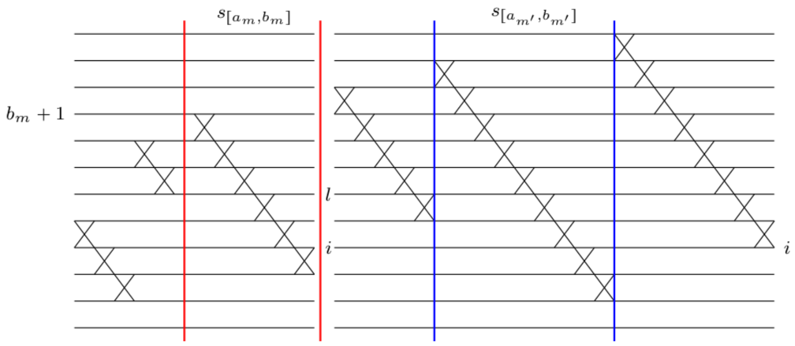
INDIAN STATISTICAL INSTITUTE

Doctoral Thesis

CRYSTALLIZATION OF THE QUANTIZED FUNCTION ALGEBRAS OF $SU_q(n+1)$



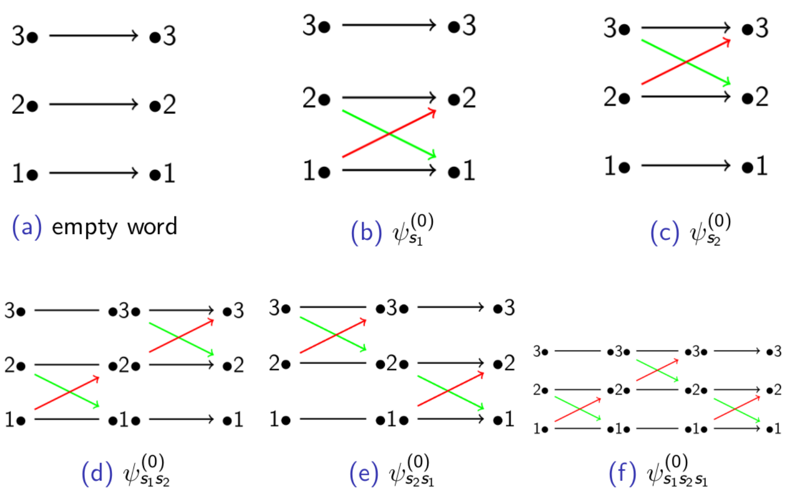


Figure: irreducible representations of $A_2(0)$

Submitted By
Manabendra Giri

Theoretical Statistics and Mathematics Unit, ISI Delhi

Indian Statistical Institute

Doctoral Thesis

Crystallization of the quantized function algebras of $SU_q(n+1)$

Manabendra Giri

Theoretical Statistics & Mathematics Unit

Indian Statistical Institute, Delhi Centre

May 2025

Indian Statistical Institute

Doctoral Thesis

Crystallization of the quantized function algebras of $SU_q(n+1)$

Author:
Manabendra Giri

Supervisor:
Prof. Arup Kumar Pal

A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the degree of Doctor of Philosophy (in Mathematics)

Theoretical Statistics & Mathematics Unit

Indian Statistical Institute, Delhi Centre

May 2025

Dedicated to my parents and grand parents

Acknowledgements

I think words are not enough to express gratitude and acknowledge the support I have received throughout this journey. I would say for me this is not just a degree, it was an enriching experience and as the saying goes "*Life is the best teacher*!".

I would take this opportunity to thank my supervisor Prof. Arup Kumar Pal for his continuous support, advice, encouragement and motivation. His guidance and knowledge helped me in pursuing good research and writing of this thesis. It is a privilege to have a supervisor who has constantly supported me, both academically and personally.

I wish to offer my gratitude to my course teachers, Prof. Maneesh Thakur, Prof. Shanta Laishram, Prof. Rahul Roy, Dr. Anish Sarkar and Dr. Tanvi Jain. I am really fortunate to have such mentors during my Ph.D. These courses will always remain precious in my academic journey.

Life is incomplete without friends, especially the crazy ones. Lucky to have Subhra, Sreyosi, Manish, Mohammad, Biman, Sayantan, Parth Da, Hemant Da as my constant support team. Thanks for those virtual meetups, endless discussion on any topic and cheering me whenever I felt disappointed. Special thanks go to all of my juniors in SMU Delhi. Thanks to Hemant Da, Partha Da, and many more who helped me at different times in different ways. They made my stay in ISI Delhi both inspiring and enjoyable.

I express my heartfelt gratitude to the academic, administrative and technical members of the Indian Statistical Institute, Delhi Centre, who have assisted me with several logistics and official work.

I wish to thank my teachers and friends from my Bishpur High School, Kalinagar High school, Barasat Govt College, and Calcutta University(Ballygunge campus).

Last but not the least, I will remain indebted to my family for their continuous support, love, and help. Special thanks go to my parents and my maternal grandparents. Without their encouragement, inspiration and motivation, I would not have got the mental strength to pursue a career in academics amidst the societal stereotypes and prejudices.

Manabendra Giri

Manabendra Giri
May 2025

Contents

Notations & Abbreviations

$\mathscr{S}_n$	Permutation group of n symbols
$\mathscr{P}_{n,i,j}$	$\{\sigma \in \mathscr{S}_n \ : \ \sigma(i) = j\}$
$(S^1)^n$	n-Torus
$\mathbb{N}$	$\{0, 1, 2, \cdots\}$
q^N	The operator $e_n \longmapsto q^n e_n$ on $\ell^2(\mathbb{N})$
$-q^{N+1}$	The operator $e_n \longmapsto -q^{n+1} e_n$ on $\ell^2(\mathbb{N})$
$\sqrt{1-q^{2N}}$	The operator $e_n \longmapsto \sqrt{1-q^{2n}} e_n$ on $\ell^2(\mathbb{N})$
S	The unilateral backward shift operator on $\ell^2(\mathbb{N})$
	as well as bilateral backward shift operator on $\ell^2(\mathbb{Z})$
$\mathscr{T}$	Toeplitz algebra
$\mathcal{H}$	Hilbert space
$B(\mathcal{H})$	The set of bounded linear operators on $\mathcal{H}$
$\mathcal{K}(\mathcal{H})$	The set of compact operators on $\mathcal{H}$
$\mathscr{E}'$	Commutant of $\mathscr{E}$, i.e. $\{T \in B(\mathcal{H}) \ : \ TR = RT \ \forall R \in \mathscr{E} \subseteq B(\mathcal{H})\}$
$\mathscr{E}''$	SOT-closure of span($\mathscr{E}$), i.e. double commutant of $\mathscr{E}$
$M(A)$	The multiplier algebra of a C^* algebra A
$\mathfrak{g}$	Lie algebra

Introduction

The class of quantum groups, unlike their classical counterparts (i.e. groups) are not yet captured by a single definition. But mathematicians broadly agree on a number of classes of objects that qualify as quantum groups. What all of these have in common is a Hopf structure, together with additional structures that can be algebraic, functional analytic or a combination of both in nature. The origin of the study of quantum groups can be traced back to the work on group duality in the early 1930's. However, work on quantum groups in their present form started in the early 1980's, by Drinfeld and Jimbo and subsequently by Woronowicz, Vaksman & Soibelman and many others. Most interesting classes of quantum groups known thus far are the deformations of classical Lie groups which were first introduced by Drinfeld and Jimbo. The q-deformation of a connected simply connected Lie group G is typically studied through two Hopf algebras associated with it: the quantized universal enveloping algebra (QUEA) $\mathcal{U}_q(\mathfrak{g})$ ([KS97]) and the quantized function algebra (QFA) $\mathcal{O}(G_q)$ ([KS98]).

Let $\mathcal{U}(\mathfrak{g})$ denote the universal envelopping algebra of a complex simple lie algebra $\mathfrak{g}$. The set of equivalence classes of finite dimensional representations has a ring structure.Any finite dimensional representation decomposes as a direct sum of irreducible representations. Therefore the tensor product of any two finite dimensional irreducible representations decomposes into a direct sum of irreducible representations. The quantized universal envelopping algebras $\mathcal{U}_q(\mathfrak{g})$, where $q \in \mathbb{C}\backslash\{0,1\}$, are objects whose finite dimensional irreducible representations share this same structure, except that unlike their classical counterparts, the tensor products $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ of two representations π_1 and π_2 are not equivalent through the flip map but a more complicated map coming from what is known as the R-matrix.

In general, the dual $\mathcal{U}_q(\mathfrak{g})^*$ is not a Hopf-algebra. But one can restrict oneself to an appropriate subspace of $\mathcal{U}_q(\mathfrak{g})^*$ to get a restricted dual $\mathcal{O}(G_q)$ that is a Hopf algebra. This gives the quantized coordinate algebra of functions on the Lie group G. If G has a compact real form K, one can use the Cartan involution to give a $*$-structure on $\mathcal{O}(G_q)$, which makes it a Hopf-*-algebra. Passing on to the envelopping C^*-algebra of $\mathcal{O}(G_q)$, one then obtains a compact quantum group, denoted usually by $C(K_q)$. Thus for example starting from the Lie algebra $sl(2,\mathbb{C})$, its quantized universal envelopping algebra $\mathcal{U}_q(sl(2,\mathbb{C}))$ and the compact real form $SU(2)$ of $SL(2,\mathbb{C})$, one obtains the compact quantum group $SU_q(2)$ studied by Woronowicz et al.

What is widely considered to be one of the major achievements of quantum group theory and has found applications in representation theory of classical groups is the theory of crystal bases, which was developed independently by Kashiwara ([Kas90], [Kas91], [Kas93], [Kas94], [Kas95]) and Lusztig ([Lus90a], [Lus90b]) in the early 1990's. For a broad class of Lie algebras $\mathfrak{g}$, they observed that even though the quantized universal enveloping algebras $U_q(\mathfrak{g})$ do not exist at $q = 0$, when one looks at a finite dimensional module over $U_q(\mathfrak{g})$, one can normalize the actions of the generating elements in such a way that they continue to make sense at $q = 0$.

The coordinate ring of regular functions $\mathcal{O}(G_q)$ for the q-deformation of a complex simple Lie group G can be viewed as a module over the corresponding quantized universal enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of the group G. Crystallization of these modules were studied by Kashiwara in [Kas93] while developing his theory of crystal bases. But $\mathcal{O}(G_q)$ also has an algebra structure. If $\mathfrak{K}$ is the compact real form of $\mathfrak{g}$ with K being the corresponding connected simply connected compact subgroup of G, then $\mathcal{O}(G_q)$ gets a *-algebra structure, and together with this *-algebra structure, it is denoted by $\mathcal{O}(K_q)$.

What we are concerned with in this thesis are the possible crystal limits of this *-algebra $\mathcal{O}(K_q)$ and its C^*-completion $C(K_q)$. For the quantum $SU(2)$ group, Woronowicz had described the C^*-algebra $C(SU_q(2))$ at $q = 0$ in [Wor87]. Very recently Chakraborty & Pal ([CP24]) demonstrated a very interesting application of this C^*-algebra in proving an approximate equivalence between the GNS representation of the Haar state of

$SU_q(2)$ and another well known faithful representation, which we call the Soibelman representation following Matassa & Yuncken ([MY23]), of the same C^*-algebra. This approximate equivalence, in turn, has interesting applications in noncommutative geometry and topology. Therefore it seems worthwhile to investigate the crystallization of quantized function algebras for the q-deformations of compact Lie groups in the C^*-algebra set up. This is the main aim of the thesis.

The notion of crystallization of quantized function algebras was first introduced for the quantum $SU(n+1)$ group in [GP22]. In Kashiwara's work, while crystallizing a $\mathcal{U}_q(\mathfrak{g})$ module, the passage to $q=0$ is done through quotienting by an appropriate ideal, but the actions of the generating elements of $\mathcal{U}_q(\mathfrak{g})$ need to be 'renormalized' before the quotienting procedure so that one gets something meaningful after quotienting. In our approach, we use a limiting procedure instead of quotienting, and similar to the Crystal bases case, we make an appropriate scaling of the genarators before taking limits. The main tool we use is a remarkable result due to Soibelman that gives a list of all irreducible representations of the C^*-algebra $C(K_q)$ for the compact real form K of G for all simple complex Lie groups G. In particular, in the type A_n case, the Weyl group is the permutation group $\mathscr{S}_{n+1}$, and the set of inequivalent irreducible representations are parametrized by (λ, ω) where $\lambda \in (S^1)^n$ and ω is a reduced word in $\mathscr{S}_{n+1}$ with respect to the generating set $\{s_1, s_2, \ldots, s_n\}$, s_k being the transposition $(k, k+1)$. Let us call these representations $\psi^{(q)}_{\lambda,\omega}$. Using Soibelman's result, we make the following crucial observation.

Theorem 0.0.1 *Let $q > 0$ and let $u_{i,j}(q)$ denote the generators of $C(SU_q(n+1))$. For any irreducible representation π of $C(SU_q(n+1))$ on a Hilbert space $\mathcal{H}$, the norm limits $\lim_{q\to 0+}(-q)^{\min\{i-j,0\}}\pi(u_{i,j}(q))$ exist for all $1 \le i, j \le n+1$.*

We then extract a set of relations satisfied by these limit operators, and define the crystal limit $C(SU_0(n+1))$ to be the universal C^*-algebra given by these relations. The crystal limit $\mathcal{O}(SU_0(n+1)))$ is similarly defined as the *-algebra given by these relations. What helps us here is the fact that these relations ensure that for each representation of them on a Hilbert space, the norm of each generating element is bounded by a fixed constant.

It should be noted here that Matassa & Yuncken ([MY23]) had subsequently came up with a slightly different definition of the crystallization of quantized function algebras, covering the q-deformations of all connected simply connected compact Lie groups. We believe that the crystallization defined in [MY23] will coincide with the one given in [GP22] for the type A case mentioned in this thesis, but we do not have a proof of this yet. We do not deal with the Matassa-Yuncken notion of crystallization in this thesis.

Going back to the representations $\psi^{(q)}_{\lambda,\omega}$, if one now defines

$$\psi_{\lambda,\omega}(z_{i,j}) := \lim_{q\to 0+}(-q)^{\min\{i-j,0\}}\psi^{(q)}_{\lambda,\omega}(u_{i,j}(q)),$$

then one gets a representation of the C^*-algebra $C(SU_0(n+1))$. We next investigate the following three problems that naturally arise at this stage:

1. whether the representations $\psi_{\lambda,\omega}$ are irreducible,
2. whether they are inequivalent,
3. if the answer to (1) above is yes, then whether these give all irreducible representations of $C(SU_0(n+1))$.

We examine the case $n = 2$ in Chapter 3, where we have the following results that answer the above questions completely.

Theorem 0.0.2 *For any $\lambda \in (S^1)^2$ and a reduced word ω in $\mathscr{S}_3$, the representation $\psi_{\lambda,\omega}$ is irreducible.*

If $(\lambda,\omega) \neq (\mu,\upsilon)$, then the representations $\psi_{\lambda,\omega}$ and $\psi_{\mu,\upsilon}$ are inequivalent.

Theorem 0.0.3 *Let π be an irreducible representation of $C(SU_0(3))$ on a Hilbert space $\mathcal{H}$. Then there exist $\lambda \in (S^1)^2$ and a reduced word $\omega \in \mathscr{S}_3$ such that π is equivalent to $\psi_{\lambda,\omega}$.*

The results above are from the paper [GP22].

We extend the above results to higher rank cases in the final chapter. More specifically, we prove the following results:

Theorem 0.0.4 *For any $\lambda \in (S^1)^n$ and a reduced word ω in $\mathscr{S}_{n+1}$, the representation $\psi_{\lambda,\omega}$ is irreducible.*

If $(\lambda,\omega) \neq (\mu,\upsilon)$, then the representations $\psi_{\lambda,\omega}$ and $\psi_{\mu,\upsilon}$ are inequivalent.

Theorem 0.0.5 *Let π be an irreducible representation of $C(SU_0(n+1))$ on a Hilbert space $\mathcal{H}$. Then there exist $\lambda \in (S^1)^n$ and a reduced word $\omega \in \mathscr{S}_{n+1}$ such that π is equivalent to $\psi_{\lambda,\omega}$.*

The proofs for the higher rank case are more involved and intricate. We first prove several relations involving the generating elements $z_{i,j}$ that play a very useful role in the proofs. Subsequently we study the image $\pi(C(SU_0(n+1)))$ and its SOT-closure for a representation π of $C(SU_0(n+1))$ on a complex separable Hilbert space. We then further specialize to an irreducible representation π, which helps us draw stronger conclusions on the operators we looked at in the image $\pi(C(SU_0(n+1)))$. In particular, we are able to decompose the operators $\pi(z_{i,j})$ in a certain way that helps us prove a factorization theorem for irreducible representations. This factorization theorem, in turn, helps us give a recursive proof of the classification theorem for all irreducible representations of $C(SU_0(n+1))$. In particular, the classification result tells us that any irreducible representation must occur as a $q \to 0+$ limit of an irreducible representation for $C(SU_q(n+1))$.

As consequences, we also prove the following.

Theorem 0.0.6 *$C(SU_0(n+1))$ is a type-I C^*-algebra.*

Theorem 0.0.7 *Let $\Delta : C(SU_0(n+1)) \to C(SU_0(n+1))\overline{\otimes}C(SU_0(n+1))$ and $\epsilon : C(SU_0(n+1)) \to \mathbb{C}$ be the C^*-homomorphisms given on the generators by*

$$\Delta(z_{i,j}) = \sum_{k=min\{i,j\}}^{max\{i,j\}} z_{i,k} \otimes z_{k,j}, \qquad \epsilon(z_{i,j}) = \delta_{i,j}.$$

Then $(C(SU_0(n+1)), \Delta, \epsilon)$ a C^*-bialgebra.

The results in the final chapter are from the paper [GP24]

CHAPTER 1

Preliminaries

In this chapter, we discuss notations, formally define quantization, and discuss background relevant to this thesis. Some notations and tools specific to a chapter are described in the respective chapter.

1.1 Compact quantum group

Gelfand and Naimark established that if A is a commutative unital C^*-algebra, then there exists a compact Hausdorff space X such that A is isometrically isomorphic to $C(X)$ with sup-norm. That means a unital C^*-algebra can be viewed as a noncommutative analogue of a compact Hausdorff topological space. One can consider that the concept of compact quantum groups is a noncommutative generalization of a compact group. Many authors tried to give an appropriate definition of quantum groups in topological or measure theoritic setting. G. I. Kac, M. Takesaki, M. Enock, J.M. Schwartz, S. L. Woronowicz, V. G. Drinfield and others have a significant role in this direction. It was Woronowicz who first defined this concept in the C^*-algebra setting. A compact group is a group where the underlying set is compact and both the group multiplication and inversion are continuous functions. We also know that compact semigroup with both sided cancellation law is a compact group. In order to generalize this concept to the noncommutative setting, one replaces the compact space by a noncommutative unital C^*-algebra, the group multiplication by a comultiplication on this C^*-algebra and cancellation property by its non-commutative version which is a density condition. Proceeding in this direction, Woronowicz axiomatized the theory of compact quantum groups in [Wor98]. In this section we review some of the basic features of this theory.

Definition 1.1.1 (Compact quantum group) *A compact quantum group is definrd as a pair (A, Δ) where A is a unital C^*-algebra and $\Delta : A \to A\overline{\otimes}A$ is a unital $*$ homomorphism such that*

- ***Coassociativity:*** $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
- ***Cancellation / Density:***
  - *Closed linear span of* $\{(a \otimes 1)\Delta(b) : a, b \in A\}$ *is* $A\overline{\otimes}A$
  - *Closed linear span of* $\{(1 \otimes a)\Delta(b) : a, b \in A\}$ *is* $A\overline{\otimes}A$

Here $\overline{\otimes}$ is the minimal tensor product.

Therefore, for any two bounded linear functionals f and g on A, we can define another bounded linear functional $f * g$ such that

$$f * g := (f \otimes g)\Delta. \tag{1.1.1}$$

Compact quantum group (A, Δ) has a unique state h (known as Haar State) such that $f * h = h * f = f(1_A)h$ for all $f \in A^*$. This state gives a Hilbert space denoted by $L^2(A, \Delta)$ and a C^* representation π_h of A on $L^2(A, \Delta)$.

1.2 Deformation of the universal enveloping algebra

1.2.1 Quantization of Lie algebra $\mathfrak{g}$

Deformation of the universal enveloping algebra of a Lie algebra has been studied extensively in the literature. For a detailed treatment, we refer to [KS97]. A finite dimensional complex simple Lie algebra $\mathfrak{g}$ is given by a finite set of generators satisfying certain relations called Weyl's relations and Serre's relation. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is an asscociative algebra which is the quotient of the tensor algebra $\mathcal{T}(\mathfrak{g})$ over $\mathfrak{g}$ by two sided ideal $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$. By Serre's theorem (see [KS98]),

$\mathcal{U}(\mathfrak{g})$ is given by a finite set of generators and relations. Lie algebra can be embedded inside universal enveloping algebra which is generated by same generators and Lie bracket of any two elements of Lie algebra is nothing but the commutator of them inside universal enveloping algebra. The algebra $\mathcal{U}(\mathfrak{g})$ together with two linear multiplicative map $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and $\varepsilon : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ and one linear antimultiplicative map $S : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ forms a Hopf structure where for all $a \in \mathfrak{g}$ we have

$$\begin{aligned} \Delta(a) &= a \otimes 1 + 1 \otimes a, \\ \varepsilon(a) &= 0, \\ \varepsilon(1) &= 1, \\ S(a) &= -a. \end{aligned}$$

One constructs the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ by replacing these relations by their q-analogues. A finite dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ can be completely determined up to equivalence by an r-tuple of complex numbers where r is the rank of $\mathfrak{g}$.

1.2.2 Quantized universal enveloping algebra

Let $((a_{i,j}))$ be Cartan matrix and $D = diag(d_1, d_2, \cdots, d_r)$ be the corresponding diagonal matrix such that $d_i \in \{1, 2, 3\}$ for all i and DA is symmetric and positive definite. The symbol $\mathbb{F}$ will stand for either the field $\mathbb{Q}(t)$ of rational functions in the variable t over $\mathbb{Q}$ or the complex field. The letter z will stand for t when $\mathbb{F} = \mathbb{Q}(t)$ and non zero complex number that is not a root of unity when $\mathbb{F} = \mathbb{C}$. The quantized universal enveloping algebra $\mathcal{U}_z(\mathfrak{g})$ is universal $\mathbb{F}$-associative algebra generated by $\{X_1, X_2, \cdots, X_r, Y_1, Y_2, \cdots, Y_r\} \cup \{K_1, K_1^{-1}, K_2, K_2^{-1}, \cdots, K_r, K_r^{-1}\}$ satisfying the following relations:

1. $K_iK_j = K_jK_i$ and $K_iK_i^{-1} = K_i^{-1}K_i = 1$,
2. $X_iY_i - Y_iX_i = \frac{1}{(z^{d_i} - z^{-d_i})}[K_i - K_i^{-1}]$ for $1 \leq i \leq r$,
3. $X_iY_j - Y_jX_i = 0$ for $i \neq j$,
4. $K_iX_jK_i^{-1} = z^{d_i a_{i,j}} X_j$,
5. $K_iY_jK_i^{-1} = z^{d_i a_{i,j}} Y_j$,

6. And quantum Serre relations for $i \neq j$

- $\sum_{s=0}^{1-a_{ij}} (-1)^s \frac{[1-a_{i,j}]_{z_i}!}{[s]_{z_i}! \ [1-a_{i,j}-s]_{z_i}!} \ X_i^s X_j X_i^{1-a_{ij}-s} = 0,$
- $\sum_{s=0}^{1-a_{i,j}} (-1)^s \frac{[1-a_{i,j}]_{z_i}!}{[s]_{z_i}! \ [1-a_{i,j}-s]_{z_i}!} \ Y_i^s Y_j Y_i^{1-a_{ij}-s} = 0,$

where $[s]_{z_i} = \frac{(z^{s.d_i})-(z^{-s.d_i})}{z^{d_i}-(z^{-d_i})}$ and $[s]_{z_i}! = [1]_{z_i} \cdot [2]_{z_i} \cdots [s]_{z_i}$.

This will be a Hopf algebra with the following linear multiplicative comultiplication Δ, linear multiplicative counit ε and linear anti-multiplicative antipode S where

$$\Delta : \begin{cases} K_i & \longmapsto & K_i \otimes K_i \\ X_i & \longmapsto & X_i \otimes K_i + 1_A \otimes X_i \\ Y_i & \longmapsto & K_i^{-1} \otimes Y_i + Y_i \otimes 1_A, \end{cases}$$

$$\varepsilon : \begin{cases} K_i & \longmapsto & 1 \\ X_i & \longmapsto & 0 \\ Y_i & \longmapsto & 0, \end{cases}$$

$$S : \begin{cases} K_i & \longmapsto & K_i^{-1} \\ X_i & \longmapsto & -X_i K_i^{-1} \\ Y_i & \longmapsto & -K_i Y_i. \end{cases}$$

When $\mathbb{F} = \mathbb{Q}(t)$ and z is t we will denote $\mathcal{U}_z(\mathfrak{g})$ by $\mathcal{U}_t(\mathfrak{g})$ and when $\mathbb{F} = \mathbb{C}$ and $z = q$, we will denote $\mathcal{U}_z(\mathfrak{g})$ by $\mathcal{U}_q(\mathfrak{g})$.

Now onwards we will use q to denote a real number with $0 < q < 1$.

1.2.3 Real forms of $\mathcal{U}_q(\mathfrak{g})$

A real Lie algebra $\mathfrak{g}_0$ is called a real form of a complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$,i.e., $\mathfrak{g} \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$. Thus a real form gives rise to a $*$ structure on $\mathcal{U}(\mathfrak{g})$. For a complex simple algebra $\mathfrak{g}$, G being the corresponding Lie group, there exists a unique real form g_0 that is the Lie algebra of a connected simply connected compact Lie subgroup of the group G. This is called the compact real form of $\mathfrak{g}$. From that complexification we get an involution $K_i^* = K_i$, $X_i^* = K_i Y_i$ and $Y_i^* = X_i K_i^{-1}$ on $\mathcal{U}_q(\mathfrak{g})$ such that the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ becomes a Hopf-$*$-algebra. Similarly we can get an involution satisfying $t^* = t$, $K_i^* = K_i$, $X_i^* = K_iY_i$ and $Y_i^* = X_iK_i^{-1}$ which turn $\mathcal{U}_t(\mathfrak{g})$ into a Hopf-$*$-algebra.

1.2.4 Finite dimensional representations of $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_t(\mathfrak{g})$

Let π be a representation of $\mathcal{U}_q(\mathfrak{g})$ ($\mathcal{U}_t(\mathfrak{g})$ respectively) on a vector space V. Given a tuple $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, \lambda_r)$, where λ_i are element in corresponding field, we define the vector subspace V_λ as follows:

$$V_\lambda = \{x \in V \ : \ \pi(K_i)x = \lambda_i x \quad \forall i = 1, 2, \cdots, r\} \tag{1.2.1}$$

If $V_\lambda \neq 0$, then the tuple λ is called a weight of π and V_λ is called the weight subspace corresponding to the weight λ of π. We call a representation π of $\mathcal{U}_q(\mathfrak{g})$ ($\mathcal{U}_t(\mathfrak{g})$ respectively) a weight representation if its underlying vector space V decomposes into a direct sum of weight subspaces.

The following theorem says that irreducible weight representations exhaust all finite dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{g})$.

Theorem 1.2.1 ([KS97]) *Every irreducible finite dimensional representation is a weight representation.*

Definition 1.2.2 (Highest weight representation) *We say a weight representation π of $\mathcal{U}_q(\mathfrak{g})$(or $\mathcal{U}_t(\mathfrak{g})$) on a vector space V a highest weight representation if there exists a weight vector v such that $\pi(X_i)v = 0$ for $i = 1, 2, ..., r$ and $\pi(\mathcal{U}_q(\mathfrak{g}))v = V$(or $\pi(\mathcal{U}_t(\mathfrak{g}))v = V$ respectively). The vector v is called highest weight vector and corresponding weight is called highest weight.*

The following theorems will completely characterize all finite dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ or $\mathcal{U}_t(\mathfrak{g})$.

Theorem 1.2.3 ([KS97]) *Every irreducible finite dimensional representaion is a highest weight representation. Such a representation is uniquely determined, up to equivalence, by its highest weight.*

The following theorems tells us about the set of all highest weights which appears in a irreducible representation.

Theorem 1.2.4 ([KS97]) *A tuple $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, \lambda_r)$ is a highest weight of an irreducible finite dimensional representation of $\mathcal{U}_q(\mathfrak{g})$(or $\mathcal{U}_t(\mathfrak{g})$) if and only if it is of the form $\lambda = (\pm q^{n_1}, \pm q^{n_2}, \cdots, \pm q^{n_{r-1}}, \pm q^{n_r})$(or $\lambda = (\pm t^{n_1}, \pm t^{n_2}, \cdots, \pm t^{n_{r-1}}, \pm t^{n_r})$ respectively) with $n_i \in \mathbb{N}$.*

Theorem 1.2.5 ([KS97]) *Every finite dimensional representation of $\mathcal{U}_q(\mathfrak{g})$(or $\mathcal{U}_t(\mathfrak{g})$) can be decomposed as a direct sum of irreducible representations.*

Definition 1.2.6 (Finite dimensional type I representation) *A finite dimensional representation π of $\mathcal{U}_q(\mathfrak{g})$(or $\mathcal{U}_t(\mathfrak{g})$) on a vector space V is called type I representation if any weight $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, \lambda_r)$ of π is of the form $\lambda = (q^{n_1}, q^{n_2}, \cdots, q^{n_{r-1}}, q^{n_r})$(or $\lambda = (t^{n_1}, t^{n_2}, \cdots, t^{n_{r-1}}, t^{n_r})$ respectively) with $n_i \in \mathbb{N}$*

Let $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$, $\omega_i = \pm 1$. Then there is a one dimensional representation χ_ω such that $\chi_\omega(X_i) = \chi_\omega(Y_i) = 0$, $\chi_\omega(K_i) = \omega_i$ where $i = 1, 2, \cdots, r$.

Therefore any finite dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{g})$(or $\mathcal{U}_t(\mathfrak{g})$) on a vector space V with highest weight $\lambda = (\omega_1 q^{n_1}, \omega_2 q^{n_2}, \cdots, \omega_{r-1} q^{n_{r-1}}, \omega_r q^{n_r})$ (or $\lambda = (\omega_1 t^{n_1}, \omega_2 t^{n_2}, \cdots, \omega_{r-1} t^{n_{r-1}}, \omega_r t^{n_r})$ respectively) can be written as tensor product $\chi_\omega \otimes L_{\widetilde{n}}$ where $L_{\widetilde{n}}$ is finite dimensional type I representation of $\mathcal{U}_q(\mathfrak{g})$(or $\mathcal{U}_t(\mathfrak{g})$) with highest weight $\widetilde{n} = (q^{n_1}, q^{n_2}, \cdots, q^{n_{r-1}}, q^{n_r})$ (or $\widetilde{n} = (t^{n_1}, t^{n_2}, \cdots, t^{n_{r-1}}, t^{n_r})$ respectively) [See [KS97]].

1.3 Quantized Function Algebra

Let π be a representation of $\mathcal{U}_q(\mathfrak{g})$ acting on a vector space V over $\mathbb{C}$ and a f be a linear functional defined on V. The functional given by $a \mapsto f(\pi(a)v)$ where $a \in \mathcal{U}_q(\mathfrak{g})$ is called a matrix element of the representation π. Note that a matrix element of a representation lies in the dual space of $\mathcal{U}_q(\mathfrak{g})$. The set of matrix elements of all finite dimensional type I representations of $\mathcal{U}_q(\mathfrak{g})$ forms a Hopf algebra which is known as the coordinate ring of regular functions $\mathcal{O}(G_q)$ where G is the Lie group of $\mathfrak{g}$.

Similarly, we have matrix elements which are $\mathbb{Q}(t)$-linear map on $\mathcal{U}_t(\mathfrak{g})$. The set of matrix elements of all finite dimensional type I representations of $\mathcal{U}_t(\mathfrak{g})$ forms a Hopf algebra which is known as the coordinate ring of regular functions $\mathcal{O}(G_t)$ where G is the Lie group of $\mathfrak{g}$. The involution on $\mathcal{U}_t(\mathfrak{g})$ given in Section 1.2.3 make $\mathcal{O}(G_t)$ a Hopf-$*$-algebra.

For a compact, simply connected Lie group K with Lie algebra $\mathfrak{k}$, let $\mathfrak{g}$ be the complexification of $\mathfrak{k}$ and G be complex simple Lie group whose Lie algebra is $\mathfrak{g}$; then this complexification induces a real form on $\mathcal{U}_q(\mathfrak{g})$. The Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ with this $*$ structure induces a Hopf -structure on $\mathcal{O}(G_q)$. The Hopf algebra $\mathcal{O}(G_q)$ together with this involution is called the quantized algebra of regular functions on K, denoted by $\mathcal{O}(K_q)$. We denote the enveloping C^*-algebra of $\mathcal{O}(K_q)$ by $C(K_q)$ (See [KS97], chapter 9). By extending the comultiplication from $\mathcal{O}(K_q)$ to $C(K_q)$, we get a compact quantum group K_q. It is called q-deformation of the compact group K.

There is another approach from Faddeev, Reshetikhin & Takhtajan to define the quantized function algebra for a simply connected compact Lie group. To define the algebra of regular functions on the quantum group, they start with a finite set of generators and relations based on the quantum R-matrix of a simple Lie algebra. Here we take classical simple Lie algebras $\mathfrak{su}(n+1)$, $\mathfrak{sp}(2n)$, $\mathfrak{so}(2n)$ and $\mathfrak{so}(2n+1)$. One can view the algebra of regular functions defined in the FRT approach as certain subalgebra of the dual algebra of a quantized universal enveloping algebra through a pairing between this algebra of regular functions and quantized universal enveloping algebra,. This relates FRT approach with the previous one.

Let $\pi^{\mathbb{1}}$ denote the fundamental representation of $\mathcal{U}_q(\mathfrak{sl}_{n+1}(\mathbb{C}))$ corresponding to the weight $\mathbb{1} := (1, 0, \ldots, 0)$. The R-matrix for this representation is given by

$$R_q = q^{-1} \sum_i E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} E_{i,i} \otimes E_{j,j} + (q^{-1} - q) \sum_{i>j} E_{i,j} \otimes E_{j,i}.$$

Let σ be the flip operator. Then $\hat{R}_q := \sigma \circ R_q$. intertwines $\pi^{\mathbb{1}} \bigcirc\!\!\!\!\top\, \pi^{\mathbb{1}}$ with itself, where $\pi^{\mathbb{1}} \bigcirc\!\!\!\!\top\, \pi^{\mathbb{1}}$ denotes the tensor product of the representation $\pi^{\mathbb{1}}$ with itself. Fix a highest weight vector e_1 and let $\{e_1, e_2, \ldots, e_{n+1}\}$ be the basis given by $e_{j+1} = \pi^{\mathbb{1}}(Y_j)e_j$. Let

$(\!(u_{i,j})\!)$ denote the matrix of $\pi^{\mathbb{1}}(\cdot)$ with respect to this basis and its dual basis. The intertwiner equation $\hat{R}_q(\pi^{\mathbb{1}} \bigoplus \pi^{\mathbb{1}})(\cdot) = (\pi^{\mathbb{1}} \bigoplus \pi^{\mathbb{1}})(\cdot)\hat{R}_q$ gives following commutation relations

$$u_{i,j}u_{i,l} = qu_{i,l}u_{i,j} \qquad \text{if } j < l, \tag{1.3.1}$$

$$u_{i,j}u_{k,j} = qu_{k,j}u_{i,j} \qquad \text{if } i < k, \tag{1.3.2}$$

$$u_{i,l}u_{k,j} = u_{k,j}u_{i,l} \qquad \text{if } i < k \text{ and } j < l, \tag{1.3.3}$$

$$u_{i,j}u_{k,l} - u_{k,l}u_{i,j} = (q - q^{-1})u_{i,l}u_{k,j} \qquad \text{if } i < k \text{ and } j < l. \tag{1.3.4}$$

The quantum determinant for the matrix $(\!(u_{i,j})\!)$ is given by (see Equation (2.7), [Koe91] or Chapter 4, [PpW91])

$$\mathscr{D}_q = \sum_{\sigma \in \mathscr{S}_{n+1}} (-q)^{\ell(\sigma)} u_{1,\sigma(1)} u_{2,\sigma(2)} \cdots u_{n+1,\sigma(n+1)}, \tag{1.3.5}$$

where $\ell(\sigma)$ denotes the length of the permutation σ.

The compact real form of $\mathcal{U}_q(\mathfrak{sl}_{n+1}(\mathbb{C}))$ induces a $*$-structure that makes the matrix $(\!(u_{i,j})\!)$ unitary.

It is known that the quantized algebra $\mathcal{O}(SU_q(n+1))$ is the $*$-algebra generated by the relations (1.3.1-1.3.4) together with the relations

$$\mathscr{D}_q = 1, \tag{1.3.6}$$

$$u_{i,j}^* = (-q)^{i-j} \sum_{\widetilde{\sigma} \in \mathscr{P}_{n+1,j,i}} (-q)^{\ell(\sigma)} u_{1,\widetilde{\sigma}(1)} u_{2,\widetilde{\sigma}(2)} \cdots u_{j-1,\widetilde{\sigma}(j-1)} u_{j+1,\widetilde{\sigma}(j+1)} \cdots u_{n+1,\widetilde{\sigma}(n+1)}. \tag{1.3.7}$$

Then $\mathcal{O}(SU_q(n+1))$ has a Hopf-$*$-structure given by the following map

- **Comultiplication:** $\Delta(u_{i,j}) = \sum_{k=1}^{n+1} u_{i,k} \otimes u_{k,j}$
- **Counit:** $\epsilon(u_{i,j}) = \delta_{i,j}$
- **Antipode:** $S(u_{i,j}) = u_{j,i}^*$

To make $\mathcal{O}(SU_q(n+1))$ a normed $*$-algebra we define

$$\| a \| = \sup \{ \| \pi(a) \| : \pi \text{ is a representation of } \mathcal{O}(SU_q(n+1)) \}$$

Then we have $\| u_{i,j} \| \leq 1$, hence for all $a \in \mathcal{O}(SU_q(n+1))$, $\| a \| < \infty$. We define by $C(SU_q(n+1))$ the completion of $\mathcal{O}(SU_q(n+1))$ and Δ extend on $C(SU_q(n+1))$. The pair $(C(SU_q(n+1)), \Delta)$ is a compact quantum group. See (Chapters 8–9, [KS97]), ([Koe91]) , (Chapters 3–5,[PpW91]), (Chapters 3, [KS98]) for details.

1.3.1 Specialization maps

Let us denote the subring $\mathbb{Q}[t, t^{-1}]$ of $\mathbb{Q}(t)$ by A. Let $U_t^A(\mathfrak{g})$ denote the A-subring of $U_t(\mathfrak{g})$ generated by the elements

$$X_i^{(r)} := \frac{X_i^r}{[r]_{t_i}!}, \quad Y_i^{(r)} := \frac{Y_i^r}{[r]_{t_i}!}, \quad K_i^{\pm 1},$$

where $[r]_{t_i}!$ stands for the t-factorial $\prod_{j=1}^r [j]_{t_i}$, with $[j]_{t_i} = \frac{t^{d_i j} - t^{-d_i j}}{t^{d_i} - t^{-d_i}}$. One then has (see Section 4.1, [Kas93])

$$U_t(\mathfrak{g}) \cong \mathbb{Q}(t) \otimes_A U_t^A(\mathfrak{g}).$$

Corresponding to $U_t^A(\mathfrak{g})$, one now defines an A-subring of $\mathcal{O}(G_t)$ as follows (see equation 7.3.5, [Kas93]):

$$\mathcal{O}^A(G_t) := \left\{ v \in \mathcal{O}(G_t) : v(x) \in A \text{ for all } x \in U_t^A(\mathfrak{g}) \right\}.$$

. One has the isomorphism (see Lemma 7.3.1, [Kas93])

$$\mathcal{O}(G_t) \cong \mathbb{Q}(t) \otimes_A \mathcal{O}^A(G_t).$$

For $q \in (0, \infty)$, through the action of A on $\mathbb{C}$ by $p[t, t^{-1}] \cdot \lambda = p[q, q^{-1}]\lambda$, one then has the specialization

$$\mathcal{O}(G_q) \cong \mathbb{C} \otimes_A \mathcal{O}^A(G_t). \tag{1.3.8}$$

Note that $U_t^A(\mathfrak{g})$ is closed under the $*$-structure on $U_t(\mathfrak{g})$ given by

$$K_i^* = K_i, \qquad X_i^* = K_i Y_i, \qquad Y_i^* = X_i K_i^{-1}, \qquad t^* = t.$$

This induces a $*$-structure on $\mathcal{O}(G_t)$. We will denote the resulting $*$-algebra over $\mathbb{Q}(t)$ by $\mathcal{O}(K_t)$. The A-subring $\mathcal{O}^A(G_t)$ is closed under the $*$-structure on $\mathcal{O}(G_t)$. We will denote by $\mathcal{O}^A(K_t)$ the A-subring $\mathcal{O}^A(G_t)$ together with this inherited $*$-structure. The isomorphism (1.3.8) now allows one to get a map

$$\theta_q : \mathcal{O}^A(K_t) \to \mathcal{O}(K_q)$$

that sends $p[t,t^{-1}]u_{i,j}(t)$ to $p[q,q^{-1}]u_{i,j}(q)$ for each i, j and for any polynomial $p[t,t^{-1}]$.

1.3.2 Soibelman Representations of the C^*-algebra $C(K_q)$

Let S be the left shift operator on $\ell^2(\mathbb{N})$:

$$Se_n = \begin{cases} e_{n-1} & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases}$$

Let $f : \mathbb{N} \to \mathbb{C}$ is a function. Then the operator $f(N)$ on $\ell^2(\mathbb{N})$ is given by $f(N)e_i = f(i)e_i$

Next theorem will talk about irreducible C^*-representations of $C(SU_q(2))$.

Theorem 1.3.1 ([CP95], Proposition 13.1.8)

1. *Suppose that $\mu \in \mathbb{C}$ and $\mid \mu \mid= 1$, there is a one-dimensional C^*-representation $(\chi_\mu, \mathbb{C})$ such that $\chi_\mu(\alpha_q) = \mu$ and $\chi_\mu(\beta_q) = 0$.*

2. *There is an infinite dimensional seperable C^*-representation $(\Im_q, \ell^2(\mathbb{N}))$ such that $\Im_q(\alpha_q) = S\sqrt{1-q^{2N}}$ and $\Im_q(\beta_q) = q^N$.*

3. *Suppose that $\mu \in \mathbb{C}$ and $\mid \mu \mid= 1$. Then $(\chi_\mu * \Im_q = (\chi_\mu \otimes \Im_q)\Delta_q, \ell^2(\mathbb{N}))$ is an irreducible C^*-representation such that $(\chi_\mu * \Im_q)(\alpha_q) = \mu S\sqrt{1-q^{2N}}$ and $(\chi_\mu * \Im_q)(\beta_q) = q^N$.*

4. *The collection $\{\chi_\mu, \chi_\mu * \Im_q : \mid \mu \mid= 1\}$ is the complete list of all inequivalent irreducible C^*-representation of $C(SU_q(2))$.*

From the canonical embeddings

$$\wp_k : \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \longrightarrow \mathcal{U}_q(\mathfrak{sl}(n+1,\mathbb{C})) \qquad \text{for } 1 \leq k \leq n,$$

we have a compact quantum group epimorphism $\wp_i^* : C(SU_q(n+1)) \longrightarrow C(SU_q(2))$. Any irreducible C^* representation π of $C(SU_q(2))$ yields an irreducible C^* representation $\pi \circ \wp_i^*$ of $C(SU_q(n+1))$.

Elementary representation of $C(SU_q(n+1))$**:** An irreducible C^* representation of $C(SU_q(n+1))$ is called elementary if it is of the form $\pi \circ \wp_i^*$ where π is an irreducible C^* representation of $C(SU_q(2))$.

Let us list elementary representations which are useful to get irreducible representations of $C(SU_q(n+1))$.

The representation $\Im_q \circ \wp_k^*$ of $C(SU_q(n+1))$ on $\ell^2(\mathbb{N})$ is given below for $1 \le k \le n$ such that

$$\Im_q \circ \wp_k^*(u_{i,j}) = \begin{cases} S\sqrt{1-q^{2N}} & \text{if } i=j=k, \\ \sqrt{1-q^{2N}}S^* & \text{if } i=j=k+1, \\ -q^{N+1} & \text{if } i=k, j=k+1, \\ q^N & \text{if } i=k+1, j=k, \\ \delta_{i,j}I & \text{otherwise.} \end{cases} \tag{1.3.9}$$

The above representation will be denoted by $\psi_{s_k}^{(q)}$.

Irreducible representations of the C^*-algebras $C(SU_q(n+1))$ are parametrized by the elements (ω, μ) where ω is an element of Weyl group and μ is an element of distinguished maximal torus. To get a representations corresponding to an element of Weyl group, we need to know its reduced word expression.

The Weyl group $W_n(A)$ for $SU_q(n+1)$ is s generated by $s_1, s_2, \cdots, s_n$ satisfying the following relations.

$$\begin{aligned} s_i^2 &= 1 && \text{for } i = 1, 2, \cdots n \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } i = 1, 2, \cdots n-1 \\ s_i s_j &= s_j s_i && \text{otherwise} \end{aligned}$$

Reduced expression of any element of the Weyl group $W_n(A)$ can be written in the form $s_{[k_1,1]} s_{[k_2,2]} \cdots s_{[k_n,n]}$. For $1 \le r \le n$, $s_{[k_r,r]}$ is given below where $1 \le k_r \le r+1$

- $s_{[r+1,r]}$= empty string
- $s_{[i,r]} = s_r s_{r-1} \cdots s_{i+1} s_i$ where $1 \le i \le r$

Here the Weyl group $W_n(A)$ is isomorphic to the permutation group $\mathscr{S}_{n+1}$ of $n+1$ symbols and s_i is the transposition $(i, i+1)$. Here the longest word w_0 of $W_n(A)$ is

$$(s_1)(s_2 s_1)\cdots(s_i s_{i-1}\cdots s_1)\cdots(s_n s_{n-1}\cdots s_1)$$

Then we can define following representations.

- For $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ reduced word expression of the elements of Weyl group

$$\psi_w^{(q)} = \psi_{s_{i_1}}^{(q)} * \psi_{s_{i_2}}^{(q)} * \cdots * \psi_{s_{i_r}}^{(q)} \quad \Big(\text{See } \textbf{Equation 1.1.1}\Big)$$

- For $\mu \equiv (\mu_1, \mu_2, \cdots \mu_n) \in (S^1)^n$

$$\Big(\Big(\chi_\mu(u_{k,l})\Big)\Big) = diag(\mu_1, \frac{\mu_2}{\mu_1}, \cdots, \frac{\mu_n}{\mu_{n-1}}, \frac{1}{\mu_n}) \tag{1.3.10}$$

- Define $\psi_{\mu,w}^{(q)} = \chi_\mu * \psi_w^{(q)}$ **(See Equation 1.1.1)**

It is well-known (Theorem 6.2.7, page 121, [KS98]) that for any reduced word w and a tuple $\mu \equiv (\mu_1, \mu_2, \cdots \mu_n) \in (S^1)^n$, the representation $\psi_{\mu,w}^{(q)}$ is an irreducible representation of $C(SU_q(n+1)$ and these give all the irreducible representations of them.

Let $(\chi, \ell^2(\mathbb{Z})^{\otimes n})$ be the representation given below:

$$\chi(u_{k,l}) = \begin{cases} 0 & \text{if } k \neq l \\ S \otimes I^{\otimes(n-1)} & \text{if } k = l = 1 \\ S^* \otimes S \otimes I^{\otimes(n-2)} & \text{if } k = l = 2 \\ I^{\otimes(i-2)} \otimes S^* \otimes S \otimes I^{\otimes(n-i)} & \text{if } 3 \leq k = l = i \leq n-1 \\ I^{\otimes(n-2)} \otimes S^* \otimes S & \text{if } k = l = n \\ I^{\otimes(n-1)} \otimes S^* & \text{if } k = l = n+1 \end{cases}$$

Note that $\psi_{\lambda,\omega}^{(q)}$ acts on the Hilbert space $\ell^2(\mathbb{N}^k)$, where k is the length of the reduced word $\omega = s_{i_1} s_{i_2} \ldots s_{i_k}$ in $\mathfrak{S}_{n+1}$. For the longest word w_0, $\phi^{(q)} := \chi * \psi_{w_0}^{(q)}$ (Equation 1.1.1) is faithful representation. We denote this representation by $\pi_S^{(q)}$ and corresponding Hilbert space by $\mathcal{H}_S$.

CHAPTER 2

The crystalized C^*-algebra

In the first section, we recall a few instances in the literature where "$q = 0$" version of family of C^*-algebras that depend on a nonzero parameter q has been studied.

2.1 Quantum algebras at $q = 0$

2.1.1 Quantum group $SU_q(2)$

For $0 < q < 1$, $C(SU_q(2))$ is the C^* algebra generated by α_q and β_q such that they satisfy the following relations

$$
\begin{aligned}
\alpha_q\alpha_q^* + q^2\beta_q\beta_q^* &= 1, \\
\alpha_q^*\alpha_q + \beta_q^*\beta_q &= 1, \\
\alpha_q\beta_q - q\beta_q\alpha_q &= 0, \\
\alpha_q\beta_q^* - q\beta_q^*\alpha_q &= 0, \\
\beta_q\beta_q^* &= \beta_q^*\beta_q.
\end{aligned}
$$

Now using $q = 0$ in the above relations, Woronowicz ([Wor87]) got a new universal C^* algebra $C(SU_0(2))$ generated by α_0 and β_0 satisfying the following relations

$$\begin{aligned}
\alpha_0\alpha_0^* &= 1, \\
\alpha_0^*\alpha_0 + \beta_0^*\beta_0 &= 1, \\
\alpha_0\beta_0 &= 0, \\
\alpha_0\beta_0^* &= 0, \\
\beta_0\beta_0^* &= \beta_0^*\beta_0.
\end{aligned}$$

We know that $C(SU_q(2))$ (for $0 < q < 1$) is isomorphic to $C(SU_0(2))$.

2.1.2 Odd dimensional quantum sphere S_q^{2n+1}

Suppose $q \in (0, 1)$. The universal C^*-algebra $C(S_q^{2n+1})$ is generated by $z_1, z_2, \cdots, z_{n+1}$ satisfying the following relations(see [HS02], [VS90]):

$$\begin{aligned}
z_i z_j &= q z_j z_i && \text{if } 1 \le j < i \le n+1, \\
z_i^* z_j &= q z_j z_i^* && \text{if } 1 \le j \ne i \le n+1, \\
z_i z_i^* - z_i^* z_i + (1 - q^2) \sum_{k>i} z_k z_k^* &= 0 && \text{if } 1 \le i \le n+1, \\
z_1 z_1^* + z_2 z_2^* + \cdots + z_{n+1} z_{n+1}^* &= 1.
\end{aligned}$$

All the irreducible representations of $C(S_q^{2n+1})$ had been studied by Vaksman & Soibelman [VS90] and it is also known that the C^*-algebra $C(S_q^{2n+1})$ is generated by the matrix elements of the first row of the fundamental matrix V of $C(SU_q(n+1))$ [PV99].

Now using $q = 0$ in the defining relations, we get the universal C^*-algebra $C(S_0^{2n+1})$ generated by $z_1, z_2, \cdots, z_{n+1}$ satisfying the following relations (see [HS02]):

$$\begin{aligned}
z_i z_j &= 0 && \text{if } 1 \le j < i \le n+1, \\
z_i^* z_j &= 0 && \text{if } 1 \le j \ne i \le n+1, \\
z_i^* z_i &= \sum_{k=i}^{n+1} z_k z_k^* && \text{if } 1 \le i \le n+1, \\
z_1 z_1^* + z_2 z_2^* + \cdots + z_{n+1} z_{n+1}^* &= 1.
\end{aligned}$$

2.1.3 Quantum quaternion sphere H_q^{2n}

Suppose

$$(\rho_1, \rho_2, \cdots, \rho_n, \rho_{n+1}, \rho_{n+2}, \cdots, \rho_{2n}) = (n, n-1, \cdots, 1, -1, -2 \cdots, -n),$$
$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \epsilon_{n+1}, \epsilon_{n+2}, \cdots, \epsilon_{2n}) = (1, 1, \cdots, 1, -1, -1 \cdots, -1).$$

Let q be a positive real number smaller than 1. The universal C^*-algebra $C(H_q^{2n})$ is generated by $z_1, z_2, \cdots, z_{2n}$ satisfying the following relations (see [Sau15], Section 4, page 15,29):

$$
\begin{aligned}
z_i z_j &= q z_j z_i && \text{if } j < i,\ i + j \neq 2n + 1,\\
q^{2n-i} z_i z_{2n+1-i} &= q^{2n+2-i} z_{2n+1-i} z_i - (1 - q^2) \sum_{k>i}^{2n} q^{2n-k} z_k z_{2n+1-k} && \text{if } n < i,\\
z_i^* z_{2n+1-i} &= q^2 z_{2n+1-i} z_i^*,\\
z_i^* z_j &= q z_j z_i^* && \text{if } 2n + 1 < i + j,\ i \neq j,\\
z_i^* z_j &= q z_j z_i^* + (1 - q^2)\epsilon_i \epsilon_j q^{\rho_i + \rho_j} z_{2n+1-i} z_{2n+1-j} && \text{if } i + j < 2n + 1,\ i \neq j,\\
z_i^* z_i &= z_i z_i^* + (1 - q^2) \sum_{k>i}^{2n} z_k z_k^* && \text{if } n < i,\\
z_i^* z_i &= z_i z_i^* + (1 - q^2) \sum_{k>i}^{2n} z_k z_k^* + (1 - q^2) q^{2\rho_i} z_{2n+1-i} z_{2n+1-i}^* && \text{if } i \leq n,\\
\sum_{k=1}^{2n} z_k z_k^* &= 1.
\end{aligned}
$$

All irreducible representations of $C(H_q^{2n})$ were obtained in [Sau15]. It is also proved that the C^*-algebra $C(H_q^{2n})$ is generated by the matrix elements of the first and last rows of the fundamental matrix V of $C(USP_q(2n))$.

The collection $\{(\pi_{r,k}^{(q)},\ \ell_2(\mathbb{N})^{\otimes(k-1)}) : | r |= 1,\ 1 \leq k \leq 2n\}$ gives a complete list of irreducible representations of $C(H_q^{2n})$ where we have

$$
\pi_{r,k}^{(q)}(z_j) = \begin{cases} r\delta_{1,j} & \text{if } k = 1 \\ r \cdot \left(\psi_{s_1} * \psi_{s_2} * \cdots * \psi_{s_{k-1}}\right)(u_{2n,2n+1-j}) & \text{if } 2 \leq k \leq n \\ r \cdot \left(\psi_{s_1} * \psi_{s_2} * \cdots * \psi_{s_{n-1}} * \psi_{s_n} * \psi_{s_{n-1}} * \cdots * \psi_{s_{2n-k+1}}\right)(u_{2n,2n+1-j}) & \text{if } n < k \leq 2n \end{cases}
$$

Here $\{u_{2n,2n+1-j} \ : 1 \leq j \leq 2n\}$ are the elements of the last row of the fundamental matrix of the quantum symplectic group $USp_q(2n)$.

It was proved in [Sau17] that $C(H_q^{2n})$ is isometrically isomorphic to $C(S_0^{4n-1})$.

We also know $\rho_i \geq 1$ for $i \leq n$ and $\rho_i + \rho_j = 2n+1-(i+j) > 0$ for $i+j < 2n+1$. It was mentioned in [Sau17] that using $q = 0$ in the defining relations, one get the universal C^*-algebra $C(H_0^{2n})$ generated by $z_1, z_2, \cdots, z_{2n}$ satisfying the following relations

$$\begin{aligned}
z_i z_j &= 0 && \text{if } 1 \leq j < i \leq 2n,\ i+j \neq 2n+1,\\
z_{2n} z_1 &= 0, &&\\
z_i^* z_j &= 0 && \text{if } i \neq j,\\
z_i^* z_i &= \sum_{k=i}^{2n} z_k z_k^*, &&\\
\sum_{k=1}^{2n} z_k z_k^* &= 1. &&
\end{aligned}$$

The next theorem shows that $\{(\pi_{r,k}^{(0)},\ \ell_2(\mathbb{N})^{\otimes(k-1)}) \ : \mid r \mid = 1,\ 1 \leq k \leq 2n\}$ is a collection of inequivalent irreducible representations of $C(H_0^{2n})$ where

- If $k = 1$,
  $\pi_{r,1}^{(0)}(z_j) = r\ \delta_{1,j}$,
- If $2 \leq k \leq n$,
$$\pi_{r,k}^{(0)}(z_j) = \begin{cases} r \cdot S^* \otimes Id^{\otimes(k-2)} & \text{if } j = 1 \\ r(-1)^{j-1} \cdot (|e_0\rangle \langle e_0|)^{\otimes(j-1)} \otimes S^* \otimes Id^{\otimes(k-j-1)} & \text{if } 2 \leq j \leq k-1 \\ r(-1)^{k-1} \cdot (|e_0\rangle \langle e_0|)^{\otimes(k-1)} & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$
- If $n+1 \leq k \leq 2n$,
$$\pi_{r,k}^{(0)}(z_j) = \begin{cases} r \cdot S^* \otimes Id^{\otimes(k-2)} & \text{if } j = 1 \\ r(-1)^{j-1} \cdot (|e_0\rangle \langle e_0|)^{\otimes(j-1)} \otimes S^* \otimes Id^{\otimes(k-j-1)} & \text{if } 2 \leq j \leq n-1 \\ r(-1)^{n-1} \cdot (|e_0\rangle \langle e_0|)^{\otimes(j-1)} \otimes S^* \otimes Id^{\otimes(k-j-1)} & \text{if } n \leq j \leq k-1 \\ r(-1)^{n-1} \cdot (|e_0\rangle \langle e_0|)^{\otimes(k-1)} & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.1.1 *For any $r \in S^1$ and $1 \leq k \leq 2n$, the representation $\pi_{r,k}^{(0)}$ given above is irreducible.*

If $(r_1, k_1) \neq (r_2, k_2)$, then the representations $\pi^{(0)}_{r_1,k_1}$ and $\pi^{(0)}_{r_2,k_2}$ are inequivalent.

Proof: Suppose

$$y_j = \begin{cases} \frac{1}{r} \cdot z_1 & \text{if } j = 1 \\ (-1)^{j-1}\frac{1}{r} \cdot z_j & \text{if } 2 \leq j \leq min(n, k) \\ (-1)^{n-1}\frac{1}{r} \cdot z_j & \text{if } n + 1 \leq j \leq k. \end{cases}$$

Therefore for any $(m_1, m_2, \cdots , m_{k-1}),\ (p_1, p_2, \cdots , p_{k-1}) \in \mathbb{N}^{k-1}$ one has

$$\begin{aligned} & |e_{m_1}\rangle \langle e_{p_1}| \otimes |e_{m_2}\rangle \langle e_{p_2}| \otimes \cdots \otimes |e_{m_{k-1}}\rangle \langle e_{p_{k-1}}| \\ = \ & \pi^{(0)}_{r,k}\Big(y_1^{m_1} y_2^{m_2} \cdots y_{k-1}^{m_{k-1}} y_k (y^*_{k-1})^{p_{k-1}} (y^*_{k_2})^{p_{k-2}} \cdots (y^*_1)^{p_1}.\Big) \end{aligned}$$

Hence $\mathcal{K}(\ell^2(\mathbb{N})^{\otimes(k-1)}) \subseteq \pi^{(0)}_{r,k}(C(H^{2n}_0))$ and that implies $\pi^{(0)}_{r,k}$ is irreducible.

Take $k_1 = k_2 = k$. Then after comparing the spectrum of $\pi^{(0)}_{r_1,k}(z_k)$ and $\pi^{(0)}_{r_2,k}(z_k)$ we can conclude that $\pi^{(0)}_{r_1,k}$ and $\pi^{(0)}_{r_2,k}$ are inequivalent.

If $k_1 < k_2$, $\pi^{(0)}_{r_1,k}(z_{k_2}) = 0$ but $\pi^{(0)}_{r_2,k_2}(z_{k_2}) \neq 0$. Therefore $\pi^{(0)}_{r_1,k_1}$ and $\pi^{(0)}_{r_2,k_2}$ are inequivalent. □

To show that the representations listed in the previous theorem are in fact all the irreducible representations of $C(H^{2n}_0)$, we further need some lemmas.

Lemma 2.1.2 *Let z_1, $z_2, \cdots , z_{2n}$ be the generating elements of $C(H^{2n}_0)$. Then we have $z_i z_j = 0$ for $j < i$.*

Proof: We have

$$\begin{aligned} (z_i z_j)^*(z_i z_j) =& z^*_j (z^*_i z_i) z_j \\ =& z^*_j (\sum_{k=i}^{2n} z_k z^*_k) z_j \\ =& z^*_j \sum_{k=i}^{2n} z_k (z^*_k z_j) \\ =& 0. \end{aligned}$$

So $z_i z_j = 0$ for $j < i$. □

Therefore we can conclude that $C(H^{2n}_0)$ is isomorphic to $C(S^{4n-1}_0)$ as a C^*-algebra.

Lemma 2.1.3 *The generating elements $z_1,\ z_2, \cdots, z_{2n}$ of $C(H_0^{2n})$ are partial isometries.*

Proof: We first show that z_1 is partial isometry. Here we have $z_1^* z_1 = \sum_{k=1}^{2n} z_k z_k^* = 1$. Assume z_i is partial isometry for some i. We want to show that z_{i+1} is partial isometry. Now we have

$$\begin{aligned} z_i^* z_i =& \sum_{k=i}^{2n} z_k z_k^* \\ =& z_i z_i^* + \sum_{k=i+1}^{2n} z_k z_k^* \\ =& z_i z_i^* + z_{i+1}^* z_{i+1} \end{aligned}$$

Therefore we can express the positive element $z_{i+1}^* z_{i+1}$ as a difference of two projections ($z_{i+1}^* z_{i+1} = z_i^* z_i - z_i z_i^*$). So the positive element $z_{i+1}^* z_{i+1}$ is also a projection. Hence z_{i+1} is partial isometry. □

Lemma 2.1.4 *The family $\{z_i z_i^* \ : \ 1 \leq i \leq 2n\}$ is a collection of mutually orthogonal projections.*

Proof: For $i \neq j$, we have $(z_i z_i^*)(z_j z_j^*) = z_i(z_i^* z_j) z_j^* = 0$. □

Lemma 2.1.5 *Let $z_1,\ z_2, \cdots, z_{2n}$ be the generating elements of $C(H_0^{2n})$. Then we have $z_j^* z_j z_i = z_i z_i^* z_i$ if $j \geq i$.*

Proof: For $j \geq i$, we have $z_j^* z_j z_i = (\sum_{k=j}^{2n} z_k z_k^*) z_i = z_i z_i^* z_i$. □

Lemma 2.1.6 *Suppose $z_1,\ z_2, \cdots, z_{2n}$ be the generating elements of $C(H_0^{2n})$ and p be a positive integer. Then we get $(z_i^*)^p z_i^p = z_i^* z_i$.*

Proof: Here we obtain

$$\begin{aligned} (z_i^*)^p z_i^p =& (z_i^*)^{p-2} z_i^* (\sum_{k=i}^{2n} z_k z_k^*) z_i z_i^{p-2} \\ =& (z_i^*)^{p-2} z_i^* (z_i z_i^* z_i) z_i^{p-2} \\ =& (z_i^*)^{p-2} z_i^* z_i z_i^{p-2} \end{aligned}$$

$$=(z_i^*)^{p-1} z_i^{p-1}.$$

Repeating this argument we get $(z_i^*)^p z_i^p = z_i^* z_i$. □

Lemma 2.1.7 *Let $z_1,\ z_2, \cdots, z_{2n}$ be the generating elements of $C(H_0^{2n})$. Suppose j be positive integer smaller than i and k and p_1, p_4 be any two positive integers and p_2, p_3 be any two non negative integers. Then we have $(z_i^*)^{p_1}(z_j^*)^{p_2} z_j^{p_3} z_k^{p_4} = \delta_{p_2,p_3}\delta_{i,k}(z_i^*)^{p_1} z_i^{p_4}$.*

Proof: If $0 < p_2 < p_3$, we have

$$\begin{aligned}
(z_i^*)^{p_1}(z_j^*)^{p_2} z_j^{p_3} z_k^{p_4} &= (z_i^*)^{p_1}(z_j^*) z_j^{p_3-p_2+1} z_k^{p_4} \\
&= (z_i^*)^{p_1} z_j^{p_3-p_2}(z_j^*)(z_j) z_k^{p_4} \\
&= 0.
\end{aligned}$$

If $p_2 > p_3 > 0$, we have

$$\begin{aligned}
(z_i^*)^{p_1}(z_j^*)^{p_2} z_j^{p_3} z_k^{p_4} &= (z_i^*)^{p_1}(z_j^*)^{p_2-p_3+1} z_j z_k^{p_4} \\
&= (z_i^*)^{p_1}(z_j^*)^{p_2-p_3} z_k z_k^* z_k^{p_4} \\
&= 0.
\end{aligned}$$

If $0 < p_2 = p_3$, we have

$$\begin{aligned}
(z_i^*)^{p_1}(z_j^*)^{p_2} z_j^{p_3} z_k^{p_4} &= (z_i^*)^{p_1}(z_j^*) z_j z_k^{p_4} \\
&= (z_i^*)^{p_1} z_k z_k^* z_k^{p_4} \\
&= \delta_{i,k}(z_i^*)^{p_1} z_i^{p_4}.
\end{aligned}$$

□

Lemma 2.1.8 *Let ψ be a representation of $C(H_0^{2n})$. Assume that there exists k such that $\psi(z_k) \neq 0$ but $\psi(z_{k+1}) = 0$. Then $\psi(z_1) \neq 0$, $\psi(z_2) \neq 0$, $\cdots$, $\psi(z_{k-1}) \neq 0$ and $\psi(z_{k+1}) = 0$, $\psi(z_{k+2}) = 0$, $\cdots$, $\psi(z_{2n}) = 0$.*

Proof: Suppose for some i, $\psi(z_i) = 0$. Then we have $\psi(z_{i+1})\psi(z_{i+1})^* = \psi(z_i^* z_i - z_i z_i^*) = 0$. Therefore we can conclude $\psi(z_{i+1}) = 0$. □

Lemma 2.1.9 *Let ψ be an irreducible representation of $C(H_0^{2n})$ on a Hilbert space $\mathcal{H}$. Assume that there exists k such that $\psi(z_k) \neq 0$ but $\psi(z_{k+1}) = 0$. Let P be a projection satisfying*

$$P \leq \psi(z_k)^*\psi(z_k).$$

Then $\mathcal{H}_P :=$ closed linear span of B is an invariant subspace for ψ where

$$B = \{\psi(z_1)^{m_1}\psi(z_2)^{m_2} \cdots \psi(z_{k-1})^{m_{k-1}}\xi : m_1, m_2, \cdots, m_{k-1} \in \mathbb{N}, \xi \in P\mathcal{H}\}.$$

If Q is another projection orthogonal to P satisfying the above conditions, $\mathcal{H}_P$ and $\mathcal{H}_Q$ are orthogonal.

Proof: Here we have $\psi(z_k^* z_k) = \psi(z_k z_k^* + z_{k+1}z_{k+1}^*) = \psi(z_k)\psi(z_k)^*$. Therefore $\psi(z_k)$ is normal.

Take $\xi \in P\mathcal{H}$. Then $\psi(z_k)\xi = \psi(z_k)P\xi = P\psi(z_k)\xi \in P\mathcal{H}$. From the generating relations, one can show that if $(m_1, m_2, \cdots, m_{j-1}) \neq (0, 0, \cdots, 0)$,

$$\psi(z_j) \cdot \psi(z_1)^{m_1}\psi(z_2)^{m_2} \cdots \psi(z_{k-1})^{m_{k-1}}\xi = 0.$$

Otherwise we have

$$\psi(z_j) \cdot \psi(z_1)^{m_1}\psi(z_2)^{m_2} \cdots \psi(z_{k-1})^{m_{k-1}}\xi = \psi(z_j)^{m_j+1} \cdots \psi(z_{k-1})^{m_{k-1}}\xi.$$

Thus we get $\mathcal{H}_P$ is invariant under ψ.

For the second part, take $\xi \in P\mathcal{H}$ and $\zeta \in Q\mathcal{H}$. Suppose $\xi' = \psi(z_k)^*\xi$ and $\zeta' = \psi(z_k)^*\zeta$. Then we have $\xi = \psi(z_k)\psi(z_k)^*\xi = \psi(z_k)\xi'$ and $\zeta = \psi(z_k)\psi(z_k)^*\zeta = \psi(z_k)\zeta'$. Therefore for $(m_1, m_2, \cdots, m_{k-1}) \in \mathbb{N}^{k-1}$ and $(p_1, p_2, \cdots, p_{k-1}) \in \mathbb{N}^{k-1}$ one has,

$$\begin{aligned}
&\langle \psi(z_1)^{m_1}\psi(z_2)^{m_2} \cdots \psi(z_{k-1})^{m_{k-1}}\xi,\ \psi(z_1)^{p_1}\psi(z_2)^{p_2} \cdots \psi(z_{k-1})^{p_{k-1}}\zeta\rangle \\
=&\langle \psi(z_{k-1}^*)^{p_{k-1}}\psi(z_{k-2}^*)^{p_{k-2}} \cdots \psi(z_2^*)^{p_2}\psi(z_1^*)^{p_1}\psi(z_1)^{m_1}\psi(z_2)^{m_2} \cdots \psi(z_{k-1})^{m_{k-1}}\xi, \zeta\rangle \\
=&\langle \psi(z_{k-1}^*)^{p_{k-1}}\psi(z_{k-2}^*)^{p_{k-2}} \cdots \psi(z_2^*)^{p_2}\psi(z_1^*)^{p_1}\psi(z_1)^{m_1}\psi(z_2)^{m_2} \cdots \psi(z_{k-1})^{m_{k-1}}\psi(z_k)\xi', \psi(z_k)\zeta'\rangle \\
=&\langle \psi(z_k^*)\psi(z_{k-1}^*)^{p_{k-1}}\psi(z_{k-2}^*)^{p_{k-2}} \cdots \psi(z_2^*)^{p_2}\psi(z_1^*)^{p_1}\psi(z_1)^{m_1}\psi(z_2)^{m_2} \cdots \psi(z_{k-1})^{m_{k-1}}\psi(z_k)\xi', \zeta'\rangle \\
=&\langle \delta_{p_1,m_1}\delta_{p_1,m_1} \cdots \delta_{p_{k-1},m_{k-1}}\ \psi(z_k^*)\psi(z_k)\xi', \zeta'\rangle
\end{aligned}$$

$$
\begin{aligned}
&=\delta_{p_1,m_1}\delta_{p_1,m_1}\cdots\delta_{p_{k-1},m_{k-1}}\ \langle\psi(z_k)\xi',\psi(z_k)\zeta'\rangle\\
&=\delta_{p_1,m_1}\delta_{p_1,m_1}\cdots\delta_{p_{k-1},m_{k-1}}\ \langle\xi,\zeta\rangle\\
&=\delta_{p_1,m_1}\delta_{p_1,m_1}\cdots\delta_{p_{k-1},m_{k-1}}\ \langle P\xi,Q\zeta\rangle\\
&=0.
\end{aligned}
$$

$\square$

Proposition 2.1.10 *Let ψ be an irreducible representation of $C(H_0^{2n})$ on a Hilbert space $\mathcal{H}$. Suppose there is a natural number k such that $\psi(z_k) \neq 0$ but $\psi(z_{k+1}) = 0$. Then*

1. *there exists a $\lambda \in S^1$ such that the spectrum $\sigma(\psi(z_k))$ is $\{0, \lambda\}$,*
2. *$\psi(z_k)^*\psi(z_k)$ is a rank-one operator,*
3. *ψ is unitarily equivalent to $\pi_{r,k}^{(0)}$ where*

$$
r = \begin{cases} \lambda(-1)^{k-1} & \text{if } k \leq n \\ \lambda(-1)^{n-1} & \text{otherwise.} \end{cases}
$$

Proof: Let E and F be two disjoint non empty closed subsets of the spectrum $\sigma(\psi(z_k))$ such that $0 \in E$ and $0 \in F$. Let P and Q be the spectral projections of $\psi(z_k)^*\psi(z_k)$ corresponding to the subsets E and F respectively. Then it follows from the previous lemma that $\mathcal{H}_P$ and $\mathcal{H}_Q$ are two invariant subspaces for ψ that are mutually orthogonal. By irreducibility of ψ, it follows that $\sigma(\psi(z_k)) = \{0, \lambda\}$ for some $\lambda \in \mathbb{C}$. Since $\psi(z_k)$ is a normal partial isometry, therefore we have $\lambda \in S^1$.

Now we will prove (2). Take a unit vector $\xi \in \psi(z_k)^*\psi(z_k)\mathcal{H}$. So we have $\psi(z_k)\xi = \lambda\xi$. Let $\mathcal{H}_\xi$ be the closed linear span of

$$
\{\psi(z_1)^{m_1}\psi(z_2)^{m_2}\cdots\psi(z_{k-1})^{m_{k-1}}\xi : m_1, m_2, \cdots, m_{k-1} \in \mathbb{N}\}.
$$

Thus $\mathcal{H}_\xi$ is an invariant subspace of $\mathcal{H}$ and hence by the irreducibility of ψ, one has $\mathcal{H}_\xi = \mathcal{H}$.

Now for the last part, consider the scalar $c_{m_1,m_2,\cdots,m_{k-1}}$ and the vector $v_{m_1,m_2,\cdots,m_{k-1}}$ where

$$c_{m_1,m_2,\cdots,m_{k-1}} = \begin{cases} \prod_{j=1}^{k}((-1)^{j-1})^{m_j} & \text{if } k \leq n, \\ \prod_{j=1}^{n}((-1)^{j-1})^{m_j} \cdot \prod_{j=n+1}^{k}((-1)^{n-1})^{m_j} & \text{otherwise.} \end{cases},$$
$$v_{m_1,m_2,\cdots,m_{k-1}} = \psi(z_1)^{m_1}\psi(z_2)^{m_2}\cdots\psi(z_{k-1})^{m_{k-1}}\xi.$$

The map

$$U : e_{m_1} \otimes e_{m_2} \otimes \cdots e_{m_{k-1}} \mapsto c_{m_1,m_2,\cdots,m_{k-1}} r^{m_1+m_2+\cdots+m_{k-1}}\, v_{m_1,m_2,\cdots,m_{k-1}}$$

from $\ell^2(\mathbb{N})^{\otimes(k-1)}$ to $\mathcal{H}$ extends to a unitary which gives us the required equivalence. □

The family of all inequivalent irreducible representations of the C^*-algebra $C(H_0^{2n})$ are naturally parametrized analogous to $q \neq 0$ case. In fact each of them can be obtained as a norm limit of the irreducible representation of the the C^*-algebra $C(H_q^{2n})$ when $q \to 0^+$.

2.2 The crystalized C^*-algebra $C(SU_0(n+1))$

The C^*-algebras $C(SU_q(n+1))$, for $q \neq 0$ are given by a set of generators and relations. In order to obtain the C^*-algebra $C(SU_0(n+1))$, we look at the 'relations at $q = 0$' and then take the universal C^*-algebra given by them. Our first aim is to obtain those relations. Recall that in Kashiwara's work, while crystallizing an $U_q(\mathfrak{g})$ module, the passage to $q = 0$ is done through quotienting by an appropriate ideal, but the actions of the generating elements of $U_q(\mathfrak{g})$ need to be 'renormalized' before the quotienting procedure so that one gets something meaningful after quotienting. We will use a limiting procedure instead of quotienting. Similar to the crystal bases case, we will make an appropriate scaling or renormalization before taking limits. Here we look at the irreducible representations of the C^*-algebra $C(SU_q(n+1))$ for $q \neq 0$ and examine the actions of the generating elements as $q \to 0+$. We arrive at a set of relations and prove the existence of a universal C^*-algebra given by these relations. We call this universal C^*-algebra the crystallization of $C(SU_q(n+1))$ and denote it by $C(SU_0(n+1))$.

We will denote the generating elements $u_{i,j}$ by $u_{i,j}(q)$ as we will be dealing with the algebras $C(SU_q(n+1))$ for different values of q. Observe that for all $q \in (0,1)$, the irreducible representations $\psi^{(q)}_{\lambda,\omega}$ of the C^*-algebras above are parametrized by the same set $(S^1)^n \times W_n$ which is independent of q, and the representation $\psi^{(q)}_{\lambda,\omega}$ acts on the Hilbert space $\mathcal{H}_{\lambda,\omega} := \ell^2(\mathbb{N}^k)$, k being the length of ω, which does not change for different values of the parameter q. Thus for any irreducible representation $\psi^{(q)}_{\lambda,\omega}$ of $C(SU_q(n+1))$, the operators $\psi^{(q)}_{\lambda,\omega}(u_{i,j}(q))$ live on the same Hilbert space $\mathcal{H}_{\lambda,\omega}$ for $q \in (0,1)$. The following is the key observation in obtaining the commutation relations used to define the crystallized C^*-algebra $C(SU_0(n+1))$.

Proposition 2.2.1 *Let $q > 0$ and let $u_{i,j}(q)$ denote the generators of $C(SU_q(n+1))$. For any irreducible representation π of $C(SU_q(n+1))$. on a Hilbert space $\mathcal{H}$,*

1. *the norm limits $\lim_{q\to 0+} \pi(u_{i,j}(q))$ exist for all $1 \le i, j \le n+1$.*
2. *the norm limits $\lim_{q\to 0+}(-q)^{i-j}\pi(u_{i,j}(q))$ exist for all $1 \le i < j \le n+1$.*

Proof: Observe that if the above limits exist for two representations π_1 and π_2, then the limits also exist for the representation $\pi_1 * \pi_2$. This is immediate from the equality

$$\pi_1 * \pi_2(u_{i,j}(q)) = \sum_{k=1}^{n+1} \pi_1(u_{i,k}(q)) \otimes \pi_2(u_{k,j}(q)).$$

Therefore the proof reduces to a simple verification of the existence of the limits for the representations χ_μ and $\psi^{(q)}_{s_k}$ using (1.3.9) and (1.3.10). □

We can observe that the limit is non-zero for some representation. Let

$$\pi^{(q)}_{i,j} = \begin{cases} \chi_{(1,1,\cdots,1)} & \text{if } i = j, \quad \text{(See 1.3.10)} \\ \psi^{(q)}_{s_i s_{i-1} \cdots s_j} & \text{if } i > j, \\ \psi^{(q)}_{s_i s_{i+1} \cdots s_j} & \text{if } i < j. \end{cases}$$

Then we have

$$\pi^{(q)}_{i,j}(u_{i,j}(q)) = \begin{cases} 1 & \text{if } i = j, \\ \underbrace{q^N \otimes q^N \otimes \cdots \otimes q^N}_{i-j} & \text{if } i > j, \\ (-1)^{j-i}\underbrace{q^{N+1} \otimes q^{N+1} \otimes \cdots \otimes q^{N+1}}_{j-i} & \text{if } i < j. \end{cases}$$

Therefore we get

$$\lim_{q\to 0^+}(-q)^{j-i}\pi_{i,j}^{(q)}(u_{i,j}(q)) = \lim_{q\to 0^+}\underbrace{q^N\otimes q^N\otimes\cdots\otimes q^N}_{j-i} \neq 0 \qquad \text{if } i<j,$$

$$\lim_{q\to 0^+}\pi_{i,j}^{(q)}(u_{i,j}(q)) = \lim_{q\to 0^+}\underbrace{q^N\otimes q^N\otimes\cdots\otimes q^N}_{i-j} \neq 0 \qquad \text{if } i>j,$$

$$\lim_{q\to 0^+}\pi_{i,i}^{(q)}(u_{i,i}(q)) = 1 \neq 0 \qquad \text{otherwise.}$$

The above proposition says that for each irreducible representation $\psi_{\lambda,\omega}^{(q)}$ of $C(SU_q(n+1))$, the norm limits $\lim_{q\to 0+}\psi_{\lambda,\omega}^{(q)}\circ\theta_q\big((-t)^{\min\{i-j,0\}}u_{i,j}(t)\big)$ where θ_q is the map defined in Subsection 1.3.2, exist and finite.

We now proceed to find the relations satisfied by the limiting operators. For an irreducible representation π of $C(SU_q(n+1))$, let us define

$$Z_{i,j}^{\pi} := \begin{cases} \lim_{q\to 0+}\pi(u_{i,j}(q)) & \text{if } i\geq j,\\ \lim_{q\to 0+}(-q)^{i-j}\pi(u_{i,j}(q)) & \text{if } i<j. \end{cases} \tag{2.2.1}$$

Theorem 2.2.2 *For any irreducible representation π of $C(SU_q(n+1))$, the operators $Z_{i,j}\equiv Z_{i,j}^{\pi}$ satisfy the following commutation relations:*

$$Z_{i,j}Z_{i,l} = 0 \qquad \textit{if } j<l, \tag{2.2.2}$$

$$Z_{i,j}Z_{k,j} = 0 \qquad \textit{if } i<k, \tag{2.2.3}$$

$$\tag{2.2.4}$$

$$Z_{i,l}Z_{k,j} - Z_{k,j}Z_{i,l} = 0 \qquad \textit{if } i<k \textit{ and } j<l. \tag{2.2.5}$$

$$Z_{i,l}Z_{k,j} = 0 \qquad \textit{if } i<k,\ j<l \textit{ and } \max\{i,j\}\geq\min\{k,l\}, \tag{2.2.6}$$

$$Z_{i,j}Z_{k,l} - Z_{k,l}Z_{i,j} = 0 \qquad \textit{if } i<k,\ j<l \textit{ and } \max\{i,j\}+1<\min\{k,l\}, \tag{2.2.7}$$

$$Z_{i,j}Z_{k,l} - Z_{k,l}Z_{i,j} = Z_{i,l}Z_{k,j} \qquad \textit{if } i<k,\ j<l \textit{ and } \max\{i,j\}+1=\min\{k,l\}, \tag{2.2.8}$$

Proof: Let us recall the commutation relations among the generators of $C(SU_q(n+1))$

$$u_{i,j}u_{i,l} = q\ u_{i,l}u_{i,j} \qquad \text{if } j < l, \tag{2.2.9}$$

$$u_{i,j}u_{k,j} = q\ u_{k,j}u_{i,j} \qquad \text{if } i < k, \tag{2.2.10}$$

$$u_{i,l}u_{k,j} = u_{k,j}u_{i,l} \qquad \text{if } i < k \text{ and } j < l, \tag{2.2.11}$$

$$u_{i,j}u_{k,l} - u_{k,l}u_{i,j} = (q - q^{-1})\ u_{i,l}u_{k,j}. \qquad \text{if } i < k \text{ and } j < l \tag{2.2.12}$$

Let $F_{i,j}(q)$ and $C_{i,j}$ be scalars such that

$$F_{i,j}(q) = \begin{cases} q^{i-j} & \text{if } i < j \\ 1 & \text{otherwise,} \end{cases} ; \qquad C_{i,j} = \begin{cases} (-1)^{i-j} & \text{if } i < j \\ 1 & \text{otherwise.} \end{cases}$$

Therefore for $i < j$ we have $\lim_{q\to 0+} \frac{1}{F_{i,j}(q)} = 0$.

Multiplying both sides of (2.2.9) with $C_{i,j}C_{i,l}F_{i,j}(q)F_{i,l}(q)$, multiplying both sides of (2.2.10) with $C_{i,j}C_{k,j}F_{i,j}(q)F_{k,j}(q)$ and multiplying both sides of (2.2.11) with $C_{i,l}C_{k,j}F_{i,l}(q)F_{k,j}(q)$ and taking limit we get the relations (2.2.2), (2.2.3) and (2.2.5). Relation (2.2.6) is obtained by multiplying both sides of (2.2.12) with $q{\cdot}C_{i,j}C_{k,l}F_{i,j}(q)F_{k,l}(q)$ and taking limit. Now multiplying both sides of the relation (2.2.12) with $C_{i,j}C_{k,l}F_{i,j}(q)F_{k,l}(q)$ and taking limit we get relations (2.2.7) and (2.2.8). □

Proposition 2.2.3 *For any irreducible representation π of $C(SU_q(n+1))$, one has*

$$Z_{1,1}Z_{2,2}\ldots Z_{n+1,n+1} = I. \tag{2.2.13}$$

Proof: From quantum determinant condition one gets $\pi(\mathscr{D}_q) = I$ and the result follows by taking limit as $q \to 0+$. □

Lemma 2.2.4 *Let $1 \le s < r \le n+1$. Let*

$$i_k = \begin{cases} k & \text{if } 1 \le k < r, \\ k+1 & \text{if } r \le k \le n, \end{cases} \qquad j_k = \begin{cases} k & \text{if } 1 \le k < s, \\ k+1 & \text{if } s \le k \le n, \end{cases}$$

Let $\sigma \in \mathscr{S}_n$. Then

$$\sum_{k:j_{\sigma(k)}>i_k} (j_{\sigma(k)} - i_k) \ge r - s.$$

Proof: The required inequality follows from the following simple computation:

$$\begin{aligned}
\sum_{k:j_{\sigma(k)}>i_k} (j_{\sigma(k)} - i_k) &\geq \sum_{k=1}^{n} (j_{\sigma(k)} - i_k) \\
&= \sum_{k=1}^{n} j_{\sigma(k)} - \sum_{k=1}^{n} i_k \\
&= \left(\frac{(n+1)(n+2)}{2} - s\right) - \left(\frac{(n+1)(n+2)}{2} - r\right) \\
&= r - s.
\end{aligned}$$

□

Proposition 2.2.5 *For any irreducible representation π of $C(SU_q(n+1))$, one has*

$$Z_{r,s}^* = \begin{cases} Z_{1,1} \ldots Z_{s-1,s-1} Z_{s,s+1} Z_{s+1,s+2} \ldots Z_{r-1,r} Z_{r+1,r+1} \ldots Z_{n+1,n+1} & \text{if } r > s, \\ Z_{1,1} \ldots Z_{r-1,r-1} Z_{r+1,r} Z_{r+2,r+1} \ldots Z_{s,s-1} Z_{s+1,s+1} \ldots Z_{n+1,n+1} & \text{if } r < s, \\ Z_{1,1} \ldots Z_{s-1,s-1} Z_{s+1,s+1} Z_{s+2,s+2} \ldots Z_{n+1,n+1} & \text{if } r = s. \end{cases} \tag{2.2.14}$$

Proof: We will denote the operator $\pi(u_{i,j}(q))$ by $Y_{i,j}^{\pi} \equiv Y_{i,j}^{\pi}(q)$ in this proof. we get

$$(Y_{r,s}^{\pi})^* = (-q)^{s-r} \sum_{\sigma \in \mathscr{S}_n} (-q)^{\ell(\sigma)} Y_{i_1, j_{\sigma(1)}}^{\pi} \ldots Y_{i_n, j_{\sigma(n)}}^{\pi} \tag{2.2.15}$$

$$= (-q)^{s-r} \sum_{\widetilde{\sigma} \in \mathscr{P}_{n+1,r,s}} (-q)^{\ell(\sigma)} Y_{1,\widetilde{\sigma}(1)}^{\pi} \ldots Y_{r-1,\widetilde{\sigma}(r-1)}^{\pi} Y_{r+1,\widetilde{\sigma}(r+1)}^{\pi} \ldots Y_{n+1,\widetilde{\sigma}(n+1)}^{\pi}, \tag{2.2.16}$$

where i_k and j_k are as in above Lemma and $\widetilde{\sigma} \in \mathscr{S}_{n+1}$ is the permutation that takes $i_k \mapsto j_{\sigma(k)}$ for all $k \in \{1, 2, \ldots, n\}$ and takes r to s. Consider the permutation $\widetilde{id_{r,s}}$

such that

$$\widetilde{id_{r,s}}(i) = \begin{cases} i & \text{if} \quad i < min\{r,s\} \\ i & \text{if} \quad i > max\{r,s\} \\ i-1 & \text{if} \quad r < i \leq s \\ i+1 & \text{if} \quad s \leq i < r \\ s & \text{if} \quad r = i \end{cases}$$

Now let us look at the case $r < s$ first. In this case, from Equation 1.3.7 one has

$$\begin{aligned} (-q)^{r-s}(Y^\pi_{r,s})^* &= \sum_{\widetilde{\sigma}\in\mathscr{P}_{n+1,r,s}} (-q)^{\ell(\sigma)} Y^\pi_{1,\widetilde{\sigma}(1)} \dots Y^\pi_{r-1,\widetilde{\sigma}(r-1)} Y^\pi_{r+1,\widetilde{\sigma}(r+1)} \dots Y^\pi_{n+1,\widetilde{\sigma}(n+1)} \\ &= Y^\pi_{1,1}Y^\pi_{2,2}\dots Y^\pi_{r-1,r-1}Y^\pi_{r+1,r}\dots Y^\pi_{s,s-1}Y^\pi_{s+1,s+1}\dots Y^\pi_{n+1,n+1} \\ &+ \sum_{\substack{\widetilde{\sigma}\in\mathscr{P}_{n+1,r,s} \\ \widetilde{\sigma}\neq\widetilde{id_{r,s}}}} (-q)^{\ell(\sigma)} Y^\pi_{1,\widetilde{\sigma}(1)} \dots Y^\pi_{r-1,\widetilde{\sigma}(r-1)} Y^\pi_{r+1,\widetilde{\sigma}(r+1)} \dots Y^\pi_{n+1,\widetilde{\sigma}(n+1)}. \end{aligned}$$

Taking limit as $q \to 0+$, one gets

$$Z^*_{r,s} = Z_{1,1}\dots Z_{r-1,r-1}Z_{r+1,r}Z_{r+2,r+1}\dots Z_{s,s-1}Z_{s+1,s+1}\dots Z_{n+1,n+1}.$$

For $r = s$, from Equation 1.3.7 one has

$$\begin{aligned} (Y^\pi_{s,s})^* &= \sum_{\widetilde{\sigma}\in\mathscr{P}_{n+1,s,s}} (-q)^{\ell(\sigma)} Y^\pi_{1,\widetilde{\sigma}(1)} \dots Y^\pi_{s-1,\widetilde{\sigma}(s-1)} Y^\pi_{s+1,\widetilde{\sigma}(s+1)} \dots Y^\pi_{n+1,\widetilde{\sigma}(n+1)} \\ &= Y^\pi_{1,1}Y^\pi_{2,2}\dots Y^\pi_{s-1,s-1}Y^\pi_{s+1,s+1}\dots Y^\pi_{n+1,n+1} \\ &+ \sum_{\substack{\sigma\in\mathscr{S}_n \\ \widetilde{\sigma}\neq\widetilde{id_{r,s}}}} (-q)^{\ell(\sigma)} Y^\pi_{1,\widetilde{\sigma}(1)} \dots Y^\pi_{s-1,\widetilde{\sigma}(s-1)} Y^\pi_{s+1,\widetilde{\sigma}(s+1)} \dots Y^\pi_{n+1,\widetilde{\sigma}(n+1)}. \end{aligned}$$

Taking limit as $q \to 0+$, one gets

$$Z^*_{s,s} = Z_{1,1}\dots Z_{s-1,s-1}Z_{s+1,s+1}Z_{s+2,s+2}\dots Z_{n+1,n+1}.$$

For $r > s$, from Equation 1.3.7 one has

$$\begin{aligned}
(Y^\pi_{r,s})^* = & \quad (-q)^{s-r} \sum_{\widetilde{\sigma}\in\mathscr{P}_{n+1,r,s}} (-q)^{\ell(\sigma)} Y^\pi_{1,\widetilde{\sigma}(1)} \dots Y^\pi_{r-1,\widetilde{\sigma}(r-1)} Y^\pi_{r+1,\widetilde{\sigma}(r+1)} \dots Y^\pi_{n+1,\widetilde{\sigma}(n+1)} \\
= & \quad (-q)^{s-r} Y^\pi_{1,1} Y^\pi_{2,2} \dots Y^\pi_{s-1,s-1} Y^\pi_{s,s+1} \dots Y^\pi_{r-1,r} Y^\pi_{r+1,r+1} \dots Y^\pi_{n+1,n+1} \\
& + (-q)^{s-r} \sum_{\substack{\widetilde{\sigma}\in\mathscr{P}_{n+1,r,s} \\ \widetilde{\sigma}\neq\widetilde{id_{r,s}}}} (-q)^{\ell(\sigma)} Y^\pi_{1,\widetilde{\sigma}(1)} \dots Y^\pi_{r-1,\widetilde{\sigma}(r-1)} Y^\pi_{r+1,\widetilde{\sigma}(r+1)} \dots Y^\pi_{n+1,\widetilde{\sigma}(n+1)}.
\end{aligned}$$

Taking limit as $q \to 0+$ and by Lemma 2.2.4, it follows that

$$Z^*_{r,s} = Z_{1,1} \dots Z_{s-1,s-1} Z_{s,s+1} Z_{s+1,s+2} \dots Z_{r-1,r} Z_{r+1,r+1} \dots Z_{n+1,n+1}.$$

This completes the proof. □

Proposition 2.2.6 *For any irreducible representation π of $C(SU_q(n+1))$, the operators $Z_{i,j} \equiv Z^\pi_{i,j}$ satisfy the following relations:*

$$Z^*_{i,j} Z_{r,s} = Z_{r,s} Z^*_{i,j}, \qquad i \neq r,\ j \neq s. \tag{2.2.17}$$

Proof: Let us recall a relation involving involution among the generators of $C(SU_q(n+1))$

$$u_{i,j} u^*_{r,s} = u^*_{r,s} u_{i,j}, \qquad i \neq r,\ j \neq s. \tag{2.2.18}$$

Therefore the result is immediate from (2.2.18) and Proposition 2.2.1. □

Theorem 2.2.7 *There is a universal C^*-algebra generated by the elements $z_{i,j}$, $1 \leq i,j \leq n+1$ satisfying the following relations:*

$$z_{i,j} z_{i,l} = 0 \qquad \text{if } j < l, \tag{2.2.19}$$

$$z_{i,j} z_{k,j} = 0 \qquad \text{if } i < k, \tag{2.2.20}$$

$$z_{i,l} z_{k,j} - z_{k,j} z_{i,l} = 0 \qquad \text{if } i < k \text{ and } j < l. \tag{2.2.21}$$

$$z_{i,l} z_{k,j} = 0 \qquad \text{if } i < k,\ j < l, \max\{i,j\} \geq \min\{k,l\}, \tag{2.2.22}$$

$$z_{i,j}z_{k,l} - z_{k,l}z_{i,j} = 0 \qquad \text{if } i<k,\ j<l, \max\{i,j\}+1 < \min\{k,l\}, \tag{2.2.23}$$

$$z_{i,j}z_{k,l} - z_{k,l}z_{i,j} = z_{i,l}z_{k,j} \qquad \text{if } i<k,\ j<l, \max\{i,j\}+1 = \min\{k,l\}, \tag{2.2.24}$$

$$z_{1,1}z_{2,2}\cdots z_{n+1,n+1} = 1, \tag{2.2.25}$$

$$z_{r,s}^* = \begin{cases} (z_{1,1}\cdots z_{s-1,s-1})\,(z_{s,s+1}z_{s+1,s+2}\cdots z_{r-1,r})\,(z_{r+1,r+1}\cdots z_{n+1,n+1}) & \text{if } r>s, \\ (z_{1,1}\cdots z_{r-1,r-1})\,(z_{r+1,r}z_{r+2,r+1}\cdots z_{s,s-1})\,(z_{s+1,s+1}\cdots z_{n+1,n+1}) & \text{if } r<s, \\ (z_{1,1}\cdots z_{s-1,s-1})\,(z_{s+1,s+1}z_{s+2,s+2}\cdots z_{n+1,n+1}) & \text{if } r=s. \end{cases} \tag{2.2.26}$$

$$z_{i,j}^* z_{r,s} = z_{r,s} z_{i,j}^*, \qquad i \neq r,\ j \neq s. \tag{2.2.27}$$

Proof: Let $\mathcal{H}$ be the Hilbert space $\ell^2(\mathbb{N})$. Let $1 \le k \le n$ and let

$$\psi_{s_k}^{(0)}(z_{i,j}) = \begin{cases} S & \text{if } i=j=k, \\ S^* & \text{if } i=j=k+1, \\ |e_0\rangle\langle e_0| & \text{if } i=k,\ \ j=k+1, \\ |e_0\rangle\langle e_0| & \text{if } i=k+1,\ \ j=k, \\ \delta_{i,j} I & \text{otherwise}, \end{cases}$$

where $|e_0\rangle\langle e_0|$ is the projection onto the span of e_0. Then $\psi_{s_k}^{(0)}$ gives a representation of the above relations. It follows from the relations (2.2.22–2.2.26) that

$$\sum_{k=1}^{j} z_{j,k} z_{j,k}^* = \sum_{k=j}^{n+1} z_{k,j}^* z_{k,j} = 1, \qquad 1 \le i \le n+1. \tag{2.2.28}$$

Hence for any representation π of the relations, we have $\|\pi(z_{i,j})\| \le 1$ for all $i \ge j$. Using (2.2.27), one then concludes that $\|\pi(z_{i,j})\| \le 1$ for all $i < j$. Thus $\|\pi(z_{i,j})\| \le 1$ for all i, j. The rest of the proof is now standard and follows from the fact that for any polynomial a in the $z_{i,j}$'s and $z_{i,j}^*$'s, there is a positive real K such that $\|\pi(a)\| \le K$ for any representation π of the relations. □

The C^*-algebra in the above theorem is defined to be the crystallization of the family of C^*-algebras $C(SU_q(n+1))$. This will be denoted by $C(SU_0(n+1))$. The *-algebra given by the above set of relations is similarly defined as the crystallization of the quantized algebra of regular functions $\mathcal{O}(SU_q(n+1))$. We will denote this *-algebra by $\mathcal{O}(SU_0(n+1))$.

In the next two chapters, we will study the irreducible representations of their crystallized algebras.

CHAPTER 3

Irreducible representations of $C(SU_0(3))$

Let us define

$$Z_{i,j} = \lim_{q\to 0+} (-q)^{\min\{i-j,0\}} \psi_{\lambda,\omega}^{(q)}(u_{i,j}(q)), \tag{3.0.1}$$

where $\psi_{\lambda,\omega}^{(q)}$ is the representation of $C(SU_q(n+1))$. It has been shown in (Proposition 2.2.1, Theorem 2.2.2, Proposition 2.2.3 and 2.2.5) that the above limits exist and the operators $Z_{i,j}$ obey the commuation relations, quantum determinant relations and involution relations.

Therefore

$$\psi_{\lambda,\omega}^{(0)}(z_{i,j}) = \lim_{q\to 0+} (-q)^{\min\{i-j,0\}} \psi_{\lambda,\omega}^{(q)}(u_{i,j}(q)), \qquad i,j \in \{1,2,\dots,n+1\} \tag{3.0.2}$$

defines a representation of $C(SU_0(n+1))$ on the Hilbert space $\ell^2(\mathbb{N}^{\ell(\omega)})$, where $\ell(\omega)$ denotes the length of the element ω. If $\lambda = (1,1,\dots,1)$, we will denote $\psi_{\lambda,\omega}^{(0)}$ by just $\psi_{\omega}^{(0)}$.

This chapter is about all irreducible representations of $C(SU_0(3))$ which is mainly from [GP22].

3.1 Irreducible representations of $C(SU_0(3))$

Throughout this chapter we consider $\mathscr{S}_3 = \{$ empty word, $s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$ where s_1 is the transposition $(1,2)$ and s_2 is the transposition $(2,3)$.

Theorem 3.1.1 *For any $(\lambda,\mu) \in (S^1)^2$ and a reduced word ω in $\mathscr{S}_3$, the representation $\psi_{(\lambda,\mu),\omega}$ given by (3.0.2) is irreducible.*

If $((\lambda,\mu),\omega) \neq ((\lambda',\mu'),\omega')$, then the representations $\psi_{(\lambda,\mu),\omega}$ and $\psi_{(\lambda',\mu'),\omega'}$ are inequivalent.

Proof: We will denote the rank one projection $|e_0\rangle\langle e_0|$ by P_0. For $\omega =$ empty word, the representation $\psi_{(\lambda,\mu),\omega}$ is one dimensional and hence irreducible.

Let $\omega = s_1$. In this case, $\psi_{(\lambda,\mu),\omega}$ is given by

$$\begin{array}{lllllllll} z_{1,1} & \mapsto & \lambda S, & z_{1,2} & \mapsto & \lambda P_0 & z_{1,3} & \mapsto & 0 \\ z_{2,1} & \mapsto & \bar{\lambda}\mu P_0 & z_{2,2} & \mapsto & \bar{\lambda}\mu S^* & z_{2,3} & \mapsto & 0 \\ z_{3,1} & \mapsto & 0 & z_{3,2} & \mapsto & 0 & z_{3,3} & \mapsto & \bar{\mu} I. \end{array} \tag{3.1.1}$$

Therefore for any $j,k \in \mathbb{N}$, one has

$$\lambda^{j-k-1}(z_{1,1}^*)^j z_{1,2} z_{1,1}^k \mapsto |e_j\rangle\langle e_k|.$$

Therefore $\mathcal{K}(\ell^2(\mathbb{N})) \subseteq \psi_{(\lambda,\mu),\omega}(C(SU_0(3)))$. This imples $\psi_{(\lambda,\mu),\omega}$ is irreducible.

Proof for $\omega = s_2$ is similar.

Next, let us take $\omega = s_1s_2$. In this case we have

$$\begin{array}{lllllllll} z_{1,1} & \mapsto & \lambda S \otimes I, & z_{1,2} & \mapsto & \lambda P_0 \otimes S & z_{1,3} & \mapsto & \lambda P_0 \otimes P_0 \\ z_{2,1} & \mapsto & \bar{\lambda}\mu P_0 \otimes I & z_{2,2} & \mapsto & \bar{\lambda}\mu S^* \otimes S & z_{2,3} & \mapsto & \bar{\lambda}\mu S^* \otimes P_0 \\ z_{3,1} & \mapsto & 0 & z_{3,2} & \mapsto & \bar{\mu} I \otimes P_0 & z_{3,3} & \mapsto & \bar{\mu} I \otimes S^*. \end{array} \tag{3.1.2}$$

In this case, for any $j,k \in \mathbb{N}$, we have

$$\lambda^{j-k-1}(z_{1,1}^*)^j z_{1,2} z_{1,1}^k \longmapsto |e_j\rangle\langle e_k| \otimes I,$$
$$\mu^{j-k+1} z_{3,3}^j z_{3,2} (z_{3,3}^*)^k \longmapsto I \otimes |e_j\rangle\langle e_k|.$$

Thus $\mathcal{K}(\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N})) \subseteq \psi_{(\lambda,\mu),\omega}(C(SU_0(3)))$, which imples $\psi_{(\lambda,\mu),\omega}$ is irreducible. Proof for $\omega = s_2 s_1$ is similar.

Finally take $\omega = s_1 s_2 s_1$. Here one has

$$\begin{cases} z_{1,1} & \mapsto & \lambda S \otimes I \otimes S, \\ z_{2,1} & \mapsto & \bar{\lambda}\mu P_0 \otimes I \otimes S \\ z_{3,1} & \mapsto & I \otimes P_0 \otimes P_0 \end{cases} \begin{cases} z_{1,2} & \mapsto & \lambda P_0 \otimes S \otimes S^* + S \otimes I \otimes P_0 \\ z_{2,2} & \mapsto & \bar{\lambda}\mu S^* \otimes S \otimes S^* \\ z_{3,2} & \mapsto & \bar{\mu} I \otimes P_0 \otimes S^* \end{cases} \begin{cases} z_{1,3} & \mapsto & \lambda P_0 \otimes P_0 \otimes I \\ z_{2,3} & \mapsto & \bar{\lambda}\mu S^* \otimes P_0 \otimes I \\ z_{3,3} & \mapsto & \bar{\mu} I \otimes S^* \otimes I \end{cases} \tag{3.1.3}$$

Now observe that for any $j, k, m, n \in \mathbb{N}$,

$$\begin{aligned} c_1 z_{3,3}^m \left(z_{2,3}^j z_{1,3} (z_{2,3}^*)^k\right) (z_{3,3}^*)^n &\longmapsto |e_j\rangle \langle e_k| \otimes |e_m\rangle \langle e_n| \otimes I, \\ c_2 z_{3,3}^j \left(z_{3,2}^m z_{3,1} (z_{3,2}^*)^n\right) (z_{3,3}^*)^k &\longmapsto I \otimes |e_j\rangle \langle e_k| \otimes |e_m\rangle \langle e_n|, \end{aligned}$$

where c_1 and c_2 are two scalars involving λ and μ. Therefore it follows that

$$\mathcal{K}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})) \subseteq \psi_{(\lambda,\mu),\omega}(C(SU_0(3))),$$

which imples that $\psi_{(\lambda,\mu),\omega}$ is irreducible.

The inequivalence of different $\psi_{(\lambda,\mu),\omega}$'s follow by comparing the spectrum of the operators $\psi_{(\lambda,\mu),\omega}(z_{i,j})$'s. $\square$

The next theorem states that the ones listed in the previous theorem are in fact all the irreducible representations of $C(SU_0(3))$. The proof is given in the next subsection.

Theorem 3.1.2 *Let π be an irreducible representation of $C(SU_0(3))$ on a Hilbert space $\mathcal{H}$. Then there exist $\lambda, \mu \in S^1$ and a reduced word $\omega \in \mathscr{S}_3$ such that π is equivalent to $\psi_{(\lambda,\mu),\omega}$.*

Corollary 3.1.3 *Theorem 3.1.1 and Theorem 3.1.2 together gives us the family of all inequivalent irreducible representations of the C^*-algebra $C(SU_0(3))$. They are naturally parametrized exactly as their counterparts for nonzero q, and in fact they arise as limits of the irreducible representations of the C^*-algebras $C(SU_q(3))$ as $q \to 0+$.*

Corollary 3.1.4 *In the course of proof of Theorem 3.1.1, we have also observed that for any irreducible representation π of $C(SU_0(3))$ on a Hilbert space $\mathcal{H}_\pi$, one has $\mathcal{K}(\mathcal{H}_\pi) \subseteq \pi(C(SU_0(3)))$. Thus $C(SU_0(3))$ is a type I C^*-algebra.*

3.1.1 Proof of Theorem 3.1.2

We first prove a few lemmas on the generating elements and certain products of them that will be needed in the proof.

We will first show that the generators $z_{i,j}$ and certain products of them are partial isometries. This will be needed in the computations that follow in obtaining all the irreducible representations.

Define $p_{i,j} = z_{i,j}^* z_{i,j}$ and $q_{i,j} = z_{i,j} z_{i,j}^*$. In terms of the $p_{i,j}$'s and $q_{i,j}$'s, the relations can be written as

$$\begin{aligned} p_{1,1} + p_{2,1} + p_{3,1} &= 1, & q_{3,1} + q_{3,2} + q_{3,3} &= 1, \\ p_{2,2} + p_{3,2} &= 1, & q_{2,1} + q_{2,2} &= 1, \\ p_{3,3} &= 1, & q_{1,1} &= 1, \end{aligned} \tag{3.1.4}$$

We have the following results.

Lemma 3.1.5 ([Erd68], [HW70]) *Let u and v be partial isometries. Then uv is a partial isometry if and only if u^*u and vv^* commute.*

Lemma 3.1.6 *The family $\{p_{i,j}, q_{i,j} : 1 \le j \le i \le 3\}$ is a commuting family of projections.*

Proof: First observe that

$$p_{1,1} = z_{1,1}^* z_{1,1} = z_{2,2} z_{3,3} z_{1,1} = z_{2,2} z_{1,1} z_{3,3} = z_{2,2} z_{2,2}^* = q_{2,2}.$$

Since $q_{1,1} = 1$, it follows that $p_{1,1} = q_{2,2}$ is a projection. From (3.1.4), it now follows that $p_{2,2}$ and $q_{2,1}$ are projections. Consequently $p_{3,2} = 1 - p_{2,2}$ is a projection. Since $q_{2,1}$ is a projection, $p_{2,1}$ also is, and hence so is $p_{3,1} = 1 - p_{2,1} - p_{1,1}$. Thus $p_{i,j}$ is a projection for all $i \ge j$. Hence $q_{i,j}$ also is a projection for all $i \ge j$.

Note that

$$p_{3,1} = z_{3,1}^* z_{3,1} = z_{1,2} z_{2,3} z_{3,1} = z_{1,2} z_{3,1} z_{2,3} = z_{3,1} z_{1,2} z_{2,3} = z_{3,1} z_{3,1}^* = q_{3,1}.$$

From the fact that $p_{1,1} = q_{2,2}$ and from the equalities (3.1.4), we get

$$p_{3,1} = q_{3,1}, \quad p_{3,2} = q_{3,2} + q_{3,1}, \quad p_{3,3} = q_{3,3} + q_{3,2} + q_{3,1}, \tag{3.1.5}$$

$$p_{1,1} = q_{2,2}, \quad p_{2,1} = q_{2,1} - q_{3,1}, \quad p_{2,2} = q_{3,3}. \tag{3.1.6}$$

Therefore it is now enough to show that the family $\{q_{i,j} : 1 \le j \le i \le 3\}$ is commuting.

Since the $q_{i,j}$'s are projections for $i \ge j$, $z_{i,j}$ is a partial isometry for $i \ge j$. Since $q_{1,1} = I$ and $q_{2,2}$ commute, it follows from the lemma above that $z_{2,2}^* z_{1,1}$ is a partial isometry. Since $z_{2,2}^* z_{1,1} = z_{1,1} z_{2,2}^*$, one more application of the lemma tells us that $p_{1,1}$ and $p_{2,2}$ commute, i.e. $q_{2,2}$ commutes with $q_{3,3}$. From the relation $z_{2,2} z_{3,1} = z_{3,1} z_{2,2}$, it follows that $q_{2,2}$ commutes with $p_{3,1} = q_{3,1}$. Hence $q_{2,2}$ commutes with $q_{3,2} = 1 - q_{3,1} - q_{3,3}$. It now follows that the family $\{q_{i,j} : 1 \le j \le i \le 3\}$ is commuting. □

From these two lemmas we have the following corollary.

Corollary 3.1.7 *Any product of the form ab, where $a, b \in \{z_{i,j} : i \ge j\} \cup \{z_{i,j}^* : i \ge j\}$, is a partial isometry.*

Now come to the proof of Theorem 3.1.2. We will split it into different cases, corresponding to the different words in the Weyl group $\mathscr{S}_3$ for the type A_2 case. Let us start with an irreducible representation π of $C(SU_0(3))$ on a Hilbert space $\mathcal{H}$. We will denote by $P_{i,j}$ and $Q_{i,j}$ the operators $\pi(p_{i,j})$ and $\pi(q_{i,j})$ respectively. It is easy to see that exactly one of the following conditions will hold for the irreducible representation π:

1. $\pi(z_{3,1}) = 0$, $\pi(z_{3,2}) = 0$, $\pi(z_{2,1}) = 0$,
2. $\pi(z_{3,1}) = 0$, $\pi(z_{3,2}) = 0$, $\pi(z_{2,1}) \neq 0$,
3. $\pi(z_{3,1}) = 0$, $\pi(z_{3,2}) \neq 0$, $\pi(z_{2,1}) = 0$,
4. $\pi(z_{3,1}) = 0$, $\pi(z_{3,2}) \neq 0$, $\pi(z_{2,1}) \neq 0$,
5. $\pi(z_{3,1}) \neq 0$, $\pi(z_{1,3}^*) = \pi(z_{2,1})\pi(z_{3,2}) = 0$,

6. $\pi(z_{3,1}) \neq 0$, $\pi(z_{1,3}^*) = \pi(z_{2,1})\pi(z_{3,2}) \neq 0$.

We will deal with the above cases separately. As we will see, these cases correspond to the six elements of the Weyl group $\mathscr{S}_3$.

Case 1: $\pi(z_{3,1}) = 0$**,** $\pi(z_{3,2}) = 0$**,** $\pi(z_{2,1}) = 0$**.**

From the commutation relations it follows that $\pi(z_{i,j}) = 0$ for all $i \neq j$. Therefore $\pi(z_{i,i})$ are unitary with $\pi(z_{1,1})\pi(z_{2,2}) = \pi(z_{3,3}^*)$ and $\pi(z_{1,1})$ commutes with $\pi(z_{2,2})$. Irreducibility of π now implies that $\mathcal{H}$ is one dimensional and there exist $\lambda, \mu \in S^1$ such that $\pi(z_{1,1}) = \lambda$, $\pi(z_{2,2}) = \mu$ and $\pi(z_{3,3}) = \overline{\lambda\mu}$. Here we take $\omega =$ empty word. Thus π is unitarily equivalent to $\psi_{(\lambda,\lambda\mu),\omega} := \chi_{(\lambda,\lambda\mu)}$.

Case 2: $\pi(z_{3,1}) = 0$**,** $\pi(z_{3,2}) = 0$**,** $\pi(z_{2,1}) \neq 0$**.**

From the commutation relations it follows that

1. $\pi(z_{1.3}) = \pi(z_{2,3}) = 0$,
2. $\pi(z_{3,3})$ is a unitary,
3. $\pi(z_{2,1})$ is normal,
4. $\pi(z_{2,1})$ commutes with $\pi(z_{3,3})$ and $\pi(z_{1,2})$.

Theorem 3.1.8 *Let P be a projection such that*

$$P \leq P_{2,1}, \qquad P\pi(z_{2,1}) = \pi(z_{2,1})P, \qquad P\pi(z_{3,3}) = \pi(z_{3,3})P.$$

Then $\mathcal{H}_P := \{\pi(z_{2,2})^n\xi : n \in \mathbb{N}, \xi \in P\mathcal{H}\}$ is an invariant subspace for π.

If Q is another projection orthogonal to P satisfying the above conditions, then $\mathcal{H}_P$ and $\mathcal{H}_Q$ are orthogonal.

Proof: From the commutation relations, it follows that

$$\pi(z_{1,1})\left(\pi(z_{2,2})^n\xi\right) = \begin{cases} \pi(z_{1,1})\xi & \text{if } n = 0, \\ \pi(z_{3,3})^*\pi(z_{2,2})^{n-1}\xi & \text{if } n > 0, \end{cases} \tag{3.1.7}$$

$$\pi(z_{1,2})\left(\pi(z_{2,2})^n\xi\right) = \begin{cases} \pi(z_{1,2})\xi & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{3.1.8}$$

$$\pi(z_{2,1})\left(\pi(z_{2,2})^n\xi\right) = \begin{cases} \pi(z_{2,1})\xi & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{3.1.9}$$

$$\pi(z_{2,2})\left(\pi(z_{2,2})^n\xi\right) = \pi(z_{2,2})^{n+1}\xi, \tag{3.1.10}$$

$$\pi(z_{3,3})\left(\pi(z_{2,2})^n\xi\right) = \pi(z_{2,2})^n\ \pi(z_{3,3})\xi. \tag{3.1.11}$$

Since $P \leq P_{2,1}$, we have $\xi = P_{2,1}\xi = Q_{2,1}\xi$ for all $\xi \in \mathcal{H}_P$. Hence

$$\pi(z_{1,1})\xi = \pi(z_{1,1})\pi(z_{2,1})\left(\pi(z_{2,1}^*)\xi\right) = 0.$$

Thus $\mathcal{H}_P$ is invariant under $\pi(z_{1,1})$. Since $z_{1,2}^* = z_{2,1}z_{3,3}$, $\pi(z_{1,2})$ commutes with P and we have $\pi(z_{1,2})\xi = \pi(z_{1,2})P\xi = P\pi(z_{1,2})\xi$. Thus $\mathcal{H}_P$ is invariant under $\pi(z_{1,2})$. Similarly, since $\pi(z_{2,1})$ and $\pi(z_{3,3})$ both commute with P, we have the invariance of $\mathcal{H}_P$ under the actions of $\pi(z_{2,1})$ and $\pi(z_{3,3})$.

From the commutation relations, it follows that $\mathcal{H}_P$ is invariant under π.

For the second part, take $\xi \in P\mathcal{H}$ and $\zeta \in Q\mathcal{H}$.

Then for $n \in \mathbb{N}$, one has

$$\begin{aligned} \langle \pi(z_{2,2})^n\zeta, \pi(z_{2,2})^n\xi\rangle &= \langle \pi(z_{2,2}^*)^n\pi(z_{2,2})^n\zeta, \xi\rangle \\ &= \langle \zeta, \xi\rangle \\ &= \langle Q\zeta, P\xi\rangle \\ &= 0. \end{aligned}$$

Next, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} \langle \pi(z_{2,2})^m\zeta, \pi(z_{2,2})^n\xi\rangle &= \langle \zeta, \pi(z_{2,2}^*)^m\pi(z_{2,2})^n\xi\rangle \\ &= \langle \zeta, \pi(z_{2,2}^*)^{m-n}\xi\rangle \end{aligned}$$

$$= \langle \zeta, \pi(z_{3,3})^{m-n}\pi(z_{1,1})^{m-n}\xi\rangle$$
$$= 0.$$

A similar calculation gives $\langle \pi(z_{2,2})^m\zeta, \pi(z_{2,2})^n\xi\rangle = 0$ for $m < n$. Thus $\mathcal{H}_P$ and $\mathcal{H}_Q$ are orthogonal. □

Corollary 3.1.9 *There exists a $\lambda \in S^1$ such that $\sigma(\pi(z_{2,1})) = \{0, \lambda\}$.*

Proof: Let E and F be two disjoint closed subsets of the spectrum $\sigma(\pi(z_{2,1}))$ such that $0 \notin E$, $0 \notin F$ and let P and Q be the spectral projections of $\pi(z_{2,1})$ corresponding to the subsets E and F respectively. Then it follows from the previous proposition that $\mathcal{H}_P$ and $\mathcal{H}_Q$ are two invariant subspaces for π that are mutually orthogonal. By irreducibility of π, it follows that $\sigma(\pi(z_{2,1})) = \{0, \lambda\}$ for some $\lambda \neq 0$ in $\mathbb{C}$. Since $\pi(z_{2,1})$ is a normal partial isometry, it follows that $\lambda \in S^1$. □

Corollary 3.1.10 *There exists a $\mu \in S^1$ such that $\sigma(\pi(z_{3,3})) = \{\mu\}$.*

Proof: From the commutation relations, it follows that $\pi(z_{3,3})$ is a unitary and commutes with $P_{2,1}$. Let P and Q be two spectral projections corresponding to two disjoint closed subsets of the spectrum of the restriction of $\pi(z_{3,3})$ to $P_{2,1}\mathcal{H}$. Then the above proposition tells us that $\mathcal{H}_P$ and $\mathcal{H}_Q$ are two invariant subspaces for π that are mutually orthogonal. Using irreducibility of π, we deduce that $\sigma(\pi(z_{3,3})|_{P_{2,1}\mathcal{H}}) = \{\mu\}$ where $\mu \in S^1$. Hence for $\xi \in P_{2,1}\mathcal{H}$, we have

$$\pi(z_{3,3}^*)(\pi(z_{2,2}^n)\xi) = (\pi(z_{2,2}^n)\pi(z_{3,3}^*)\xi = \bar{\mu}\pi(z_{2,2}^n)\xi.$$

Since $\mathcal{H}_{P_{2,1}} = \mathcal{H}$, we have the result. □

Theorem 3.1.11 *The projection $P_{2,1}$ is of rank one and π is unitarily equivalent to the representation $\psi_{(\bar{\lambda}\bar{\mu},\bar{\mu}),s_1}$.*

Proof: Take a unit vector $\xi \in P_{2,1}\mathcal{H}$. From the previous two corollaries, it follows that $\pi(z_{3,3})\xi = \mu\xi$ and $\pi(z_{2,1})\xi = \lambda\xi$. Therefore $\pi(z_{1,2})\xi = \pi(z_{3,3}^* z_{2,1}^*)\xi = \bar{\lambda}\bar{\mu}\xi$. Let

$$\mathcal{H}_\xi = \text{span}\,\{\pi(z_{2,2})^n\xi : n \in \mathbb{N}\}.$$

From equations, it now follows that

$$\pi(z_{1,1})\left(\pi(z_{2,2})^n\xi\right) = \begin{cases} 0 & \text{if } n = 0, \\ \bar{\mu}\pi(z_{2,2})^{n-1}\xi & \text{if } n > 0, \end{cases} \tag{3.1.12}$$

$$\pi(z_{1,2})\left(\pi(z_{2,2})^n\xi\right) = \begin{cases} \bar{\lambda}\bar{\mu}\xi & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{3.1.13}$$

$$\pi(z_{2,1})\left(\pi(z_{2,2})^n\xi\right) = \begin{cases} \lambda\xi & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{3.1.14}$$

$$\pi(z_{2,2})\left(\pi(z_{2,2})^n\xi\right) = \pi(z_{2,2})^{n+1}\xi, \tag{3.1.15}$$

$$\pi(z_{3,3})\left(\pi(z_{2,2})^n\xi\right) = \mu\pi(z_{2,2})^n\xi. \tag{3.1.16}$$

Thus $\mathcal{H}_\xi$ is an invariant subspace of $\mathcal{H}$ and hence by irreducibility of π, one has $\mathcal{H}_\xi = \mathcal{H}$.

The map

$$U : e_n \mapsto \lambda^{-n}\pi(z_{2,2})^n\xi$$

from $\ell^2(\mathbb{N})$ to $\mathcal{H}$ now extends to a unitary and gives us the required unitary equivalence.

□

Case 3: $\pi(z_{3,1}) = 0$**,** $\pi(z_{3,2}) \neq 0$**,** $\pi(z_{2,1}) = 0$**.**

Analogous to the results in the previous case, here we have the following results. The proofs are similar.

Theorem 3.1.12 *Let P be a projection such that*

$$P \leq P_{3,2}, \qquad P\pi(z_{3,2}) = \pi(z_{3,2})P, \qquad P\pi(z_{1,1}) = \pi(z_{1,1})P.$$

Then $\mathcal{H}_P := \{\pi(z_{3,3})^n\xi : n \in \mathbb{N}, \xi \in P\mathcal{H}\}$ is an invariant subspace for π.

If Q is another projection orthogonal to P satisfying the conditions above, then $\mathcal{H}_P$ and $\mathcal{H}_Q$ are orthogonal.

Corollary 3.1.13 *There exists a $\lambda \in S^1$ such that $\sigma(\pi(z_{3,2})) = \{0, \lambda\}$.*

Corollary 3.1.14 *There exists a $\mu \in S^1$ such that $\sigma(\pi(z_{1,1})) = \{\mu\}$.*

Theorem 3.1.15 *The projection $P_{3,2}$ is of rank one and π is unitarily equivalent to the representation $\psi_{(\mu,\bar{\lambda}),s_2}$.*

Case 4: $\pi(z_{3,1}) = 0$, $\pi(z_{3,2}) \neq 0$, $\pi(z_{2,1}) \neq 0$.

Let us first prove the following lemma.

Lemma 3.1.16 *If $\pi(z_{3,1}) = 0$ and $\pi(z_{2,1})\pi(z_{3,2}) = 0$, then either $\pi(z_{2,1}) = 0$ or $\pi(z_{3,2}) = 0$.*

Proof: Let us assume that $\pi(z_{3,1}) = 0$, $\pi(z_{2,1})\pi(z_{3,2}) = 0$ and $\pi(z_{2,1}) \neq 0$. We will show that $\pi(z_{3,2}) = 0$. Since $\pi(z_{3,1}) = 0$, it follows from commutation relations that $\pi(z_{1,1})\pi(z_{3,2}) = \pi(z_{3,2})\pi(z_{1,1})$. Since $z_{2,3}$ commutes with $z_{3,2}$, we get

$$\pi(z_{3,2})\pi(z_{3,2}^*) = \pi(z_{3,2}z_{1,1}z_{2,3}) = \pi(z_{1,1}z_{2,3}z_{3,2}) = \pi(z_{3,2}^* z_{3,2}),$$

i.e. $\pi(z_{3,2})$ is normal. Since $\pi(z_{2,1})^*$ commutes with $\pi(z_{3,2})$, it follows that $\pi(z_{2,1})$ commutes with $\pi(z_{3,2})$.

Let $\mathcal{H}_0$ be the closed linear span of $\{\pi(z_{2,2}^k)\pi(z_{2,1})\xi : k \in \mathbb{N}, \xi \in \mathcal{H}\}$. Using the commutation relations and the fact that $\pi(z_{3,1}) = 0$, we then have

$$\pi(z_{1,1})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = \begin{cases} 0 & \text{if } k = 0, \\ \pi(z_{2,2}^{k-1})\pi(z_{2,1})\left(\pi(z_{3,3}^*)\xi\right) & \text{if } k > 0. \end{cases}$$

$$\pi(z_{1,2})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = \begin{cases} \pi(z_{2,1})\left(\pi(z_{1,2})\xi\right) & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

$$\pi(z_{1,3})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = \begin{cases} \pi(z_{2,1})\left(\pi(z_{1,3})\xi\right) & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

$$\pi(z_{2,1})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = \begin{cases} \pi(z_{2,1})\left(\pi(z_{2,1})\xi\right) & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

$$\pi(z_{2,2})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = \pi(z_{2,2}^{k+1})\pi(z_{2,1})\xi.$$

$$\pi(z_{2,3})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = 0.$$

$$\pi(z_{3,2})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = 0.$$

$$\pi(z_{3,3})\left(\pi(z_{2,2}^k)\pi(z_{2,1})\xi\right) = \pi(z_{2,2}^k)\pi(z_{2,1})\left(\pi(z_{3,3})\xi\right).$$

Therefore $\mathcal{H}_0$ is invariant subspace. Since π is irreducible, we have $\mathcal{H} = \mathcal{H}_0$. Therefore $\pi(z_{3,2}) = 0$. □

Theorem 3.1.17 *Let P be a projection such that*

$$P \leq P_{1,3}, \qquad P\pi(z_{2,1}) = \pi(z_{2,1})P, \qquad P\pi(z_{3,2}) = \pi(z_{3,2})P.$$

Then the subspace $\mathcal{H}_P := \{\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi : m,n \in \mathbb{N}, \xi \in P\mathcal{H}\}$. is invariant for π.

If Q is another projection orthogonal to P satisfying the same conditions above, then $\mathcal{H}_P$ and $\mathcal{H}_Q$ are orthogonal.

Proof: Let $\xi \in P\mathcal{H}$. Then we have

$$\pi(z_{1,1})\xi = 0 = \pi(z_{1,2})\xi = \pi(z_{2,2})\xi = \pi(z_{2,2}^*)\xi = \pi(z_{2,3}^*)\xi = \pi(z_{3,3}^*)\xi. \tag{3.1.17}$$

Therefore the actions of the $\pi(z_{i,j})$'s on $\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi$ are given by

$$\pi(z_{1,1})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} 0 & \text{if } m = 0, \\ \pi(z_{1,1}^*)^{m-1}\pi(z_{3,3})^n\xi & \text{if } m > 0. \end{cases} \tag{3.1.18}$$

$$\pi(z_{1,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} 0 & \text{if } m = 0, n = 0, \\ \pi(z_{3,3})^{n-1}\left(\pi(z_{2,1}^*)\xi\right) & \text{if } m = 0, n > 0, \\ 0 & \text{if } m > 0. \end{cases} \tag{3.1.19}$$

$$\pi(z_{1,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \pi(z_{3,2})^*\pi(z_{2,1}^*)\xi & \text{if } m = 0, n = 0, \\ 0 & \text{if } m = 0, n > 0, \\ 0 & \text{if } m > 0. \end{cases} \tag{3.1.20}$$

$$\pi(z_{2,1})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \pi(z_{3,3})^n\left(\pi(z_{2,1})\xi\right) & \text{if } m=0,\\ 0 & \text{if } m>0. \end{cases} \tag{3.1.21}$$

$$\pi(z_{2,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} 0 & \text{if } n=0,\\ \pi(z_{1,1}^*)^{m+1}\pi(z_{3,3})^{n-1}\xi & \text{if } n>0. \end{cases} \tag{3.1.22}$$

$$\pi(z_{2,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \pi(z_{1,1}^*)^{m+1}\left(\pi(z_{3,2}^*)\xi\right) & \text{if } n=0,\\ 0 & \text{if } n>0. \end{cases} \tag{3.1.23}$$

$$\pi(z_{3,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \pi(z_{1,1}^*)^{m}\left(\pi(z_{3,2})\xi\right) & \text{if } n=0,\\ 0 & \text{if } n>0. \end{cases} \tag{3.1.24}$$

$$\pi(z_{3,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \pi(z_{1,1}^*)^m\pi(z_{3,3})^{n+1}\xi. \tag{3.1.25}$$

As P commutes with $\pi(z_{2,1})$ and $\pi(z_{3,2})$, for any $\xi \in P\mathcal{H}$ we have $\pi(z_{2,1})\xi$, $\pi(z_{2,1}^*)\xi$, $\pi(z_{3,2})\xi$ and $\pi(z_{3,2}^*)\xi$ lies in $P\mathcal{H}$. From above actions we get that $\mathcal{H}_P$ is invariant subspace for π. As π is irreducible, $\mathcal{H} = \mathcal{H}_P$

For the second part, Let us compute the inner product $\langle \pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi,$ $\pi(z_{1,1}^*)^{m'}\pi(z_{3,3})^{n'}\xi'\rangle$ where $\xi \in P\mathcal{H}$ and $\xi' \in Q\mathcal{H}$. In the case $m \neq m'$ or $n \neq n'$, Using the relations $z_{1,1}z_{1,1}^* = 1 = z_{3,3}^*z_{3,3}$, $z_{1,1}z_{3,3} = z_{3,3}z_{1,1}$ and $z_{1,1}z_{3,3}^* = z_{3,3}^*z_{1,1}$, one arrives at a vector of the form $\pi(a)\zeta$ where a is $z_{1,1}^*$ or $z_{3,3}^*$ and ζ is either ξ or ξ', and by (3.1.17), $\pi(a)\zeta = 0$. If $m = m'$ and $n = n'$, the above inner product reduces to $\langle \xi, \xi'\rangle$, is zero. □

Proof of the next two corollaries are similar to the earlier cases and hence omitted.

Corollary 3.1.18 *The operator $\pi(z_{2,1})$ is normal, with $P_{1,3} \le P_{2,1}$ and there exists a $\lambda \in S^1$ such that the restriction of $\pi(z_{2,1})$ to $P_{1,3}\mathcal{H}$ is λI.*

Corollary 3.1.19 *The operator $\pi(z_{3,2})$ is normal, with $P_{1,3} \le P_{3,2}$ and there exists a $\mu \in S^1$ such that the restriction of $\pi(z_{3,2})$ to $P_{1,3}\mathcal{H}$ is μI.*

Theorem 3.1.20 *The representation π is unitarily equivalent to $\psi_{(\bar{\lambda}\bar{\mu},\bar{\mu}),s_1s_2}$.*

Proof: Take a unit vector $\xi \in P_{1,3}\mathcal{H}$. From the previous two corollaries, it follows that $\pi(z_{3,2})\xi = \mu\xi$ and $\pi(z_{2,1}z_{3,2})\xi = \lambda\mu\xi$. Let

$$\mathcal{H}_\xi = \{\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi : m, n \in \mathbb{N}\}.$$

From equations(3.1.18-3.1.25), it now follows that

$$\pi(z_{1,1})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} 0 & \text{if } m=0, \\ \pi(z_{1,1}^*)^{m-1}\pi(z_{3,3})^n\xi & \text{if } m>0. \end{cases} \quad (3.1.26)$$

(3.1.27)

$$\pi(z_{1,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} 0 & \text{if } m=0, n=0, \\ \pi(z_{3,3})^{n-1}\left(\bar{\mu}\xi\right) & \text{if } m=0, n>0, \\ 0 & \text{if } m>0. \end{cases} \quad (3.1.28)$$

(3.1.29)

$$\pi(z_{1,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \bar{\mu}\bar{\lambda}\xi & \text{if } m=0, n=0, \\ 0 & \text{if } m=0, n>0, \\ 0 & \text{if } m>0. \end{cases} \quad (3.1.30)$$

(3.1.31)

$$\pi(z_{2,1})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \pi(z_{3,3})^n\left(\lambda\xi\right) & \text{if } m=0, \\ 0 & \text{if } m>0. \end{cases} \quad (3.1.32)$$

(3.1.33)

$$\pi(z_{2,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} 0 & \text{if } n=0, \\ \pi(z_{1,1}^*)^{m+1}\pi(z_{3,3})^{n-1}\xi & \text{if } n>0. \end{cases} \quad (3.1.34)$$

(3.1.35)

$$\pi(z_{2,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \pi(z_{1,1}^*)^{m+1}\left(\bar{\mu}\xi\right) & \text{if } n=0, \\ 0 & \text{if } n>0. \end{cases} \quad (3.1.36)$$

(3.1.37)

$$\pi(z_{3,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \begin{cases} \pi(z_{1,1}^*)^{m}\left(\bar{\mu}\xi\right) & \text{if } n=0, \\ 0 & \text{if } n>0. \end{cases} \quad (3.1.38)$$

(3.1.39)

$$\pi(z_{3,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi\right) = \pi(z_{1,1}^*)^m\pi(z_{3,3})^{n+1}\xi. \quad (3.1.40)$$

Thus $\mathcal{H}_\xi$ is an invariant subspace of $\mathcal{H}$ and hence by irreducibility of π, one has $\mathcal{H}_\xi = \mathcal{H}$.

The map

$$U : e_{m,n} \mapsto \mu^{-m-n}\lambda^{-m}\pi(z_{1,1}^*)^m\pi(z_{3,3})^n\xi$$

from $\ell^2(\mathbb{N}^2)$ to $\mathcal{H}$ gives us the required unitary. □

Case 5: $\pi(z_{3,1}) \neq 0$, $\pi(z_{2,1})\pi(z_{3,2}) = 0$.

Note that in both Case 5 and Case 6, one has $\pi(z_{3,1}^*) = \pi(z_{1,2})\pi(z_{2,3}) \neq 0$. This implies that $\pi(z_{2,3}) \neq 0$, which in turn implies that $\pi(z_{3,2}) \neq 0$ because $\pi(z_{2,3}^*) = \pi(z_{1,1})\pi(z_{3,2})$.

Theorem 3.1.21 *Let P be a projection such that $P \leq P_{3,1}$ and $P\pi(z_{1,2}) = \pi(z_{1,2})P$, $P\pi(z_{2,3}) = \pi(z_{2,3})P$. Let $\mathcal{H}_P := \{\pi(z_{1,1}^*)^n \pi(z_{3,3})^m \xi : n, m \geq 0, \xi \in P\mathcal{H}\}$. Then $\mathcal{H}_P$ is an invariant subspace for π.*

If Q is another projection orthogonal to P such that $Q\pi(z_{1,2}) = \pi(z_{1,2})Q$ and $Q\pi(z_{2,3}) = \pi(z_{2,3})Q$, then $\mathcal{H}_P$ and $\mathcal{H}_Q$ are orthogonal.

Corollary 3.1.22 *The operator $\pi(z_{1,2})$ is normal, with $P_{3,1} \leq P_{1,2}$ and there exists a $\lambda \in S^1$ such that the restriction of $\pi(z_{1,2})$ to $P_{3,1}\mathcal{H}$ is λI.*

Corollary 3.1.23 *The operator $\pi(z_{2,3})$ is normal, with $P_{3,1} \leq P_{2,3}$ and there exists a $\mu \in S^1$ such that the restriction of $\pi(z_{2,3})$ to $P_{3,1}\mathcal{H}$ is μI.*

Theorem 3.1.24 *The representation π is unitarily equivalent to $\psi_{(\lambda,\lambda\bar{\mu}),s_2 s_1}$.*

Case 6: $\pi(z_{3,1}) \neq 0$, $\pi(z_{2,1})\pi(z_{3,2}) \neq 0$.

Theorem 3.1.25 *If $\pi(z_{3,1}) \neq 0$ and $\pi(z_{2,1})\pi(z_{3,2}) \neq 0$, then $\pi(z_{3,1}z_{2,1}z_{3,2}) \neq 0$.*

Proof: Let us assume that $\pi(z_{3,1}) \neq 0$ and $\pi(z_{3,1}z_{2,1}z_{3,2}) = 0$. We will show that this implies $\pi(z_{2,1})\pi(z_{3,2}) = 0$.

Let $\mathcal{H}_{3,1}$ be the closed linear span of $\{\pi(z_{1,1}^*)^m \pi(z_{3,3}^n)\xi : m, n \in \mathbb{N}, \xi \in P_{3,1}\mathcal{H}\}$. We will show that $\mathcal{H} = \mathcal{H}_{3,1}$ and $\pi(z_{2,1})\pi(z_{3,2}) = 0$.

Let $\xi \in P_{3,1}\mathcal{H}$. Then one has $\xi = P_{3,1}\xi = \pi(z_{3,1}^*)\xi'$ for some $\xi' \in \mathcal{H}$. Since $z_{3,1}$ is normal, one also has $\xi = \pi(z_{3,1})\xi''$ for some $\xi'' \in \mathcal{H}$. Using this observation, the relation $z_{1,3}^* = z_{2,1}z_{3,2}$ and the condition $\pi(z_{3,1}z_{2,1}z_{3,2}) = 0$, we now get

$$\pi(z_{1,3})\xi = 0 \qquad \pi(z_{1,3}^*)\xi = 0, \tag{3.1.41}$$

$$\pi(z_{1,1})\xi = 0 \qquad \pi(z_{2,1})\xi = 0, \tag{3.1.42}$$

$$\pi(z_{3,2}^*)\xi = 0 \qquad \pi(z_{3,3}^*)\xi = 0. \tag{3.1.43}$$

Since $z_{1,2}$ and $z_{2,3}$ commute with $z_{3,1}^*$ and $z_{3,1}$ is normal, it also follows that

$$\pi(z_{1,2})\xi \in \mathcal{H}_{3,1} \qquad \pi(z_{1,2}^*)\xi \in \mathcal{H}_{3,1}, \tag{3.1.44}$$

$$\pi(z_{2,3})\xi \in \mathcal{H}_{3,1} \qquad \pi(z_{2,3}^*)\xi \in \mathcal{H}_{3,1}. \tag{3.1.45}$$

$$\pi(z_{2,3}^*)\pi(z_{1,1}^*)\xi = \pi(z_{1,1}^*)\pi(z_{2,3}^*)\xi, \quad \xi \in P_{3,1}\mathcal{H}. \tag{3.1.46}$$

$$\pi(z_{1,1})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \begin{cases} 0 & \text{if } m = 0, \\ \pi(z_{1,1}^*)^{m-1}\pi(z_{3,3}^n)\xi & \text{if } m > 0. \end{cases}$$

$$\pi(z_{1,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \begin{cases} 0 & \text{if } m > 0, \\ \pi(z_{3,3})^n\left(\pi(z_{1,2})\xi\right) & \text{if } m = 0. \end{cases}$$

$$\pi(z_{1,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = 0.$$

$$\pi(z_{2,1})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \begin{cases} 0 & \text{if } m > 0, \\ \pi(z_{3,3})^{n-1}\left(\pi(z_{1,2}^*)\xi\right) & \text{if } m = 0, n > 0, \\ 0 & \text{if } m = 0, n = 0. \end{cases}$$

$$\pi(z_{2,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \begin{cases} \pi(z_{1,1}^*)^{m+1}\pi(z_{3,3}^{n-1})\xi & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

$$\pi(z_{2,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \begin{cases} \pi(z_{1,1}^*)^m\left(\pi(z_{2,3})\xi\right) & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

$$\pi(z_{3,1})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \begin{cases} 0 & \text{if } m > 0, \\ 0 & \text{if } m = 0, n > 0, \\ \pi(z_{3,1})\xi & \text{if } m = 0, n = 0. \end{cases}$$

$$\pi(z_{3,2})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \begin{cases} 0 & \text{if } n > 0, \\ \pi(z_{1,1}^*)^{m+1}\left(\pi(z_{2,3}^*)\xi\right) & \text{if } n = 0. \end{cases}$$

$$\pi(z_{3,3})\left(\pi(z_{1,1}^*)^m\pi(z_{3,3}^n)\xi\right) = \pi(z_{1,1}^*)^m\pi(z_{3,3})^{n+1}\xi.$$

Thus $\mathcal{H}_{3,1}$ is an invariant subspace for π. Irreducibility of π gives us $\mathcal{H} = \mathcal{H}_{3,1}$. Since $\pi(z_{1,3}) = 0$ on $\mathcal{H}_{3,1}$ from the above computation, it follows that $\pi(z_{2,1})\pi(z_{3,2}) = \pi(z_{1,3}) = 0$.

Observe that the commutation relations imply that $z_{1,3}$ and $z_{3,1}$ are both normal, i.e. $P_{1,3} = Q_{1,3}$ and $P_{3,1} = Q_{3,1}$. Using the commutation relations again, we now conclude that $z_{3,1}z_{2,1}z_{3,2}$ is a normal partial isometry, with initial and final projection given by $P_{1,3}P_{3,1}$ as the following computations show

$$\begin{aligned} \left(z_{3,1}z_{2,1}z_{3,2}\right)\left(z_{3,1}z_{2,1}z_{3,2}\right)^* &= \left(z_{3,1}z_{1,3}^*\right)\left(z_{3,1}z_{1,3}^*\right)^* \\ &= z_{3,1}z_{1,3}^*z_{1,3}z_{3,1}^* \\ &= z_{1,3}^*z_{1,3}z_{3,1}z_{3,1}^* \\ &= z_{1,3}^*z_{1,3}z_{3,1}^*z_{3,1}, \end{aligned}$$

and similarly,

$$\begin{aligned} \left(z_{3,1}z_{2,1}z_{3,2}\right)^*\left(z_{3,1}z_{2,1}z_{3,2}\right) &= \left(z_{3,1}z_{1,3}^*\right)^*\left(z_{3,1}z_{1,3}^*\right) \\ &= z_{1,3}z_{3,1}^*z_{3,1}z_{1,3}^* \\ &= z_{1,3}z_{1,3}^*z_{3,1}^*z_{3,1} \\ &= z_{1,3}^*z_{1,3}z_{3,1}^*z_{3,1}. \end{aligned}$$

□

Theorem 3.1.26 *Let P be a projection such that*

$$P \leq P_{1,3}P_{3,1}, \qquad P\pi(z_{3,1}) = \pi(z_{3,1})P, \qquad P\pi(z_{1,3}) = \pi(z_{1,3})P.$$

Then $\mathcal{H}_P := \left\{\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi : k,m,n \geq 0, \xi \in P\mathcal{H}\right\}$ is an invariant subspace for π.

If Q is another projection orthogonal to P satisfying the same conditions above, then $\mathcal{H}_P$ and $\mathcal{H}_Q$ are orthogonal.

Proof:

Since $P \leq P_{1,3}P_{3,1}$, one has $\xi = P_{1,3}P_{3,1}\xi$ for $\xi \in P\mathcal{H}$. Therefore

$$\pi(z_{1,1})\xi = 0, \qquad \pi(z_{1,2})\xi = 0, \qquad \pi(z_{2,1})\xi = 0, \tag{3.1.47}$$

$$\pi(z_{2,3}^*)\xi = 0, \qquad \pi(z_{3,2}^*)\xi = 0, \qquad \pi(z_{3,3}^*)\xi = 0. \tag{3.1.48}$$

Consequently, we also have

$$\pi(z_{2,2})\xi = \pi(z_{3,3}^*)\pi(z_{1,1}^*)\xi = \pi(z_{1,1}^*)\pi(z_{3,3}^*)\xi = 0. \tag{3.1.49}$$

From (3.1.47-3.1.48) and (2.2.19–2.2.22), it follows that for all $n \in \mathbb{N}$.

$$\pi(z_{2,1})\pi(z_{2,3}^n)\xi = 0, \qquad \pi(z_{1,2})\pi(z_{3,2}^n)\xi = 0. \tag{3.1.50}$$

Since P commutes with $\pi(z_{1,3})$ and $\pi(z_{3,1})$, we have

$$\xi \in P\mathcal{H} \Longrightarrow \pi(z_{1,3})\xi, \pi(z_{1,3}^*)\xi, \pi(z_{3,1})\xi, \pi(z_{3,1}^*)\xi \in P\mathcal{H}. \tag{3.1.51}$$

Let $\xi \in P\mathcal{H}$.

$$\pi(z_{1,1})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} 0 & \text{if } k = 0, \\ \pi(z_{3,3}^m)\pi(z_{2,3}^{k-1})\left(\pi(z_{3,2}^*)\pi(z_{3,2}^n)\xi\right) & \text{if } k > 0. \end{cases} \tag{3.1.52}$$

$$\pi(z_{1,2})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} 0 & \text{if } m = 0, k = 0, \\ \pi(z_{2,3}^{k-1})\left(\pi(z_{3,1}^*)\pi(z_{3,2}^n)\xi\right) & \text{if } m = 0, k > 0, \\ \pi(z_{3,3}^{m-1})\pi(z_{3,2}^{n+1})\left(\pi(z_{1,3})\xi\right) & \text{if } m > 0, k = 0, \\ \pi(z_{3,3}^m)\pi(z_{2,3}^{k-1})\left(\pi(z_{3,1}^*)\pi(z_{3,2}^n)\xi\right) & \text{if } m > 0, k > 0. \end{cases} \tag{3.1.53}$$

$$\pi(z_{1,3})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} \pi(z_{3,2}^n)\left(\pi(z_{1,3})\xi\right) & \text{if } m = 0, k = 0, \\ 0 & \text{if } m > 0 \text{ or } k > 0. \end{cases} \tag{3.1.54}$$

$$\pi(z_{2,1})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} 0 & \text{if } m=0, k=0, n=0, \\ \pi(z_{3,2}^{n-1})\left(\pi(z_{1,3}^*)\xi\right) & \text{if } m=0, k=0, n>0, \\ 0 & \text{if } m=0, k>0, \\ \pi(z_{3,3}^{m-1})\pi(z_{2,3}^{k+1})\left(\pi(z_{3,1})\xi\right) & \text{if } m>0, n=0 \\ \pi(z_{3,3}^{m})\pi(z_{3,2}^{n-1})\left(\pi(z_{1,3}^*)\xi\right) & \text{if } m>0, n>0, k=0, \\ 0 & \text{if } m>0, n>0, k>0. \end{cases} \tag{3.1.55}$$

$$\pi(z_{2,2})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} 0 & \text{if } m=0, \\ \pi(z_{3,3}^{m-1})\pi(z_{2,3}^{k+1})\pi(z_{3,2}^{n+1})\xi & \text{if } m>0. \end{cases} \tag{3.1.56}$$

$$\pi(z_{2,3})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} \pi(z_{2,3}^{k+1})\pi(z_{3,2}^{n})\xi & \text{if } m=0, \\ 0 & \text{if } m>0. \end{cases} \tag{3.1.57}$$

$$\pi(z_{3,1})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} \pi(z_{2,3}^{k})\left(\pi(z_{3,1})\xi\right) & \text{if } m=0, n=0, \\ 0 & \text{if } m+n>0. \end{cases} \tag{3.1.58}$$

$$\pi(z_{3,2})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \begin{cases} \pi(z_{2,3}^{k})\pi(z_{3,2}^{n+1})\xi & \text{if } m=0, \\ 0 & \text{if } m>0. \end{cases} \tag{3.1.59}$$

$$\pi(z_{3,3})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) = \pi(z_{3,3}^{m+1})\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi. \tag{3.1.60}$$

Now observe that

$$\pi(z_{3,2}^*)\pi(z_{3,2}^n)\xi = \begin{cases} \pi(z_{3,2}^*)\xi = \pi(z_{3,2}^*)\pi(z_{3,1}^*)\pi(z_{3,1})\xi = 0 & \text{if } n=0, \\ (I-\pi(z_{2,2}^*)\pi(z_{2,2}))\pi(z_{3,2}^{n-1})\xi = \pi(z_{3,2}^{n-1})\xi & \text{if } n>0. \end{cases}$$

$$\pi(z_{3,1}^*)\pi(z_{3,2}^n)\xi = \begin{cases} \pi(z_{3,1}^*)\xi & \text{if } n=0, \\ \pi(z_{1,2})\pi(z_{2,3})\pi(z_{3,2}^n)\xi = \pi(z_{1,2})\pi(z_{3,2}^n)\pi(z_{2,3})\xi = 0 & \text{if } n>0. \end{cases}$$

$$\pi(z_{2,1})\pi(z_{3,2}^n)\xi = \begin{cases} \pi(z_{2,1})\xi = 0 & \text{if } n=0, \\ \pi(z_{1,3}^*)\pi(z_{3,2}^{n-1})\xi = \pi(z_{3,2}^{n-1})\pi(z_{1,3}^*)\xi & \text{if } n>0. \end{cases}$$

Therefore it follows that $\pi(z_{i,j})\left(\pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi\right) \in \mathcal{H}_P$ for all i,j. Thus $\mathcal{H}_P$ is an invariant subspace for π.

For the second part, take $\xi \in P\mathcal{H}$ and $\xi' \in Q\mathcal{H}$. Let

$$\xi_{k,m,n} = \pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi, \quad k,m,n \in \mathbb{N},$$
$$\xi'_{k,m,n} = \pi(z_{3,3}^m)\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi', \quad k,m,n \in \mathbb{N}.$$

Since $z_{3,3}^* z_{3,3} = 1$, we have

$$\langle \xi_{k,m,n}, \xi'_{k',m',n'} \rangle = \begin{cases} \langle \pi(z_{3,3}^*)^{m'-m}\pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi \ , \ \pi(z_{2,3}^{k'})\pi(z_{3,2}^{n'})\xi' \rangle & \text{if } m < m', \\ \langle \pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi \ , \ \pi(z_{3,3}^*)^{m-m'}\pi(z_{2,3}^{k'})\pi(z_{3,2}^{n'})\xi' \rangle & \text{if } m > m', \\ \langle \pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi \ , \ \pi(z_{2,3}^{k'})\pi(z_{3,2}^{n'})\xi' \rangle & \text{if } m = m'. \end{cases}$$

Using the equalities $z_{3,3}^* = z_{1,1}z_{2,2}$, $z_{2,2}z_{2,3} = 0 = z_{2,2}z_{3,2}$ and $\pi(z_{2,2})\xi = 0 = \pi(z_{2,2})\xi'$, we get

$$\langle \xi_{k,m,n}, \xi'_{k',m',n'} \rangle = \begin{cases} 0 & \text{if } m \neq m', \\ \langle \pi(z_{2,3}^k)\pi(z_{3,2}^n)\xi \ , \ \pi(z_{2,3}^{k'})\pi(z_{3,2}^{n'})\xi' \rangle & \text{if } m = m'. \end{cases} \tag{3.1.61}$$

Next, note that from the equality $z_{2,2}^* z_{2,2} + z_{3,2}^* z_{3,2} = 1$, it follows that

$$(z_{3,2}^*)^m (z_{3,2}^n) = (z_{3,2}^*)^{m-1}(z_{3,2}^{n-1}) \quad \text{if } m \geq 1, n \geq 1, \text{ and } m+n > 2.$$

Therefore using the equalities $\pi(z_{3,2}^*)\xi = 0 = \pi(z_{3,2}^*)\xi'$, we get

$$\langle \xi_{k,m,n}, \xi'_{k',m',n'} \rangle = \begin{cases} 0 & \text{if } m \neq m' \text{ or } n \neq n', \\ \langle \pi(z_{2,3}^k)\xi \ , \ \pi(z_{2,3}^{k'})\xi' \rangle & \text{if } m = m', n = n'. \end{cases} \tag{3.1.62}$$

Finally, note that

$$z_{2,3}^* z_{2,3} = z_{1,1}z_{3,2}z_{2,3} = z_{1,1}z_{2,3}z_{3,2} = z_{3,2}^* z_{3,2}.$$

This together with the relations $z_{2,3}z_{3,2} = z_{3,2}z_{2,3}$ and $z_{2,3}z_{3,2}^* = z_{3,2}^* z_{2,3}$ give us

$$\langle \xi_{k,m,n}, \xi'_{k',m',n'} \rangle = \begin{cases} 0 & \text{if } m \neq m' \text{ or } n \neq n' \text{ or } k \neq k', \\ \langle \xi, \xi' \rangle & \text{if } m = m', n = n', k = k', \end{cases} \tag{3.1.63}$$

Since $PQ = 0$, we have $\langle \xi_{k,m,n}, \xi'_{k',m',n'} \rangle = 0$. $\square$

The following two corollaries are now immediate.

Corollary 3.1.27 *There exists a* $\lambda \in S^1$ *such that the restriction of* $\pi(z_{1,3})$ *to* $P_{1,3}P_{3,1}\mathcal{H}$ *is* λI.

Corollary 3.1.28 *There exists a* $\mu \in S^1$ *such that the restriction of* $\pi(z_{3,1})$ *to* $P_{1,3}P_{3,1}\mathcal{H}$ *is* μI.

Theorem 3.1.29 *The representation* π *is unitarily equivalent to* $\psi_{(\lambda,\bar{\mu}),s_1 s_2 s_1}$.

Proof: In view of the above corollaries, it follows that

$$\pi(z_{1,1})\xi_{k,m,n} = \begin{cases} \xi_{k-1,m,n-1} & \text{if } k > 0, n > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1.64}$$

$$\pi(z_{1,2})\xi_{k,m,n} = \begin{cases} \lambda\xi_{0,m-1,n+1} & \text{if } k = 0, m > 0 \\ \bar{\mu}\xi_{k-1,m,0} & \text{if } k > 0, n = 0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1.65}$$

$$\pi(z_{1,3})\xi_{k,m,n} = \begin{cases} \lambda\xi_{0,0,n} & \text{if } m = 0, k = 0 \\ 0 & \text{otherwise}, \end{cases} \tag{3.1.66}$$

$$\pi(z_{2,1})\xi_{k,m,n} = \begin{cases} \bar{\lambda}\xi_{0,m,n-1} & \text{if } k = 0, n > 0, \\ \mu\xi_{k+1,m-1,0} & \text{if } m > 0, n = 0 \\ 0 & \text{otherwise}, \end{cases} \tag{3.1.67}$$

$$\pi(z_{2,2})\xi_{k,m,n} = \begin{cases} 0 & \text{if } m = 0, \\ \xi_{k+1,m-1,n+1} & \text{if } m > 0. \end{cases} \tag{3.1.68}$$

$$\pi(z_{2,3})\xi_{k,m,n} = \begin{cases} \xi_{k+1,0,n} & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases} \tag{3.1.69}$$

$$\pi(z_{3,1})\xi_{k,m,n} = \begin{cases} \mu\xi_{k,0,0} & \text{if } m = 0, n = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{3.1.70}$$

$$\pi(z_{3,2})\xi_{k,m,n} = \begin{cases} \xi_{k,0,n+1} & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases} \tag{3.1.71}$$

$$\pi(z_{3,3})\xi_{k,m,n} = \xi_{k,m+1,n}. \tag{3.1.72}$$

Thus if we define $\mathcal{H}_\xi$ to be the clsoed linear span of $\{\xi_{k,m,n} : k, m, n \in \mathbb{N}\}$, then $\mathcal{H}_\xi$ is an invariant subspace for π. Since π is irreducible, we have $\mathcal{H} = \mathcal{H}_\xi$. By the computations in the proof of the second part of Theorem 3.1.26, it follows that $\{\xi_{k,m,n} : k, m, n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$. The map

$$e_{m,k,n} \mapsto \lambda^k \mu^{k-m-n} \xi_{k,m,n}$$

from $\ell^2(\mathbb{N} \times \mathbb{N} \times \mathbb{N})$ to $\mathcal{H}$ now sets up a unitary equivalence between π and $\psi_{(\lambda,\bar{\mu}),s_1 s_2 s_1}$.

$\square$

CHAPTER 4

Irreducible representations of $C(SU_0(n+1))$

First section of this chapter is about some properties of $C(SU_0(n+1))$ which are essential to get all irreducible representations. Second section is about all irreducible representations of $C(SU_0(n+1))$ which is mainly from [GP24]. Last section is about a bialgebra structure on $C(SU_0(n+1))$, faithful representation of $C(SU_0(n+1))$ related to Soibelman representation of $C(SU_q(n+1))$ and relation among irreducible C^*-representations of $C(SU_0(n+1))$ and $C(SU_q(n+1))$(for positive q).

4.1 Projections in $C(SU_0(n+1))$

In this section we will restrict ourselves to the $*$-subalgebra of $C(SU_0(n+1))$ generated by the $z_{i,j}$'s and derive certain relations that will in particular show that the generators $z_{i,j}$ are all partial isometries.

Theorem 4.1.1 *Let $1 \leq i \leq j \leq n+1$. Then*

$$(z_{i,i}z_{i+1,i+1}\dots z_{j,j})^* = (z_{1,1}z_{2,2}\dots z_{i-1,i-1})(z_{j+1,j+1}z_{j+2,j+2}\dots z_{n+1,n+1}). \tag{4.1.1}$$

Proof: Note that for $i = j$, it follows. We will assume that the equality holds for $i \le j < n+1$ and prove that it holds for $i < j+1$ also. Using above relaions, we get

$$\begin{aligned}
(z_{i,i}z_{i+1,i+1}\cdots z_{j+1,j+1})^* &= z^*_{j+1,j+1}\left(z_{i,i}z_{i+1,i+1}\cdots z_{j,j}\right)^* \\
&= (z_{1,1}z_{2,2}\cdots z_{j,j})(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1}) \\
&\quad\ (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}) \\
&= (z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})(z_{1,1}z_{2,2}\cdots z_{j,j}) \\
&\quad\ (z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})(z_{1,1}z_{2,2}\cdots z_{i-1,i-1}) \\
&= (z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})(z_{1,1}z_{2,2}\cdots z_{i-1,i-1}) \\
&= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1}).
\end{aligned}$$

This completes the proof. $\square$

Lemma 4.1.2 *Let us denote by $p_{i,j}$ the element $z^*_{i,j}z_{i,j}$ and by $q_{i,j}$ the element $z_{i,j}z^*_{i,j}$. For $1 \le i \le j \le n+1$, one has the following:*

$$\sum_{k=1}^{i} q_{k,j} = \sum_{k=j}^{n+1} p_{i,k} \tag{4.1.2}$$

$$\sum_{k=1}^{j} q_{k,j} = \sum_{k=j}^{n+1} p_{j,k} = 1. \tag{4.1.3}$$

Proof: We will prove that

$$\sum_{k=1}^{i} q_{k,j} = \begin{cases} (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,j}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}\cdots z_{n+1,n+1}) & \text{if } i < j, \\ 1, & \text{if } i = j \end{cases} \tag{4.1.4}$$

$$\sum_{k=j}^{n+1} p_{i,k} = \begin{cases} (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,j}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}\cdots z_{n+1,n+1}) & \text{if } i < j, \\ 1, & \text{if } i = j. \end{cases} \tag{4.1.5}$$

Note that for $i = 1$, one has

$$\begin{aligned}
q_{1,j} &= z_{1,j}z^*_{1,j} \\
&= z_{1,j}(z_{2,1}z_{3,2}\cdots z_{j,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}).
\end{aligned}$$

Thus the first equality in (4.2.4) holds for $i = 1$. Next assume that it holds for $i-1$ and $1 \le i < j \le n+1$. Then using equation (2.2.25) we get

$$\begin{aligned}
q_{i,j} + \sum_{k=1}^{i-1} q_{k,j} & \\
&= z_{i,j}(z_{1,1}z_{2,2}\cdots z_{i-2,i-2}z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}\cdots z_{n+1,n+1})\\
&\quad + (z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i-1,j}z_{i,i-1}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}\cdots z_{n+1,n+1})\\
&= (z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i,j}z_{i-1,i-1}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}\cdots z_{n+1,n+1})\\
&\quad + (z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i-1,j}z_{i,i-1}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&= (z_{1,1}\cdots z_{i-2,i-2}z_{i-1,i-1})z_{i,j}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}\cdots z_{n+1,n+1}).
\end{aligned}$$

Hence the first equality in (4.2.4) follows. Using this, one now gets

$$\begin{aligned}
\sum_{k=1}^{j} q_{k,j} &= \sum_{k=1}^{j-1} q_{k,j} + q_{j,j}\\
&= (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{j-1,j}z_{j,j-1}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&\quad + z_{j,j}(z_{1,1}z_{2,2}\cdots z_{j-2,j-2}z_{j-1,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&= (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{j-1,j}z_{j,j-1}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&\quad + (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{j,j}z_{j-1,j-1}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&= z_{1,1}z_{2,2}\cdots z_{j-2,j-2}z_{j-1,j-1}z_{j,j}z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}\\
&= 1
\end{aligned}$$

Thus we get (4.2.4). Proof of (4.2.5) is similar. First observe that for $j = n+1$, one has

$$\begin{aligned}
p_{i,n+1} &= z_{i,n+1}^* z_{i,n+1}\\
&= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{n+1,n})z_{i,n+1}\\
&= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,n+1}(z_{i+1,i}z_{i+2,i+1}\cdots z_{n+1,n}).
\end{aligned}$$

Thus the first equality in (4.2.5) holds for $j = n+1$. Now, assuming that it holds for $j+1$, we have

$$p_{i,j} + \sum_{k=j+1}^{n+1} p_{i,k}$$

$$
\begin{aligned}
&= (z_{1,1}z_{2,2}\cdots z_{i-2,i-2}z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})z_{i,j}\\
&\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,j+1}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})z_{j+1,j}(z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&= (z_{1,1}z_{2,2}\cdots z_{i-2,i-2}z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})z_{j+1,j+1}z_{i,j}(z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})z_{i,j+1}z_{j+1,j}(z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})z_{i,j}(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})\\
&= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,j}(z_{i+1,i}z_{i+2,i+1}\cdots z_{j,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}).
\end{aligned}
$$

This proves the first equality in (4.2.5). Using this we now get

$$
\begin{aligned}
\sum_{k=i}^{n+1} p_{i,k} &= \sum_{k=i+1}^{n+1} p_{i,k} + p_{i,i}\\
&= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,i+1}z_{i+1,i}(z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})\\
&\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})(z_{i+1,i+1}z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})z_{i,i}\\
&= (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i,i+1}z_{i+1,i}(z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})\\
&\quad + (z_{1,1}z_{2,2}\cdots z_{i-1,i-1})z_{i+1,i+1}z_{i,i}(z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1})\\
&= z_{1,1}z_{2,2}\cdots z_{i-1,i-1}z_{i,i}z_{i+1,i+1}z_{i+2,i+2}z_{i+3,i+3}\cdots z_{n+1,n+1}\\
&= 1,
\end{aligned}
$$

which completes the proof. □

Lemma 4.1.3 *The elements $p_{i,j}$ and $q_{i,j}$ are projections for $1 \le i \le j \le n$.*

Proof: We will show that for any $1 \le j \le n$, if $\left\{\sum_{k=1}^{i} q_{k,j} : 1 \le i \le j\right\}$ is a family of projections then $\left\{\sum_{k=1}^{i} q_{k,j+1} : 1 \le i \le j+1\right\}$ is also a family of projections. Then by an application of equation (4.2.3) in Lemma 4.1.2, we will successively conclude for $i = 1, 2, \ldots, n+1$ that the sums $\sum_{k=1}^{i} q_{k,j}$ are all projections for $i \le j \le n+1$. This will imply that all $q_{i,j}$'s (and hence all $p_{i,j}$'s) for $1 \le i \le j \le n+1$ are projections.

Now fix $1 \le j \le n$ and assume that $\left\{\sum_{k=1}^{i} q_{k,j} : 1 \le i \le j\right\}$ is a family of projections. Note that by this assumption, it follows that $q_{i,j} = \sum_{k=1}^{i} q_{k,j} - \sum_{k=1}^{i-1} q_{k,j}$, being positive and a difference of two projections, is also a projection. Hence $p_{i,j}$ is also a projection. Now from Lemma 4.1.2, for $i \le j$, we get

$$
\sum_{k=1}^{i} q_{k,j+1} = \sum_{k=j+1}^{n+1} p_{i,k} = \sum_{k=j}^{n+1} p_{i,k} - p_{i,j} = \sum_{k=1}^{i} q_{k,j} - p_{i,j}.
$$

Thus the element $\sum_{k=1}^{i} q_{k,j+1}$ is positive and is a difference of two projections. Therefore it is a projection. For $i = j+1$, $\sum_{k=1}^{j+1} q_{k,j+1} = 1$ and hence is a projection. □

Lemma 4.1.4 *For $1 \le i \le j \le n+1$, one has the following:*

$$\sum_{k=1}^{j} q_{i,k} = \sum_{k=i}^{n+1} p_{k,j}, \tag{4.1.6}$$

$$\sum_{k=1}^{i} q_{i,k} = \sum_{k=i}^{n+1} p_{k,i} = 1 \tag{4.1.7}$$

Proof: We will prove that

$$\sum_{k=1}^{j} q_{i,k} = \begin{cases} (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}\cdots z_{n+1,n+1}), & \text{if } j < i, \\ 1 & \text{if } j = i, \end{cases} \tag{4.1.8}$$

$$\sum_{k=i}^{n+1} p_{k,j} = \begin{cases} (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}\cdots z_{n+1,n+1}), & \text{if } i > j, \\ 1 & \text{if } i = j. \end{cases} \tag{4.1.9}$$

For $j = 1$, the first equality in (4.2.8) holds as one has

$$\begin{aligned} q_{i,1} =& z_{i,1}z_{i,1}^{*} \\ =& z_{i,1}(z_{1,2}z_{2,3}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}). \end{aligned}$$

Next, assuming that it holds for $j-1$, we have

$$\begin{aligned} & q_{i,j} + \sum_{k=1}^{j-1} q_{i,k} \\ =& z_{i,j}(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\ &+ (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{i,j-1}z_{j-1,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \end{aligned}$$

$$\begin{aligned} =& (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{i,j}z_{j-1,j-1}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \\ &+ (z_{1,1}z_{2,2}\cdots z_{j-2,j-2})z_{i,j-1}z_{j-1,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}) \end{aligned}$$

$$= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}).$$

Thus the equality holds. For $i=j$, one has

$$\begin{aligned}
&\sum_{k=1}^{i} q_{i,k}\\
=&\sum_{k=1}^{i-1} q_{i,k} + q_{i,i}\\
\\
=&(z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i,i-1}z_{i-1,i}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})\\
+&z_{i,i}(z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i-1,i-1}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})\\
\\
=&(z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i,i-1}z_{i-1,i}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})\\
+&(z_{1,1}z_{2,2}\cdots z_{i-2,i-2})z_{i,i}z_{i-1,i-1}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})\\
\\
=&z_{1,1}z_{2,2}\cdots z_{i-2,i-2}z_{i-1,i-1}z_{i,i}z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}\\
=&1.
\end{aligned}$$

Thus we have (4.2.8)

Next, note that

$$\begin{aligned}
p_{n+1,j} &= z^*_{n+1,j}z_{n+1,j}\\
&= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\dots z_{n,n+1})z_{n+1,j}\\
&= (z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{n+1,j}(z_{j,j+1}z_{j+1,j+2}\dots z_{n,n+1}).
\end{aligned}$$

Thus the first equality in (4.2.9) holds for $i=n+1$.

Assume it holds for $i+1$. Then

$$\begin{aligned}
&p_{i,j} + \sum_{t=i+1}^{n+1} p_{t,j}\\
=&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})z_{i,j}\\
+&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i+1,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})z_{i,i+1}(z_{i+2,i+2}\cdots z_{n+1,n+1})
\end{aligned}$$

$$
\begin{aligned}
=&(z_{1,1}z_{2,2}\cdots z_{j-2,j-2}z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})z_{i+1,i+1}z_{i,j}(z_{i+2,i+2}\cdots z_{n+1,n+1})\\
+&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})z_{i,i+1}z_{i+1,j}(z_{i+2,i+2}\cdots z_{n+1,n+1})\\
\\
=&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})z_{i,j}(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1})\\
=&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{i,j}(z_{j,j+1}z_{j+1,j+2}\cdots z_{i-1,i})(z_{i+1,i+1}z_{i+2,i+2}\cdots z_{n+1,n+1}).
\end{aligned}
$$

Thus we have the first equality in (4.2.9).

For $i=j$,

$$
\begin{aligned}
&p_{j,j}+\sum_{k=j+1}^{n+1}p_{k,j}\\
\\
=&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{j+1,j}z_{j,j+1}(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})\\
+&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})(z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1})z_{j,j}\\
\\
=&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{j+1,j}z_{j,j+1}(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})\\
+&(z_{1,1}z_{2,2}\cdots z_{j-1,j-1})z_{j+1,j+1}z_{j,j}(z_{j+2,j+2}z_{j+3,j+3}\cdots z_{n+1,n+1})\\
\\
=&z_{1,1}z_{2,2}\cdots z_{j-1,j-1}z_{j,j}z_{j+1,j+1}z_{j+2,j+2}\cdots z_{n+1,n+1}\\
=&1.
\end{aligned}
$$

Thus the proof is complete. □

Lemma 4.1.5 *The elements $p_{i,j}$ and $q_{i,j}$ are projections for $1\le j\le i\le n$.*

Proof: By using Above Lemma, one can prove exactly as in the proof of Lemma 4.1.4 that if $\left\{\sum_{k=1}^{j}q_{i,k}:1\le j\le i\right\}$ is a family of projections then $\left\{\sum_{k=1}^{j}q_{i+1,k}:1\le j\le i+1\right\}$ is also a family of projections. This leads to the required conclusion. □

Corollary 4.1.6 *The elements $z_{i,j}$ are partial isometries for all $1\le i,j\le n+1$.*

Lemma 4.1.7 *Let $i<k$ and $j<\ell$. Assume that either $i\geq\ell$ or $k\leq j$. Then*

$$z_{i,\ell}z_{k,j}^*=0 \tag{4.1.10}$$

$$z_{i,\ell}^*z_{k,j}=0 \tag{4.1.11}$$

Proof: Assume that $i\geq\ell$. Then using Lemma 4.1.4, we have

$$\begin{aligned}\sum_{k=i+1}^{n+1}\big(z_{i,\ell}z_{k,j}^*\big)\big(z_{i,\ell}z_{k,j}^*\big)^* &= z_{i,\ell}\left(\sum_{k=i+1}^{n+1}p_{k,j}\right)z_{i,\ell}^*\\ &= z_{i,\ell}\Big(\sum_{s=1}^{j}q_{i+1,s}\Big)z_{i,\ell}^*\\ &= \sum_{s=1}^{j}z_{i,\ell}z_{i+1,s}z_{i+1,s}^*z_{i,\ell}^*\\ &= 0,\end{aligned}$$

since $z_{i,\ell}z_{i+1,s}=0$ for all $1\leq s\leq\ell-1$. Therefore $z_{i,\ell}z_{k,j}^*=0$ for all $j<\ell$ and $i<k$.

Similarly if $k\leq j$ then using Lemma 4.1.2, we get

$$\begin{aligned}\sum_{\ell=j+1}^{n+1}\big(z_{i,\ell}z_{k,j}^*\big)^*\big(z_{i,\ell}z_{k,j}^*\big) &= z_{k,j}\sum_{\ell=j+1}^{n+1}\big(p_{i,\ell}\big)z_{k,j}^*\\ &= z_{k,j}\Big(\sum_{s=1}^{i}q_{s,j+1}\Big)z_{k,j}^*\\ &= 0.\end{aligned}$$

Therefore as before, $z_{i,\ell}z_{k,j}^*=0$.

The second equality can be proved in a similar way using above Lemmas 4.1.2 and 4.1.4 $\square$

Lemma 4.1.8 *Let $k>max\{i,j\}$. Then*

$$q_{i,j}z_{n+1,k}=z_{n+1,k}q_{i,j}, \qquad z_{i,j}p_{n+1,k}=p_{n+1,k}z_{i,j}.$$

Proof: If $\max\{i,j\}+1<k$, then one has $z_{i,j}z_{n+1,k}=z_{n+1,k}z_{i,j}$ and hence the equality follows. Assume $\max\{i,j\}+1=k$ (i.e. $k=s+1$ and $\max\{i,j\}=s$). Then

$$z_{i,j}z_{n+1,k}=z_{n+1,k}z_{i,j}+z_{i,k}z_{n+1,j}.$$

It follows from Lemmas 4.1.2 and 4.1.4 that if $i\geq j$, then $p_{n+1,j}p_{i,j}=0$ and if $i<j$, then $p_{i,k}p_{i,j}=0$. Therefore in either case we get from the above equation that

$$\begin{aligned} q_{i,j}z_{n+1,k} &= z_{i,j}z_{n+1,k}z_{i,j}^* \\ &= z_{n+1,k}z_{i,j}z_{i,j}^*+z_{i,k}z_{n+1,j}z_{i,j}^* \\ &= z_{n+1,k}q_{i,j}. \end{aligned}$$

Similarly, We have

$$\begin{aligned} z_{i,j}p_{n+1,k} &= z_{i,j}z_{n+1,k}^*z_{n+1,k} \\ &= z_{n+1,k}^*z_{i,j}z_{n+1,k} \\ &= p_{n+1,k}z_{i,j}+z_{n+1,k}^*z_{i,k}z_{n+1,j} \\ &= p_{n+1,k}z_{i,j}+z_{n+1,k}^*z_{n+1,j}z_{i,k} \\ &= p_{n+1,k}z_{i,j}. \end{aligned}$$

□

Lemma 4.1.9 *Let $1\leq i\leq j\leq n+1$. Then $z_{i,j+1}z_{n+1,j+1}^*-z_{n+1,j}^*z_{i,j}=0$*

Proof: We have

$$\begin{aligned} &\left(z_{i,j+1}z_{n+1,j+1}^*-z_{n+1,j}^*z_{i,j}\right)\left(z_{i,j+1}z_{n+1,j+1}^*-z_{n+1,j}^*z_{i,j}\right)^* \\ &=z_{i,j+1}p_{n+1,j+1}z_{i,j+1}^*+z_{n+1,j}^*q_{i,j}z_{n+1,j}-z_{i,j+1}z_{n+1,j}^*z_{i,j}^*z_{n+1,j}-z_{n+1,j}^*z_{i,j}z_{n+1,j+1}z_{i,j+1}^* \end{aligned}$$

$$\begin{aligned} =&z_{i,j+1}(p_{n+1,j}+q_{n+1,j+1})z_{i,j+1}^*+z_{n+1,j}^*q_{i,j}z_{n+1,j} \\ &-z_{i,j+1}(z_{n+1,j+1}z_{i,j}+z_{n+1,j}z_{i,j+1})^*z_{n+1,j}-z_{n+1,j}^*(z_{n+1,j+1}z_{i,j}+z_{n+1,j}z_{i,j+1})z_{i,j+1}^* \\ =&z_{i,j+1}p_{n+1,j}z_{i,j+1}^*+z_{n+1,j}^*q_{i,j}z_{n+1,j}-z_{i,j+1}z_{i,j+1}^*z_{n+1,j}^*z_{n+1,j}-z_{n+1,j}^*z_{n+1,j}z_{i,j+1}z_{i,j+1}^* \\ =&p_{n+1,j}q_{i,j+1}+z_{n+1,j}^*q_{i,j}z_{n+1,j}-p_{n+1,j}q_{i,j+1}-z_{n+1,j}^*q_{i,j+1}z_{n+1,j} \end{aligned}$$

$$=z_{n+1,j}^*(q_{i,j}-q_{i,j+1})z_{n+1,j}.$$

Summing over i we get

$$\begin{aligned}
&\sum_{i=1}^{j}\left(z_{i,j+1}z_{n+1,j+1}^*-z_{n+1,j}^*z_{i,j}\right)\left(z_{i,j+1}z_{n+1,j+1}^*-z_{n+1,j}^*z_{i,j}\right)^*\\
=&z_{n+1,j}^*\left(\sum_{i=1}^{j}q_{i,j}-\sum_{i=1}^{j}q_{i,j+1}\right)z_{n+1,j}\\
=&z_{n+1,j}^*\left(I-\sum_{k=j+1}^{n+1}p_{j,k}\right)z_{n+1,j}\\
=&z_{n+1,j}^*p_{j,j}z_{n+1,j}\\
=&0.
\end{aligned}$$

Therefore we have the result. □

4.2 Irreducible representations of $C(SU_0(n+1))$

The remaining part of this section is devoted to the proof of irreducibility of $\psi_{\lambda,\omega}$ and their inequivalence for different pairs (λ,ω).

Let us start with the observation that the representation ψ_{s_r} is given by

$$\psi_{s_r}(z_{i,j})=\begin{cases}S & \text{if } i=j=r,\\ S^* & \text{if } i=j=r+1,\\ P_0 & \text{if } i\neq j \text{ and } i,j\in\{r,r+1\},\\ \delta_{i,j}I & \text{otherwise,}\end{cases} \tag{4.2.1}$$

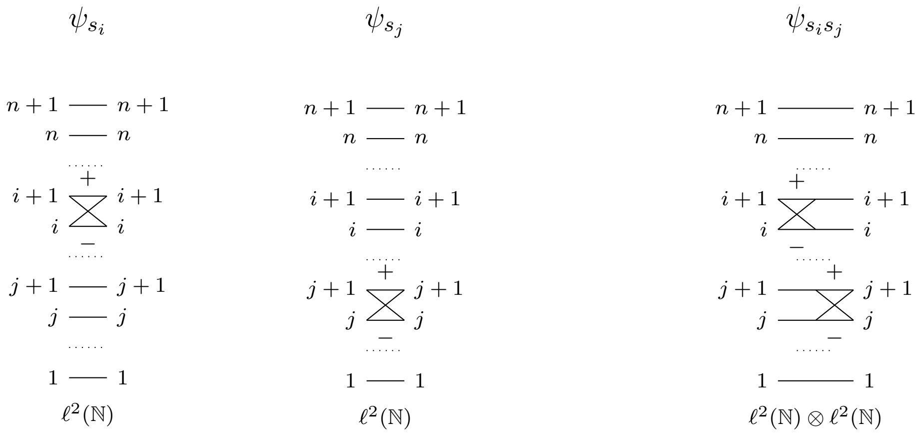
where P_0 denotes the rank one projection $|e_0\rangle\langle e_0|=I-S^*S$. The following lemma is an important observation that will be used in the proof of the main theorem in this section. The proof is straightforward and hence omitted.

Lemma 4.2.1 *Let $\omega=s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\ldots s_{[a_1,b_1]}$ and let $\omega'=\omega s_{a_1-1}$. Then*

$$\psi_{\lambda,\omega'}(z_{i,j})=\sum_{m=\min\{i,j\}}^{\max\{i,j\}}\psi_{\lambda,\omega}(z_{i,m})\otimes\psi_{s_{a_1-1}}(z_{m,j}). \tag{4.2.2}$$

Diagrams for the representations $\psi_{\lambda,\omega}$. We will next introduce certain diagrams for the representations $\psi_{\lambda,\omega}$ similar to those introduced in [CP03] for the case of $C(SU_q(n+1))$ for nonzero q. These diagrams will be very helpful in understanding the images of the generatoring elements $z_{i,j}$ of the C^*-algebra $C(SU_0(n+1))$. In particular, they will play a very important role in proving the irreducibility of the representations $\psi_{\lambda,\omega}$ in the present section and later on while characterising all irreducible representations, these diagrams will once again he helpful by providing us the necessary ideas and intuition.

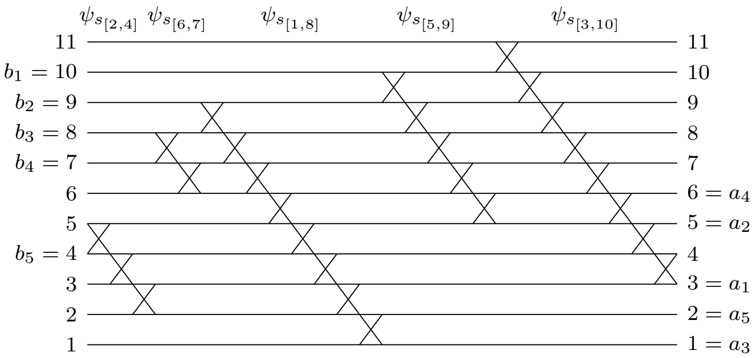
Let us start with the diagrams for the representations ψ_{s_i} and $\psi_{s_i s_j}$.



In the leftmost diagram above, each path from a node k on the left to a node l on the right stands for an operator on $\ell^2(\mathbb{N})$. A horizontal unlabelled line stands for the identity operator, a horizontal line labelled with a '+' sign stands for the unilateral forward shift S^* and one labelled with a '−' sign stands for the unilateral backward shift S. Diagonal lines going upward or downward represent the projection P_0. The image $\psi_{s_i}(z_{m,l})$ is the operator represented by the path from m on the left to l on the right, and is zero if there is no such path. Similarly the diagram in the middle stands for the representation ψ_{s_j}.

The diagram on the right above is for the representation $\psi_{s_i s_j}$, where one simply puts the two diagrams representing ψ_{s_i} and ψ_{s_j} adjacent to each other, with the one for ψ_{s_j} to the right of that for ψ_{s_i}. The image $\psi_{s_i s_j}(z_{m,l})$ of $z_{m,l}$ is an operator on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ given by *the sum of the paths from the node m on the extreme left to the node l on the extreme right that do not contain both upward and downward diagonals.* It would be zero if there is no such path.

Next, we come to the description of ψ_ω. As we have already remarked, we will take a reduced word for ω of the form $\omega = s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\ldots s_{[a_1,b_1]}$ where $1 \leq k \leq n$, $1 \leq b_k < b_{k-1} < \cdots < b_1 \leq n$, and $1 \leq a_i \leq b_i$ for $1 \leq i \leq k$. To get the diagram for ψ_ω, we simply put the diagrams for the ψ_{s_r}'s appearing in this word adjacent to each other in the order they appear in the word from left to right. Thus for example, if $\omega = s_{[2,4]}s_{[6,7]}s_{[1,8]}s_{[5,9]}s_{[3,10]}$, then the following is the diagram for ψ_ω:



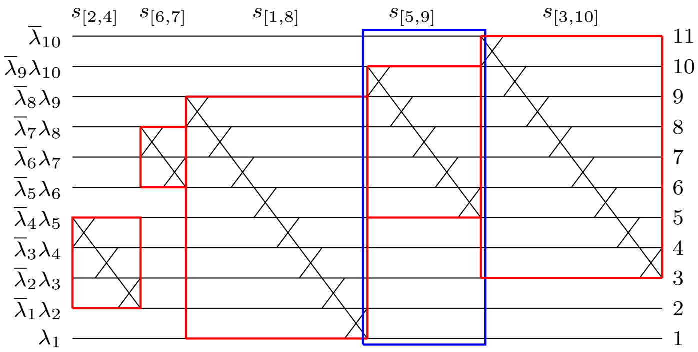
Finally, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in S^1$, the diagram for $\psi_{\lambda,\omega}$ is the same as that for ψ_ω, except that *the paths starting at m on the left get multiplied by a scalar* which is $\overline{\lambda}_{m-1}\lambda_m$ if $1 \leq m \leq n+1$ (with the convention that $\lambda_0 = \lambda_{n+1} = 1$).

Remark 4.2.2 *In view of the above, except for the last theorem in this section (Theorem 4.2.9) where we prove that $\psi_{\lambda,\omega}$'s are inequivalent for different pairs (λ, ω), we will not need to keep track of the λ_i's. Accordingly, all the operator equalities we will write before Theorem 4.2.9 in this section will be valid only modulo a complex number of modulus 1.*

In the diagram for $\psi_{\lambda,\omega}$, we will refer to the part corresponding to $\psi_{s_{[a_j,b_j]}}$ as the $s_{[a_j,b_j]}$**-section** and the smallest rectangle containing the diagonal lines in the $s_{[a_j,b_j]}$-section as the $s_{[a_j,b_j]}$**-rectangle**. For example, the different $s_{[a_j,b_j]}$-rectangles have been shown in red colour and the $s_{[5,9]}$-section has been shown in blue colour in the diagram below. Observe that if $i > b_j + 1$, then the horizontal line starting at node i on the left passes above the $s_{[a_m,b_m]}$-rectangles for $m \geq j$. For an operator R given by a path, we will denote by R_j the operator given by the $s_{[a_j,b_j]}$-section of that path. In such a case one has

$$R = R_k \otimes R_{k-1} \otimes \ldots \otimes R_1,$$

where R_j acts on $\ell^2(\mathbb{N})^{\otimes \ell_j}$ for all j.



The diagram for $\psi_{\lambda,\omega}$ and the terminology introduced above will be helpful in understanding the operators in the image $\psi_{\lambda,\omega}(C(SU_0(n+1)))$ as well as certain quantities associated with them that we will define in the remaining part of this section. They are also going to be helpful in getting insight into some of the proofs in sections 5 and 6, though we will not explicitly use or refer to the diagrams there.

Given a reduced word $\omega = s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\ldots s_{[a_1,b_1]}$, we will next define certain elements in the C^*-algebra $A_n(0)$ that will play a crucial role in the proof of irreducibility of the representation $\psi_{\lambda,\omega}$. For $2 \le j \le k$ and $a_j \le i \le b_j + 1$, define two integers $n(j,i)$ and $n'(j,i)$ as follows:

$$n(j,i) = \max\{m : 1 \le m \le j-1 \text{ and } a_m \le i\}, \tag{4.2.3}$$

$$n'(j,i) = \max\{m : 1 \le m \le j-1 \text{ and } a_m + 1 \le i\}, \tag{4.2.4}$$

with the usual convention that maximum of an empty set will be taken as $-\infty$. Thus $n(j,i)$ is the maximum of all those m for which the i^{th} horizontal line (from the bottom) intersects the $s_{[a_m,b_m]}$-rectangle, and $n(j,i) = -\infty$ when the i^{th} horizontal line is below the $s_{[a_m,b_m]}$-rectangles for $1 \le m \le j-1$.

Next, for $1 \le j \le k$ and for $a_j \le i \le b_j + 1$, define operators $v_{j,i}(\omega) \equiv v_{j,i}$ as follows:

$$v_{1,i} = z_{b_1+1,i}, \tag{4.2.5}$$

$$v_{j,i} = \begin{cases} z_{b_j+1,i}v_{n(j,i),i+1}, & \text{if } n(j,i) \text{ is finite} \\ z_{b_j+1,i} & \text{if } n(j,i) = -\infty \end{cases} \qquad 2 \le j \le k. \tag{4.2.6}$$

Let us denote by $Z_{i,j}$ and $V_{i,j}$ the operators $\psi_{\lambda,\omega}(z_{i,j})$ and $\psi_{\lambda,\omega}(v_{i,j})$ respectively. We will next study the operators $V_{j,i}$. In general, $Z_{b_j+1,i}$ is given by a sum of paths from b_j+1 to i going downwards. The operator $V_{j,i}$ will be given by the lowest among these paths (appearing in the sum for $Z_{b_j+1,i}$) with some of its $s_{[a_r,b_r]}$-sections slightly modified for $1 \le r \le j-1$ so that at most one shift operator occurs in each such section. This is made more precise in the next two propositions.

Proposition 4.2.3 *Let $\ell_r = b_r - a_r + 1$ for $1 \le r \le k$. Then for $a_j \le i \le b_j + 1$, one has*

$$V_{j,i} = T^{(j)}_{k,i} \otimes T^{(j)}_{k-1,i} \otimes \cdots \otimes T^{(j)}_{1,i}, \tag{4.2.7}$$

where $T^{(j)}_{r,i}$'s are of the following form:

$$I^{\otimes \ell_r} \qquad \text{if } r > j, \tag{4.2.8}$$

$$P_0^{\otimes(b_r+1-i)} \otimes S^* \otimes I^{\otimes(i-a_r-1)} \qquad \text{if } r = j \text{ and } i \ge a_r+1, \tag{4.2.9}$$

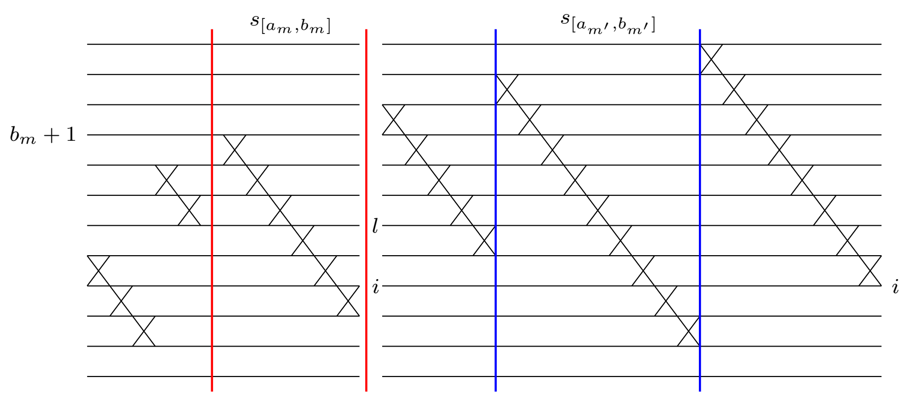
$$P_0^{\otimes(b_r-a_r+1)} \qquad \text{if } r = j \text{ and } i = a_r, \tag{4.2.10}$$

$$P_0^{\otimes(b_r-a_r-1)} \otimes I \otimes S^* \qquad \text{if } r < j \text{ and } i = a_r+1, \tag{4.2.11}$$

$$P_0^{\otimes(b_r-i)} \otimes I \otimes S^* \otimes I^{\otimes(i-a_r-1)} \qquad \text{if } r < j \text{ and } a_r+1 < i < b_r, \tag{4.2.12}$$

$$\text{tensor product of } P_0\text{'s and } I\text{'s} \qquad \text{if } r < j \text{ and } i \le a_r. \tag{4.2.13}$$

Proof: For $j = 1$, it is easy to check from the diagram for $\psi_{\lambda,\omega}$ that (4.2.7) holds for $a_1 + 1 \le i \le b_1 + 1$. To help follow the argument, let us keep the following diagram before us in which the $s_{[a_m,b_m]}$-section and the the $s_{[a_{m'},b_{m'}]}$-section have been enclosed by pairs of coloured vertical lines.



Let us now assume that (4.2.7) holds for $j \le m-1$ and $a_j+1 \le i \le b_j+1$. Take an i such that $a_m+1 \le i \le b_m+1$. Let us look at the case $n(m,i) = -\infty$ first. In this case, $i < a_j$ for all $1 \le j \le m-1$, i.e. the i^{th} horizontal line is below the $s_{[a_j,b_j]}$-rectangles for $1 \le j \le m-1$. Therefore $T_{j,i} = I^{\otimes(\ell_j)}$ for $1 \le j \le m-1$. Since the horizontal line starting at b_m+1 on the left passes above the $s_{[a_j,b_j]}$-rectangles for $j > m$, there is exactly one path going from b_m+1 on the left to i on the right and it does not intersect the $s_{[a_j,b_j]}$-rectangles for $1 \le j \le m-1$. Thus

$$\begin{aligned} Z_{b_m+1,i} &= I^{\otimes(\sum_{r=m+1}^{k}\ell_r)} \otimes \left(P_0^{\otimes(b_m+1-i)} \otimes S^* \otimes I^{\otimes(i-a_m-1)}\right) \otimes I^{\otimes(\sum_{r=1}^{m-1}\ell_r)} \\ &= T_{k,i}^{(m)} \otimes T_{k-1,i}^{(m)} \otimes \cdots T_{1,i}^{(m)}. \end{aligned}$$

Next assume $m' := n(m,i)$ is finite. We have $V_{m,i} = Z_{b_m+1,i}V_{m',i+1}$. Since $m' < m$, by our hypothesis,

$$V_{m',i+1} = T_{k,i+1}^{(m')} \otimes T_{k-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}.$$

Note that since $m' < m$, one has $T_{r,i+1}^{(m')} = I^{\otimes(\ell_r)}$ for $m \le r \le k$. It is also immediate that $Z_{b_m+1,i}$ is of the form $I^{\otimes(\sum_{r=m+1}^{k}\ell_r)} \otimes R$ for some operator R that acts on the last $\sum_{r=1}^{m}\ell_r$ copies of $\ell^2(\mathbb{N})$. The operator R is given by $\sum_{l=i}^{b_m+1} R(l)$ where $R(l)$ is the sum of operators given by the paths that go from b_m+1 to l in the $s_{[a_m,b_m]}$-section and then from l to i in the $s_{[a_m,b_m]}\ldots s_{[a_1,b_1]}$-section.

From the diagram, we conclude that for $l > i$, the $s_{[a_{m'},b_{m'}]}$-section of $R(l)$ is either of the form $I^{\otimes(b_{m'}-l)} \otimes S \otimes S^* \otimes I^{\otimes(l-a_{m'}-1)}$ or of the form $I^{\otimes(b_{m'}-l)} \otimes S \otimes I^{\otimes(l-a_{m'})}$, while $T_{m',i+1}^{(m')} = P_0^{\otimes(b_{m'}-i)} \otimes S^* \otimes I^{\otimes(i-a_{m'})}$. Therefore

$$R(l)\left(T_{m,i+1}^{(m')} \otimes T_{m-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}\right) = 0$$

Hence

$$R\left(T_{m,i+1}^{(m')} \otimes T_{m-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}\right) = R(i)\left(T_{m,i+1}^{(m')} \otimes T_{m-1,i+1}^{(m')} \otimes \cdots T_{1,i+1}^{(m')}\right).$$

Let us write $T_{r,i} = R(i)_r T_{r,i+1}^{(m')}$ for $1 \le r \le m$. We will show that $T_{r,i}$ is given by (4.2.8–4.2.13) for $j = m$. Note that the path $R(i)$ consists of the diagonal line coming down from b_m+1 to i is the $s_{[a_m,b_m]}$-section and its horizonal continuation in the $s_{[a_{m-1},b_{m-1}]}\ldots s_{[a_1,b_1]}$-section. Let us now look at the following three cases separately.

Case I: $1 \leq r \leq m-1$ and $a_r < i$.

Since $a_r < i$, we have $r \leq m'$. Therefore in this case, $R(i)_r = I^{\otimes(b_r - i)} \otimes S \otimes S^* \otimes I^{\otimes(i-a_r-1)}$, and $T^{(m')}_{r,i+1} = P_0^{\otimes(b_r-i)} \otimes S^* \otimes I^{\otimes(i-a_r)}$. Therefore $T_{r,i} = R(i)_r T^{(m')}_{r,i+1} = P_0^{\otimes(b_r-i)} \otimes I \otimes S^* \otimes I^{\otimes(i-a_r-1)}$. Since $r < m$ and $i > a_r$, from (4.2.12) it follows that this is of the form $T^{(m)}_{r,i}$.

Case II: $1 \leq r \leq m-1$ and $a_r = i$.

As in the earlier case, we have $r \leq m'$. Hence $R(i)_r = I^{\otimes(\ell_r - 1)} \otimes S$. Now if $r = m'$, then by (4.2.9) we have $T^{(m')}_{r,i+1} = P_0^{\otimes(\ell_r-1)} \otimes S^*$ and if $r < m'$, then by (4.2.11), we have $T^{(m')}_{r,i+1} = P_0^{\otimes(\ell_r-1)} \otimes I \otimes S^*$ Thus $T_{r,i} = R(i)_r T^{(m')}_{r,i+1}$ is either $P_0^{\otimes(\ell_r-1)} \otimes I$ or $P_0^{\otimes(\ell_r-2)} \otimes I \otimes I$. Since $r < m$ and $i = a_r$, by (4.2.13) $T_{r,i}$ is of the form $T^{(m)}_{r,i}$, as required.

Case III: $1 \leq r \leq m-1$ and $i < a_r$.

In this case, $R(i)_r = I^{\otimes(\ell_r)}$, so that $T_{r,i} = T^{(m')}_{r,i+1}$. If $r > m'$, then $T^{(m')}_{r,i+1} = I^{\otimes \ell_r}$. Since $i+1 \leq a_r$, if $r = m'$, we must have $i+1 = a_r$. Hence by (4.2.12) $T^{(m')}_{r,i+1}$ is a tensor product of P_0's and I's. If $r < m'$ and $i+1 \leq a_r$, then also by (4.2.13), $T^{(m')}_{r,i+1}$ is a tensor product of P_0's and I's. Since $T_{r,i} = T^{(m')}_{r,i+1}$, in all these subcases, it is of the required form. □

We now list a few important corollaries of the above propsition.

Corollary 4.2.4

$$|V_{j,a_j}| \cdot |V_{j-1,a_{j-1}}| \cdots |V_{1,a_1}| = T_k \otimes T_{k-1} \otimes \cdots \otimes T_1,$$

where

$$T_r = \begin{cases} I^{\otimes \ell_r} & \text{if } r > j, \\ P_0^{\otimes(b_r+1-a_r)} & \text{if } r \leq j. \end{cases}$$

Corollary 4.2.5 *Let $a_j \leq i \leq b_j + 1$. Then*

$$V^*_{n'(j,i),i} V_{j,i} = T_k \otimes T_{k-1} \otimes \cdots \otimes T_1, \tag{4.2.14}$$

where

$$T_r = \begin{cases} I^{\otimes \ell_r} & \textit{if } r > j, \\ P_0^{\otimes(b_r+1-a_r)} & \textit{if } r = j \textit{ and } i = a_r, \\ P_0^{\otimes(b_r+1-i)} \otimes S^* \otimes I^{\otimes(i-a_r-1)} & \textit{if } r = j \textit{ and } i \geq a_r+1, \\ \textit{tensor product of } P_0\textit{'s and } I\textit{'s} & \textit{if } r < j. \end{cases} \tag{4.2.15}$$

Proof: Observe that for $r < j$, the section $T_{r,i}^{(j)}$ is of the form (4.2.11) or (4.2.12) (i.e. contains the right shift) exactly when $r < j$ and $a_r < i$. Let $n'(j,i)$ be as defined in (4.2.4). Then for the operator $V_{n'(j,i),i}$ also, the corresponding sections $T_{r,i}^{(s)}$ exactly match $T_{r,i}^{(j)}$ and the remaining ones are tensor products of I and P_0's. Therefore the result follows. $\square$

Corollary 4.2.6 *Let $a_j \leq i \leq b_j+1$. Define*

$$E_{j,i}(\omega) := \begin{cases} V_{n'(j,i),i}^* V_{j,i}|V_{j-1,a_{j-1}}| \cdot |V_{j-2,a_{j-2}}| \cdots |V_{1,a_1}| & \textit{if } n'(j,i) > -\infty \\ V_{j,i}|V_{j-1,a_{j-1}}| \cdot |V_{j-2,a_{j-2}}| \cdots |V_{1,a_1}| & \textit{if } n'(j,i) = -\infty \end{cases}$$

Then

$$E_{j,i}(\omega) = T_k \otimes T_{k-1} \otimes \cdots \otimes T_1, \tag{4.2.16}$$

where

$$T_r = \begin{cases} I^{\otimes \ell_r} & \textit{if } r > j, \\ P_0^{\otimes(b_r+1-i)} \otimes S^* \otimes I^{\otimes(i-a_r-1)} & \textit{if } r = j \textit{ and } i \geq a_r+1, \\ P_0^{\otimes(b_r+1-a_r)} & \textit{if } r = j \textit{ and } i = a_r, \\ P_0^{\otimes(b_r+1-a_r)} & \textit{if } r < j. \end{cases} \tag{4.2.17}$$

Proposition 4.2.7 *For $1 \leq j \leq k$ and $a_j+1 \leq i \leq b_j+1$, let*

$$R_{j,i} = I^{\otimes(\sum_{r=j+1}^{k} \ell_r)} \otimes \left(P_0^{\otimes(b_j+1-i)} \otimes S^* \otimes I^{\otimes(i-a_j-1)}\right) \otimes I^{\otimes(\sum_{r=1}^{j-1} \ell_r)},$$
$$S_{j,i} = I^{\otimes(\sum_{r=j+1}^{k} \ell_r)} \otimes \left(I^{\otimes(b_j+1-i)} \otimes S^* \otimes I^{\otimes(i-a_j-1)}\right) \otimes I^{\otimes(\sum_{r=1}^{j-1} \ell_r)}.$$

Then $R_{j,i}$ and $S_{j,i}$ belong to $\psi_{\lambda,\omega}(C(SU_0(n+1)))''$ for $1 \leq j \leq k$ and $a_j+1 \leq i \leq b_j+1$.

Proof: This is easy to see for $j=1$, as one has $R_{1,i}=V_{1,i}$ for $a_1+1\le i\le b_1+1$, $S_{1,1+b_1}=V_{1,1+b_1}$ and

$$S_{1,i}=\sum_{k_i,\ldots k_{b_j}\in\mathbb{N}} S_{1,1+b_1}^{k_{b_1}}\ldots S_{1,i+1}^{k_i}V_{1,i}(S_{1,1+b_1}^{k_{b_1}})^*\ldots(S_{1,i+1}^{k_i})^* \quad \text{if } a_1\le i\le b_1.$$

Fix a $j<k$. Assume that $R_{r,i},S_{r,i}\in\psi_{\lambda,\omega}(C(SU_0(n+1)))''$ for $1\le r\le j$. Take i such that $a_{j+1}+1\le i\le b_{j+1}+1$. Let us denote the operators in Corollary 4.2.6 by $E_{j,i}$. Then

$$R_{j+1,i}=\sum_{r=1}^{j-1}\sum_{k_{r,a_r},\ldots k_{r,b_r}\in\mathbb{N}} S_{r,1+b_r}^{k_{r,b_r}}\ldots S_{r,1+a_r}^{k_{r,a_r}}E_{j+1,i}(S_{r,1+a_r}^{k_{r,a_r}})^*\ldots(S_{r,1+b_r}^{k_{r,b_r}})^*.$$

Since $E_{j+1,i}$ belongs to the image $\psi_{\lambda,\omega}(C(SU_0(n+1)))$, we have $R_{j+1,i}\in\psi_{\lambda,\omega}(C(SU_0(n+1)))''$. Now note that

$$S_{j+1,i}=\sum_{k_i,\ldots k_{b_j}\in\mathbb{N}} R_{j+1,1+b_j}^{k_{b_j}}\ldots R_{j+1,i+1}^{k_i}R_{j+1,i}(R_{j+1,i+1}^{k_i})^*\ldots(R_{j+1,1+b_j}^{k_{b_j}})^*,$$

Therefore it follows that $S_{j+1,i}$ also belongs to the strong operator closure of $\psi_{\lambda,\omega}(C(SU_0(n+1)))$. □

Theorem 4.2.8 *Let $\lambda\in(S^1)^n$ and $\omega=s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\ldots s_{[a_1,b_1]}$ be a reduced word in $\mathfrak{S}_{n+1}$. Then the representation $\psi_{\lambda,\omega}$ is an irreducible representation.*

Proof: This is an immediate consequence of the previous proposition. □

Theorem 4.2.9 *$\lambda,\lambda'\in(S^1)^n$ and let*

$$\omega=s_{[a_k,b_k]}s_{[a_{k-1},b_{k-1}]}\ldots s_{[a_1,b_1]}, \qquad \omega'=s_{[a'_{k'},b'_{k'}]}s_{[a'_{k'-1},b'_{k'-1}]}\ldots s_{[a'_1,b'_1]}$$

be two reduced words in $\mathfrak{S}_{n+1}$. If $(\lambda,\omega)\neq(\lambda',\omega')$, then the representations $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent.

Proof: Let $1\le j\le n+1$. Define $m_{k+1}(j,\omega)=j$ and for $1\le i\le k$, define $m_i(j,\omega)$ recursively as follows:

$$m_i(j,\omega) = \begin{cases} m_{i+1}(j,\omega)+1 & \text{if } a_i \le m_{i+1}(j,\omega) \le b_i, \\ m_{i+1}(j,\omega) & \text{if } m_{i+1}(j,\omega) \ge b_i+1 \text{ or } m_{i+1}(j,\omega) < a_i. \end{cases}$$

In the diagram for $\psi_{\lambda,\omega}$, one starts at j and proceeds horizontally and goes one step up whenever one encounters a diagonal going upward. The numbers $m_r(j,\omega)$ simply keep track of the horizontal line one is on when one completes the $s_{[a_r,b_r]}$-section. Thus $m_1(j,\omega)$ is the maximum k such that $\psi_{\lambda,\omega}(z_{j,k}) \neq 0$.

Let us first assume that $\omega = \omega'$ and $\lambda \neq \lambda'$. Let

$$j := \max\{1 \le i \le n : \lambda_i \neq \lambda'_i\}.$$

If $b_{i+1} < j < b_i$ for some i, then

$$\operatorname{spec}(\psi_{\lambda,\omega}(z_{j+1,m_1(j+1,\omega)})) = \{0, \overline{\lambda_j}\lambda_{j+1}\},$$
$$\operatorname{spec}(\psi_{\lambda',\omega}(z_{j+1,m_1(j+1,\omega)})) = \{0, \overline{\lambda'_j}\lambda'_{j+1}\},$$

i.e. the spectrums of the same element are different for the two representations. Therefore $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent. Similarly if $j = b_i$ for some i, then write $m = n'(i,a_i)$. By Corollary 4.2.5 and keeping track of the effect of λ and λ', we get that if $m > -\infty$, then

$$\begin{aligned}\operatorname{spec}\big(\psi_{\lambda',\omega}(E_{i,a_i}(\omega))\big) &= \Big\{0, \overline{\lambda}'_{b_m}\overline{\lambda}'_j\lambda'_{b_m+1}\lambda'_{j+1}\Big\} \\ &= \Big\{0, \overline{\lambda}_{b_m}\overline{\lambda}'_j\lambda_{b_m+1}\lambda_{j+1}\Big\}, \\ \operatorname{spec}\big(\psi_{\lambda,\omega}(E_{i,a_i}(\omega))\big) &= \Big\{0, \overline{\lambda}_{b_m}\overline{\lambda}_j\lambda_{b_m+1}\lambda_{j+1}\Big\},\end{aligned}$$

and if $m = -\infty$, then

$$\operatorname{spec}\big(\psi_{\lambda',\omega}(E_{i,a_i}(\omega))\big) = \{0, \overline{\lambda'_j}\lambda'_{j+1}\} = \{0, \overline{\lambda'_j}\lambda_{j+1}\}$$
$$\operatorname{spec}\big(\psi_{\lambda,\omega}(E_{i,a_i}(\omega))\big) = \{0, \overline{\lambda_j}\lambda_{j+1}\}.$$

Thus $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent in this case also.

Let us next assume that $\omega \neq \omega'$. We will split this into different cases and in each case, produce an element of $C(SU_0(n+1))$ whose image under one representation will be zero while the image under the other is nonzero, thus proving that the two representations are not equivalent.

Case I:

There exists $j \leq \min\{k, k'\}$ such that $b_i = b'_i$, $a_i = a'_i$ for $1 \leq i \leq j-1$ and $b_j \neq b'_j$.
Assume without loss in generality that $b_j > b'_j$. Then $m_1(b_j, \omega) > m_1(b_j, \omega')$. Therefore

$$\psi_{\lambda',\omega'}(z_{b_j,m_1(b_j,\omega)}) = 0,$$
$$\psi_{\lambda,\omega}(z_{b_j,m_1(b_j,\omega)}) \neq 0.$$

Case II:

There exists $j \leq \min\{k, k'\}$ such that $b_i = b'_i$, $a_i = a'_i$ for $1 \leq i \leq j-1$, $b_j = b'_j$ and $a_j \neq a'_j$.
Assume without loss in generality that $a_j > a'_j$. Once again using Corollary 4.2.6, we observe that modulo multiplication by complex numbers of modulus one, $\psi_{\lambda,\omega}(E_{j,a_j}(\omega))$ is a projection, while $\psi_{\lambda',\omega'}(E_{j,a_j}(\omega))$ is a tensor product of shifts and projections. Therefore $\psi_{\lambda,\omega}$ and $\psi_{\lambda',\omega'}$ are inequivalent.

Case III:

$b_i = b'_i$, $a_i = a'_i$ for $1 \leq i \leq k'$ and $k' < k$.
In this case $m_1(b_k, \omega) > m_1(b_k, \omega')$. Therefore

$$\psi_{\lambda',\omega'}(z_{b_k,m_1(b_k,\omega)}) = 0,$$
$$\psi_{\lambda,\omega}(z_{b_k,m_1(b_k,\omega)}) \neq 0.$$

Case IV:

$b_i = b'_i$, $a_i = a'_i$ for $1 \leq i \leq k$ and $k < k'$.
This case is identical to Case III with k and k' interchanged. □

4.2.1 Operators in $\pi(C(SU_0(n+1)))''$

In this section, we will study the actions of the generating elements $z_{i,j}$ on a Hilbert space and construct certain operators using the strong operator topology available there. These will turn out to be very useful tool in analysing the irreducible representations. Let us start with a representation π of $C(SU_0(n+1))$ on a Hilbert space $\mathcal{H}$. For all

i and j, we will denote the operators $\pi(z_{i,j})$, $\pi(p_{i,j})$ and $\pi(q_{i,j})$ by $Z_{i,j}$, $P_{i,j}$ and $Q_{i,j}$ respectively.

Proposition 4.2.10 *Let $1 \leq s \leq m \leq n$. Assume $Z_{m+1,s} = 0$ where $s < m+1$. Then $Z_{i,j} = 0$ for $i \geq m+1$ and $j \leq s$.*

Proof: Let us assume $s > 1$ and take $i = m+1$, $j = s-1$. From relation (2.2.21), we have $Z_{m+1,s-1}Z_{s,s} = 0$, so that

$$Z_{m+1,s-1}Q_{s,s} = 0.$$

For $1 \leq k \leq s-1$, the commutation relation (2.2.22) gives us

$$Z_{k,s-1}Z_{m+1,s} - Z_{m+1,s}Z_{k,s-1} = Z_{m+1,s-1}Z_{k,s}.$$

Since $Z_{m+1,s} = 0$, we get $Z_{m+1,s-1}Z_{k,s} = 0$ which implies

$$Z_{m+1,s-1}Q_{k,s} = 0, \qquad 1 \leq k \leq s-1.$$

Therefore

$$Z_{m+1,s-1} = Z_{m+1,s-1}\left(\sum_{k=1}^{s} Q_{k,s}\right) = 0. \tag{4.2.18}$$

Next, let us assume $m < n$ and take $i = m+2$, $j = s$. Using relation (2.2.21) we get $Z_{m+1,m+1}Z_{m+2,s} = 0$, so that

$$P_{m+1,m+1}Z_{m+2,s} = 0.$$

For $m+2 \leq k \leq n+1$, we get from (2.2.22)

$$Z_{m+1,s}Z_{m+2,k} - Z_{m+2,k}Z_{m+1,s} = Z_{m+1,k}Z_{m+2,s}.$$

Since $Z_{m+1,s} = 0$, we get $Z_{m+1,k}Z_{m+2,s} = 0$ which implies

$$P_{m+1,k}Z_{m+2,s} = 0, \qquad m+2 \leq k \leq n+1.$$

Therefore

$$Z_{m+2,s} = \left(\sum_{k=m+1}^{n+1} P_{m+1,k}\right) Z_{m+2,s} = 0. \tag{4.2.19}$$

By repeated application of (4.2.18) and (4.2.19), we get $Z_{i,j}=0$ for $i\geq m+1$ and $j\leq s$. $\square$

Lemma 4.2.11 *Assume $i\neq m$ and $j\neq k$. If $Z_{i,j}Q_{m,k}=Q_{m,k}Z_{i,j}$ and $Z_{m,k}Q_{i,j}=Q_{i,j}Z_{m,k}$, then one has $Z_{i,j}Z_{m,k}=Z_{m,k}Z_{i,j}$.*

Proof: Computing $\left(Z_{i,j}Z_{m,k}-Z_{m,k}Z_{i,j}\right)\left(Z_{i,j}Z_{m,k}-Z_{m,k}Z_{i,j}\right)^*$ using the given relations we get the required equality. $\square$

Lemma 4.2.12 *Let F be a subset of $\{1,2,\ldots,n+1\}$. Let $j\notin F$ and $i\neq m$. Assume $Z_{m,k}Q_{i,j}=Q_{i,j}Z_{m,k}$ for all $k\in F$ and $Z_{i,j}\left(\sum_{k\in F}Q_{m,k}\right)=\left(\sum_{k\in F}Q_{m,k}\right)Z_{i,j}$. Then*

$$Z_{i,j}Z_{m,k}=Z_{m,k}Z_{i,j} \qquad \textit{for all } k\in F.$$

Proof: Note that

$$\begin{aligned}
&\sum_{k\in F}\left(Z_{i,j}Z_{m,k}-Z_{m,k}Z_{i,j}\right)\left(Z_{i,j}Z_{m,k}-Z_{m,k}Z_{i,j}\right)^*\\
=&Z_{i,j}\left(\sum_{k\in F}Q_{m,k}\right)Z_{i,j}^*+\sum_{k\in F}\left(Z_{m,k}Q_{i,j}Z_{m,k}^*\right)-Z_{i,j}\sum_{k\in F}\left(Z_{m,k}Z_{i,j}^*Z_{m,k}^*\right)-\sum_{k\in F}\left(Z_{m,k}Z_{i,j}Z_{m,k}^*\right)Z_{i,j}^*\\
=&\left(\sum_{k\in F}Q_{m,k}\right)Z_{i,j}Z_{i,j}^*+Q_{i,j}\sum_{k\in F}\left(Z_{m,k}Z_{m,k}^*\right)-Z_{i,j}Z_{i,j}^*\sum_{k\in F}\left(Z_{m,k}Z_{m,k}^*\right)-\sum_{k\in F}\left(Z_{m,k}Z_{m,k}^*\right)Z_{i,j}Z_{i,j}^*\\
=&0.
\end{aligned}$$

Therefore we have the result. $\square$

Given a representation π of $C(SU_0(n+1))$, we will denote by $r(\pi)\equiv r$ the following

$$r(\pi)\equiv r:=\min\{i:1\leq i\leq n+1, Z_{n+1,i}\neq 0\}. \tag{4.2.20}$$

Proposition 4.2.13 *Assume $2\leq r\leq n$. Then one has*

$$Z_{i,j}Z_{n+1,k}=Z_{n+1,k}Z_{i,j} \quad \textit{for } 1\leq i\leq n,\ 1\leq j\leq r-1 \textit{ and } r\leq k\leq n+1. \tag{4.2.21}$$

Proof: Observe that for $1\leq i\leq r-1$, $1\leq j\leq r-1$ and $r\leq k\leq n+1$, one has $\max\{i,j\}+1\leq r\leq\min\{n+1,k\}$. If $\max\{i,j\}+1<\min\{n+1,k\}$, then $Z_{i,j}Z_{n+1,k}=$

$Z_{n+1,k}Z_{i,j}$. Since by Proposition 4.2.10, we have $Z_{n+1,j} = 0$ for $1 \leq j \leq r-1$, if $\max\{i,j\}+1 = \min\{n+1,k\}$, then one has

$$Z_{i,j}Z_{n+1,k} = Z_{n+1,k}Z_{i,j} + Z_{n+1,j}Z_{i,k} = Z_{n+1,k}Z_{i,j}.$$

Let us next assume that $Z_{i,j}Z_{n+1,k} = Z_{n+1,k}Z_{i,j}$ for all $1 \leq j \leq r-1$, $r \leq k \leq n+1$ and for all $1 \leq i \leq t < n$. Then $P_{i,j}Z_{n+1,k} = Z_{n+1,k}P_{i,j}$ for all such j, k and i. Therefore

$$Z_{n+1,k} = Z_{n+1,k}\left(\sum_{i=j}^{n} P_{i,j}\right) = \left(\sum_{i=j}^{n} P_{i,j}\right) Z_{n+1,k}.$$

Since $P_{i,j}Z_{n+1,k} = Z_{n+1,k}P_{i,j}$ for $1 \leq i \leq t$ and $\sum_{i=t+1}^{n} P_{i,j} = \sum_{m=1}^{j} Q_{t+1,m}$, it follows that

$$Z_{n+1,k}\left(\sum_{m=1}^{j} Q_{t+1,m}\right) = \left(\sum_{m=1}^{j} Q_{t+1,m}\right) Z_{n+1,k}.$$

Therefore

$$Z_{n+1,k}Q_{t+1,j} = Q_{t+1,j}Z_{n+1,k}$$

for $1 \leq j \leq r-1$. On the other hand, we also have

$$Z_{t+1,j} = Z_{t+1,j}\left(\sum_{k=r}^{n+1} Q_{n+1,k}\right) = \left(\sum_{k=r}^{n+1} Q_{n+1,k}\right) Z_{t+1,j}.$$

Therefore by Lemma 4.2.12, we have

$$Z_{t+1,j}Z_{n+1,k} = Z_{n+1,k}Z_{t+1,j}$$

for $r \leq k \leq n+1$ and $1 \leq j \leq r-1$.

By repeated application, we now get the result. $\square$

We next look at certain infinite sums of elements from $\pi(C(SU_0(n+1)))$ that converge in the strong operator topology (SOT) so that they belong to the SOT-closure $\pi(C(SU_0(n+1)))''$ of $\pi(C(SU_0(n+1)))$.

Proposition 4.2.14 *Let r be as in (4.2.20) and let $r \leq k \leq n+1$. If T_k is a partial isometry with $T_k^* T_k \leq P_{n+1,k}$ and $T_k T_k^* \leq P_{n+1,k}$, then the operator*

$$T_{k+1} := \sum_{m \in \mathbb{N}} Z_{n+1,k+1}^m T_k (Z_{n+1,k+1}^m)^*$$

exists as a strong operator convergent sum and is a partial isometry with $T_{k+1}^* T_{k+1} \leq P_{n+1,k+1}$ and $T_{k+1} T_{k+1}^* \leq P_{n+1,k+1}$.

Proof: Define the operators

$$R_{k,m} := Z_{n+1,k+1}^m P_{n+1,k} (Z_{n+1,k+1}^m)^*, \qquad m \in \mathbb{N}.$$

Then $R_{k,m}$'s form a family of orthogonal projections. Observe that

$$P_{n+1,k+1} - \sum_{j=0}^{m} R_{k,j} = Z_{n+1,k+1}^{m+1} (Z_{n+1,k+1}^{m+1})^*$$

and the sum $\sum_{j \in \mathbb{N}} R_{k,j}$ converges in SOT. Therefore we have $\sum_{j \in \mathbb{N}} R_{k,j} \leq P_{n+1,k+1}$. Hence

$$P_{n+1,k+1} - \sum_{j \in \mathbb{N}} R_{k,j} = \text{SOT-}\lim_{m \to \infty} Z_{n+1,k+1}^m (Z_{n+1,k+1}^m)^*. \tag{4.2.22}$$

Next, write

$$T_{k,m} = Z_{n+1,k+1}^m T_k (Z_{n+1,k+1}^m)^*, \qquad m \in \mathbb{N}.$$

Then one has $T_{k,m} = R_{k,m} T_{k,m} R_{k,m}$ for all $m \in \mathbb{N}$. It is easy to check that $T_{k,m}^* T_{k,m'} = 0 = T_{k,m} T_{k,m'}^*$ for $m \neq m'$ and one has

$$T_{k,m}^* T_{k,m} \leq P_{n+1,k+1},$$
$$T_{k,m} T_{k,m}^* \leq P_{n+1,k+1}.$$

Therefore the sum $T := \sum_{m \in \mathbb{N}} T_{k,m}$ converges in SOT and it is a partial isometry with both initial and range spaces contained in $P_{n+1,k+1}\mathcal{H}$. $\square$

Using the above proposition, one can now define recursively the operators $W_{i,j}$ for $1 \leq j \leq n+1$ and $j \leq i \leq n+1$ and the operators U_i for $r+1 \leq i \leq n+1$ as follows:

$$W_{i,j} := \begin{cases} Z_{n+1,j}, & \text{if } i = j, \\ \sum_{k\in\mathbb{N}} Z_{n+1,i}^k W_{i-1,j} (Z_{n+1,i}^k)^*, & \text{if } j+1 \leq i \leq n+1. \end{cases} \tag{4.2.23}$$

$$U_i := \begin{cases} Z_{n+1,r}, & \text{if } i = r+1, \\ \sum_{k\in\mathbb{N}} Z_{n+1,i}^k U_{i-1} (Z_{n+1,i}^k)^*, & \text{if } r+2 \leq i \leq n+1. \end{cases} \tag{4.2.24}$$

Thus we have

$$W_{i,r+1} = \sum_{k_{r+2},\ldots,k_i\in\mathbb{N}} Z_{n+1,i}^{k_i} Z_{n+1,i-1}^{k_{i-1}} \cdots Z_{n+1,r+2}^{k_{r+2}} Z_{n+1,r+1} (Z_{n+1,r+2}^{k_{r+2}})^* \cdots (Z_{n+1,i-1}^{k_{i-1}})^* (Z_{n+1,i}^{k_i})^*, \tag{4.2.25}$$

$$W_{i,r} = \sum_{k_{r+2},\ldots,k_i\in\mathbb{N}} Z_{n+1,i}^{k_i} Z_{n+1,i-1}^{k_{i-1}} \cdots Z_{n+1,r+2}^{k_{r+2}} Z_{n+1,r} (Z_{n+1,r+2}^{k_{r+2}})^* \cdots (Z_{n+1,i-1}^{k_{i-1}})^* (Z_{n+1,i}^{k_i})^*, \tag{4.2.26}$$

$$U_i = \sum_{k_{r+2},\ldots,k_i\in\mathbb{N}} Z_{n+1,i}^{k_i} Z_{n+1,i-1}^{k_{i-1}} \cdots Z_{n+1,r+2}^{k_{r+2}} Z_{n+1,r} (Z_{n+1,r+2}^{k_{r+2}})^* \cdots (Z_{n+1,i-1}^{k_{i-1}})^* (Z_{n+1,i}^{k_i})^*. \tag{4.2.27}$$

We will use these three operators repeatedly in the rest of the paper. We start with a few observations on these operators.

Proposition 4.2.15 *Let r be as in (4.2.20). Then*

1. *U_i is a normal partial isometry for $r+2 \leq i \leq n+1$,*
2. *$W_{i,r}$ is a normal partial isometry for all $r \leq i \leq n+1$,*
3. *$W_{i,r+1}$ is a partial isometry with source projection and range projection contained in $P_{n+1,i}$ for $r+2 \leq i \leq n+1$,*

Proof: By Proposition 4.2.14, the operators U_i, $W_{i,r}$ and $W_{i,r+1}$ are all partial isometries. Normality of U_i and $W_{i,r}$ follow from the fact that $P_{n+1,r} = Q_{n+1,r}$. It is easy to see from (4.2.25) and (4.2.27) that

$$W_{i,r+1}^* W_{i,r+1} \leq P_{n+1,i}, \tag{4.2.28}$$

$$W_{i,r+1}^* W_{i,r+1} - W_{i,r+1} W_{i,r+1}^* = U_i^* U_i. \tag{4.2.29}$$

Therefore part 3 follows. $\square$

Proposition 4.2.16 *Let r be as in (4.2.20). Then we have the following relations:*

$$Z_{i,r}W_{n+1,r+1} - W_{n+1,r+1}Z_{i,r} = U_{n+1}Z_{i,r+1}, \quad i \le r, \tag{4.2.30}$$

$$Z_{i,r}W^*_{n+1,r+1} - W^*_{n+1,r+1}Z_{i,r} = 0, \quad i \le r. \tag{4.2.31}$$

Proof: Since $i \le r$, the operator $Z_{i,r}$ commutes with $Z_{n+1,j}$ for $r+2 \le j \le n+1$. Using this and the relation (2.2.22), we get (4.2.30). Equation (4.2.31) follows by using (2.2.22) and (2.2.26). $\square$

Theorem 4.2.17 *Let r be as in (4.2.20). Then the operator $W_{n+1,r}$ commutes with all the $Z_{i,j}$'s.*

Proof: For $1 \le j \le r-1$, the commutaions follow from Proposition 4.2.13.

Assume next that $1 \le i \le j = r$. Then

$$\begin{aligned} Z_{i,r}W_{r+1,r} &= Z_{i,r}\sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}(Z^k_{n+1,r+1})^* \\ &= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{i,r}Z_{n+1,r}(Z^k_{n+1,r+1})^* \\ &+ \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}Z_{i,r+1}Z_{n+1,r}(Z^{k+1}_{n+1,r+1})^* \\ &= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z^2_{n+1,r}Z_{i,r+1}(Z^{k+1}_{n+1,r+1})^*, \end{aligned}$$

and

$$\begin{aligned} W_{r+1,r}Z_{i,r} &= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}(Z^*_{n+1,r+1})^kZ_{i,r} \\ &= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}Z_{i,r}(Z^k_{n+1,r+1})^*. \end{aligned}$$

Thus we have

$$Z_{i,r}W_{r+1,r} - W_{r+1,r}Z_{i,r} = \sum_{k\in\mathbb{N}} Z^k_{n+1,r+1}Z_{n+1,r}\left(Z_{n+1,r}Z_{i,r+1}Z^*_{n+1,r+1} - Z_{i,r}\right)(Z^k_{n+1,r+1})^*.$$

Now

$$
\begin{aligned}
&\left(Z_{n+1,r}^2 Z_{i,r+1} Z_{n+1,r+1}^* - Z_{n+1,r} Z_{i,r}\right)\left(Z_{n+1,r}^2 Z_{i,r+1} Z_{n+1,r+1}^* - Z_{n+1,r} Z_{i,r}\right)^* \\
=& Z_{n+1,r} Q_{i,r} Z_{n+1,r}^* - Q_{n+1,r} Q_{i,r+1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_{i=1}^{r}\left(Z_{n+1,r}^2 Z_{i,r+1} Z_{n+1,r+1}^* - Z_{n+1,r} Z_{i,r}\right)\left(Z_{n+1,r}^2 Z_{i,r+1} Z_{n+1,r+1}^* - Z_{n+1,r} Z_{i,r}\right)^* \\
=& Z_{n+1,r}\left(\sum_{i=1}^{r} Q_{i,r}\right) Z_{n+1,r}^* - Q_{n+1,r}\left(\sum_{i=1}^{r} Q_{i,r+1}\right) \\
=& Q_{n+1,r} - Q_{n+1,r}\left(\sum_{k=r+1}^{n+1} P_{r,k}\right) \\
=& Q_{n+1,r} P_{r,r} \\
=& P_{n+1,r} P_{r,r} \\
=& 0.
\end{aligned}
$$

Thus we have $Z_{i,r} W_{r+1,r} = W_{r+1,r} Z_{i,r}$ for $1 \leq i \leq r$. Since $Z_{i,r}$ commutes with $Z_{n+1,j}$ for all $j \geq r+2$, we get

$$
Z_{i,r} W_{n+1,r} = W_{n+1,r} Z_{i,r}.
$$

Next, let $1 \leq i \leq r+1$. Then one has

$$
\begin{aligned}
Z_{i,r+1}W_{r+2,r} =& Z_{i,r+1}\sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r+1}^{s} Z_{n+1,r} (Z_{n+1,r+1}^{s})^{*} (Z_{n+1,r+2}^{k})^{*} \\
=& \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{i,r+1} Z_{n+1,r+1}^{s} Z_{n+1,r} (Z_{n+1,r+1}^{s})^{*} (Z_{n+1,r+2}^{k})^{*} \\
&+ \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r+1} Z_{i,r+2} Z_{n+1,r+1}^{s} Z_{n+1,r} (Z_{n+1,r+1}^{s})^{*} (Z_{n+1,r+2}^{k+1})^{*} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{i,r+1} Z_{n+1,r} (Z_{n+1,r+2}^{k})^{*} \\
&+ \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r+1}^{s+1} Z_{i,r+2} Z_{n+1,r} (Z_{n+1,r+1}^{s})^{*} (Z_{n+1,r+2}^{k+1})^{*} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r} (Z_{n+1,r+2}^{k})^{*} Z_{i,r+1} \\
&+ \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r+1} W_{r+1,r} Z_{i,r+2} (Z_{n+1,r+2}^{k+1})^{*},
\end{aligned}
$$

and

$$
\begin{aligned}
W_{r+2,r}Z_{i,r+1} =& \sum_{k\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r} (Z_{n+1,r+2}^{k})^{*} Z_{i,r+1} \\
&+ \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r+1}^{s+1} Z_{n+1,r} (Z_{n+1,r+1}^{s+1})^{*} (Z_{n+1,r+2}^{k})^{*} Z_{i,r+1} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r} (Z_{n+1,r+2}^{k})^{*} Z_{i,r+1} \\
&+ \sum_{k\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r+1} W_{r+1,r} Z_{n+1,r+1}^{*} Z_{i,r+1} (Z_{n+1,r+2}^{k})^{*}.
\end{aligned}
$$

Using Lemma 4.2.9, we now get

$$
\begin{aligned}
& Z_{i,r+1}W_{r+2,r} - W_{r+2,r}Z_{i,r+1} \\
&= \sum_{k\in\mathbb{N}} Z_{n+1,r+2}^{k} Z_{n+1,r+1} W_{r+1,r} \left(Z_{i,r+2} Z_{n+1,r+2}^{*} - Z_{n+1,r+1}^{*} Z_{i,r+1} \right) Z_{n+1,r+1}^{*} Z_{i,r+1} (Z_{n+1,r+2}^{k})^{*} \\
&= 0.
\end{aligned}
$$

Thus we have $Z_{i,r+1}W_{r+2,r} = W_{r+2,r}Z_{i,r+1}$. Since $Z_{i,r+1}$ commutes with $Z_{n+1,j}$ for all $j \geq r+3$, we get

$$
Z_{i,r+1}W_{n+1,r} = W_{n+1,r}Z_{i,r+1}.
$$

Let $j \geq r+2$ and $1 \leq i \leq j$. Then one has

$$\begin{aligned}
Z_{i,j}W_{j+1,r} &= Z_{i,j}\sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} W_{j,r}(Z_{n+1,j+1}^{*})^{k} \\
&= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} Z_{i,j} W_{j,r}(Z_{n+1,j+1}^{*})^{k} \\
&\quad + \sum_{k\geq 1} Z_{n+1,j+1}^{k-1} Z_{n+1,j} Z_{i,j+1} W_{j,r}(Z_{n+1,j+1}^{*})^{k} \\
&= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} W_{j-1,r}(Z_{n+1,j+1}^{*})^{k} Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} Z_{n+1,j} W_{j,r} Z_{i,j+1}(Z_{n+1,j+1}^{*})^{k+1},
\end{aligned}$$

and

$$\begin{aligned}
W_{j+1,r}Z_{i,j} &= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} W_{j,r}(Z_{n+1,j+1}^{*})^{k} Z_{i,j} \\
&= \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,j+1}^{k} Z_{n+1,j}^{s} W_{j-1,r}(Z_{n+1,j}^{*})^{s}(Z_{n+1,j+1}^{*})^{k} Z_{i,j} \\
&= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} W_{j-1,r}(Z_{n+1,j+1}^{*})^{k} Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}}\sum_{s\geq 1} Z_{n+1,j+1}^{k} Z_{n+1,j}^{s} W_{j-1,r}(Z_{n+1,j}^{*})^{s}(Z_{n+1,j+1}^{*})^{k} Z_{i,j} \\
&= \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} W_{j-1,r}(Z_{n+1,j+1}^{*})^{k} Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} Z_{n+1,j} W_{j,r} Z_{n+1,j}^{*}(Z_{n+1,j+1}^{*})^{k} Z_{i,j}.
\end{aligned}$$

Use Lemma 4.2.9 to get

$$\begin{aligned}
&Z_{i,j}W_{j+1,r} - W_{j+1,r}Z_{i,j} \\
=&\sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} Z_{n+1,j} W_{j,r} Z_{i,j+1}(Z_{n+1,j+1}^{*})^{k+1} \\
&\quad - \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} Z_{n+1,j} W_{j,r} Z_{n+1,j}^{*}(Z_{n+1,j+1}^{*})^{k} Z_{i,j} \\
=&\sum_{k\in\mathbb{N}} Z_{n+1,j+1}^{k} Z_{n+1,j} W_{j,r}\left(Z_{i,j+1}Z_{n+1,j+1}^{*} - Z_{n+1,j}^{*}Z_{i,j}\right)(Z_{n+1,j+1}^{*})^{k} \\
=&0.
\end{aligned}$$

Thus we have $Z_{i,j}W_{j+1,r} = W_{j+1,r}Z_{i,j}$. Since $Z_{i,j}$ commutes with $Z_{n+1,k}$ for all $k \geq j+2$, we get

$$Z_{i,j}W_{n+1,r} = W_{n+1,r}Z_{i,j}.$$

Thus $W_{n+1,r}$ commutes with $Z_{i,j}$ for all $i \leq j$. Therefore using (2.2.25), we conclude that $W_{n+1,r}$ commutes with $Z^*_{i,j}$ for all $i > j$. Since $W_{n+1,r}$ is normal, it follows that it commutes with all $Z_{i,j}$'s. ☐

Proposition 4.2.18 *Let r be as in (4.2.20). Then*

$$W^*_{n+1,r}U_{n+1} = U^*_{n+1}U_{n+1}. \tag{4.2.32}$$

Proof: We have

$$\begin{aligned}
&W^*_{n+1,r}U_{n+1}\\
&= \left(\sum_{k_i\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}Z^{k_n}_{n+1,n}\cdots Z^{k_{r+1}}_{n+1,r+1}Z^*_{n+1,r}(Z^*_{n+1,r+1})^{k_{r+1}}\cdots(Z^*_{n+1,n})^{k_n}(Z^*_{n+1,n+1})^{k_{n+1}}\right)\\
&\quad\times\left(\sum_{k_i\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}Z^{k_n}_{n+1,n}\cdots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n})^{k_n}(Z^*_{n+1,n+1})^{k_{n+1}}\right)\\
&= \left(\sum_{k_i\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}Z^{k_n}_{n+1,n}\cdots Z^{k_{r+2}}_{n+1,r+2}\right.\\
&\quad\times\left.\left(\sum_{k_{r+1}\in\mathbb{N}} Z^{k_{r+1}}_{n+1,r+1}Z^*_{n+1,r}(Z^*_{n+1,r+1})^{k_{r+1}}\right)Z_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n})^{k_n}(Z^*_{n+1,n+1})^{k_{n+1}}\right)\\
&= \left(\sum_{k_i\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}Z^{k_n}_{n+1,n}\cdots Z^{k_{r+2}}_{n+1,r+2}\left(Z^*_{n+1,r}Z_{n+1,r}\right)(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n})^{k_n}(Z^*_{n+1,n+1})^{k_{n+1}}\right).
\end{aligned}$$

$$\begin{aligned}
&U^*_{n+1}U_{n+1}\\
&= \left(\sum_{k_i\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}Z^{k_n}_{n+1,n}\cdots Z^{k_{r+2}}_{n+1,r+2}Z^*_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n})^{k_n}(Z^*_{n+1,n+1})^{k_{n+1}}\right)\\
&\quad\times\left(\sum_{k_i\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}Z^{k_n}_{n+1,n}\cdots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n})^{k_n}(Z^*_{n+1,n+1})^{k_{n+1}}\right)\\
&= \left(\sum_{k_i\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}Z^{k_n}_{n+1,n}\cdots Z^{k_{r+2}}_{n+1,r+2}\left(Z^*_{n+1,r}Z_{n+1,r}\right)(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n})^{k_n}(Z^*_{n+1,n+1})^{k_{n+1}}\right).
\end{aligned}$$

Thus we have the required equality. □

4.2.2 Implications of irreducibility

In this section, we will further restrict our attention to an irreducible representation π. Irreducibility will help us extract more information on the operators $W_{i,j}$'s, which will be needed in analysing all irreducible representations.

Proposition 4.2.19 *Assume π is irreducible. Then there is a $\lambda \in S^1$ such that $W_{n+1,r} = \lambda I$.*

Proof: Since $W_{n+1,r}$ is normal, by Theorem 4.2.17 and by the irreducibility of π it is a scalar operator. But it is also a partial isometry and

$$W^*_{n+1,r}W_{n+1,r} \geq P_{n+1,r} \neq 0.$$

Therefore we have the result. □

Our next goal is to show that for an irreducible representation π, the operator $W_{n+1,r+1}$ is an isometry. For that, we start with a few commutation relations.

Proposition 4.2.20 *Let $1 \leq j \leq r-1$ and $1 \leq i \leq n$. Then*

$$W_{n+1,r+1}Z_{i,j} = Z_{i,j}W_{n+1,r+1},$$
$$W^*_{n+1,r+1}Z_{i,j} = Z_{i,j}W^*_{n+1,r+1}$$

Proof: This is an immediate consequence of Proposition 4.2.13. □

Proposition 4.2.21

$$W_{i,r+1}Z_{n+1,j} = 0, \qquad r+2 \leq i < j. \tag{4.2.33}$$
$$Z_{n+1,j}W_{i,r+1} = W_{i,r+1}Z_{n+1,j}, \qquad j \leq i \leq n+1. \tag{4.2.34}$$
$$Z_{i,j}W_{j,r+1} = Z_{i,j}W_{j-1,r+1} \tag{4.2.35}$$
$$= W_{j-1,r+1}Z_{i,j}, \qquad i \leq j,\ r+2 \leq j. \tag{4.2.36}$$

Proof: The first equality follows from the fact that $Q_{n+1,i}Q_{n+1,j} = 0$ for $i \neq j$.

For the second equality, observe that if $j < i$, then

$$
\begin{aligned}
Z_{n+1,j}W_{i,r+1} =& Z_{n+1,j}\sum_{k\in\mathbb{N}} Z_{n+1,i}^{k}W_{i-1,r+1}(Z_{n+1,i}^{k})^{*} \\
=& Z_{n+1,j}W_{i-1,r+1}, \\
W_{i,r+1}Z_{n+1,j} =& \sum_{k\in\mathbb{N}} Z_{n+1,i}^{k}W_{i-1,r+1}(Z_{n+1,i}^{k})^{*}Z_{n+1,j} \\
=& W_{i-1,r+1}Z_{n+1,j}.
\end{aligned}
$$

By repeating the above, we get $Z_{n+1,j}W_{i,r+1} = Z_{n+1,j}W_{j,r+1}$ and $W_{i,r+1}Z_{n+1,j} = W_{j,r+1}Z_{n+1,j}$. Finally, for $j = i$ one has

$$
\begin{aligned}
Z_{n+1,j}W_{j,r+1} =& Z_{n+1,j}\sum_{k\in\mathbb{N}} Z_{n+1,j}^{k}W_{j-1,r+1}(Z_{n+1,j}^{k})^{*} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,j}^{k+1}W_{j-1,r+1}(Z_{n+1,j}^{k})^{*} \\
=& Z_{n+1,j}W_{j-1,r+1} + \sum_{k\geq 1} Z_{n+1,j}^{k+1}W_{j-1,r+1}(Z_{n+1,j}^{k})^{*} \\
=& Z_{n+1,j}W_{j-1,r+1}Z_{n+1,j}^{*}Z_{n+1,j} + \sum_{k\geq 1} Z_{n+1,j}^{k+1}W_{j-1,r+1}(Z_{n+1,j}^{k+1})^{*}Z_{n+1,j} \\
=& \sum_{k\geq 1} Z_{n+1,j}^{k}W_{j-1,r+1}(Z_{n+1,j}^{k})^{*}Z_{n+1,j}, \\
=& W_{j,r+1}Z_{n+1,j} - W_{j-1,r+1}Z_{n+1,j} \\
=& W_{j,r+1}Z_{n+1,j}.
\end{aligned}
$$

The last equation follows from the observation that $Z_{i,i}Z_{n+1,i} = 0$. □

Proposition 4.2.22 *Let $j \geq r+2$ and $1 \leq i \leq j$. Then*

$$
\begin{aligned}
Z_{i,j}W_{j+1,r+1} &= W_{j+1,r+1}Z_{i,j}, \\
Z_{i,j}W_{j+1,r+1}^{*} &= W_{j+1,r+1}Z_{i,j}^{*}.
\end{aligned}
$$

Proof: Let $j \geq r+2$ and $1 \leq i \leq j$. Then one has

$$\begin{aligned}
Z_{i,j}W_{j+1,r+1} =& Z_{i,j}\sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j,r+1}(Z_{n+1,j+1}^*)^k \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{i,j} W_{j,r+1}(Z_{n+1,j+1}^*)^k \\
&\quad + \sum_{k\geq 1} Z_{n+1,j+1}^{k-1} Z_{n+1,j} Z_{i,j+1} W_{j,r+1}(Z_{n+1,j+1}^*)^k \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j-1,r+1}(Z_{n+1,j+1}^*)^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r+1} Z_{i,j+1}(Z_{n+1,j+1}^*)^{k+1},
\end{aligned}$$

and

$$\begin{aligned}
W_{j+1,r+1}Z_{i,j} =& \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j,r+1}(Z_{n+1,j+1}^*)^k Z_{i,j} \\
=& \sum_{k\in\mathbb{N}}\sum_{s\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j}^s W_{j-1,r+1}(Z_{n+1,j}^*)^s(Z_{n+1,j+1}^*)^k Z_{i,j} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j-1,r+1}(Z_{n+1,j+1}^*)^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}}\sum_{s\geq 1} Z_{n+1,j+1}^k Z_{n+1,j}^s W_{j-1,r+1}(Z_{n+1,j}^*)^s(Z_{n+1,j+1}^*)^k Z_{i,j} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k W_{j-1,r+1}(Z_{n+1,j+1}^*)^k Z_{i,j} \\
&\quad + \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r+1}(Z_{n+1,j}^*)(Z_{n+1,j+1}^*)^k Z_{i,j}.
\end{aligned}$$

Use Lemma 4.2.9 to get

$$\begin{aligned}
& Z_{i,j}W_{j+1,r+1} - W_{j+1,r+1}Z_{i,j} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r+1} Z_{i,j+1}(Z_{n+1,j+1}^*)^{k+1} \\
&\quad - \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r+1}(Z_{n+1,j}^*)(Z_{n+1,j+1}^*)^k Z_{i,j} \\
=& \sum_{k\in\mathbb{N}} Z_{n+1,j+1}^k Z_{n+1,j} W_{j,r+1}\left(Z_{i,j+1}Z_{n+1,j+1}^* - Z_{n+1,j}^* Z_{i,j}\right)(Z_{n+1,j+1}^*)^k \\
=& 0.
\end{aligned}$$

Thus we have the required relation. $\square$

Proposition 4.2.23 *Let $1 \leq i \leq r$. Then $Z_{i,r}$ commutes with $W^*_{n+1,r+1}W_{n+1,r+1}$.*

Proof: From (4.2.25), we have for $r+2 \leq j \leq n+1$,

$$W^*_{j,r+1}W_{j,r+1} = \sum_{k_{r+2},\ldots,k_j \in \mathbb{N}} Z^{k_j}_{n+1,j} \ldots Z^{k_{r+2}}_{n+1,r+2} \left(Z^*_{n+1,r+1}Z_{n+1,r+1}\right) (Z^{k_{r+2}}_{n+1,r+2})^* \ldots (Z^{k_j}_{n+1,j})^*.$$

Hence

$$\begin{aligned}
&Z_{i,r}W^*_{n+1,r+1}W_{n+1,r+1} - W^*_{n+1,r+1}W_{n+1,r+1}Z_{i,r} \\
= &\sum_{k_{r+2},\ldots,k_{n+1} \in \mathbb{N}} Z^{k_{n+1}}_{n+1,n+1} \ldots Z^{k_{r+2}}_{n+1,r+2} \\
&\quad \left(Z_{i,r}Z^*_{n+1,r+1}Z_{n+1,r+1} - Z^*_{n+1,r+1}Z_{n+1,r+1}Z_{i,r}\right) (Z^{k_{r+2}}_{n+1,r+2})^* \ldots (Z^{k_{n+1}}_{n+1,n+1})^*.
\end{aligned}$$

since $Q_{n+1,r+1}Q_{n+1,r} = 0$, we have

$$\begin{aligned}
Z_{i,r}Z^*_{n+1,r+1}Z_{n+1,r+1} &= Z^*_{n+1,r+1}Z_{i,r}Z_{n+1,r+1} \\
&= Z^*_{n+1,r+1}\left(Z_{n+1,r+1}Z_{i,r} + Z_{n+1,r}Z_{i,r+1}\right) \\
&= Z^*_{n+1,r+1}Z_{n+1,r+1}Z_{i,r}.
\end{aligned}$$

Therefore we have the equality $Z_{i,r}W^*_{n+1,r+1}W_{n+1,r+1} = W^*_{n+1,r+1}W_{n+1,r+1}Z_{i,r}$. $\square$

Proposition 4.2.24 *Let $1 \leq i \leq r+1$. Then $Z_{i,r+1}$ commutes with $W^*_{n+1,r+1}W_{n+1,r+1}$.*

Proof: We have

$$
\begin{aligned}
& Z_{i,r+1} W^*_{r+2,r+1} W_{r+2,r+1} \\
=& Z_{i,r+1} \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z^*_{n+1,r+1} Z_{n+1,r+1}\right) (Z^k_{n+1,r+2})^* \\
=& \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z_{i,r+1} Z^*_{n+1,r+1} Z_{n+1,r+1}\right) (Z^k_{n+1,r+2})^* \\
& \qquad + \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z_{n+1,r+1} Z_{i,r+2} Z^*_{n+1,r+1} Z_{n+1,r+1}\right) (Z^{k+1}_{n+1,r+2})^* \\
=& \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{i,r+1} \left(Q_{n+1,r+1} + Q_{n+1,r}\right) (Z^k_{n+1,r+2})^* \\
& \qquad + \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z_{n+1,r+1} Z_{i,r+2} Z^*_{n+1,r+2}\right) (Z^k_{n+1,r+2})^* \\
=& \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{i,r+1} Q_{n+1,r} (Z^k_{n+1,r+2})^* \\
& \qquad + \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z_{n+1,r+1} Z_{i,r+2} Z^*_{n+1,r+2}\right) (Z^k_{n+1,r+2})^*,
\end{aligned}
$$

and

$$
\begin{aligned}
& W^*_{r+2,r+1} W_{r+2,r+1} Z_{i,r+1} \\
& \qquad = \left(\sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z^*_{n+1,r+1} Z_{n+1,r+1}\right) (Z^k_{n+1,r+2})^*\right) Z_{i,r+1} \\
& \qquad = \left(\sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z_{n+1,r} Z^*_{n+1,r} + Z_{n+1,r+1} Z^*_{n+1,r+1}\right) Z_{i,r+1} (Z^k_{n+1,r+2})^*\right) \\
& \qquad = \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Q_{n+1,r} Z_{i,r+1}\right) (Z^k_{n+1,r+2})^* \\
& \qquad\qquad + \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z_{n+1,r+1} Z^*_{n+1,r+1} Z_{i,r+1}\right) (Z^{k+1}_{n+1,r+2})^*.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& Z_{i,r+1} W^*_{r+2,r+1} W_{r+2,r+1} - W^*_{r+2,r+1} W_{r+2,r+1} Z_{i,r+1} \\
=& \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} \left(Z_{i,r+1} Q_{n+1,r} - Q_{n+1,r} Z_{i,r+1}\right) (Z^k_{n+1,r+2})^* \\
& \qquad + \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r+1} \left(Z_{i,r+2} Z^*_{n+1,r+2} - Z^*_{n+1,r+1} Z_{i,r+1}\right) (Z^k_{n+1,r+2})^* \\
=& \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2} Z_{n+1,r+1} \left(Z_{i,r+2} Z^*_{n+1,r+2} - Z^*_{n+1,r+1} Z_{i,r+1}\right) (Z^k_{n+1,r+2})^*.
\end{aligned}
$$

The right hand side above vanishes by Lemma 4.2.9. Since $Z_{i,r+1}$ commutes with $Z_{n+1,j}$ and $Z^*_{n+1,j}$ for $j \geq r+3$, it follows that $Z_{i,r+1}$ commutes with $W^*_{n+1,r+1}W_{n+1,r+1}$. $\square$

Proposition 4.2.25 *Let π be irreducible. Then $W^*_{n+1,r+1}W_{n+1,r+1} = I$.*

Proof: From propositions 4.2.20, 4.2.22, 4.2.23 and 4.2.24, it follows that $W^*_{n+1,r+1}W_{n+1,r+1}$ commutes with $Z_{i,j}$ for all $1 \leq i \leq j \leq n+1$. Therefore using (2.2.25) we conclude that $W^*_{n+1,r+1}W_{n+1,r+1}$ commutes with $Z_{i,j}$ for all i and j. Since $W^*_{n+1,r+1}W_{n+1,r+1}$ is a projection, and $W^*_{n+1,r+1}W_{n+1,r+1} \geq P_{n+1,r+1} \neq 0$, we get the result. $\square$

The above result tells us that the operator $W^*_{n+1,r}W_{n+1,r+1}$ is also an isometry. We next express the operators $Z_{i,j}$ using this operator. This will be the main tool in proving the main theorems in the next subsection.

Theorem 4.2.26 *Assume π is irreducible. Define*

$$W := W^*_{n+1,r}W_{n+1,r+1} = \bar{\lambda}W_{n+1,r+1}, \tag{4.2.37}$$

$$Y^{(1)}_{i,j} = \begin{cases} Z_{i,j} & \text{if } j \notin \{r, r+1\}, \\ Z_{i,r}W & \text{if } j = r, \\ W^*Z_{i,r+1} & \text{if } j = r+1. \end{cases} \tag{4.2.38}$$

$$Y^{(2)}_{i,j} = \begin{cases} W & \text{if } i = j = r+1, \\ W^* & \text{if } i = j = r, \\ I - WW^* & \text{if } i,j \in \{r, r+1\} \text{ and } i \neq j, \\ I & \text{if } i = j \notin \{r, r+1\}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.2.39}$$

Then $Y^{(1)}_{i,j} \in \{W, W^*\}'$ for all i,j and

$$Z_{i,j} = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} Y^{(1)}_{i,k} Y^{(2)}_{k,j}. \tag{4.2.40}$$

Before getting into the proof of the above theorem, we first prove two propositions where we show that the elements $Z_{i,r}W$ and $W^*Z_{i,r+1}$ are in the commutant $\{W, W^*\}'$.

Proposition 4.2.27 *Let W be as defined in Theorem 4.2.26. Then $Z_{i,r}W \in \{W, W^*\}'$ for $1 \le i \le r$.*

Proof: It is enough to show that $Z_{i,r}W_{n+1,r+1} \in \{W_{n+1,r+1}, W^*_{n+1,r+1}\}'$ From Proposition 4.2.16, we have

$$(Z_{i,r}W_{n+1,r+1} - W_{n+1,r+1}Z_{i,r})W_{n+1,r+1} = U_{n+1}Z_{i,r+1}W_{n+1,r+1}.$$

Now note that

$$\begin{aligned}
&U_{n+1}Z_{i,r+1}W_{n+1,r+1}\\
=&\Big(\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\Big)\\
&\quad Z_{i,r+1}\Big(\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r+1}(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\Big)\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}}\Big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}\left(Z_{n+1,r}Z_{i,r+1}Z_{n+1,r+1}\right)(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\Big)\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}}\Big(Z^{k_{n+1}}_{n+1,n+1}\cdots Z^{k_{r+2}}_{n+1,r+2}\left(Z_{i,r+1}Z_{n+1,r}Z_{n+1,r+1}\right)(Z^*_{n+1,r+2})^{k_{r+2}}\cdots(Z^*_{n+1,n+1})^{k_{n+1}}\Big)\\
=&0.
\end{aligned}$$

Thus $Z_{i,r}W_{n+1,r+1}$ commutes with $W_{n+1,r+1}$. One also has

$$\begin{aligned}
(Z_{i,r}W_{n+1,r+1})W^*_{n+1,r+1} &= Z_{i,r}(I - U_{n+1}U^*_{n+1}) = Z_{i,r},\\
W^*_{n+1,r+1}(Z_{i,r}W_{n+1,r+1}) &= Z_{i,r}W^*_{n+1,r+1}W_{n+1,r+1} = Z_{i,r}.
\end{aligned}$$

Thus $Z_{i,r}W_{n+1,r+1} \in \{W_{n+1,r+1}, W^*_{n+1,r+1}\}'$ for $1 \le i \le r$. □

Proposition 4.2.28 *Let $i \le r+1$. Then $W^*Z_{i,r+1} \in \{W, W^*\}'$.*

Proof: Let us first prove that $W^*_{n+1,r+1}Z_{i,r+1}W_{n+1,r+1} = W_{n+1,r+1}W^*_{n+1,r+1}Z_{i,r+1}$. We have

$$\begin{aligned}
&Z_{i,r+1}W_{r+2,r+1}\\
&\qquad= \sum_{k\in\mathbb{N}} Z^k_{n+1,r+2}Z_{i,r+1}Z_{n+1,r+1}(Z^*_{n+1,r+2})^k + \sum_{k\ge 1} Z^{k-1}_{n+1,r+2}Z_{i,r+2}Z^2_{n+1,r+1}(Z^*_{n+1,r+2})^k
\end{aligned}$$

$$=\sum_{k\in\mathbb{N}} Z^k_{n+1,r+2}Z_{i,r+2}Z^2_{n+1,r+1}(Z^*_{n+1,r+2})^{k+1}.$$

$$\begin{aligned}
&W^*_{n+1,r+1}Z_{i,r+1}W_{n+1,r+1}\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}\left(Z^*_{n+1,r+1}Z_{i,r+2}Z^2_{n+1,r+1}Z^*_{n+1,r+2}\right)(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}\left(Z^*_{n+1,r+1}Z^2_{n+1,r+1}Z_{i,r+2}Z^*_{n+1,r+2}\right)(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}\left(P_{n+1,r+1}Z_{n+1,r+1}Z_{i,r+2}Z^*_{n+1,r+2}\right)(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}\left(Z_{n+1,r+1}Z_{i,r+2}Z^*_{n+1,r+2}\right)(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*.
\end{aligned}$$

$$\begin{aligned}
&W_{n+1,r+1}W^*_{n+1,r+1}Z_{i,r+1}\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}\left(Z_{n+1,r+1}Z^*_{n+1,r+1}Z_{i,r+1}\right)(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*.
\end{aligned}$$

Therefore using Lemma 4.2.9,

$$\begin{aligned}
&W^*_{n+1,r+1}Z_{i,r+1}W_{n+1,r+1}-W_{n+1,r+1}W^*_{n+1,r+1}Z_{i,r+1}\\
=&\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r+1}\left(Z_{i,r+2}Z^*_{n+1,r+2}-Z^*_{n+1,r+1}Z_{i,r+1}\right)\\
&\qquad(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\\
=&0.
\end{aligned}$$

Since W commutes with all the $Z_{i,j}$'s, we get $W^*Z_{i,r+1}W=WW^*Z_{i,r+1}$. It remains to prove that $W^*Z_{i,r+1}W^*=(W^*)^2Z_{i,r+1}$. Since $W_{n+1,r+1}$ is an isometry, we have

$$\begin{aligned}
W^*_{n+1,r+1}Z_{i,r+1}W^*_{n+1,r+1}W_{n+1,r+1}&=W^*_{n+1,r+1}Z_{i,r+1}\\
&=W^*_{n+1,r+1}W_{n+1,r+1}W^*_{n+1,r+1}Z_{i,r+1}\\
&=(W^*_{n+1,r+1})^2Z_{i,r+1}W.
\end{aligned}$$

Therefore it is enough to prove that

$$W^*_{n+1,r+1}Z_{i,r+1}W^*_{n+1,r+1}(I-W_{n+1,r+1}W^*_{n+1,r+1}) = (W^*_{n+1,r+1})^2 Z_{i,r+1}(I-W_{n+1,r+1}W^*_{n+1,r+1}).$$

Note that the left hand side above is 0. Since by (4.2.29) we have $I - W_{n+1,r+1}W^*_{n+1,r+1} = U_{n+1}U^*_{n+1}$, it suffices to show that $(W^*)^2 Z_{i,r+1}U_{n+1} = 0$. Now

$$\begin{aligned}
& Z_{i,r+1}U_{n+1} \\
=& Z_{i,r+1}\left(\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r}(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\right) \\
=& \sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+3}}_{n+1,r+3}Z_{i,r+1}Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r}(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^* \\
=& \sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+3}}_{n+1,r+3}Z^{k_{r+2}}_{n+1,r+2}Z_{i,r+1}Z_{n+1,r}(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^* \\
&+ \sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r+1}Z_{i,r+2}(Z^{1+k_{r+2}}_{n+1,r+2})^*(Z^{k_{r+3}}_{n+1,r+3})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*.
\end{aligned}$$

Hence

$$\begin{aligned}
& (W^*)^2 Z_{i,r+1}U_{n+1} \\
=& \left(\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}(Z^*_{n+1,r+1})^2(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\right) \\
& \left(\sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+3}}_{n+1,r+3}Z^{k_{r+2}}_{n+1,r+2}Z_{i,r+1}Z_{n+1,r}(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\right. \\
& \left.+ \sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}Z_{n+1,r+1}Z_{i,r+2}(Z^{1+k_{r+2}}_{n+1,r+2})^*(Z^{k_{r+3}}_{n+1,r+3})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^*\right) \\
=& \sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+3}}_{n+1,r+3}Z^{k_{r+2}}_{n+1,r+2}(Z^*_{n+1,r+1})^2 Z_{i,r+1}Z_{n+1,r}(Z^{k_{r+2}}_{n+1,r+2})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^* \\
&+ \sum_{k,k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}(Z^*_{n+1,r+1})^2(Z^{k_{r+2}}_{n+1,r+2})^*(Z_{n+1,r+2})^{k+k_{r+2}}Z_{n+1,r+1}Z_{i,r+2} \\
&\qquad (Z^{k+1+k_{r+2}}_{n+1,r+2})^*(Z^{k_{r+3}}_{n+1,r+3})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^* \\
=& \sum_{k,k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z^{k_{n+1}}_{n+1,n+1}\ldots Z^{k_{r+2}}_{n+1,r+2}(Z^*_{n+1,r+1})^2(Z_{n+1,r+2})^{k}Z_{n+1,r+1}Z_{i,r+2} \\
&\qquad (Z^{k+1+k_{r+2}}_{n+1,r+2})^*(Z^{k_{r+3}}_{n+1,r+3})^*\ldots(Z^{k_{n+1}}_{n+1,n+1})^* \\
=& 0.
\end{aligned}$$

Thus the proof is complete. □

For the remaining operators, i.e. $W^*Z_{i,r+1}$ for $i \geq r+2$ and $Z_{i,r}W$ for $i \geq r+1$, we now invoke the relation (2.2.25). Observe that for $i \geq r+2$, one has

$$\begin{aligned}
&\left(W^*_{n+1,r+1}Z_{i,r+1}\right)^* \\
&= \left(Z_{1,1}Z_{2,2}\dots Z_{r,r}\right)\left(Z_{r+1,r+2}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right)W_{n+1,r+1} \\
&= \left(Z_{1,1}Z_{2,2}\dots Z_{r-1,r-1}\right)\left(Z_{r,r}W_{n+1,r+1}\right)\left(Z_{r+1,r+2}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right),
\end{aligned}$$

and for $i \geq r+1$, one similarly has

$$\begin{aligned}
&\left(Z_{i,r}W_{n+1,r+1}\right)^* \\
&= W^*_{n+1,r+1}\left(Z_{1,1}Z_{2,2}\dots Z_{r-1,r-1}\right)\left(Z_{r,r+1}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right) \\
&= \left(Z_{1,1}Z_{2,2}\dots Z_{r-1,r-1}\right)W^*_{n+1,r+1}Z_{r,r+1}\left(Z_{r+1,r+2}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right).
\end{aligned}$$

Therefore using the earlier results (Propositions 4.2.20, 4.2.27 and 4.2.22), we now obtain the following proposition.

Proposition 4.2.29 *Let W be as in Theorem 4.2.26. Then*

$$\begin{aligned}
W^*Z_{i,r+1} &\in \{W, W^*\}' && \textit{for } i \geq r+2. \\
Z_{i,r}W &\in \{W, W^*\}' && \textit{for } i \geq r+1.
\end{aligned}$$

Proof of Theorem 4.2.26:

From Propositions 4.2.20, 4.2.22 and 4.2.27–4.2.29, it follows that $Y^{(1)}_{i,j} \in \{W, W^*\}'$ for all i, j. So it remains to prove (4.2.40). For that, we need to prove the following equalities

$$Z_{i,r}WW^* = Z_{i,r} \qquad i \leq r, \tag{4.2.41}$$
$$Z_{i,r}W(I - WW^*) + W^*Z_{i,r+1}W = Z_{i,r+1} \qquad i \leq r, \tag{4.2.42}$$
$$W^*Z_{i,r+1}W = Z_{i,r+1} \qquad i \geq r+1, \tag{4.2.43}$$
$$W^*Z_{i,r+1}(I - WW^*) + Z_{i,r}WW^* = Z_{i,r} \qquad i \geq r+1 \tag{4.2.44}$$

For (4.3.43), note that

$$Z_{i,r}(I - WW^*) = Z_{i,r}(I - W_{n+1,r+1}W^*_{n+1,r+1}) = Z_{i,r}U_{n+1}U^*_{n+1}.$$

Since $i \leq r$, we have

$$Z_{i,r}U_{n+1} = \sum_{k_{r+2},\ldots,k_{n+1}\in\mathbb{N}} Z_{n+1,n+1}^{k_{n+1}} \ldots Z_{n+1,r+3}^{k_{r+3}} Z_{n+1,r+2}^{k_{r+2}} \left(Z_{i,r}Z_{n+1,r}\right) (Z_{n+1,r+2}^{k_{r+2}})^* \ldots (Z_{n+1,n+1}^{k_{n+1}})^* = 0.$$

Therefore

$$\begin{aligned} Z_{i,r}WW^* =& Z_{i,r} - Z_{i,r}(I - WW^*) \\ =& Z_{i,r} - Z_{i,r}U_{n+1}U_{n+1}^* \\ =& 0. \end{aligned}$$

Next let us prove (4.2.42). Use Proposition 4.2.16 to get

$$\begin{aligned} Z_{i,r}W(I - WW^*) =& W_{n+1,r}^* Z_{i,r} W_{n+1,r+1}(I - WW^*) \\ =& W_{n+1,r}^* \left(W_{n+1,r+1}Z_{i,r} + U_{n+1}Z_{i,r+1}\right)(I - WW^*) \\ =& WZ_{i,r}(I - WW^*) + W_{n+1,r}^* U_{n+1} Z_{i,r+1}(I - WW^*) \\ =& WZ_{i,r}U_{n+1}U_{n+1}^* + U_{n+1}U_{n+1}^* Z_{i,r+1}(I - WW^*) \\ =& (I - WW^*)Z_{i,r+1}(I - WW^*). \end{aligned}$$

Since $W^*Z_{i,r+1}$ commutes with W, we get

$$\begin{aligned} Z_{i,r}W(I - WW^*) + W^*Z_{i,r+1}W =& (I - WW^*)Z_{i,r+1}(I - WW^*) + WW^*Z_{i,r+1} \\ =& Z_{i,r+1} - (I - WW^*)Z_{i,r+1}WW^* \\ =& Z_{i,r+1} - U_{n+1}^*U_{n+1}Z_{i,r+1}WW^*. \end{aligned}$$

By (4.2.41), the second term in the right hand side above vanishes. Therefore we have (4.2.42).

For proving (4.2.43) and (4.2.44), we will need the following equality.

$$W^*Z_{i,r+1} - Z_{i,r+1}W^* = Z_{i,r}(I - WW^*) \qquad \text{if } i \geq r+1. \tag{4.2.45}$$

Here

$$\begin{aligned} & Z_{i,r+1}^* W_{n+1,r+1} - W_{n+1,r+1} Z_{i,r+1}^* \\ =& \left(Z_{1,1}Z_{2,2}\ldots Z_{r,r}\right)\left(Z_{r+1,r+2}\ldots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\ldots Z_{n+1,n+1}\right) W_{n+1,r+1} \end{aligned}$$

$$
\begin{aligned}
&- W_{n+1,r+1}\left(Z_{1,1}Z_{2,2}\dots Z_{r,r}\right)\left(Z_{r+1,r+2}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right)\\
=&\left(Z_{1,1}Z_{2,2}\dots Z_{r-1,r-1}\right)\left(Z_{r,r}W_{n+1,r+1} - W_{n+1,r+1}Z_{r,r}\right)\\
&\qquad\qquad\left(Z_{r+1,r+2}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right)\\
=&\left(Z_{1,1}Z_{2,2}\dots Z_{r-1,r-1}\right)\left(U_{n+1}Z_{r,r+1}\right)\left(Z_{r+1,r+2}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right)\\
=&U_{n+1}\left(Z_{1,1}Z_{2,2}\dots Z_{r-1,r-1}\right)\left(Z_{r,r+1}Z_{r+1,r+2}\dots Z_{i-1,i}\right)\left(Z_{i+1,i+1}\dots Z_{n+1,n+1}\right)\\
=&U_{n+1}Z_{i,r}^*.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
W^*Z_{i,r+1} - Z_{i,r+1}W^* &= Z_{i,r}U_{n+1}^*W_{n+1,r}\\
&= Z_{i,r}(I - WW^*).
\end{aligned}
$$

Using the above equality, we get

$$
\begin{aligned}
W^*Z_{i,r+1}W - Z_{i,r+1} &= \left(W^*Z_{i,r+1} - Z_{i,r+1}W^*\right)W\\
&=Z_{i,r}(I - WW^*)W\\
&=0.
\end{aligned}
$$

Thus we have (4.2.43). Next,

$$
\begin{aligned}
&W^*Z_{i,r+1}(I - WW^*) + Z_{i,r}WW^*\\
=&\left(Z_{i,r+1}W^* + Z_{i,r}(I - WW^*)\right)(I - WW^*) + Z_{i,r}WW^*\\
=&Z_{i,r}.
\end{aligned}
$$

Thus we have proved (4.2.44). This completes the proof of Theorem 4.2.26.

4.2.3 Classification of irreducible representations

We are now in a position to prove the following factorisation theorem for irreducible representations of the C^*-algebra $C(SU_0(n+1))$. This will subsequently help us identify all the irreducible representations of $C(SU_0(n+1))$.

Theorem 4.2.30 (Factorization theorem) *Let π be an irreducible representation of $C(SU_0(n+1))$ on a Hilbert space $\mathcal{H}$. Assume $r \equiv r(\pi) \leq n$ where r is as in (4.2.20).*

Then there is an irreducible representation π_1 of $C(SU_0(n+1))$ acting on a Hilbert space $\mathcal{H}_1$ such that $r(\pi_1) = r+1$ and π is unitarily equivalent to the representation $\pi_1 * \psi_{s_r}$ acting on $\mathcal{H}_1 \otimes \ell^2(\mathbb{N})$ given by

$$\pi_1 * \psi_{s_r}(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} \pi_1(z_{i,k}) \otimes \psi_{s_r}(z_{k,j}), \quad 1 \le i,j \le n+1.$$

Proof: From Proposition 4.2.25, we know that W is an isometry. Since $U_{n+1}^* U_{n+1} \ge P_{n+1,r}$, it follows from (4.2.29) that W is not a unitary. By Wold decomposition, there are Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and a unitary U on $\mathcal{H}_0$ such that $(\mathcal{H}, W)$ is unitarily equivalent to $((\mathcal{H}_1 \otimes \ell^2(\mathbb{N})) \oplus \mathcal{H}_0, (I \otimes S^*) \oplus U)$. By Theorem 4.2.26, the subspace $\mathcal{H}_1 \otimes \ell^2(\mathbb{N})$ is kept invariant by $Y_{i,j}^{(1)}$, $Y_{i,j}^{(2)}$, $(Y_{i,j}^{(1)})^*$ and $(Y_{i,j}^{(2)})^*$ for all i,j. Therefore the subspace $\mathcal{H}_1 \otimes \ell^2(\mathbb{N})$ is an invariant subspace for π. By irreducibility of π, it follows that $\mathcal{H}_0 = \{0\}$. Thus there are operators $Z_{i,j}^{(1)} \in \mathcal{L}(\mathcal{H}_1)$ and $Z_{i,j}^{(2)} \in \mathcal{L}(\ell^2(\mathbb{N}))$ such that

$$Y_{i,j}^{(1)} = Z_{i,j}^{(1)} \otimes I, \qquad Y_{i,j}^{(2)} = I \otimes Z_{i,j}^{(2)}, \qquad 1 \le i,j \le n+1,$$

and we have

$$Z_{i,j} = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} Z_{i,k}^{(1)} \otimes Z_{k,j}^{(2)} \qquad 1 \le i,j \le n+1.$$

In fact one has

$$Z_{i,j}^{(2)} = \begin{cases} S^* & \text{if } i=j=r+1, \\ S & \text{if } i=j=r, \\ P_0 & \text{if } i,j \in \{r,r+1\} \text{ and } i \ne j, \\ I & \text{if } i=j \notin \{r,r+1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.46}$$

Thus the map $z_{i,j} \mapsto Z_{i,j}^{(2)}$ defines a representation of $C(SU_0(n+1))$ on $\ell^2(\mathbb{N})$ which is in fact the representation ψ_{s_r} defined in section 3. Observe that the $Z_{i,j}^{(2)}$'s belong to the Toeplitz algebra $\mathscr{T} \subseteq \mathcal{L}(\ell^2(\mathbb{N}))$. Therefore

$$\pi(z_{i,j}) \in \mathcal{L}(\mathcal{H}_1) \otimes \mathscr{T} \quad \text{for all } i,j. \tag{4.2.47}$$

Let $\sigma : \mathscr{T} \to \mathbb{C}$ be the $*$-homomorphism given by $S \mapsto 1$. Then one has

$$Z_{i,j}^{(1)} = (id \otimes \sigma) Z_{i,j} = (id \otimes \sigma)\pi(z_{i,j}).$$

Thus the map $\pi_1 : z_{i,j} \mapsto Z_{i,j}^{(1)}$ gives a representation of $C(SU_0(n+1))$ on $\mathcal{H}_0$ and we have

$$\pi(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} \pi_1(z_{i,k}) \otimes \psi_{s_r}(z_{k,j}), \quad 1 \le i, j \le n+1.$$

If T belongs to the commutant of $\pi_1(C(SU_0(n+1)))$, then $T \otimes I \in \pi(C(SU_0(n+1)))'$. By irreducibility of π, it follows that $T = I$. Thus π_1 is irreducible. Therefore we get $\pi_1(z_{n+1,r}) = Z_{n+1,r} W = 0$. Also, note that

$$\begin{aligned}
\left(W^* Z_{n+1,r+1}\right)^* \left(W^* Z_{n+1,r+1}\right) &= Z_{n+1,r+1}^* W W^* Z_{n+1,r+1} \\
&= P_{n+1,r+1} - Z_{n+1,r+1}^* (I - WW^*) Z_{n+1,r+1} \\
&= P_{n+1,r+1} - Z_{n+1,r+1}^* U_{n+1}^* U_{n+1} Z_{n+1,r+1} \\
&= P_{n+1,r+1} \neq 0.
\end{aligned}$$

Thus we have

$$\min\{i : 1 \le i \le n+1, \pi_1(z_{n+1,i}) \neq 0\} = r+1.$$

This completes the proof. □

We next come to the main result that gives a parametrisation of all the irreducible representations.

Theorem 4.2.31 *Let π be an irreducible representation of $C(SU_0(n+1))$ on a Hilbert space $\mathcal{H}$. Then there is a $\lambda = (\lambda_1, \ldots, \lambda_n) \in (S^1)^n$ and a reduced word $\omega = s_{[a_k,b_k]} s_{[a_{k-1},b_{k-1}]} \ldots s_{[a_1,b_1]}$ in $\mathfrak{S}_{n+1}$ such that $\pi \cong \psi_{\lambda,\omega}$, where $1 \le b_k < b_{k-1} < \cdots < b_1 \le n$, and $1 \le a_i \le b_i$ for $1 \le i \le k$.*

Proof: Let r is as in (4.2.20). Note that when $\pi \cong \psi_{\lambda,\omega}$, then $r \le n$ implies $b_1 = n$ and $a_1 = r$.

It was proved in [GP22] that the statement holds for $n = 2$.

Let us assume that it holds for $n-1$. Let us first deal with the case $r = n+1$. In this case, from Proposition 4.2.10 it follows that $Z_{n+1,j} = 0$ for $1 \leq j \leq n$. Therefore we have

1. $P_{n+1,n+1} = Q_{n+1,n+1} = I$,
2. $Z_{i,n+1} = 0$ for $1 \leq i \leq n$,
3. $Z_{i,j} Z_{n+1,n+1} = Z_{n+1,n+1} Z_{i,j}$ for all i, j.

Therefore $Z_{n+1,n+1}$ is a unitary and any spectral subspace of $Z_{n+1,n+1}$ is an invariant subspace for π. By irreducibility of π, it follows that $Z_{n+1,n+1} = \mu_0 I$ for some $\mu_0 \in S^1$. Now define

$$\pi_0(z_{i,j}^{(n-1)}) = \begin{cases} Z_{i,j} & \text{if } 1 \leq j \leq n \text{ and } 1 \leq i \leq n-1, \\ \mu_0 Z_{i,j} & \text{if } 1 \leq j \leq n \text{ and } i = n, \end{cases}$$

where $z_{i,j}^{(n-1)}$'s denote the generators for $C(SU_0(n))$. Then π_0 is an irreducible representation of $C(SU_0(n))$ on $\mathcal{H}$. Therefore there is a $\mu = (\mu_1, \ldots, \mu_{n-1}) \in (S^1)^{n-1}$ and a reduced word $\omega = s_{[a_k,b_k]} \ldots s_{[a_1,b_1]} \in \mathfrak{S}_n \subseteq \mathfrak{S}_{n+1}$ where $1 \leq b_k < b_{k-1} < \cdots < b_1 \leq n-1$ such that $\pi_0 \cong \psi_{\mu,\omega}$. We thus have

$$\pi \cong \psi_{\lambda,\omega},$$

where $\lambda = (\mu_1, \mu_2, \ldots, \mu_{n-1}, \overline{\mu_0})$. Next, assume $1 \leq r \leq n$. By an application of the second factorisation result proved earlier (Theorem 4.2.30), it follows that there is an irreducible representation π_1 acting on a Hilbert space $\mathcal{H}_1$ such that $r(\pi_1) = r+1$ and $\pi = \pi_1 * \psi_{s_r}$ i.e.

$$\pi(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} \pi_1(z_{i,k}) \otimes \psi_{s_r}(z_{k,j}).$$

Using Lemma 4.2.1 and by repeated application of Theorem 4.2.30, we conclude that there is an irreducible representation π_0 acting on a Hilbert space $\mathcal{H}_0$ such that $r(\pi_0) = n+1$ and $\pi = \pi_0 * \psi_{s_{[r,n]}}$. By the previous case, the result now follows. $\square$

4.3 Corollaries of the classification theorem

In this section, we present a few important consequences of the classification theorem.

Proposition 4.3.1 *The set*

$$\left\{\psi_{\lambda,\omega} : \lambda \in (S^1)^n,\ \omega \text{ a reduced word in } \mathfrak{S}_{n+1}\right\}$$

gives all inequivalent irreducible representations of $C(SU_0(n+1))$.

This follows from Theorems 4.2.8 and 4.2.9 in Section 2 of Chapter 4 and Theorem 4.2.31. Thus irreducible representations of $C(SU_0(n+1))$ are precisely the ones that occur as $q \to 0+$ limits of irreducible representations of the C^*-algebras associated with the corresponding quantum groups.

Recall that $\mathcal{O}(SU_0(n+1))$ denotes the $*$-algebra given by the relations 2.2.19–2.2.28 in Theorem 2.2.7. Thus $\mathcal{O}(SU_0(n+1))$ can be viewed as a dense $*$-subalgebra of $C(SU_0(n+1))$, and the family $\psi_{\lambda,\omega}$ gives all the irreducible $*$-representations of this algebra.

Let σ_λ denote the C^*-homomorphism from $\mathscr{T}$ to $\mathbb{C}$ obtained by composing the evaluation map $ev_\lambda : C(S^1) \to \mathbb{C}$ with the canonical projection map $\eta : \mathscr{T} \to C(S^1) \cong \mathscr{T}/\mathcal{K}$.

Let $\omega_0 = (s_1)(s_2 s_1)\cdots(s_i s_{i-1}\cdots s_1)\cdots(s_n s_{n-1}\cdots s_1)$ be the longest word for $\mathscr{S}_{n+1}$. Let $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in (S^1)^n$. Since $\psi_{\mu,\omega_0}(C(SU_0(n+1))) \subseteq \mathscr{T}^{\otimes \frac{1}{2}n(n+1)}$, one can view the irreducible representation ψ_{μ,ω_0} as a C^*-homomorphism into the C^*-algebra $\mathscr{T}^{\otimes \frac{1}{2}n(n+1)}$. Let $u \in C(S^1)$ denote the function $\lambda \mapsto \lambda$ and $\mathbb{1} \in C(S^1)$ denote the function $\lambda \mapsto 1$. Let us define a map ϕ from the set $\{z_{i,j} : 1 \le i,j \le n+1\}$ of generators of $C(SU_0(n+1))$ to the C^*-algebra $C(S^1)^{\otimes n} \otimes \mathscr{T}^{\otimes \frac{1}{2}n(n+1)}$ as follows:

$$z_{i,j} \mapsto \begin{cases} u \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \mathbb{1} \otimes \psi_\omega(z_{i,j}) & \text{if } i = 1, \\ u^* \otimes u \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \mathbb{1} \otimes \psi_\omega(z_{i,j}) & \text{if } i = 2, \\ \mathbb{1} \otimes u^* \otimes u \otimes \cdots \otimes \mathbb{1} \otimes \mathbb{1} \otimes \psi_\omega(z_{i,j}) & \text{if } i = 3, \\ \vdots & \\ \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes u^* \otimes u \otimes \psi_\omega(z_{i,j}) & \text{if } i = n-1, \\ \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes u^* \otimes \psi_\omega(z_{i,j}) & \text{if } i = n \end{cases}$$

Theorem 4.3.2 *The map ϕ defined above extends to an injective $*$-homomorphism from $C(SU_0(n+1))$ to $C(S^1)^{\otimes n} \otimes \mathscr{T}^{\otimes \frac{1}{2}n(n+1)}$.*

Proof: Since the elements $\phi(z_{i,j})$ obey the defining relations (Equation 2.2.19 to Equation 2.2.27), the map ϕ extends to a C^*-homomorphism from $C(SU_0(n+1))$ into $C(S^1)^{\otimes n} \otimes \mathscr{T}^{\otimes \frac{1}{2}n(n+1)}$. For injectivity, note that $\psi_{\mu,\tau}$, τ is a reduced word in $\mathscr{S}_{n+1}$, constitute all irreducible representations of $C(SU_0(n+1))$. Therefore it is enough to show that each $\psi_{\mu,\tau}$ factorises through ϕ. Let $\sigma_\lambda : \mathscr{T} \to \mathbb{C}$ be given above. Define

$$\theta_{[k,r]} = \begin{cases} \sigma_1^{\otimes r} & \text{if } k = r+1 \\ id \otimes \sigma_1^{\otimes(r-1)} & \text{if } k = r \\ id^{\otimes(r-k+1)} \otimes \sigma_1^{\otimes(k-1)} & \text{if } 2 \le k \le r \\ id^{\otimes r-1} \otimes \sigma_1 & \text{if } k = 2 \\ id^{\otimes r} & \text{if } k = 1 \end{cases}$$

For any word $\omega = s_{[k_1,1]} s_{[k_2,2]} \cdots s_{[k_n,n]}$ and any tuple $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in (S^1)^n$. Then $\psi_{\mu,\omega} = (ev_{\mu_1} \otimes ev_{\mu_2} \otimes \cdots \otimes ev_{\mu_n} \otimes \theta_{[k_1,1]} \otimes \theta_{[k_2,2]} \otimes \cdots \otimes \theta_{[k_n,n]}) \circ \phi$.

Through the embeddings $C(S^1)^{\otimes n} \hookrightarrow \mathcal{L}(\ell^2(\mathbb{Z})^{\otimes n})$ and $\mathscr{T}^{\otimes \frac{1}{2}n(n+1)} \hookrightarrow \mathcal{L}(\ell^2(\mathbb{N})^{\otimes \frac{1}{2}n(n+1)})$, the above map ϕ gives a faithful representation of $C(SU_0(n+1))$ acting on the Hilbert space $\ell^2(\mathbb{Z})^{\otimes n} \otimes \ell^2(\mathbb{N})^{\otimes \frac{1}{2}n(n+1)}$, denoted by $\pi_S^{(0)}$. This is the analogue of the representation π_0 of $C(SU_0(2))$ used by Chakraborty and Pal in [CP03]. $\square$

Theorem 4.3.3 *$C(SU_0(n+1))$ is a type I C^*-algebra.*

Proof: For each $1 \le r \le k$, let i_s and j_s be nonnegative integers for $a_r \le s \le b_r$. Let $|e_i\rangle\langle e_j|$ be the rank one operator $x \mapsto \langle e_j, x\rangle e_i$. Define the operator

$$T_r := |e_{i_{b_r}}\rangle\langle e_{j_{b_r}}| \otimes |e_{i_{b_r-1}}\rangle\langle e_{j_{b_r-1}}| \otimes \cdots \otimes |e_{i_{a_r}}\rangle\langle e_{j_{a_r}}|.$$

It is enough to prove that $T_k \otimes T_{k-1} \otimes \cdots \otimes T_1$ belongs to the image $\psi_{\lambda,\omega}(C(SU_0(n+1)))$.

We will use the operators

$$E_{j,i} := V^*_{n'(j,i),i} V_{j,i} |V_{j-1,a_{j-1}}| \cdot |V_{j-2,a_{j-2}}| \cdots |V_{1,a_1}|$$

described in Corollary 4.2.6 for this. Recall that

$$E_{j,i} = I^{\otimes \sum_{s=j+1}^k (b_s - a_s + 1)} \otimes \left(P_0^{\otimes(b_j+1-i)} \otimes S^* \otimes I^{\otimes(i-a_j-1)}\right) \otimes P_0^{\otimes \sum_{s=1}^{j-1}(b_s-a_s+1)}.$$

Now observe that $|e_i\rangle\langle e_j| = (S^i)^* P_0 S^j$ and hence

$$E_{k,1+b_k}^{i_{b_k}} \ldots E_{k,1+a_k}^{i_{a_k}} E_{k,a_k} (E_{k,1+a_k}^{j_{a_k}})^* \ldots (E_{k,1+b_k}^{j_{b_k}})^* = T_k \otimes P_0^{\otimes(b_{k-1}-a_{k-1}+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}.$$

Thus $T_k \otimes P_0^{\otimes(b_{k-1}-a_{k-1}+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}$ belongs to $\psi_{\lambda,\omega}(C(SU_0(n+1)))$. Having proved that

$$T := T_k \otimes T_{k-1} \otimes \cdots \otimes T_{s+1} \otimes P_0^{\otimes(b_s-a_s+1)} \otimes P_0^{\otimes(b_{s-1}-a_{s-1}+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}$$

belongs to $\psi_{\lambda,\omega}(C(SU_0(n+1)))$, note that

$$\begin{aligned} & E_{s,1+b_s}^{i_{b_s}} \ldots E_{s,1+a_s}^{i_{a_s}} T (E_{s,1+a_s}^{j_{a_s}})^* \ldots (E_{s,1+b_s}^{j_{b_s}})^* \\ =& T_k \otimes T_{k-1} \otimes \cdots \otimes T_{s+1} \otimes T_s \otimes P_0^{\otimes(b_{s-1}-a_{s-1}+1)} \otimes \cdots \otimes P_0^{\otimes(b_1-a_1+1)}. \end{aligned}$$

Thus by repeating the arguent, it follows that $T_k \otimes T_{k-1} \otimes \cdots \otimes T_1 \in \psi_{\lambda,\omega}(C(SU_0(n+1)))$.

□

Theorem 4.3.4 *The map $\Delta_n : C(SU_0(n+1)) \to C(SU_0(n+1)) \overline{\otimes} C(SU_0(n+1))$ given on the generators by*

$$\Delta_n(z_{i,j}) = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} z_{i,k} \otimes z_{k,j}$$

together with $\epsilon = \psi_{(1,1,\cdots,1),id}$ from $C(SU_0(n+1))$ to $\mathbb{C}$ makes $(C(SU_0(n+1)), \Delta_n, \epsilon)$ a C^*-bialgebra.

Proof: Note that since $C(SU_0(n+1))$ is of type I, the tensor product C^*-algebra $C(SU_0(n+1)) \otimes C(SU_0(n+1))$ is unique. In order that Δ_n defined on the generating elements as above extend to a C^*-homomorphism from $C(SU_0(n+1))$ to $C(SU_0(n+1)) \otimes C(SU_0(n+1))$, it is enough to verify that the elements $\Delta(z_{i,j})$ obey the same relations as the $z_{i,j}$'s. Recall from Section 2 that for any irreducible representation π of $C(SU_q(n+1))$, the operators $Y_{i,j} = \lim_{q \to 0+} (-q)^{\min\{i-j,0\}} \pi(t_{i,j}(q))$ satisfy these same relations.

Let $q \in (0,1)$. Since Δ_q is a C^*-homomorphism from $C(SU_q(n+1))$ to $C(SU_q(n+1)) \overline{\otimes} C(SU_q(n+1))$, the elements $\Delta_q(u_{i,j}) = \sum_{k=1}^{n+1} u_{i,k} \otimes u_{k,j}$ obey the same commutation relations as the $u_{i,j}$'s. Hence for $\lambda, \mu \in (S^1)^n$ and ω_1, ω_2 reduced words in $\mathscr{S}_{n+1}$, the elements

$$\left(\psi_{\lambda,\omega_1}^{(q)} \otimes \psi_{\mu,\omega_2}^{(q)}\right) (\Delta_q(u_{i,j}))$$

also satisfy the same relations. Now observe that

$$\lim_{q\to 0+}\left(\psi^{(q)}_{\lambda,\omega_1}\otimes\psi^{(q)}_{\mu,\omega_2}\right)\left(\Delta_q((-q)^{\min\{i-j,0\}}u_{i,j})\right)=(\psi_{\lambda,\omega_1}\otimes\psi_{\mu,\omega_2})\left(\sum_{k=\min\{i,j\}}^{\max\{i,j\}} z_{i,k}\otimes z_{k,j}\right).$$

Since $C(SU_0(n+1))$ is of type I, the representations $\psi_{\lambda,\omega_1}\otimes\psi_{\mu,\omega_2}$ give all irreducible representations of the C^*-algebra $C(SU_0(n+1))\overline{\otimes}C(SU_0(n+1))$ (See Proposition 11.3.2 of [KR97]). Therefore it follows that the $\Delta(z_{i,j})$'s obey the relations.

Since $C(SU_0(n+1))$ is of type I, the representations $\psi_{\lambda,\omega_1}\otimes\psi_{\mu,\omega_2}$ give all irreducible representations of the C^*-algebra $C(SU_0(n+1))\otimes C(SU_0(n+1))$. Therefore it follows that the $\Delta_n(z_{i,j})$'s obey the relations .

The identities $(\Delta_n\otimes id)\Delta_n=(id\otimes\Delta_n)\Delta_n$ and $(\epsilon\otimes id)\Delta_n=(id\otimes\epsilon)\Delta_n=id$ follow by evaluating both sides on the generating elements. □

Theorem 4.3.5 $(C(SU_0(n+1)),\Delta_n)$ *is not a compact quantum group.*

Proof: We will show that the density condition is violated for $C(SU_0(n+1))$.

Let us denote the generators of $C(SU_0(2))$ by $y_{i,j}$. Then one can directly verify that the elements $\sum_{k=\min\{i,j\}}^{\max\{i,j\}} y_{i,k}\otimes y_{k,j}$ in the tensor product $C(SU_0(2))\otimes C(SU_0(2))$ satisfy the same relations as the $y_{i,j}$'s and hence $\Delta_1 : C(SU_0(2))\to C(SU_0(2))\otimes C(SU_0(2))$ given by $y_{i,j}\mapsto\sum_{k=\min\{i,j\}}^{\max\{i,j\}} y_{i,k}\otimes y_{k,j}$ makes $(C(SU_0(2)),\Delta_1)$ a compact quantum semigroup. Now let

$$w_{i,j}=\begin{cases} y_{i,j} & \text{if } 1\le i,j\le 2,\\ \delta_{i,j} & \text{if } i>2 \text{ or } j>2.\end{cases}$$

Then one can verify that the $w_{i,j}$'s obey the relations for $C(SU_0(n+1))$. Therefore the map $\varphi : C(SU_0(n+1))\to C(SU_0(2))$ given by

$$\varphi(z_{i,j})=w_{i,j}$$

extends to a surjective C^*-homomorphism and it satisfies $(\varphi\otimes\varphi)\Delta_n=\Delta_1\varphi$. Thus $(C(SU_0(2)),\Delta_1)$ is a quantum subsemigroup of $(C(SU_0(2)),\Delta_n)$.

Finally, assume that the density condition in the definition of a compact quantum group holds for $(C(SU_0(n+1)),\Delta_n)$. Then applying the homomorphism $\varphi\otimes\varphi$ and using the fact that it is surjective, it follows that $\{(1\otimes a)\Delta_1(b) : a,b\in C(SU_0(2))\}$ and

$\{(a\otimes 1)\Delta_1(b) : a,b\in C(SU_0(2))\}$ are total in $C(SU_0(2))\otimes C(SU_0(2))$. But this implies that $(C(SU_0(2)),\Delta_1)$ is a compact quantum group which is false. Thus $(C(SU_0(n+1)),\Delta_n)$ is not a compact quantum group. □

Theorem 4.3.6 *The crystallized algebra $C(SU_0(n+1))$ is isomorphic to the C^*-subalgebra of the space of bounded operators on $\ell^2(\mathbb{Z})^{\otimes n}\otimes \ell^2(\mathbb{N})^{\otimes \frac{n(n+1)}{2}}$ generated by the limits*

$$\lim_{q\to 0+}\pi_S^{(q)}\left((-q)^{\min\{i-j,0\}}u_{i,j}(q)\right),$$

where $\pi_S^{(q)}$ is the Soibelman representation of $C(SU_q(n+1))$.

Proof: Similar to the case of nonzero q, one can prove that the direct integral $\pi_S^{(0)}$ of the representations ψ_{λ,ω_0} over $\lambda\in(S^1)^n$, where ω_0 is the longest word in $\mathscr{S}_{n+1}$, is a faithful representation of $C(SU_0(n+1))$ acting on $\ell^2(\mathbb{Z})^{\otimes n}\otimes \ell^2(\mathbb{N})^{\otimes \frac{n(n+1)}{2}}$. It is simple to check that the limits $\lim_{q\to 0+}\pi_S^{(q)}\left((-q)^{\min\{i-j,0\}}u_{i,j}(q)\right)$ exist and one has

$$\pi_S^{(0)}(z_{i,j})=\lim_{q\to 0+}\pi_S^{(q)}\left((-q)^{\min\{i-j,0\}}u_{i,j}(q)\right).$$

Therefore the result follows. □

Note that the C^*-algebra described in the above theorem resembles the Matassa-Yuncken crystallization in the type A_n case. Finally, we have the following important property of our crystallized algebra.

Theorem 4.3.7 *Let $\pi^{(0)}$ be an irreducible representation of the crystallization $C(SU_0(n+1))$ (respectively $\mathcal{O}(SU_0(n+1))$) on a Hilbert space $\mathcal{H}$. Then there exist irreducible representations $\pi^{(q)}$ of $C(SU_q(n+1)$ (respectively $\mathcal{O}(SU_q(n+1))$) on the same Hilbert space $\mathcal{H}$ such that*

$$\pi^{(0)}(z_{i,j})=\lim_{q\to 0+}\pi^{(q)}\left((-q)^{\min\{i-j,0\}}u_{i,j}(q)\right),\qquad i,j\in\{1,2,\dots,n+1\}.$$

Proof: This follows from Theorem 4.2.9, Proposition 4.3.1 and Soibelman's result (Theorem 6.2.7, page 121, [KS98]). □

For the Lie groups of type B, C, D etc., a similar definition of crystallized quantized function algebras seem possible. It will be interesting to see if analogous of the main results (Theorems 4.2.31) hold for them as well.

Bibliography

[BS23] Akshay Bhuva and Bipul Saurabh. Computation of Gelfand-Kirillov dimension for B-type structures. *https://arxiv.org/abs/2310.12163*, 2023.

[Con04] Alain Connes. Cyclic cohomology, quantum group symmetries and the local index formula for $SU_q(2)$. *(arXiv:math/0209142 [math.QA] 12 Sep 2002) J. Inst. Math. Jussieu 3*, (01):17–68, 2004.

[CP95] Vyjayanthi Chari and Andrew N. Pressley. *A Guide to Quantum Groups.* Cambridge University Press, Cambridge, 1995.

[CP03] Partha Sarathi Chakraborty and Arup Kumar Pal. Characterization of spectral triples: A combinatorial approach. *https://arxiv.org/abs/math/0305157*, 2003.

[CP08] Partha Sarathi Chakraborty and Arup Kumar Pal. Characterization of $SU_q(\ell + 1)$-equivariant spectral triples for the odd dimensional quantum spheres. *https://arxiv.org/abs/math/0701694*, 2007. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, Volume 2008, Issue 623, Pages 25-42.

[CP24] Partha Sarathi Chakraborty and Arup Kumar Pal. An approximate equivalence of the GNS representation for the haar state of $SU_q(2)$. *https://arxiv.org/abs/2203.06908*, 2021. Indian Journal of Pure and Applied Mathematics, Volume 55, Pages 879-892, 2024

[CS18] Partha Sarathi Chakraborty and Bipul Saurabh. Gelfand-Kirillov dimension of some simple unitarizable modules. *arXiv:1709.08586 [math.OA]*, 2017. *Journal of Algebra*, Volume 514, Pages 199-218, 2018

[DHHMW12] Ludwik Dabrowski, Tom Hadfield, Piotr M Hajac, Rainer Matthes, Elmar Wagner. Index pairings for pullbacks of C^*-algebras, Banach Center Publ.,98 Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2012, 67-84

[Erd68] Ivan Erdélyi. Partial isometries closed under multiplication on Hilbert spaces. *J. Math. Anal. Appl.*, 22:546–551, 1968.

[GP22] Manabendra Giri and Arup Kumar Pal. Quantized function algebras at $q = 0$: type A_n case. *(arXiv:2203.14665 [math.QA]*, 2022.) Proceedings - Mathematical Sciences, Volume 134, Pages 1-31, 2024. https://doi.org/10.1007/s12044-024-00792-4

[GP24] Manabendra Giri and Arup Kumar Pal. Irreducible representations of the crystallization of the quantized function algebras $C(SU_q(n+1))$. *arXiv:2402.09347 [math.OA]*, 2024. Journal of Noncommutative Geometry, 2025, https://ems.press/journals/jncg/articles/14298497 (DOI 10.4171/JNCG/608)

[HK02] Jin Hong and Seok-Jin Kang. *Introduction to Quantum Groups and Crystal Bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, 2002.

[HS02] Jeong Hee Hong and Wojciech Szymański. Quantum spheres and projective spaces as graph algebras. *Communications in Mathematical Physics*, 232(1):(157–188), 2002.

[HW70] P. R. Halmos and L. J. Wallen. Powers of partial isometries. *Indiana Univ. Math. J.*, 19(8):657–663, 1970.

[Kas90] Masaki Kashiwara. Crystalizing the q-analogue of universal enveloping algebras. *Communications in Mathematical Physics*, 133:(249–260), 1990.

[Kas91] Masaki Kashiwara. On crystal bases of the Q-analogue of universal enveloping algebras. *(https://api.semanticscholar.org/CorpusID:17526555) Duke Mathematical Journal*, 63:(465–516), 1991.

[Kas93] Masaki Kashiwara. Global crystal bases of quantum groups. *Duke Mathematical Journal*, 69:(455–485), 1993.

[Kas94] Masaki Kashiwara. Crystal bases of modified quantized enveloping algebra. *Duke Mathematical Journal*, 73:(383–413), 1994.

[Kas95] Masaki Kashiwara. On crystal bases. in representations of groups (banff, ab, 1994). *CMS Conf. Proc., pages 155–197. Amer. Math. Soc.*, 16:(155–197), Providence, RI, 1995.

[Koe91] H. Tjerk Koelink. On *ast*-representations of the hopf star-algebra associated with the quantum group $U_q(n)$. Compositio Math., 77(2):(199–231), 1991.

[KR97] Richard V. Kadison and John R. Ringrose. *Fundamentals of the Theory of Operator Algebras. Volume II*, volume 16 of *Graduates Studies in Mathematics*. American Mathematical Society, 1997.

[KS97] Anatoli Klimyk and Konrad Schmüdgen. *Quantum Groups and Their Representations*. Texts and Monographs in Physics. Springer Berlin, Heidelberg, first edition, 1997.

[KS98] Leonid I. Korogodski and Yan S. Soibelman. *Algebras of Functions on Quantum Groups: Part I*, volume 56 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1998.

[Lus90a] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):(447–498), 1990.

[Lus90b] G. Lusztig. Canonical bases arising from quantized enveloping algebras. II. *Progr. Theoret. Phys. Suppl.*, 102:(175–201), 1990.

[MY23] Marco Matassa and Robert Yuncken. Crystal limits of compact semisimple quantum groups as higher-rank graph algebras. *(arXiv:2208.13201v2 [math.QA] 2022) J. Reine Angew. Math.*, 802:(173–221), 2023.

[NT10] Sergey Neshveyev and Lars Tuset. The Dirac operator on compact quantum groups. *J. Reine Angew. Math.*, 641:(1–20), 2010.

[PpW91] Brian Parshall and Jian pan Wang. Quantum linear groups. *Memoirs of the American Mathematical Society*, 89, January 1991.

[PS10] Arup Kumar Pal and S. Sundar. Regularity and dimension spectrum of the equivalent spectrum triple for the odd-dimensional quantum spheres. *J. Noncommut. Geom.*, 4(3):389–439, 2010.

[PV99] G. B. Podkolzin and L. I. Vainerman. Quantum Stiefel manifold and double cosets of quantum unitary group. *Pacific J. Math*, 188(1):179–199, 1999.

[Sau15] Bipul Saurabh. On quantum quaternion spheres. *arXiv:1401.3503 [math.OA]*, 2015. *Proceedings-Mathematical Sciences*, Volume 127, Pages 133-164, 2017.

[Sau17] Bipul Saurabh. Topological invariance of quantum quaternion spheres. *Pacific J. Math*, 288(2):435–452, 2017.

[VS90] L. L. Vaksman and Ya. S. Soibel'man. Algebra of functions on the quantum group $SU(n+1)$ and odd-dimensional quantum spheres. *Algebra i Analiz*, 2(5):101–120, 1990.

[Wor87] S. L. Woronowicz. Twisted $SU(2)$ group. An example of a noncommutative differential calculus. Publ. Res. Inst. Math. Sci., 23:(117–181), 1987.

[Wor98] S. L. Woronowicz. Compact quantum groups. *In* Symétries quantiques (Les Houches, 1995), pages 845–884, 1998.

List of Papers

1. Manabendra Giri and Arup Kumar Pal
   Quantized function algebras at "$q = 0$": type A_n case.(arXiv:2203.14665 [math.QA])
   Proceedings - Mathematical Sciences, Volume 134, Pages 1-31, 2024.
   https://doi.org/10.1007/s12044-024-00792-4

2. Manabendra Giri and Arup Kumar Pal
   Irreducible representations of the crystallization of the quantized function algebras
   $C(SU_q(n+1))$.(arXiv:2402.09347 [math.OA] 2024)
   Journal of Noncommutative Geometry, 2025. https://ems.press/journals/jncg/articles/14298497
   (DOI 10.4171/JNCG/608)