0.1991	7.924	2	2	13	0.046	2.3

Table 1.2.5 (contd.): Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p_{0.10}^{(3)}$ /Total AQL
79	0.1256	3.155	5	2	8	13	0.046	2.3
80	0.1991	3.155	5	2	8	13	0.050	2.3
81	0.1256	0.3155	7.924	2	3	13	0.047	2.3
82	0.1991	0.3155	7.924	2	4	13	0.050	2.2
83	0.3155	3.155	5	2	7	14	0.047	2.3
84	0.1256	0.5	7.924	1	3	14	0.038	2.3
85	0.1991	0.5	7.924	1	3	14	0.049	2.3
86	0.5	3.155	5	2	8	14	0.050	2.3
87	0.3155	0.5	7.924	2	3	14	0.043	2.3
88	0.1256	0.7924	7.924	2	3	14	0.048	2.3
89	0.1991	0.7924	7.924	2	4	14	0.042	2.3
90	0.7924	3.155	5	3	9	14	0.049	2.2
91	0.3155	0.7924	7.924	2	4	14	0.049	2.2
92	0.5	0.7924	7.924	4	6	14	0.049	2.2
93	0.1256	1.256	7.924	1	4	15	0.045	2.3
94	0.1991	1.256	7.924	2	4	15	0.044	2.3
95	1.256	3.155	5	4	9	15	0.045	2.2
96	0.3155	1.256	7.924	2	5	15	0.040	2.2
97	0.5	1.256	7.924	3	5	15	0.046	2.2
98	0.7924	1.256	7.924	4	7	15	0.049	2.1
99	0.1256	1.991	7.924	2	5	16	0.045	2.2
100	0.1991	1.991	7.924	2	5	16	0.049	2.2
101	1.991	3.155	5	5	10	16	0.049	2.2
102	0.3155	1.991	7.924	2	6	16	0.042	2.2
103	0.5	1.991	7.924	3	6	16	0.048	2.1
104	0.7924	1.991	7.924	4	8	16	0.048	2.1
105	1.256	1.991	7.924	5	7	17	0.049	2.1
106	0.1256	3.155	7.924	1	8	17	0.047	2.1
107	0.1991	3.155	7.924	2	8	17	0.044	2.1
108	0.3155	3.155	7.924	3	8	17	0.048	2.1
109	0.5	3.155	7.924	3	10	17	0.050	2
110	0.7924	3.155	7.924	4	8	18	0.048	2.1
111	1.256	3.155	7.924	6	10	18	0.049	2
112	0.1256	0.1991	12.56	1	2	19	0.049	2
113	0.1256	0.3155	12.56	1	3	19	0.050	2
114	0.1256	5	7.924	2	11	19	0.047	2
115	1.991	3.155	7.924	6	11	19	0.050	2
116	0.1991	0.3155	12.56	2	3	19	0.047	2
117	0.1991	5	7.924	2	11	19	0.050	2

Table 1.2.5 (contd.) : Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p^{(3)}_{0.10}$ /Total AQL
118	0.1256	0.5	12.56	2	4	19	0.048	2
119	0.3155	5	7.924	3	13	19	0.050	2
120	0.1991	0.5	12.56	1	3	20	0.049	2
121	0.3155	0.5	12.56	2	3	20	0.043	2
122	0.5	5	7.924	4	10	20	0.050	2
123	0.1256	0.7924	12.56	2	3	20	0.047	2
124	0.1991	0.7924	12.56	2	4	20	0.040	2
125	0.3155	0.7924	12.56	2	4	20	0.046	2
126	0.7924	5	7.924	4	11	20	0.049	2
127	0.5	0.7924	12.56	3	5	20	0.046	2
128	0.1256	1.256	12.56	2	5	20	0.049	1.9
129	0.1991	1.256	12.56	2	6	20	0.050	1.9
130	0.3155	1.256	12.56	2	5	21	0.039	2
131	1.256	5	7.924	6	11	21	0.049	2
132	0.5	1.256	12.56	3	5	21	0.043	2
133	0.7924	1.256	12.56	4	6	21	0.047	1.9
134	0.1256	1.991	12.56	2	6	21	0.048	1.9
135	0.1991	1.991	12.56	2	7	21	0.048	1.9
136	0.3155	1.991	12.56	3	8	21	0.050	1.9
137	1.991	5	7.924	6	12	22	0.050	1.9
138	0.5	1.991	12.56	3	6	22	0.045	1.9
139	0.7924	1.991	12.56	4	7	22	0.046	1.9
140	1.256	1.991	12.56	5	7	23	0.047	1.9
141	0.1256	3.155	12.56	2	7	23	0.048	1.9
142	0.1991	3.155	12.56	2	8	23	0.041	1.9
143	0.3155	3.155	12.56	2	8	23	0.047	1.9
144	3.155	5	7.924	9	14	23	0.049	1.9
145	0.5	3.155	12.56	3	9	23	0.045	1.9
146	0.7924	3.155	12.56	5	11	23	0.049	1.8
147	1.256	3.155	12.56	7	9	24	0.050	1.9
148	0.1256	5	12.56	2	10	25	0.048	1.8
149	1.991	3.155	12.56	7	10	25	0.049	1.8
150	0.1991	5	12.56	2	11	25	0.044	1.8
151	0.3155	5	12.56	2	11	25	0.050	1.8
152	0.5	5	12.56	3	12	25	0.049	1.8
153	0.7924	5	12.56	4	11	26	0.046	1.8
154	1.256	5	12.56	5	13	26	0.048	1.8
155	1.991	5	12.56	7	13	27	0.049	1.8
156	0.1256	0.1991	19.91	1	2	28	0.048	1.8

Table 1.2.5 (contd.) : Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p_{0.10}^{(3)}$ /Total AQL
157	0.1256	0.3155	19.91	1	3	28	0.048	1.8
158	0.1991	0.3155	19.91	2	3	28	0.045	1.8
159	0.1256	0.5	19.91	2	3	28	0.049	1.8
160	0.1991	0.5	19.91	2	4	28	0.048	1.8
161	0.1256	7.924	12.56	2	16	28	0.048	1.8
162	0.1991	7.924	12.56	3	16	28	0.050	1.7
163	3.155	5	12.56	10	17	28	0.050	1.7
164	0.3155	0.5	19.91	3	5	28	0.050	1.7
165	0.3155	7.924	12.56	3	14	29	0.048	1.8
166	0.1256	0.7924	19.91	2	3	29	0.047	1.8
167	0.1991	0.7924	19.91	2	4	29	0.039	1.8
168	0.5	7.924	12.56	3	15	29	0.045	1.8
169	0.3155	0.7924	19.91	2	4	29	0.045	1.8
170	0.5	0.7924	19.91	3	5	29	0.044	1.8
171	0.7924	7.924	12.56	4	16	29	0.048	1.7
172	0.1256	1.256	19.91	2	5	29	0.046	1.8
173	0.1991	1.256	19.91	2	5	29	0.049	1.7
174	0.3155	1.256	19.91	3	6	29	0.048	1.7
175	0.5	1.256	19.91	3	5	30	0.043	1.8
176	1.256	7.924	12.56	4	16	30	0.050	1.8
177	0.7924	1.256	19.91	4	6	30	0.044	1.7
178	0.1256	1.991	19.91	2	6	30	0.046	1.7
179	0.1991	1.991	19.91	2	6	30	0.049	1.7
180	0.3155	1.991	19.91	3	7	30	0.047	1.7
181	0.5	1.991	19.91	4	8	30	0.050	1.7
182	1.991	7.924	12.56	7	16	31	0.050	1.7
183	0.7924	1.991	19.91	4	7	31	0.044	1.7
184	1.256	1.991	19.91	5	9	31	0.049	1.7
185	0.1256	3.155	19.91	2	9	31	0.049	1.7
186	0.1991	3.155	19.91	3	10	31	0.050	1.7
187	0.3155	3.155	19.91	2	8	32	0.045	1.7
188	0.5	3.155	19.91	3	8	32	0.049	1.7
189	3.155	7.924	12.56	9	18	32	0.050	1.7
190	0.7924	3.155	19.91	4	9	32	0.050	1.7
191	1.256	3.155	19.91	5	9	33	0.049	1.7
192	0.1256	5	19.91	2	10	34	0.047	1.7
193	1.991	3.155	19.91	6	10	34	0.049	1.7
194	0.1991	5	19.91	2	10	34	0.050	1.7
195	0.3155	5	19.91	2	11	34	0.047	1.7

Table 1.2.5 (contd.) : Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p^{(3)}_{0.10}$ /Total AQL
196	0.5	5	19.91	3	11	34	0.050	1.7
197	5	7.924	12.56	12	21	34	0.048	1.7
198	0.7924	5	19.91	4	13	34	0.049	1.7
199	1.256	5	19.91	5	12	35	0.048	1.7
200	1.991	5	19.91	6	13	36	0.048	1.7
201	0.1256	7.924	19.91	2	15	37	0.046	1.7
202	0.1991	7.924	19.91	2	15	37	0.048	1.6
203	3.155	5	19.91	9	15	37	0.049	1.6
204	0.3155	7.924	19.91	3	16	37	0.047	1.6
205	0.5	7.924	19.91	4	17	37	0.049	1.6
206	0.7924	7.924	19.91	4	15	38	0.050	1.6
207	1.256	7.924	19.91	6	17	38	0.049	1.6
208	1.991	7.924	19.91	6	18	39	0.050	1.6
209	3.155	7.924	19.91	10	21	40	0.050	1.6
210	0.1256	0.1991	31.55	2	3	41	0.049	1.6
211	0.1256	0.3155	31.55	2	2	42	0.046	1.6
212	0.1991	0.3155	31.55	2	3	42	0.040	1.6
213	0.1256	0.5	31.55	1	3	42	0.049	1.6
214	0.1991	0.5	31.55	2	3	42	0.046	1.6
215	0.3155	0.5	31.55	2	4	42	0.047	1.6
216	0.1256	0.7924	31.55	2	4	42	0.046	1.6
217	0.1991	0.7924	31.55	2	4	42	0.049	1.6
218	0.1256	12.56	19.91	2	22	42	0.049	1.6
219	0.3155	0.7924	31.55	3	5	42	0.048	1.6
220	0.1991	12.56	19.91	3	22	42	0.050	1.6
221	0.3155	12.56	19.91	4	24	42	0.050	1.6
222	5	7.924	19.91	11	21	43	0.046	1.6
223	0.5	0.7924	31.55	3	4	43	0.046	1.6
224	0.1256	1.256	31.55	2	4	43	0.049	1.6
225	0.5	12.56	19.91	3	21	43	0.048	1.6
226	0.1991	1.256	31.55	2	5	43	0.043	1.6
227	0.3155	1.256	31.55	2	5	43	0.048	1.6
228	0.7924	12.56	19.91	4	22	43	0.048	1.6
229	0.5	1.256	31.55	3	6	43	0.046	1.6
230	0.7924	1.256	31.55	5	7	43	0.049	1.6
231	0.1256	1.991	31.55	3	8	43	0.050	1.6
232	1.256	12.56	19.91	5	22	44	0.046	1.6
233	0.1991	1.991	31.55	2	6	44	0.044	1.6
234	0.3155	1.991	31.55	2	6	44	0.049	1.6

Table 1.2.5 (contd.) : Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p_{0.10}^{(3)}$ /Total AQL
235	0.5	1.991	31.55	3	7	44	0.046	1.6
236	0.7924	1.991	31.55	4	8	44	0.049	1.6
237	1.991	12.56	19.91	8	25	44	0.050	1.6
238	1.256	1.991	31.55	5	8	45	0.045	1.6
239	0.1256	3.155	31.55	2	8	45	0.045	1.6
240	0.1991	3.155	31.55	2	8	45	0.048	1.6
241	0.3155	3.155	31.55	2	9	45	0.049	1.6
242	0.5	3.155	31.55	4	9	45	0.049	1.6
243	0.7924	3.155	31.55	3	9	46	0.050	1.6
244	3.155	12.56	19.91	9	24	46	0.050	1.6
245	1.256	3.155	31.55	5	10	46	0.049	1.6
246	0.1256	5	31.55	2	11	47	0.046	1.6
247	1.991	3.155	31.55	6	11	47	0.049	1.6
248	0.1991	5	31.55	2	11	47	0.049	1.6
249	0.3155	5	31.55	3	12	47	0.047	1.6
250	0.5	5	31.55	3	13	47	0.050	1.5
251	0.7924	5	31.55	4	12	48	0.044	1.6
252	5	12.56	19.91	11	27	48	0.050	1.6
253	1.256	5	31.55	6	13	48	0.049	1.5
254	1.991	5	31.55	7	14	49	0.047	1.5
255	0.1256	7.924	31.55	2	16	50	0.049	1.5
256	0.1991	7.924	31.55	3	16	50	0.050	1.5
257	3.155	5	31.55	9	17	50	0.050	1.5
258	0.3155	7.924	31.55	3	18	50	0.050	1.5
259	0.5	7.924	31.55	3	15	51	0.049	1.5
260	0.7924	7.924	31.55	4	16	51	0.049	1.5
261	7.924	12.56	19.91	17	31	51	0.049	1.5
262	1.256	7.924	31.55	7	22	51	0.050	1.5
263	1.991	7.924	31.55	7	19	52	0.050	1.5
264	3.155	7.924	31.55	8	19	54	0.049	1.5
265	0.1256	12.56	31.55	3	24	55	0.050	1.5
266	0.1991	12.56	31.55	2	21	56	0.046	1.5
267	0.3155	12.56	31.55	3	21	56	0.047	1.5
268	5	7.924	31.55	12	21	56	0.049	1.5
269	0.5	12.56	31.55	3	22	56	0.048	1.5
270	0.7924	12.56	31.55	4	23	56	0.050	1.5
271	1.256	12.56	31.55	6	22	57	0.049	1.5
272	1.991	12.56	31.55	7	23	58	0.048	1.5
273	3.155	12.56	31.55	9	25	59	0.049	1.5

Table 1.2.5 (contd.) : Three attribute A kind single sampling plans for normal inspection

SL. No.	n.(AQL $_1$)	n.(AQL $_2$)	n.(AQL $_3$)	a_1	a_2	a_3	Over all Prod. risk	$p_{0.10}^{(3)}$ /Total AQL
274	5	12.56	31.55	13	27	61	0.050	1.5
275	0.1256	19.91	31.55	2	30	64	0.048	1.5
276	0.1991	19.91	31.55	2	30	64	0.050	1.5
277	0.3155	19.91	31.55	3	31	64	0.047	1.5
278	0.5	19.91	31.55	4	31	64	0.050	1.5
279	7.924	12.56	31.55	17	32	64	0.049	1.5
280	0.7924	19.91	31.55	5	34	64	0.050	1.5
281	1.256	19.91	31.55	5	32	65	0.049	1.5
282	1.991	19.91	31.55	8	32	66	0.050	1.5
283	3.155	19.91	31.55	9	35	67	0.049	1.4
284	5	19.91	31.55	12	37	69	0.050	1.4
285	7.924	19.91	31.55	18	42	72	0.050	1.4
286	12.56	19.91	31.55	26	49	77	0.050	1.4

1.3 Multiattribute sampling plans of given strength

1.3.1 Scope

We considered in the last chapter two schemes: one, consisting of plans of A kind and, another consisting of plans of C kind, which differed in terms of acceptance criteria. In this chapter we try to construct MASSP's defined by two equations $Q(\boldsymbol{p}) = \alpha$ and $P(\boldsymbol{p'}) = \beta$, where $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ is the process average vector for the satisfactory quality level and $\boldsymbol{p'} = (p'_1, p'_2, ..., p'_r)$ is the process average vector for an unsatisfactory quality level; $Q(\boldsymbol{p})$is the probability of rejection at $\boldsymbol{p}$ and $P(\boldsymbol{p'})$ is the probability of acceptance at $\boldsymbol{p'}$. Further, α and β are the stipulated producer's and consumer's risks, respectively.

1.3.2 Single attribute situation

Before we start, we look at the method of obtaining the Poisson solution for sampling plan of a given strength for the case of single attribute as outlined by Hald (1981). For the case of single attribute, we define a sampling plan of strength (p, α, p', β) as the plan satisfying two equations $Q(p) = \alpha$, and $P(p') = \beta$, p and p' denote the process average values for the satisfactory and unsatisfactory quality levels, respectively. Allowing c and n to be real numbers, these two equations determine (c, n) uniquely. However, in practice c and n are integers. We therefore find a sampling plan such that the two conditions are satisfied as nearly as possible and actual risks are less than or equal to the stipulated risks.

For this we define $m_P(c)$ as the value of m such that $G(c, m_P(c)) = P$. Under Poisson conditions the problem boils down to solving the two equations: $G(c, np) = 1 - \alpha$ and $G(c, np') = \beta$. Let $G(c, m_P(c)) = P$ then we solve the equations

$$np = m_{1-\alpha}(c)$$

and

$$np' = m_\beta(c).$$

...(1.3.1)

The ratio $R(c, \alpha, \beta) = m_\beta(c)/m_{1-\alpha}(c)$ is a decreasing function of c for given (α, β).
We may therefore obtain c uniquely from the equation:

$$R(c, \alpha, \beta) = p'/p$$

...(1.3.2)

since c is an integer we obtain c from : $R(c-1, \alpha, \beta) > p'/p \geq R(c, \alpha, \beta)$

...(1.3.3)

and obtain an integer n from $m_\beta(c)/p' \leq n \leq m_{1-\alpha}(c)/p$

...(1.3.4)

Note that the smallest value of n (as a real number) is obtained as $n = m_\beta(c)/p'$ as the plan giving the smallest sample size.

1.3.3 Construction of C kind plans of given strength

The Problem

We consider the case r = 2 and we want to find a C kind plan satisfying the equations:

$$G(c_1, np_1).G(c_2, np_2) = 1 - \alpha$$

$$G(c_1, np'_1).G(c_2, np'_2) = \beta$$

...(1.3.5)

Restricting to the situation where $p_1/(p_1 + p_2) = p'_1/(p'_1 + p'_2) = \rho$ we write $p = p_1 + p_2, \quad p' = p'_1 + p'_2, m = np, m' = np'$. For a given ρ, we define $m_P(c_1, c_2, \rho)$ as the value of m satisfying the equation

$$G(c_1, m\rho)G(c_2, m(1 - \rho)) = P$$

...(1.3.6)

Note: The function $m_P(c_1, c_2, \rho)$ used for the C kind MASSP is similar to the function $m_P(c)$ defined in the context to the single sampling plan for single attribute.

We must have n, c_1, c_2 such that

$$m_{1-\alpha}(c_1, c_2, \rho) = m$$

$$m_\beta(c_1, c_2, \rho) = m'$$

...(1.3.7)

Lemma

Let

$$G(c_1, M_1\rho)G(c_2, M_1(1-\rho)) = G(c_1, M_2\rho)G(c_2+1, M_2(1-\rho)) = G(c_1+1, M_3\rho).G(c_2, M_3(1-\rho)) = P.$$

Then $M_2 \geq M_1$ and $M_3 \geq M_1$

...(1.3.8)

Proof

Let $M_2 < M_1$. Then $G(c_1, M_1\rho) < G(c_1, M_2\rho)$ and $G(c_2, M_1(1-\rho)) < G(c_2, M_2(1-\rho)) < G(c_2+1, M_2(1-\rho))$.
This makes $G(c_1, M_1\rho).G(c_2, M_1(1-\rho)) < G(c_1, M_2\rho).G(c_2+1, M_2(1-\rho))$ which is not true. Hence $M_2 \geq M_1$.

If $M_3 < M_1$ then $G(c_1+1, M_3\rho)) > G(c_1, M_1\rho))$ and $G(c_2, M_3(1-\rho)) > G(c_2, M_1(1-\rho))$
This makes

$$G(c_1, M_1\rho).G(c_2, M_1(1-\rho)) < G(c_1, M_3\rho).G(c_2+1, M_3(1-\rho))$$

which is not true. Hence $M_3 \geq M_1$.

We now introduce the auxiliary function

$$R(c_1, c_2, \rho, \alpha, \beta) = m_\beta(c_1, c_2, \rho)/m_{1-\alpha}(c_1, c_2, \rho)$$

...(1.3.9)

$R(c_1, c_2, \rho, \alpha, \beta)$ is conveniently taken to ∞ when $c_2 = -1$

Theorem 1.3.1
If we increase c_2 keeping c_1 as fixed then as $c_2 \to \infty$, $R(c_1, c_2, \rho, \alpha, \beta)$ first decreases to a minimum and then increases so that the function has a unique minimum in the range $m > 0$

We show this by the following partly demonstrative arguments which are supported by extensive numerical computations (not included in the dissertation for the sake of brevity).

1. From the lemma preceeding the theorem, we observe that if we increase c_2 keeping c_1 as fixed i.e. when $c_2 \to \infty$ the function $m_{1-\alpha}(c_1, c_2, \rho)$ increases. But $G(c_1, m\rho)$ must be greater than or equal to $1-\alpha$, else $G(c_1, m\rho).G(c_2, m(1-\rho))$ becomes less than $1-\alpha$.

2. It also follows that $m_{1-\alpha}(c_1, c_2, \rho) \to m_{1-\alpha}(c_1)/\rho$ as $c_2 \to \infty$. Similarly, $m_\beta(c_1)/\rho[m_{1-\alpha}(c)$ and $m_\beta(c)$ are the Poisson fractiles of appropriate strengths as defined in section 1.3.1]

3. However, $m_\beta(c_1, c_2, \rho) > m_{1-\alpha}(c_1, c_2, \rho)$ always. And for a small increase of c_2 both $m_\beta(c_1, c_2, \rho)$ and $m_{1-\alpha}(c_1, c_2, \rho)$ will increase by a small amount relative to their original values and therfore their ratio will decrease in general. Thus for small values of c_2, $R(c_1, c_2, \rho, \alpha, \beta)$ is a decreasing function of c_2.[Reference figure 1.3.1]

4. With the increase in c_2 beyond some point the increase in $m_{1-\alpha}(c_1, c_2, \rho)$ will be very small but increase in $m_\beta(c_1, c_2, \rho)$ will still be finite. As c_2 increases the denominator of $R(c_1, c_2, \rho, \alpha, \beta)$ approximates its limiting value $m_{1-\alpha}(c_1)/\rho$ much earlier usually than

the numerator approximating its limiting value $m_\beta(c_1)/\rho$ and hence after a certain point $R(c_1, c_2, \rho, \alpha, \beta)$ becomes an increasing function. [Reference Figure 1.3.1]

5. But the rate of increase will decrease so that $R(c_1, c_2, \rho, \alpha, \beta) \to m_\beta(c_1)/m_{1-\alpha}(c_1)$ as $c_2 \to \infty$.

(The arguments provided above do not constitute a proof of the theorem. But, what is being claimed is simple enough and the supporting arguments help in understanding the main point.)

Thus, for a given c_1 there is a unique c_2 for which $R(c_1, c_2, \rho, \alpha, \beta)$ will be minimum. We call this as $c_2^{(c_1)}$.

Note that $c_2^{(c_1)} > c_2^{(c_1+1)}$

...(1.3.10)

and $R(c_1, c_2^{(c_1)}, \rho, \alpha, \beta) > R(c_1, c_2^{(c_1+1)}, \rho, \alpha, \beta)$

...(1.3.11)

Figure 1.3.1 presents the graph of $R(c_1, c_2^{(c_1)}, \rho, \alpha, \beta)$ as a function of c_2 for $\alpha = 0.05, \beta = 0.10, \rho = 0.1$ and $c_1 = 1, 2, 3, 4$.

Since c_1, c_2 are all integers we must consider the set of c_1, c_2 values for which $R(c_1, c_2, \rho, \alpha, \beta)$ is less than p'/p and choose the c_1 and c_2 for which $m_\beta(c_1, c_2, \rho)$ is minimum. This will ensure the minimum sample size satisfying

$$m_{1-\alpha}(c_1, c_2, \rho) = m$$

$$m_\beta(c_1, c_2, \rho) = m'$$

...(1.3.12)

Construction Algorithm

Precisely we shall adopt the following steps.
Step 1. Choose c_1

Step 2. For $c_2 < c_2^{(c_1)}$, we check if $R(c_1, c_2 - 1, \rho, \alpha, \beta) \geq p'/p > R(c_1, c_2, \rho, \alpha, \beta)$
If it holds, then choose $n = m_\beta(c_1, c_2, \rho)/p'$. Else,

Step 3. increase c_1 by one and go to step 1.

To facilitate the above tasks we may construct a table containing
$c_1, c_2, m_\beta(c_1, c_2, \rho), m_{1-\alpha}(c_1, c_2, \rho)$ arranged in descending order of $R(c_1, c_2, \rho, \alpha, \beta)$ for a given

$\alpha, \beta, and\, \rho.$
For $\alpha = 0.05, \beta = 0.10$ and $\rho = 0.1$ this table is given below as table 1.3.1. Similar tables for other values of α, β and ρ can be constructed. With the tables provided, construction of fixed risk C kind plan for $r = 2$ as demonstrtrated is simple.

No attempts are made to invistigate the case $r > 2$. The approach will be similar in essence but numerically the problem is sure to turn more clumsy for $r > 2$.

Table:1.3.1 Construction parameters for fixed risk C kind MASSP's for
$\boldsymbol{\alpha = 0.05}, \quad \boldsymbol{\beta = 0.1}, \quad \boldsymbol{\rho = 0.1}$.

Sl.	c_1	c_2	$m_{0.95}(c_1, c_2, \rho)$	$m_{0.10}(c_1, c_2, \rho)$	$R(c_1, c_2, \rho, \alpha, \beta)$
1	0	0	0.051	2.303	44.89
2	0	1	0.266	3.786	14.23
3	0	2	0.437	5.075	11.61
4	1	1	0.392	4.224	10.78
5	1	2	0.882	5.72	6.49
6	1	3	1.423	7.11	5.00
7	1	4	1.958	8.427	4.30
8	1	5	2.449	9.688	3.96
9	1	6	2.867	10.904	3.80
10	1	7	3.184	12.081	3.79
11	2	5	2.858	10.125	3.54
12	2	6	3.554	11.447	3.22
13	2	7	4.243	12.734	3.00
14	2	8	4.914	13.991	2.85
15	2	9	5.551	15.22	2.74
16	2	10	6.143	16.424	2.67
17	2	11	6.676	17.604	2.64
18	2	12	7.137	18.763	2.63
19	3	9	5.957	15.591	2.62
20	3	10	6.735	16.865	2.50
21	3	11	7.504	18.12	2.41
22	3	12	8.26	19.356	2.34
23	3	13	8.994	20.574	2.29
24	3	14	9.7	21.776	2.24
25	3	15	10.371	22.96	2.21
26	3	16	10.999	24.13	2.19
27	3	17	11.574	25.283	2.18
28	4	14	10.159	22.134	2.18
29	4	15	10.983	23.373	2.13
30	4	16	11.797	24.599	2.09
31	4	17	12.598	25.813	2.05
32	4	18	13.381	27.014	2.02

Similar tables for other values of α, β, and ρ can be constructed. With the tables provided construction of fixed risk C kind plans for r =2 as demonstrared is simple. No attempts are made to investigate the case $r > 2$. The approach will be similar in essence, but numerically the problem is sure to turn clumsy for $r > 2$

1.3.4 Construction of A kind plans of given strength

In this case we want to satisfy the equations :

$$PA(a_1, a_2; np_1, np_2) = 1 - \alpha$$

$$PA(a_1, a_2; np'_1, np'_2) = \beta$$

Restricting to the situation where $p_1/(p_1 + p_2) = p'_1/(p'_1 + p'_2) = \rho$, for a given ρ we define $ma_P(a_1, a_2, \rho)$ as the value of m satisfying the equation $PA(a_1, a_2; m\rho, m(1 - \rho)) = P$.

We must have n, a_1, a_2 such that,

$$ma_{1-\alpha}(a_1, a_2, \rho) = m$$

$$ma_\beta(a_1, a_2, \rho) = m'$$

...(1.3.14)

Introducing the auxiliary function $Ra(a_1, a_2, \rho, \alpha, \beta) = ma_\beta(a_1, a_2, \rho)/ma_{1-\alpha}(a_1, a_2, \rho)$, we note that $ma_P(a_1, a_2, \rho)$ is an increasing function of a_1 and a_2, $Ra(a_1, a_2, \alpha, \beta, \rho)$ is a decreasing function of a_1 and a_2 and We obtain the smallest $a_2 = a_2^*$ for which,

$$Ra\,(0, a_2^* - 1, \rho, \alpha, \beta) > p'/p \geq Ra(a_2^*, a_2^*, \rho, \alpha, \beta)$$

...(1.3.15)

then find a_1^* such that

$$Ra(a_1^* - 1, a_2^*, \rho, \alpha, \beta) > p'/p \geq Ra(a_1^*, a_2^*, \rho, \alpha, \beta)$$

...(1.3.16)[5pt]

To facilitate the above tasks we may construct a table containing $a_1, a_2, ma_\beta(a_1, a_2, \rho), ma_{1-\alpha}(a_1, a_2, \rho)$ arranged in descending order of $Ra(a_1, a_2, \rho, \alpha, \beta)$ for

a given set of α, β, and ρ.
For $\alpha = 0.05, \beta = 0.10$ and $\rho = 0.1$, the result is presented in table 1.3.2. Similar tables can easily be constructed for other possible values of α, β and ρ

Table:1.3.2: Construction parameters for fixed risk A kind MASSP's for $\boldsymbol{\alpha = 0.05}$, $\boldsymbol{\beta = 0.1}$, $\boldsymbol{\rho = 0.1}$.

Sl.	a_2	a_1	$m_{0.95}(a_2, a_1, \rho)$	$m_{0.10}(a_1, a_2, \rho)$	$Ra\,(a_1, a_2, \rho, \alpha, \beta)$
1	0	0	0.05	2.3	45.16
1	0	0	0.05	2.3	44.89
2	1	0	0.26	3.79	14.58
3	1	1	0.35	3.89	10.96
4	2	1	0.8	5.32	6.65
5	2	2	0.81	5.33	6.58
6	3	1	1.31	6.65	5.08
7	3	2	1.36	6.68	4.91
8	4	1	1.83	7.93	4.33
9	4	2	1.96	7.99	4.08
10	4	3	1.97	8	4.06
11	5	1	2.31	9.16	3.97
12	5	2	2.59	9.27	3.58
13	5	3	2.61	9.28	3.56
14	6	2	3.24	10.52	3.25
15	6	3	3.28	10.54	3.21
16	7	2	3.88	11.74	3.03
17	7	3	3.97	11.77	2.96
18	7	4	3.98	11.78	2.96
19	8	2	4.52	12.94	2.86
20	8	3	4.68	12.99	2.78

Table:1.3.2 (Contd.): Construction parameters for fixed risk A kind MASSP's for $\alpha = 0.05, \quad \beta = 0.1, \quad \rho = 0.1$.

Sl.	a_2	a_1	$ma_{0.95}(a_1, a_2, \rho)$	$ma_{0.10}(a_1, a_2, \rho)$	$Ra\,(a_1, a_2, \rho, \alpha, \beta)$
21	8	4	4.69	13	2.77
22	9	2	5.15	14.12	2.74
23	9	3	5.39	14.2	2.63
24	9	4	5.42	14.21	2.62
25	10	3	6.11	15.39	2.52
26	10	4	6.16	15.41	2.50
27	11	3	6.83	16.57	2.43
28	11	4	6.91	16.6	2.40
29	11	5	6.92	16.6	2.40
30	12	3	7.55	17.74	2.35
31	12	4	7.67	17.78	2.32
32	12	5	7.68	17.79	2.32
33	13	3	8.25	18.9	2.29
34	13	4	8.43	18.95	2.25
35	13	5	8.46	18.96	2.24
36	14	4	9.19	20.12	2.19
37	14	5	9.24	20.13	2.18
38	15	4	9.96	21.27	2.14
39	15	5	10.02	21.29	2.12

1.3.5 D kind plans of given strength

In the chapter 1.2 we have introduced MASSP of D kind as the one with the following rule. From each lot of size N, take a sample of size n, accept if total number of defectives or defects of all types put together is less than or equal to k i.e. $x_1 + x_2 + ... + x_r \leq k$, otherwise reject the lot. For obtaining the D type MASSP of given strength we use the fact that under Poisson condition the OC at $(p_1, p_2, ..., p_r)$ is identical with that of single sampling plan for a single attribute with sample size n and with acceptance number = k at $p = p_1 + p_2 + .. + p_r$. Thus the problem boils down to solving the pair of equations:

$G(k, np) = 1 - \alpha$ and $G(k, np') = \beta$

where $p = p_1 + p_2 + .. + p_r$ and $p' = p'_1 + p'_2 + .. + p'_r$

...(1.3.17)

Let $G(k, md_P(k)) = P$ then we solve the equations:
$np = md_{1-\alpha}(k)$ and $np' = md_\beta(k)$.

...(1.3.18)

To find a plan $D(k, n)$ of a given strength we need to find a single sampling plan such that k satisfies $Rd(k-1) > p'/p \geq Rd(k)$ where $Rd(k)$is defined as $md_\beta(k)/md_{1-\alpha}(k)$. We obtain n from,
$n = md_\beta(k)/p'$

...(1.3.19)

These plans have been extensively tabulated by Hald (1981) for different values α, β, and p'/p for integer as well as noninteger values of k.

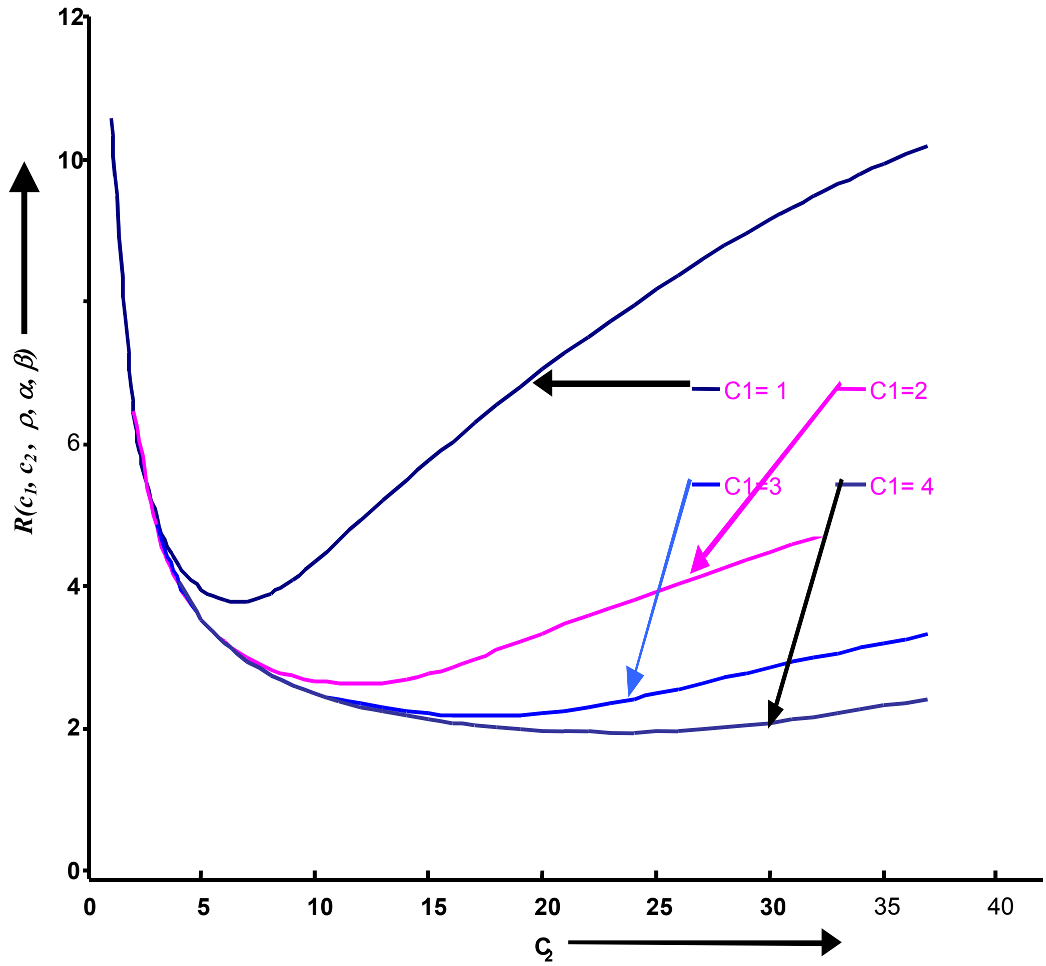
1.3.6 Comparing sample sizes of A kind C kind and D kind MASSP's of given strength

We are now in a position to compare the sample size of an A kind plan and that of a C kind plan of the same strength. We may do this taking the sample size of a D kind plan of the same strength as the datum. For example, if we take $\alpha = 0.05$, $\beta = 0.10$ and $\rho = 0.1$ and find the value of np' for a given p'/p for all the three plans obtained by the procedures explained. It may be noted from the Table 1.3.1 and 1.3.2 that the sample size for plan A is more or less same as that of Plan D even for small value of ρ. The sample size for Plan C is significantly more than that of the equivalent Plan D. It is also more than that of the equivalent plan A.

1.3.7 Conclusion

To summarize the results of this chapter we note that for r = 2 it is possible to uniquely determine the fixed risk minimum sample size MASSP's for a given value of ρ. Taking note of the results of chapter 1.2 that the A kind plans also satisfy the condition of higher absolute slope of the OC for the lower AQL attribute and the plans of C kind and D kind do not, and the fact the sample size of the A kind plan is likely to be less than the equivalent C kind plan we might choose an A kind plan as a better alternative.

Figure 1.3.1: The sketch of the function $R(c_1, c_2, \rho, \alpha, \beta)$ plotted against c_2. $\alpha = 0.05$, $\beta = 0.10$, $\rho = 0.1$ and $c_1 = 1, 2, 3, 4$.



Part 2: Bayesian multiattribute sampling inspection plans - general cost models and discrete prior distributions

2.1 General cost models

2.1.1 Scope

So far we have discussed the comparative features of A kind, C kind and D kind multiattribute acceptance sampling schemes based on OC function. We now attempt to do this on the basis of economic considerations. We first review the various cost models for dealing simultaneously with more than one attribute. We further develop a generalized cost model as appropriate to compare the costs of different sampling schemes proposed by us, under the assumptions of independence of defect occurrences and also under the assumption of mutually exclusive occurrences of defect in a lot/sample. We do this under Poisson conditions.

2.1.2 Existing cost models for multiattribute acceptance sampling

Hald [See Chapter 0.2.7] has been the major contributor in the field of economic design of acceptance sampling plans. He obtained general solutions for linear cost models under discrete and continuous prior distribution of process average for single attribute. Hald's major emphasis has been on finding asymptotic relationships between n and c, and between N and n for a single attribute single sampling plan.

The economic design of multiattribute sampling scheme was considered by Schimdt and Bennett(1972), and further by Case, Schimdt and Bennett(1975), Ailor, Schimdt and Bennet(1975), Majumdar(1980), Moskowitz, Plante, Tang and Ravindran(1984), Tang, Plante and Mokowitz(1986), and Majumdar(1997). We discuss some of them and examine their relevance in the context of our enquiries.

i) Case, Schimdt and Bennett(1975)

This model is developed under following assumptions:

(a) Each attribute is assumed to have its own sample size n_i and an acceptance number c_i for $i = 1, 2, ..., r$; r being the number of attributes.

(b) Any item inspected on one attribute may be inspected on all other attributes thus resulting in the total member of items sampled being maximum of $(n_1, n_2, ..., n_r)$. Acceptance is realized only if and only if $x_i \leq c_i$; $i = 1, 2, ..., r$

(c) The number of items inspected for the i th attribute is without exception n_i. No screening/sorting is made on the rejected lots.

(d) Irrespective of the lot size a rejected lot is 'scrapped' at a fixed cost.

(e) The sampled items are replaced in the lot by additional items and are taken from a lot of the same overall quality as the sampled lots.

(f) The number of defectives $X_1, X_2, ..., X_r$ in the lot are independently distributed.

(g) The number of defectives, $x_1, x_2, ..., x_r$ in the sample are independently distributed.

Comparison is made (on specific cases) between the optimal plans constructed under (i) assumption of approximate continuous distribution of lot quality and (ii) assumption of discrete prior distribution of lot quality consistent with the resulting continuous distribution of process average. All computations are made using a search algorithm.

ii) Moskowitz, Plante, Tang and Ravindran(1984)

In this model the authors introduced the concept of multiattribute utility theory into quality control acceptance schemes and developed a general multiattribute sampling scheme. They used the criterion of maximizing the total utility. The process averages are considered as independently distributed beta variable. The cost model considered is general.

iii) Tang, Plante and Moskowitz (1986)

The basic tenets of this model are :

(a) There are two classes of attributes viz. scrappable and screenable.

(b) For the rejection due to scrappable defects, the cost of rejection is proportional to the lot sizes irrespective of number of defects present in the remainder of the lot.

(c) For rejection due to screenable defects, the cost of rejection is proportional to the number of items screened (inspected) and no fixed cost is incurred.

(d) Whenever a lot is rejected due to a scrappable defect, the remainder of the lot is not at all inspected for any other attribute(s), scrappable or screenable.

(e) In the event the lot is not rejected on scrappable attribute(s) but on screenable attributes(s), the lot is not tested for scrappable attribute(s) at all.

(f) Effectively, the number of defects observed on any member of the screenable set of attributes does not affect the decision (or the cost) in case the number of defects on any member of the scrappable set violates the acceptance criteria. Only when all the members of the scrappable set satisfy the acceptance criteria, the observations on any member of screenable set may affect the decision.

(g) The random variables $X_1, X_2, ..., X_r$ are independently distributed.

(h) The random variables $x_1, x_2, ..., x_r$ are independently distributed.

(i) Sampling plans for screenable attributes can be obtained by solving a set of independent single attributes models.

(j) Interaction of scrappable attributes on screenable attributes and conversely result in smaller sample sizes for screenable attributes than in single attribute plans.

(k) Interaction among scrabble attributes result in either smaller sample sizes, lower acceptance probabilities or both, relative to single attribute plans.

A heuristic solution procedure is developed to obtain near optimal multiattribute acceptance sampling plans.

2.1.3 Cost model - some practical considerations

i) In today's industrial scenario we notice that screening is becoming more and more feasible due to rapid growth in computerized testing and inspection system. In many fields today, the sampling inspection is relevant only for deciding whether to accept or to screen.The basis for differentiating the attributes in these situations depends on their contribution to cost components. In any ease even for a scrappable attribute, one may not be willing to reject the whole lot at a fixed cost, as assumed by Case et al.(1975) or at a cost proportional to the lot size as assumed by Tang et al.(1986), more so, in the situation of nondestructive testing.

ii) Moreover, the assumption that a lot rejected for a single scrappable attribute is not screened at all for other attributes (scrappable or screenable) may not hold in many situations. For example a lot rejected for a scrappable attribute like undersize diameter may be screened for a defect like oversize diameter for which one can rework the item.

However, this is not to say that the assumptions made in these models do not hold in some situations.

iii) There are many practical situations, where testing is done on all attributes for all the pieces in a sample. Examples can be cited for finished garment checking, visual inspection of plastic containers, regulatory testing for packaged commodities like biscuits, nondestructive testing for foundry and forged items etc. For testing of components of assembled units the general practice is to test all components for all the sampled items. Similar practices are considered as practical for screening/sorting the rejected lot. In these cases $n_i = n; i = 1, 2, ..., r$. Moreover, in some situations, if we use different sample sizes for different attributes, we may save only testing cost and not on sampling cost, as we may have to draw a sample of size n as the maximum of all $n_i; i = 1, 2, ..., r$ to enable testing for all attributes.

(iv) The assumption that a lot rejected for a single scrappable attribute is not screened at all for other attributes (scrappable or screenable) may not hold in many situations.

(e) The defect occurrences in the lot and sample are considered as jointly independent for each attribute. There are many situations where defect occurrences in the lot/sample may be mutually exclusive. This may happen due to the very nature of defect occurrences (e.g.

a shirt with a button missing or a shirt with a wrong button), or when defects are classified in mutually exclusive classes e.g. critical, major or minor. It would therefore be necessary to take care of both the situations.

As stated earlier that the primary focus of the present enquiry is to examine different alternative acceptance criteria rather than to locate an optimal plan in a given set up and to develop search alogorithm for the mentioned purpose. We make use of the cost models developed by Majumdar(1980), Majumdar(1990) and Majumdar (1997). The cost models mentioned along with some elementary results in the following chapter of the part have already been published as noted here.

2.1.4 A generalized cost model for nondestructive testing

In our case we take a sample of size n and inspect each item for all the attributes. We observe $x_i; i = 1, 2, .., r.$ as the number of defectives on the i th attribute. We denote the vector $(x_1, x_2, ..., x_r)$ as $\boldsymbol{x}$. We define set A as the set of $\boldsymbol{x}$ for which we declare the lot as acceptable. Let $\bar{A}$ be the complementary set for which we reject the lot. Let X_i; denotes the number of defectives on i th characteristic in the lot, $i = 1, 2, ..., r$.
Let the costs be

$$C(\boldsymbol{x}) = nS_0 + \sum_{i=1}^{r} x_i S_i + (N-n)A_0 + \sum_{i=1}^{r}(X_i - x_i)A_i$$

when $\boldsymbol{x} \in A$

... (2.1.1)

$$C(\boldsymbol{x}) = nS_0 + \sum_{i=1}^{r} x_i S_i + (N-n)R_0 + \sum_{i=1}^{r}(X_i - x_i)R_i$$

when $\boldsymbol{x} \in \bar{A}$ ...(2.1.2)

The interpretations of cost parameters are as follows: S_0 is the common cost of inspection i.e. sampling and testing cost per item in the sample for all the characterstics put together; $x_i S_i$ the cost proportional to the number of defectives of i th type in the sample which is the additional cost for an inspected item containing defects of i th type.

The cost of acceptance is composed of two parts; $(N-n)A_0$ is cost proprtional to the items in the remainder of the lot, and another part $\sum_{i=1}^{r}(X_i - x_i)A_i$ where A_i is the cost of accepting an item containing defective for i th attribute. We assume the loss due to use of defective item is additive over all the characterstics. This means if an item contains more than one defect say for $i = 1$ and 2 the loss will be the sum of the damages for both the characterstics put together. The assumption is reasonably valid under many situations. However, proportion of items containing more than one category of defects will usually be small.

Costs of rejection consists of a part $(N - n)R_0$ proportional to the number of items in the remainder of the lot and another part, $\sum_{i=1}^{r}(X_i - x_i)R_i$ proportional to the number of defective items rejected for all the attribute put together. If rejection means sorting, R_0 will give the sorting cost /item for all category of defects put together. The R_i denotes the additional cost for items found with defective of i th category (for example, cost of repair) and is additive over different category of defects.

Note that there is only one sample of size n to be taken for inspection for all the characteristics and the cost model given here is only a multiattribute analogue of the cost model considered by Hald (1965) in the case of a single attribute.

The average costs

A) When defect occurrences are independent.
In this case the probability of getting x_i defectives in a sample of size n from a lot of size N containing X_i defectives, for the i th attribute, $i = 1, 2, ..., r$,is given by hypergeometric probability as

$$Pr(x_i \mid X_i) = \binom{n}{x_i}\binom{N-n}{X_i - x_i}/\binom{N}{X_i}$$

X_i being the number of defectives of type i in a lot $i = 1, 2, ..., r$.

...(2.1.3)

Further,

$$P_r(X_1, X_2, ..., X_r) = \prod_{i=1}^{r} P_r(X_i).$$

If the lot quality $X_i\,; i = 1, 2, ...r$ is distributed as binomial, then

$$Pr(X_i) = \binom{N}{X_i} p_i^{X_i}(1 - p_i)^{N - X_i}.$$

...(2.1.4)

The average cost per lot at a process average $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ for the lots of size N is

$$K(N, n, \boldsymbol{p}) = \sum_{\boldsymbol{x}\in A} C(\boldsymbol{x}) \prod_{i=1}^{r} Pr(x_i|X_i) + \sum_{\boldsymbol{x}\in \bar{A}} C(\boldsymbol{x}) \prod_{i=1}^{r} Pr(x_i|X_i)$$

From (2.1.3) and (2.1.4), it can be easily shown that:

$$K(N,n,\boldsymbol{p}) = n\left(S_0+\sum_i S_i p_i\right)+(N-n)\left[\left(A_0+\sum_i A_i p_i\right)P(\boldsymbol{p})+\left(R_0+\sum_i R_i p_i\right)Q(\boldsymbol{p})\right],$$

where P ($\boldsymbol{p}$) denotes the average probability of acceptance at $\boldsymbol{p}$.

$$P(\boldsymbol{p}) = \sum_{\boldsymbol{x}\in A}\prod_{i=1}^{r}\binom{n}{x_i}p_i^{x_i}(1-p_i)^{n-x_i},$$

and $Q(\boldsymbol{p}) = 1-P(\boldsymbol{p})$.

...(2.1.5)

B) When the defect occurrences are mutually exclusive.

In this situation the expression for the probability of observing $(x_1, x_2, ..., x_r)$ defective in a sample of size n from a lot of size N, containing $(X_1, X_2, ..., X_r)$ defectives of types $i = 1, 2, ..., r$, will be multivariate hypergeometric as

$$Pr(x_1,x_2,...,x_r \mid X_1,...,X_r) = \binom{X_1}{x_1}\binom{X_2}{x_2}...\binom{X_r}{x_r}\binom{N-X_1-X_2...-X_r}{n-x_1-x_2...-x_r}/\binom{N}{n}.$$

...(2.1.6)

At any process average the joint probability distribution of $(X_1, X_2, ..., X_r)$ can be assumed to be multinomial such that

$$P_r(X_1,X_2,...,X_r \mid p_1,p_2,...,p_r) = \binom{N}{X_1}\binom{N-X_{(1)}}{X_2}...\binom{N-X_{(r-1)}}{X_r}p_1^{X_1}...p_r^{X_r}\left(1-p_{(r)}\right)^{(N-X_{(r)})}.$$

.

$$X_{(i)} = X_1+X_2+...+X_i \quad \text{and} \quad p_{(i)} = p_1+p_2+...+p_i; \quad i=1,2,...,r.$$

...(2.1.7)

From (2.1.6) and (2.1.7) it follows that average cost at $\boldsymbol{p}$ can be expressed as (2.1.5) replacing P ($\boldsymbol{p}$) by

$$P(\boldsymbol{p}) = \sum_{x_1,x_2,...,x_r\in A}\binom{n}{x_1}\binom{n-x_{(1)}}{x_2}...\binom{n-x_{(r-1)}}{x_r}p_1^{x_1}...p_r^{x_r}\left(1-p_{(r)}\right)^{(n-x_{(r)}}.$$

$x_{(i)} = x_1 + x_2 + ... + x_i$ for $i = 1, 2, ..., r$.

...(2.1.8)

2.1.5 Approximation under Poisson conditions

As discussed in chapter 1.1 we use the phrase 'Poisson conditions' when Poisson probability can be used in the expressions of type B OC function in the two situations.

i) Poisson as approximation to binomial and multinomial.

If $p_i \to 0, \quad n \to \infty, \quad$ and $np_i \to m_i$ then the binomial probability $b(x_i, n, p_i)$ tends to Poisson probability $g(x_i, np_i)$ where

$$g(x_i, np_i) = e^{-np_i}(np_i)^{x_i}/(x_i)!.$$

Under this condition the expression $P(\boldsymbol{p})$ given in equation (2.1.5) can be modified as :

$$P(\boldsymbol{p}) = \sum_{x_1,x_2,...,x_r \in A} Pr(x_1, x_2, ..., x_r \mid p_1, p_2, ..., p_r) = \prod_{i=1}^{r} g(x_i, np_i).$$

...(2.1.9)

If we also make an additional assumption that $\sum_{i=1}^{r} p_i \to 0$ then the expression $P(\boldsymbol{p})$ given in equation (2.1.8) can also be modified as (2.1.9).

(ii)Poisson as an exact distribution and the occurrences of defetc types independent.

We assume the number of defects for r distinct characteristics in a unit are independently distributed. The output of such a process is called a product of quality $(p_1, p_2, ..., p_r)$, the parameter vector representing the mean occurrence rates of defects per observational unit. The total number of defects for any characteristic in a lot of size N drawn from such a process will vary at random according to a Poisson law with parameter Np_i for the i th characteristics under usual circumstances. Similarly the distribution of number of defects on attribute i, in a random sample of size n drawn from a typical lot will be a Poisson variable with parameter np_i. Independence of the different characteristics will be naturally maintained in the sample so that the type B probability of acceptance will be given by the expression of $P(\boldsymbol{p})$ given as equation (2.1.9). In this situation we will be dealing with defects rather than defectives.

2.1.6 The genaralized cost model

Thus under Poisson condition described as above the expression of $K(N, n, \boldsymbol{p})$ is same as (2.1.5) with

$P(\boldsymbol{p}) = \sum_{s \in A} \prod_{i=1}^{r} g(x_i, np_i)$ and $Q(\boldsymbol{p}) = 1 - P(p)$

We have therefore arrived at a model as applicable to a multiattribute situation discussed in section 2.1.3. for $r > 1$. This may be considered as a generalization of the cost model of Hald(1965) for the single attribute situation i.e. for $r = 1$

2.2 The expected cost model for discrete prior distributions

2.2.1 Assumptions

By Bayesian plans we understand the plans obtained by minimizing average costs which has three identifiable components viz. inspection costs, acceptance costs, and rejection costs. For these plans the process average defective is taken to be a random variable. In our present context the prior distribution (i.e. the distribution of process average) is the expected distribution of lot quality vector on which the sampling plan is going to operate. For the multiattribute Bayesian plans considered by others [See chapter 2.1], the process average for each attribute has been assumed to follow a beta distribution so that the lot quality distribution for each attribute becomes a beta binomial.Thus, in a situation when defect occurrences are jointly independent the product of individual beta distributions are chosen as an appropriate prior.

We may, however, note that even when the process is in control with respect to such a prior, the process will occasionally go out of control and some lots of poorer quality will be produced before the process gets corrected. We then have a situation of a beta prior with outliers.

As an alternative to these models, consider the process average vector as a random variable which may take on two values,$(p_1, p_2, ..., p_r)$ and $(p'_1, p'_2, ..., p'_r)$, a satisfactory and an unsatisfactory quality level with given probabilities. This two point prior may be considered as a simplification of the one point prior with outliers, because the model contains some information about the distribution of the outlier. When the process performs at an unsatisfactory level it generally happens (e.g. for a manufacturing operation) that the quality level is poor for all the attributes. We cite one example from a real life situation.

2.2.2 A real life example

The example given below constitutes a typical example picked up from the author's list of similar applications in factories rendered by him as a QC professional. For the general discussion intended here, the details relating to the particular application are not included and only the relevant calculations are presented.

A company manufactures about 1.5 lakhs of 25 mm RS closures in a shift. A shift's production by a group is being packed in cartons, each carton is considered as an inspection lot for verification after the end of the shift. The two important sets of attributes for the product are functional defects and surface defects. On the spot observations have been made using a p-chart data format on these attributes.

In a typical scenario one observes, that most of the time the process is stable at a certain process average. However the process works unsatisfactorily for about 1.5 hours on an average in a day (24 hours). During this time both types of defects occur more frequently. The process average during this phase although not quite stable, hovers around a higher level. Thus we make an attempt to approximate the process variation due to common causes by a discrete two point prior distribution.

Figure 2.2.1 and Figure 2.2.2 depict the frequency distributions of defectives based on 100% inspection of 552 cartons from about 3 consecutive days of production. It can be seen that both the type of defects have occured at two distinctly different defect levels. We, therefore, presume that there are two sets of process average vectors, (p_1, p_2) and (p'_1, p'_2) occuring with probability w_1 and w_2, respectively such that $w_1 + w_2 = 1$.
From the observations made, we now estimate p_i, p'_i and w_i for $i = 1, 2$. We find w_1 is more or less same for both the attributes, so that we can approximate the process as proposed. In this case the estimates are obtained as (0.0065, 0.0450) and (.0800, 0.1200) for (p_1, p_2) and (p'_1, p'_2) respectively. Further we obtain the estimates of $w_1 = .94$ and that of $w_2 = 0.06$. The estimate of w_2 roughly agrees with the estimate made from on the spot shop floor observation as $w_2 = 1.5$ hrs/24 hrs $= 0.0625$.

Further, from the the results of the χ^2 test for goodness of fit [see Table 2.2.1 and Table 2.2.2] we may justify our assumptions of the two point prior distribution for the process average vector.

2.2.3 The average costs

To start with, we may define a q point prior distribution for the process average vector $\boldsymbol{p}$ such that the process average vector value at state j is denoted as $\boldsymbol{p}^{(j)}$ and the corresponding probability as w_j for $j = 1, 2, ..., q$ and

$$\boldsymbol{p}^{(j)} = (p_1^{(j)}, p_2^{(j)}, ...p_r^{(j)}); \quad j = 1, 2, ..., q$$

and

$$\sum_{j=1}^{q} w_j = 1, w_j \geq 0, j = 1, 2, ..., q.$$

...(2.2.1)

In the last chapter we defined the average costs for lots of size N at a given process average $\boldsymbol{p}$ as equation 2.1.9. Using the notation for a q point prior we rewrite this as :

$$K(N, n, \boldsymbol{p}^{(j)}) = n\left(S_0 + \sum_{i=1}^{r} S_i p_i^{(j)}\right) + (N-n)\left[\left(A_0 + \sum_{i=1}^{r} A_i p_i^{(j)}\right) P(\boldsymbol{p}^{(j)}) + \left(R_0 + \sum_{i=1}^{r} R_i p_i^{(j)}\right) Q(\boldsymbol{p}^{(j)})\right]$$

$P(\boldsymbol{p}^{(j)})$ denotes the probability of acceptance at $\boldsymbol{p}^{(j)}$ and $Q(\boldsymbol{p}^{(j)}) = 1 - P(\boldsymbol{p}^{(j)})$.

...(2.2.2)

For a q point prior distribution, the cost averaged over the prior become

$$K(N,n) = \sum_{j=1}^{q} w_j K(N,n,\boldsymbol{p}^{(j)}).$$

...(2.2.3)

For $r = 1$ and $q = 2$, the model is identical to the cost model developed by Hald(1965) for discrete prior distribution for the single attribute. As discussed in section 2.2.1, we will consider the case $q = 2$ i.e. the situation where we can use a two point discrete prior distribution.

2.2.4 The regret function for the two point prior.

Starting from (2.2.2), we introduce cost functions for $j = 1,2$ and for $i = 1,2,...,r$

$$k_s(\boldsymbol{p}^{(j)}) = S_0 + \sum_{i=1}^{r} S_i p_i^{(j)}$$

...(2.2.4)

$$k_a(\boldsymbol{p}^{(j)}) = A_0 + \sum_{i=1}^{r} A_i p_i^{(j)}$$

...(2.2.5)

$$k_r(\boldsymbol{p}^{(j)}) = R_0 + \sum_{i=1}^{r} R_i p_i^{(j)}$$

...(2.2.6)

Let $k_a(\boldsymbol{p}^{(1)}) < k_r(\boldsymbol{p}^{(1)})$ and $k_a(\boldsymbol{p}^{(2)}) > k_r(\boldsymbol{p}^{(2)})$. [The assumption is obviously reasonable and the rational, one is referred to Hald (1981)]

We now define the function $k_m(p^{(j)})$ which stands for the unavoidable (minimum) cost as

$$k_m(\boldsymbol{p}^{(j)}) = k_a(\boldsymbol{p}^{(j)})$$

for j = 1 and

$$k_m(\boldsymbol{p}^{(j)}) = k_r(\boldsymbol{p}^{(j)})$$

for j = 2.

...(2.2.7)

We assume that all the above functions are nonnegative and none are identical to 0. We also assume that

$$k_s(\boldsymbol{p}^{(j)}) \geq k_m(\boldsymbol{p}^{(j)}), j = 1, 2.$$

Further, let k_s, k_a, k_r and k_m denote the expected values of the corresponding cost functions defined in (2.2.4), (2.2.5), (2.2.6) and (2.2.7) w.r.t. the prior. Note that these functions are expressed as costs per unit. The average costs for the *three cases without sampling inspection* i.e. the cases where,

(a) all lots are classified correctly,
(b) all lots are accepted, and
(c) all lots are rejected

then become k_m, k_a and k_r respectively. Sampling inspection should only be taken recourse to if

$k_{Avg} - k_m < min[k_a - k_m, k_r - k_m]$ where $k_{Avg} = K(N, n)/N$.

Taking case (a) as the reference case we define the regret function $R(N, n)$ as

$$R(N, n) = [K(N, n) - K_m(N, n)]/(k_s - k_m)$$

where $K_m(N, n)$ is the average minimum unavoidable cost per lot a multiattribute analogue of what is given by Hald (1965).

...(2.2.8)

We may further simplify the above expression for $R(N, n)$ as

$$R(N, n) = n + (N - n)[\gamma_1 Q(\boldsymbol{p}^{(1)}) + \gamma_2 P(\boldsymbol{p}^{(2)})]$$

$$\gamma_j = w_j |k_a(\boldsymbol{p}^{(j)}) - k_r(\boldsymbol{p}^{(j)})|/(k_s - k_m); \quad j = 1, 2.$$

...(2.2.9)

Note that if $R_0 = S_0$ and $R_i = S_i$; $\quad i = 1, 2, \quad$ then $k_s = k_r$ and $\gamma_1 = 1$.

2.2.5 Using the model

In the above model γ_j's are functions of the cost parameters and the prior distributions. The change of acceptance criteria affects the cost function through the probability of acceptance. This model therefore enables us to consider different kinds of MASSP's and compare them in terms of costs, which is the primary focus of our inquiry. We shall continue our discussions to this end in the next chapters.

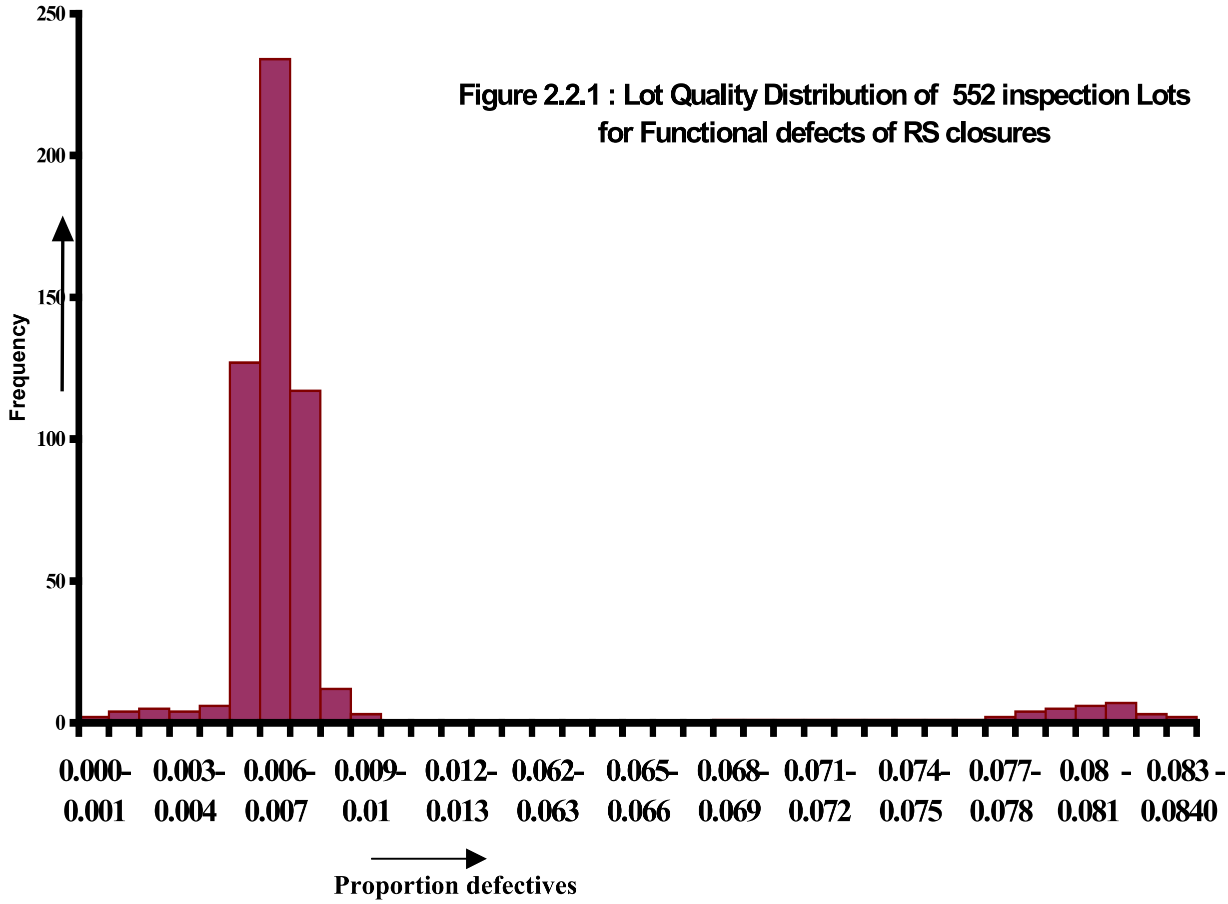


Figure 2.2.1 : Lot Quality Distribution of 552 inspection Lots
for Functional defects of RS closures
Frequency
250
200
150
100
50
0
0.000- 0.001
0.003- 0.004
0.006- 0.007
0.009- 0.01
0.012- 0.013
0.062- 0.063
0.065- 0.066
0.068- 0.069
0.071- 0.072
0.074- 0.075
0.077- 0.078
0.08 - 0.081
0.083 - 0.0840
Proportion defectives

Figure 2.2.2 : Lot Quality Distribution of Visual Defects for 552 lots of RS closures

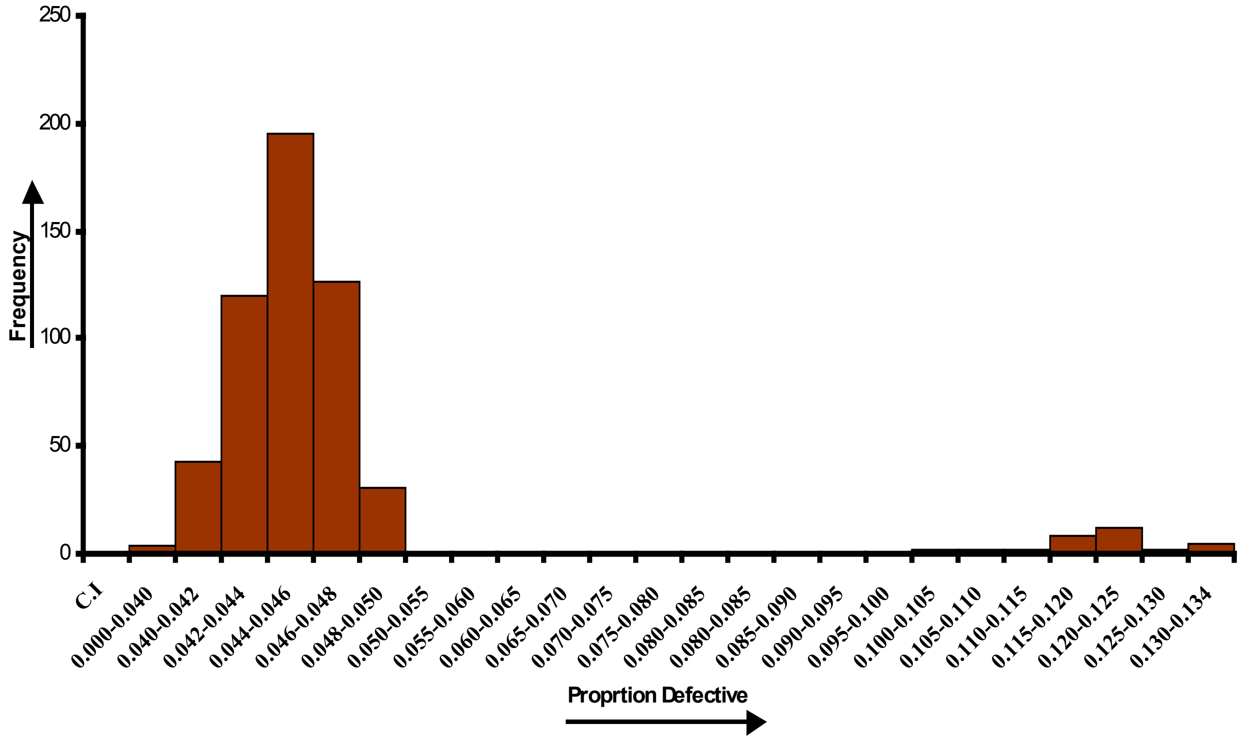


Table 2. 2. 1 Testing goodness of fit of two point discrete distribution for the functional defects for 552 inspection lots of RS closures of size 10000 each.

Number of defects /unit	Observed Frequency (O_i)	Estimated Frequency (E_i)	$(O_i - E_i)^2 / E_i$
0-0.005	20	16.07	0.959
0.005-0.006	127	122.42	0.172
0.006-0.007	235	241.90	0.197
0.007-0.008	117	122.42	0.240
0.008-0.009	15	15.59	0.022
0.009-0.078	11	8.12	1.023
0.078-0.080	9	8.93	0.001
0.080-0.0825	13	10.65	0.518
More	5	5.90	0.139
TOTAL	552.00	552.00	3.27

Estimated (p_1, p'_1, w_1) = (0.0065, 0.08, 0.94)

Pr. ($\chi 2_{df=5}$ >=3.27)= 0.659

Table 2. 2. 2 Testing goodness of fit of two point discrete distribution for the Visual defects for 552 inspection Lots of RS closures of size 10000

Number of defects /unit	Observed Frequency (O_i)	Estimated Frequency (E_i)	$(O_i - E_i)^2 / E_i$
0.000-0.040	4	4.12	0.003
0.040-0.042	43	34.24	2.240
0.042-0.044	120	124.97	0.197
0.044-0.046	195	192.23	0.040
0.046-0.048	126	124.97	0.009
0.048-0.050	31	34.24	0.307
0.050-0.115	6	6.17	0.005
0.115--0.120	8	14.51	2.920
0.120-0.124	12	12.94	0.069
More	7	3.62	3.167
Total	552	552	8.956

Estimated (p_2, p'_2, w_1) = (0.0450, 0.1200, 0.94)

Prob (($\chi 2_{df = 6}$ >= 8.95) 0.176

2. 3 Cost of MASSP's of A kind

2.3.1 Scope

In this chapter we examine the properties of optimal A kind MASSP's and also examine how the ordering of the attributes affects the costs. We evaluate the costs in terms of the regret function as defined in the last chapter for discrete two point prior distribution of the process average vector.

We do this under the assumption that the Poisson conditions hold so that the type B OC function can be expressed in terms of the sum of products of the individual Poisson probability terms with parameter np_i , $\boldsymbol{i} = 1,2,...,r$ over the domain of x_i values satisfying the acceptance criterion.

At the outset we recall that given two levels of the process average vectors, $\boldsymbol{p}$ and $\boldsymbol{p'}$ and the values of γ_1, γ_2 and N, the regret is a function of sample size (n), and the acceptance numbers as applicable to a specific acceptance criterion. The expression for the regret function has been given in (2.2.9).

In the discussions that follow we consider the optimality of a sampling scheme in terms of n and the acceptance parameters given $\boldsymbol{p}, \boldsymbol{p'}$, γ_1, γ_2 and N.

Notations

We introduced in Part 1 the MASSP's of A kind as the one where we take a sample of size n, observe the number of defectives (defects) for the i th attribute as x_i for i = 1, 2,..., r and apply the following acceptance criterion : accept if $x_1 \le a_1$; $x_1 + x_2 \le a_2$; ... ; $x_1 + x_2 + ... + x_r \le a_r$ and reject otherwise. Similarly for a MASSP of D kind we take a sample of size n, observe the number of defectives (defects) s for the i th attribute as x_i for all i = 1, 2,...,r and apply the following acceptance criteria : accept if $x_1 + x_2 + \ldots + x_r \le k$ and reject otherwise. We use the following notation as before :

$x_{(j)} = x_1 + ... + x_j$; j = 1, 2, ..., r. For the prior the process average vector can take two values viz. $\boldsymbol{p}$ and $\boldsymbol{p'}$, where $\boldsymbol{p} = (p_1, p_2,..., p_r)$ $\boldsymbol{p'} = (p'_1, p'_2,..., p'_r)$; $p'_i > p_i$ for $i = 1,2,...,r$.

Let ,

$$p_{(j)} = p_1 + ... + p_j \;; j = 1, 2,...,r;$$
$$p = p_1 + p_1 + ... + p_r;$$
$$p' = p'_1 + p'_2 + ... + p'_r \;;$$
$$m_i = np_i \;, \; i = 1, 2,..., r;$$
$$m'_i = np'_i, i = 1,2,...,r \;; \; m = np \;;$$
$$m' = np'.$$

The individual Poisson term is denoted as $g(x,m) = Exp(-m)\, m^x / x!$ and the cumulative Poisson as $G(c,m) = \sum_{x=0}^{c} Exp(-m)\, m^x / x!$. We also introduce $h(x,m) = m^x / x!$. Further the Poisson probability of acceptance at $(p_1, p_2, ..., p_r)$, for the Plan A with sample size *n* and acceptance number $a_1, a_2, ..., a_r$ is denoted as: $PA(a_1, a_2, ..., a_r; m_1, m_2, ..., m_r)$ and the corresponding probability of rejection at $(p_1, p_2, ..., p_r)$ as $QA(a_1, a_2, ..., a_r; m_1, m_2, ..., m_r)$ $= 1 - PA(a_1, a_2, ..., a_r; m_1, m_2, ..., m_r)$.

For the D kind plan with sample size *n* and acceptance number *k* the probability of acceptance at $(p_1, p_2, ..., p_r)$ is denoted as $PD(k,m)$. The corresponding probability of rejection at $(p_1, p_2, ..., p_r)$ is denoted by $QD(k,m)$.
For given values of γ_1, γ_2 the regret function defined in equation (2.2.9) for lots of size *N* using an A kind, with sample size *n* and with acceptance numbers $(a_1, a_2, ..., a_r)$ is written as :

$RA(n; a_1, a_2, ..., a_r) =$
$n + (N-n)[\gamma_1 QA(a_1, a_2, ..., a_r; m_1, m_2, a_1, a_2, ..., a_r m_r) + \gamma_2 PA(a_1, a_2, ..., a_r; m'_1, m'_2, ..., m'_r)]$
Similarly the regret function of the plan D with sample size *n* and acceptance number *k* :
$RD(n,k) = n + (N-n)[\gamma_1 QD(k,m) + \gamma_2 PD(k,m')]$

2.3.2 Situation 1

Theorem (2. 3.1)
Assume γ_1, γ_2, N, ***p, p'*** to be given. Given an optimal D plan with parameters (*k, n*), the regret function value of the plan A with acceptance criterion:
$x_{(i)} \le k-1$; for *i* = 1, 2,...,*r*-1; is less than the regret function value of the plan D (*k, n*) if

$$[p'_{(r-1)} / p_{(r-1)}]^k > (p'/p)^{k+1} \qquad \text{.... (2.3.1)}$$

Proof:
Since D (*n*, *k*) is an optimal plan *RD* (*n, k*+1) - *RD* (*n, k*) ≥ 0

$$\left(\frac{\gamma_2}{\gamma_1}\right) e^{-(m'-m)} \left(\frac{m'}{m}\right)^{k+1} > 1 \qquad \text{.... (2.3.2)}$$

Now *RD* (*n*; *k*) - *RA* (*n*; a_1 = *k*-1*,* a_2= *k*-1 ,..., a_{r-1} = *k*-1*,* a_r = *k*) / (*N*-*n*) =
$\gamma_2 [g(k, m'_{(r-1)}) g(0, m'_r)] - \gamma_1 [g(k, m_{(r-1)}) g(0, m_r)]$

From (2.3.2) we observe that

$$\frac{\gamma_2 g(k, m'_{(r-1)})g(0,m')}{\gamma_1 g(k, m_{(r-1)})\, g(0,m))} > \frac{\gamma_2}{\gamma_1} e^{-(m'-m)} [m'/m]^{k+1} \geq 1$$

This implies , *RD* (*n*; *k*) > *RA* (*n*; $a_1 = k$-1,..., $a_{r-1} = k$-1, $a_r = k$). **(Proved)**

Remarks:
A. The condition (2.3.1) is a sufficient condition and holds if
$(p'_r / p'_{(r-1)}) < (p_r / p_{(r-1)})$
B. For r = 2 the condition (2.3.1) becomes
$(p'_1 / p_1)^k > (p'/p)^{k+1}$
For r = 2 this implies
$\frac{p'_1}{p'} > \frac{p_1}{p}$.
We now try to obtain more general results.

2.3.3 Situation 2

Let us consider the set of A plans with acceptance criteria
$x_{(i)} \leq a_{r-1}$ for $i = 1, 2, ..., r$-1
$x_{(r)} \leq a_r$.

Note that the probability of acceptance at a process average $\boldsymbol{p} = (p_1,...,p_r)$ for this plan is
$PA(a_1, a_2, ..., a_r; m_1, m_2, ..., m_r) =$

$$= \sum_{x_1=0}^{a_{r-1}} .... \sum_{x_r=0}^{a_r - x_{(r-1)}} \prod_{i=1}^{r} g(x_i, m_i)$$

$$= \sum_{x_{(r-1)}=0}^{a_{r-1}} g(x_{(r-1)}, m_{(r-1)}) \sum_{x_r=0}^{a_r - x_{(r-1)}} g\ (x_r, m_r)$$

... (2.3.3)

Theorem (2.3.2)
Let *n*, a_{r-1}, a_r be the optimal parameters with acceptance criterion specified at the beginning of the section 2.3.4, for a given lot size *N*, γ_1, γ_2, *p*, $\boldsymbol{p'}$ etc. and assumption $a_{r-1} \geq 1$. If we now construct a plan with the same sample size but with acceptance criterion changed to: $x_{(r-2)} \leq a_{r-1} - 1$, $x_{(r-1)} \leq a_{r-1}, x_{(r)} \leq a_r$ then the latter plan will have lesser regret value than the former plan, optimal in the specified set up, if

$$[p'_{(r-2)} / p_{(r-2)}]^{a_{r-1}} > [p'_{(r-1)} / p_{(r-1)}]^{a_{r-1}+1}$$

... (2.3.4)

Proof:

Denoting the regret function value of the 1st plan and 2nd plan by R_1 & R_2 respectively we get

$$R_1 - R_2 = \gamma_2 \, g(a_{r-1}, m'_{(r-2)}) \;\; g(0, m'_{(r-1)}) \sum_{x_r=0}^{a_r - a_{r-1}} g_{(x_r, m'_r)}$$

$$- \; \gamma_1 g(a_{r-1}, m_{(r-2)}) \;\; g(0, m_{(r-1)}) \sum_{x_r=0}^{a_r - a_{r-1}} g(x_r, m_r)$$

To show that $R_1 - R_2 > 0$ we show that,

$$(\gamma_2 / \gamma_1) \;\; e^{-(m'-m)} \left[m'_{(r-2)} / m_{(r-2)} \right]^{a_{r-1}} . F_1 \;\; > 1$$

where,

$$F_1 = \sum_{x_r=0}^{a_r - a_{r-1}} [m'^{\,x_r}_r / x_r!] / \sum_{x_r=0}^{a_r - a_{r-1}} [m_r^{\,x_r} / x_r!]$$

… (2.3.5)

In order to prove theorem 2.3.2, we first prove the lemma stated below.

Lemma:

$$\sum_{x=0}^{c} (m'^{\,x} / x!) / \sum_{x=0}^{c-1} (m'^{\,x} / x!) > \sum_{x=0}^{c} (m^x / x!) / \sum_{x=0}^{c-1} (m^x / x!); \text{ for } c \geq 1 \text{ and } m' > m.$$

….(2.3.6)

Proof of the lemma: The function $\sum_{x=0}^{c-1} m^{x-c} / x!$ $c \geq 1$, decreases as m increases . Thus,

$$\sum_{x=0}^{c} (m^x / x!) / \sum_{x=0}^{c-1} (m^x / x!) = 1 + \frac{1}{c! \sum_{x=0}^{c-1} m^{x-c} / x!}$$ is an increasing function of m and therefore (2.3.6) holds.

Let us come back to theorem 2.3.2. We continue with the proof in the following lines:

Since optimal a_{r-1} must satisfy the inequality $\Delta^{(r-1)}_{a_{r-1}} R_1 > 0$ for the first optimal plan, where $\Delta^{(r-1)}_{a_{r-1}} R_1$ is the standard notation for the increase in R_1 achieved when a_{r-1} is replaced by $a_{r-1} + 1$ keeping all other parameters unaltered.

The condition:

$$(\gamma_2 / \gamma_1)\ e^{-(m'-m)} \left[m'_{(r-1)} / m_{(r-1)} \right]^{a_{r-1}+1} .F > 1 \ \text{ implies } \ \Delta^{(r-1)}_{a_{r-1}} R_1 > 0.$$

Here,

$$F = \left[\sum_{x_r=0}^{a_r - a_{r-1} - 1} m'^{\,x_r}_r / x_r! \right] \Big/ \left[\sum_{x_r=0}^{a_r - a_{r-1} - 1} m_r^{\,x_r} / x_r! \right]$$

We note that F > F1 by (2.3.6).

Now R_1 - R_2 > 0 if

$$\left[m'_{(r-2)} / m_{(r-2)} \right]^{a_{r-1}} > \left[m'_{(r-1)} / m_{(r-1)} \right]^{a_{r-1}+1}$$

and therefore we get (2.3.4) and therefore the theorem 2.3.2 is proved.

2.3.4 Situation 3

Consider a set of A plans such that for some $j\ (1 \leq j < r)$, $a_i = a_j < a_{j+1}$ $i = 1,2,\ldots, j-1$ i.e. to say that all acceptance numbers are all equal for $i \leq j$ but $a_{j+1} > a_j$.

Theorem (2.3.3)

Let the optimal plan from the set specified at the beginning of the section 2.3.5 has the parameter n, $a_1 = a_2 = .. = a_j, a_{j+1}, \ldots, a_r$ for a given N, γ_1, γ_2, $\boldsymbol{p}$ and $\boldsymbol{p'}$. If we now construct a plan with acceptance numbers b_i such that $b_1 = b_2 = \ldots b_{j-1} = a_j - 1$, (assuming $a_j \geq 1$), and $b_i = a_i$ for $i = j, j+1, \ldots, r$ then the latter plan will have a lesser regret value if:

$$\left[p'_{(j-1)} / p_{(j-1)} \right]^{a_j} > \left[p'_{(j)} / p_{(j)} \right]^{a_j+1} \qquad \ldots\ (2.3.7)$$

Proof:

We denote the regret function value of the optimal plan in the specified set up as R_1 and that of the constructed second plan as R_2.

$(R_1 - R_2) / (N\text{-}n)$

$$= \gamma_2\, g(a_j, m'_{(j-1)})\ g(0, m'_j)\ N_1 - \gamma_1\, g(a_j, m_{(j-1)})\ g(0, m_j)\ D_1$$

where

$$N_1 = \sum_{x_{j+1}=0}^{a'_{j+1}} \cdots \sum_{x_r=0}^{a'_r - (x_{j+1}+\ldots+x_{r-1})} \prod_{i=j+1}^{r} g(x_i, m'_i)$$

$$= S_0 \prod_{i=j+1}^{r} g(x_i, m'_i)\ .$$

S_0 denotes the summation with respect to x_i' s over the domain indicated in the line preceding line and, $a'_{j+l} = a_{j+l} - a_j$ for $l = 1, 2, ..., r - j$ and

D_1 is obtained from N_1 by replacing m'_i by m_i.

…. (2.3.8)

Let,

$$W = \frac{S_0 \prod_{i=j+1}^{i=r} h(x_i, m'_i)}{S_0 \prod_{i=j+1}^{i=r} h(x_i, m_i)}.$$

The ratio of the first term of to the second term of (2.3.8) can be written as

$= (\gamma_2 / \gamma_1) Exp(-(m'-m)).(m'_{(j-1)} / m_{(j-1)})^{a_j}.W$. The theorem is proved if we can show that this quantity is greater than 1 under the given condition.

Now,

$$\Delta_{a_j}^{(j)} PA = PA(a_1 = ...a_{j-1}, a_j + 1, a_{j+1}, a_{j+1}, ..., a_r; m_1, ..., m_r)$$

$$- PA(a_1 = ... = a_j; a_{j+1}, ..., a_r; m_1, ..., m_r)$$

$$= g(a_j + 1, m_{(j)}) \sum_{x_{j+1=0}}^{a'_{j+1}-1} \sum_{x_r=0}^{a'_r-1-(x_{j+1}+...+x_{r-1})} \prod_{i=j+1}^{r} g(x_i, m_i) \quad ...(2.3.9)$$

$\Delta_{a_j}^{(j)} PA$ and $\Delta_{a_j}^{(j)} QA$, when used in the regret function lead to the regret difference denoted as $\Delta_{a_j}^{(j)} (R_1)$

$$\Delta_{a_j}^{(j)} (R_1) \geq 0 \Rightarrow$$

$$(\gamma_2 / \gamma_1).Exp(-(m'-m)).(m'_{(j)} / m_{(j)})^{j+1} S_1 \prod_{i=j+1}^{i=r} h(x_i, m'_i) / S_1 \prod_{i=j+1}^{i=r} h(x_i, m_i) \geq 1.$$

…(2.3.9a)

S_1 denotes the summation with respect to x_i's over the domain indicated in (2.3.9).

Note that the set of tuples $(x_{j+1}, ..., x_r)$ coming under S_1 (call this set as T_1) is a subset of x_i's coming under S_0 (call this set as T_0).

Let $T_2 = T_0 - T_1$. Let S_2 denote the summation on all x_i combinations appearing in T_2.

We now consider the following function :

$Z(m_{j+1}, m_{j+2}, \ldots, m_r)$

$= S_2 \prod_{i=j+1}^{r} h(x_i, m_i) / \prod_{i=j+1}^{r} h(x'_i, m_i)$

where $(x'_{j+1}, \ldots, x'_r)$ is any fixed member of the set T_1. Note that in the Numerator summation there is at least one product term represented by the tuple $(x_{j+1}, \ldots, x_r)$ for which, $x_i > x'_i$, for $i = j+1, \ldots, r$. Hence, the function $Z(m_{j+1} \ldots m_r)$ is increasing in m_i, $i = j+1, \ldots, r$.

It therefore follows that,

$$Z(m'_{j+1}, m'_{j+2}, \ldots, m'_r) > Z(m_{j+1}, m_{j+2}, \ldots, m_r)$$

since $m'_i > m_i$ for all i.

Consider now the following expression.

$$[S_o \prod_{i=j+1}^{r} h(x_i, m_i)] \; / \; [S_1 \prod_{i=j+1}^{r} h(x_i, m_i)]$$

$$= 1 + 1/S_1 \left[\prod_{i=j+1}^{r} h(y_i, m_i) \; / S_2 \prod_{i=j+1}^{r} h(x_i, m_i) \right] = 1 + 1/B \text{ (say).}$$

Let us recall the each tuple $(x_{j+1}, \ldots, x_r)$ in the denominator product of B$\in S_2$ and each tuple $(y_{j+1} \ldots y_r)$ in the numerator product of B $\in S_1$. Hence, the above function is also an increasing function in m_i, $i = j+1, \ldots, r$.

$$\therefore S_o \prod_{i=j+1}^{r} h(x_i, m'_i) \; / \; S_o \prod_{i=j+1}^{r} h(x_i, m_i) \; \geq S_1 \prod_{i=j+1}^{r} h(x_i, m'_i) \; / \; S_1 \prod_{i=j+1}^{r} h(x_i, m_i)$$

when $m'_i > m_i$, $i = j+1, \ldots, r$.

$$\therefore (\gamma_2 \; /\gamma_1).\text{Exp}(-(m'\text{-}m)).(m'_{(j-1)} \; / \; m_{(j-1)})^{a_j} \; W$$

$$\geq (\gamma_2 \; /\gamma_1).Exp \; (-(m'\text{-}m)).(m'_{(j)}/m_{(j)})^{a_j+1} \; [S_1 \prod_{i=j+1}^{i=r} h(x_i, m'_i) \; /S_1 \prod_{i=j+1}^{i=r} h(x_i, m_i) \;]$$

when :

$$\left[\frac{(m'_{(j-1)})^{a_j}}{(m_{(j-1)})^{a_j}} \right] > \left[\frac{(m'_{(j)})^{a_j+1}}{(m_{(j)})^{a_j+1}} \right]$$

Thus, the theorem is proved from 2.3.9a.

2.3.5 Situation 4

Theorem (2.3.4) :

Assume N, γ_1, γ_2, $\boldsymbol{p}, \boldsymbol{p'}$ are given.

There exists a minimum regret A plan with $a_{j-1} = a_j$, if

$$p'_{j-1} / p_{j-1} < p_j ' / p_j \qquad \text{....(2.3.10)}$$

Proof:

Suppose there does not exist an optimal A plan with $a_{j-1} = a_j$, but the given condition is satisfied.

Let us suppose $(a_1, \ldots, a_{j-1}, a_j, \ldots, a_r)$ is an optimal A plan with $a_{j-1} < a_j$.

Let us use the notation

$$\Delta_{a_{j-1}}^{(j)} [\, PA(a_1, a_2, \ldots, a_{j-1}, a_j, \ldots, a_r; m_1, m_2, \ldots, m_{j-1}, m_j, \ldots, m_r)] =$$

$$= PA(a_1, \ldots, a_{j-1}, a_j, \ldots, a_r; m_1, \ldots, m_r) - PA(a_1, \ldots, a_{j-1}, a_{j-1}, a_{j+1}, \ldots, a_r; m_1, \ldots, m_r)$$

$$= \sum_{x_1=0}^{a_1} \sum_{x_2=0}^{a_2 - x_{(1)}} \cdots \sum_{x_{j-2}=0}^{a_{j-2} - x_{(j-3)}} \prod_{i=1}^{j-2} g(x_i, m_i) \sum_{x_{j-1}=0}^{a_{j-1} - x_{(j-2)}} g(x_{j-1}, m_{j-1})\, g(a_j - x_{(j-1)}, m_j).$$

$$\sum_{x_{j+1}=0}^{a'_{j+1}} \sum_{x_{j+2}=0}^{a'_{j+2} - x_{j+1}} \cdots\cdots \sum_{x_r=0}^{a'_r - (x_{j+1} + x_{j+2} + \ldots + x_{r-1})} \prod_{i=j+1}^{r} g(x_i, m_i)$$

Where ,

$$a_i' = a_i - a_j, \forall i \geq j+1. \qquad \text{...(2.3.11)}$$

Since $(a_1, \ldots, a_j, \ldots, a_r)$ must satisfy the optimal criteria, $\Delta_{a_j - 1}^{(j)} RA, \leq 0$ where the notation represents the difference of the regret function value for the A plan as explained in the proofs of theorems 2.3.2 and 2.3.3. The condition is equivalent to,

$$(\gamma_2 / \gamma_1).Exp(-(m'-m)). \frac{S_A \prod_{i=1}^{i=j-2} h(x_i, m_i') \sum_{x_{j-1}=0}^{a_{j-1} - x_{(j-2)}} h(x_{j-1}, m'_{j-1}) h(a_j - x_{(j-1)}, m'_j) S_B \prod_{i=j+1}^{i=r} h(x_i, m_i')}{S_A \prod_{i=1}^{i=j-2} h(x_i, m_i) \sum_{x_{j-1}=0}^{A_{j-1} - x_{(j-2)}} h(x_{j-1}, m_{j-1}) h(A_j - x_{(j-1)}, m_j) S_B \prod_{i=j+1}^{i=r} h(x_i, m_i)} \leq 1$$

….(2.3.12)

Here S_A denotes the summation with respect to $x_1, x_2, \ldots, x_{j-1}$ over the specified domain in (2.3.11) and S_B denotes the summation with respect to $x_{j+1}, x_{j+2}, \ldots, x_r$ over the specified domain on the same expression. The respective domains are indicated in (2.3.11)

For any set of $x_1, x_2, \ldots, x_{j-1}$ included in LHS of (2.3.12) we find that the following fraction:

$$\frac{\sum_{x_{j-1}=0}^{a_{j-1}-x_{(j-2)}} h(x_{j-1}, m'_{j-1})h(a_j - x_{(j-1)}, m'_j)}{\sum_{x_{j-1}=0}^{a_{j-1}-x_{(j-2)}} h(x_{j-1}, m_{j-1})h(a_j - x_{(j-1)}, m_j)} \geq \left(\frac{m'_{j-1}}{m_{j-1}}\right)^{a_{j-1}-x_{(j-2)}} \left(\frac{m'_j}{m_j}\right)^{a_j - a_{j-1}}$$

$$\geq \left(\frac{m'_{j-1}}{m_{j-1}}\right)^{a_j - x_{(j-2)}} .$$

$\therefore$ LHS of (2.3.12) $\geq$

$$(\gamma_2 \;/\gamma_1).Exp(\text{-}(m'\text{-}m)).(m'_{j\text{-}1}/m_{j\text{-}1})^{a_j - x_{(j-2)}} \frac{S_A \prod_{i=1}^{i=j-2} h(x_i, m'_i) S_B \prod_{i=j+1}^{i=r} h(x_i, m'_i)}{S_A \prod_{i=1}^{i=j-2} h(x_i, m_i) S_B \prod_{i=j+1}^{i=r} h(x_i, m_i)} .$$

Thus the value of the above expression is less than or equal to 1.
… (2.3.13)

We now compare the regret function of two A kind plans, one with (j-1) th acceptance number raised to a_j and the other with (j-1) th acceptance number as a_j-1, whereas the other acceptance number are retained unchanged in both the plans. Since,

$$\text{PA}(a_1, a_2, \ldots, a_j, a_j, \ldots, a_r, m_1, m_2, \ldots, m_j, m_{j+1}, \ldots, m_r)$$

$$- \quad \text{PA}(a_1, a_2, \ldots, a_j - 1, a_j, \ldots, a_r, m_1, m_2, \ldots, m_j, m_{j+1}, \ldots, m_r) =$$

$$= \sum_{x_1=0}^{a_1} \sum_{x_2=0}^{a_2-x_{(1)}} \cdots \sum_{x_{j-2}=0}^{a_{j-2}-x_{(j-3)}} \prod_{i=1}^{j-2} g(x_i, m_i)\, g(a_j - x_{(j-2)}, m_{j-1})\, g(0, m_j) \,.$$

$$\sum_{x_{j+1}=0}^{a'_{j+1}} \sum_{x_{j+2}=0}^{a'_{j+1}-x_{j+1}} \cdots \sum_{x_r=0}^{a'_r-(x_{j+1}+\ldots+x_{r-1})} \prod_{i=j+1}^{r} g(x_i, m_i) \,.$$

It becomes obvious by arguments similar to those used in theorems 2.3.2 and 2.3.3, from equation (2.3.13) that

$$RA(n; a_1, a_2, \ldots, a_{j-1} = a_j, a_j, \ldots, a_r) - RA(n; a_1, a_2, \ldots, a_{j-1} = a_j - 1, a_j \ldots a_r) \le 0 \qquad \ldots(2.3.14)$$

if the given condition of the theorem is satisfied.

Thus, we see that if the condition of the theorem is satisfied, then

$RA\ (n\ ;\ a_1 \ldots, a_{j-1}, a_j, a_j, a_{j+1}, \ldots, a_r) \le RA\ (n\ ;\ a_1, \ldots, a_{j-1}, a_{j-1}, a_j, a_{j+1}, \ldots, a_r)$.

Let us consider the feasible interval for the (j-1) th acceptance number, denoted by c_{j-1}, viz., $a_{j-1} \le c_{j-1} \le a_j$ (recall $a_{j-1} < a_j$ given) and keep all other acceptance numbers fixed as in the given optimal A plan.

Then, $RA\ (n\ ;\ a_1 \ldots, a_{j-1}, c_{j-1}, a_j, a_{j+1}, \ldots, a_r)$ treated as a function of a single variable c_{j-1} keeping all other acceptance numbers fixed at their optimal values, has to exhibit one of the following three features in the interval $a_{j-1} \le c_{j-1} \le a_j$:

I. The function is monotonically decreasing in the interval.
II. The function is monotonically increasing in the interval.
III. The function first decreases monotonically and then increases monotonically.

[In the definition monotonically decreasing (increasing) includes the possibility of equality also and does not mean strictly monotonically decreasing (increasing).]

The only possibility ruled out is first increasing and then decresing or a feature of multiple waves. This is clearly not possible for a realistic regret function which can be supposed to possess one of the three properties stated for any feasible segment of values of c_{j-1}.

Now because of (2.3.14), the only possibility is:
$RA\ (n\ ;\ a_1 \ldots, a_{j-1}, a_{j-1}, a_j, \ldots, a_r)$

$= RA\ (n\ ;\ a_1 \ldots, a_{j-1}, a_{j-1}+1, a_j, \ldots, a_r)$

$= RA\ (n\ ;\ a_1, \ldots, a_{j-1}, a_j, a_j, \ldots, a_r)$

∴ There exists an optimal A plan for which $c_{j-1} = a_j$.

Thus, the original supposition is contradicted.

∴ The theorem is proved.

2.3.6 Conclusion

From the above results, we now know how the ordering of attributes and the process average values at the two states effect the cast of an A kind plan . In the chapter 2.5 we attempt to illustrate numerically the results obtained in this chapter.

2.4 Cost of MASSP's of C kind

2.4.1 Approach

In this chapter we first introduce the concept of the OC distribution and the OC moments of the C kind plans. The approach of moment equivalent plans was originally used by Hald (1981) for finding the approximate OC curve for a double sampling plan (in a single attribute situation) from the OC of the moment equivalent single sampling plan. We use this approach in the case of MASSP of C kind to investigate some of its optimality properties.

2.4.2 The OC distribution and the OC moments of the C plan

We reacll that for the C kind plan we take a random sample of size n, observe the number of defectives/defects on each of r attributes as $x_i; \quad i = 1, 2, ..., r$; we accept the lot if $x_i \leq c_i; i = 1, 2, ..., r$; and reject the lot otherwise. The type B probability of acceptance at $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ under Poisson conditions can be written as

$$P(\boldsymbol{p}) = \sum_{x_1=0}^{c_1} .. \sum_{x_r=0}^{c_r} \prod_{i=1}^{r} g(x_i, m_i),$$

where $m_i = np_i; i = 1, 2, ..., r$.

...(2.4.1)

We now define,
$p = p_1 + ... + p_r$;
$m = m_1 + m_2 + ... + m_r$;
$\rho_i = p_i/p = m_i/m, \quad i = 1, 2, ..., r$ so that $\rho_1 + \rho_2 + ... + \rho_r = 1$. Further,
$x = x_1 + x_2 + ... + x_r$;
$x_{(j)} = x_1 + ... + x_j$ and
$\rho_{(j)} = \rho_1 + ... + \rho_j$.

If we now treat $\rho_1, \rho_2, ..., \rho_r$ as fixed then we can consider the OC as a function of m only. We denote this function as $P(m)$.

$$P(m) = \sum_{x_1=0}^{c_1} ... \sum_{x_{r-1}=0}^{c_{r-1}} \sum_{x=x_{(r-1)}}^{c_r+x_{(r-1)}} g(x, m) \frac{x!}{x_1!..x_{r-1}!(x - x_{(r-1)})!} \rho_1^{x_1} ... \rho_{r-1}^{x_{r-1}} \left(1 - \rho_{(r-1)}\right)^{x-x_{(r-1)}}.$$

....(2.4.2)

The $P(m)$ is a monotonically decreasing function of m with the property $P(m) = 1$ for $m = 0$ and $P(m) = 0$ for $m = \infty$.

Hence, $Q(m) = 1 - P(m)$ has the same properties as that of a distribution function, even if we do not regard m as random variable in the present context. For a given $(\rho_1, \rho_2, ..., \rho_r)$ the OC distribution for a C kind plan is given by $1 - P(m)$.

2.4.3 OC moments

Theorem 2.4.1
For the OC distribution defined as above,

$$E\left(m^{k+1}\right) = (k+1)S\left[(x+k)^{(k)}t(x_1, ..., x_{r-1}, x, \rho_1, ..., \rho_{r-1})\right]$$

t ( ) denotes the individual term of the multinomial distribution viz.

$$t(x_1, x_2, ..., x_{r-1}, x, \rho_1, \rho_2, ..., \rho_{r-1}) = \frac{x!}{x_1!x_2!...x_r!(x - x_{r-1})!}\rho_1^{x_1}\rho_2^{x_2}...\rho_{r-1}^{x_{r-1}}(1 - \rho_{(r-1)})^{x - x_{(r-1)}})$$

and $(x+k)^{(k)} = (x+k)(x+k-1)...(x+1)$.

....(2.4.3)

We use here the symbol S for the summation with respect to $x_1, x_2, ..., x_{r-1}$ and x over the domain indicated in (2.4.2).

Proof

Let $\gamma(m, \alpha, \beta) = \beta e^{-m\beta}(m\beta)^{\alpha-1}/\Gamma(\alpha), 0 \leq m < \infty,\ \alpha > 0,\ \beta > 0$

then,

$g(x, m) = \gamma(m, x+1, 1)$

$-g'(x, m) = \gamma(m, x+1, 1) - \gamma(m, x, 1)$.

Therefore, $-P'(m) = S\, t(x_1, ..., x_{r-1}, x, \rho_1, ..., \rho_{r-1})g'(x, m)$
$= S\, t(x_1, ..., x_{r-1}, x, \rho_1, ..., \rho_{r-1})[\gamma(m, x+1, 1) - \gamma(m, x, 1)]$

....(2.4.4)

and using the fact

$$\int_0^\infty (m^k)\gamma(m, x, 1) = x(x+1)...(x+k-1),$$

we get (2.4.3).

2.4.4 The moment equivalent plans

From the above results we may now construct a moment equivalent single sampling plan with parameter (a_0, n_0) for a given vector of $(\rho_1, \rho_2, ..., \rho_r)$ such that the single sampling plan has the same mean and variance of the OC distribution for a given C kind plan. For the single sampling plan, let $m_0 = n_0 p$ denote the parameter and let $b = m_0/m$.

Then

$$E(m_0) = bE(m); \qquad Var(m_0) = b^2 V(m).$$

Equating the first two moments we get

$$a_0 + 1 = E^2(m)/V(m); \qquad n_0 = nE(m)/V(m).$$

....(2.4.5)

It may be noted here that, for a fixed ρ_i; $i = 1, 2, ..., r$ the OC curve for a multiattribute C kind plan is looked upon as a function of np since in this case $(p_1, p_2, ..., p_r) = (\rho_1 p, \rho_2 p, ..., \rho_r p)$.

In the following discussions we treat ρ_i, $i = 1, 2, ..., r$ as fixed.

2.4.5 Sample size of the moment equivalent plan

Hald (1981) assumes near identity of the OC curves of the moment equivalent single sampling plan for estimating the probability of acceptance in case of a double sampling plan for different process average values. Hald's assumption has been corroborated by numerical computations. The expression used by Hald may be considered as the first term of series expansion of Khamis (1960) and has the same accuracy as the Edgeworth and Cornish-Fisher approximation. Following Hald's argument we may also assume the near identity of the OC curves i.e. the OC curve as of the given C kind plan as a function of p for a given vector $(\rho_1, \rho_2, ..., \rho_r)$ and that of the OC curve of the moment equivalent single sampling plan(SSP).

We use the above logic (although mathematically non-rigorous) and numerically justify the near identity of the OC curves of a MASSP of C kind and that of the moment equivalent SSP (under the restriction of a specified set of values for the ρ_i's). As we shall see later that we require effectively this near identity only for the tail probabilities.

Theorem 2.4.2

Let $\boldsymbol{p}^{[r]} = (p_1, p_2, ..., p_r)$ and $P(\boldsymbol{p}^{[r]})$ denote the OC function value at $\boldsymbol{p}^{[r]}$ where there are r

attributes.

Further,

$$P(\boldsymbol{p}^{[r]}) = G(c_1, np_1)G(c_2, np_2)...G(c_r, np_r) \simeq G(a_0, n_0(p_1 + p_2 + ... + p_r))$$

where a_0 and n_0 are the parameters of the moment equivalent plans obtained from (2.4.5) for $\rho_i = p_i/(p_1 + p_2 + ... + p_r)$ for $i = 1, 2, ..., r$,

then $n_0 < n$.

....(2.4.6)

Proof

For a fixed $s \geq 2$ let (2.4.6) hold for all r, $2 \leq r \leq s$ and we take $r = s$.

Consider a C kind plan with sample size n and acceptance numbers $c_1, c_2, ..., c_s$.

Now,

$$P(\boldsymbol{p}^{[s]}) = G(c_1, np_1)G(c_2, np_2)...G(c_s, np_s) = G(a_0, n_0 p_{(s)}),$$

where $p_{(s)} = p_1+p_2+...+p_s$ and a_0 and n_0 have been obtained from (2.4.5) for $\rho_i = p_i/p_{(s)} i = 1, 2, ...s$.

Adding one more attribute to the study system,

$P(\boldsymbol{p}^{[s+1]}) = G(c_1, np_1)G(c_2, np_2)....G(c_s, np_s)G(c_{s+1}, np_{s+1})$

$= G(a_0, n_0 p_{(s)})G(c_{s+1}, np_{s+1})$

$= G(a_0, n(n_0/n)p_{(s)})G(c_{s+1}, np_{s+1})$

$= G(a_0, n.p'_{(s)})G(c_{s+1}, np_{s+1})$, where $p'_{(s)} = (n_0/n)p_{(s)} \quad < 1$.

Note that $n_0/n < 1$ by supposition and therefore $p'_{(s)} < p_{(s)} < 1$.

We may now construct a SSP with parameter (a_1, n_1) for the C kind plan with number of attributes, r =2 using (2.4.5) such that,

$$G(a_1, n_1(p_{s+1} + p'_{(s)}) = G(a_0, n.p'_{(s)})G(c_{s+1}, np_{s+1}).$$

Further let this be equal to $G(a_1, n_2.(p_{s+1} + p_{(s)}))$.

But $p'_{(s)} < p_{(s)}$ and, therefore, $n_2 < n_1 < n$.

This implies that if (2.4.6) holds for $r = 2$ then it holds for all $r \geq 2$, by mathmatical induction.

We now prove that (2.4.6) holds for $r = 2$.

To prove that $n_0 < n$ we must show that

$E(m^2) - E^2(m) > E(m)$ or $E(m^2) - E(m) > E^2(m)$.

Defining $\rho = m_1/m = p_1/p$, and $b(x_1, x, \rho) = \binom{x}{x_1}\rho^{x_1}(1-\rho)^{x-x_1}$,we get

$$E(m) = \sum_{x_1=0}^{c_1}\sum_{x=x_1}^{c_2+x_1} b(x_1, x, \rho) = \sum_{x=0}^{c_1}\sum_{x_1=0}^{x} b(x_1, x, \rho) + \sum_{x=c_1+1}^{c_2+c_1}\sum_{x_1=0}^{c_1} b(x_1, x, \rho) = (c_1+1) + \sum_{i=1}^{c_2} B(c_1, c_1+i, \rho).$$

$B()$ denotes the cumulative binomial probability. From (2.4.3) we get

$$E(m^2) = \sum_{x_1=0}^{c_1} \sum_{x=x_1}^{c_2+x_1} 2(x+1)b(x_1, x, \rho).$$

Therefore,

$$E(m^2) - E(m) = \sum_{x_1=0}^{c_1} \sum_{x=x_1}^{c_2+x_1} (2x+1)b(x_1, x, \rho)$$

$$= \sum_{x=0}^{c_1} \sum_{x_1=0}^{x} (2x+1)b(x_1, x, \rho) + \sum_{x=c_1+1}^{c_2+c_1} \sum_{x_1=0}^{c_1} (2x+1)b(x_1, x, \rho) = (c_1+1)^2 + \sum_{i=1}^{c_2} [2(c_1+i)+1]B(c_1, c_1+i, \rho).$$

Further, $E^2(m) = \left[(c_1+1) + \sum_{i=1}^{c_2} B(c_1, c_1+i, \rho)\right]^2$

$= (c_1+1)^2 + 2(c_1+1)\sum_{i=1}^{c_2} B(c_1, c_1+i, \rho) + \left[\sum_{i=1}^{c_2} B(c_1, c_1+i, \rho)\right]^2.$

Writing $\left[\sum_{i=1}^{c_2} B(c_1, c_1+i, \rho)\right]^2 = \sum_{i=1}^{c_2} B(c_1, c_1+i, \rho) \sum_{j=1}^{c_2} B(c_1, c_1+j, \rho)$
and noting that $B(c_1, c_1+j, \rho) \geq B(c_1, c_1+i, \rho)$ for $j \leq i$, we have

$$\left[\sum_{i=1}^{c_2} B(c_1, c_1+i, \rho)\right]^2 \leq B(c_1, c_1+1, \rho) + 3B(c_1, c_1+2, \rho) + ... + (2c_2-1)B(c_1, c_1+c_2, \rho) = \sum_{i=1}^{c_2}(2i-1)B(c_1, c_1-$$

Thus,

$$E^2(m) \leq (c_1+1)^2 + \sum_{i=1}^{c_2}(2(c_1+i)+1)B(c_1, c_1+2, \rho) \leq E(m^2) - E(m).$$

Hence, $V(m) > E(m)$ and therefore $n_0 < n$.

It therefore follows by mathematical induction that $n_0 < n$ for all $r \geq 2$.

2.4.6 Obtaining a D kind plan with regret value lesser than that of the optimal C plan

Using (2.2.9) the regret function of a C kind plan in the present context can be written as

$$R(N, n) = n + (N-n)[\gamma_1 Q(\boldsymbol{p}) + \gamma_2 P(\boldsymbol{p'})]$$

where, $\boldsymbol{p} = (p_1, p_2, ..., p_r)$ and $\boldsymbol{p'} = (p'_1, p'_2, ..., p'_r)$ are the process average vectors at state 1(satisfacory) and state 2 (unsatisfactory) respectively for the two point prior distribution of the process average. $P()$ and $Q()$ are probabilitiy of acceptance and the probability of rejection respectively for the C plan for which the sample size is n and acceptance numbers are $(c_1, c_2, ..., c_r)$ with acceptance criteria as defined.
We now consider the optimal C kind plan obtained by minimizing the regret function value for given $\gamma_1, \gamma_2, \boldsymbol{p}, \boldsymbol{p'}$ and N.

Having obtained an optimal C kind Plan we construct two moment equivalent single sampling plans (SSP) (a_0, n_0) and (a'_0, n'_0) such that, $P(\boldsymbol{p}) \simeq G(a_0, n_0 p)$ and $P(\boldsymbol{p'}) \simeq G(a'_0, n'_0 p')$ where, $p = p_1 + p_2 + ... + p_r$ and $p' = p'_1 + p'_2 + ... + p'_r$.
Note that $n_0 < n$ and $n'_0 < n$ from the results of section 2.2.5.

Case I

$a_0 > a'_0, n_0 < n'_0$

In this case suppose we choose SSP(a_0, n_0).

Then $Q(\boldsymbol{p}) \simeq 1 - G(a_0, n_0 p)$ and $P(\boldsymbol{p}) \simeq G(a'_0, n'_0) > G(a_0, n_0 p')$

Given $\gamma_1, \gamma_2, \boldsymbol{p}, \boldsymbol{p'}$ and N, let RE and RC denote the regret function values for the chosen SSP and the optimal C plan respectively. Then

$$RC = n + (N - n)[\gamma_1 Q(\boldsymbol{p}) + \gamma_2 P(\boldsymbol{p'})] = n + (N - n)\gamma_1 d$$

where,

$$d = Q(\boldsymbol{p}) + (\gamma_2/\gamma_1)P(\boldsymbol{p'})$$

and,

$$RE = n_0 + (N - n_0)[\gamma_1(1 - G(a_0, n_0 p)) + \gamma_2 G(a_0, n_0 p')] = n_0 + (N - n_0)\gamma_1 d_1$$

where

$$d_1 = 1 - G(a_0, n_0 p) + (\gamma_2/\gamma_1)G(a_0, n_0 p').$$

Then $d - d_1 > 0$ in the region of our interest, $d < 1$ for $\gamma_2 < \gamma_1$. The regret function value for the SSP (a_0, n_0) is

$$RE = n_0 + (N - n_0)\gamma_1 d_1 \leq n_0 + (N - n_0)\gamma_1 d \leq n + (N - n)\gamma_1 d.$$

Hence, $RE \leq RC$.
Thus the SSP with a_0 and n_0 as acceptance number and sample size respectively will have lesser regret value than the optimal C kind plan.

Case II

$a_0 < a'_0, n_0 > n'_0$

In this case we choose the SSP (a'_0, n'_0) as a competitor to the C kind optimal plan. We note, $Q(\boldsymbol{p}) \simeq 1 - G(a_0, n_0 p) \geq 1 - G(a'_0, n'_0 p)$ and,

$P(\boldsymbol{p'}) \simeq G(a'_0, n'_0 p')$. Thus by the same logic the regret function for the SSP (a'_0, n'_0) will be less than the optimal C plan.

Case III

$a_0 > a'_0, n_0 > n'_0$

Consider the function $d(m) = G(a_0, m\lambda) - G(a'_0, m)$. For $\lambda > 1$, d(m) has one change of sign only from positive to negative for $m > 0$. [This result is due to T.Tjur as quoted by Hald(1981)]

If now these two OC's for the two SSP's intersect at p_0 such that $p < p_0 < p'$ then we choose the plan with parameter a_0, n_0 so that

$$P(\boldsymbol{p}) \simeq G(a_0, n_0 p) \qquad Q(\boldsymbol{p}) \simeq 1 - G(a_0, n_0 p)$$

and

$$P(\boldsymbol{p'}) \simeq G(a'_0, n'_0 p') > G(a_0, n_0 p').$$

Thus we get $RE < RC$ as in case I.

In the event $p_0 < p$ we also choose (a_0, n_0) as the SSP parameters as before.

If $p_0 > p'$ neither of the SSP (a_0, n_0) and the SSP (a'_0, n'_0) may have lesser regret than the optimal C plan. We may therefore choose a plan (a''_0, n''_0); such that $a'' > a_0, n''_0 > n_0$ such the OC's of plan (a_0, n_0) and that of (a''_0, n''_0) intersect at p_0 and $p < p_0 < p'$.

In this case the the OC of the plan (a''_0, n''_0) and that of the plan (a'_0, n'_0) will intersect at a point $p'_0 < p_0$; $P(\boldsymbol{p}) \simeq G(a_0, n_0 p) < G(a''_0, n''_0 p)$ and $P(\boldsymbol{p'}) = G(a'_0, n'_0) > G(a''_0, n''_0 p')$.

This plan will have lesser regret if $n > n''_0$.

Obtaining the D kind plan

We note that the OC of the D kind plan (k, n) at $P(\boldsymbol{p})$ and at $P(\boldsymbol{p'})$ under Poisson conditions is identical with that of a SSP at p and at p', where $p = p_1 + p_2 + ... + p_r$, $p' = p'_1 + p'_2 + ... + p'_r$ with acceptance criterion: $x \leq k$. It, therefore, follows that given an optimal MASSP of C kind it should be possible to construct a D kind plan with lesser regret values in first two situations as above. For the third situation we may have to satisfy the additional condition mentioned as above to obtain such a D kind plan.

In the next chapter we will demonstrate numerically the comparative superiority of the D kind plan to the C kind plan.

2.5 Results of numerical investigations

2.5.1 Introduction

This chapter presents examples comparing the regret value of a) A, D and C type plans for different situations using the results obtained in earlier chapters and b) proposes algorithm for construction of A kind optimal plans. In chapter 2.2, we have noted that for a given two point discrete prior distribution of $\boldsymbol{p}$ the regret function is

$$R(N,n) = n + (N-n)[\gamma_1 Q(\boldsymbol{p}) + \gamma_2 P(\boldsymbol{p'})]$$

Where γ_j's are functions of i) cost parameters $(A_0, A_1, A_2, ..., A_r), (S_0, S_1, S_2, ..., S_r), (R_0, R_1, R_2, ..., R_r)$ and ii) the parameters of prior distributions $w_1, w_2, \boldsymbol{p}$ and $\boldsymbol{p'}$ as defined in section 2.1.3. Moreover, the $P(\boldsymbol{p})$ denotes the type B probability of acceptance at $\boldsymbol{p}$ and $Q(\boldsymbol{p}) = 1 - P(\boldsymbol{p})$ Further, if $R_0 = S_0$ and $R_i = S_i$, $i = 1, 2, ..., r$ then $\gamma_1 = 1$.

We address the problem of obtaining the optimal plans under different acceptance criteria, given the values of γ_1, γ_2, $\boldsymbol{p}$ and $\boldsymbol{p'}$.

2.5.2 Optimal Bayesian plans in situation $p'_1/p_1 = p'_2/p_2 = ... = p'_r/p_r$

Table 2.5.1

In this case, let $\rho_i = p_i/p$ and $\rho'_i = p'_i/p'$ for $i = 1, 2..., r$. Here $\rho_i = \rho'_i, \forall i$. From the results of chapter 2.4 it follows that for the optimal C plan with sample size n, acceptance parameters $c_1, c_2, ..., c_r$ and the values of ρ_i we can construct a moment equivalent D plan with sample size n_0 and acceptance number k_0, so that the two plans will have approximately same probabilities of acceptance at $\boldsymbol{p}$ and at $\boldsymbol{p'}$. Further, this D plan has smaller sample size and lesser regret value than the optimal C plan.

Let $r = 3$. For a typical value of $\gamma_1 = 1, \gamma_2 = 0.7, (p_1, p_2, p_3) = (0.002, 0.008, 0.020)$ and $p'_1/p_1 = p'_2/p_2 = p'_3/p_3 = 5$, we compute the optimal parameters of the C plan and those of the equivalent D type plan (which is effectively a single sampling plan for single attribute). Parameters of these plans have been retained in real numbers. Integer appeoximation will affect the equivalence to some extent.

We note from table 2.5.1 for lot sizes 1000 (1000) 10000 (5000) (50000) the regret value of the moment equivalent D plan is less than that of optimal C plans. Moreover, the regret value of the optimal D plans is still less than that of both these plans.

2.5.3 Optimal Bayesian plans in situation where p'_i/p_i's are not same for all i

Table 2.5.2

For r = 3, let $p'_1 = 0.01, p'_2 = 0.04, p'_3 = 0.10, p'_1/p_1 = 5, p'_2/p_2 = 5, p'_3/p_3 = 3$ and we take $\gamma_1 = 1, \gamma_2 = 0.7$.

Since $p'_1/p_1 = p'_2/p_2 = 5$ from the results of Theorem 2.3.4, we must have $a_1 = a_2$ for the optimal A plan.

Note that in this case $p'_{(2)}/p_{(2)} = 5$ and $p'/p = 3.462$. Thus, for $k \geq 4$ we find the inequality $(p'_{(2)}/p_{(2)})^k > (p'/p)^{(k+1)}$ satisfied. This means there exists an A plan cheaper than the optimal D plan for which the k value is greater than or equal to 4. (Theorem 2.3.1).
Table 2.5.2 presents the parameters of the optimal D plan, optimal A plan and the corresponding regret values for lot sizes 1000 (1000) 10000. Note that for all the optimal A plans, $a_1 = a_2$ and $a_3 > a_2$. This table also gives the parameters of the optimal C plan and the corresponding regret value, which is higher than that of the optimal D plan and the optimal A plan.

Table 2.5.3:

For r = 3 let $p'_1 = 0.01, p'_2 = 0.04, p'_3 = 0.10, p'_1/p_1 = 8, p'_2/p_2 = 5$, and $p'_3/p_3 = 3$. Let, $\gamma_1 = 1$, and $\gamma_2 = 0.7$

In this case $p'_{(1)}/p_{(1)} = 8$ and $p'_{(2)}/p_{(2)} = 5.405$ and $p'/p = 3.523$. For $k > 2$ the inequality $(p'_{(2)}/p_{(2)})^k > (p'/p)^{(k+1)}$ is satisfied. This means there exists an A plan cheaper than the optimal D plan for which the k value is greater than 2.(Theorem 2.3.1)
Also by Theorem 2.3.3, $a_3 > a_2$ for $a_3 > 2$.
Further, $(p'_{(1)}/p_{(1)})^a_2 > (p'_{(2)}/p_{(2)})^{a_2+1}$ for $a_2 > 4$ and hence $a_2 > a_1$ for $a_2 > 4$.
Table 2.5.3 presents the parameters of the optimal D plan, optimal A plan and the corresponding regret values for lot sizes 1000 (1000) 10000. Note that for all the optimal A plans $a_2 > a_1$ and $a_3 > a_2$.
This table also gives the parameters of the optimal C plan and the corresponding regret value, which is higher than that of the optimal D plan and the optimal A plan.

The annexures contains Microsoft Visual Basic Programmes used to construct the optimal plans.

Table 2.5.1 : Optimal multi attribute Bayesian plans for two point discrete prior Distribution

$r = 3, \; p_1'/p_1 = p_2'/p_2 = p_3'/p_3 = 5, \; p'_1 = 0.01, \; p'_2 = 0.04, \; p'_3 = 0.10, \; \gamma_1 = 1.0, \; \gamma_2 = 0.7$

Optimal C plan						Moment equivalent SSP			Optimal D plan		
Lot Size	c_1	c_2	c_3	n	Regret	a_0	n_0	Regret	k	n	Regret
1000	2	3	5	95	139.31	5.50	77.13	136.29	6	84	118.44
2000	3	4	6	119	175.46	6.75	94.72	172.17	8	111	145.09
3000	3	5	8	153	196.09	8.94	120.81	193.59	9	124	160.73
4000	3	5	8	155	210.66	8.94	122.38	206.77	10	138	172.09
5000	4	6	9	177	223.45	10.06	136.38	221.71	10	138	180.97
6000	4	6	9	178	233.01	10.06	137.16	230.61	11	151	187.80
7000	4	6	10	192	240.35	11.12	149.80	237.23	11	151	194.06
8000	4	6	10	193	247.36	11.12	150.58	243.38	12	165	199.25
9000	4	6	10	194	254.25	11.12	151.36	249.33	12	165	203.68
10000	4	7	11	214	259.82	12.27	164.48	258.30	12	165	208.10
15000	5	7	12	230	281.67	13.29	177.87	278.06	14	191	224.33
20000	5	8	13	251	297.10	14.45	192.42	296.42	14	191	235.45
25000	5	8	13	253	308.41	14.45	193.95	306.33	15	205	244.01
30000	5	8	13	254	319.43	14.45	194.71	316.21	16	218	251.75
35000	5	9	14	275	327.71	15.50	207.52	327.72	16	218	257.37
40000	5	9	15	289	334.96	16.66	220.95	334.01	16	218	263.00
45000	6	9	15	290	340.69	16.63	221.05	340.00	17	232	267.56
50000	6	9	15	291	346.27	16.63	221.81	344.73	17	232	271.57

[Note : The acceptance numbers of the equivalent SSP's or the D kind have been computed as real number . The costs of the optimal D kind plans is less than the costs of the equivalent D kind plans .]

Table 2.5.2 : Optimal multi attribute Bayesian plans for two point discrete prior distribution

$r = 3,\ p_1'/p_1 = p_2'/p_2 = 5,\ p_3'/p_3 = 3,\ p'_1 = 0.01,\ p'_2 = 0.04,\ p'_3 = 0.10,\ \gamma_1 = 1.0,\ \gamma_2 = 0.7$

Optimal A plan						Optimal D Plan			Optimal C plan				
Lot Size	a_1	a_2	a_3	n	Regret	k	n	Regret	c_1	c_2	c_3	n	Regret
1000	5	5	9	106	151.32	9	106	151.71	2	3	8	116	178.14
2000	6	6	12	141	188.99	12	141	189.55	3	4	11	160	230.51
3000	7	7	14	164	211.64	13	155	212.12	3	5	13	192	259.99
4000	7	7	15	176	227.38	15	176	227.98	3	6	14	212	283.90
5000	8	8	16	188	239.85	16	188	240.45	4	6	15	226	300.05
6000	8	8	17	199	250.11	17	200	250.75	4	6	16	237	314.81
7000	8	8	17	201	258.82	17	202	259.43	4	7	17	258	326.71
8000	9	9	18	211	266.23	18	211	266.83	4	7	18	268	336.65
9000	9	9	19	223	273.01	19	223	273.63	4	7	18	269	345.39
10000	9	9	19	223	278.67	19	223	279.42	4	7	18	270	354.05

Table 2.5.3 : Optimal multiattribute Bayesian plans for two point discrete prior Distribution

$r = 3, \; p_1'/p_1 = 8, \; p_2'/p_2 = 5, \; p_3'/p_3 = 3, \; p'_1 = 0.01, \; p'_2 = 0.04, \; p'_3 = 0.10, \; \gamma_1 = 1.0, \; \gamma_2 = 0.7$

Optimal A plan						Optimal D Plan			Optimal C plan				
Lot Size	a_1	a_2	a_3	n	Regret	k	n	Regret	c_3	c_2	c_1	n	Regret
1000	3	4	9	106	149.16	8	97	149.86	8	3	1	112	176.07
2000	3	5	11	130	186.09	11	133	186.83	11	4	2	158	226.34
3000	4	6	13	155	207.76	13	155	208.67	13	5	2	189	255.87
4000	4	7	15	177	223.73	14	168	224.62	13	5	2	191	279.44
5000	4	7	15	179	235.91	15	179	236.85	15	6	3	224	295.66
6000	4	7	16	189	245.75	16	190	246.70	16	6	3	235	310.04
7000	4	8	17	202	254.10	17	202	255.09	17	7	3	256	321.57
8000	4	8	18	212	261.54	18	213	262.67	17	7	3	257	331.33
9000	4	8	18	212	267.93	18	214	268.97	18	7	3	267	339.87
10000	4	9	19	224	274.02	19	225	274.97	18	7	3	268	348.19

Annexure

Microsoft Visual Basic programme as Excel Macro "OptimalAPlan()" for obtaining the Bayesian Optimal A plan for discrete prior Distribution for a given lot size, and values of (p_1, p_2, p_3), (p'_2, p'_2, p'_3), γ_1, γ_2

'The following array variables are used ;
'MINREGRET () is an array variable denoting the regret value for a given c_3
'Optc2() is an array variable denoting the optimum c_2 *value for a given* c_3
'*Optc1() is an array variable denoting the optimum* c_1 *value for a given* c_3
Optcn() is an array variable denoting the optimum n value for a given c_3
'For all other variables see the explanations given as comment

```
Sub optimalAthree()
Dim MINREGRET(50)
Dim optc2(50)
Dim optc1(50)
Dim optn(50)

J = 1 'j is a variable used to define the row of the output sheet

'Print Header for the output worksheet "Sheet 1"
        Worksheets("sheet1").Cells(1,1).Value="LOT"

        Worksheets("Sheet1").Cells(1,2).Value="Optimum c3"
        Worksheets("Sheet1").Cells(1,3).Value="Optimum c2"
        Worksheets("Sheet1").Cells(1,4).Value="Optimum c1"
        Worksheets("Sheet1").Cells(1,5).Value="Optimum n"

        Worksheets("sheet1").Cells(1,6).Value="Regret Value"

'Set the process average values at stage 2 for the first, second and the third attribute

        p1dash=1/100
        p2dash=4/100
        p3dash=10/100
        pdash  =p1dash+p2dash+p3dash

'Set the ratios of the process averages at stage two and stage one for attribute 1,2 and three

        onerat=8
        tworat=5
        threerat=3
```

```
    p1=p1dash/onerat
    p2=p2dash/tworat
    p3=p3dash/threerat
    p=(p1+p2+p3)



    'Set the Gamma1 and Gamma2 Values
        gamma1=1.0
        gamma2=0.7

    'Initialize for Minregret ()
        For k = 0 To 50
        MINREGRET(i)=10000
        Next
    'Set Lotsize
        Lot=1000

    'Start iteration w.r.t. c3
        For c3=0 To 30

    'Calculate lower bound and upper bound for the sample size for a given c3
        nupbound=(Log(gamma2)+((c3+1)*Log(onerat))/(pdash-p))
        n1pbound=(Log(gamma2)+(c3*Log(threerat))/(pdash-p))

    'Start iteration w.r.t c1,c2
        For c2=0 To c3
        For c1=0 To c2

'PrevRegret is a variable used to limit the n value not to exceed the optimum n. To start with we
set a large value for PrevRegret
   prevRegret=10000
    For n = n1pbound To nupbound Step 2

'Calculate Regret using the Pa function to obtain Probability of acceptance
    Regret = n+(lot – n)*(gamma1*(1-PA(c1, c2, c3, n, p1, p2, p3)) + gamma2*
    PA(c1, c2, c3, n, p1dash, p2dash, p3dash))
    Regret<MINREGRET(c3) Then
    MINREGRET(c3)=Regret
    Optc2(c3)=c2
    Optc1(c3)=c1
    Optn(c3)=n
    End If
    If Regret>prevRegret Then GoTo 100
'n need not be incremented beyond this point
```

```
If Regret<prevRegret Then prevRegret
Next n
100 Next c1
Next c2
```

‘Compare minimum regret for a given c3 with that of the previous c3-1 if the earlier has exceeded then we have reached the optimum c3

```
If MINREGRET(c3)>,OMREGRET(c3-1) Tjem
Prevoptc3=c3-1
```

‘Go to print the output sheet

```
GoTo 200
End If
Next c3


200 ‘Print output Sheet
Worksheets(“sheet1”).Cells(2+j,1).Value=lot
Worksheets(“sheet1”).Cells(2+j,2).Value=prevoptc3
Worksheets(“sheet1”).Cells(2+j,3).Value=optc2(prevoptc3)
Worksheets(“sheet1”).Cells(2+j,4).Value=optc1(prevoptc3)
Worksheets(“sheet1”).Cells(2+j,5).Value=optn(prevoptc3)
Worksheets(“sheet1”).Cells(2+j,6).Value=MINREGRET(prevoptc3)
```

Part3 Bayesian multiattribute sampling inspection plans for continuous prior distribution

3.1 Bayesian single sampling multiattribute plans for continuous prior distribution

3.1.1 Scope

In this chapter it will be assumed that the process average for each attribute has a continuous prior distribution. We shall examine in particular the problems of choice of a theoretical distribution as relevant to a multiattribute situation.

In chapter 2.1 we have developed the expression for the generalized cost function at a given process average. We make use of this to obtain the expression for the average cost when the process averages follow independent continuous distributions. In particular, we consider the case when the process average for each attribute can be assumed to follow a gamma distribution. Further, we demonstrate how this expression can be used to compare the expected costs of an A kind, D kind and C kind plan.

3.1.2 The cost model for continuous prior

We recall from chapter 2.1 that the average costs for accepted and rejected product at $\boldsymbol{p}$ is expressed as:

$$K(N,n,\boldsymbol{p}) = n(S_0 + \sum_{i=1}^{r} S_i) + (N-n)\left[(A_0 + \sum_{i=1}^{r} A_i p_i)P(\boldsymbol{p}) + (R_0 + \sum_{i=1}^{r} R_i p_i)Q(\boldsymbol{p})\right]$$

The notation and interpretation used for the above expression are to be found in section 2.1.4. For our present discussion we rewrite the same as:

$$K(N,n,\boldsymbol{p}) = n(S_0 + \sum_{i=1}^{r} S_i p_i) + (N-n)\left[\left(R_0 + \sum_{i=1}^{r} R_i p_i\right) + (A_1 - R_1)\left(\sum_{i=1}^{r} d_i p_i - d_0\right)P(\boldsymbol{p})\right]$$

where, $d_0 = (R_0 - A_0)/(A_1 - R_1)$; $d_i = (A_i - R_i)/(A_1 - R_1)$ for $i = 1, 2, ..., r$.

...(3.1.1)

Let p_i be distributed from lot to lot according to the prior distribution $w_i(p_i), i = 1, 2, ..., r$ and the p_i's are jointly independent. Then,

$$K(N,n) = nk_s + (N-n)k_r + (N-n)(A_1 - R_1)\int_{p_1}\int_{p_2}...\int_{p_r}\left[\left(\sum_{i=1}^{r} d_i p_i - d_0\right)P(\boldsymbol{p})dw_1(p_1)dw_2(p_2)...dw_r(p_r)\right]$$

where k_s is the average cost of sampling over the prior, i.e.

$$k_s = \int_{p_1}\int_{p_2}...\int_{p_r}\left[(S_0 + \sum_{i=1}^{r} S_i p_i)dw_1(p_1)dw_2(p_2)...dw_r(p_r)\right]$$

and

$$k_r = \int_{p_1}\int_{p_2}...\int_{p_r}\left[(R_0 + \sum_{i=1}^{r} R_i p_i)dw_1(p_1)dw_2(p_2)...dw_r(p_r)\right]$$

...(3.1.2)

3.1.3 The distribution of process average

The most widely used continuous prior distribution for the process average quality p_i is the beta distribution,

$$\beta(p_i, s_i, t_i) = {p_i}^{s_i-1}(1-p_i)^{t_i-1}/\beta(s_i, t_i), \quad s_i > 0, \quad t_i > 0, i = 1, 2, ..r,$$

and $0 < p_i < 1$
where $\beta(s_i, t_i) = \Gamma(s_i)\Gamma(t_i)/\Gamma(s_i + t_i)$.

...(3.1.3)

The mean equals to $E(p_i) = \bar{p_i} = s_i/(s_i + t_i)$ and the variance is $V(p_i) = \bar{p_i}(1 - \bar{p_i})/(s_i + t_i + 1) = (\bar{p_i})^2(1 - \bar{p_i})/(s_i + \bar{p_i})$. We, therefore, can also use the parameters $(\bar{p_i}, s_i)$, instead of the parameters (s_i, t_i).

Corresponding to a beta distribution (3.1.3) of the single attribute process average of quality p_i, the distribution of the lot quality denoted by X_i becomes a beta-binomial distribution:

$$b(X_i, N, \bar{p_i}, s_i) = \binom{N}{X_i}\beta(s_i + X_i, t_i + N - X_i)/\beta(s_i, t_i),$$

X_i non-negative integer.

...(3.1.4)

It follows that the distribution of the number of defectives x_i in the sample is $b(x_i, n, \bar{p_i}, s_i)$.

Examples of fitting beta-binomial to the observed quality distribution have been given in Hopkins (1955), Hald (1960) and B.E. Smith (1965). Chiu (1974) has however demonstrated that beta distribution is not always an adequate substitute for any reasonable prior distribution. He has used the data reported by Barnard (1954) on 226 batches of sizes 24000-135000 to substantiate his claim.

For the situation when the process average follows beta binomial with parameters $\bar{p_i}, s_i, t_i$, we may use the gamma distribution with parameter $\bar{p_i}, s_i$ as an approximation.The gamma distribution in the present context is defined as :

$$f(p_i, \bar{p_i}, s_i)dp_i = e^{-v_i}(v_i)^{s_i-1}dv_i/\Gamma(s_i); \quad v_i = s_ip_i/\bar{p_i}, \quad p_i > 0$$

with mean $E(p_i) = \bar{p_i}$ and the shape parameter, s_i. The variance is $V(p_i) = (\bar{p_i}^2)/s_i$. ...(3.1.5)

As pointed out by Hald (1981) that this gamma distribution gives fairly accurate approximation to the beta distribution with same s_i when both $\bar{p_i}$ and $\bar{p_i}/s_i$ are small; more precisely, if $\bar{p_i} < 0.1$ and $\bar{p_i}/s_i < 0.2$. Hald (1981) used the gamma distribution to tabulate the optimal single sampling plans. Most of his results are based on assuming gamma as the right prior distribution.

Moreover corresponding to a beta distribution of the single attribute process average of quality p_i, the distribution of the lot quality denoted by X_i as well as sample quality x_i become a beta-binomial distribution which can similarly be approximated as a gamma-Poisson distribution.

It was pointed out in Chapter 1.1 that if for a single attribute the process is stable at a given p_i and we count the number of defects per item with reference to the characteristic, we may construct a model for which the number of defects for each unit for the characteristic equals to p_i in the long run. In case of r such characteristics, we assume in addition that the number of defects with reference to different characteristics observed in a unit are jointly independent. The outputs of such a process of r characteristics are called product of quality $(p_1, p_2, ..., p_r)$. The vector $(p_1, p_2, ..., p_r)$ is also the mean occurrence rate (of defects) vector per observational unit. Dividing the outputs of the process successively into inspection lots of size N each, the quality of the lot expressed by total number of defects for i th characteristic will vary at random according to the Poisson law with parameter Np_i $i = 1, 2, ..., r$. and the distribution of defects for the ith charactersticsdefects in a lot of size N drawn from this process will be similarly a Poisson distribution with parameter Np_i, $i = 1, 2, ...r$. In this situation if the process average for the i th characteristic is distributed as a gamma distribution for the i th attribute, the distribution of the lot quality becomes a gamma-Poisson distribution with reference to the i th attribute.

The distribution of lot quality X_i for the i th characteristic which holds, either approximately or exactly, as the case may be in these two situations (as explained above) can be expressed as :

$$g(X_i, N\bar{p_i}, s_i) = \frac{\Gamma(s_i + X_i)}{X_i!\Gamma s_i}\theta_i^{s_i}.(1-\theta_i)^{X_i}; \quad \theta_i = \frac{s_i}{(s_i + N\bar{p_i})}$$

and X_i non-negative integer.

...(3.1.6)

[Note that we use $g(x, \theta)$ to denote the Poisson distribution term and $g(x, \theta_1, \theta_2,)$ to denote the gamma-Poisson term.]

It follows that the distribution of the number of defectives or defects x_i in the sample folows gamma-Poisson law with probability mass function given by $g(x_i, n\bar{p_i}, s_i)$. Further there is stochastic independence of $x_1, x_2, ..., x_r$. This is the framework for development of cost models in the present chapter. In the sections which follow, we use gamma prior distribution which work either approximately accurately or as exact distributions under different situations as explained.

3.1.4 The verification for the appropriate prior distribution

Table 3.1.1 presents the inspection data for 86 lots containing about 25500 pieces of filled vials of an eye drop produced by an established pharmaceutical company based at Kolkata. Each vial is inspected for six attributes. From the criticality point of view, however, the defects can be grouped in two categories. The first category of defects is due to the presence of foreign matters viz. glass, fiber or impurities, which are critical from the user's point of view. The other category of defects consists of breakage, defective sealing and leakage, reasonably obviously detectable and can be easily discarded by the user.

Since the lot size (25500) is quite large, the observed lot quality variation approximates almost exactly the distribution of process average. We have therefore, instead of a gamma-Poisson distribution, fitted a gamma distribution with mean and the variance estimated from the observed data for both types of defects.The results are presented in Table 3.1.2 and table 3.1.3.

Note that for both the attributes the $\bar{p_i}$'s are less than 0.1 ($\bar{p_1} = 0.01708; \bar{p_2} = 0.02302$). The estimated $s_i = \bar{p_i}^2/Var(p_i)$ are 3.4532 and 0.6229 respectively so that $\bar{p_i}/s_i$ are much less than 0.2 justifying the use of gamma distribution as a substitute of beta distribution.

The tables also present the usual χ^2 goodness of fit analysis. It can be seen that the computed χ^2 as a measure of the goodness of fit values are small (they are not statistically significant) enough for both types of attributes to justify the assumption that the prior distributions follow the assumed theoretical gamma distributions.

Further, the scatter plot (see figure 3.1.1) of the observed numbers of defects of the second category against those of the first category exhibits no specific pattern. It would be therefore reasonable to assume that the p_i's are independently distributed in the present context.

It should be pointed out at this stage that the beta and gamma distribution are, however, not always appropriate. For example, we could not fit the gamma or beta distribution in case of quality variation of ceiling fans, garments and cigarettes the relevent data on which were collected. In such cases we will have to take recourse to direct computation of the cost function derived from the empirical distributions as observed. Numerically, the difficulty level in computation will not increase significantly. Nevertheless, since the gamma (or beta) distribution is likely to be appropriate at least in some situations and neat theoretical expressions can be obtained in such cases, we will study the cost functions under such assumptions.

3.1.5 The expression of average costs under the assumption of independent gamma prior distributions of the process average vector

Theorem 3.1.1

Let each p_i be distributed with probability density function $f(p_i, \bar{p}_i, s_i)$ for $i = 1, 2, ..., r$; p_i's are jointly independent; the lot quality X_i is distributed as $g(X_i, Np_i)$ $\forall i$ and X_i 's are jointly independent. The optimal plan in this situation for a specified acceptance criteria $\boldsymbol{x} = (x_1, x_2, ..., x_r) \in A$ is obtained by minimizing the function :

$$K(N,n)/(A_1 - R_1) = nk'_s + (N-n)k'_r + (N-n)\left[\sum_{\boldsymbol{x}\in A}\left\{\sum_{i=1}^{r} d_i\bar{p}_i(s_i + x_i)/(s_i + n\bar{p}_i) - d_0\right\}\prod_{i=1}^{r} g(x_i, n\bar{p}_i, s_i)\right]$$

where $g(x_i, n\bar{p}_i, s_i)$ is a gamma-Poisson density given by,

$$g(x_i, n\bar{p}_i, s_i) = \frac{\Gamma(s_i + x_i)}{x!\Gamma s}\theta_i^{s_i}.(1-\theta_i)^{x_i}; \quad \theta_i = \frac{s_i}{(s_i + n\bar{p}_i)}$$

and x_i non-negative integer. $k'_s = k_s/(A_1 - R_1), k'_r = k_r/(A_1 - R_1)$.

...(3.1.7)

Proof:
Since,

$$\int_{p_1}\int_{p_2}...\int_{p_r} p_i \prod_{j=1}^{r} g(x_j, np_j) f(p_j, \bar{p}_j, s_j) dp_j = [\bar{p}_i(s_i + x_i)/(s_i + n\bar{p}_i) \prod_{j=1}^{r} g(x_j, n\bar{p}_j, s_j)$$

We get,

$$\int_{p_1}\int_{p_2}...\int_{p_r} d_i p_i P(\boldsymbol{p}) dw(p_1) dw(p_2)..dw(p_r) = \sum_{\boldsymbol{x}\in A} d_i\bar{p}_i(s_i + x_i)/(s_i + n\bar{p}_i)\prod_{i=1}^{r} g(x_i, n\bar{p}_i, s_i).$$

Further,

$$\int_{p_1}\int_{p_2}...\int_{p_r} P(\boldsymbol{p})dw(p_1)dw(p_2)...dw(p_r) = \sum \boldsymbol{x} \in A \prod_{i=1}^{r} g(x_i, n\bar{p_i}, s_i).$$

Using the above we get the result.

3.1.6 Bayesian Plans

Using the expression (3.1.7), we may now construct optimal A kind, C kind and D kind plans for a given lot size N, cost parameters and the parameters of the prior distributions and compare their relative merit in a given situation. We demonstrate this with one real life example obtained in respect of plastic containers used for cosmetics.
In this case the defects which can be categorized as major type are defects like colour variation, improper neck finishing, prominent marks, weak body. The cost of testing this is approximately around Rs. 0.06 per unit. The minor type of defects are black spot, shrink marks, less visible parting lines which can be verified at a cost of around Rs 0.14 per unit. (Note that verification for critical defect is less costly.) An item containing the major defects will be discarded at a cost of Rs.1.50 and an item containing minor defects can be resold to another consumer at a reduced price such that the cost of rejection in this case is around Rs.0.50. An item containing defects of major category, if found during filling at the consumer's end, costs the producer around Rs.5.50 (price of the product) and a product containing minor defects will cost the producer around Rs.3.20 (manufacturing cost + transportation costs).

Using the notation of cost model we find that the cost parameters are $S_0 = R_0 = 0.20, S_1 = R_1 = 1.50, R_2 = S_2 = 0.50, A_1 = 5.50$ and $A_2 = 3.20$. From the past data we compute the $\bar{p_1} = 0.0105$, $\bar{p_2} = 0.035$ and Standard deviations as 0.0091 and 0.0055 respectively, so that the parameters of the gamma priors s_1 and s_2 work out to be around 1.2 and 40 respectively. This gives $k'_s = 0.05, d_0 = 0.05, d_1 = 1, d_2 = 0.675$.

We now construct optimal A kind, D kind and C kind plans minimizing the cost function $[K(N,n)/(A_1 - R_1)]$. The table 3.1.4 gives the optimal A, C and D plans for lot sizes 20000 (10000) 50000.

We observe that optimal A kind plans are cheaper than the optimal C kind plans and the optimal D kind plans. The optimal C kind plans are cheaper than the optimal D kind plans.

For a lot of size 30000 the behaviour of cost function near the neighbourhood of the optimum k and n for the Plan D is presented in figure 3.1.2. For the optimum plan, $k = 25$ and $n = 319$.

For the same lot size, 30000, the optimum A kind plan has $a_2 = 25$ and $n = 250$. If we vary a_1 from 0 to 25, we may see from figure 3.1.3 that the minimum cost is obtained at $a_1 = 8$. If we choose $n = 319$ (the sample size of optimal D) plan and $a_2 = 25$ and vary a_1,

the minimum cost is obtained at $a_1 = 10$.
The Visual Basic program as Excel Macro "Costcalculation" for obtaining the cost as relevant to Figure 3.1.3 has been included at the end of this chapter. The output of the programme is the cost of an A kind plan with $a_2 = 25, n = 250, N = 30000$, and for $a_1 = 5(1)20$ for the cost parameters and parameters of the prior gamma distribution as given in this section.

For $N = 30000$, the optimal C kind plan has sample size $n = 250$ and acceptance numbers $c_1 = 8, c_2 = 30$.

We have therefore, demonstrated that it should be possible to obtain an optimal MASSP by the above methods. While doing so, it should be noted that the Bayesian solutions rest on the assumption that the prior distribution is stable and that no outliers occur, so that for small and medium lot sizes the Bayesian plans will often reduce to 'accept without inspection' [Hald (1981)]. It is, however, clear that the choice of acceptance criterion does affect the costs of optimal MASSP's. However, in the present chapter we do not attempt at obtaining general theoretical results for choosing the acceptance criteria as in the case of discrete prior distributions.

Table 3.1.1 : Results of 100% QC checks on 86 lots each of size 25500 pieces (approximate) of eye drop vials . The table presents the number of different types of defects observed.

Day	Glass	Fibre	Impurity	Breakage	Defective Sealing	Leakage	Total
1	94	357	131	18	20	13	633
2	82	178	141	22	102	191	716
3	72	400	115	10	3	89	689
4	25	66	42	16	42	31	222
5	37	281	58	5	33	168	582
6	53	226	78	8	61	142	568
7	40	123	136	4	30	75	408
8	74	238	98	16	74	802	1302
9	48	348	94	9	59	1207	1765
10	77	544	135	12	3	44	815
11	58	253	68	11	49	302	741
12	42	120	29	11	45	28	275
13	60	406	123	18	62	353	1022
14	75	477	154	15	59	505	1285
15	60	205	54	16	83	297	715
16	78	471	91	14	65	296	1015
17	69	356	92	11	48	1471	2047
18	48	363	76	6	53	43	589
19	58	204	86	11	35	211	605
20	46	171	73	9	34	160	493
21	49	180	79	7	69	96	480
22	45	154	83	8	31	85	406
23	41	155	70	7	36	69	378
24	36	180	56	8	44	36	360
25	13	60	43	18	43	66	243
26	71	297	59	11	19	72	529
27	48	218	50	15	81	111	523
28	48	193	83	15	102	117	558
29	53	180	66	8	92	27	426
30	46	280	79	18	34	55	512

Table 3.1.1 (Contd.) : Results of 100% QC checks on 86 lots each of size 25500 pieces (approximate) of eye drop vials . The table presents the number of different types of defects observed.

Day	Glass	Fibre	Impurity	Breakage	Defective Sealing	Leakage	Total
31	53	57	45	17	47	95	314
32	35	140	38	19	128	110	470
33	43	198	57	22	37	117	474
34	46	103	33	2	43	171	398
35	64	305	85	21	197	858	1530
36	71	162	70	26	95	365	789
37	52	270	73	18	198	424	1035
38	56	473	110	24	464	841	1968
39	44	205	219	19	259	1807	2553
40	61	235	37	44	266	2257	2900
41	69	192	38	15	168	2025	2507
42	170	410	113	29	228	3521	4471
43	117	257	55	22	216	2887	3554
44	50	235	44	14	120	1967	2430
45	81	201	42	13	114	1594	2045
46	44	202	43	16	129	1768	2202
47	50	99	35	15	120	1752	2071
48	67	244	57	16	112	1631	2127
49	70	283	57	11	86	927	1434
50	49	260	151	16	108	1403	1987
51	81	1561	177	17	98	784	2718
52	49	676	148	13	88	470	1444
53	75	441	69	9	33	60	687
54	106	564	133	11	110	459	1383
55	41	526	55	9	65	502	1198
56	72	376	82	11	94	252	887
57	56	350	62	12	79	478	1037
58	90	398	80	33	100	896	1597
59	70	120	40	12	55	287	584
60	58	281	48	15	22	40	464

Table 3.1.1 (Contd.) : Results of 100% QC checks on 86 lots each of size 25500 pieces (approximate) of eye drop vials . The table presents the number of different types of defects observed.

Day	Glass	Fibre	Impurity	Breakage	Defective Sealing	Leakage	Total
61	60	230	61	8	11	88	458
62	67	204	56	9	49	577	962
63	58	302	37	5	68	103	573
64	74	665	104	10	12	85	950
65	54	400	70	10	64	162	760
66	45	193	56	5	69	150	518
67	54	131	62	10	58	164	479
68	93	406	132	11	86	661	1389
69	54	243	82	5	64	82	530
70	54	144	85	11	46	60	400
71	53	176	60	10	61	63	423
72	54	316	75	11	46	69	571
73	72	245	109	11	49	20	506
74	54	189	60	32	30	58	423
75	64	256	109	12	31	59	531
76	79	480	209	16	45	424	1253
77	78	170	70	15	44	219	596
78	87	617	210	16	42	116	1088
79	162	594	122	18	64	452	1412
80	65	312	83	29	77	254	820
81	69	163	79	18	46	205	580
82	59	300	77	12	56	76	580
83	52	361	100	16	79	244	852
84	9	107	78	11	42	73	320
85	66	182	33	9	43	61	394
86	53	125	56	4	36	139	413

Table 3.1.2 : Testing goodness of fit of gamma distribution for the observed critical defects in lots containing about 25500 eye drop vials each

Number of defects /unit (p_1)	Observed Frequency (O_i)	Estimated Frequency (E_i)	$(O_i - E_i)^2 / E_i$
<= 0.008	7	12.3940	2.3475
0.008- 0.01	4	7.7206	1.7929
0.01 - 0.012	15	8.4379	5.1034
0.12 - 0.14	13	8.4918	2.3933
0.014-0.016	9	8.0577	0.1102
0.016 -0.018	9	7.3158	0.3877
0.018- 0.02	6	6.4183	0.0273
0.02 - 0.022	4	5.4786	0.3990
0.022-0.024	5	4.5728	0.0399
0.024 - 0.026	4	3.7461	0.0172
0.026- 0.03	3	5.4239	1.0833
0.030-0.034	3	3.3600	0.0386
>0.034	4	4.5827	0.0704
Totals	**86**	**86**	**13.811**

$\overline{p_1} = 0.01708$, Var $(p_1) = 8.44821E-05$, $s_1 = \overline{p_1}.^2 \; / \; /Var(p_1) = 3.4532$

Prob [$\chi^2_{(10)} > 13.811$] = 0.182

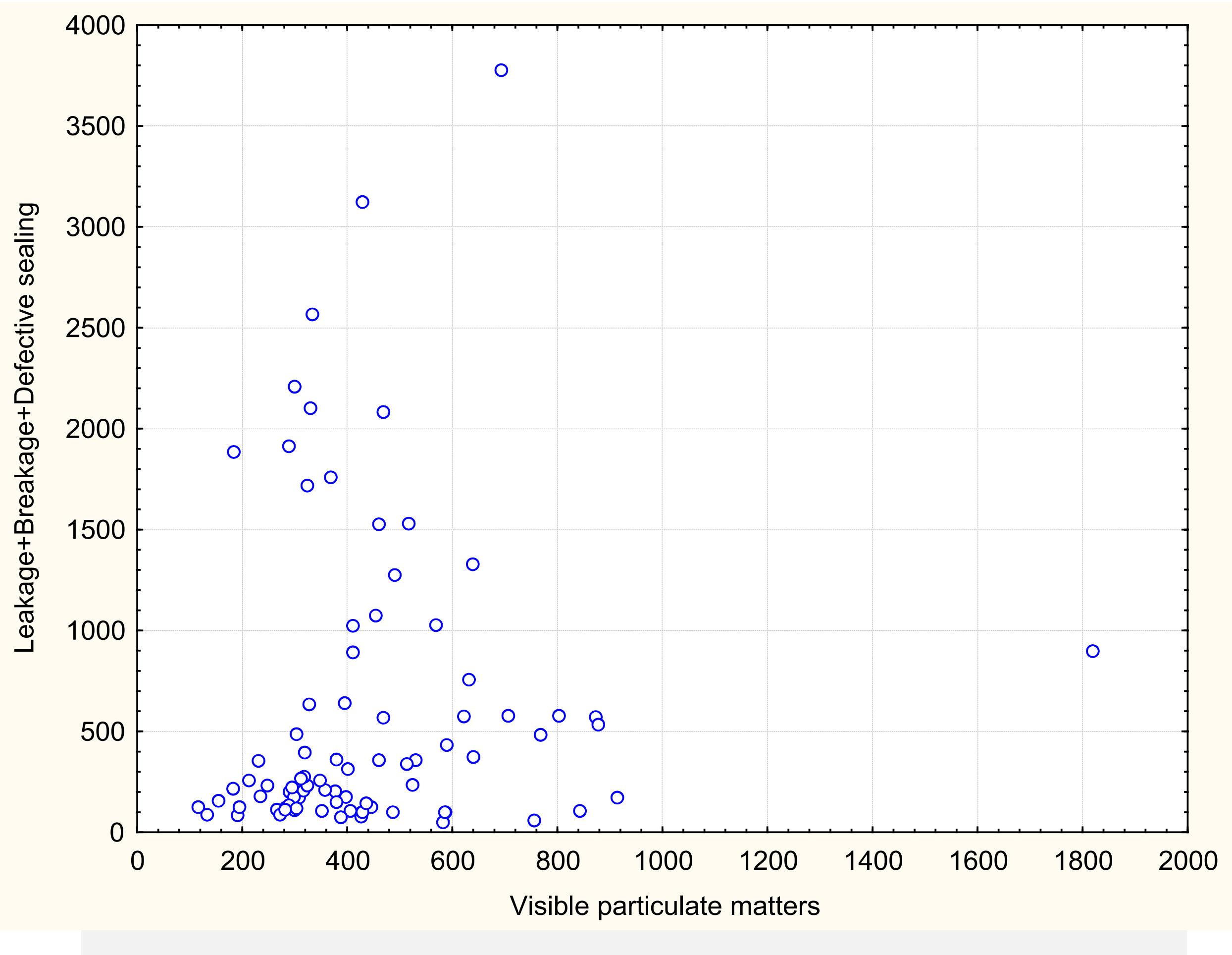
Table 3.1.3 : Testing goodness of fit of gamma distribution for the observed visual defects in lots containing about 25500 Eye drop vials each

Number of defects /unit	Observed Frequency (O_i)	Estimated Frequency (E_i)	$(O_i - E_i)^2 / E_i$
<=0.005	25	26.2307	0.0577
0.005 -0.0075	9	6.7010	0.7888
0.0075- 0.01	9	5.5052	2.2186
0.001-0.0125	5	4.6758	0.0225
0.0125-0.015	6	4.0497	0.9393
0.0125-0.020	4	6.6984	1.0870
0.020-0.0225	3	2.8040	0.0137
0.0225- 0.03	6	6.8164	0.0978
0.03-0.0425	5	7.7194	0.9580
0.03-0.0675	5	8.1248	1.2018
0.0675-0.08	3	2.1343	0.3511
>0.08	6	4.5404	0.4692
Totals	86	86	8.2054

$\overline{p_2} = 0.0230$, $Var(p_2) = 0.000852$, $s_2 = \overline{p_2}.^2 / /Var(p_2) = 0.622$

Prob [$\chi_{(9)}^2 > 8.205$]= 0.513

Figure 3.1.1 : Scatter plots of number of observed defects in 86 lots (each of size 25000) of eye drop. [X axis : number of occurrences of visible particulate matters . Y axis : Number of occurrences of Leakage, breakage and Defective sealing.]



Conclusion : No pattern of dependence is visible. The two types of defects may be assumed to be independently distributed.

Table: 3.1.4 : Optimal MASSP's under assumption that the process average follows gamma distribution for the costs and other parameters given in section 3.1.2

Optimal D kind MASSP

Lot Size	n	k	Costs
20000	254	21	673.582
30000	319	25	1008.179
40000	430	32	1342.191
50000	525	38	1675.764

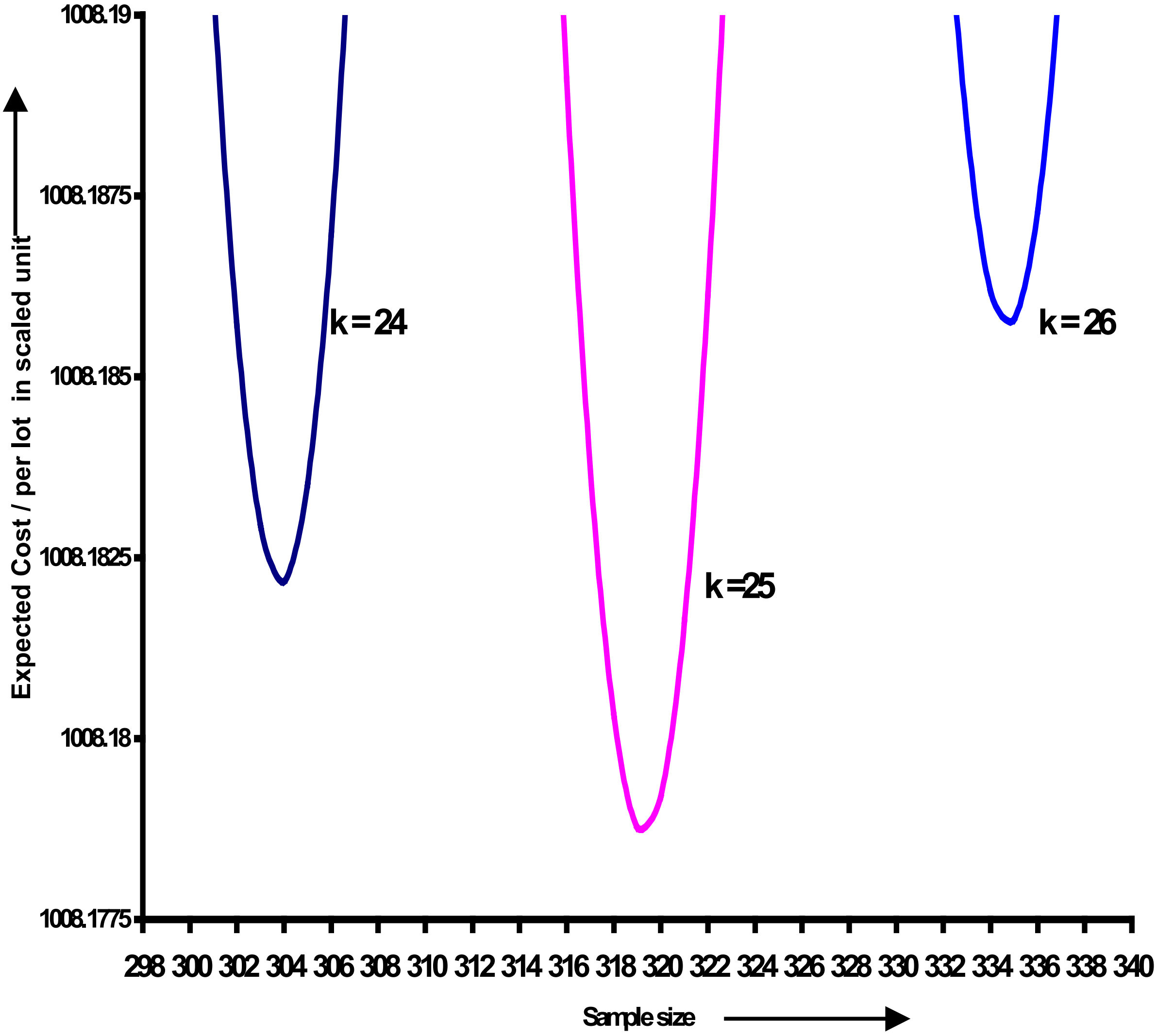
Optimal A kind optimal MASSP

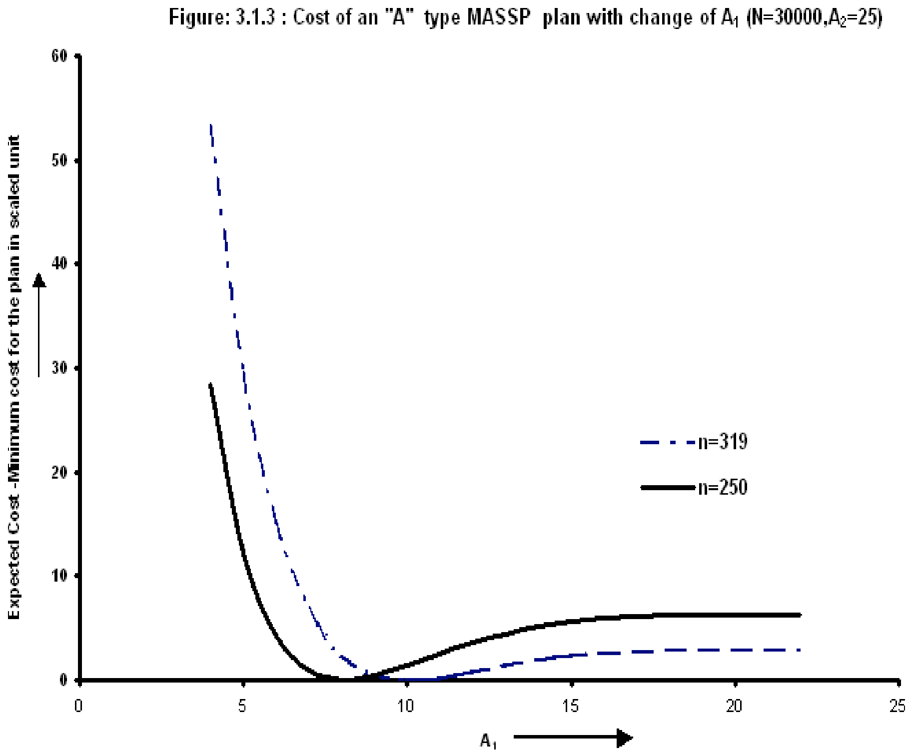
Lot Size	Optimal n	a_1	a_2	Costs
20000	175	6	23	670.5942
30000	250	8	25	1004.088
40000	358	11	34	1337.171
50000	396	12	37	1669.948

Optimal C kind MASSP

Lot Size	n	c_1	c_2	Costs
20000	175	6	24	670.5945
30000	250	8	30	1004.073
40000	398	12	37	1337.315
50000	399	12	38	1669.978

Figure 3.1.2: Sketch of the cost function for the D kind plans for different values of *k* and *n* for the example of section 3.1.6 . The costs are given in scaled unit of $A_2 - R_2$. Lot size = 30000, all other cost parameters are as given in the example. The optimal plan is obtained as k = 25 and n = 319.





Note : The ordinate is 0 for the respective minimum costs for the plan.

The minimum costs for the two plans :

n	**Cost in unit of (A_1 - R_1)**
250	**1004.08843**
319	1005.2731

Annexure

Microsoft Visual Basic program as Excel Macro "Costcalculation" for obtaining the costs as relevant to Figure 3.1.3. The output of the program is the costs of an A kind plan with $a_2 = 25, n = 250, N = 30000,$ and for $a_1 = 5\,(1)\,20$ and the given cost parameters and parameters of the prior gamma distribution in section 3.1.6.

```
Sub costcalculation()

'Fix lotsize
Lotsize = 30000

'Input cost parameters and the parameters of the prior gamma distribution
p1bar = 0.01
p2bar = 0.035
s1 = 1.2
s2 = 40

'Fix a2 and Sample size(n)
a2 =25
n = 250

'Define the range for a1
For a1 = 5 To 20
'Cost calculation using the function gammapoisson
For x1 = 0 To a1
For x2 = 0 To a2 - x1 'This is the range of acceptance
Z1 = p1bar * (s1 + x1) / (s1 + n * p1bar)
Z2 = 0.675 * P2bar * (s2 + x2) / (s2 + n * P2bar)
Z = (z1 + z2 - 0.05)
Sum = sum + z * gammapoisson (x1, n * p1bar, s1) * gammapoisson (x2, n * P2bar, s2)
Next x2
Next x1
Cost = lot * 0.05 + (lot - n) * z
Next n
```

'Print Output as worksheet

```
Worksheets("Sheet1").Cells (i , 1).Value = a1
Worksheets ("Sheet1").Cells (i, 2).Value = cost
i = i + 1
Next a1
End Sub
```

'Computation of the value of the gamma Poisson for a given set of parameters using the Excel worksheet function

```
Function gammapoisson (x, npbar, s)

theta = s / (s + npbar)
Gammanum = Exp(-1) / Application.WorksheetFunction.GammaDist(1, s + x, 1, False)
onedeno = Exp(-1) / Application.WorksheetFunction.GammaDist(1, s, 1, False)
Twodeno = Exp(-1) / Application.WorksheetFunction.GammaDist(1, x + 1, 1, False)
gammapoisson = (Gammanum / (onedeno * Twodeno)) * (theta ŝ) * ((1 - theta) x̂)
End Function
```

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