INDIAN STATISTICAL INSTITUTE
B. Stat. (H) 2024-2025
Special Topics on Algorithms
Semestral Examination

Date: 25. 04. 2025 | Marks: 100 | Time: 3 Hours

Answer any five questions. Each question carries 20 marks.
Write the part answers of each question at the same place.

1. Outline efficient deterministic classical algorithms and deduce the time complexity of such algorithms for the following problems.
   (a) To find the minimum as well as the maximum from an unsorted array of n positive integers.
   (b) To find the maximum and the second maximum from an unsorted array of n positive integers.
   (c) To find the number of 1's in the bit representation of an unsigned integer represented with n bits.
   (d) To calculate the Greatest Common Divisor of two integers p, q, where each of them is at most n-bit long.

   $[5 + 5 + 5 + 5 = 20]$

2. Consider an undirected graph $G = (V, E)$, with $|V| = n$.
   (a) Can this problem be solved in a classical computer with a time complexity polynomial in n? Provide a complexity theoretic argument.
   (b) Write an algorithm (may be exponential in n) to obtain a vertex cover of G, having the minimum cardinality (say q). Deduce the time complexity of your algorithm.
   (c) Write an algorithm (polynomial in n) to obtain a vertex cover of size αq of G, where α is a constant and q is as above. Explain the time complexity of your algorithm.

   $[8 + (3 + 3) + (3 + 3) = 20]$

3. Consider an n-input 1-output Boolean function f.
   (a) Define the Walsh Spectrum of f.
   (b) How to calculate the Walsh spectrum from the truth table? Provide an algorithm and explain the time complexity with step-by-step demonstration for $n = 4$ taking a specific Boolean function of your choice (say g).
   (c) How can you check whether a Boolean function is balanced given it's Walsh spectrum? Is the function g balanced?
   (d) How the nonlinearity can be defined given the Walsh spectrum? What is the nonlinearity of g?

   $[2 + (5 + 3) + (2 + 2) + (3 + 3) = 20]$

4. Consider an n-input 1-output Boolean function f, which is either constant or balanced. You can only query the function by providing an input to obtain the output.
   (a) What is the worst case query complexity to deduce the result in a classical computer? How a randomised algorithm will perform in such a scenario?
   (b) Explain in detail how the Deutsch-Jozsa algorithm can be applied to solve this problem in a quantum set-up. How the internal state before the measurement is related to the Walsh spectrum of f?

   $[(3 + 3) + (10 + 4) = 20]$

5. Consider an n-input n-output Boolean function F such that there exists only a single non-zero $s \in \{0,1\}^n$ for which $F(x \oplus s) = F(x)$, $\forall x \in \{0,1\}^n$. You can only query the function by providing an input to obtain the output.

   (a) Explain in detail how the Simon's algorithm can be applied to solve this problem in a quantum set-up.

   (b) Provide an expected estimate of how many times the quantum algorithm has to be executed.

   (c) Describe what is the classical part of the algorithm to obtain the final result from the set of outcomes after the quantum part.

   [10 + 5 + 5 = 20]

6. (a) Consider the single qubit state:

$$|\psi\rangle = \frac{\sqrt{7}}{2\sqrt{3}}|0\rangle - \frac{2+i}{2\sqrt{3}}|1\rangle .$$

   Determine the probabilities when measured in the $\{|+\rangle, |-\rangle\}$ basis.

   (b) Prove that an entangled state remains entangled irrespective of the choice of basis.

   (c) Explain how the Grover's algorithm can amplify the amplitude of a marked state.

   [5 + 5 + 10 = 20]

7. (a) Let $P = p_1, p_2, \ldots, p_n$ be a set of n points distributed on a 2D plane. The objective is to test whether the convex hull of P is a quadrilateral or not. Write a linear time implementable algorithm (with pseudo-code) to solve this problem. Justify the time complexity of your algorithm.

   (b) Consider the planar map $VOR(P)$ induced by the Voronoi diagram of a point set $P = \{p_1, p_2, \ldots, p_n\}$. The objective is to report the nearest point $p \in P$ that is closest to a query point q. Suggest a data structure to store $VOR(P)$ in the memory such that the said query can be answered in $O(\log n)$ time. You are allowed to use space that is polynomial in n, the size of P. The preprocessing time is also ensured to be polynomial in n. Justify the time and space complexities of your proposed method.

   [(4 + 3) + (7 + 6) = 20]

8. (a) What is the objective of the art gallery problem? State (only) the computational status of the art gallery problem.

   (b) State the upper bound on the number of guards required to watch an arbitrary shaped simple polygon in 2D. Justify that the proposed upper bound is achievable, i.e., there exists some simple polygon, where these many guards are necessary.

   (c) A star shaped polygon is a polygon in 2D with non-empty kernel, where the kernel of a polygon is a region where from every point inside the polygon is visible.
   (i) Show that the kernel of a star-shaped polygon is a convex region.
   (ii) Formulate a method of finding the minimum number of guards to cover the walls of an art gallery (an arbitrary simple polygon) through the *Set Cover* problem as follows:
   Given a set $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ of elements and a set $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of subsets of $\mathcal{U}$, i.e., $S_i \subseteq \mathcal{U}$ for each $i = 1, 2, \ldots, m$, the objective is to choose the minimum number of subsets of $\mathcal{S}$ such that their union is $\mathcal{U}$.

   [5 + 6 + 9 = 20]