Design of Experiments

Program: B.Stat Third Year

Academic Year: 2024-25, Second Semester

End-Semester Exam, Date: 09 May, 2025

Duration: 3 hours, Total Points: 100 105

This is a closed book, closed notes exam. Calculators without communication features are allowed. Unless otherwise mentioned, follow the notations used in class. Unless specifically asked to prove, you may use any result that was proved or stated in class. In that case, you must clearly state the result that you are using. Statistical Tables will be provided. Answer all questions. However, the maximum you can score is 100.

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1. [5 + 5 = 10] Consider block designs with v treatments and b blocks.
   (a) If N is the $v \times b$ incidence matrix of an equi-replicate, binary, and proper design and $N^* = (J_{v\times b} - N)$, then show that the design whose incidence matrix is N^* is an equi-replicate, binary, and proper design.
   (b) Find the C-matrix of the design N^* in terms of the C-matrix of the design N.

2. [5 × 3 = 15] Determine which of the following designs are connected:
   Design I: (C,A,B) (C,B,D) (A,C,D) (A,B,D) (D,F,G) (E,F,G) (D,E,F) (D,E,G)
   Design II: (A,B,G) (A,F,G) (A,E,F) (A,D,E) (A,C,D) (A,B,C)
   Design III: (B,E,G) (A,F,G) (C,E,F) (A,D,E) (C,D,G) (B,D,F) (A,B,C)

3. [15 + 5 = 20] Consider the following models:

$$\begin{aligned}
\text{Model } M_1: \quad & y_{n\times 1} = X_{n\times p}\beta_{p\times 1} + z_{n\times 1}\gamma_{1\times 1} + \varepsilon_{n\times 1} \\
\text{Model } M_2: \quad & y_{n\times 1} = X_{n\times p}\beta_{p\times 1} + \varepsilon_{n\times 1} \\
\text{Model } M_3: \quad & z_{n\times 1} = X_{n\times p}\beta_{p\times 1} + \varepsilon_{n\times 1}
\end{aligned}$$

   where X is a matrix with all entries either 0 or 1, $z_i \neq z_j$ for $i \neq j$. Moreover, $1 < \rho(X) < p < n$, and $\rho(X : z) = \rho(X) + 1$.
   (a) Find $\widehat{\gamma}_{LSE}^{(M_1)}$ and $\widehat{\beta}_{LSE}^{(M_1)}$, where the subscript LSE denotes least squares estimate, and the superscript indicates the concerned model.
   (b) Express $\widehat{\beta}_{LSE}^{(M_1)}$ in terms of $\widehat{\beta}_{LSE}^{(M_2)}$, $\widehat{\beta}_{LSE}^{(M_3)}$, and $\widehat{\gamma}_{LSE}^{(M_1)}$.

4. [10 × 2 = 20] How would you construct the blocks in the following situations?
   (a) Confound a 2^5 factorial design in four blocks using ADE and BCE of the three-factor interactions.
   (b) Confound a 3^3 design in three blocks using the AB^2C component of the three-factor interactions.

5. [$2+6+6+6=20$] Consider a two-stage balanced nested model with factor B nested in factor A.

   (a) Write down the model.

   (b) Write down the expressions of the sum of squares used in the ANOVA table.

   (c) Write down the expressions of the expected mean squares under each of the following cases:

      i. A and B both are fixed.

      ii. A is fixed, B is random.

      iii. A and B both are random.

   (d) How would you test the effect of factor A under each of the above cases?

6. [20] Consider a 2^3 factorial experiment with two replicates each run in two blocks with ABC confounded in the first replicate and AB confounded in the second replicate. The data are summarized below:

$$\begin{gathered}\text{Replicate I, Block 1: } (1)=550,\ ab=642,\ ac=749,\ bc=1075\\ \text{Replicate I, Block 2: } a=669,\ b=633,\ c=1037,\ abc=729\\ \text{Replicate II, Block 1: } (1)=604,\ c=1052,\ ab=635,\ abc=860\\ \text{Replicate II, Block 2: } a=650,\ b=601,\ ac=868,\ bc=1063\end{gathered}$$

Write down the ANOVA table with sources of variation, sums of squares, and the degrees of freedom.
