Indian Statistical Institute
M. Math 2nd year
Academic year 2025-2026
Third semester
Backpaper Examination
Course: Riemann surfaces
3 hours

Date : 22 . 12 . 2025

- *Answer as many questions as you can.*
- *You may use results proved in class, but make sure to state them clearly.*
- *Maximum marks is 100.*
- *Unless otherwise mentioned, all Riemann surfaces are assumed to be connected.*

1. For an irreducible nonconstant polynomial $P \in \mathbb{C}[z, w]$ let $C_P = \{(z, w) \in \mathbb{C}^2 \mid P(z, w) = 0\} \subset \mathbb{C}^2, \overline{C_P} \subset \mathbb{CP}^2$ denote the corresponding affine and projective algebraic curves respectively, and let X_P denote the normalization of $\overline{C_P}$. For the polynomial $P(z, w) = w^2 - Q(z)$, where Q is a polynomial of even degree $2n$ with all zeroes distinct:

   a) Find the points at infinity and singular points of $\overline{C_P}$.

   b) Compute the genus of X_P.

   (10+10 = 20 marks)

2. Let f be a meromorphic function on a Riemann surface X such that f is not identically zero. Show that the 1-form $\omega = df/f$ is a meromorphic 1-form on X whose only poles are simple poles at the zeroes and poles of f, and

$$\mathrm{Res}(df/f, p) = \mathrm{ord}_p(f)$$

   for all $p \in X$.

   (10 marks)

3. Let X be a compact Riemann surface of genus zero and let ω be a meromorphic 1-form on X. Show that $\omega = df$ for some meromorphic function f on X if and only if all residues of ω are equal to zero.

(10 marks)

4. Let X be a Riemann surface, let $p \in X$, and let z be a local coordinate of X in a neighbourhood of p. Define an endomorphism $J_p : T_pX \to T_pX$ by $J_p := (dz_p)^{-1} \circ M_i \circ dz_p$ where $M_i : \mathbb{C} \to \mathbb{C}$ denotes multiplication by i.

a) Show that the endomorphism J_p is independent of the choice of local coordinate z near p.

b) Show that a C^1 function $f : U \subset X \to \mathbb{C}$ defined on an open subset $U \subset X$ is holomorphic if and only if $df_p \circ J_p = i \cdot df_p$ for all $p \in U$.

c) Given a local coordinate $z = x + iy$ near $p \in X$, let $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ denote the corresponding basis of T_pX. Show that $J_p(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$, $J_p(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$.

d) Let $J_p^* : T_p^*X \to T_p^*X$ be defined by $J_p^*\alpha = \alpha \circ J_p$ for $\alpha \in T_p^*X$. Show that for any local coordinate $z = x + iy$ near p we have $J_p^*dx = -dy$, $J_p^*dy = dx$.

(5+5+5+5 = 20 marks)

5. Recall that a function $f : X \to \mathbb{C}$ on a Riemann surface X is said to be harmonic if $f \circ z^{-1}$ is harmonic for any local coordinate z on X.

a) Show that if a real-valued harmonic function $f : X \to \mathbb{R}$ on X attains its maximum then it is constant.

b) Show that a smooth function $f : X \to \mathbb{C}$ is harmonic if and only if $d * df = 0$, where $*$ denotes the Hodge star operator acting on 1-forms.

c) Show that if X is simply connected, then any real-valued harmonic function u on X has a harmonic conjugate v, i.e. there exists a real-valued harmonic function v on X such that the function $f = u + iv$ is holomorphic.

(8+6+6 = 20 marks)

6. Recall that a 1-form on a Riemann surface X is said to be harmonic if locally it is given by the derivative of a harmonic function.

a) Show that a smooth 1-form ω on X is harmonic if and only if $d\omega = d * \omega = 0$.

b) Show that any harmonic 1-form on X can be written uniquely as the sum of a holomorphic 1-form and an anti-holomorphic 1-form.

c) Let ω be a harmonic 1-form on a compact Riemann surface X. Show that if ω is exact, then $\omega \equiv 0$.

(6+6+8 = 20 marks)

7. Show that a smooth 1-form ω on a Riemann surface X is harmonic if and only if

$$\int_X \omega \wedge df = \int_X \omega \wedge *df = 0$$

for all $f \in C_c^\infty(X)$.

(10 marks)