Probability Theory III
B. Stat. IInd Year Semester 1
Indian Statistical Institute

Mid-semestral Examination

Time: 2 Hours Full Marks: 40

Date: September 8, 2025

1. If X is a binomial random variable with parameters n and p, show that, for $p < \alpha < 1$, we have
$$\mathrm{P}[X > \alpha n] \leq \left(\frac{1-p}{1-\alpha}\right)^{(1-\alpha)n} \left(\frac{p}{\alpha}\right)^{\alpha n}.$$
[8]

2. For the sample space $\Omega = \mathbb{N}$, consider the class of subsets given by $\mathcal{C} = \{\{2n\} : n \in \mathbb{N}\}$. Show that the σ-field generated by $\mathcal{C}$, $\sigma(\mathcal{C})$ is given by
$$\sigma(\mathcal{C}) = \{2A : A \subseteq \mathbb{N}\} \cup \{2A \cup S : A \subseteq \mathbb{N}\},$$
where S is the set of the odd positive integers and for any subset $A \subset \mathbb{N}$, we have $2A = \{2k : k \in A\}$, when $A \neq \emptyset$ and otherwise $2A = \emptyset$. [8]

3. For a (nondefective) nonnegative random variable X, show that
$$\mathrm{E}[X] = \lim_{n\to\infty} \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathrm{P}\left[\frac{k-1}{2^n} \leq X < \frac{k}{2^n}\right].$$
For a nonnegative random variable X, show that
$$\mathrm{E}[X^2] = \int_0^\infty 2x\, \mathrm{P}[X > x] dx.$$
(You will be penalized if you assume density or probability mass function of the random variable X.) [3+5=8]

4. For a random variable X with $\mathrm{E}[|X|^k] < \infty$ for some $k > 0$, show that the function
$$f(t) = \mathrm{E}\left[X^l \exp(itX)\right]$$
is uniformly continuous for any $l \leq k$. [4]

5. Which of the following four functions $\cos t^3$, $\cos^3 t$, $\cos |t|^3$ and $|\cos t|^3$ are characteristic function(s) of some random variable(s)? [12]