INDIAN STATISTICAL INSTITUTE
End-Semester Examination
M. Tech. CS I year, 1st Sem, AY 2025–2026
Probability and Stochastic Processes

Date: 21. 11. 2025 | Time: 3:00 Hours | Total Marks: 100

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1. Answer **all** the questions from **Group-A**.
2. You are **not allowed to answer more than six questions** from **Group-B**. If you answer six questions:
   (a) the best five will be considered.

Group-A

1. A sequence of events $\{E_n, n \geq 1\}$ is said to be an increasing sequence if

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots,$$

and we define a new event, $\lim_{n\to\infty} E_n$, by

$$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i.$$

Prove that

$$\lim_{n\to\infty} \Pr(E_n) = \Pr\left(\lim_{n\to\infty} E_n\right).$$

[6]

2. Mohun Bagan and East Bengal are the two top football clubs of Kolkata. They are scheduled to play n matches (where n is odd) against each other to determine the Champion of the National Football League. Each match produces a decisive result: no draws are allowed. Mohun Bagan wins any individual match independently with probability p. For any $k > 0$, determine the values of p for which Mohun Bagans overall probability of winning the championship (that is, winning more than half of the matches) increases as n increases.

[6]

3. Let X be a random variable that takes values in the range $[a, b]$. Show that, for any $c > 0$, we have

$$\Pr(|X - \mathbf{E}[Z]| \geq c) \leq \frac{(b-a)^2}{4c^2}.$$

[8]

4. Let X be a continuous random variable with the PDF

$$f_X(x) = \frac{\lambda}{2}e^{-\lambda|x|}, \infty < x < \infty,$$

where λ is a positive real number. Verify that $f_X(x)$ satisfies the normalization condition. Evaluate the mean and variance of X. [3+3+1=10]

5. (a) Let X and Y be two independent Geometric random variables with common parameter p. Derive

$$\Pr[X = x | X + Y = n].$$

(b) Let X and Y be two independent exponential random variables with common parameter λ. Derive the PDF of $Z = X + Y$.

[5+5=10]

6. Show that the expected value of a continuous random variable X satisfies

$$\mathbf{E}[X] = \int_0^\infty \Pr(X > x)dx - \int_0^\infty \Pr(X < -x)dx.$$

[10]

Group-B

7. (a) On a rainy day, n employees arrive at an office and place their wet umbrellas in a common box. In the evening, as they leave for home, each employee randomly picks one umbrella from the box. Let X be the random variable denoting the number of employees who pick their own umbrella. Find the $\mathbf{Var}[X]$.

(b) Let $X_1, \ldots, X_n$ be i.i.d. geometric random variables, each with success probability p. Prove that $\Pr(\max_i X_i > t) \leq n(1-p)^t$.

[5+5=10]

8. Let X and Y be two random variables that are uniformly distributed over the triangle formed by the points $(0, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Using the total expectation theorem, find $\mathbf{E}[X]$ in terms of $\mathbf{E}[Y]$.

(b) Compute the $\mathbf{E}[X]$.

[6+4=10]

9. (a) Let X be a random variable that assumes only non-negative values. Show that for all $a > 0$,

$$\Pr(X \geq a) \leq \frac{\mathbf{E}[X^m]}{a^m},$$

where m is a positive integer ≥ 2.

(b) Show that for a random variable X with variance σ^2 and any positive real number t,

$$\Pr\left(X - \mathbf{E}[X] \geq t\sigma\right) \leq \frac{1}{1+t^2}.$$

Prove all results that you need.

[5+5=10]

10. Let $Y_1, \ldots, Y_n$ be independent random variables with

$$\Pr(Y_i = 1) = \Pr(Y_i = 0) = \frac{1}{2}.$$

Let $Y = \sum_{i=1}^{n} Y_i$. Prove that, for any $a > 0$,

$$\Pr\left(Y \geq \frac{n}{2} + a\right) \leq e^{-2a^2/n}.$$

Prove all results that you need. [10]

11. Prove that there is a two-coloring of edges of K_n with at most

$$\binom{n}{a} 2^{1-\binom{a}{2}}$$

monochromatic K_a. [10]

12. Establish the validity of the expected value rule

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx,$$

where X is a continuous random variable with PDF f_X. [10]

13. Compute the transform (MGF) associated with continuous random variable $Y \sim N(\mu, \sigma^2)$. Prove all results that you need. [10]

14. A factory produces X_n gadgets on day n, where the X_n are independent and identically distributed random variables, with mean 5 and variance 9.

(a) Find an approximation to the probability that the total number of gadgets produced in 100 days is less than 440.

(b) Find (approximately) the largest value of n such that

$$\Pr(X_1 + \cdots + X_n \geq 200 + 5n) \leq 0.05.$$

(c) Let N be the first day on which the total number of gadgets produced exceeds 1000. Calculate an approximation to the probability that $N \geq 220$.

Provide all answers in terms of standard normal CDF Φ. [4+4+3=10]