INDIAN STATISTICAL INSTITUTE

B.Stat. III Year: 2025-26 (Semester I)

Parametric Inference: End-Semester Exam

Date: 25/11/2025 Duration: 3 hours Full Marks: 50

Note: Show necessary steps to solve all of the following problems. There is no credit for a solution if the appropriate work is not shown even if the answer is correct. The total marks allotted in this question paper are 55. However, if a candidate scores x marks out of 55, the final score will be recorded as $\min\{x, 50\}$. Here, $n > 2$ is a natural number and $\mathbb{R}$ is the set of real numbers. $\mathbb{I}_{\{A\}}$ denotes the indicator variable of an event A. Use of calculator is allowed in this exam.

---

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with probability density function

$$f(x|\theta) = \frac{e^{x-\theta}}{1 - e^{-\theta}} \mathbb{I}_{\{0<x<\theta\}}, \qquad \text{where } \theta > 0.$$

   (a) Show that $T = X_{(n)}$ is a minimal sufficient statistic for θ. [3]

   (b) Determine whether $\sin(T)$ is a sufficient statistic for θ. Justify your answer. [3]

2. A bivariate random vector (X, Y) follows uniform distribution on the square region with four vertices $(\theta, 0), (0, \theta), (-\theta, 0)$ and $(0, -\theta)$, where $\theta > 0$. Suppose that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are n independent copies of (X, Y).

   (a) Find a complete sufficient statistic T_n for θ. [3]

   (b) Derive the uniformly minimum variance unbiased estimator δ_n of $\psi(\theta) = \theta^2$. [2]

   (c) Suggest a pivotal quantity for θ based on T_n obtained in (a). Justify your answer. [2]

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables with density

$$f(x|\theta) = \frac{1}{2\theta} \exp\left(-\frac{x-\theta}{2\theta}\right) \mathbb{I}_{\{x \geq \theta\}}, \text{ where unknown parameter } \theta > 0.$$

   (a) Prove or disprove whether $T_1 = X_{(1)}$ is independent of $T_2 = \sum_{i=1}^{n}(X_{(n)} - X_i)$. [4]

   (b) Derive the maximum likelihood estimator δ_n of θ and determine whether δ_n is a consistent estimator of θ. [3 + 3 = 6]

4. Suppose X is a single observation with density $f_\theta(x) = (1-\theta) f_0(x) + \theta f_1(x)$, where $\theta \in \{0, 1\}$, $x > 0$.

$$f_0(x) = \mathbb{I}_{\{0 < x < 1\}} \text{ and } f_1(x) = \frac{2}{3}\,\mathbb{I}_{\{0 < x \le 1/2\}} + \frac{4}{3}\,\mathbb{I}_{\{1/2 < x < 1\}}.$$

Derive (i) a *randomized* and (ii) a *non-randomized* most powerful test of size $\alpha = 0.1$ for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. [3 + 3 = 6]

5. Suppose that X has density $f_\theta(x) = \dfrac{\theta}{(\theta + x)^2}$ for $x > 0$ and $\theta > 0$. Suppose that $\theta_0 > 0$ and $\alpha \in (0, 1)$ are prefixed.

   (a) Construct a uniformly most powerful test $\phi^\star$ of size α for testing $H_0 : \theta \le \theta_0$ against $H_1 : \theta > \theta_0$. Justify your construction. [3]

   (b) Compute the p-value of $\phi^\star$. [2]

   (c) Consider testing $H_0 : \theta = \theta_0$ against $H_1 : \theta < \theta_0$. Prove that $\mathbb{E}_\theta[\phi^\star] \le \mathbb{E}_\theta[\phi]$ for all $\theta < \theta_0$ and for any test ϕ satisfying $\mathbb{E}_{\theta_0}[\phi] = \alpha$. [4]

6. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having the distribution with density function

$$f(x|\mu) = \frac{1}{x\sqrt{2\pi}} \exp\left[-\frac{(3\log x + 2\mu)^2}{18}\right] \text{ where } x > 0 \text{ and } \mu \in \mathbb{R}.$$

Suppose $\mu_0 \in \mathbb{R}$ and $\alpha \in (0, 1)$ are prefixed.

   (a) Derive the uniformly most powerful unbiased (UMPU) test of size α for testing [4]

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \ne \mu_0.$$

   (b) What is the definition of a uniformly most accurate unbiased (UMAU) $(1-\alpha)$ confidence interval for μ? [2]

   (c) If M denotes the median of the above distribution with density $f(x|\mu)$, then construct a UMAU $(1-\alpha)$ confidence interval for M. Justify your answer. [4]

7. Suppose that $X_1, \ldots, X_n$ are i.i.d. Uniform$(0, \theta)$, and that θ has a Pareto(c, s) prior distribution with density function

$$\pi(t) = \frac{sc^s}{t^{s+1}}\,\mathbb{I}_{\{t > c\}} \text{ where } s > 0 \text{ and } c > 0 \text{ are known constants.}$$

   (a) Derive the posterior distribution of θ. [3]

   (b) Find the Bayes estimator of θ under absolute error loss function $L(\theta, \delta) = |\delta - \theta|$. [2]

   (c) For prefixed $\alpha \in (0, 1)$, derive the $(1-\alpha)$ level highest posterior density (HPD) credible interval for θ. [2]