INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2025-26 (First Semester)
B. Stat. I Year **Vectors & Matrices-I**
Date: 08/09/2025 Maximum Marks: 30 Duration: 2 Hours

Note: Give proper justification to all your answers.
State clearly all the results you are using.

Notation.

null T: the null space of a linear map T
$B_{j,\cdot}$: the jth row of a matrix B
$\mathcal{P}_m(F)$: the vector space of polynomials of degree less than or equal to m with coefficients in F over the field F

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(1) (a) Suppose

$$U = \{(x, y, x, y) : x, y \in \mathbb{R}\}.$$

Find a subspace W of $\mathbb{R}^4$ such that $\mathbb{R}^4 = U + W$.

(b) Let F be a field and S be the subspace of F^5 defined by

$$S = \{(x_1, x_2, x_3, x_4, x_5) \in F^5 : 2x_1 = x_3 \text{ and } x_2 + 3x_4 - x_5 = 0\}.$$

Find a basis of S.

(c) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear map such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 3x_2 \text{ and } x_2 + 4x_4 = 0\}.$$

Is T onto (surjective)?

(d) Let $a = (a_1\ a_2\ \cdots\ a_n)$ be a $1 \times n$ matrix and B be an $n \times k$ matrix. Show that

$$aB = a_1 B_{1,\cdot} + a_2 B_{2,\cdot} + \cdots + a_n B_{n,\cdot}.$$

(e) Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}.$$

Show that A has a right inverse and find <u>two</u> right inverses of A. **[10]**

(2) (a) Let V be a vector space over a field F and $x_1, x_2, \ldots, x_n \in V$. For $k = 1, 2, \ldots, n$, define $y_k = x_1 + x_2 + \cdots + x_k$. Show that $\{x_1, x_2, \ldots, x_n\}$ is linearly independent if and only if $\{y_1, y_2, \ldots, y_n\}$ is linearly independent.

(b) Let F be a field and m be a positive integer. Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}_m(F)$ are such that each p_k has degree k. Prove that $\{p_0, p_1, \ldots, p_m\}$ is a basis of $\mathcal{P}_m(F)$. **[6+4]**

(3) (a) Let A be an $m \times n$ matrix with rank r and $1 \le k \le r$. Show that A has a $k \times k$ submatrix with rank k.

(b) Let M be a square matrix with complex entries. Show that there exists $\alpha \in \mathbb{C}$ such that $\alpha I + M$ is invertible. **[4+4]**

[*Hint. A strictly diagonally dominated complex matrix is invertible.*]

(4) For $x \in \mathbb{R}$, let

$$p_1(x) = 1, \quad p_2(x) = x, \quad p_3(x) = x^2,$$

and

$$q_1(x) = 1 + x, \quad q_2(x) = 2 + x, \quad q_3(x) = 1 + x + x^2.$$

(a) Let $f(x) = \alpha + \beta x + \gamma x^2$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Express f as a linear combination of q_1, q_2 and q_3.

(b) Show that $\{q_1, q_2, q_3\}$ is a basis of $\mathcal{P}_2(\mathbb{R})$.

(c) Let $V = W = \mathcal{P}_2(\mathbb{R})$. Consider the bases $\mathcal{B} = \{p_1, p_2, p_3\}$ and $\mathcal{B}' = \{q_1, q_2, q_3\}$ of V and W respectively. Let $T : V \to W$ be the linear map

$$(Tp)(x) = 2xp'(x) - p(x), \quad p \in \mathcal{P}_2(\mathbb{R}).$$

Compute the matrix of T with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$.

[2+2+4]