

Essays in Mechanism Design

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Thesis submitted to the Indian Statistical Institute in
partial fulfilment of the requirements for the award of
the degree of Doctor of Philosophy

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July 1998

Acknowledgements

First and foremost, my family: Appa, Amma and Rohini. They had their doubts about this venture, but once I made the decision, they gave me their whole-hearted support. A special thanks to Rohini, who lent her support whilst in the middle of her own Ph.D program.

My supervisor, Professor Bhaskar Dutta. For all that he has given me over the last four years, words are not enough. I will not even attempt to thank him here. Paraphrasing the American patriot Nathan Hale, I only regret that I was not able to meet his expectations all of the time.

My teachers at the Indian Statistical Institute from whom I learnt so much: Professors Shurojit Chatterjee, Dipankar Dasgupta, Shubhashis Gangopadhyay, Omkar Goswami, Dilip Mookherjee, Bharat Ramaswami and Arunava Sen; and from the Stat-Math Unit, Professors Rajeeva Karandikar and Rahul Roy. It has been a pleasure learning from all of you. I also thank Rahulda for the generous offer of his room during the time I was writing *this* dissertation. It made my job that much easier.

The people at the Computer Centre: R.C. Satija, B. Ganeshan, B. M. Juyal and C. D. Banerjee. Without their help, work would have proceeded more smoothly, but it wouldn't have been nearly as much fun.

Finally, my friends at the Indian Statistical Institute: in alphabetical order, they are Bappa, Lakshmi, Manipushpak, Mausumi, Sukanta and Vinay. Thanks to all of you for a most memorable time spent whilst working on my dissertation.

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Chapter 1

Introduction

The theory of implementation or mechanism design had its origins in the debates in the 1930s between Hayek, Lange and Lerner on the informational efficiency of the market economy. However, it was the work of Hurwicz in the 1950s and the 1960s which formalised the insights of Hayek, Lange and Lerner and paved the way for the body of work that followed his pioneering effort.

In addition to the considerable theoretical literature on mechanism design¹, there also exists a body of literature which uses the mechanism design approach to address specific problems. Some examples of work in this vein include Laffont and Tirole [34] on the economics of regulation, Mirrlees [39] on optimal income tax schemes and Moulin [42] on cost sharing. The three essays in this thesis are also examples of such work. While two of the essays are formally problems in mechanism design, the third essay is heavily influenced by this literature.

In the rest of this chapter, we shall formally introduce the problem of mechanism design and discuss some theoretical results in this area. We shall then provide an overview of the work contained in this thesis by discussing briefly each of the essays, and the results contained in them.

¹See the surveys of Moore [40] and Palfrey [49].

1.1 Some Results on Mechanism Design

Formally, a non-cooperative game in strategic form is $G \equiv \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, where N is the (finite) set of players, S_i is the strategy set of player i and $u_i : \prod_{i \in N} S_i \rightarrow \mathcal{R}$ is the payoff function of player i . Let $S \equiv \prod_{j \in N} S_j$ and $S_{-i} \equiv \prod_{j \in N \setminus \{i\}} S_j$. A generic element of S_{-i} shall be denoted s_{-i} . An equilibrium concept e for the game G is a selection from S and is denoted $e(G)$. We now define some specific equilibrium concepts.

Definition 1.1.1 *The strategy s_i^* is said to be a dominant strategy for player i if $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_i \in S_i$ and all $s_{-i} \in S_{-i}$.*

Definition 1.1.2 *The strategy profile $s^* = (s_i^*)_{i \in N}$ is a dominant strategy equilibrium of the game G if for each player $i \in N$, s_i^* is a dominant strategy for i . The set of all dominant strategy equilibria of G will be denoted by $DS(G)$.*

Obviously, a dominant strategy equilibrium need not exist for a given game G . The next equilibrium concept to which we turn to is Nash equilibrium.

Definition 1.1.3 *The strategy profile s^* is a Nash equilibrium of the game G if $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$. The set of all Nash equilibria of G will be denoted by $NE(G)$.*

In contrast to the previous concepts which only considered individual incentives to deviate, the next two concepts will require that the equilibrium situation be immune against coalitional attempts to deviate. Given a coalition $T \subseteq N$, the terms s_T and s_{-T} will refer to $(s_i)_{i \in T}$ and $(s_i)_{i \in N \setminus T}$ respectively.

Definition 1.1.4 *The strategy profile s^* is a Strong Nash equilibrium of the game G if there does not exist $T \subseteq N$, a strategy profile s_T for T such that $u_i(s_T, s_{-T}^*) > u_i(s^*)$ for all $i \in T$.*

To define the next equilibrium concept, we need some notation. For any $T \subset N$, and $s_{N \setminus T}^* \in S_{N \setminus T}$, let $\Gamma(G, s_{N \setminus T}^*)$ denote the game induced on T by $s_{N \setminus T}^*$. So, $\Gamma(G, s_{N \setminus T}^*) = \langle T, (S_i)_{i \in T}, (u_i^j)_{i \in T} \rangle$ where for all $j \in T$, for all $s_T \in S_T$, $u_j^j(s_T) = u_j(s_T, s_{N \setminus T}^*)$. Observe that if $N \setminus T = \emptyset$, then $u_j^j(s_T) = u_j(s_T)$. The set of Coalition-proof Nash equilibria (CPNE) of G is defined inductively on the set of players.

Definition 1.1.5 *In a single-player game, s^* is a CPNE of G iff s_i^* maximises $u_i^j(s)$ over $s \in S$. Let G be a game with n players, where $n > 1$. Suppose CPNE have been defined for all games with less than n players. Then,*

1. $s^* \in S$ is self-enforcing if for all $T \subset N$, s_T^* is a CPNE of $\Gamma(G, s_{N \setminus T}^*)$; and
2. $s^* \in S$ is a CPNE of G if it is self-enforcing and moreover there does not exist another self-enforcing strategy vector $s \in S$ such that $u_i(s) > u_i(s^*)$ for all $i \in N$.

We are now in a position to understand the problem of mechanism design. The typical mechanism design scenario envisages a situation where there is a group of agents, along with a social planner. There is informational asymmetry between the agents on the one hand and the social planner on the other: while the agents know the “environment,”² the planner does not. However, depending on the actual “environment,” the planner wants a particular decision to be taken. The problem for the planner then, is to find a way of ensuring that the agents reveal their information regarding the environment truthfully. One way to do this is to have the agents play a suitable game which will give the agents incentives to reveal their information truthfully. The problem, as can be seen, is one of “designing” an appropriate game form; hence the term mechanism design. It is important to observe that the

²This is the complete information framework.

informational asymmetry between the agents and the planner is what gives the mechanism design problem its bite.

Formally, the mechanism design problem can be set out, following Moore [40] as follows. Let $N = \{1, \dots, n\}$ be the set of agents. Let A be the set of feasible decisions. The environment or state of the world is denoted by $\theta \in \Theta$. Assume that in state θ , agent i has preferences $R_i(\theta)$ on A . Note here that we are assuming that the actual state θ is known to all the agents: this is the *complete information* scenario. The scenario is one of *incomplete information* if the agents do not know the environment.

The planner's objective, which we alluded to earlier, is embodied in a social choice correspondence (scc) which is a set-valued mapping from Θ to A . Formally, the scc f associates to each $\theta \in \Theta$, $f(\theta) \subset A$. The social choice correspondence is referred to as a social choice function (scf) if it is single-valued. The problem of mechanism design can now be posed as follows: Given a scc f , does there exist a game G such that the equilibrium³ outcomes of G for any $\theta \in \Theta$ coincide with $f(\theta)$? (A weaker condition wherein we only require that the equilibrium outcomes of G contain the set $f(\theta)$ is referred to as partial implementation or truthful implementation.) The answer to this question depends both on the equilibrium concept used and the nature of the environment.

The most satisfying situation is when we can implement a scc in dominant strategy equilibrium. Implementation under this concept is convincing for the simple reason that truthtelling is optimal for an agent *irrespective* of what others do. Unfortunately, while implementation in dominant strategy equilibrium is very appealing, we run into a basic impossibility result known as the Gibbard-Satterthwaite theorem.

Theorem 1.1.1 *Suppose Θ includes all possible strict preference orderings over A . Then there is no social choice function (scf) such that $|f(\Theta)| \geq 3$*

³The equilibrium concept can be any of those referred to earlier.

which can be truthfully implemented in dominant strategy equilibrium unless it is dictatorial.

Proof: See Gibbard [23], also Satterthwaite [53].

The Gibbard-Satterthwaite theorem implies that if we want positive results with dominant strategy implementation, then the domain restrictions will have to be weakened. Moulin [41] is an early example of such work where he obtains positive results by assuming that the set of possible preferences are single-peaked. The second essay of this thesis (Chapter 3) is also an example where one obtains positive results using dominant strategy implementation on restricted domains.

One other way of getting around the Gibbard-Satterthwaite theorem is to work with a notion of equilibrium which is weaker than that of dominant strategy equilibrium. Nash equilibrium is one such concept. The primary result in this area is due to Maskin [38]. We need the following definitions.

Definition 1.1.6 *The scc f is monotonic if, whenever $a \in f(\theta)$, and for some state ϕ , $a \notin f(\phi)$, then there exists an agent $i \in N$ and an outcome $y \in A$ such that $aR_i(\theta)y$ and $yP_i(\phi)a$ where $P_i(\cdot)$ is the strict ordering associated with the preference $R_i(\cdot)$.*

Definition 1.1.7 *The scc f satisfies No Veto Power if whenever the outcome a is top-ranked in A by $(n-1)$ agents in state θ , then $a \in f(\theta)$.*

Theorem 1.1.2 *If the scc f is implementable in Nash equilibrium, then it must be monotonic. Furthermore if $n \geq 3$ and f is monotonic and satisfies No Veto Power, then f is implementable in Nash equilibrium.*

Proof: See Maskin [38].

One can consider further refinements along this line, for instance, one can ask what social choice correspondences are implementable under subgame perfect Nash equilibrium (see Abreu and Sen [1]) or what social choice correspondences are implementable in undominated strategies (see Palfrey and

Srivastava [50]). Rather than proceeding along this line which is tangential to my purpose (see Moore [40] for a discussion of these issues), we shall proceed to discuss implementation under equilibrium concepts which explicitly take into consideration coalitional attempts to manipulate the mechanism. The two concepts that we shall deal with in this regard are Strong Nash equilibrium and Coalition-proof Nash equilibrium.

With regard to implementation under Strong Nash equilibrium, the central result is due to Dutta and Sen [19] who provide a condition which is both necessary and sufficient for implementation in Strong Nash equilibrium. This condition, which the authors call Condition γ is complicated to state. But it requires the scc to be non-empty valued, Pareto optimal (otherwise all or a subgroup of agents can deviate) and also monotonic (a condition also necessary for Nash implementation). The heart of Condition γ is a property called Generalized Unanimity which, roughly speaking, is an extension of the monotonicity condition to allow for deviations by coalitions. Dutta and Sen [19] establish the following.

Theorem 1.1.3 *The scc f is implementable in Strong Nash equilibrium if and only if it satisfies Condition γ .*

Proof: See Dutta and Sen [19].⁴

There has not been much work characterising the social choice correspondences which are implementable in Coalition-proof Nash equilibria. However, in a recent work, Keiding and Peleg [31] have obtained partial characterizations in this regard. The characterization, however, is in terms of the effectivity function induced by the social choice correspondence. This is, as the authors themselves recognize, less useful than having a characterization in terms of the social choice correspondence itself. Given a set X , let $P(X)$ denote the power set of X .

⁴See also Suh [59].

Definition 1.1.8 *An effectivity function is a map $E : P(N) \rightarrow P^2(A)$ satisfying the following conditions:*

1. *for all $B \in P(A)$, $B \notin E(\emptyset)$,*
2. *for all $B \in 2^A$, $B \in E(N)$,*
3. *for all $S \in P(N)$, $\emptyset \notin E(S)$,*
4. *for all $S \in 2^N$, $A \in E(S)$.*

Formally, Keiding and Peleg [31] establish:

Theorem 1.1.4 *The effectivity function induced by a scc f which is truthfully implementable in CPNE must be superadditive, monotonic and maximal. Conversely, given an arbitrary effectivity function E which is superadditive and maximal, one can find a scc f which is truthfully implementable in CPNE and has E as its associated effectivity function.*

Proof: See Keiding and Peleg [31].

The above completes the brief discussion of mechanism design that is essential for our purpose. We now turn to the work done in this thesis. As mentioned before, two of the essays in this thesis are formally problems in mechanism design. They are discussed in Chapters 3 and 4 respectively. However, before addressing them, we first discuss a problem, which is strictly speaking, not a problem in mechanism design but is nonetheless, highly influenced by that literature.

1.2 Stable Networks

1.2.1 Networks in General

The first essay in this thesis is a problem on network formation.⁵ It can also be thought of as a contribution to the theory of coalition formation. Before discussing the problem in particular, we discuss the economic analysis of networks. A lot of recent work has focussed on the topic of networks. One aspect of this work has been in the context of industrial organization (on this, see the survey by Economides [20]). In contrast to the work relating to Industrial Organization which uses non-cooperative game theory, another branch, taking the lead from the early work of Myerson [45], has used cooperative game theory to examine a different set of issues. The original work of Myerson [45] was concerned with examining the effect of communication restrictions on the division of surplus. The starting point of his analysis was the familiar characteristic function transferable utility game. A graph, which was specified externally represented the communication possibilities to the players. Myerson then defined a new game taking into account the externally defined communication possibilities, suggested a solution for this game, and also characterised it. Further work has been done along this line, and the entire body of work is surveyed in van den Nouweland's doctoral thesis [47].⁶

As van den Nouweland notes, this body of work assumes that the communication structure is exogeneously given. Our work, however, fits into a body of work which endogenizes the formation of links. The work of Aumann and Myerson [2] was an early attempt in this direction. They specified an extensive form game to endogenize link formation. Assume that there is a fixed order on the set of all possible links. The game consists of a pair of players being offered the chance to form a link. A link is formed only if

⁵This work, which constitutes Chapter 2 of the thesis, is based on Dutta and Mutsaers [16]

⁶See also the doctoral thesis of Feltkamp [21].

both players involved in the link agree to it. The offers are made according to the prespecified order. The game ends when all possible pairs have been offered the chance to form a link. Aumann and Myerson [2] obtain results for particular games when the allocation rule is chosen to be the Myerson value.

Myerson [46] also suggested a strategic form game, called the *link formation game* to study the endogeneous formation of links. This game is described in detail in the next chapter. Qin [48] has used this game to study the endogeneous formation of coalition structures. This game has also been used by Dutta, van den Nouweland and Tijs [17] to study network formation. It should be noted that the starting point in all these papers is a game in characteristic form. The problem is one of studying what network structures (coalition structures) form when agents have the freedom to break or form links. The answer clearly depends on the allocation rule.

Jackson and Wolinsky [30] modify the above framework in a small but crucial way. Note that the value function in the characteristic function framework specifies the “worth” of each coalition. In the Jackson-Wolinsky framework, a value function specifies for each *network* or graph, the corresponding worth of that network. As will be clear in the next chapter, the worth of a set of players in the Myerson framework depends only on whether they are “connected,” whereas in the Jackson-Wolinsky framework, it can depend on how they are “connected” as well. The Jackson-Wolinsky framework is a generalization of the Myerson framework, and it is the one we will adopt as well.

A similar approach to network formation has been adopted by Bala and Goyal [4]. The difference between their work and that of Jackson and Wolinsky [30] is their adoption of a *dynamic* framework. They study the formation of networks when agents interact over time and learn from the past history. They show that under some conditions, the learning process they study leads to an efficient network, that is, one which maximises total worth.

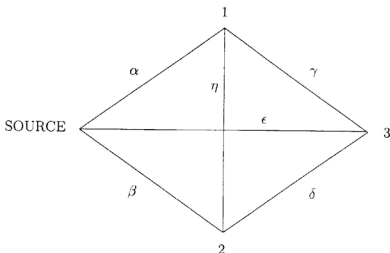


Figure 1.2.1: A Minimum Cost Spanning Tree Problem

In addition to the above mentioned work, there is a considerable literature in the Operations Research area on specific problems like the minimum cost spanning tree problem. Some of these also involve the endogenous formation of links.⁷

It is worth mentioning that in all these problems, a central insight has been the realization that unless the allocation rule which specifies how the total surplus from a network is to be shared is “properly” designed, the efficient network may not be stable. To illustrate this point, consider the following example of a minimum cost spanning tree problem taken from Sharkey [56].

As depicted in Figure 1.2.1, the problem involves a source and three customers. Each of the customers has to be supplied an input from the source. The cost of establishing each link is depicted in the diagram. Assume

⁷See the survey by Sharkey [56] on the use of network models in economics.

that the parameters take on the following values: $\alpha = 2.7, \beta = \gamma = 100, \eta = 4, \delta = 2$ and $\epsilon = 4.1$. It is easy to check that the cost efficient network is $\{(01), (12), (23)\}$ where 0 represents the Source. Consider the allocation rule which divides the cost of each individual link equally among the agents using it (this is the Shapley Value in operation). In this case, agent 3 would be required to pay $\alpha/3 + \eta/2 + \delta = 4.9$. However, as the direct cost of linking 3 to the source is only 4.1, agent 3 would prefer not to be a part of the efficient network. Thus the efficient network turns out not to be stable under this “reasonable” cost allocation scheme. It is this tension between efficiency and stability that we address in our work, albeit in a setting different from the minimum cost spanning tree problem.

1.2.2 The Problem and its Resolution

The framework within which we address this problem is an extension of the transferable utility characteristic function framework of cooperative game theory. Observe that in the characteristic function framework, a value function specifies the “worth” of each coalition. Jackson and Wolinsky [30], drawing on the work of Myerson [45], observed that one could extend this framework by having the value function specify for each network or graph, the total value of the graph. A graph can be thought of as a set of bilateral links between the players. (We will be more specific in the next chapter.) The value function then specifies the total surplus that can be achieved by the players conditional on the bilateral relationship that they choose to establish. Within this framework, Jackson and Wolinsky [30] showed a basic impossibility result. Assume that an *allocation rule* specifies for every network, how the surplus generated from that network should be divided amongst the players. Define a network to be efficient if the value of that network is at least as much as any other network. A network is *stable* if no set of players can deviate from that network (by either forming new links amongst themselves or breaking existing links) and make themselves better off. One can then

ask whether there exists an allocation rule which ensures that at least one efficient graph is stable? Jackson and Wolinsky [30] show that the answer is no if the allocation rule is constrained to be both anonymous (the names of players should not matter) and component balanced (the surplus from a disjoint subnetwork should be shared amongst the players in that subnetwork).⁸ Jackson and Wolinsky's result thus shows that if one wants to reconcile stability and efficiency, then either anonymity or component balancedness will have to be given up. In their work, Jackson and Wolinsky pursue the former option. In our work, we pursue the latter option.

It should be obvious that the problem is not formally one of mechanism design. We had observed earlier that the mechanism design problem gets its bite from the fact that there is asymmetric information between the players and a "social planner." However, here the social planner - who is assumed to have the power to decide which allocation rule will be followed - knows all that the players themselves know, namely, the value function itself. Rather, what renders the network problem non-trivial is the fact that the planner cannot control the formation of links which is under the control of the players themselves. The planner can influence the formation of links only indirectly through the allocation rule.

Even though the problem itself is not one of mechanism design, it is nonetheless similar to a mechanism design problem. The problem in mechanism design is one of designing a game such that in all states of the world, agents find it in their own self-interest to report truthfully. The problem before us is one of designing an allocation rule such that the agents find it in their own self-interest to establish the links associated with an efficient graph. The analogy is clear: just as the planner in a mechanism design problem cannot compel the agents to report truthfully, so the planner in the

⁸Jackson and Wolinsky actually prove the result using a weaker notion of stability called pair-wise stability. This, as well as other notions of stability are defined formally in Chapter 2.

network problem cannot compel the agents to form the efficient network.

We thus address ourselves to the problem of designing an allocation rule which ensures that at least one efficient network is stable. We also require that the allocation rule ensure that inefficient networks are *not* stable. Finally, we require that violations of anonymity be restricted to those networks which are not formed in equilibrium. Our first significant result is to demonstrate the existence of a component balanced allocation rule satisfying all of the above conditions.

To understand the other main result, we need to understand the process of link formation. We basically model the process as a *link formation game* following Myerson [46]. In this game, agents announce the set of players with whom they want to form links, and a link is established between two players if and only if they both announce that they want to establish a link between themselves. Imagine that after the allocation rule has been selected by the planner, the players play the link formation game which determines the network that actually gets formed. In the first result, we assume that the network that actually gets formed is one that is induced by some Strong Nash equilibrium of the link formation game. So, if a network is not formed, it follows that there does not exist any Strong Nash equilibrium of the link formation game which can give rise to that particular network.

The problem with Strong Nash equilibrium, as has been pointed out by Bernheim, Peleg and Whinston [9] is that a strategy profile is ruled out as a Strong Nash equilibrium if there exists a deviation for some coalition which makes all members of that coalition better off. However, it is easy to construct examples where the proposed deviation is not "credible." An example of this is the familiar Prisoners' Dilemma game where the unique Nash equilibrium is ruled out as a Strong Nash equilibrium by a joint deviation of the two players to the (cooperate, cooperate) profile. However, this deviation is not credible since if it were carried out, a player would at once deviate to play non-cooperatively. In order to avoid this problem where potential equi-

libria are ruled out by incredible deviations, Bernheim, Peleg and Whinston [9] propose a modification which they dub Coalition-proof Nash equilibrium.

In the context of our problem, it is reasonable to require that the strategy profile underlying an efficient graph be a Strong Nash equilibrium - that is, proof against all deviations, even incredible ones. However, there is a problem with requiring that there not exist any Strong Nash equilibria giving rise to inefficient graphs. Indeed, if a strategy profile giving rise to an inefficient graph is ruled out as a Strong Nash equilibrium by an "incredible" deviation, then we have no reason to believe that inefficient graphs will not form. It would thus be more reasonable to require that there not exist any Coalition-proof Nash equilibria of the link formation game giving rise to inefficient graphs. We show that under a domain restriction, there do exist allocation rules meeting the modified criteria. Taken together, the results suggest that the conflict between efficiency and stability can be reconciled after all.

1.3 Strategyproof Rules for Cost Sharing

The second essay of this thesis looks at a problem of cost sharing. Consider the case where a set of agents desire an indivisible public good like network access. The cost of supplying the good depends on the set of agents to whom the good is to be supplied. If the willingness to pay for the public good for each individual were known to the planner, then she could optimally provide the service to the efficient coalition, that is, the coalition maximizing the difference between the total willingness to pay and the cost of supplying the good. In the absence of such information, a mechanism has to be designed which will give the agents incentives to reveal their information truthfully. In this context, Green and Laffont [25] show that there do not exist any mechanism satisfying efficiency, strategyproofness and budget balance.

This setup has been examined recently by Moulin and Shenker [44] and Deb and Razzolini [14] among others. Assuming that the cost function is

submodular and that preferences are quasi-linear, Moulin and Shenker first consider a mechanism which is strategyproof, efficient and satisfying Voluntary Participation. They then proceed to demonstrate that such a mechanism is vulnerable to manipulations by coalitions. Next, they restrict attention to Budget Balanced and Group Strategyproof mechanisms.⁹ They show that any mechanism satisfying five axioms, viz. Strong Group Strategyproofness, Voluntary Participation, No Subsidy, Consumer Sovereignty and Budget Balance must be characterised by the property of Cross-Monotonicity, that is, an individual's cost share is non-increasing in the set of agents given access to the good. They also establish a converse to the above. Moulin and Shenker [44] then focus on a particular cross-monotonic mechanism based on the Shapley value. They show that this mechanism has a special property in that it minimizes the maximum loss caused by the non-formation of the efficient coalition.

Deb and Razzolini [14] consider a model very similar to that of Moulin and Shenker. They restrict attention to the case where the cost of supplying the good is the same for all coalitions (this is the case for an excludable public good), but in contrast to Moulin and Shenker, they allow for a wider range of preferences, and also use Strategyproofness instead of Group Strategyproofness. The main result of their paper demonstrates that any Strategyproof and Budget Balanced mechanism satisfying Voluntary Participation and a condition called "Voluntary Non-Participation" must be characterised by an "Equal Price" theorem: any agent given access must be charged the same amount.

The differing approaches of Moulin and Shenker who use Group Strategyproofness and Deb and Razzolini who use Strategyproofness prompts the question: how different are the set of Group Strategyproof mechanisms from the set of Strategyproof mechanisms? We answer this question by linking

⁹Group Strategyproofness requires that the mechanism be immune against manipulations by coalitions. We state this precisely later.

Strategyproofness and Group Strategyproofness through the concept of *Non-Bossiness*. Non-Bossiness was introduced by Satterthwaite and Sonnenschein [54] and basically requires that a unilateral deviation by an agent which leaves the agent herself indifferent not affect any other agent. We define two variants of this concept: Outcome Non-Bossiness and Utility Non-Bossiness. Outcome Non-Bossiness requires that a deviation by an agent which gives the agent himself the same outcome, not change the outcome to any other agent. Another version is Strong Utility Non-Bossiness. This requires that a deviation by an agent which gives the agent herself the same utility, not change the utility level of any other agent. We then define a weaker variant of Strong Utility Non-Bossiness called Weak Utility Non-Bossiness: this requires that if an agent deviates in a manner which gives herself the same utility level, then no other agent should be strictly better off.

We show that any Strategyproof mechanism satisfying Voluntary Participation, No Subsidy and Outcome Nonbossiness must be at least Weak Group Strategyproof.¹⁰ We show through an example that the four conditions together do not imply Strong Group Strategyproofness. Next, we show that the combination of Strategyproofness, Voluntary Participation, No Subsidy, Outcome Nonbossiness and Weak Utility Nonbossiness implies Strong Group Strategyproofness. We also show that if the cost function is submodular, then any Strongly Group Strategyproof mechanism satisfying Voluntary Participation, No Subsidy, Consumer Sovereignty and Budget Balance must also satisfy Outcome Nonbossiness and Weak Utility Nonbossiness. Taken together, the results here suggest that the class of Strategyproof and Strong Group Strategyproof mechanisms are essentially the same if the cost function is submodular.

¹⁰Weak Group Strategyproofness requires that no coalition should be able to deviate in a manner which leaves all members of the coalition strictly better off. Strong Group Strategyproofness requires that no coalition should be able to deviate in a manner which leaves all members weakly better off and at least one member strictly better off. These concepts are defined precisely later.

We then go on to characterise a particular Strong Group Strategyproof and Budget Balanced mechanism under the assumption that the cost function is submodular. This is the *Egalitarian cost allocation rule*, and is based on the egalitarian solution for characteristic function TU-games which was proposed by Dutta and Ray [18]. Assume that agents' valuations are independent of each other and distributed according to a common distribution function which satisfies the *monotone hazard rate* condition. Then, we are able to show that the Egalitarian cost allocation rule maximises the probability that all members of a given coalition accept the cost shares imputed to them in the class of all Strong Group Strategyproof and Budget Balanced rules satisfying Voluntary Participation, No Subsidy and Consumer Sovereignty. Under a stronger condition on the distribution function, we show that the Egalitarian cost allocation rule maximizes the expected surplus from the *ex ante* efficient coalition. We also use the results obtained on the link between Strategyproofness and Strong Group Strategyproofness to give alternate characterisations of the Shapley value and the Egalitarian rules in terms of Strategyproofness and Nonbossiness.

1.4 Multi-Good Cost Sharing

The third essay deals with an extension of the model examined in Chapter 3. The motivation for this problem comes from the literature on club goods. As Cornes and Sandler [12] observe, the literature on club goods has for the most part confined itself to the analysis of the single club good (or excludable public good) model.¹¹ However, most clubs in real life offer a range of products to their members. For example, a club might offer a swimming pool, a golf course and tennis courts to its members. Typically, there are economies of scale associated with offering multiple services; this externality

¹¹The literature on multiproduct clubs include Berglas [7], [8], Brueckner and Lee [11], Sandler and Tschirhart [52] and Wooders [61].

in production is what renders the multiproduct case different from the single good case. To the best of our knowledge, there have been no results obtained on the Strategyproof cost sharing of multiple excludable public goods.

Moulin [42] offers a model of cost sharing which can be interpreted to accomodate the multiple excludable public good case. In his model, there are n agents, labelled for convenience as $\{1, \dots, n\}$. Agent i is assumed to be able to consume discrete units of good i . The cost function defined on the vector of quantities $(q_1, \dots, q_n)$ is assumed to be supermodular (increasing marginal costs). To interpret the model as one involving multiple excludable public goods, assume that agent i can consume k units of good i where k is the number of excludable public goods. Unfortunately, Moulin's assumption of increasing marginal costs cannot be satisfied because with multiple excludable public goods, the cost function displays neither increasing marginal costs nor decreasing marginal costs. To see this, observe that once the first good has been produced, the marginal cost of accomodating an additional person is zero unless the additional person demands a unit of the second good. So, the marginal cost is zero if a good has been produced and is increasing only at those points where an additional good has to be produced. The consequence of this is that Moulin's model cannot really address the multiple excludable goods problem.

The work closest to what we are doing are two papers - one by van Kolpin [32] and the other by Koster [33] - which aim to extend the serial cost sharing rule proposed by Moulin and Shenker [43] for the homogeneous good case to multiple goods. The paper by Kolpin is basically an impossibility result showing that the additivity axiom is incompatible with other axioms in the multiple goods scenario. Koster [33] proposes a way of extending the serial cost sharing rule to the multiple goods case, but the analysis is from an axiomatic viewpoint. Other related work include the paper by Ausubel [3] on efficient auctions for multiple objects and the paper by Bikhchandani and Mamer [10] on competitive equilibria in economies with indivisibilities.

The situation with multiple excludable goods is more complicated than the single good case. In the single good case, the mechanism had to decide whether the good was to be produced, and if the good was produced, the set of agents getting access and their cost shares. With multiple goods, the mechanism has to first decide which subset of goods is to be produced, then for each agent the set of goods to which he is to be given access, and finally the cost shares. For instance, with just two goods and two agents, there are sixteen potential cases to consider ranging from the one where both agents get access to both goods to the one where no production takes place and so neither agent gets access to any good.

Suppose we are interested in mechanisms which satisfy the axioms postulated for the single good case. That is, we want mechanisms which are Strategyproof, Outcome Non-Bossy, Budget Balanced and satisfy No Subsidy, Voluntary Participation and Consumer Sovereignty. (The Consumer Sovereignty axiom has to be reformulated to handle the multiple good case.) We show, as our main result, that at least for the special case of two agents, there exist no mechanism satisfying all of the above axioms. We do identify mechanisms violating one or more of the above axioms.

One issue which is of great interest is to evaluate the welfare implications of these mechanisms. The result of Green and Laffont [25] still goes through. So we know that any budget balanced and strategyproof mechanism must, in general, be inefficient. However, the computations of welfare loss become much more complicated, and we have not been able to obtain any results which would be the natural analogue of the characterisations of the Shapley Value rule and Egalitarian Cost Allocation rule for the single good case.

Chapter 2

Stable Networks

2.1 Introduction

The interaction between agents can often be fruitfully described by a *network structure* or *graph*, where the nodes represent the agents and an arc exists between two nodes if the corresponding agents interact bilaterally. Network structures have been used in a wide variety of contexts ranging from social networks (Wellman and Berkowitz [60]), information transmission (Goyal [24]), internal organization of firms (Marschak and Reichelstein [37]), cost allocation schemes (Henriet and Moulin [29]), to the structure of airline routes (Hendricks, Piccione and Tan [28]).

In a recent paper, Jackson and Wolinsky [30] focus on the *stability* of networks. Their analysis is designed to give predictions concerning which networks are likely to form when self-interested agents can choose to form new links or sever existing links. They use a specification where a *value function* gives the value (or total product) of each graph or network, while an allocation rule gives the distribution of value amongst the agents forming the network. A principal result of their analysis shows that efficient graphs (that is, graphs of maximum value) may not be stable when the allocation rule treats individuals symmetrically.

In this chapter, the potential conflict between stability and efficiency of graphs is subject to further scrutiny. In order to do this, we follow Dutta, et. al. [17] and assume that agents' decisions on whether or not to form a link with other agents can be represented as a game in strategic form.¹ In this "link formation" game, each player announces a set of players with whom he or she wants to form a link. A link between two players is formed if both players want the link. This rule determines the graph corresponding to any n -tuple of announcements. The value function and the allocation rule then give the payoff function of the strategic form game.

Since the link formation game is a well-defined strategic form game, one can use any equilibrium concept to analyze the formation of networks. We will define a graph to be *strongly stable* (respectively *weakly stable*) if it corresponds to a *Strong Nash* equilibrium (respectively *Coalition-proof Nash equilibrium*) of the link formation game. Although Jackson and Wolinsky [30] did not use the link formation game, their specification assumed that only *two-person* coalitions can form; their notion of *pairwise stability* is implied by our concept of strong stability. Hence, it follows straightaway from their analysis that there is a conflict between strong stability and efficiency if the allocation rule is symmetric.

How can we ensure that efficient graphs will form? One possibility is to use allocation rules which are *not* symmetric. For instance, fix a vector of weights $w = (w_1, w_2, \dots, w_n)$. Call an allocation rule *w-fair* if the gains or losses to players i and j from the formation of the *new* link (ij) is proportional to $\frac{w_i}{w_j}$. *w-fair* rules are symmetric only if $w_i = w_j$ for all i and j . However, the vector of weights w can be chosen so that there is only a 'slight' departure from symmetry. We first show that the class of *w-fair* rules coincides with the class of *weighted Shapley values* of an appropriately defined transferable

¹This game was originally suggested by Myerson [46] and subsequently used by Qin [48]. See also Hart and Kurz [27] who discuss a similar strategic form game in the context of the endogenous formation of coalition structures.

utility game. We then go on to construct a value function under which *no* efficient graph is strongly stable for any *w*-fair allocation rule. Thus, the relaxation of symmetry in this direction does not help.

A second possibility is to use weak stability instead of strong stability. However, again we demonstrate a conflict between efficiency, symmetry and (weak) stability.

We then go on to adopt an *implementation* or *mechanism design* approach. Suppose the implicit assumption or prediction is that only those graphs which correspond to Strong Nash equilibria of the link formation game will form. Then, our interest in the *ethical* properties of the allocation rule should be restricted only to how the rule behaves *on the class of these graphs*. Hence, if we want symmetry of the allocation rule, we should be satisfied if the allocation rule is symmetric on the *subdomain* of strongly stable graphs.

We analyse *two* specific problems within this general approach. In the first *design* problem, we construct an allocation rule which ensures that (i) the class of strongly stable graphs is a *nonempty* subset of the set of efficient graphs, and (ii) satisfies the restriction that the rule is symmetric on the class of strongly stable graphs. This result is proved under a very mild restriction on the class of value functions. The second result is much stronger, but is proved for a *more* restrictive class of value functions. More specifically, we construct an allocation rule which (given the restrictions on the class of value functions) guarantees that (i) there is at least *one* strongly stable graph, (ii) all *weakly stable* graphs are efficient, and (iii) the allocation rule is symmetric on the class of weakly stable graphs. Thus, this achieves a kind of 'double' implementation in Strong Nash and Coalition-proof Nash equilibrium.

A common feature of the allocation rules constructed by us is that these distribute the value of stable graphs *equally* amongst all agents. Obviously, this ensures symmetry of the allocation rules on the class of stable graphs. Of course, the rules do not treat agents symmetrically on some graphs which

are not stable. Indeed, the asymmetries are carefully constructed so as to ensure that the other requirements of the design problem(s) are satisfied.

2.2 Some Definitions

Let $N = \{1, 2, \dots, n\}$ be a finite set of agents with $n \geq 3$. Interactions between agents are represented by graphs whose vertices represent the players, and whose arcs depict the pairwise relations. The *complete* graph, denoted g^N , is the set of *all* subsets of N of size 2. G is the set of *all* possible graphs on N , so that $G = \{g | g \subset g^N\}$. Given any $g \in G$, let $N(g) = \{i \in N | \exists j \text{ such that } (ij) \in g\}$.

The link (ij) is the subset of N containing i, j . $g + (ij)$ and $g - (ij)$ are the graphs obtained from g by adding and subtracting the link (ij) respectively.

Two agents i and j are *connected* in g if there is a sequence $\{i_0, i_1, \dots, i_K\}$ such that $i_0 = i$, $i_K = j$ and $(i_k i_{k+1}) \in g$ for all $k = 0, 1, \dots, K - 1$. We will use $C(g)$ to denote the set of connected components of g . The graph g is said to be *fully connected* (respectively *connected* on S) if all pairs of agents in N (respectively in S) are connected. The graph g is *totally disconnected* if $g = \{\emptyset\}$. If h is a component of g , then $N(h) = \{i | (ij) \in h \text{ for some } j \in N \setminus \{i\}\}$, and n_h denotes the cardinality of $N(h)$.

The value of a graph is represented by a function $v : G \rightarrow \mathbb{R}$. We will only be interested in the set V of such functions satisfying *Component Additivity*.

Definition 2.2.1 *A value function is component additive if $v(g) = \sum_{h \in C(g)} v(h)$.*

We interpret the value function to indicate the total “output” produced by agents in N when they are “organised” according to a particular graph. For instance, the members of N may be workers in a firm. The graph g then represents the internal organization of the firm, that is the structure of communication amongst the workers. Alternatively, N could be a set of (tax) auditors and supervisors, and g could represent a particular hierarchical

structure of auditors and supervisors. In this case, $v(g)$ is the (expected) tax revenue realized from a population of tax payers when g is in “operation.”

It is worth emphasizing at this point that the value function is a very general concept. In particular, it is more general than Myerson’s [45] games with cooperation structure. A cooperation structure is a graph in our terminology. Given any exogeneously specified transferable utility game (N, u) and a graph g , we define for each $S \subset N$, the restricted graph on S as $g|S \equiv \{(ij) \in g | i, j \in S\}$. The *graph-restricted* game (N, u^g) specifies the *worth* of a coalition as follows.

$$\text{For all } S \subset N, \quad u^g(S) = \sum_{h \in C(g|S)} u(N(h)) \quad (2.2.1)$$

The value or worth of a given set of agents in Myerson’s formulation depends on whether they are connected or not, whereas in the Jackson-Wolinsky approach, the value of a coalition can in principle depend on *how* they are connected. The following example illustrates this.

Example 2.2.1 Let $N = \{1, 2, 3\}$. Let the characteristic function v be defined as $v(N) = 2$, $v(S) = 1$ for all $S \subset N$ such that $|S| = 2$, and $v(S) = 0$ otherwise. Consider two graphs $g = g^N$ and $h = \{(12), (13)\}$. It is easy to see that the value function for graph-restricted game corresponding to the graph g is the same as the characteristic function v . The characteristic function corresponding to h is given as $v^h(N) = 2$, $v^h(\{1, 2\}) = v^h(\{1, 3\}) = 1$, and $v^h(S) = 0$ for all other $S \subset N$. Note that the worth of the grand coalition, and the subcoalitions $\{1, 2\}$, $\{1, 3\}$ are the same under the characteristic functions v^g and v^h . However, since 2 and 3 are not connected directly in h , the worth of $\{2, 3\}$ in the graph-restricted game v^h is zero. This illustrates the point that the value of a set of agents depends only on whether they are connected. In the Jackson-Wolinsky framework, this is precisely what is different.

Given v , g is *strongly efficient* if $v(g) \geq v(g')$ for all $g' \in G$. Let $E(v)$ denote the set of strongly efficient graphs.

Finally, an *allocation rule* $Y : V \times G \rightarrow \mathbb{R}^N$ describes how the value associated with each network is distributed to the individual players. $Y_i(v, g)$ will denote the payoff to player i from graph g under the value function v . Clearly, an allocation rule corresponds to the concept of a *solution* in cooperative game theory.

Given a permutation $\pi : N \rightarrow N$, let $g^\pi = \{(ij) | i = \pi(k), j = \pi(\ell), (k\ell) \in g\}$. Let v^π be defined by $v^\pi(g^\pi) = v(g)$.

The following condition imposes the restriction that all agents should be treated symmetrically by the allocation rule. In particular, *names* of the agents should not determine their allocation.

Definition 2.2.2 Y is *anonymous on* $G' \subseteq G$ if for all pairs $(v, g) \in V \times G'$, and for all permutations π , $Y_{\pi(i)}(v^\pi, g^\pi) = Y_i(v, g)$.

Remark 2.2.1 If Y is anonymous on G , we say that Y is *fully anonymous*.

Definition 2.2.3 Y is *component balanced* if $\sum_{i \in N(h)} Y_i(v, g) = v(h)$ for every $g \in G$, $h \in C(g)$.

Component balance implies that cross-subsidization is ruled out. We will restrict attention to component balanced allocation rules throughout this chapter.²

2.3 The Link Formation Game

In this section, we describe the strategic form game which will be used to model the endogenous formation of networks or graphs³. The following

²Jackson and Wolinsky [30] point out that the conflict between anonymity, stability and efficiency disappears if the rule is not component balanced.

³Aumann and Myerson [2] use an *extensive* form approach in modeling the endogenous formation of cooperation structures.

description of the link formation game assumes a specific value function v and an allocation rule Y . Let $\gamma \equiv (v, Y)$.

The linking game $\Gamma(\gamma)$ is given by the $(n + 2)$ - tuple $(N; S_1, \dots, S_n, f^\gamma)$, where for each $i \in N$, S_i is player i 's *strategy set* with $S_i = 2^{N \setminus \{i\}}$ and the *payoff function* is the mapping $f^\gamma : S \equiv \prod_{i \in N} S_i \Rightarrow \mathbb{R}^N$ given by

$$f_i^\gamma(s) = Y_i(v, g(s)) \quad (2.3.2)$$

for all $s \in S$, with

$$g(s) = \{(ij) | j \in s_i, i \in s_j\} \quad (2.3.3)$$

So, a typical strategy of player i in $\Gamma(\gamma)$ consists of the set of players with whom i wants to form a link. Then, (2.3.3) states that a link between i and j forms if and only if they both want to form this link. Hence, each strategy vector gives rise to a unique graph $g(s)$. Finally, the payoff to player i associated with s is simply $Y_i(v, g(s))$, the payoff that is given by the allocation rule for the graph *induced by* s .⁴

Our interest lies in the *graphs* which are induced by the strategy vectors corresponding to either the Strong Nash equilibria or Coalition-proof Nash equilibria of the link formation game. This motivates the following definition.

Definition 2.3.4 g^* is *strongly stable* [respectively *weakly stable*] for $\gamma = (v, Y)$ if g^* is induced by some s which is a SNE [respectively CPNE] of $\Gamma(\gamma)$.

Hence, a strongly stable graph is induced or supported by a strategy vector which is a Strong Nash equilibrium of the linking game. Of course, a strongly stable graph must also be weakly stable.

Finally, in order to compare the Jackson-Wolinsky notion of *pairwise stability*, suppose the following constraints are imposed on the set of possible deviations in the definition of Strong Nash equilibrium (see Definition 1.1.4.).

⁴We will say that g is induced by s if $g \equiv g(s)$, where $g(s)$ satisfies (2.3.3).

First, the deviating coalition can contain at most *two* agents. Second, the deviation can consist of severing just *one* existing link or forming *one additional* link. Then, the set of graphs which are immune to such deviations is called pairwise stable. Obviously, if g^* is strongly stable, then it must be pairwise stable.

2.4 The Results

Notice that strong stability (as well as weak stability) has been defined for a *specific* value function v and allocation rule Y . Of course, which network structure is likely to form must depend upon both the value function as well as on the allocation rule. Here, we adopt the approach that the value function is given *exogeneously*, while the allocation rule itself can be “chosen” or “designed.”

Within this general approach, it is natural to seek to construct allocation rules which are (ethically) attractive and which also lead to the formation of stable network structures which maximize output, no matter what the exogeneously specified value function. This is presumably the underlying motive behind Jackson and Wolinsky’s search for a symmetric allocation rule under which at least *one* strongly efficient graph would be pairwise stable for every value function. Jackson and Wolinsky [30] however showed that this is not possible.

Proposition 2.4.1 *There does not exist any anonymous and component balanced allocation rule such that for all value functions, at least one efficient graph is pairwise stable.*

Proof: The proof is by a counterexample. Let $N = \{1, 2, 3\}$ and consider the following value function: $v(\{i, j\}) = 1$ for all $\{i, j\} \subset N$, $v(g^N) = 1$, $v(\{i, j\}, \{j, k\}) = 1 + \epsilon$ for all $i, j, k \in N$ such that $i \neq j, j \neq k, k \neq i$. Observe that the efficient graphs are of the form $(\{i, j\}, \{j, k\})$. We shall show that such graphs cannot be stable for certain values of ϵ .

Note that by anonymity we have $Y_i(\{i, j\}) = Y_j(\{i, j\}) = 1/2$ for all $\{i, j\} \subset N$, and also $Y_i(g^N) = 1/3$ for all $i \in N$.

Next, observe that $Y_j(\{i, j\}, \{j, k\}) \geq 1/2$, otherwise j will break links with either k or i to induce the graph $\{i, j\}$ or $\{j, k\}$. By component balancedness and anonymity, we thus have $Y_i(\{i, j\}, \{j, k\}) = Y_k(\{i, j\}, \{j, k\}) \leq 1/4 + \epsilon/2$. But if $\epsilon < 1/6$, then both i and k are better off deviating from the graph $(\{i, j\}, \{j, k\})$ by forming the link $\{i, k\}$ and moving to g^N . ■

Given the negative result, we initially impose weaker requirements. First, instead of *full anonymity*, we only ask that the allocation rule be *w-fair*, a condition which is defined presently. However, we show that there can be value functions under which *no* strongly efficient graph is strongly stable⁵. Second, we retain full anonymity but replace strong stability by weak stability. Again, we construct a value function under which the unique strongly efficient graph is *not* weakly stable.

Our final results, which are the main results of the chapter, explicitly adopt an implementation approach to the problem. Assuming that Strong Nash equilibrium is the “appropriate” concept of equilibrium and that the individual agents decide to form network relations through the link formation game is equivalent to predicting that only strongly stable graphs will form. Let $S(\gamma)$ be the set of strongly stable graphs corresponding to $\gamma \equiv (v, Y)$. Instead of imposing full anonymity, we only require that the allocation rule be anonymous on the *restricted* domain $S(\gamma)$. However, we now require that for all permissible value functions, $S(\gamma)$ is contained in the set of strongly efficient graphs, instead of merely *intersecting* with it, which was the “target” sought to be achieved in the earlier results. We are able to construct an allocation rule which satisfies these requirements.

Suppose, however, that the designer has some doubt whether Strong Nash equilibrium is really the “appropriate” notion of equilibrium. In particular, she apprehends that weakly stable graphs may also form. Then, she would

⁵We point out below that strong stability can be replaced by pairwise stability.

want to ensure anonymity of the allocation rule over the larger class of weakly stable graphs, as well as efficiency of these graphs. Assuming a stronger restriction on the class of permissible value functions, we are able to construct an allocation rule which satisfies these requirements. In addition, the allocation rule also guarantees that the set of strongly stable graphs is nonempty.

Our first result uses w -fairness. Fix a vector $w = (w_1, \dots, w_n) \gg 0$.

Definition 2.4.5 *An allocation rule Y is w -fair if for all $v \in V$, for all $g \in G$, for all $i, j \in N$,*

$$\frac{1}{w_i}[Y_i(v, g) - Y_i(v, g - (ij))] = \frac{1}{w_j}[Y_j(v, g) - Y_j(v, g - (ij))].$$

In Proposition 2.4.2 below, we show that the unique allocation rule which satisfies w -fairness and component balance is the *Weighted Shapley value* of the following characteristic function game.

Take any $(v, g) \in V \times G$. Recall that for any $S \subseteq N$, the restricted graph on S is denoted $g|_S$. Then, the TU game $u_{v,g}$ is given by :

$$\text{For all } S \subseteq N, \quad u_{v,g}(S) = \sum_{h \in C(g|_S)} v(h).$$

Proposition 2.4.2 *For all $v \in V$, the unique w -fair allocation rule Y which satisfies component balance is the weighted Shapley value of $u_{v,g}$.*

Proof : The proof is omitted since it is a straightforward extension of the corresponding result in Dutta et.al. [17]. ■

Remark 2.4.2 This proposition is similar to corresponding results of Dutta, et. al. [17] and Jackson and Wolinsky [30]. The former proved that w -fair allocation rules satisfying component balance are the weighted Shapley values (also called weighted Myerson values) of the *graph-restricted* game given by any exogenous TU game and any graph g . Of course, the set of graph-restricted games is a strict subset of V , and hence Proposition 2.4.2 is a formal generalization of their result. Jackson and Wolinsky [30] show that when $w_i = w_j$ for all i, j , the unique w -fair allocation rule satisfying component balance is the Shapley value of $u_{v,g}$.

Our first result on stability follows. The motivation for proving this result is the following. Since the weight vector w can be chosen to make the allocation rule ‘approximately’ anonymous (by choosing w to be very close to the unit vector $(1, \dots, 1)$), we may ‘almost’ resolve the tension between stability, efficiency and symmetry unearthed by Jackson-Wolinsky by using such a w -fair allocation rule. However, the next result rules out this possibility.

Theorem 2.4.1 *Suppose $w \gg 0$. Then, there is no w -fair allocation rule Y satisfying component balance and such that for each $v \in V$, at least one strongly efficient graph is strongly stable.*

Proof : Let $N = \{1, 2, 3\}$, and choose any w such that $w_1 \geq w_2 \geq w_3 > 0$ and $\sum_{i=1}^3 w_i = 1$.

Now, consider the (component additive) v such that $v(\{(ij)\}) = 1$, $v(\{(ij), (jk)\}) = 1 + \varepsilon$, and $v(g^N) = 1 + 2\varepsilon$, where $\varepsilon \in (0, \frac{1}{2}(1 - \frac{w_2}{w_1 + w_3}))$.

Using Proposition 2.4.2, the unique w -fair allocation rule Y satisfying component balance is the weighted Shapley value of $u_{v,g}$. Routine calculation yields the following.

$$Y_i(v, g^N) = 2\varepsilon w_i + w_i \left(\frac{w_k}{w_i + w_j} + \frac{w_j}{w_i + w_k} \right) \text{ for all } i, j, k \in N \quad (2.4.4)$$

$$Y_i(v, \{(ij)\}) = \frac{w_i}{w_i + w_j} \text{ for all } i, j \in N \quad (2.4.5)$$

From 2.4.4 and 2.4.5, and using $\sum_{i=1}^3 w_i = 1$, we get

$$Y_i(v, \{(ij)\}) - Y_i(v, g^N) = w_i \left(1 - 2\varepsilon - \frac{w_j}{w_i + w_k} \right) \quad (2.4.6)$$

Remembering that $w_1 \geq w_2 \geq w_3$ and that $\varepsilon < \frac{1}{2}(1 - \frac{w_2}{w_1 + w_3})$, (2.4.6) yields

$$Y_i(v, \{(ij)\}) - Y_i(v, g^N) > 0 \text{ for } i \in \{2, 3\} \quad (2.4.7)$$

(2.4.7) implies that g^N is not strongly stable since $\{2, 3\}$ will break links with 1 to move to the graph $\{(2, 3)\}$. Since g^N is the unique strongly efficient graph, the theorem follows. ■

Remark 2.4.3 Note that since only a pair of agents need form a coalition to “block” g^N , the result strengthens the intuitive content of the Jackson-Wolinsky result.

Our next result uses weak stability instead of strong stability.

Theorem 2.4.2 *There is no fully anonymous allocation rule Y satisfying component balance such that for each $v \in V$, at least one strongly efficient graph is weakly stable.*

Proof : Let $N = \{1, 2, 3\}$, and consider v such that $v(g^N) = 1 = v(\{(ij)\})$ and $v(\{(ij), (jk)\}) = 1 + 2\varepsilon$. Assume that $0 < \varepsilon < \frac{1}{12}$.

Since Y is fully anonymous and component balanced, $Y_i(v, \{(ij)\}) = Y_j(v, \{(ij)\}) = \frac{1}{2}$. Let $g^j \equiv \{(ij), (jk)\}$. Note that $\{g^j | j \in N\}$ is the set of strongly efficient graphs. Choose any $j \in N$. Then, $Y_j(v, g^j) \geq \frac{1}{2}$. For, suppose $Y_j(v, g^j) < \frac{1}{2}$. Then, j can deviate *unilaterally* to change g^j to $\{(ij)\}$ or $\{(jk)\}$ by breaking the link with i or k respectively. So, if $Y_j(v, g^j) < \frac{1}{2}$ and g^j is induced by s , then s is not a Nash equilibrium, and hence not a CPNE.

So, $Y_j(v, g^j) \geq \frac{1}{2}$. Since Y is fully anonymous and component balanced, $Y_i(v, g^j) = Y_k(v, g^j) \leq \frac{1}{4} + \varepsilon$. Again, full anonymity of Y ensures that $Y_i(v, g^N) = \frac{1}{3}$ for all $i \in N$.

Hence, $\{i, k\}$ can deviate from g^j and form the additional link (ik) . This will precipitate the complete graph. From preceding arguments, the deviation is profitable if $\varepsilon < \frac{1}{12}$.

Letting s^N denote the strategy n -tuple which induces g^N , one notes that s^N is a Nash equilibrium. Hence, the deviation of $\{i, k\}$ to s^N is not deterred by the possibility of a further deviation by either i or k . So, g^j is *not* weakly stable. This completes the proof of the theorem. ■

Remark 2.4.4 Again note that only a 2-person coalition has to form to block g^j . So, the result could have been proved in terms of “pairwise weak

stability," which is strictly weaker than pairwise stability. Hence, this generalizes Jackson and Wolinsky's basic result.

Definition 2.4.6 *v satisfies monotonicity if for all $g \in G$, for all $i, j \in N$, $v(g + (ij)) \geq v(g)$.*

Thus, a monotonic value function has the property that additional links never decrease the value of the graph⁶. A special class of monotonic value functions, the class considered by Dutta, et. al. [17], is the set of graph-restricted games derived from superadditive TU games. Of course, there are also other contexts which might give rise to monotonic value functions. Dutta, et. al proved that for the class of graph-restricted games derived from superadditive TU games, a large class of component balanced allocation rules (including all w -fair rules with $w \gg 0$) has the property that the set of weakly stable graphs is a *subset* of the set of graphs which are *payoff-equivalent* to g^N . Moreover, g^N itself is weakly stable⁷. Their proof can be easily extended to cover *all* monotonic value functions. We state the following result.

Theorem 2.4.3 *Suppose v is monotonic. Let Y be any w -fair allocation rule with $w \gg 0$, and satisfying component balance. Then, g^N is weakly stable for (v, Y) . Moreover, if g is weakly stable for (v, Y) , then g is payoff - equivalent to g^N .⁸*

Proof : The proof is omitted since it is almost identical to that of Dutta, et. al. [17]. ■

Remark 2.4.5 Note that Theorem 2.4.3 ensures that *only* strongly efficient graphs are weakly stable. Thus, if our prediction is that only weakly stable

⁶Hence, g^N is strongly efficient.

⁷ g and g' are payoff-equivalent under (v, Y) if $Y(v, g) = Y(v, g')$. Also, note that if v is monotonic, then g^N and hence graphs which are payoff-equivalent to g^N are strongly efficient.

⁸The result is true for a larger class of allocation rules which is not being defined here.

graphs will form, then this result guarantees that there will be no loss in efficiency. This guarantee is obviously stronger than that provided if *some* stable graph is strongly efficient. In the latter case, there is the possibility that *other* stable graphs are inefficient, and since there is no reason to predict that *only* the efficient stable graph will form, inefficiency can still occur.

Unfortunately, monotonicity of the value function is a stringent requirement. There are a variety of problems in which the optimum network is a *tree* or a *ring*. For example, cost allocation problems give rise to the minimum-cost spanning tree. Efficient airline routing or optimal trading arrangements may also imply that the *star* or *ring* is the efficient network.⁹ Indeed, in cases where there is a (physical) cost involved in setting up an additional link, g^N will seldom be the optimal network.

This provides the motivation to follow the “implementation approach” and prove results similar to that of Theorem 2.4.3, but covering *nonmonotonic* value functions. First, we construct a component balanced allocation rule which is anonymous on the set of *strongly stable graphs* and which ensures that *all* strongly stable graphs are strongly efficient. In order to prove this result, we impose a restriction on the class of value functions.

Definition 2.4.7 *The set of admissible value functions is $V^* = \{v \in V \mid v(g) > 0 \text{ iff } g \text{ is not totally disconnected}\}$.*

So, a value function is admissible if *all* graphs (except the trivial one in which no pair of agents is connected) have positive value.¹⁰

⁹See Hendricks et.al. [28] on the “hub and spokes” model of airline routing. See also Landa [35] for an interesting account of why the ring is the efficient institutional arrangement for organization of exchange amongst tribal communities in East Papua New Guinea.

¹⁰In common with Jackson and Wolinsky [30], we are implicitly assuming that the value of a disconnected player is zero. This assumption can be dropped at the cost of some complicated notation.

Before we formally define the particular allocation rule which will be used in the proof of the next theorem, we discuss briefly the key properties which will have to be satisfied by the rule.

Choose some efficient $g^* \in G$. Suppose s^* induces g^* , and we want to ensure that g^* is strongly stable. Now, consider any g which is different from g^* , and let s induce g . Then, the allocation rule must *punish* at least one agent who has deviated from s^* to s . This is possible only if a deviant can be *identified*. This is trivial if there is some $(ij) \in g \setminus g^*$, because then both i and j must concur in forming the extra link (ij) . However, if $g \subset g^*$, say $(ij) \in g^* \setminus g$, then *either* i *or* j can unilaterally break the link. The only way to ensure that the deviant is punished, is to punish *both* i and j .

Several simple punishment schemes can be devised to ensure that at least two agents who have deviated from s^* are punished sufficiently to make the deviation unprofitable. However, since the allocation rule has to be component balanced, these punishment schemes may result in some other agent being given more than the agent gets in g^* . This possibility creates a complication because the punishment scheme has to *satisfy an additional property*. Since we also want to ensure that inefficient graphs are *not* strongly stable, agents have to be provided with an incentive to deviate from any graph which is not strongly efficient. Hence, the punishment scheme has to be relatively more sophisticated.

Choose some strongly efficient g^* with $C(g^*) = \{h_1^*, \dots, h_J^*\}$, and let $\succ$ be a *strict* ordering on arcs of g^* . Consider any other graph g , and let $C(g) = \{h_1, \dots, h_K\}$.

The first step in the construction of the allocation rule is to choose agents who will be punished in some components $h_k \in C(g)$. For reasons which will become clear later on, we only need to worry about components h_k such that $D(h_k) = \{i \in N(h_k) | (ij) \in g^* \text{ for some } j \notin N(h_k)\}$ is nonempty. For such components, choose $i(h_k) \equiv i_k$ such that $\forall j \in N(h_k) \setminus \{i_k\}, \forall m \notin N(h_k)$:

$$(jm) \in g^* \Rightarrow (i_k l) \succ (jm) \text{ for some } (i_k l) \in g^*, l \notin N(h_k). \quad (2.4.8)$$

We will say that g is a **-supergraph* of g^* if for each $h^* \in C(g^*)$, there is $h \in C(g)$ such that $N(h^*) \subseteq N(h)$. Note that the fully connected graph is a **-supergraph* of every graph.

Lemma 2.4.1 *Suppose $g = (h_1, \dots, h_K)$ is not a *-supergraph of g^* . Then, $\exists k, l \in \{1, \dots, K\}$ such that $(i_k i_l) \in g^*$.*

Proof : Since g is not a **-supergraph* of g^* , it follows that g is not fully connected, and that there exists a component h and players i, j such that $i \in N(h), j \notin N(h)$ and $(ij) \in g^*$. Indeed, assume that for each $h_k \in C(g)$, the set $D(h_k)$ is nonempty.¹¹ For every $k = 1, 2, \dots, K$, there is $j_k \notin N(h_k)$ such that $(i_k j_k) \in g^*$ and $(i_k j_k) \succ (ij)$ for all $i \in D(h_k) \setminus \{i_k\}$ and for all $j \notin N(h_k)$ with $(ij) \in g^*$. Let the $\succ$ -maximal element within the set $\{(i_1, j_1), \dots, (i_K, j_K)\}$ be (i_{k^*}, j_{k^*}) . Let $j_{k^*} \in N(h_l)$. Note that from the definition of the pair (i_{k^*}, j_{k^*}) , it follows that $l \neq k^*$. Also, $(i_{k^*}, j_{k^*}) \in g^*$. It therefore follows that $i_l = j_{k^*}$ and $j_l = i_{k^*}$. Hence, $(i_k i_l) \in g^*$. This completes the proof of the lemma. ■

The implication of Lemma 2.4.1 is the following. Suppose one or more agents deviate from g^* to some $g \in G$ (such that g is not a **-supergraph* of g^*) with components $\{h_1, \dots, h_K\}$. Then, the set of agents $\{i(h_1), \dots, i(h_K)\}$ must contain a deviator. This property will be used intensively in the proof of the next theorem.

Theorem 2.4.4 *Let $v \in V^*$. Then, there is a component balanced allocation rule Y^* such that the set of strongly stable graphs is nonempty and contained in $E(v)$. Moreover, Y^* is anonymous on the set of strongly stable graphs.*

Proof : Choose any $v \in V^*$. Fix $g^* \in E(v)$. Let $C(g^*) = \{h_1^*, \dots, h_K^*\}$. An allocation rule Y^* satisfying the required properties will be constructed which ensures that g^* is strongly stable. Moreover, no g will be strongly stable unless it is in $E(v)$.

¹¹Otherwise, we can restrict attention to those components for which $D(h_k)$ is nonempty.

For any $S \subseteq N$ with $|S| \geq 2$, let G_S be the set of graphs which are connected on S , and have no arcs outside S . So,

$$G_S = \{g \in G \mid g \text{ is connected on } S \text{ and } N(g) = S\}.$$

Let $a_S = \min_{g \in G_S} \frac{v(g)}{|S|(n-2)}$. Choose any ϵ such that

$$0 < \epsilon < \min_{S \subseteq N, |S| \geq 2} a_S \quad (2.4.9)$$

The allocation rule Y^* is defined by the following rules. Choose any g .

(Rule 1) For any $h \in C(g)$, suppose $N(h) = \cup_{i \in I} N(h_i^*)$ for some (non-empty) $I \subseteq \{1, \dots, K\}$. Then,

$$Y_i^*(v, g) = \frac{v(h)}{n_h} \text{ for all } i \in N(h).$$

(Rule 2) Suppose $N(h) \neq \cup_{i \in I} N(h_i^*)$ for all $I \subseteq \{1, \dots, K\}$. Then, g is not a $*$ -supergraph of g^* . Choose $j_h \in N(h)$ such that $j_h \neq i_h$. Then,

$$Y_i^*(v, g) = \begin{cases} (n_h - 1)\epsilon & \text{if } i \neq j_h \\ v(h) - (n_h - 1)^2\epsilon & \text{otherwise} \end{cases}$$

Clearly, the rule defined above is component balanced. We will show later that Y^* is anonymous on the set of strongly stable graphs. We first show that the efficient graph g^* is strongly stable under the above allocation rule.

Let s^* be the strategy profile defined as follows: For all $i \in N$, $s_i^* = \{j \in N \mid (ij) \in g^*\}$. Clearly, s^* induces g^* in $\gamma = (v, Y^*)$. We need to show that s^* is a SNE of $\Gamma(\gamma)$.

Consider any $s \neq s^*$, and let g be induced by s . Also, let $T = \{i \in N \mid s_i \neq s_i^*\}$. Suppose $h \in C(g)$. If $N(h) = \cup_{i \in I} N(h_i^*)$ for some nonempty subset I of $\{1, \dots, K\}$, then $Y_i^*(v, g) = \frac{v(h)}{n_h}$ for all $i \in N(h)$. However, since g^* is efficient, there exists some $i \in I$ such that $\frac{v(h_i^*)}{n_{h_i^*}} \geq \frac{v(h)}{n_h}$. So, no member of h_i^* is better-off as a result of the deviation. Also, note that $T \cap N(h_i^*) \neq \emptyset$. So, T does not have a profitable deviation in this case.

Suppose there is $h \in C(g)$ such that $N(h) \neq \cup_{i \in I} N(h_i^*)$ for any nonempty subset I of $\{1, \dots, K\}$. Then, g is not a $*$ -supergraph of g^* , and let $C(g) = \{h_1, \dots, h_L\}$. From Lemma 2.4.1, there exists $(i_k, i_l) \in g^*$ where i_k and i_l are the players who are punished in h_k and h_l respectively. Obviously, $T \cap \{i_k, i_l\} \neq \emptyset$. But from Rule (2), it follows that $Y_{i_k}^*(v, g) = (n_{h_k} - 1)\epsilon$ and $Y_{i_l}^*(v, g) = (n_{h_l} - 1)\epsilon$. Given the value of ϵ , it follows that both i_k and i_h are worse-off from the deviation.

We now show that if g is strongly stable, then $g \in E(v)$. So suppose that g is an inefficient graph.

(i) If g is an inefficient graph which is a $*$ -supergraph of g^* , then there exist $h \in C(g)$, $h^* \in C(g^*)$ such that $N(h^*) \subseteq N(h)$ and

$$Y_i^*(v, g) = \frac{v(h)}{n_h} < Y_i^*(v, g^*) = \frac{v(h^*)}{n_{h^*}} \text{ for all } i \in N(h^*).$$

So, each $i \in N(h^*)$ can deviate to the strategy s_i^* . This will induce the component h^* where they are all strictly better off.

(ii) Suppose that g is not a $*$ -supergraph of g^* . Let $C(g) = (h_1, \dots, h_K)$. Without loss of generality, let $n_{h_1} \leq \dots \leq n_{h_K}$. Since g is not a $*$ -supergraph of g^* , Rule (2) of the allocation rule applies and we know that there exist $h_k, h_l \in C(g)$, and $i_{h_k} \in N(h_k)$, $i_{h_l} \in N(h_l)$ such that $Y_{i_{h_k}}^*(v, g) = (n_{h_k} - 1)\epsilon$ and $Y_{i_{h_l}}^*(v, g) = (n_{h_l} - 1)\epsilon$. Let $\bar{s}$ be such that

- (i) $\forall j \notin \{i_{h_k}, i_{h_l}\}, \bar{s}_j = s_j$.
- (ii) $\bar{s}_{i_{h_k}} = \{j | j \in s_{i_{h_k}} \text{ or } j = i_{h_l}\}$.
- (iii) $\bar{s}_{i_{h_l}} = \{j | j \in s_{i_{h_l}} \text{ or } j = i_{h_k}\}$.

Let $\bar{g}$ be the graph induced by $\bar{s}$. Notice that $\bar{g} = g + (i_{h_k}, i_{h_l})$. We claim that

$$Y_j^*(v, \bar{g}) > Y_j^*(v, g) \quad \forall j \in \{i_{h_k}, i_{h_l}\} \quad (2.4.10)$$

Let $\bar{h} \in C(\bar{g})$ be the component containing players i_{h_k} and i_{h_l} . Notice that $n_{\bar{h}} > \max(n_{h_k}, n_{h_l})$. Given the value of ϵ , it follows that

$$Y_j^*(v, \bar{g}) \geq (n_{\bar{h}} - 1)\epsilon \quad \forall j \in \{i_{h_k}, i_{h_l}\}$$

This shows that the coalition $\{i_{h_k}, i_{h_l}\}$ has a deviation which makes both players better off.

The second half of the proof also shows that g is strongly stable only if g is a $\ast$ -supergraph of $g^\ast$. From Rule (1), it is clear that $Y^\ast$ is anonymous on all such graphs. This observation completes the proof of the theorem. ■

We have remarked earlier that we need to restrict the class of permissible value functions in order to prove the analogue of a “double implementation” in Strong Nash and Coalition-proof Nash equilibrium. In order to explain the motivation underlying the restricted class of value functions, we first show in Example 2.4.2 below that the allocation rule used in Theorem 2.4.4 *cannot* be used to prove the double implementation result. In particular, this allocation rule does not ensure that weakly stable graphs are efficient.

Example 2.4.2 Let $N = \{1, 2, 3, 4\}$. Consider a value function such that $v(g^\ast) = 4$, $v(g_1) = 3.6$, $v(g_2) = v(g_3) = 2.9$, where $g^\ast = \{(14), (13), (23), (12)\}$, $g_1 = \{(12), (13), (34)\}$, $g_2 = \{(12), (13)\}$ and $g_3 = \{(13), (34)\}$. Also, $v(\{(ij)\}) = 1.6$ for all $\{i, j\} \subset N$. Finally, the value of other graphs is such that $\epsilon = 0.4$ satisfies (2.4.9). Note that $g^\ast$ is the unique efficient graph. Let the strict order on links (used in the construction of the allocation rule in Theorem 2.4.4) be given by

$$(13) \succ (23) \succ (14) \succ (12)$$

Consider the graph $g = \{(12), (34)\}$. Then, from (Rule 2) and the specification of $\succ$, we have $Y_2^\ast(v, g) = Y_4^\ast(v, g) = 1.2$, $Y_1^\ast(v, g) = Y_3^\ast(v, g) = 0.4$. Now, g is weakly stable, but not efficient.

To see that g is weakly stable, notice first that agents 2 and 4 have no profitable deviation. Second, check using the particular specification of $\succ$ that $Y_3^\ast(v, g_2) = 1.3 > Y_3^\ast(v, g_1) = 0.9$, $Y_1^\ast(v, \{(13)\}) > Y_1^\ast(v, g_2)$ and $Y_3^\ast(v, g_3) = 0.8 > Y_3^\ast(v, \{(13)\}) = 0.4$.

Finally, consider the 2-person link formation game T with player set $\{1, 3\}$ generated from the original game by fixing the strategies of players 2 and 4

at $s_2 = \{1\}, s_4 = \{3\}$. Routine inspection yields that there is *no* Nash equilibrium in this game. This shows that g is weakly stable. ■

In order to rule out inefficient graphs from being stable, we need to give some coalition the ability to deviate credibly. However, the allocation rule constructed earlier fails to ensure this essentially because agents can sever links and become the “residual claimant” in the new graph. For instance, notice that in the previous example, if 3 “deviates” from g_1 to g_2 by breaking ties with 4, then 3 becomes the residual claimant in g_2 . Similarly, given g_2 , 1 breaks links with 2 to establish $\{(13)\}$, where she is the residual claimant.

To prevent this jockeying for the position of the residual claimant, one can impose the condition that on all inefficient graphs, players are punished according to a *fixed* order. Unfortunately, while this does take care of the problem mentioned above, it gives rise to a new problem. It turns out that in some cases the *efficient* graph itself may not be (strongly) stable. The following example illustrates this.

Example 2.4.3 Let $N = \{1, 2, 3, 4\}$. Let g^* , the unique efficient graph be $\{(12), (23), (34), (41)\}$, let $g = \{(12), (34)\}$. Assume that $v(g^*) = 4$ and $v(\{(ij)\}) = 1.5$ for all $\{i, j\} \subset N$. The values of the other graphs are chosen so that

$$\min_{S \subseteq N} \min_{g \in G_S} \frac{v(g)}{(|N| - 2)|S|} = 0.25$$

Choose $\epsilon = 0.25$ and let $\succ_p$ be an ordering on N such that $1 \succ_p 2 \succ_p 3 \succ_p 4$. Applying the allocation rule specified above, it follows that

$$Y_i(v, g^*) = 1 \text{ for all } i \in N$$

$$Y_2(v, g) = Y_4(v, g) = 1.25 \text{ and } Y_1(v, g) = Y_3(v, g) = 0.25$$

One easily checks that the coalition $\{2, 4\}$ can deviate from the graph g^* to induce the graph g . This deviation makes both deviators better off. The symmetry of the value function on graphs of the form $\{(ij)\}$ now implies that *no* fixed order will work here. ■

This explains why we need to impose a restriction on the class of value functions. We impose a restriction which ensures that for some efficient graph g^* , there is a “central” agent within each component, that is, an agent who is connected to every other agent in the component. This restriction is defined formally below.

Definition 2.4.8 *A graph g is focussed if for each $h \in C(g)$, there is $i_h \in N(h)$ such that $(i_h j) \in h$ for all $j \in N(h) \setminus \{i_h\}$.*

Let $\bar{V}$ be the set of all value functions v such that

- (i) $v(g) = 0$ only if g is completely disconnected.
- (ii) There exists $g^* \in E(v)$ such that g^* is focussed.

We now assume that the class of permissible value functions is $\bar{V}$. This is a much stronger restriction than the assumption used in the previous theorem. However, there are several interesting problems which give rise to such value functions. Indeed, the two special models discussed in Jackson and Wolinsky [30] (the *symmetric connections* and *coauthor* models) both give rise to value functions in $\bar{V}$.

Choose some $v \in \bar{V}$, and let $g^* \in E(v)$ be focussed. Assume that $(h_1^*, \dots, h_K^*)$ are the components of g^* , and let i_k be the player who is connected to all other players in $N(h_k^*)$.¹²

Let $\succ_p$ be a strict order on the player set N satisfying the following conditions:

- (i) $\forall i, j \in N$, if $i \in N(h_k)$, $j \in N(h_l)$ and $k < l$, then $i \succ_p j$.
- (ii) $i_k \succ_p j$ for all $j \in N(h_k^*) \setminus \{i_k\}$, $k = 1, \dots, K$.

So, $\succ_p$ satisfies two properties. First, all agents in $N(h_k^*)$ are ranked above agents in $N(h_{k+1}^*)$. Second, *within* each component, the player who is connected to all other players is ranked first. Finally, choose any ϵ satisfying (2.4.9).

¹²If more than one such player exists, then any selection rule can be employed.

The allocation rule Y^* is defined by the following rules. Choose any g and $h \in C(g)$.

(Rule 1) Suppose $N(h) = N(h^*)$ for some $h^* \in C(g^*)$. Then,

$$Y_i^*(v, g) = \frac{v(h)}{n_h} \text{ for all } i \in N(h).$$

(Rule 2) Suppose $N(h) \subset N(h^*)$ for some $h^* \in C(g^*)$. Let j_h be the “minimal” element of $N(h)$ under the order $\succ_p$. Then, for all $i \in N(h)$,

$$Y_i^*(v, g) = \begin{cases} (n_h - 1)\epsilon & \text{if } i \neq j_h \\ v(h) - (n_h - 1)^2\epsilon & \text{if } i = j_h \end{cases}$$

(Rule 3) Suppose $N(h) \not\subseteq N(h^*)$ for any $h^* \in C(g^*)$. Let j_h be the “minimal” element of $N(h)$ under the order $\succ_p$. Then, for all $i \in N(h)$,

$$Y_i^*(v, g) = \begin{cases} \frac{\epsilon}{2} & \text{if } i \neq j_h \\ v(h) - \frac{(n_h - 1)\epsilon}{2} & \text{if } i = j_h \end{cases}$$

The allocation rule has the following features. First, provided a component consists of the same set of players as some component in g^* , the value of the component is divided equally amongst all the agents in the component. Second, punishments are levied in all other cases. The punishment is more severe if players form links *across* components in g^* .

Let s^* be the strategy profile given by $s_i^* = \{j \in N | (ij) \in g^*\}$ for all $i \in N$, and let $C(g^*) = \{h_1^*, \dots, h_K^*\}$. We first show that if agents in components $h_1^*, \dots, h_k^*$ are using the strategies s_i^* , then no group of agents in h_k^* will find it profitable to deviate. Moreover, this is independent of the strategies chosen by agents in components coming ‘after’ h_k^* .

Lemma 2.4.2 *Let $v \in \bar{V}$. Suppose s is the strategy profile given by $s_i = s_i^*$ for all $i \in N(h_k^*)$, $\forall k = 1, \dots, \bar{K}$ where $\bar{K} \leq K$. Then, there is no s' such that $f_i'(s') > f_i'(s)$ for all $i \in T$ where $s'_i = s_i^*$ for all $i \in N(h_k^*)$, $k < \bar{K}$ and $T = \{i \in N(h_{\bar{K}}^*) | s'_i \neq s_i\}$.*

Proof : Consider the case $\bar{K} = 1$. Let g be the graph induced by s . Note that $h_1^* \in C(g)$.

Consider any s' , and let g' be the graph induced by s' . Suppose that $T = \{i \in N(h_1^*) \mid s_i \neq s'_i\} \neq \emptyset$.

Case (1): There exists $h \in C(g')$ such that $N(h) = N(h_1^*)$.

In this case, Rule (1) applies, and we have

$$Y_i^*(v, g') = \frac{v(h)}{|N(h)|} \leq \frac{v(h_1^*)}{|N(h_1^*)|} = Y_i^*(v, g) \quad \forall i \in N(h_1^*).$$

So no $i \in N(h_1^*)$ benefits from the deviation.

Case (2): There exists $h \in C(g')$ such that $N(h) \cap N(h_1^*) \neq \emptyset$, and $N(h) \not\subseteq N(h_1^*)$.

In this case, Rule (3) applies, and we have

$$Y_i^*(v, g') = \frac{\epsilon}{2} < Y_i^*(v, g) \quad \forall i \in N(h_1^*) \cap N(h).$$

Noting that $N(h_1^*) \cap N(h) \cap T \neq \emptyset$, we must have $f_i^\gamma(s) > f_i^\gamma(s')$ for some $i \in T$.

Case (3): There exists $h \in C(g')$ such that $N(h) \subset N(h_1^*)$.

Noting that there is i_1 who is connected to everyone in $N(h_1^*)$, either $i_1 \in T$ or $T = N(h)$. If $i_1 \in T$, then since $f_{i_1}^\gamma(s') \leq (n_h - 1)\epsilon < f_{i_1}^\gamma(s)$, the lemma is true. Suppose $i_1 \notin T$. Ruling out the trivial case where a single agent breaks away,¹³ we have $|T| \geq 2$. From Rule 2 or Rule 3, at least one of the agents must be worse off.

Hence, in all possible cases, there is some $i \in T$ who does not benefit from the deviation.

The proof can be extended in an obvious way for all values of $\bar{K}$. ■

Lemma 2.4.3 *Let $v \in \bar{V}$. Let g be the graph induced by a strategy profile s . Suppose there exists $h \in C(g)$ such that $N(h) \subset N(h_1^*)$. Then, g is not weakly stable.*

¹³The agent then gets 0.

Proof: If s is not a Nash equilibrium of $\Gamma(\gamma)$, then there is nothing to prove. So, assume that s constitutes a Nash equilibrium.

We will prove the lemma by showing that there is a credible deviation from s for a coalition $D \subset N(h_1^*)$, $|D| = 2$. The game induced on the coalition D is defined as $\Gamma(\gamma, s_{N \setminus D}) = \langle D, \{S_i\}_{i \in D}, \bar{f}^\gamma \rangle$ where $\bar{f}_j^\gamma(s'_D) = Y_j^*(v, g(s'_D, s_{N \setminus D}))$ for all $j \in D$. We show that there is a Nash equilibrium in this two-person game which Pareto-dominates the payoff corresponding to s .

Suppose there is $i \in N(h_1^*) \setminus N(h)$, $j \notin N(h_1^*)$ such that $(ij) \in g$. Then, $Y_i^*(v, g) = \frac{c}{2}$. Since s is a Nash equilibrium, this implies that i by a unilateral deviation cannot induce a graph g' in which i will be in some component such that $N(h') \subseteq N(h_1^*)$.

Now, let j be the $\succ_p$ -maximal agent in $N(h)$. Consider the coalition $D = \{i, j\}$. Choose $s'_i = \{j\}$, and let s'_j be the best response to s'_i in the game $\Gamma(\gamma, s_{N \setminus D})$. Then, (s'_i, s'_j) must be a Nash equilibrium in $\Gamma(\gamma, s_{N \setminus D})$.¹⁴ Using Rule (2), it is trivial to check that both i and j gain by deviating to s' from s .

Hence, we can now assume that if $N(h) \subset N(h_1^*)$, then there exist $\{h_1, \dots, h_L\} \subseteq C(g)$ such that $N(h_1^*) = \bigcup_{i=1, \dots, L} N(h_i)$.¹⁵ Note that $L \geq 2$.

Without loss of generality, let 1 be the $\succ_p$ -maximal agent in $N(h_1^*)$, and $1 \in N(h_1)$. Let i be the $\succ_p$ -maximal agent in $N(h_2)$, and let $D = \{1, i\}$.

Suppose $L > 2$. Then, consider $\bar{s}_1 = s_1 \cup \{i\}$, and let $\bar{s}_i$ be the best response to $\bar{s}_1$ in the game $\Gamma(\gamma, s_{N \setminus D})$. Note that 1 can never be the residual claimant in any component, and that $1 \in \bar{s}_i$. It then follows that $(\bar{s}_1, \bar{s}_i)$ is a Nash equilibrium in $\Gamma(\gamma, s_{N \setminus D})$ which Pareto-dominates the payoffs (of D) corresponding to the original strategy profile s .

Suppose $L = 2$. Let $\bar{S} = \{\bar{s} | \bar{s} = (\bar{s}_1, \bar{s}_i, s_{-D}) \text{ for some } (\bar{s}_1, \bar{s}_i) \in S_1 \times S_i\}$. Let $\bar{G}$ be the set of graphs which can be induced by D subject to the

¹⁴The fact that i has no profitable deviation from s'_i follows from the assumption that the original strategy profile is a Nash equilibrium.

¹⁵Again, we are ignoring the possible existence of isolated individuals.

restriction that both 1 and i belong to a component which is connected on $N(h_1^*)$. Let $\bar{g}$ be such that $v(\bar{g}) = \max_{g \in \bar{G}} v(g)$, and suppose that $\bar{s}$ induces $\bar{g}$. Then, note that $i \in \bar{s}_1$ and $1 \in \bar{s}_i$.

Now, $Y_1^*(v, \bar{g}) = Y_i^*(v, \bar{g}) = \frac{v(\bar{g})}{n_h}$. Clearly, $Y_j^*(v, \bar{g}) > Y_j^*(v, g)$ for $j \in D$. If $(\bar{s}_1, \bar{s}_i)$ is a Nash equilibrium in $\Gamma(\gamma, s_{N \setminus D})$, then this completes the proof of the Lemma. Suppose $(\bar{s}_1, \bar{s}_i)$ is not a Nash equilibrium of $\Gamma(\gamma, s_{N \setminus D})$. Then, the only possibility is that i has a profitable deviation since 1 can never become the residual claimant. Let $\tilde{s}_i$ be the best response to $\bar{s}_1$ in $\Gamma(\gamma, s_{N \setminus D})$. Note that $1 \in \tilde{s}_i$. Let $\tilde{g}$ denote the induced graph. We must therefore have $Y_1^*(v, \tilde{g}) > Y_1^*(v, g)$.¹⁶ Obviously, $Y_i^*(v, \tilde{g}) > Y_i^*(v, g)$. Since $\bar{s}_1$ is also a best response to $\tilde{s}_i$ in $\Gamma(\gamma, s_{N \setminus D})$, this completes the proof of the lemma. ■

We can now prove the following.

Theorem 2.4.5 Let $v \in \bar{V}$. Then, there exists a component balanced allocation rule Y satisfying the following

- (i) The set of strongly stable graphs is nonempty.
- (ii) If g is weakly stable, then $g \in E(v)$.
- (iii) Y is anonymous over the set of weakly stable graphs.

Proof : Clearly, the allocation rule Y defined above is component balanced. We first show that the efficient graph g^* is strongly stable by showing that s^* is a Strong Nash equilibrium of $\Gamma(\gamma)$.

Let $C(g^*) = \{h_1^*, \dots, h_K^*\}$.

Let $s \neq s^*$, g be the graph induced by s , and $T = \{i \in N \mid s_i \neq s_i^*\}$. Let $t^* = \operatorname{argmin}_{1 \leq j \leq K} s_j \neq s_j^*$ for some $i \in N(h_{t^*}^*)$.

¹⁶This follows since 1 is now in a component containing more agents.

By Lemma 2.4.2, it follows that at least one member in $N(h_{t^*}) \cap T$ does not profit by deviating from the strategy s^* . This shows that the graph g^* is strongly stable.

We now show that if g is not efficient, then it cannot be weakly stable. Let s be a strategy profile which induces the graph g . We have the following cases.

Case (1a): There exists $h \in C(g)$ such that $N(h_1^*) = N(h)$ and $v(h) < v(h_1^*)$.

Suppose all individuals i in $N(h_1^*)$ deviate to s_i^* . Clearly, all individuals in $N(h_1^*)$ gain from this deviation. Moreover, Lemma 2.4.2 shows that no subcoalition of $N(h_1^*)$ has any further profitable deviation. Hence, s cannot be a CPNE of $\Gamma(\gamma)$ in this case.

Case (1b): There does not exist $h \in C(g)$ such that $N(h) \subseteq N(h_1^*)$.

In this case all players in $N(h_1^*)$ are either isolated (in which case they get zero) or they are in (possibly different) components which contain players not in $N(h_1^*)$. Using Rule (3) of the allocation rule, it follows that

$$Y_i(v, g) \leq \frac{\epsilon}{2} \quad \forall i \in N(h_1^*).$$

So all players in $N(h_1^*)$ can deviate to the strategy s_i^* . Obviously, this will make them strictly better off. That this is a credible deviation follows from Lemma 2.4.2.

Case (1c): There exists $h \in C(g)$ such that $N(h) \subset N(h_1^*)$.

In this case, it follows from Lemma 2.4.3 that there is a credible deviation for a coalition $D \subset N(h_1^*)$.

Case (2): If there exists $h \in C(g)$ such that $N(h) = N(h_1^*)$ and $v(h) = v(h_1^*)$, then apply the arguments of Case 1 to h_2^* and so on.

The preceding arguments show that if g is weakly stable, then :

$$(i) \quad N(h_i) = N(h_i^*) \text{ for each } i \in \{1, \dots, K\}.$$

$$(ii) \quad v(h_i) = v(h_i^*) \text{ for each } i \in \{1, \dots, K\}.$$

These show that all weakly stable graphs must be efficient. Furthermore, it follows from Rule (1) that Y is anonymous on all such graphs. This completes the proof of the theorem. ■

Notice that in both Theorems 2.4.4 and 2.4.5, we have imposed the requirement that the allocation rule satisfy component balance on *all* graphs, and not just on the set of stable (or weakly stable) graphs. This raises the obvious question as to why we have treated the two properties of component balance and anonymity asymmetrically.

The answer lies in the fact that component balance has a *strategic* role, while anonymity is a purely ethical property. Consider, for instance, the “equal division” allocation rule which specifies that each agent gets $\frac{v(g)}{n}$ on all graphs g . This rule violates component balance. Let the value function be such that $\frac{v(\{12\})}{2} > \frac{v(g^*)}{n}$ where g^* is some efficient graph. Then, given the equal division rule, agents i and j both do strictly better by breaking away from the rest of the society since the total reward given to them by this allocation rule is less than what they can get by themselves. On the other hand, Theorems 2.4.4 and 2.4.5 show that *some* allocation rules which are component balanced ensure that no set of agents wants to break away.

There is an obvious analogy with the literature on implementation in this regard. In the implementation context, mechanisms which waste resources “out of equilibrium” will not be renegotiation-proof since all agents can move away to a Pareto-superior outcome. Here, the violation of component balance implies that *all* agents in some *component* can agree on a jointly better outcome.

There is also another logical motivation which can be provided for this asymmetric treatment of component balance and anonymity. In view of the Jackson-Wolinsky result, one or both the conditions have to be relaxed in order to resolve the tension between stability and efficiency. Here, we show that *simply relaxing anonymity out of equilibrium is sufficient*. Since we have also argued that the violation of ethical conditions such as anonymity

on graphs which are not likely to be observed is not a matter for concern, our results suggest an interesting avenue for avoiding the potential conflict between stability and efficiency in the context of this framework.

2.5 Conclusion

The central theme in this chapter has been to examine the possibility of constructing allocation rules which will ensure that efficient networks of agents form when the individual agents decide to form or sever links amongst themselves. Exploiting the insights provided by Jackson and Wolinsky [30], it is shown that in general it may not be possible to reconcile efficiency with stability if the allocation rule is required to be anonymous on *all* graphs.

However, we go on to argue that if our prediction is that *only* efficient graphs will form, then the requirement that the allocation rule be anonymous on all graphs is unnecessarily stringent. We suggest that a “mechanism design” approach is more appropriate, and show that under almost all value functions, the nonempty set of (strongly) stable graphs will be a subset of the efficient graphs under an allocation rule which is anonymous on the domain of strongly stable graphs. A stronger domain restriction allows us to prove that the above result also holds when strong stability is replaced by a condition requiring efficient graphs to be strongly stable and inefficient graphs to be *not* weakly stable. Since these allocation rules will treat agents symmetrically on the graphs which are “likely to be observed,” it seems that stability can be reconciled with efficiency after all.

Chapter 3

Strategyproof Rules for Cost Sharing

3.1 Introduction

We consider a situation where an excludable public good like network access is desired by a group of agents. The cost of supplying the service depends on the set of agents to whom the service is to be provided. If the willingness to pay of each individual for the service is known to the planner, then he can optimally provide the service to the efficient coalition, that is, the coalition which maximises the difference between the total willingness to pay and the cost of supplying the service. In the absence of such information, the planner has to design a suitable mechanism which will give the individuals incentives to reveal their private information truthfully. A mechanism is called *Strategyproof* if telling the truth is a dominant strategy for all agents. However, a price has to be paid for ensuring Strategyproofness: either efficiency or budget balance has to be given up. This result is well-known in the implementation literature.¹

¹See Green and Laffont [25].

Moulin and Shenker [44] have examined this problem from a non-Bayesian perspective.² They first characterise an efficient and Strategyproof mechanism called the marginal contribution mechanism and show that it is not immune against coalitional deviations. They then consider Budget Balanced mechanisms and characterise a class of mechanisms which are Budget Balanced and Strong Group Strategyproof³ under the assumption that the cost function is *submodular*. These mechanisms are characterised by the condition of cross monotonicity, that is, the cost imputation to an individual does not increase when she becomes part of a larger coalition. Moulin and Shenker also characterise a particular cross monotonic cost allocation rule which is based on the Shapley value. This rule is characterised by the condition that it minimises the maximum loss resulting from the non-formation of the efficient coalition.

Deb and Razzolini [14] have also examined this problem for a particular specification of the submodular cost function. This is the case where the cost of supplying the service to a coalition is independent of its size.⁴ In contrast to Moulin and Shenker who use Strong Group Strategyproofness, they use the weaker axiom of Strategyproofness. They prove that all Strategyproof and Budget Balanced mechanisms satisfying a condition they call “Voluntary Non-Participation” must be characterised by an “equal price theorem”: all agents given access must be charged the same amount. The different approaches of Deb and Razzolini [14] and Moulin and Shenker [44] raise the question of whether there are significant differences between the class of Strategyproof mechanisms and the class of Strong Group Strategyproof

²By a non-Bayesian perspective, we mean that no assumptions are made about the planner’s beliefs concerning the distribution of the agents’ valuations.

³The mechanism is Strong Group Strategyproof if there does not exist a deviation which makes all members of the deviating coalition weakly better-off and at least one member strictly better-off.

⁴Deb and Razzolini, however, work with a broader group of utility functions than followed here or in Moulin and Shenker [44].

mechanisms. Since the class of mechanisms identified by Deb and Razzolini can also be characterised in terms of Strong Group Strategyproofness⁵, this question remains to be answered.

In this chapter, we retain the focus on Budget Balanced mechanisms. We first examine the relationship between Strategyproofness and Group Strategyproofness under different definitions of *Non-bossiness*. The condition of Non-bossiness is used in many contexts, and it amounts to saying that if an agent's unilateral change of announcement leaves him indifferent, then all other agents must be indifferent as well. We give three different definitions of this concept. One version of this concept is labelled *Outcome Non-bossiness*, and means that if one agent's unilateral change of announcement gives him the same *outcome*, then the outcome must be the same for all other agents. Another one is *Strong Utility Non-bossiness* which requires that if the deviating agent is indifferent in terms of *utility*, then all other agents must also be deriving the same utility. A weaker version of Strong Utility Non-bossiness is then defined: this requires that if an agent's unilateral change of announcement leaves him indifferent in utility terms, then no other agent should be strictly better-off.

We show that any Strategyproof mechanism satisfying the conditions of Voluntary Participation, No Subsidy and Outcome Non-bossiness must be at least Weak Group Strategyproof. We show through an example that the four conditions together do not imply Strong Group Strategyproofness. Next, we show that the combination of Strategyproofness, Voluntary Participation, No Subsidy, Outcome Non-bossiness and Weak Utility Non-bossiness implies Strong Group Strategyproofness. We also show that if the cost function is submodular, then any Strongly Group Strategyproof mechanism satisfying Voluntary Participation, No Subsidy, Consumer Sovereignty and Budget Balance must also satisfy Outcome Non-bossiness and Weak Utility Non-bossiness. Taken together, the results here suggest that the class of Strat-

⁵See Moulin [42].

egyproof and Strong Group Strategyproof mechanisms are essentially the same if the cost function is submodular.

We go on to characterise a particular Strong Group Strategyproof and Budget Balanced mechanism under the assumption that the cost function is submodular. This is the *Egalitarian cost allocation rule*, and is based on the egalitarian solution for characteristic function TU-games which was proposed by Dutta and Ray [18]. Assume that agents' valuations are independent of each other and distributed according to a common distribution function which satisfies the *monotone hazard rate* condition. Then, we are able to show that the Egalitarian cost allocation rule maximises the probability that all members of a given coalition accept the cost shares imputed to them in the class of all Strong Group Strategyproof and Budget Balanced rules satisfying Voluntary Participation, No Subsidy and Consumer Sovereignty. Under a stronger condition on the distribution function, we show that the Egalitarian cost allocation rule maximizes the expected surplus from the *ex ante* efficient coalition. We also use the results obtained on the link between Strategyproofness and Strong Group Strategyproofness to give alternate characterisations of the Shapley value and the Egalitarian rules in terms of Strategyproofness and Non-bossiness.

3.2 The Model

Let N be the set of agents, and let u_i denote individual i 's valuation for the service. The utility function of each individual is quasi-linear:

$$U_i(q_i, x_i) = u_i q_i - x_i \quad (3.2.1)$$

where $q_i \in \{0, 1\}$ and x_i is the cost imputed to i .

The cost function $C : 2^N \rightarrow \mathcal{R}$ is assumed to be nondecreasing and nonnegative:

$$C(\emptyset) = 0; S \subseteq T \Rightarrow C(S) \leq C(T) \quad (3.2.2)$$

In Section 3.4, we shall also assume that the cost function is submodular:

$$C(S \cup T) + C(S \cap T) \leq C(S) + C(T) \text{ for all } S, T \subseteq N \quad (3.2.3)$$

A *cost allocation mechanism* associates to each announcement $v = (v_i)_{i \in N}$ a tuple $(q_i(v), x_i(v))$ for each $i \in N$.

A *cost allocation rule* ξ associates to each coalition S , a corresponding non-negative $|S|$ -tuple $(\xi_i(S))_{i \in S}$ such that $\sum_{i \in S} \xi_i(S) = C(S)$.

The relationship between the two concepts will be made clear in Section 3.4. We now define a few axioms for cost allocation mechanisms. In defining these axioms, we follow Moulin and Shenker [44]. Let $u = (u_i)_{i \in N}$ be the “true” profile of the agents, and let $v = (v_i)_{i \in N}$ be an announcement.

Definition 3.2.1 *The mechanism satisfies No Subsidy (NS) if $x_i(v) \geq 0$ for all v and all $i \in N$.*

Definition 3.2.2 *The mechanism satisfies Voluntary Participation (VP) if $U_i(q_i(v), x_i(v)) \geq 0$ for all v such that $v_i = u_i$.*

Voluntary Participation says that the mechanism must guarantee any agent who reports truthfully at least the utility level corresponding to not getting the service and not paying anything either.

Definition 3.2.3 *The mechanism satisfies Consumer Sovereignty (CS) if there exists a message v'_i for individual i such that $q_i(v) = 1$ for all v such that $v_i = v'_i$.*

Let $S(v) = \{i \in N | q_i(v) = 1\}$ denote the set of agents who are given access to the service when the announcement is v .

Definition 3.2.4 *The mechanism is Budget Balanced if for each announcement v , $\sum_{i \in S(v)} x_i(v) = C(S(v))$.*

The following three axioms are all variants of the idea that the mechanism be immune against attempts to manipulate it.

Definition 3.2.5 *The mechanism $F = (q, x)$ is Strong Group Strategyproof if for all v, v' such that $v_i = v'_i$ for all $i \notin T \subseteq N$ and $v_i = u_i$ for all $i \in T$, $[U_i(q_i(v'), x_i(v')) \geq U_i(q_i(v), x_i(v)) \text{ for all } i \in T] \Rightarrow [U_i(q_i(v'), x_i(v')) = U_i(q_i(v), x_i(v)) \text{ for all } i \in T]$.*

Definition 3.2.6 *The mechanism $F = (q, x)$ is Weak Group Strategyproof if for all v, v' such that $v_i = v'_i$ for all $i \notin T \subseteq N$ and $v_i = u_i$ for all $i \in T$, it is not true that $U_i(q_i(v'), x_i(v')) > U_i(q_i(v), x_i(v))$ for all $i \in T$.*

Definition 3.2.7 *The mechanism $F = (q, x)$ is Strategyproof if for all v, v' such that $v_j = v'_j$ for all $j \neq i$ and $v_i = u_i$, it is not true that $U_i(q_i(v'), x_i(v')) > U_i(q_i(v), x_i(v))$.*

Weak and Strong Group Strategyproofness are both expressions of the idea that the mechanism be proof against coalitional attempts to manipulate. Weak Group Strategyproofness only requires that there not exist a deviation for any coalition which makes *all* members of the deviating coalition strictly better off. Strong Group Strategyproofness, on the other hand, requires that there not exist a deviation for any coalition which makes *at least one* member strictly better off, and none worse-off. This notion of group Strategyproofness is used by Moulin and Shenker [44] in establishing their characterisation result for Strong Group Strategyproof and Budget Balanced mechanisms.⁶ Strategyproofness is the weakest of the above three axioms since it only requires that the mechanism be proof against individual attempts to manipulate.

In most of the results that we obtain, we will assume that the mechanism satisfies No Subsidy, Voluntary Participation, Consumer Sovereignty and Budget Balance. This motivates the following definition.

⁶See Theorem 1 in Moulin and Shenker [44]; also Moulin [42].

Definition 3.2.8 *The mechanism is said to be admissible if it satisfies No Subsidy, Voluntary Participation, Consumer Sovereignty and Budget Balance.*

3.3 Strategyproofness and Non-bossiness

In this section, we discuss and propose various ways of formalising the concept of Non-bossiness. We then go on to prove results establishing the relationship between the different concepts of non-manipulability under alternative definitions of Non-bossiness.

The notion of Non-bossiness was introduced by Satterthwaite and Sonnenschein [54]. The basic idea behind this notion is that the mechanism should not allow the outcome to any agent to be changed by unilateral deviations by some other agent which leaves the deviator indifferent. The use of Non-bossiness to explore the link between Strategyproof and Group Strategyproof mechanisms has been done in other contexts, for instance, Barberá and Jackson [5] for exchange economies and Serizawa [55] for the case of public good economies.

We now define the concept of Outcome Non-bossiness. This says that if an agent changes his announcement unilaterally in a manner that gives him the same *outcome*, then the other agents must also be getting the same *outcome*.

Let $F_i(v) = (q_i(v), x_i(v))$ denote the outcome to individual i when the announcement vector is v .

Definition 3.3.9 *Let v and v' be two announcements such that $v_j = v'_j$ for all $j \neq i$. The mechanism $F = (q, x)$ is said to satisfy Outcome Non-bossiness (ONB) if $F_i(v) = F_i(v') \Rightarrow [F_j(v) = F_j(v') \text{ for all } j \neq i]$.*

As will become clear later on, Outcome Non-bossiness is not sufficient to establish the connection between Strategyproof and Strong Group Strategyproof mechanisms. We now turn to a different version of Non-bossiness,

one which is defined in terms of *utilities* rather than outcomes. Let $u_i(v)$ denote the utility that individual i gets from the mechanism when his true type is u_i and the announcement vector is v :

$$u_i(v) \equiv U_i(q_i(v), x_i(v)) = u_i q_i(v) - x_i(v) \quad (3.3.4)$$

Definition 3.3.10 Let u and u' be profiles such that $u_j = u'_j$ for all $j \neq i$. The mechanism $F = (q, x)$ is said to satisfy *Strong Utility Non-bossiness (SUNB)* if $u_i(u) = u_i(u') \Rightarrow [u_j(u) = u_j(u') \text{ for all } j \neq i]$.

Strong Utility Non-bossiness says that if an agent's unilateral change of announcement leaves him indifferent in utility terms, then the other agents must also be indifferent in utility terms. While this is the "natural" definition if Non-bossiness is to be defined in utility terms, it turns out that this definition is far too strong. We show below that there exist virtually no Strategyproof and admissible mechanisms satisfying SUNB. For the purposes of the next two results, let $N = \{1, 2\}$.

Lemma 3.3.1 Let $F = (q, x)$ be a Strategyproof mechanism satisfying NS, VP and SUNB. Let u and u' be two profiles such that $S(u) = S(u')$. Then, $x_i(u) = x_i(u')$ for all $i \in S(u)$.

Proof: Note that VP and NS together imply that $i \notin S(v) \Rightarrow x_i(v) = 0$. So the lemma follows trivially if $S(u) = S(u') = \emptyset$. So, suppose that $S(u) = S(u') = \{1, 2\}$ and without loss of generality, assume that $x_1(u) \neq x_1(u')$. Let $z^1 = (\bar{u}_1, u_2)$ and $z^2 = (\bar{u}_1, u'_2)$ where $\bar{u}_1 > \max[u_1, u'_1]$.

By Strategyproofness, VP and NS, it follows that $1 \in S(z^1) \cap S(z^2)$, $x_1(z^1) = x_1(u) < \bar{u}_1$ and $x_1(z^2) = x_1(u') < \bar{u}_1$. Suppose first that $2 \in S(z^1) \cap S(z^2)$. By Strategyproofness, $x_2(z^1) = x_2(z^2)$ and hence, $u_2(z^1) = u_2(z^2)$. By SUNB,

$$\bar{u}_1(z^1) = \bar{u}_1(z^2) \quad (3.3.5)$$

However, since $x_1(z^1) = x_1(u) \neq x_1(u') = x_1(z^2)$ (3.3.5) cannot be true. This is a contradiction.

Next suppose that $2 \notin S(z^1) \cap S(z^2)$. NS implies that $x_2(z^1) = x_2(z^2) = 0$. Hence,

$$u_2(z^1) = u_2(z^2) = 0 \quad (3.3.6)$$

Using SUNB, (3.3.5) follows. We again have a contradiction.

Finally, suppose that $2 \in S(z^1) \setminus S(z^2)$. Strategyproofness implies that $u'_2 < u_2$. If $u_2 = x_2(z^1)$, then (3.3.6) must be true. Once again, using SUNB, (3.3.5) follows and yields a contradiction.

So, suppose that $u_2 > x_2(z^1)$. Consider the profile $\tilde{z} = (\tilde{u}_1, \tilde{u}_2)$ where $\tilde{u}_2 = x_2(z^1)$. If $2 \in S(\tilde{z})$, then Strategyproofness and NS imply that $x_2(\tilde{z}) = \tilde{u}_2 = x_2(z^1)$. SUNB implies that $1 \in S(\tilde{z})$ and $x_1(\tilde{z}) = x_1(z^1) = x_1(u)$. Suppose now that 2 deviates at $\tilde{z}$ to u'_2 . Then, she is indifferent but since $x_1(z^1) \neq x_1(z^2)$, $\bar{u}_1(z^1) \neq \bar{u}_1(z^2)$. This contradicts SUNB.

Suppose now that $2 \notin S(\tilde{z})$. Then, since $2 \notin S(z^2)$ also, $u'_2(\tilde{z}) = u'_2(z^2) = 0$. SUNB implies that $1 \in S(\tilde{z})$ and $x_1(\tilde{z}) = x_1(z^2)$. Consider the deviation of 2 at $\tilde{z}$ to u_2 . Then, 2 is indifferent since $x_2(z^1) = \tilde{u}_2$, but since $x_1(z^1) \neq x_1(z^2)$, $\bar{u}_1(z^1) \neq \bar{u}_1(z^2)$. Again, we have a contradiction.

The remaining case, $2 \in S(z^2) \setminus S(z^1)$, is dealt with in a similar manner. This completes the proof of the lemma. ■

Proposition 3.3.1 *Suppose the cost function C satisfies $C(12) \neq C(1) + C(2)$. Then, there does not exist any admissible mechanism $F = (q, x)$ satisfying Strategyproofness and SUNB.*

Proof: By CS, there exists a message $\hat{u}_i$ for agent i such that $q_i(\hat{u}_i, u_j) = 1$ for all u_j . So, there exists a profile $\hat{u}$ such that $S(\hat{u}) = N$. Using the previous lemma, let (c_1, c_2) be the cost shares whenever $S(u) = S(u') = N$.

Claim: $S(u) = N$ whenever $u_i > c_i$ for all $i \in N$.

Proof: Let u and $\bar{u}$ be profiles such that $u_i > c_i$ and $\bar{u}_i > \max[\hat{u}_i, u_i]$ for all $i \in N$. By CS and the previous lemma, $S(\hat{u}) = N$ and $x_i(\hat{u}) = c_i$ for all $i \in N$. By Strategyproofness and NS, $S(\bar{u}) = N$ and $x_i(\bar{u}) = c_i$ for all $i \in N$.

It now follows from Strategyproofness and NS that

$$1 \in S(u_1, \bar{u}_2) \text{ and } x_1(u_1, \bar{u}_2) = c_1 \quad (3.3.7)$$

By SUNB, $\bar{u}_2(\bar{u}) = \bar{u}_2(u_1, \bar{u}_2)$. Using the fact that $\bar{u}_2 > \max[\hat{u}_2, u_2] > c_2$, we have that $2 \in S(u_1, \bar{u}_2)$ and $x_2(u_1, \bar{u}_2) = c_2$. Using Strategyproofness, NS and $u_2 > c_2$, we get

$$2 \in S(u_1, u_2) \text{ and } x_2(u_1, u_2) = x_2(u_1, \bar{u}_2) = c_2 \quad (3.3.8)$$

Using (3.3.7) and SUNB, we get

$$1 \in S(u_1, u_2) \text{ and } x_1(u_1, u_2) = c_1 \quad (3.3.9)$$

(3.3.8) and (3.3.9) establish the claim. ■

Returning to the proof of the proposition, since $C(12) \neq C(1) + C(2)$, there exists $i \in N$ such that $c_i \neq C(i)$. Without loss of generality, assume that $c_1 \neq C(1)$. Consider the profile $u' = (u'_1, u'_2)$ where $u'_1 = \hat{u}_1$ and $u'_2 = c_2$. By CS, $1 \in S(u')$. Suppose $2 \in S(u')$. Then, $S(u') = N$ and $x_i(u') = c_i$ for all $i \in N$. Suppose now that 2 deviates by announcing $\bar{u}_2 < u'_2$. By Strategyproofness, VP and CS, $S(u'_1, \bar{u}_2) = \{1\}$. Since $x_2(u') = c_2$, we have $u'_2(u') = u'_2(u'_1, \bar{u}_2)$. However, since $C(1) \neq c_1$, $u'_1(u') \neq u'_1(u'_1, \bar{u}_2)$. This violates SUNB.

Suppose that $2 \notin S(u')$. By Budget Balance, $x_1(u') = C(1)$. Suppose now that 2 deviates to $\bar{u}_2 > u'_2$. By the above claim, $S(u'_1, \bar{u}_2) = N$, and thus, $x_i(u'_1, \bar{u}_2) = c_i$ for all $i \in N$. Hence, $u'_2(u'_1, \bar{u}_2) = u'_2(u')$. However, since $x_1(u') = C(1) \neq c_1 = x_1(u'_1, \bar{u}_2)$, SUNB is violated. This contradiction completes the proof of the lemma. ■

Remark 3.3.1 The above lemma suggests that one needs a definition of Non-bossiness which is weaker than SUNB in order to explore the link between Strategyproofness and Strong Group Strategyproofness.

Definition 3.3.11 *The mechanism $F = (q, x)$ satisfies Weak Utility Non-bossiness (WUNB) if for all profiles u, u' such that $u_j = u'_j$ for all $j \neq i$, $[u_i(u') = u_i(u)] \Rightarrow [u_j(u') \leq u_j(u) \text{ for all } j \neq i]$.*

Weak Utility Non-bossiness only requires that an agent's unilateral change of announcement which leaves him indifferent not make any other agent *strictly* better-off. Clearly, SUNB implies WUNB. The converse is not true as the following example shows.

Example 3.3.1 Let $N = \{1, 2\}$, and let the cost function C be strictly submodular.

$$\begin{array}{lll} v_1 \geq C(1), v_2 \geq C(12) - C(1) & S(v) = N & x_1(v) = C(1), x_2(v) = C(12) - C(1) \\ v_1 \geq C(1), v_2 < C(12) - C(1) & S(v) = \{1\} & x_1(v) = C(1) \\ v_1 < C(1), v_2 \geq C(2) & S(v) = \{2\} & x_2(v) = C(2) \\ v_1 < C(1), v_2 < C(2) & S(v) = \emptyset & \end{array}$$

The above mechanism satisfies WUNB. However, SUNB is violated at the profile $u = (C(1), v_2)$ where $v_2 > C(2) > C(12) - C(1)$. If 1 deviates at u by announcing $v_1 < C(1)$, then he is indifferent, but 2 is worse-off since $C(2) > C(12) - C(1)$. ■

The following two examples show that WUNB and ONB are independent.

Example 3.3.2 Let N and C be as in the previous example. Consider the following mechanism.

$$\begin{array}{lll} v_1 > C(1), v_2 \geq C(12) - C(1) & S(v) = N & x_1(v) = C(1), x_2(v) = C(12) - C(1) \\ v_1 = C(1), v_2 > C(12) - C(1) & S(v) = N & x_1(v) = C(1), x_2(v) = C(12) - C(1) \\ v_1 = C(1), v_2 \leq C(12) - C(1) & S(v) = \{1\} & x_1(v) = C(1) \\ v_1 < C(1), v_2 \geq C(2) & S(v) = \{2\} & x_2(v) = C(2) \\ v_1 < C(1), v_2 < C(2) & S(v) = \emptyset & \end{array}$$

This mechanism satisfies WUNB. However, ONB is violated at all profiles $(u_1, C(12) - C(1))$ where $u_1 > C(1)$. If 1 changes her announcement to $u'_1 = C(1)$, then her outcome remains the same, but 2's outcome changes. ■

Example 3.3.3 Let N and C be as in the previous two examples. Consider the following mechanism.

$$\begin{array}{lll}
v_1 > C(1), v_2 \geq C(12) - C(1) & S(v) = \emptyset & x_1(v) = C(1), x_2(v) = C(12) - C(1) \\
v_1 > C(1), v_2 < C(12) - C(1) & S(v) = \{1\} & x_1(v) = C(1) \\
v_1 \leq C(1), v_2 \geq C(2) & S(v) = \{2\} & x_2(v) = C(2) \\
v_1 \leq C(1), v_2 < C(2) & S(v) = \emptyset &
\end{array}$$

This mechanism satisfies ONB. However, WUNB is violated at the profile $(C(1), v_2)$ where $v_2 > C(2) > C(12) - C(1)$. If 1 deviates at the profile $(C(1), v_2)$ by announcing $v_1 > C(1)$, then she is indifferent but 2 is better-off. ■

We first show that a Strategyproof mechanism satisfying No Subsidy, Voluntary Participation and Outcome Non-bossiness must also be Weak Group Strategyproof. In what follows, the notation (v_j, u_{-j}) refers to the profile where the j th component of u has been replaced by v_j , all other components remaining the same.

Theorem 3.3.1 *Suppose the Strategyproof mechanism $F = (q, x)$ satisfies NS, VP and ONB. Then, it is also Weak Group Strategyproof.*

Proof: Throughout the proof, assume that F satisfies NS, VP, ONB and is Strategyproof. Consider two profiles u and $\bar{u}$ such that for some $T \subseteq N$, $u_i = \bar{u}_i$ for all $i \notin T$. We need to prove that it *cannot* be the case that

$$u_i(\bar{u}) > u_i(u) \text{ for all } i \in T \quad (3.3.10)$$

(3.3.10) cannot be true when $|T| = 1$ since F is Strategyproof. Assume that (3.3.10) is not true for all T such that $|T| \leq k - 1$. Suppose, however that (3.3.10) is true when $|T| = k$. Let $u^i \equiv (u_i, \bar{u}_{-i})$ for all $i \in T$.

By Strategyproofness, VP and (3.3.10), we have

$$u_i(u^i) \geq u_i(\bar{u}) > u_i(u) \geq 0 \text{ for all } i \in T \quad (3.3.11)$$

Observe that if $q_j(\bar{u}) = 0$ for some $j \in T$, then $u_j(\bar{u}) = 0$ by VP and NS. This contradicts (3.3.11). So $j \in S(\bar{u})$. (3.3.11) implies that $u_j(u^j) > 0$. Hence, $q_j(u^j) = q_j(\bar{u}) = 1$. But then, by Strategyproofness, $x_j(u^j) = x_j(\bar{u})$. By ONB, it follows that $q_i(u^j) = q_i(\bar{u})$ and $x_i(u^j) = x_i(\bar{u})$ for all $i \in T \setminus \{j\}$. So, the coalition $T \setminus \{j\}$ can deviate from u to u^j in a manner satisfying (3.3.10). This contradicts the induction hypothesis and concludes the proof of the theorem. ■

Remark 3.3.2 In the exchange economy setting with strict quasi-concave utility functions, Barberá and Jackson [5] show that Strategyproofness together with Outcome Non-Bossiness implies Weak Group Strategyproofness. Their result does not use any other assumption. The additional conditions needed here are on account of the fact that the domain of quasi-linear preferences is not monotonically closed (unlike the domain of strict quasi-concave utility functions). Moulin and Shenker [44] observe that the set of quasi-linear preferences does become monotonically closed if the domain is restricted to $(0, 0) \cup (1, x)$. Imposing the combination of No Subsidy and Voluntary Participation essentially accomplishes this.

The following example shows that the combination of Strategyproofness and Outcome Non-bossiness is weaker than Strong Group Strategyproofness.

Example 3.3.4 Consider the same mechanism as in Example 3.3.3. It is easily verified that that mechanism satisfies all the properties in the hypothesis of Theorem 3.3.1. However, it is not Strong Group Strategyproof at the profile $(C(1), C(2))$ as the coalition N can deviate by announcing $(v_1, C(2))$ where $v_1 > C(1)$. This leaves 1 indifferent, but 2 better off since $C(2) > C(12) - C(1)$.

We now show that if we include the additional condition of Weak Utility Non-bossiness in the above theorem, then the resulting mechanism is Strong Group Strategyproof. Before that, we prove the following lemma which will be invoked repeatedly in the proof of the next theorem.

Lemma 3.3.2 *Let $F = (q, x)$ be a Strategyproof mechanism satisfying NS and VP. Let u and u' be profiles such that $u_j = u'_j$ for all $j \neq i$.*

1. *Suppose that $q_i(u) = 1$ and $u_i < u'_i$. Then, $q_i(u') = 1$ and $x_i(u') = x_i(u)$.*
2. *Suppose that $q_i(u) = 0$ or $[q_i(u) = 1 \text{ and } x_i(u) = u_i]$. If $u'_i < u_i$, then $q_i(u') = 0$.*

Proof: If $q_i(u) = 1, u'_i > u_i$ and $q_i(u') = 0$, then $x_i(u') = 0$ by NS and VP. In this case, i will manipulate the mechanism at u' by announcing u_i . Since $x_i(u) \leq u_i < u'_i$ by VP, it follows that this deviation is profitable and contradicts Strategyproofness. If $q_i(u') = 1$ and $x_i(u') \neq x_i(u)$, then i will manipulate the mechanism at either u or u' which again violates Strategyproofness.

If $q_i(u) = 0, u'_i < u_i$ and $q_i(u') = 1$, then $x_i(u') \leq u'_i < u_i$ by VP. It follows therefore, that i will manipulate the mechanism at u by announcing u'_i . A similar argument takes care of the case where $q_i(u) = 1, x_i(u) = u_i$ and $u'_i < u_i$. ■

Theorem 3.3.2 *Suppose the mechanism $F = (q, x)$ satisfies Strategyproofness, NS, VP, ONB and WUNB. Then, it is also Strong Group Strategyproof.*

Proof: Let u and $\bar{u}$ be two profiles such that $u_j = \bar{u}_j$ for all $j \notin T \subseteq N$. We need to show that

$$[u_i(\bar{u}) \geq u_i(u) \text{ for all } i \in T] \Rightarrow [u_i(\bar{u}) = u_i(u) \text{ for all } i \in T] \quad (3.3.12)$$

(3.3.12) is obviously true when $|T| = 1$ since F is Strategyproof. Assume that it is true for all T such that $|T| \leq k - 1$. Suppose that is not true for some $T \subset N$ such that $|T| = k$. Without loss of generality, assume $1 \in T$ and $u_1(\bar{u}) > u_1(u)$.

Consider any $j \in T \setminus \{1\}$. Let $u^j = (u_j, \bar{u}_{-j})$. By Strategyproofness, we have

$$u_j(u^j) \geq u_j(\bar{u}) \quad (3.3.13)$$

Suppose that (3.3.13) is an equality. There are three possible cases here.

Case 1: $q_j(u^j) = q_j(\bar{u})$.

Strategyproofness implies that $x_j(u^j) = x_j(\bar{u})$. ONB implies that

$$q_k(u^j) = q_k(\bar{u}) \text{ and } x_k(u^j) = x_k(\bar{u}) \text{ for all } k \neq j \quad (3.3.14)$$

Since $j \neq 1$, (3.3.14) implies that the deviation of $T \setminus \{j\}$ from u to u^j violates (3.3.12) and contradicts the induction hypothesis.

Case 2: $q_j(u^j) = 1$ and $q_j(\bar{u}) = 0$.

Strategyproofness implies that $\bar{u}_j < u_j$. It follows from NS and VP that $x_j(\bar{u}) = 0$ and hence, $u_j(\bar{u}) = 0$. (3.3.12) and VP imply that $u_j(u) = 0$. First, suppose that $q_j(u) = 0$. Since $\bar{u}_j < u_j$, it follows from Lemma 3.3.2 that $q_j(u^{(j)}) = 0$ where $u^{(j)} = (\bar{u}_j, u_{-j})$. Using ONB, we conclude that

$$q_k(u^{(j)}) = q_k(u) \text{ and } x_k(u^{(j)}) = x_k(u) \text{ for all } k \neq j \quad (3.3.15)$$

It follows from (3.3.15) that the deviation of the coalition $T \setminus \{j\}$ from $u^{(j)}$ to $\bar{u}$ violates (3.3.12) and contradicts the induction hypothesis.

Next, suppose that $[q_j(u) = 1 \text{ and } x_j(u) = u_j]$. Since $\bar{u}_j < u_j$, Lemma 3.3.2 implies that $q_j(u^{(j)}) = 0$. By NS, $x_j(u^{(j)}) = 0$. Thus, $u_j(u) = u_j(u^{(j)}) = 0$. Using WUNB, we conclude that

$$u_k(u^{(j)}) \leq u_k(u) \text{ for all } k \neq j \quad (3.3.16)$$

(3.3.16) implies that the deviation of $T \setminus \{j\}$ from $u^{(j)}$ to $\bar{u}$ violates (3.3.12) and contradicts the induction hypothesis.

Case 3: $q_j(u^j) = 0$ and $q_j(\bar{u}) = 1$.

Strategyproofness implies that $\bar{u}_j > u_j$ and $x_j(\bar{u}) \geq u_j$. (3.3.12) implies that $x_j(\bar{u}) = u_j$. Thus, $u_j(\bar{u}) = 0$. Using (3.3.12) once again, we conclude

that $u_j(u) = 0$. There are two possibilities here. First, suppose that $q_j(u) = q_j(u^{(j)})$. By Strategyproofness, $x_j(u) = x_j(u^{(j)})$. (3.3.15) follows from ONB and implies the same contradiction as in Case 2.

So, suppose that $q_j(u) \neq q_j(u^{(j)})$. Since $\bar{u}_j > u_j$, the only possible case is where $q_j(u) = 0, q_j(u^{(j)}) = 1$ and $x_j(u^{(j)}) \geq u_j$. Suppose now that $x_j(u^{(j)}) = u_j$. Here, we have $u_j(u^{(j)}) = u_j(u) = 0$. From WUNB, (3.3.16) follows, and hence here also, we reach a contradiction. Suppose that $x_j(u^{(j)}) > u_j$. Let $\bar{u}_j$ be such that $u_j < \bar{u}_j < x_j(u^{(j)})$ and let $v = (\bar{u}_j, u_{-j})$. Lemma 3.3.2 implies that $q_j(v) = 0$. It follows from ONB that

$$q_k(v) = q_k(u) \text{ and } x_k(v) = x_k(u) \text{ for all } k \neq j \quad (3.3.17)$$

Now, consider the deviation of T from v to $\bar{u}$. (3.3.17) and $[q_j(u) = q_j(v) = 0]$ imply that this deviation violates (3.3.12). Since $\bar{u}_j > v_j > u_j^7$ and $[q_j(\bar{u}) = 1, x_j(\bar{u}) = u_j]$, Lemma 3.3.2 implies that $[q_j(v^j) = 1 \text{ and } x_j(v^j) = u_j]$ where $v^j = (\bar{u}_j, \bar{u}_{-j})$. We can now use the analysis in Case 1 to show that the deviation of $T \setminus \{j\}$ from v to v^j also violates (3.3.12) and contradicts the induction hypothesis.

The above three cases show that $u_j(u^j) = u_j(\bar{u})$ cannot be true. So, $u_j(u^j) > u_j(\bar{u}) \geq 0$. This now implies that $[q_j(u^j) = 1, q_j(\bar{u}) = 0 \text{ and } \bar{u}_j < u_j]$. To see this, note that $u_j(u^j) > 0$ implies $q_j(u^j) = 1$ by NS. If $q_j(\bar{u}) = 1$, then $x_j(u^j) = x_j(\bar{u})$ by Strategyproofness, thus implying $u_j(u^j) = u_j(\bar{u})$, a contradiction. Therefore, $q_j(\bar{u}) = 0$. Strategyproofness now requires that $\bar{u}_j < u_j$. Since $q_j(\bar{u}) = 0$, it follows from NS and VP that $x_j(\bar{u}) = 0$ and hence, $u_j(\bar{u}) = 0$. (3.3.12) implies that $u_j(u) = 0$. There are two possibilities.

Case 1: $q_j(u) = 0$

Since $\bar{u}_j < u_j$, it follows from Lemma 3.3.2 that $q_j(u^{(j)}) = 0$. Using

⁷If $\bar{u}_j \leq v_j$, then by Lemma 3.3.2, $q_j(v^{(j)}) = q_j(v) = 0$. By NS and VP, $x_j(v^{(j)}) = x_j(v) = 0$. By ONB, $q_k(v^{(j)}) = q_k(v)$ and $x_k(v^{(j)}) = x_k(v)$ for all $k \neq j$. Thus, the coalition $T \setminus \{j\}$ can deviate from $v^{(j)}$ to $\bar{u}$ in a manner violating (3.3.12). This contradicts the induction hypothesis.

ONB, (3.3.15) follows. It follows that the deviation of $T \setminus \{j\}$ from $u^{(j)}$ to $\bar{u}$ contradicts (3.3.12) and violates the induction hypothesis.

Case 2: $q_j(u) = 1$ and $x_j(u) = u_j$

As in the previous case, it follows from Lemma 3.3.2 that $q_j(u^{(j)}) = 0$. Hence, $u_j(u^{(j)}) = u_j(u) = 0$. (3.3.16) follows from WUNB. It follows that the deviation by $T \setminus \{j\}$ from $u^{(j)}$ to $\bar{u}$ violates (3.3.12) and contradicts the induction hypothesis. This concludes the proof of the theorem. ■

3.4 Cost Sharing under Submodularity

In this section, we confine ourselves to the case where the cost function is *submodular*. Our main purpose here is to define the Egalitarian cost allocation rule and characterise it. We first show that the Egalitarian cost allocation rule maximises the probability of acceptance for each coalition in the class of all admissible mechanisms satisfying Strong Group Strategyproofness. Then, we give an alternate characterisation by showing that the Egalitarian cost allocation rule maximises the probability of acceptance for each coalition in the class of all admissible mechanisms satisfying Strategyproofness, ONB and WUNB. We also provide an alternate characterisation of the Shapley value rule by showing that it minimises the maximum loss from the non-formation of the efficient coalition in the class of all admissible mechanisms satisfying Strategyproofness, ONB and WUNB.

It is worth pointing out here the importance of the alternative characterisations of the Shapley value and the Egalitarian rules in terms of Strategyproofness. The Moulin-Shenker result as well as our own characterisation result (Theorem 3.4.3 below) rely extensively on the link between Strong Group Strategyproofness of the mechanism and Cross Monotonicity of the cost allocation rule. However, Strong Group Strategyproofness is a very demanding requirement. First, it assumes that agents can collude, which may not always be possible in situations where agents do not know each other's

private information. Second, Strong Group Strategyproofness assumes that coalitions will deviate even if all members are not strictly better off. It is clear that in some situations, an agent would not agree to be part of a deviation from which he does not gain strictly. If the planner has reasons to believe that such a strong type of cooperation is not possible or likely, then she would be interested in mechanisms which satisfy non-manipulability conditions which are weaker than Strong Group Strategyproofness. The results obtained here, though, suggest that the class of mechanisms satisfying Strong Group Strategyproofness is the same as the class of mechanisms satisfying Strategyproofness.

We now examine the relationship between cost allocation mechanisms and cost allocation rules. A cost allocation rule ξ induces a corresponding cost allocation mechanism $RM(\xi)$ in the following way. For any announcement v , the set of agents who are given access to the service is given by the limit of the following inclusion sequence: $S_0 = N; S_{t+1} = \{i | v_i \geq \xi_i(S_t)\}$. The cost shares for the agents given access is given by the cost sharing rule ξ . We now define a property of cost allocation rules which will link the concepts of cost allocation mechanisms and cost allocation rules.

Cross monotonicity: The cost allocation rule ξ is cross monotonic if for all $S, T \subset N$ such that $S \subset T$ and all $i \in S$, $\xi_i(S) \geq \xi_i(T)$.

Cross monotonicity says an individual's cost share does not increase when she becomes part of a larger coalition. Moulin and Shenker [44] establish the following:

Proposition 3.4.2 *Suppose the cost function C is submodular. For any cross monotonic cost allocation rule ξ , the mechanism $RM(\xi)$ is admissible and Strongly Group Strategyproof. Conversely, pick a revelation mechanism M which is admissible and Strongly Group Strategyproof. Then, there exists a cross monotonic cost allocation rule ξ such that $RM(\xi)$ is welfare equivalent to M .*

Proof: See Moulin and Shenker [44].

The above proposition allows us to concentrate on cross monotonic cost allocation rules instead of cost allocation mechanisms if the mechanisms are constrained to be admissible.⁸ We shall follow Moulin and Shenker [44] in focussing only on such mechanisms. Thus, in the remainder of this chapter, we shall be concerned only with cross monotonic cost allocation rules.

With any cross monotonic cost allocation rule, what is sacrificed is efficiency. To see what this entails, fix a profile u . Define for each coalition S , the surplus $w(S, u) \equiv \sum_{i \in S} u_i - C(S)$. The efficient coalition is the one that maximises $w(S, u)$ over all $S \subseteq N$. Since efficiency, Strategyproofness and Budget Balance cannot hold together,⁹ it follows that at any given profile, the set of agents getting access to the service will typically be a strict subset of the efficient coalition.

Moulin and Shenker [44] show that the Shapley value rule minimises the maximum loss that can result from the efficient coalition not forming. We now describe another cross monotonic rule, the Egalitarian cost allocation rule, and characterise it. This scheme is based on the egalitarian solution proposed for characteristic function TU-games by Dutta and Ray [18].

We now introduce some notation needed to define the Egalitarian cost allocation rule. Let x, y be two vectors in $\mathcal{R}^n$. Let $\tilde{x}$ and $\tilde{y}$ be the vectors obtained by rearranging the vectors x and y such that $\tilde{x}_1 \leq \dots \leq \tilde{x}_n$ and $\tilde{y}_1 \leq \dots \leq \tilde{y}_n$. The vector x is said to Lorenz dominate y if for all $j = 1, \dots, n-1$, $\sum_{i=1}^j \tilde{x}_i \geq \sum_{i=1}^j \tilde{y}_i$ with strict inequality for some j , and $\sum_{i=1}^n \tilde{x}_i = \sum_{i=1}^n \tilde{y}_i$.

Dutta and Ray [18] define the egalitarian solution for all TU-games and show that when it exists, it is unique. Instead of giving a general definition, we will use the following characterisation of the egalitarian solution which is valid for convex games. This is sufficient for our purpose because the

⁸In light of Proposition 3.4.2, the mechanism $RM(\xi)$ will be referred to as the “ascending bid auction” mechanism.

⁹See Green and Laffont [25].

submodularity of the cost function will imply that the TU-games to be considered in defining the Egalitarian cost allocation rule are convex. We state the following result.

Proposition 3.4.3 *Let $\Gamma = (N, v)$ be a TU-game. If Γ is convex, then the egalitarian solution is in the core and Lorenz dominates all other core points.*

Proof: See Dutta and Ray [18].

Definition 3.4.12 *Given a submodular cost function C and a set of agents N , the Egalitarian cost allocation rule ξ^e associates to each $S \subset N$, a corresponding $|S|$ -tuple $(\xi_i^e(S))_{i \in S}$ which is the egalitarian solution of the cost sharing game (S, C_1) ¹⁰ where $C_1(T) = C(T)$ for all $T \subseteq S$.*

The submodularity of the cost function and Proposition 3.4.3 imply that the egalitarian cost allocation rule is well-defined.

Remark 3.4.3 Dutta [15] shows that the egalitarian solution in convex games satisfies the property of cross monotonicity.¹¹ It follows from his result that the egalitarian cost allocation rule is cross monotonic.

Our characterisation of the Egalitarian cost allocation rule requires assumptions on the distribution of the agents' valuations. We state these below.

Assumptions: The valuations of all individuals lie in the interval $[0, M]$ according to a common distribution function F with corresponding density function f . The valuations of i and j , u_i and u_j , are independent for all $i, j \in N$.

¹⁰The egalitarian solution of the cost sharing game (N, C) is defined as the egalitarian solution of the game (N, v) where for each $S \subset N$, $v(S) = -C(S)$.

¹¹Cross monotonicity as defined by Dutta [15] requires that the payoff to an individual be non-decreasing as she becomes part of a larger coalition. In the context of cost-sharing this means that the share of an individual is non-increasing as she becomes part of a larger coalition.

Definition 3.4.13 *The distribution function F satisfies the monotone hazard rate condition if*

$$\frac{d}{du} \frac{f(u)}{1 - F(u)} \geq 0$$

The monotone hazard rate condition constitutes a mild restriction on the distribution function F . It is a standard assumption in much of the (Bayesian) mechanism design literature and is satisfied by most of the standard distribution functions.¹²

Let ξ be a cost allocation rule and let $S \subset N$. Let x denote the cost allocation for S under ξ . The probability that all members of S accept the cost imputed to them is given by $P(x) \equiv \prod_{i \in S} \Pr(u_i \geq x_i) = \prod_{i \in S} [1 - F(x_i)]$. We now turn to our main characterisation result.

Theorem 3.4.3 *Let C be a submodular cost function and let $S \subset N$. Let F satisfy the monotone hazard rate condition. Then, the egalitarian cost allocation rule ξ^e maximises the probability that all members of S accept the cost imputed to them in the class of all cross monotonic cost allocation rules.*

We first prove the following lemmas.

Lemma 3.4.3 *Let ξ be a cost allocation rule, let $S \subset N$ and let x denote the cost allocation for S . Suppose there exist $i, j \in S$ such that $x_i < x_j$. Then,*

$$dP(x) = \sum_{k \in S} \frac{\partial P(x)}{\partial x_k} dx_k \geq 0$$

where $0 < dx_i = -dx_j = dx$ and $dx_k = 0$ if $k \neq i, j$.

Proof: One computes directly,

$$\frac{dP(x)}{dx} = \left\{ \prod_{k \in S \setminus \{i, j\}} (1 - F(x_k)) \right\} [f(x_j)(1 - F(x_i)) - f(x_i)(1 - F(x_j))] \geq 0 \quad (3.4.18)$$

¹²See Fudenberg and Tirole [22].

The inequality follows from the monotone hazard rate condition. Note that the inequality is strict if the monotone hazard rate condition is strict. ■

Lemma 3.4.4 *Let ξ be a cross monotonic cost allocation rule, let $S \subset N$ and let x denote the cost allocation for coalition S . Then, x is a core-point of the cost sharing game (S, C_1) .*

Proof: We need to show that $\sum_{i \in T} x_i \leq C_1(T)$ for all $T \subset S$. Let y be the cost allocation for T under ξ . Cross monotonicity implies that $x_i \leq y_i$ for all $i \in T$. The lemma now follows trivially since $\sum_{i \in T} y_i = C(T) = C_1(T)$. ■

Proof of the theorem: Let x^e be the egalitarian cost allocation and let x be the cost allocation for S under some other cross monotonic cost allocation rule. From Lemma 3.4.4, it follows that both x^e and x are core points of the cost-sharing game (S, C_1) . Let σ, σ' be permutations of the player set S such that $x_{\sigma(1)}^e \leq \dots \leq x_{\sigma(|S|)}^e$ and $x_{\sigma'(1)} \leq \dots \leq x_{\sigma'(|S|)}$. Let y be the allocation for S defined by $y_{\sigma(j)} = x_{\sigma'(j)}$ for $j = 1, \dots, |S|$. Note that y is a permutation of a core allocation (which may itself not be in the core) and that $\prod_{i \in S} [1 - F(x_i)] = \prod_{i \in S} [1 - F(y_i)]$.

Since the allocation x^e Lorenz dominates all other core allocations, it follows that $\sum_{i=1}^k x_{\sigma(i)}^e \geq \sum_{i=1}^k y_{\sigma(i)}$ for all $k = 1, \dots, n$. From a well-known result due to Hardy, Littlewood and Polya [26]¹³, it follows that we can get from y to x^e through a finite series of transformations of the type

$$\begin{aligned} y_{\sigma(i)}^{\alpha+1} &= y_{\sigma(i)}^{\alpha} + e^{\alpha}, & j > i \\ y_{\sigma(j)}^{\alpha+1} &= y_{\sigma(j)}^{\alpha} - e^{\alpha}, & e^{\alpha} > 0 \\ y_{\sigma(k)}^{\alpha+1} &= y_{\sigma(k)}^{\alpha}, & \text{if } k \neq i, j; \end{aligned}$$

It follows from Lemma 3.4.3 that $\prod_{i \in N} (1 - F(y_i^{\alpha+1})) \geq \prod_{i \in N} (1 - F(y_i^{\alpha}))$. Since the value of the objective function is non-decreasing at each stage of

¹³The version of this result that we use is to be found in Dasgupta, Sen and Starrett [13].

the algorithm, it follows that

$$\prod_{i \in N} (1 - F(x_i^e)) \geq \prod_{i \in N} (1 - F(y_i)) = \prod_{i \in N} (1 - F(x_i)) \quad (3.4.19)$$

We have shown that the value of the objective function at the egalitarian solution is no lower than the value at any other feasible solution. This completes the proof of the theorem. ■

Remark 3.4.4 The egalitarian cost allocation rule is the unique scheme maximising the probability of acceptance if the monotone hazard rate condition is strict.

The following corollary whose proof we omit, follows straightforwardly from Theorem 3.4.3.¹⁴

Corollary 3.4.1 *The egalitarian cost allocation rule maximises the expected size of the coalition getting service in the class of all cross monotonic cost allocation rules.*

We now turn to proving a lemma which will give us alternate characterisations of both the Shapley value and the Egalitarian rules. Recall that the revelation mechanism associated with a cost allocation rule ξ chooses the coalition to be given access to the service as the limit of the *inclusive* sequence $S_0 = N; S_{t+1} = \{i \in S_t | u_i \geq \xi_i(S_t)\}$. The cost shares for the agents given access are given by the cost allocation rule. Recall also that $S(u)$ denotes the agents given access to the service when the announcement is u .

Lemma 3.4.5 *The revelation mechanism associated with a cross monotonic cost allocation rule ξ satisfies ONB and WUNB.*

Proof: Let u and u' be two profiles such that $u_j = u'_j$ for all $j \neq i$. Let $(q_i(u), x_i(u)) = (q_i(u'), x_i(u'))$. Suppose that ONB is violated. Then, since

¹⁴I am grateful to Professor Hervé Moulin for pointing this out to me.

the cost shares are determined by the cost allocation rule ξ , it must be that $S(u) \neq S(u')$. Consider the operation of the revelation mechanism when the profile is u . Let $S = S(u) \cup S(u')$. Since the cost allocation rule is cross monotonic, we have

$$\xi_k(S) \leq \begin{cases} \xi_k(S(u)) & \text{if } k \in S(u) \\ \xi_k(S(u')) & \text{if } k \in S(u') \end{cases}$$

Observe that $u_k \geq \xi_k(S)$ for all $k \in S$. We claim that $S \subset S_t$ for all t where S_t is defined by the operation of the algorithm mentioned above. Using the cross monotonicity of the cost allocation rule and the fact that $u_j = u'_j$ for all $j \neq i$, we have $\xi_k(N) \leq \xi_k(S) \leq u_k$ for all $k \in S$. The claim is thus true at $t = 1$. Assume that it is true for all $t < T$. At stage T , the set S_T is defined as $S_T = \{k \in S_{T-1} | u_k \geq \xi_k(S_{T-1})\}$. Using $S \subset S_{T-1}$ and cross monotonicity, it follows that $S \subset S_T$. Thus, $S \subset S(u)$. This is a contradiction and hence, $S(u) = S(u')$.

We now show that WUNB is also satisfied. If $q_i(u) = q_i(u')$, then the analysis in the above paragraph shows that $u_j(u') = u_j(u)$ for all $j \neq i$. Thus, WUNB is trivially satisfied. Suppose that $q_i(u') \neq q_i(u)$ but $u_i(u') = u_i(u)$. Note that if $q_i(u') = 1$ and $q_i(u) = 0$, then by definition of the revelation mechanism, $u_i < x_i(u')$. Thus, in this case, $u_i(u') \neq u_i(u)$.

The only way that $q_i(u') \neq q_i(u)$ and $u_i(u') = u_i(u)$ can occur together is when $q_i(u) = 1, q_i(u') = 0$ and $x_i(u) = u_i$. In this case, $S(u') \subset S(u)$. To see this, suppose that $S(u') \not\subset S(u)$. Let $S = S(u) \cup S(u')$. We can now replicate the argument used above in showing that the revelation mechanism satisfies ONB to demonstrate that $S \subset S(u)$. This is a contradiction and shows that $S(u') \subset S(u)$. It follows from cross monotonicity that $x_k(u') = \xi_k(S(u')) \geq \xi_k(S(u)) = x_k(u)$ for all $k \in S(u')$. Hence, $u_k(u') \leq u_k(u)$ for all $k \neq i$. This shows that WUNB is satisfied and completes the proof of the lemma. ■

We have as a corollary to the above, the following alternate characterisations of the Shapley value and the Egalitarian cost allocation rules.

Corollary 3.4.2 *The Shapley value rule minimises the maximum loss resulting from the non-formation of the efficient coalition in the class of all admissible mechanisms satisfying Strategyproofness, ONB and WUNB.*

Corollary 3.4.3 *The egalitarian cost allocation rules maximises the probability of acceptance for each $S \subseteq N$ in the class of all admissible mechanisms satisfying Strategyproofness, ONB and WUNB.*

We now assume that the density function f is uniform. This is a stronger assumption than the monotone hazard rate condition. The main virtue of this assumption is that it gives us a stronger characterisation of the Egalitarian cost allocation rule which enables us to compare it with the Shapley value rule.

Let $S \subseteq N$ and let x denote the cost imputation for S under ξ . Suppose that the planner has decided to offer the service first to the coalition S and to offer it others only after S has rejected the offer. Then, the expected surplus from S is

$$W(S) = \prod_{i \in S} (1 - F(x_i)) \left[\sum_{i \in S} \int_{x_i}^M \frac{u_i}{1 - F(x_i)} dF - C(S) \right] \quad (3.4.20)$$

Theorem 3.4.4 *Let the cost function C be submodular. Suppose the density function f is uniform. Then, the Egalitarian cost allocation rule maximizes $W(S)$ for each $S \subset N$ in the class of all admissible mechanisms satisfying Strong Group Strategyproofness.*

As before, we first prove a small lemma.

Lemma 3.4.6 *Let ξ be a cost allocation rule and let x be the cost allocation for $S \subset N$. Assume that the density function f is uniform. Suppose there exist $i, j \in S$ such that $x_i < x_j$. Then,*

$$dW(S) = \sum_{k \in S} \frac{\partial W(S)}{\partial x_k} dx_k \geq 0$$

where $0 < dx_i = -dx_j = dx$ and for all $k \neq i, j$, $dx_k = 0$

Proof: Using the equality $\sum_{k \in S} x_k = C(S)$, we can rewrite $W(S)$ as:

$$\begin{aligned} W(S) = & \left[\prod_{k \in S \setminus \{i, j\}} [1 - F(x_k)] \right] \left\{ [1 - F(x_j)] \int_{x_i}^M (u_i - x_i) dF(u_i) \right. \\ & + [1 - F(x_i)] \int_{x_j}^M (u_j - x_j) dF(u_j) \\ & \left. + [1 - F(x_i)][1 - F(x_j)] \sum_{k \in S \setminus \{i, j\}} \int_{x_k}^M \frac{u_k - x_k}{1 - F(x_k)} dF(u_k) \right\} \end{aligned}$$

Routine computations yield:

$$\begin{aligned} \frac{dW(S)}{dx} = & \left[\prod_{k \in S \setminus \{i, j\}} [1 - F(x_k)] \right] \left\{ f(x_j) \int_{x_i}^M (u_i - x_i) dF(u_i) - f(x_i) \int_{x_j}^M (u_j - x_j) dF(u_j) \right. \\ & \left. + [f(x_j)(1 - F(x_i)) - f(x_i)(1 - F(x_j))] \sum_{k \in S \setminus \{i, j\}} \int_{x_k}^M \frac{u_k - x_k}{1 - F(x_k)} dF(u_k) \right\} \end{aligned}$$

Using the facts that $f(x_i) = f(x_j)$ and $x_i < x_j$, it follows that $\frac{dW(S)}{dx} \geq 0$. ■

Proof of the theorem: The proof of this theorem is omitted since it is exactly the same as the proof of the previous theorem with Lemma 3.4.6 playing the role of Lemma 3.4.3. Since the uniform distribution satisfies the monotone hazard rate condition strictly, it also follows that the Egalitarian cost allocation rule is the unique cross monotonic rule satisfying the premises of the theorem. ■

Remark 3.4.5 Since the operation of the revelation mechanism offers the service first to the grand coalition, the above theorem implies that the Egalitarian rule maximises the expected surplus from the grand coalition in the class of all admissible rules satisfying Strong Group Strategyproofness. Another way of interpreting the theorem is as follows. Define the *ex ante* efficient coalition S^* to be the coalition that maximises $\sum_{i \in S} E(u_i) - C(S)$ over all $S \subseteq N$. Suppose the planner decides that she wants to maximise

the expected surplus from the *ex ante* efficient coalition. Then, she would first want to offer the service to members of S^* , and then to others.¹⁵ The above theorem tells us that the rule which maximises the expected surplus from S^* in the class of all admissible mechanisms satisfying Strong Group Strategyproofness is the Egalitarian rule.

Remark 3.4.6 Note that the Egalitarian cost allocation rule “equalises” the cost shares for the grand coalition subject to the core constraints. The above theorem tells us that the Egalitarian cost allocation rule also maximises the expected surplus from the grand coalition. The theorem is thus interesting because it shows that egalitarianism is not incompatible with efficiency.

Remark 3.4.7 As before, one can replace Strong Group Strategyproofness by Strategyproofness, ONB and WUNB in the statement of the above theorem.

Remark 3.4.8 The above theorem can be proved by assuming that the density function is non-decreasing. This relaxation, however, does not amount to much. We conjecture that the theorem can be proved under a weaker condition than the one stated in the theorem. However, we have not been able to either prove the result with a weaker assumption or find a counterexample which would illustrate the necessity of our assumption.

Remark 3.4.9 The risk-averse planner would prefer the Shapley value rule minimises the maximum loss resulting from the non-formation of the efficient coalition. On the other hand, a risk-neutral planner will prefer the Egalitarian rule as it maximizes the expected surplus from the *ex ante* efficient coalition. The characterisations obtained here in tandem with Moulin and Shenker’s results will thus help the planner in deciding what mechanism to select for implementation.

¹⁵In this case, treat S^* as the grand coalition.

3.5 Conclusion

We have examined the link between Strategyproofness and Group Strategyproofness in the context of cost sharing of an excludable public good. We show that the combination of Strategyproofness, Outcome Non-bossiness, Voluntary Participation and No Subsidy imply Weak Group Strategyproofness. By including the additional axiom of Weak Utility Non-bossiness, we are able to show that the combination implies Strong Group Strategyproofness.

In the case where the cost function is submodular, we have also provided a characterisation of the Egalitarian rule. This rule is based on the egalitarian solution for TU-games proposed by Dutta and Ray [18]. Assuming that agents' valuations are independent and distributed according to a common distribution function, we show that the Egalitarian rule maximises the probability that agents of a given coalition accept the cost shares imputed to them. This is proved under only a mild restriction on the distribution function. Assuming that the density function is *uniform*, we show that the Egalitarian rule maximises the expected surplus from the *ex ante* efficient coalition. We also use the results on the connection between Strategyproofness, Non-bossiness and Strong Group Strategyproofness to obtain alternate characterisations of both the Shapley value and the Egalitarian rules.

Moulin and Shenker [44] show that the Shapley value rule minimises the maximum loss that can occur from the non-formation of the efficient coalition. Clearly, such a scheme would be preferred by a risk-averse planner. On the other hand, we argue that the Egalitarian rule would be preferred by a risk-neutral planner since it maximises the expected surplus from the formation of the *ex ante* efficient coalition.

Chapter 4

Multi-Good Cost Sharing

4.1 Introduction

In this chapter, we consider an extension of the model examined in Chapter 3 to allow for the presence of multiple excludable public goods. The motivation for this problem comes from the literature on club goods. As Cornes and Sandler [12] observe, the literature on club goods has for the most part confined itself to the analysis of the single club good (or excludable public good) model.¹ However, most clubs in real life offer a range of products to their members. For example, a club might offer a swimming pool, a golf course and tennis courts to its members. Typically, there are economies of scale associated with offering multiple services; this externality in production is what renders the multiproduct case different from the single good case. To the best of our knowledge, there have been no results obtained on the strategyproof cost sharing of multiple excludable public goods.

Moulin [42] offers a model of cost sharing which can be interpreted to accomodate the multiple excludable public good case. In his model, there are n agents, labelled for convenience as $\{1, \dots, n\}$. Agent i is assumed to

¹The literature on multiproduct clubs include Berglas [7], [8], Brueckner and Lee [11], Sandler and Tschirhart [52] and Wooders [61].

be able to consume discrete units of good i . The cost function defined on the vector of quantities $(q_1, \dots, q_n)$ is assumed to be supermodular (increasing marginal costs). To interpret the model as one involving multiple excludable public goods, assume that agent i can consume k units of good i where k is the number of excludable public goods. Unfortunately, Moulin's assumption of increasing marginal costs cannot be satisfied because with multiple excludable public goods, the cost function displays neither increasing marginal costs nor decreasing marginal costs. To see this, observe that once the first good has been produced, the marginal cost of accomodating an additional person is zero unless the additional person demands a unit of the second good. So, the marginal cost is zero if a good has been produced and is increasing only at those points where an additional good has to be produced. The consequence of this is that Moulin's model cannot really address the excludable goods problem.

The work closest to what we are doing are two papers - one by van Kolpin [32] and the other by Koster [33] - which aim to extend the serial cost sharing rule proposed by Moulin and Shenker [43] for the homogeneous good case to multiple goods. The paper by Kolpin is basically an impossibility result showing that the additivity axiom is incompatible with other axioms when one tries to extend the serial cost sharing rule of Moulin and Shenker [44] to the multiple goods case. Koster [33] proposes a way of extending the serial cost sharing rule to the multiple goods case, but the analysis is from an axiomatic viewpoint. Other related work include the paper by Ausubel [3] on efficient auctions for multiple objects and the paper by Bikhchandani and Mamer [10] on competitive equilibria in economies with indivisibilities.

The situation with multiple excludable goods is more complicated than the single good case. In the single good case, the mechanism had to decide whether the good was to be produced, and if the good was produced, the set of agents getting access and their cost shares. With multiple goods, the mechanism has to first decide which subset of goods is to be produced, then

for each agent the set of goods to which he is to be given access, and finally the cost shares. For instance, with just two goods and two agents, there are sixteen potential cases to consider ranging from the one where both agents get access to both goods to the one where no production takes place and so neither agent gets access to any good.

Suppose we are interested in mechanisms which satisfy the axioms postulated for the single good case. That is, we want mechanisms which are Strategyproof, Outcome Non-Bossy, Budget Balanced and satisfy No Subsidy, Voluntary Participation and Consumer Sovereignty. (The Consumer Sovereignty axiom has to be reformulated to handle the multiple good case.) We show, as our main result, that at least for the special case of two agents, there exist no mechanisms satisfying all of the above axioms. We do identify mechanisms violating some of the axioms.

One issue which is of great interest is to evaluate the welfare implications of these mechanisms. The result of Green and Laffont [25] still goes through. So we know that any budget balanced and strategyproof mechanism must, in general, be inefficient. However, the computations of welfare loss become much more complicated, and we have not been able to obtain any results which would be the natural analogue of the characterisations of the Shapley Value rule and Egalitarian Cost Allocation rule for the single good case.

4.2 The Model

There are a fixed number of agents $N = \{1, \dots, n\}$. There are two excludable public goods, labelled for convenience as good 1 and good 2 respectively. The cost of producing good i , ($i = 1, 2$) singly is $C_i > 0$ and the cost of producing the two goods jointly is C_{12} . We assume that $C_{12} < C_1 + C_2$ as well as $C_{12} > C_1, C_{12} > C_2$.² Agent i consumes a bundle of the two goods, $q_i \equiv (q_{i1}, q_{i2})$ where q_{ik} denotes agent i 's consumption of the k th good. We

²Without this cost dependency, the problem essentially reduces to the single good case.

assume that each good is indivisible, so $q_{ik} \in \{0, 1\}$ for all i and k .

The preferences of agent i are assumed to be quasi-linear:

$$U_i(q_i, x_i) = u_{i1}q_{i1} + u_{i2}q_{i2} - x_i$$

where x_i is the cost imputation of agent i , and u_{ik} is the valuation of good k to agent i . In what follows, we let u_i denote the tuple (u_{i1}, u_{i2}) .

Let $v_i = (v_{i1}, v_{i2})$ be an announcement of agent i . That is, agent i 's announcement is a tuple consisting of his willingness to pay for the two goods.

Definition 4.2.1 A cost allocation mechanism $F = (Q, x)$ associates to each vector of announcements $v = (v_i)_{i \in N}$, a corresponding tuple $(Q(v), x(v))$ where $Q(v) \equiv (q_1(v), \dots, q_n(v))$ is the vector of access decisions for the agents, and $x(v) \equiv (x_1(v), \dots, x_n(v))$ is the corresponding vector of cost imputations for the agents.

Definition 4.2.2 A cost allocation mechanism $F = (Q, x)$ is said to be a cost sharing rule if for any two profiles v and w , $x_i(v) = x_i(w)$ for all $i \in N$ whenever $q_i(v) = q_i(w)$ for all $i \in N$.

We now define the set of axioms that we use to establish our main result. The axioms of Strategyproofness, No Subsidy, Voluntary Participation and Outcome Non-Bossiness are defined in the same way as before. Budget Balance, though, changes slightly. Let v be an announcement, and let $N_1(v) \equiv \{i \in N | q_{i1}(v) = 1\}$ and $N_2(v) \equiv \{i \in N | q_{i2}(v) = 1\}$.

Definition 4.2.3 The cost sharing mechanism $F = (Q, x)$ is Budget Balanced if for all announcements v ,

$$\sum_{i \in N} x_i(v) = \begin{cases} C_{12} & \text{if } N_1(v) \neq \emptyset \text{ and } N_2(v) \neq \emptyset. \\ C_k & \text{if } N_k(v) \neq \emptyset \text{ and } N_l(v) = \emptyset, k, l = 1, 2, k \neq l. \\ 0 & \text{otherwise.} \end{cases}$$

In the last chapter, we had defined the axiom of Consumer Sovereignty. With multiple goods, this axiom has to be generalised. There are obviously many ways of doing this, but the one we adopt is the following.

Definition 4.2.4 *Let v be an announcement. The mechanism $F = (Q, x)$ satisfies Consumer Sovereignty if for all $i \in N$, for $j, k = 1, 2, j \neq k$,*

$$\begin{aligned} A. \quad & v_{ij} \geq C_j \Rightarrow q_i(v) = 1 \\ B. \quad & v_{i1} + v_{i2} \geq C_{12} \text{ and } v_{ij} \geq C_{12} - C_k \Rightarrow q_i(v) = (1, 1). \end{aligned}$$

We shall explain the Consumer Sovereignty axiom briefly. This axiom says that if any agent is willing to pay the entire cost of a particular bundle of goods, then he should be granted access to that bundle irrespective of what others announce. However, it is quite possible that an agent is willing to pay more than C_{12} for *one* particular good, say good 1, and nothing for good 2. In this situation, it is unreasonable for the mechanism to *guarantee* the agent access to both goods. We therefore restrict the mechanism to guarantee an agent access to both goods only if he is willing to pay a minimum positive amount for both goods.

Remark 4.2.1 This version of Consumer Sovereignty rules out the “bundling” mechanism which takes the total willingness to pay, that is, $v_{i1} + v_{i2}$ for each agent and operates the “ascending bid” auction discussed in the last chapter with the cost of production equal to C_{12} . Thus, in this mechanism, either the two goods are produced or nothing is produced. All agents given access get access to both goods, and are charged an identical amount of C_{12}/n_a where n_a is the number of agents given access.

The “bundling” mechanism essentially treats the two goods as one, so all the results obtained for the one good case go through here as well. In particular, it is Strong Group Strategyproof, and satisfies No Subsidy, Voluntary Participation and Budget Balance. However, it does not satisfy Consumer Sovereignty as defined here. To see this, suppose that $N = \{1, 2\}$, and let the

two agents send reports of $v_1 = (C_1, 0)$ and $v_2 = (0, 0)$. Since $C_1 < C_{12}$, it follows that $q_1(v_1, v_2) = q_2(v_1, v_2) = (0, 0)$. However, this violates Consumer Sovereignty.

We now formally define the Anonymity axiom. Let v be an announcement, and let π be a permutation of the player set N . Let v^π be the announcement defined by $v_{\pi(i)}^\pi = v_i$ for all $i \in N$.

Definition 4.2.5 *The mechanism $F = (Q, x)$ satisfies Anonymity, if for all permutations π of N , $q_{\pi(i)}(v^\pi) = q_i(v)$ and $x_{\pi(i)}(v^\pi) = x_i(v)$.*

Finally, we make the following definition.

Definition 4.2.6 *The cost sharing mechanism $F = (Q, x)$ is said to be **acceptable** if it satisfies Strategyproofness, Anonymity, Outcome Non-Bossiness, No Subsidy, Consumer Sovereignty and Budget Balance.*

4.3 The Results

Our first step in proving the impossibility result for the case where $|N| = 2$ is to show that any acceptable cost sharing mechanism must also be a cost sharing rule. *Note that we shall assume that $N = \{1, 2\}$ in all the results that follow.*

Lemma 4.3.1 *Let $F = (Q, x)$ be an acceptable cost sharing mechanism. Then, F is also a cost sharing rule.*

Proof: Let F be an acceptable cost sharing mechanism. Let v^1 and v^2 be two announcements such that $Q(v^1) = Q(v^2)$. We want to show that $x(v^1) = x(v^2)$ as well.

Step 1: Let v and w be two announcements such that $v_j = w_j$ for $j \neq i$. Let $q_i(v) = (1, 1)$, $(w_{i1}, w_{i2}) \gg (v_{i1}, v_{i2})$. Then, $q_i(w) = (1, 1)$ and $x_i(w) = x_i(v)$.

Suppose to the contrary that $q_i(w) \neq (1, 1)$. First, suppose that $q_i(w) = (1, 0)$. Then, strategyproofness at v implies that $v_{i1} + v_{i2} - x_i(v) \geq v_{i1} - x_i(w)$ or,

$$v_{i2} \geq x_i(v) - x_i(w) \quad (4.3.1)$$

Similarly, strategyproofness at w implies that

$$w_{i2} \leq x_i(v) - x_i(w) \quad (4.3.2)$$

(4.3.1) and (4.3.2) imply a contradiction since $w_{i2} > v_{i2}$. We reach a similar contradiction if $q_i(w) = (0, 1)$. Lastly, if $q_i(w) = (0, 0)$,³ then strategyproofness at v yields

$$v_{i1} + v_{i2} \geq x_i(v) \quad (4.3.3)$$

However, strategyproofness at w yields

$$w_{i1} + w_{i2} \leq x_i(v) \quad (4.3.4)$$

Again, we have a contradiction. Now, suppose that $q_i(w) = (1, 1)$ but $x_i(v) \neq x_i(w)$. Then, i will manipulate the mechanism at either v or w . This completes the proof of Step 1.

Step 2: Let v and w be two announcements such that $v_j = w_j$ for $j \neq i$. Suppose that $q_{ik}(v) = 1, q_{il} = 0, k \neq l, w_{ik} > v_{ik}$ and $w_{il} < v_{il}$. Then, $q_{ik}(w) = 1, q_{il}(w) = 0$ and $x_i(w) = x_i(v)$.

Proof: We proceed by contradiction. First, suppose that $q_i(w) = (1, 1)$. Strategyproofness at v gives us $v_{ik} - x_i(v) \geq v_{ik} + v_{il} - x_i(w)$, or

$$v_{il} \leq x_i(w) - x_i(v) \quad (4.3.5)$$

On the other hand, strategyproofness at w implies $w_{ik} + w_{il} - x_i(w) \geq w_{ik} - x_i(v)$, or,

$$w_{il} \geq x_i(w) - x_i(v) \quad (4.3.6)$$

³Note that Voluntary Participation and No Subsidy together imply that $x_i(w) = 0$ at all profiles w such that $q_i(w) = (0, 0)$.

(4.3.5) and (4.3.6) gives the required contradiction. The cases where $q_i(w) = (0, 0)$ and $[q_{il}(w) = 1, q_{ik}(w) = 0]$ are handled similarly. Finally, if $[q_{ik}(w) = 1, q_{il}(w) = 0]$ but $x_i(v) \neq x_i(w)$, then i will manipulate the mechanism at either v or w . This completes Step 2.

Step 3: Suppose that $q_i(v) = (1, 1)$ for all i . Then, $x_i(v) = C_{12}/2$ for all i .

Proof: Let v^m be a profile such that $v_1^m = v_2^m$ and $v_{ik}^m > \max[v_{1k}, v_{2k}]$ for $i, k = 1, 2$. By Step 1 and Outcome Non-Bossiness, $q_i(v_1^m, v_2) = (1, 1)$ and $x_i(v_1^m, v_2) = x_i(v)$ for all i . Once again applying Step 1 and Outcome Non-Bossiness, we get $q_i(v^m) = q_i(v_1^m, v_2) = (1, 1)$ and $x_i(v^m) = x_i(v_1^m, v_2)$ for all i . Since $v_1^m = v_2^m$ it follows from Anonymity that $x_i(v_1^m, v_2^m) = C_{12}/2$ for all i . Since $x_i(v) = x_i(v_1^m, v_2) = x_i(v_1^m, v_2^m)$ for all i , the result follows.

Returning to the proof of the theorem, observe that if $q_i(v) = (0, 0)$, then No Subsidy and Voluntary Participation imply that $x_i(v) = 0$. Also if $q_i(v^k) = (1, 1)$ for all i and k , then it follows from Step 3 that $x_i(v^k) = C_{12}/2$ for all i and k . So, we are left with the following cases.

Case 1: $q_{ik}(v^1) = q_{ik}(v^2) = 1, q_{il}(v^1) = q_{il}(v^2) = 0, q_j(v^1) = q_j(v^2) = (1, 1)$.

Let $\bar{v}_j$ be an announcement for agent j such that $\bar{v}_{jk} \gg \max[v_{jk}^1, v_{jk}^2]$ for $k = 1, 2$. By Step 1 and Outcome Non-Bossiness, $q_l(v_i^1, \bar{v}_j) = q_l(v^1) = q_l(v^2) = q_l(v_i^2, \bar{v}_j)$, $x_l(v_i^1, \bar{v}_j) = x_l(v^1)$ and $x_l(v_i^2, \bar{v}_j) = x_l(v^2)$ for all l . If $x_i(v_i^1, \bar{v}_j) \neq x_i(v_i^2, \bar{v}_j)$ then i will manipulate at either (v_i^1, v_j) or $(v_i^2, \bar{v}_j)$. Thus, $x_i(v^1) = x_i(v_i^1, \bar{v}_j) = x_i(v_i^2, \bar{v}_j) = x_i(v^2)$. By Budget Balance, it follows that $x_j(v^1) = x_j(v^2)$.

Case 2: $q_i(v^1) = q_i(v^2) = (1, 0), q_j(v^1) = q_j(v^2) = (0, 1)$.

By Step 2 and Outcome Non-bossiness, we can set $v_{j1}^1 = v_{j1}^2 = 0$ and $v_{i2}^1 = v_{i2}^2 = 0$. By strict submodularity of the cost function, either $x_i(v^1) < C_1$ or $x_j(v^1) < C_2$. Suppose $x_i(v^1) < C_1$. Let $\bar{v}_i$ be an announcement for agent i such that $\bar{v}_{i1} > \max[v_{i1}^1, v_{i1}^2, C_1]$ and $\bar{v}_{i2} = 0$. Strategyproofness, No Subsidy and Voluntary Participation imply that $q_{i1}(\bar{v}_i, v_j^1) = 1$ and $x_i(\bar{v}_i, v_j^1) = x_i(v^1)$. Since $x_i(v^1) < C_1$ and the mechanism satisfies Voluntary Participation and Budget Balance, it follows that $q_{j2}(\bar{v}_i, v_j^1) = 1$ and $x_j(\bar{v}_i, v_j^1) = x_j(v^1)$. Let

$\bar{v}_j$ be an announcement by agent j such that $\bar{v}_{j2} > \max[v_{j2}^1, v_{j2}^2]$ and $\bar{v}_{j1} = 0$. Again, strategyproofness implies that $q_{j2}(\bar{v}_i, \bar{v}_j) = 1$ and $x_j(\bar{v}_i, \bar{v}_j) = x_j(\bar{v}_i, v_j^1) = x_j(v^1)$. By Consumer Sovereignty, $q_{i1}(\bar{v}_i, \bar{v}_j) = 1$. Thus, by Budget Balance, it follows that $x_i(\bar{v}_i, \bar{v}_j) = C_{i2} - x_j(\bar{v}_i, \bar{v}_j) = C_{i2} - x_j(v^1) = x_i(v^1)$.

Now, one can start from v^2 and using the same logic as in the previous paragraph, conclude that $x_l(\bar{v}_i, \bar{v}_j) = x_l(v^2)$ for all l . It follows now that $x_l(v^1) = x_l(v^2)$ for all l .

Case 3: $q_{ik}(v^j) = 1, q_{il}(v^j) = 0, k \neq l$ for all i and j .

Here, we shall provide a direct proof that $x_i(v) = C_k/2$ for all i at any profile v such that $q_{ik}(v) = 1, q_{il}(v) = 0, k \neq l, i = 1, 2$. By Step 2 and Outcome Non-Bossiness, we may assume that $v_{il} = 0$ for all i . Suppose that $x_1(v) \neq x_2(v)$. Without loss of generality, assume that $x_1(v) < x_2(v)$. By Voluntary Participation, we know that $v_{1k} \geq x_1(v)$ and $v_{2k} \geq x_2(v)$. By Strategyproofness, we can assume that $x_1(v) < v_{1k} < x_2(v) \leq v_{2k}$.⁴ Suppose that agent 1 deviates at v by announcing v_2 . By Strategyproofness, $q_{1k}(v_2, v_2) = 1$ and $x_1(v_2, v_2) = x_1(v)$. By Anonymity, $x_2(v_2, v_2) = x_1(v_2, v_2) = x_1(v)$. Since $x_1(v) < C_k/2$ (this follows from $x_1(v) + x_2(v) = C_k$ and $x_1(v) < x_2(v)$), it follows that Budget Balance is violated at (v_2, v_2) . This contradiction shows that $x_i(v) = C_k/2$ for all i .

We have covered all possible contingencies. This concludes the proof of the lemma. ■

The following corollary follows directly from the above lemma.

Corollary 4.3.1 *Let $F = (Q, x)$ be an acceptable cost sharing mechanism. Then, for all announcements v , $[q_1(v) = q_2(v) \Rightarrow x_1(v) = x_2(v)]$.*

⁴Suppose at the profile v' such that $x_1(v) < v'_{1k} < x_2(v)$, $v'_{1l} = v_{1l} = 0$ and $v'_2 = v_2$, it happens that $q_{1k}(v') = 0$. Since $v'_{1l} = 0$, it follows from No Subsidy and Voluntary Participation that $U_1(q_1(v'), x_1(v')) = 0$. Then, agent 1 can deviate by announcing v_1 ; in this case, $q_{1k}(v) = 1$ and $x_1(v) < v'_{1l}$. This deviation makes agent 1 strictly better off and contradicts strategyproofness. Thus $q_{1k}(v') = 1$ and $x_1(v') = x_1(v)$.

$q_i(v) = (1, 1)$ for all i	$x_i(v) = 8$ for all i	$v_{i1} + v_{i2} \geq 8$ for all i , $v_{ik} \geq 1$ for all i and k .
$q_i(v) = (1, 1), q_j(v) = (1, 0)$	$x_i(v) = 9, x_j(v) = 7$	$v_{i1} + v_{i2} \geq 9, v_{ik} \geq 1$ for all k . $v_{j1} \geq 7, v_{j2} < 1$
$q_i(v) = (1, 1), q_j(v) = (0, 1)$	$x_i(v) = 9, x_j(v) = 7$	$v_{i1} + v_{i2} \geq 7, v_{ik} \geq 1$ for all k . $v_{j1} < 1, v_{j2} \geq 7$
$q_i(v) = (1, 1), q_j(v) = (0, 0)$	$q_i(v) = 16, q_j(v) = 0$	$v_{i1} + v_{i2} \geq 16, v_{j1} < 7, v_{j2} < 7, v_{j1} + v_{j2} < 8$
$q_i(v) = (1, 0)$ for all i	$x_i(v) = 8$ for all i	$v_{i1} \geq 8, v_{i2} < 1$ for all i .
$q_i(v) = (0, 1)$ for all i	$x_i(v) = 8$ for all i	$v_{i2} \geq 8, v_{i1} < 1$ for all i .
$q_i(v) = (1, 0), q_j(v) = (0, 1)$	$x_l(v) = 8$ for $l = i, j$	$v_{i1} \geq 8, v_{i2} < 1, v_{j2} \geq 8, v_{j1} < 1$
$q_i(v) = (0, 0)$ for all i	$x_i(v) = 0$ for all i	All other profiles.

Table 4.3.1: Mechanism for Example 4.3.1.

Proof: Suppose $q_i(v) = (1, 1)$ for all i . Then, it follows from Step 3 of Lemma 4.3.1 that $x_i(v) = C_{12}/2$.

Suppose that $q_{ik}(v) = 1, q_{il}(v) = 0, k \neq l$ for all i . Here, it follows directly from Case 3 of Lemma 4.3.1 that $x_i(v) = C_k/2$ for all i . ■

Recall that in the single good case, we had shown that Strategyproofness and Outcome Non-Bossiness along with Voluntary Participation and No Subsidy implied Weak Group Strategyproofness. Such a result is not true in the multiple excludable goods setting as the following counterexample shows.

Example 4.3.1 Let C be a cost function such that $C_1 = C_2 = 10$, and $C_{12} = 16$. Consider the following mechanism depicted in Table 4.3.1. It can be checked that this mechanism satisfies Strategyproofness, Outcome Non-Bossiness, No Subsidy and Voluntary Participation. However, it is not Weak Group Strategyproof: at the profile v where $v_1 = (0.5, 9)$ and $v_2 = (9, 0.5)$ the mechanism will choose $q_1(v) = (1, 0), q_2(v) = (0, 1)$ and $x_i(v) = 8$ for all i . However the coalition consisting of $\{1, 2\}$ can deviate to v' where $v'_1 = v'_2 = (8, 8)$. In this case, $q_i(v') = (1, 1)$ and $x_i(v') = 8$ for all i , and so both agents are strictly better off.

Before proving our main result, we establish a few preliminary lemmas. Observe that with two goods and two agents, a mechanism can provide sixteen

A.	$q_i(v) = (1, 1)$ for all i	$x_i(v) = C_{12}/2$ for all i
B.	$q_i(v) = (1, 1), q_j(v) = (1, 0), i \neq j$	$x_i(v) = c_{1b}, x_j(v) = c_{2b}$
C.	$q_i(v) = (1, 1), q_j(v) = (0, 1), i \neq j$	$x_i(v) = c_{1c}, x_j(v) = c_{2c}$
D.	$q_i(v) = (1, 1), q_j(v) = (0, 0), i \neq j$	$x_i(v) = C_{12}, x_j(v) = 0$
E.	$q_i(v) = (1, 0)$ for all i	$x_i(v) = C_1/2$ for all i
F.	$q_i(v) = (0, 1)$ for all i	$x_i(v) = C_2/2$ for all i
G.	$q_i(v) = (1, 0), q_j(v) = (0, 0), i \neq j$	$x_i(v) = C_1, x_j(v) = 0$
H.	$q_i(v) = (0, 1), q_j(v) = (0, 0), i \neq j$	$x_i(v) = C_2, x_j(v) = 0$
I.	$q_i(v) = (1, 0), q_j(v) = (0, 1), i \neq j$	$x_i(v) = c_{1n}, x_j(v) = c_{2n}$
J.	$q_i(v) = (0, 0)$ for all i	$x_i(v) = 0$ for all i

Table 4.3.2: Possible Cases with two agents and two goods.

possible combinations of the two goods. Table 4.3.2 illustrates all the possible cases allowing for symmetry.

Remark 4.3.2 Corollary 4.3.1. Consumer Sovereignty and Voluntary Participation imply that in any acceptable cost sharing mechanism, Cases A, B, D, G, H and J necessarily occur. One can see this by illustrating a profile at which that particular case occurs. Consider for instance, Case A. Let $v = (v_1, v_2)$ be a profile such that $v_1 = v_2 = (C_{12}, C_{12})$. It follows from Consumer Sovereignty that $q_i(v) = (1, 1)$ for all i . Thus, v is our desired profile. Next, consider Case B. Let $w = (w_1, w_2)$ be such that $w_1 = (C_{12}, C_{12})$ and $w_2 = (C_1, 0)$. Again, Consumer Sovereignty implies that $q_1(w) = (1, 1)$ and $q_{21}(w) = 1$. If $q_{22}(w) = 1$ as well, then Case A occurs, and from Corollary 4.3.1 it follows that $x_i(w) = C_{12}/2$ for all i . However, since $C_{12}/2 > C_1$, Voluntary Participation is violated for agent 2. Therefore, $q_{22}(w) = 0$ and Case B occurs at the profile w . One can easily confirm that Cases D, G, H and J occur in any acceptable mechanism.

However, Case C need *not* occur in an acceptable mechanism. Suppose that $v_1 = (C_{12}, C_{12})$. Assume also that $v_2 = (0, C_2)$. Consumer Sovereignty implies that $q_1(v_1, v_2) = (1, 1)$. However, since $C_2 > C_{12}/2$, it does not

follow that $q_2(v_1, v_2) = (0, 1)$. It is possible that the mechanism chooses $q_i(v_1, v_2) = (1, 1)$ and $x_i(v_1, v_2) = C_{12}/2$ for all i . This does not violate Voluntary Participation (in contrast to Case B).

Lemma 4.3.2 *Let $F = (Q, x)$ be an acceptable cost sharing mechanism. Then,*

1. *Case A occurs at $v = (v_1, v_2)$ only if $v_{i1} + v_{i2} \geq C_{12}/2$ and $v_{i2} \geq C_{12}/2 - c_{2b}$ for all $i \in N$. If Case C occurs at some profile w , then in addition, it must also be true that $v_{i1} \geq C_{12}/2 - c_{2c}$ for all $i \in N$.*
2. *Case E occurs at $v = (v_1, v_2)$ only if $v_{i1} \geq C_1/2$ and $v_{i2} \leq c_{1b} - C_1/2$ for all $i \in N$.*

Proof: Suppose that Case A occurs at $v = (v_1, v_2)$. Voluntary Participation implies that $q_i(v) = (1, 1)$ only if $v_{i1} + v_{i2} \geq C_{12}/2$. Using Step 1 of Lemma 4.3.1, we may assume that $v_1 = (C_{12}, C_{12})$ so that $q_1(v_1, v'_2) = (1, 1)$ for all v'_2 . If $v_{22} < C_{12}/2 - c_{2b}$, then agent 2 can deviate to Case B by announcing $(C_1, 0)$. This is profitable since $v_{21} + v_{22} - C_{12}/2 < v_{21} - c_{2b}$. By symmetry, $v_{12} \geq C_{12}/2 - c_{2b}$ also. Suppose now that Case C occurs at the profile w . By Step 2 of Lemma 4.3.1, Outcome Non-Bossiness and Consumer Sovereignty, we may assume that $w_1 = (C_{12}, C_{12})$ and $w_2 = (0, C_2)$. If $v_{11} < C_{12}/2 - c_{2c}$, then agent 1 can deviate at v by announcing $(0, C_2)$. This is profitable because $v_{11} + v_{12} - C_{12}/2 < v_{12} - c_{2c}$. Thus, strategyproofness requires $v_{11} \geq C_{12}/2 - c_{2c}$. By symmetry, $v_{21} \geq C_{12}/2 - c_{2c}$.

Suppose that Case E occurs at the profile $v = (v_1, v_2)$. Voluntary Participation implies that $v_{i1} \geq C_1/2$ for all i . By Step 2 of Lemma 4.3.1 and Consumer Sovereignty, we may assume that $v_1 = (C_1, 0)$. Thus, $q_{11}(v_1, v'_2) = 1$ for all v'_2 . Now, if $v_{22} > c_{1b} - C_1/2$ then agent 2 would deviate to Case B by reporting (C_{12}, C_{12}) . This is profitable if $v_{21} + v_{22} - c_{1b} > v_{21} - C_1/2$. Therefore $v_{22} \leq c_{1b} - C_1/2$. By symmetry, $v_{11} \leq c_{1b} - C_1/2$. ■

Lemma 4.3.3 *Let $F = (Q, x)$ be an acceptable cost sharing mechanism. Then, $c_{2b} \geq C_1/2$.*

Proof: Suppose $c_{2b} < C_1/2$. Let $v = (v_1, v_2)$ be an announcement where $v_1 = (C_1, C_{12} - C_1)$ and $v_2 = (C_1, 0)$. Consumer Sovereignty implies that $q_1(v) = (1, 1)$ and $q_{21}(v) = 1$. Since $C_1 < C_{12}/2$, it follows from Voluntary Participation that $q_{22}(v) = 0$. Therefore, $x_1(v) = c_{1b}$ and $x_2(v) = c_{2b}$. Let agent 1 deviate at v by announcing $v'_1 = v_2 = (C_1, 0)$. By Consumer Sovereignty $q_{11}(v'_1, v_2) = q_{21}(v'_1, v_2) = 1$. Since $C_1 < C_{12}/2$, it follows from Voluntary Participation that $q_{12}(v'_1, v_2) = q_{22}(v'_1, v_2) = 0$. Case E now implies that $x_1(v'_1, v_2) = x_2(v'_1, v_2) = C_1/2$. The deviation is profitable for agent 1 since $v_{11} + v_{12} - c_{1b} = C_1 + C_{12} - C_1 - c_{1b} = c_{2b} < C_1/2 = v_{11} - C_1/2$. Thus, Strategyproofness requires $c_{1b} \leq C_{12} - C_1/2$ or $c_{2b} \geq C_1/2$. ■

Lemma 4.3.4 *Let $F = (Q, x)$ be an acceptable cost sharing mechanism. Suppose Case C occurs in the mechanism at the profile v . Let $q_i(v) = (1, 1)$ and $q_j(v) = (0, 1)$. Then, the following are true.*

1. Suppose that $c_{1c} = c_{2c} = C_{12}/2$. Then, $v_{j1} = 0$ and $v_{j2} \geq c_{2c}$.
2. $c_{2c} > C_{12}/2 - C_1$.

Proof: Suppose that $c_{1c} = c_{2c} = C_{12}/2$ and let $v = (v_i, v_j)$ be a profile at which Case C occurs. By Voluntary Participation, $v_{j2} \geq c_{2c}$. Suppose now that $v_{j1} > 0$. By Step 1 of Lemma 4.3.1 and Outcome Non-Bossiness, we can assume that $v_i = (C_{12}, C_{12})$. Let agent j deviate at v by announcing $v'_j = (C_{12}, C_{12})$. By Consumer Sovereignty and Anonymity, $q_j(v_i, v'_j) = (1, 1)$ and $x_j(v_i, v'_j) = C_{12}/2$. The deviation is strictly profitable for agent j since $v_{j1} > 0$. Thus, Case C can occur only when $v_{j1} = 0$.

Now, suppose that $c_{2c} \leq C_{12}/2 - C_1$. By Step 1 of Lemma 4.3.1 and Outcome Non-Bossiness, we can assume that $v_i = (C_{12}, C_{12})$. Using Step 2 of Lemma 4.3.1 and Outcome Non-Bossiness, we may assume that $v_j = (0, x)$ where x is some number such that $x \geq c_{2c}$. By submodularity of the cost

function, $C_1 > C_{12} - C_2$. Therefore, $c_{2c} \leq C_{12}/2 - C_1 < C_2 - C_{12}/2$. Consider a profile $w = (w_1, w_2)$ such that $w_1 = (C_{12}, C_{12})$ and $w_2 = (C_{12} - C_2, C_2)$. By Consumer Sovereignty and Corollary 4.3.1, $q_l(w) = (1, 1)$ and $x_l(w) = C_{12}/2$ for all l . Agent j 's net utility is thus $C_{12} - C_2 + C_2 - C_{12}/2 = C_{12}/2$. Suppose that agent j deviates at w by announcing $w'_j = (0, x)$. Then, as shown before, Case C occurs, and j 's net utility is $C_2 - c_{2c} > C_2 + C_{12}/2 - C_2 = C_{12}/2$. Agent j is strictly better off from the deviation. This violates Strategyproofness and proves the lemma. ■

We can now prove the following.

Theorem 4.3.1 *There exists no acceptable cost sharing mechanism $F = (Q, x)$.*

Proof: Assume that $C_1 < C_{12}/2 < C_2$ and that $C_2 < C_{12} - C_1/2$. From Lemma 4.3.1 we know that the mechanism is a cost sharing rule. Table 4.3.2 illustrates all possible cases allowing for symmetry. The cost shares for the cases where they have been specified explicitly have been computed using Anonymity and Corollary 4.3.1.

We already know from the previous lemmas that $c_{2b} \geq C_1/2$ and $c_{2c} \geq C_{12}/2 - C_1$. In addition, $c_{1n} \leq C_1$ (by Consumer Sovereignty) so that $c_{2n} \geq C_{12} - C_1 > C_{12}/2$.

We have the following cases.

Case 1: $c_{2b} < C_1$.

Let $v = (v_1, v_2)$ be a profile such that $v_{11} = c_{2b} + \epsilon + \delta$, $v_{12} = C_{12}/2 - c_{2b} - \epsilon$, $v_{21} = (C_{12} - C_2)/2 + \epsilon + \delta$, $v_{22} = C_2/2 - \epsilon$ where $0 < \epsilon < (C_1 + C_2 - C_{12})/2$. The value of $\delta > 0$ is chosen so that

$$\delta < \begin{cases} \min[c_{1b} - C_{12}/2, c_{1c} - C_{12}/2] & \text{if } c_{1c} > C_{12}/2 \\ c_{1b} - C_{12}/2 & \text{otherwise} \end{cases}$$

We now verify that for ϵ and δ small, $q_i(v) = (0, 0)$ for all i . Indeed, Case A cannot occur since that requires $v_{12} \geq C_{12}/2 - c_{2b}$ by Lemma 4.3.2. Cases B

and D cannot occur since $v_{i1} + v_{i2} < c_{1b}$ for all i . Case E cannot occur because $v_{21} < C_1/2$ for small ϵ and δ . Case F cannot occur because $v_{22} < C_2/2$. Case G cannot occur because $v_{11} < C_1$ for small ϵ and δ and $v_{21} < C_1/2$. Case H cannot occur because $v_{12} < C_{12}/2 < C_2$ and $v_{22} < C_2/2$. Case I cannot occur because $v_{i2} < C_{12}/2$ for all i (note that $c_{2n} > C_{12}/2$). Consider now Case C. If Case C occurs at v , then since $v_{i1} > 0$ for all i , it follows from Lemma 4.3.4 that $c_{1c} > C_{12}/2 > c_{2c}$. However, the restriction on δ imply that $v_{i1} + v_{i2} < c_{1c}$ for all i and so Case C cannot occur.

Let agent 1 deviate at v by announcing v'_1 such that $v'_{11} + v'_{12} = C_{12}/2 + \delta$, $C_{12}/2 > v'_{12} > C_{12}/2 - c_{2b}$ and $v'_{11} < C_1$ is chosen so that

$$v'_{11} > \begin{cases} C_{12}/2 - c_{2c} & \text{if Case C occurs in the mechanism} \\ 0 & \text{otherwise} \end{cases}$$

Observe that by Lemma 4.3.4, it is possible to select v'_{11} such that $C_{12}/2 - c_{2c} < v'_{11} < C_1$. Also, observe that $C_{12}/2 - c_{2c} + C_{12}/2 - c_{2b} < \min[c_{1b}, c_{1c}]$. Indeed, $C_{12}/2 - c_{2c} + C_{12}/2 - c_{2b} \geq c_{1b}$ implies that $c_{2c} \leq 0$ which is a contradiction since we have shown that $c_{2c} \geq C_{12}/2 - C_1$. Thus, $C_{12} - c_{2b} - c_{2c} < c_{1b}$. A similar argument also shows that $C_{12} - c_{2b} - c_{2c} < c_{1c}$. Thus, our choice of v'_1 is justified. The value of δ is chosen in the same way as before. Let $w = (w_1, w_2)$ where $w_1 = v'_1$, $w_2 = v_2$.

One can check that Cases B to I cannot occur at w . Indeed, Cases B and D will not occur because $w_{i1} + w_{i2} < c_{1b} < C_{12}$ for all i . Case E will not occur since $w_{21} < C_1/2$ for small ϵ and δ . Case F will not occur since $w_{22} < C_2/2$. Case G will not occur since $w_{i1} < C_1$ for all i . Case H will not occur since $w_{i2} < C_2$ for all i . Case I will not occur since $w_{i2} < C_{12}/2 < c_{2n}$ for all i . Now, consider Case C. Suppose Case C occurs at w . Since $w_{i1} > 0$ for all i , it follows from Lemma 4.3.4 that $c_{1c} > C_{12}/2 > c_{2c}$. However, since $w_{i1} + w_{i2} < c_{1c}$ for all i by our choice of δ , it follows that Case C cannot occur at w .

If Case A occurs at w , then Strategyproofness is violated at v . So, suppose that Case A does not occur at w and $q_i(w) = (0, 0)$ for all i . Let player 2

deviate at w by announcing $w'_2 = (C_{12}, C_{12})$. Consumer Sovereignty implies that $q_2(w_1, w'_2) = (1, 1)$. We shall now show that $q_1(w_1, w'_2) = (1, 1)$ also.

Suppose that $q_1(w_1, w'_2) \neq (1, 1)$. If $q_1(w_1, w'_2) = (0, 0)$ then 1 will also announce w'_2 . By Anonymity and Consumer Sovereignty, $q_i(w'_2, w'_2) = (1, 1)$ and $x_i(w'_1, w'_2) = C_{12}/2$ for all i . Thus, 1's deviation at (w_1, w'_2) is strictly beneficial which contradicts strategyproofness. Suppose $q_1(w_1, w'_2) = (1, 0)$. Then, by Case B, $x_1(w_1, w'_2) = c_{2b}$ and 1's net utility is $w_{11} - c_{2b} = v'_{11} - c_{2b}$. If 1 deviates by announcing w'_2 , then by Anonymity and Consumer Sovereignty, $x_1(w'_2, w'_2) = C_{12}/2$. The deviation is profitable because $v'_{12} > C_{12}/2 - c_{2b}$. This contradicts strategyproofness. A similar argument takes care of the case where $q_1(w_1, w'_2) = (0, 1)$. Thus, $q_i(w_1, w'_2) = (1, 1)$ and $x_i(w_1, w'_2) = C_{12}/2$.

Since $w_{21} + w_{22} = v_{21} + v_{22} = C_{12}/2 + \delta > C_{12}/2$, it follows now that 2's deviation at w is strictly beneficial. Thus, Case A must occur at w . We have a contradiction since we started with the assumption that Case A does not occur at w .

Case 2: $c_{2b} = C_1$

Let $v = (v_1, v_2)$ be such that $v_{i1} = C_1/2 + \epsilon$ for all i , $0 < v_{12} < C_{12}/2 - C_1 < C_2/2$, $v_{11} + v_{12} < C_{12}/2$, $C_{12} - 3C_1/2 < v_{22} < C_2$, $v_{21} + v_{22} < C_{12}$. Again, one can confirm that $q_i(v) = (0, 0)$ for all i for small ϵ . Indeed, Case A cannot occur since $v_{11} + v_{12} < C_{12}/2$. Case B cannot occur since $v_{11} + v_{12} < C_{12}/2 < c_{1b}$ and $v_{11} = C_1/2 + \epsilon < C_1$ for small ϵ . Case D cannot occur because $v_{i1} + v_{i2} < C_{12}$ for all i . Since $v_{22} > c_{1b} - C_1/2 = C_{12} - 3C_1/2$, Case E cannot occur by Lemma 4.3.4. Case F cannot occur since $v_{12} < C_2/2$. Case G cannot occur since $v_{i1} < C_1$ for all i when ϵ is small enough. Case H cannot occur because $v_{i2} < C_2$ for all i . Next, consider Case I. Suppose Case I occurs at the profile v . First assume that $q_1(v) = (1, 0)$ and $q_2(v) = (0, 1)$. Then, $c_{1n} \leq v_{11} = C_1/2 + \epsilon$ by Voluntary Participation. By Budget Balance, $c_{2n} \geq C_{12} - C_1/2 - \epsilon$. Voluntary Participation implies that $v_{22} \geq c_{2n} \geq C_{12} - C_1/2 - \epsilon$. However, $v_{22} < C_2$. We have a contradiction if $C_2 < C_{12} - C_1/2 - \epsilon$ or $\epsilon < C_{12} - C_1/2 - C_2$. Since $C_2 < C_{12} - C_1/2$,

a value of ϵ satisfying the above inequality can always be found. The other possibility, that is, $q_1(v) = (0, 1)$ and $q_2(v) = (1, 0)$ cannot occur because $v_{12} < C_2/2 < C_{12}/2 \leq c_{2n}$. Finally, consider Case C. Suppose that Case C occurs at v . Since $v_{i1} > 0$ for all i , it follows from Lemma 4.3.4 that $c_{1c} > C_{12}/2 > c_{2c}$. Since $v_{12} < C_{12}/2 - C_1$ (note that $c_{2c} > C_{12}/2 - C_1$ by Lemma 4.3.4) and $v_{11} + v_{12} < C_{12}/2 < c_{1c}$, it follows that Case C cannot occur.

Suppose now that agent 2 deviates at v by announcing $v'_2 = (C_1, 0)$. By Consumer Sovereignty $q_{21}(v_1, v'_2) = 1$. Routine inspection shows that Cases A, B, C, D, F, H, I and J cannot occur. If Case E occurs, then $x_i(v_1, v'_2) = C_1/2$ for all i , and it follows that Strategyproofness is violated at v . So, suppose that Case G occurs at (v_1, v'_2) . Let agent 1 deviate at (v_1, v'_2) by also announcing $v'_2 = (C_1, 0)$. By Consumer Sovereignty, Anonymity and the fact that $C_1 < C_{12}/2$, it follows that $q_i(v'_2, v'_2) = (1, 0)$ and $x_i(v'_2, v'_2) = C_1/2$ for all i . The deviation is profitable for agent 1 since $\epsilon > 0$. This violates Strategyproofness. We thus have an unavoidable contradiction. ■

This concludes the proof of the theorem.

Note that we used the restriction $C_1 < C_{12}/2 < C_2$ in proving the above theorem. The following example shows that if $C_k \geq C_{12}/2$ for $k = 1, 2$, then there exists a mechanism satisfying all the axioms of the above theorem.

Example 4.3.2 Consider the following mechanism depicted in Table 4.3.3. It is easy to check that this mechanism - which is just a small modification of the familiar “ascending bid auction” mechanism - is strategyproof. This is seen most conveniently through the following algorithm:

1. If $v_{i1} + v_{i2} \geq C_{12}/2$ for all i , then $q_i(v) = (1, 1)$ and $x_i(v) = C_{12}/2$ for all i . Otherwise set $N_1 = \{i \in N | v_{i1} + v_{i2} \geq C_{12}/2\}$. Set $q_j(v) = (0, 0)$, $x_j(v) = 0$ for $j \notin N_1$. If $N_1 = \emptyset$, stop.⁵

⁵Clearly, N_1 can at best be a singleton set.

$q_i(v) = (1, 1)$ for all i	$x_i(v) = C_{12}/2$ for all i	$v_{i1} + v_{i2} \geq C_{12}/2$ for all i .
$q_i(v) = (1, 0), q_j(v) = (0, 0)$	$x_i(v) = C_1, x_j(v) = 0$	$v_{i1} \geq C_1, v_{i2} < C_{12} - C_1,$ $v_{j1} + v_{j2} < C_{12}/2$
$q_i(v) = (0, 1), q_j(v) = (0, 0)$	$x_i(v) = C_2, x_j(v) = 0$	$v_{i2} \geq C_2, v_{i1} < C_{12} - C_1,$ $v_{j1} + v_{j2} < C_{12}/2$
$q_i(v) = (1, 1), q_j(v) = (0, 0)$	$x_i(v) = C_{12}, x_j(v) = 0$	$v_{i1} + v_{i2} \geq C_{12}, v_{ik} \geq C_{12} - C_l, k, l = 1, 2,$ $k \neq l, v_{j1} + v_{j2} < C_{12}/2$
$q_i(v) = (0, 0)$ for all i	$x_i(v) = 0$ for all i	All other profiles.

Table 4.3.3: Mechanism for Example 4.3.2.

2. Let $i \in N_1$. If $v_{i1} + v_{i2} \geq C_{12}$ and $v_{ik} \geq C_{12} - C_l, k, l = 1, 2, k \neq l$, then set $q_i(v) = (1, 1), x_i(v) = C_{12}$. If $v_{i1} \geq C_1$ and $v_{i2} < C_{12} - C_1$, then set $q_i(v) = (1, 0)$ and $x_i(v) = C_1$. Similarly, if $v_{i2} \geq C_2$ and $v_{i1} < C_{12} - C_2$, set $q_i(v) = (0, 1)$ and $x_i(v) = C_2$. For other profiles, set $q_i(v) = (0, 0)$ and $x_i(v) = (0, 0)$.

It is easy to see that at no stage can any agent deviate to make herself better off. The other axioms can be verified easily.

This mechanism does *not* work if $C_1 < C_{12}/2 < C_2$ because it violates Consumer Sovereignty. If agent 1 reports $v_1 = (C_1, 0)$ and agent 2 reports $v_2 = (0, C_2)$, then Consumer Sovereignty would imply that $q_{11}(v_1, v_2) = q_{22}(v_1, v_2) = 1$. However, note that since $C_1 < C_{12}/2$, agent 1 would get “knocked off” at the first stage of the algorithm. Thus, $q_1(v_1, v_2) = (0, 0)$. This violates Consumer Sovereignty.

4.4 Conclusion

In this chapter, we have considered an extension of the model in Chapter 2 to allow for the presence of multiple excludable public goods. We have noted that while most of the axioms presented for the single good case continue to hold for the multiple good case, the axioms of Budget Balance and Consumer

Sovereignty have to be recast. We propose a particular way of extending the Consumer Sovereignty axiom to the multiple good case.

Our main result demonstrates that, at least in the special case of two agents, there are no mechanisms satisfying Strategyproofness, Outcome Non-Bossiness, Anonymity, No Subsidy, Voluntary Participation, Budget Balance and Consumer Sovereignty. We also show that, unlike the single good case, Strategyproofness, Outcome Non-Bossiness, No Subsidy and Voluntary Participation do *not* imply Weak Group Strategyproofness.

Two areas of work remain to be explored. One is to see whether there are interesting non-anonymous mechanisms in this context. Moulin [42] has shown the existence of an interesting class of non-anonymous mechanisms for the case where the cost function displays increasing marginal costs. However, as we observed before, the cost function considered here does not have increasing marginal costs. So one cannot directly invoke Moulin's characterization results. The other area concerns the welfare properties of the various Strategyproof mechanisms that one obtains in this context. The idea is to obtain a characterization result along the lines that Moulin and Shenker [44] obtain for the Shapley value rule in the single good case. However, the computations of loss become extremely complicated and we have not been able to obtain any results in this regard.

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