On Univariate Process Capability Indices: Some Issues and Applications

A Thesis Submitted in Partial Fulfilment of the Requirements for the Award of the Degree of Doctor of Philosophy in Quality, Reliability & Operations Research

BY

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Declaration

I do hereby declare that this thesis entitled **"On Univariate Process Capability Indices: Some Issues and Applications"** is a bonafide record of the research work carried out by me during my fellowship tenure and that the thesis has not previously formed the basis for the award of any degree, diploma, associateship, fellowship or other similar title or recognition of any University or Society. I declare, to the best of my knowledge and belief, that this work contains no materials previously published by any other person, except where due references are made within the thesis.

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This is to certify that the thesis entitled **"On Univariate Process Capability Indices: Some Issues and Applications**" being submitted by **Kuntal Bera** to the Indian Statistical Institute Kolkata, India, for the award of the degree of Doctor of Philosophy in Quality, Reliability & Operations Research is a record of bonafide research work carried out by him under my supervision in the department of SQC & OR, Kolkata. To the best of my knowledge, the results presented in this thesis have not been submitted for the award of any other degree or diploma anywhere. In my view, the thesis meets the necessary criteria for being awarded the degree.

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Dedicated To

My Beloved Parents

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Chapter 1

Introduction

'*Improve quality, you automatically improve productivity.*'

– ***W. Edwards Deming***

QUALITY is defined as the '*fitness for use*' of a product according to Juran [59]. It has become one of the most significant factors influencing consumer decisions when selecting among competing products and services. The only principle that customers and producers both value is quality. The producer always attempts to create defect-free products that conform to preset specifications, while the consumer seek products that maximize her satisfaction at the lowest cost. Therefore, consistently producing high-quality items at a reasonable cost is essential for survival in today's competitive market. To ensure consistent quality, it is essential to properly monitor the manufacturing process. In today's highly competitive market, accurately assessing a process's ability to produce items within specified limits is increasingly vital. Enhancing quality is essential for achieving business success and growth while building a strong competitive position. Statistical quality control (SQC) methods are commonly used for these purposes.

The concept of SQC was first introduced by Shewhart [102] in his book, '*Economic Control of Quality of Manufactured Product*'. Shewhart established principles for monitoring a process and developed the concept of control charts that are still in use today. Shewhart's principles and tools have led to various methods for monitoring and controlling processes, which are essential to today's quality revolution in industry. SQC is one of the most crucial applications of statistical methods in the industry. It consists of various statistical tools utilized by quality professionals, primarily based on the principles of probability and sampling.

Process measurements are assumed to be connected to process quality. Process observations provide a measure of some quality characteristic that reflects the fitness for use of the end product. Generally, the quality characteristics have a '*target value*' representing an ideal state, and deviations from this target indicate reduced quality. In reality, every real process shows some form of variation. This variation indicates a loss of quality resulting from deviations from the target value of the quality characteristic. To understand the reasons for a loss of quality in a process, it is useful to partition the total variation of process observations in the process based on its sources of variation.

The American Society for Quality Control (ASQC) advocated using quality improvement techniques for various products and services. Shewhart [102] categorizes process variation sources into two types: chance cause variation and assignable cause variation. Chance cause variability arises from the cumulative effects of many unidentified sources, which cannot be eliminated without excessive costs or redesigning the process. Assignable cause variability arises from identifiable sources that can be systematically recognized and eliminated using various process control methods. A process is considered to be in a state of statistical control if only chance causes of variability are present. At this stage, process variation is minimal because there are no assignable causes of variability present. This indicates that the process is functioning in a predictable state, making it suitable for forecasting its ability to meet the requirements. SQC has developed effective tools, such as control charts, to monitor and stabilize the process after identifying and eliminating all assignable causes.

In addition to the previously mentioned target value, there are specification limits, such as the upper specification limit (USL) and lower specification limit (LSL), associated with the quality characteristics. These specification limits are often set by the process owners or product designers based on customer requirements. The control chart shows if a process is statistically under control, but it does not assess how well the output meets its intended purpose. That is, being in statistical control does not ensure that a process is capable. Even for a statistically controlled process, it may be observed that (i) the process is off-centered with respect to the specified target, (ii) the variability in the process is too large compared to its specifications, and (iii) the process is both off-centered and has a large variation. Thus, stability is a necessary condition of a capable process but not a sufficient one. The capability of a process is typically evaluated using process capability indices (PCIs). A process capability index (PCI) measures how well a process can produce items within predefined specification limits. It is generally a higher-the-better type of index, with the 'high' value indicating that the process is capable of producing items that will, in all likelihood, meet customers' requirements. But, before calculating the PCI of a process, it is crucial to ensure that the process is statistically controlled; without stability, the process becomes unpredictable. In these instances, PCI values may not accurately represent the true capability level of the process.

The importance of measuring the capabilities of a process has been recognized for a long time. The concept of process capability was likely developed by American statisticians; however, the use of process capability indices in the U.S. manufacturing sector did not gain popularity until reports on Japanese manufacturing methods were published in American trade and professional journals. Feigenbaum [39] and Juran [59] suggested 6σ as a measure of process capability, where σ is the standard deviation of the corresponding quality characteristic. The logic behind their suggestion is that, under the assumption of normality of the concerned quality characteristic, 6σ represents the inherent variability of a process (refer Palmer and Tsui [86]). This metric operates without considering any specifications of the quality characteristics. Due to the growing interaction between customers and producers, it has become important to incorporate customer specifications into the definition of PCIs.

To incorporate customer specifications into the definition of PCI, Juran [60] compared the 6σ to the width of the specification range and thus established a stronger relationship between process variability and customer specifications. Juran and Gryna [61] proposed a metric the so-called '*Capability Ratio*' that directly relates process variability to specification limits, defined by

$$\text{Capability Ratio} = \frac{6\sigma}{\text{Specification Width}} \tag{1.1}$$

Since this capability ratio provides a relationship between process variability and customer specifications, it emphasizes the suppliers' responsibility to meet those specifications (refer Anis [2]). Thus, the capability ratio, defined in equation (1.1) established the background of process capability analysis.

One of the earliest descriptions of capability indices published in America was by Sullivan [107]. Kane [62], in his pioneering paper, first mentioned some process capability indices that already had been utilized in industries for quite some time and discussed the importance of these PCIs to assess process capability. This work of Kane [62] brings an evolution in the area of SPC and inspired many statisticians and industrial engineers to pursue additional research on PCIs due to its unquestionable significance in the context of manufacturing industries. Kotz and Johnson [65] have reviewed about 170 such high-quality research papers published within just 15 years of the publication of Kane [62]. Yum and Kim [129] have enlisted almost 530 research articles and books written during 2000-2009 on univariate and multivariate PCIs and discussed their applications in various areas include acceptance sampling plans, supplier selection, and tolerance design. Yum [127], Yum [128] have enlisted the research articles and books written during 2010-2021 on univariate and multivariate PCIs respectively.

According to Palmer and Tsui [86], almost all the PCIs available in the literature have been developed from their common ancestor C_p, which is defined as

$$C_p = \frac{U - L}{6\sigma} = \frac{d}{3\sigma} \tag{1.2}$$

where, U is the upper specification limit (USL) and L is the lower specification limit (LSL), $d = (U - L)/2$. It is evident from equations (1.1) and (1.2), that C_p is merely the reciprocal of the '*capability ratio*'. In fact, all the PCIs have an advantage over the capability ratio because by definition PCI values are of the '*higher is better*' type. This means as the performance of the process improves, the PCI values will increase accordingly. Although this has limited significance, Palmer and Tsui [86] have rightly observed that this provides psychological relief to PCI users, as any improvement in the level of capability of a process is directly reflected by an increment in the value of the PCI, while the capability ratio has an inverse effect.

PCIs have been developed to assess the capability of processes that are associated with various quality characteristics having different types of specification limits. The specification limits set by the process owner, product designer or other stakeholders, play a major role in the assessment of process capability. Depending upon the customer's requirements, design of intent and application area, quality characteristics are typically categorized into three groups, viz., (i) nominal the best, i.e., quality characteristics with both *USL* and *LSL*; (ii) smaller the better, i.e., quality characteristics with only *USL*; (iii) larger the better, i.e., quality characteristics with only *LSL*.

The type of specification limit that is most commonly observed in manufacturing industries is bilateral, which corresponds to the nominal best quality characteristics. For a specific quality characteristic of an item, an ideal target value is established. Based on this target, a proposed set of lower and upper specification limits is provided. The observed value of a quality characteristic for a produced item should ideally remain within the specified limits. If the item falls outside these specification limits, then it is considered non-conforming to the specifications and will either be scrapped or sent for rework.

Even for nominal the best quality characteristics the specification limits may be either symmetric or asymmetric in relation to the target. A bilateral specification limit is said to be symmetric when both the specification limits are equidistant from the target, i.e., the target value (T) coincides with the midpoint of the specification ($M = (U + L)/2$). Quality characteristics such as height, length, weight, volume, temperature, etc. usually have symmetrical bilateral specification limits. On the contrary, a bilateral specification limit is said to be asymmetric if the target value (T) differs from the specification mid-point, i.e., when the U and the L are not at equal distance from 'T.' Generally, specification limits become asymmetric when, in order to control production cost or due to various design aspects, either the producer or the customer or both intend to approve more deviation from the target on one side of the specification limits than towards the other. Therefore, the quality characteristics of the nominal better type may have either symmetric or asymmetric specification limits, depending on production cost, design intent, and application area.

Unilateral or one-sided specification limits occur when the quality characteristics of interest are either of the '*larger the better*' or '*smaller the better*' types. Quality characteristics like surface roughness and flatness, degree of radiation, noise, pollution etc. are some examples of '*smaller the better type*' in the sense that their values should be as minimal as possible. Hence, only an upper specification limit U is set for such quality characteristics. If the observed quality characteristic of some item falls beyond U, then that item will be removed from the production line. On the other hand, quality characteristics like tensile strength, compressive strength, durability, hardness, longevity etc. are some examples of '*larger the better type*' where the corresponding quality characteristic values should be as high as possible. Only a lower specification limit L is set for such quality characteristics, to maintain the proper quality level of the concerned item.

1.1 Abbreviations, Notations and Terminologies Used in this Thesis

Before we review the extensive literature on the existing process capability indices, let us consider the following abbreviations, notations and terminologies that are often used in process capability studies.

1.1.1 List of Abbreviations and Notations

1. PCI: Process capability index;
2. U: Upper specification limit (USL);
3. L: Lower specification limit (LSL);
4. T: Target value;
5. AR(1): Autoregressive order 1;
6. LCB: Lower confidence bound;
7. Pdf: Probability density function;
8. Cdf: Cumulative distribution function;
9. CP: Coverage probability;
10. AW: Average width;
11. AV: Average value;
12. MSE: Means square error;
13. FGPQ: Fiducial generalized pivotal quantity;
14. FGCI: Fiducial generalized confidence interval;
15. BCI: Bootstrap confidence interval;
16. CI: Confidence interval;
17. SB: Standard bootstrap;
18. PB: Percentile bootstrap;
19. BCPB: Bias corrected percentile bootstrap;
20. Q_p: p-th percentile;
21. n: Sample size;
22. ϕ: Autoregressive parameter;
23. Φ= Cdf of standard normal distribution;
24. $C_{p;a}$= The index C_p in presence of autocorrelation;
25. $C^e_{p;a}$= The index C_p in presence of autocorrelation and measurement error;
26. $C_{pm;a}$= The index C_{pm} in presence of autocorrelation;
27. $C^e_{pm;a}$= The index C_{pm} in presence of autocorrelation and measurement error;

28. $C''_{pp;a}$= The index C''_{pp} in presence of autocorrelation;
29. $C''^{e}_{pp;a}$= The index C''_{pp} in presence of autocorrelation and measurement error;
30. $M = \frac{U+L}{2}$;
31. $d = \frac{U-L}{2}$;
32. $D_u = U - T$;
33. $D_l = T - L$;
34. $A = \max\left\{\frac{d(\mu-T)}{D_u}, \frac{d(T-\mu)}{D_l}\right\}$;
35. $\xi = \frac{\mu-T}{\sigma_X}$;
36. $\tau_a = \frac{\sigma_E}{\sigma_a}$;
37. $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$;
38. $S^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$;
39. $S_n^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}$;
40. $\widetilde{S}^2 = \frac{\sum_{i=1}^{n}(X_i-T)^2}{n}$;
41. $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$;
42. $S_Y^2 = \frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{n-1}$;
43. $S_{n;Y}^2 = \frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{n}$;
44. $\widetilde{S}_Y^2 = \frac{\sum_{i=1}^{n}(Y_i-T)^2}{n}$;
45. $\mathcal{R}_\mu$: FGPQ of the parameter μ;
46. $\mathcal{R}_\sigma$: FGPQ of the parameter σ;
47. $\delta_{pk} = C_{N_{pk1}} - C_{N_{pk2}}$;
48. $\xi_{pk} = C_{N_{pk1}}/C_{N_{pk2}}$;
49. $\delta_{py} = C_{py1} - C_{py2}$;
50. $\mathcal{R}_{C_{N_{pk}}}$: FGPQ of $C_{N_{pk}}$;
51. $\mathcal{R}_{C_{py}}$: FGPQ of C_{py};
52. $\mathcal{R}_{C_L}$: FGPQ of C_L;
53. $\mathcal{R}_{\delta_{pk}}$: FGPQ of δ_{pk};
54. $\mathcal{R}_{\xi_{pk}}$: FGPQ of ξ_{pk};
55. $\mathcal{R}_{Q_p}$: FGPQ of Q_p;
56. $\mathcal{R}_{\delta_{py}}$: FGPQ of δ_{py};

1.1.2 Some Popular Terminologies

Some terminology used throughout this thesis are listed below.

Specification Limits: Specification limits are established from a technical perspective prior to the start of production (during the design phase). These limits are defined to establish the extreme allowable values for ensuring the satisfactory performance of individual units. Specification limits are typically established by either the producer or the customer to ensure quality control for each individual unit. The upper specification limit (USL) refers to the higher value, while the lower specification limit (LSL) indicates the lower value among the specification limits.

Target Value: The ideal value for a quality characteristic is referred to as its nominal or target value. In the case of two-sided symmetric specification limits, the midpoint of the specification serves as the target value. However, the target value is typically provided by the process owner based on its intended purpose.

Proportion of Non-conformance (PNC): An item is considered non-conforming with respect to the specified limits if the corresponding quality characteristic value lies outside the specification limits. The proportion of non-conforming (PNC) is the probability of producing non-conforming items under the given set-up.

Threshold Value: The term 'threshold value' of a process capability index implies a particular value of the index such that a process with a PCI value greater than or equal to this value is considered capable, while a process having a PCI value smaller than this value is considered as incapable. Also, a process whose PCI value coincides with the threshold value, is considered as 'marginally capable'. Thus, the threshold value plays a key role in the justification of any process. Sometimes threshold values are used to make a purchase decision. If the PCI value is greater than the threshold value then the product will be purchased otherwise the product will be rejected.

Estimator: Most of the PCIs are defined as functions of the parameters like population mean (μ) and population standard deviation (σ) of the concerned quality characteristic. However, in practice, often these parameters are not observable and hence are estimated based on the information gathered from random sample(s). In the literature on process capability analysis, the common practice is to define an estimator of a PCI by merely replacing the parameter(s) of the quality characteristics with the corresponding sample statistic(s). The estimators thus obtained are known as natural estimators.

Autocorrelation: Autocorrelation refers to the correlation of a variable with itself over successive time intervals based on related observations. It contradicts the assumption of independence, which is a fundamental principle of most conventional models. Positive autocorrelation between consecutive observations indicates that they tend to have similar values. In negative autocorrelation, successive values tend to be different from one another. The autocorrelation generates patterns that may cause the process to drift over time and increase its variability.

Gauge: A gauge is an instrument used for taking measurements; it often specifically refers to devices used at the shop floor.

Nuisance Parameter & Parameters of Interest: In a statistical study, the parameters on which inferences are made are called the parameter of interest. A nuisance parameter is any parameter that is unspecified but must be accounted for in the inference of the parameters of interest. For example, in a normal distribution with unknown variance, if one wants to perform hypothesis testing on the mean, then the variance will be regarded as a nuisance parameter.

Censoring: Censoring in statistics refers to a condition where the value of a measurement or observation is only partially known. The observation is incomplete because the event of interest was not fully observed during the study period. Censoring occurs frequently in survival analysis, reliability testing, and various other fields where the complete outcome may not be observed for all subjects. There are several types of censoring: right censoring, left censoring, interval censoring, progressive censoring, etc.

1.2 Overview of Some Existing Process Capability Indices

Now we briefly mention some of the most commonly used process capability indices already available in the literature and discuss the motivational background for their proposal.

1.2.1 Univariate Process Capability Indices for Symmetric Specification Limits

Assume that the measurable quality characteristic under consideration, say 'X', follows normal distribution i.e., $X \sim N(\mu, \sigma^2)$. Some classical PCIs, which are most commonly used for symmetric bi-lateral specification limits are,

C_p **Index:** The history of PCIs began with the introduction of the C_p index by Juran and Gryna [61]. Juran recognized the industry's need for a single ratio or index to compare the specification interval with the actual process variation. They defined the index as equation (1.2), which compares the allowable process spread (voice of the customer) to the actual process spread (voice of the process). Since C_p only includes process dispersion (σ) and not process centering (μ), it accurately represents the capability of a process only when that process is centered on the target. Otherwise, it measures the potential capability of that process (Palmer and Tsui [86]), i.e., the level of capability that a process can attain under the present dispersion scenario.

C_{pk} **Index:** The index C_{pk} was introduced to take care of the limitations of the C_p index. A PCI that takes into account the changes in both the process centering (μ)

and process dispersion (σ), was felt necessary by many, and that resulted in the proposal of the index C_{pk} by Kane [62]. C_{pk} is said to be a second-generation index. The index C_{pk} is defined as follows:

$$C_{pk} = \min\left\{\frac{U-\mu}{3\sigma}, \frac{\mu-L}{3\sigma}\right\} = \frac{d-|\mu-M|}{3\sigma}. \tag{1.3}$$

The index C_{pk} and C_p are related by the equation

$$C_{pk} = \left(1 - \frac{|\mu-M|}{d}\right) C_p = (1-k)\, c_p$$

where,

$$k = \frac{|\mu-M|}{d}.$$

C_{pm} **Index:** A good PCI should be able to capture the proximity of the process centering (i.e. μ) towards the target (T). However, both C_p and C_{pk} ignore this factor. To address this issue, Taguchi [109] and Chan et al. [17] studied the problem independently, and C_{pm} was developed that incorporates a target value into its formula. The index proposed by them can be taken as a 'measure of process centering' and it is defined by

$$C_{pm} = \frac{U-L}{6\sqrt{\sigma^2+(\mu-T)^2}} = \frac{d}{3\sqrt{\sigma^2+(\mu-T)^2}}. \tag{1.4}$$

C_{pm} and C_p are related by the equation

$$C_{pm} = \frac{C_p}{\sqrt{1+\left(\frac{\mu-T}{\sigma}\right)^2}}.$$

Boyles [14] has shown that C_{pm}, actually measures the proximity of the process mean towards 'T' and thus provides a measure of evaluating the capability of a process to center around the target properly.

C_{pmk} **Index:** The C_{pmk} index was proposed by Pearn et al. [89] that combines the feature of indices C_p, C_{pk} and C_{pm} and is often referred to as the 'Third generation PCI'. It is defined by the equation,

$$C_{pmk} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}} = \frac{C_{pk}}{\sqrt{1+\left(\frac{\mu-T}{\sigma}\right)^2}}. \tag{1.5}$$

Kotz and Johnson [65] have later admitted that being a mixture of C_{pk} and C_{pm}, C_{pmk} is neither a yield-based PCI like C_{pk} nor does it provide an assessment of process centering like C_{pm}. As a result, C_{pmk} values are not easily interpretable, and hence, this PCI finds very limited applications in practice. Moreover, if the proportion of non-conformance is regarded as the most important feature of a process, then the performances of C_{pm} and C_{pmk} are often found to be unreliable (refer Kotz and Johnson [65]).

Super-structure of PCIs: Vännman [115] unified these PCIs and proposed the following super-structure of PCIs for symmetric bi-lateral specification limits:

$$C_p(u,v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \quad u, v \geq 0. \tag{1.6}$$

For $u = 0, 1$ and $v = 0, 1$, $C_p(u,v)$ gives the four basic PCIs for example, $C_p(0,0) = C_p$, $C_p(1,0) = C_{pk}$, $C_p(0,1) = C_{pm}$ and $C_p(1,1) = C_{pmk}$. Vännman and Kotz [116] have thoroughly studied the distributional properties of $C_p(u,v)$.

The threshold value is crucial to a PCI, particularly in terms of interpretation. A process is defined as capable if the capability index exceeds a threshold value, which can never be less than one. The threshold values of PCIs may vary based on the quality characteristics considered and their respective specification limits. Manufacturing products with high PCI values can help to achieve international standards, a goal of the Six Sigma philosophy and various ISO standards. The production processes are also graded based on the PCI value as mentioned in Table 1.1.

Table 1.1: Classification of processes based on PCI values.

PCI value	Gradation
Less than 1	Inadequate
Greater equal to 1 and less than 1.33	Capable
Greater equal to 1.33 and less than 1.5	Satisfactory
Greater equal to 1.5 and less than 2	Excellent
Greater than 2	Super

Apart from these PCIs, a considerably large number of other PCIs are also available in the literature to address the different challenging situations encountered in practice. Some helpful reviews of such PCIs have been given by Palmer and Tsui [86], Kotz and Johnson [65], Wu et al. [124], Deleryd [29], de Felipe and Benedito [28], Pearn and Kotz [88] among others.

1.2.2 Univariate Process Capability Indices for Unilateral (One-sided) Specification Limits

Interestingly, the majority of studies on PCIs for unilateral specification limits rely on C_{PU} and C_{PL}, which have the following definitions:

$$C_{PU} = \frac{U - \mu}{3\sigma} \qquad \text{and} \qquad C_{PL} = \frac{\mu - L}{3\sigma}. \tag{1.7}$$

The distributional properties of the estimators of C_{PU} and C_{PL} as well as their uniformly minimum variance unbiased estimators (UMVUE), have been thoroughly discussed by Kane [62], Chou and Owen [24] and Pearn and Chen [87], under the assumption that the quality characteristic under consideration follows a normal distribution.

Krishnamoorthi [66] addressed the issue of computing PCIs for processes with unilateral specifications from a different point of view. According to him, the '*ideal condition*' of a process with unilateral specification changes with process variability. He defined $C_{PT} = \frac{\mu - T}{3\sigma}$ and suggested that for pushing a process to its ideal location, one should have $1.0 \leq C_{PT} \leq C_{PL}$. However, it should be noted that a high PCI value may be obtained by simply altering 'T', even if σ is high.

The lifetime of a product is 'larger the better' type of quality characteristics. To assess the performance of a process based on the lifetime of products, Montgomery [82] proposed the index C_L, known as the *lifetime performance index*, defined as follows:

$$C_L = \frac{\mu_{\mathrm{X}} - L}{\sigma_{\mathrm{X}}} \tag{1.8}$$

where μ_{X} is the process mean, σ_{X} is the process standard deviation, and L is the lower specification limit.

1.2.3 Univariate Process Incapability Indices

Greenwich and Jahr-Schaffrath [45] introduced an *incapability* index C_{pp} by taking a simple transformation on the index C^*_{pm}, defined as follows:

$$C_{pp} = \left(\frac{1}{C^*_{pm}}\right)^2 = \left(\frac{\mu - T}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2 = C_{ia} + C_{ip} \tag{1.9}$$

where $C^*_{pm} = D/\sqrt{\sigma^2 + (\mu - T)^2}$, $D = \min\{D_u, D_l\}/3$, $D_u = U - T$, $D_l = T - L$ is a generalized version of the index C_{pm} where μ is the process mean, and σ is the process standard deviation. Greenwich and Jahr-Schaffrath [45] denoted $C_{ia} = \left(\frac{\mu - T}{D}\right)^2$ as '*inaccuracy index*', measures the relative process departure and $C_{ip} = \left(\frac{\sigma}{D}\right)^2$ called '*imprecision index*', measures the process variance relative to the tolerance specifications. However the index C_{pp} does not provide the expected measure when the tolerance specification is asymmetric. Therefore, for processes with asymmetric tolerance, Chen [19] proposed a generalized version of C_{pp} as:

$$C''_{pp} = \left(\frac{A}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2 = C''_{ia} + C_{ip} \tag{1.10}$$

where $C''_{ia} = (A/D)^2$ and $A = \max\left\{\frac{(\mu - T)d}{D_u}, \frac{(T - \mu)d}{D_l}\right\}$. Note that when the specifications are symmetric i.e., $T = (U + L)/2$ then $C''_{ia} = C_{ia}$ and hence $C''_{pp} = C_{pp}$. Contrary to other PCIs, increasing values of C_{pp} indicates that the capability of the process is decreasing. That is why C_{pp} is called an 'incapability index'.

1.2.4 Univariate Process Capability Indices for Non-normal Distributions

PCIs are primarily proposed under the assumption that quality characteristics follow a normal distribution. Many applications exhibit non-normal characteristics; for example, nonnegative quality metrics in manufacturing processes, such as roundness, taper, ovality, concentricity, run-out etc., are frequently nonnormal and skewed (refer Anis [2]). When process characteristics are non-normal, a serious consequence is that the process capability, evaluated by the classical PCIs, becomes misleading. One strategy for dealing with non-normality in the distribution of a quality characteristic is to transform the actual distribution into a normal distribution. This can be achieved using techniques such as the Box-Cox transformation or the Johnson transformation. Once the transformed variable becomes normal, then classical PCIs can be used to assess the process performance. The main challenge of the transformation method is that the transformed data's proximity to normality cannot be ensured. Therefore, interpreting the computed PCI values from transformed processes can be challenging.

For the case of a normal distribution, the quantity 3σ corresponds to the distance of the median (which, particularly in the case of a normal distribution, is equal to the mean) from both the 99.865 percentile and the 0.135 percentile. Therefore, to evaluate the capability of a process with the quality characteristic having a non-normal distribution, we can modify the formulas for classical indices by replacing μ and 3σ with relevant alternatives, such as appropriate percentiles.

Clements [25] proposed replacing 6σ with the range between the 99.865 and 0.135 percentile points of the respective distribution. For non-normal distributions, the author defined PCIs that are comparable to C_p and C_{pk} as follows:

$$C_p' = \frac{U-L}{Q_{p_3}-Q_{p_1}} \qquad \text{and} \qquad C_{pk}'' = \frac{d-|Q_{p_2}-M|}{(Q_{p_3}-Q_{p_1})/2}$$

where Q_p denotes the p-th percentile of the distribution corresponding to the random variable 'X' and $p_1 = 0.135$, $p_2 = 50.0$, $p_3 = 99.865$.

Wright [121] proposed the following PCI which is sensitive to skewness:

$$C_s = \frac{\frac{d}{\sigma} - \frac{|\mu-M|}{\sigma}}{3\sqrt{1+\left(\frac{\mu-T}{\sigma}\right)^2 + |\sqrt{\beta_1}|}}$$

where, $\sqrt{\beta_1} = \frac{\mu_3}{\sigma^{3/2}}$ is a commonly used skewness measure, and μ_3 is the third-order raw moment of the related random variable 'X'.

Chen and Pearn [20] proposed a percentile based PCI $C_{N_p}(u,v)$ which is simply a generalisation of the index $C_p(u,v)$ for non-normal distributions, defined as

$$C_{N_p}(u,v) = \frac{d-u|Q_{p_2}-M|}{3\sqrt{\left[\frac{Q_{p_3}-Q_{p_1}}{6}\right]^2 + v\left(Q_{p_2}-T\right)^2}}, \quad u,v \geq 0 \tag{1.11}$$

By putting $(u,v) = (0,0),\ (1,0),\ (0,1)\ \textit{and}\ (1,1)$ we obtain $C_{N_p}(0,0) = C_{N_p}$, $C_{N_p}(1,0) = C_{N_{pk}}$, $C_{N_p}(0,1) = C_{N_{pm}}$ and $C_{N_p}(1,1) = C_{N_{pmk}}$. Note that $C_p' = C_{N_p}$ and $C_{pk}'' = C_{N_{pk}}$. The percentile function of a distribution is a function of unknown parameters of the distribution.

Distribution families, such as Pearson and Johnson, can be utilized to deal with the non-normality of quality characteristic distributions. Additionally, one can select a process distribution from a more specific family of distributions, such as gamma, lognormal, or Weibull, to simplify the corresponding inferential problem. According to Rodriguez [95], utilizing distribution families for estimating PCIs of non-normal processes has the following advantages:

1. The maximum likelihood method can be utilized to estimate the parameters in a stable straightforward manner.
2. The maximum likelihood approach, which provides asymptotic variances for parameter estimates, may be utilized to construct confidence intervals for the PCI estimators.
3. Goodness-of-fit tests based on empirical distribution functions are available for many distribution families, including gamma, lognormal, and Weibull.
4. For standard distribution families, estimated percentile values and proportions of non-conformance associated to PCI estimators can be simply obtained using standard results.

Rodriguez [95] also recommended utilizing methods such as goodness-of-fit, percentile-percentile plot, kernel density estimation, and comparative histograms to evaluate the capabilities of non-normal processes. For a detailed discussion of the PCIs for non-normal distributions, see Pearn and Kotz [88], Tang and Than [110], Somerville and Montgomery [105] and the references therein.

1.2.5 Univariate Process Capability Indices Based on Proportion of Non-conformance

A parallel approach of assessing process performance, apart from using PCIs, is to compute the proportion of non-conformance (PNC). Yeh and Bhattcharya [126] proposed an index defined as:

$$C_f = \min\left\{\frac{p_0^L}{p^L}, \frac{p_0^U}{p^U}\right\} \tag{1.12}$$

where, p_0^L, p_0^U are the expected proportion of non-conforming that the manufacturer can tolerate and $p^L = P(X < L)$, $p^U = P(X > U)$ are the actual proportion of non-conforming that the process produces.

Perakis and Xekalaki [92] proposed an index based on process conformance as follows:

$$C_{pc} = \frac{1 - p_0}{1 - p}$$

where p_0 is the minimum allowable proportion of conformance and p is the actual proportion of conformance.

A generalized PCI, C_{py}, has been defined by Maiti et al. [75] as a ratio of the proportion of the observed process yield to the desired process yield as follows:

$$C_{py} = \frac{P(L \leq X \leq U)}{P(LDL \leq X \leq UDL)} = \frac{p}{p_0}. \tag{1.13}$$

where, LDL and UDL are the lower and upper desired limits. The advantage of using C_{py} is that it does not require the distribution of the relevant quality characteristic to be assumed to be normal. It can also be used to assess the capability of a process having a discrete distribution.

1.3 Process Capability Index and Process Yield

Process yield has been one of the standard criteria for justifying the performance of manufacturing processes. A produced item is identified as a conforming item if it falls within the specification limits; otherwise, it is called a non-conforming item. One of the important measures of process performance is the proportion of non-conformance (PNC). Proportions of non-conformance are often measured in ppm (parts per million). To increase the quality of products, manufacturers attempt to produce items with extremely low PNC. The process yield is defined as the proportion of conforming items, that is proportion of items produced within specification limits. If the quality characteristics 'X' follow a continuous distribution, then the process yield and PNC can be estimated as

$$\begin{aligned}\text{Process Yield} &= P\left[L \leq X \leq U\right] \\ &= \int_L^U f(x)dx\end{aligned}$$

$$\text{PNC} = 1 - \text{Process Yield}$$

where L, U are lower and upper specification limits, respectively and $f(x)$ is the probability density function (pdf). A process with high yield implies that the number of rejected items that cost the manufacturer an additional amount due to scrap or repair will be minimal.

We assume that the quality characteristics of interest 'X' follows a normal distribution $N(\mu, \sigma^2)$. In the following we briefly discuss the relationship of process yield with some PCIs that were previously discussed.

1.3.1 Proportion of Non-Conformance Based on C_p

Proportion of non-conformance (PNC) has a one-to-one relationship with C_p. For a centered process, the PNC can be expressed as,

$$\begin{aligned}\text{PNC} &= 1 - P[L \leq X \leq U] \\ &= 1 - \Phi\left(\frac{U-\mu}{\sigma}\right) + \Phi\left(\frac{L-\mu}{\sigma}\right) \qquad (1.14)\end{aligned}$$

where Φ is the cumulative distribution function (cdf). Since, $U = M + d$ and $L = M - d$, from equation (1.14) we have,

$$\begin{aligned} \text{PNC} &= 1 - \Phi\left(\frac{(M + d - \mu)}{d}.\frac{d}{\sigma}\right) + \Phi\left(\frac{(M - d - \mu)}{d}.\frac{d}{\sigma}\right) \\ &= \Phi\left[\left(-1 + \frac{(\mu - M)}{d}\right).\frac{d}{\sigma}\right] + \Phi\left[\left(-1 - \frac{(\mu - M)}{d}\right).\frac{d}{\sigma}\right] \\ &= \Phi\left[(-1 + |k|).\frac{d}{\sigma}\right] + \Phi\left[(-1 - |k|)\frac{d}{\sigma}\right] \\ &= \Phi\left[3(-1 + |k|)C_p\right] + \Phi\left[3(-1 - |k|)C_p\right]. \end{aligned} \tag{1.15}$$

1.3.2 Proportion of Non-Conformance Based on C_{pk}

The index C_{pk} also has a one-to-one relationship with the PNC. The index C_{pk} defined by equation (1.3). If $L \leq \mu \leq M$, then

$$C_{pk} = \frac{\mu - L}{3\sigma}$$

and

$$\frac{U - \mu}{3\sigma} = \frac{U - L + L - \mu}{3\sigma} = 2C_p - C_{pk}.$$

Therefore, using the equation (1.14), the PNC can be obtained as:

$$\text{PNC} = \Phi\left[-3(2C_p - C_{pk})\right] + \Phi\left[-3C_{pk}\right]. \tag{1.16}$$

A similar relation can be obtained if $M \leq \mu \leq U$. Since $C_p \geq C_{pk}$ then, using equation (1.16) we can obtain two-sided bounds for the PNC in terms of C_{pk} as $\Phi(-3C_{pk}) \leq PNC \leq 2\Phi(-3C_{pk})$.

1.3.3 Proportion of Non-Conformance and the Index C_{pm}, C_{pmk}

Like the index, C_p and C_{pk} there is no such direct relationship of PNC with the index C_{pm} and C_{pmk}. But under certain conditions, these indices provide an upper bound for the PNC as follows:

$$\text{PNC} \leq 2\Phi(-3C_{pm}), \quad \text{if} \quad C_{pm} \geq \sqrt{3}/3 \tag{1.17}$$

and

$$\text{PNC} \leq 2\Phi(-3C_{pmk}), \quad \text{if} \quad C_{pmk} \geq \sqrt{2}/3. \tag{1.18}$$

1.3.4 Proportion of Non-Conformance Based on C_{PU}, C_{PL}

PNC has a one-to-one relationship with the index C_{PU} and C_{PL}. For the quality characteristic, with *larger the better* type

$$\begin{aligned} \text{PNC} &= P[X < L] \\ &= P[\frac{X-\mu}{\sigma} < \frac{L-\mu}{\sigma}] \\ &= \Phi(-3C_{PL}). \end{aligned} \tag{1.19}$$

For the quality characteristics with *smaller the better* type

$$\begin{aligned} \text{PNC} &= P[X > U] \\ &= P[\frac{X-\mu}{\sigma} > \frac{U-\mu}{\sigma}] \\ &= 1-\Phi(3C_{PU}). \end{aligned} \tag{1.20}$$

1.4 Some Applications of Process Capability Indices

Process capability indices (PCIs) are important statistical quality control tool use to measure the performance of a process. PCIs have several applications in the industry. Kane [62], Palmer and Tsui [86] and Deleryd [29] have discussed a wide range of applications of PCIs. Some common applications of PCIs are mentioned below:

1. **Measure of Performance:** The most common use of PCI is the performance measure of a process. The information obtained from a process capability analysis is used to improve the process.

2. **Continuous Improvement:** PCI values can be monitored over time in order to get information of relative improvement. In such a way, the PCI value works as an alarm clock because if the process suddenly change for the worse then that is reflected by PCIs value. The process owner can take immediate action to detect the cause of deterioration and improve the process performance.

3. **Certificate for Customers:** The suppliers may attach the results obtained from the process capability studies that was performed at the time of the production process. This information will help the customers to trust the delivery products. Based on this information the customers will be able to know if it is possible to directly use the products or some sort of inspection is required.

4. **A Basic for New Constructions:** By getting information of the capability of a production process, the product engineers determine how to set the specification tolerance in order to produce conforming products. If more narrow specification is needed than their own process is able to meet, then either the performance of the present machine has to improve or to purchase a new one.

5. **Certificates at Audit:** When audits like ISO 9000 are conducted, the results of process capability analysis are utilized.

6. **Comparson of Processes:** PCIs are used to compare the performance of two competitive processes or the same processes at different time points. Selection of a better supplier is always a challenging problem. PCIs are often utilized to select a reliable supplier with better process capability.

1.5 A Brief Literature Review

Note that all the PCIs discussed so far are written as functions of the parameters μ, σ and percentiles of the relevant quality characteristic and are thus typically unobservable. The common procedure is to evaluate a process's capability using the estimated value obtained from one or more randomly selected samples from the process under consideration. As a result, these estimators are susceptible to fluctuations caused by sampling variability. The distributional properties and estimation procedures of these basic PCIs are thoroughly studied by several researchers (refer Pearn et al. [89], Kotz and Johnson [64], Kotz and Johnson [65], Vännman and Kotz [116], Pearn and Kotz [88] and the references therein).

It is not sufficient to merely suggest a point estimate of a PCI. It is crucial to investigate interval estimators and conduct appropriate hypothesis tests. A significant amount of research has been done to design confidence intervals for PCIs under various circumstances (refer Chou et al. [23], Hoffman [55], Kushler and Hurley [70], Zimmer et al. [132], Pearn and Kotz [88] and the references therein). Numerous researchers have proposed various hypothesis testing procedures for C_p, C_{pk}, C_{pm} and C_{pmk}. For instance, one may refer to Lin and Pearn [73], Hoffman [54], Chen and Hsu [22] and Pearn and Lin [90] among others.

Most of the statistical properties of PCIs have been studied under the simple assumption that the observed quality characteristic is normally distributed and independent of each other. However, it is not uncommon to encounter auto-correlated data in process industries. Vännman and Kulahci [117] point out that the use of online data acquisition systems in industry induces autocorrelation in the data. According to Runger and Willemain [96], the increased use of online data collecting technologies is reducing the gap between the process observations thus producing positive autocorrelation. These tendencies are especially common in the chemical and process industries. Even in discrete parts manufacturing, it is routine practice to measure each part manufactured. First-order autocorrelation can be detected using the well-known Durbin-Watson test (Durbin and Watson [34]). The Bartlett test (Bartlett [6]) can be used to test higher-order autocorrelation; see also Box et al. [13]. If process capability is measured for an autocorrelated process while autocorrelation is ignored, the results may be erroneous and misrepresent the true health of the process. Before estimating PCIs, it is important to test whether the process exhibits autocorrelation. Variance of a process increases in the presence of autocorrelation, resulting in a decrease in the PCI value. The impact of autocorrelation on PCIs has not been significantly discussed due to the complexities involved in calculating the statistical properties of the estimator. Shore [103] first studied PCIs in the presence of an autocorrelated sample. For other notable work refer Zhang [131], Wallgren [118], Vännman and Kulahci [117].

The majority of studies on the distributional and inferential aspects of PCIs assume that the estimated values of the parameters of the corresponding quality characteristics are free of measurement error. As a result, the observed estimated values, based on the available data, are regarded as the desired estimated PCI values. However, variations in measurement systems are inherent and have been studied extensively. These are broadly referred to as measurement system analysis. The effectiveness of measurement systems depends upon accurate gauges and proper gauge use. An ideal measurement system must consistently deliver accurate measurements. However, in practical reality, this is almost unachievable because measurements contain both systematic and random errors. Zappa and Deldossi [130] emphasize that measurement systems capability analysis attempts to assess if the measurement process is capable, that is, if the variability of the measurement system is small relative to the variability of the monitored process. Gauge repeatability and reproducibility (GR & R) studies are used to act as an audit tool and as a source of feedback to improve the measurement procedure. When the observations are independent, statistical tools like ANOVA (e.g., Antony et al. [4]) and the design of experiments (e.g., Montgomery and Runger [83]) may be used for GR & R studies. Burdick et al. [16] provides a helpful review of measurement systems capability analysis methods.

We now turn our attention to the problem of constructing confidence intervals for the PCIs. This necessarily involves the distribution of the estimated PCIs. However, obtaining the distribution of the estimated PCIs is quite challenging in most cases. We establish confidence interval by fiducial approach. The ideas of fiducial inference and fiducial distribution were proposed by Fisher [40]. Fisher attempted to overcome what he saw as a serious deficiency of the Bayesian approach that infer about the distribution of model parameters by using prior information even when no such informations are available. Fiducial inference derives a measure on a parameter space similar to a Bayesian approach but with fewer assumptions (no prior). In subsequent studies, Fisher further developed the fiducial inference concept (Fisher [41]). The concept of fiducial inference faces severe criticisms concerning the interpretation of fiducial distribution. However, Efron [36] mentioned in Section 8 of his paper that "Maybe Fisher's biggest blunder will become a big hit in the 21st century!" The idea of fiducial inference resurfaced by the name of the 'generalized variable approach' proposed by Tsui and Weerahandi [114]. The notions of generalized p-values and generalized confidence intervals (GCIs) have shown to be particularly useful in getting tests and confidence intervals for some difficult problems where standard procedures are difficult to implement. Tsui and Weerahandi [114] had introduced the idea of generalised p value and proposed generalised hypothesis testing based on them. The concepts of generalized pivotal quantity (GPQ) and generalized confidence interval (GCI) were first proposed by Weerahandi [119]. Since then, the idea has been applied in numerous practical areas. Hannig [50] noted that the generalized variable approach is a special case of fiducial inference. Hannig et al. [52] discussed an important subclass of GPQs known as fiducial generalized pivotal quantities (FGPQs). The confidence interval estimated based on FGPQ is known as fiducial generalized confidence intervals (FGCIs). One of the important desirable properties of the FGCIs is that under some reasonable assumptions, they have asymptotically correct frequentist coverage. For more details about fiducial generalized pivotal quantities and fiducial inference, we refer to Hannig et al. [51]. The fiducial approach have been successfully used in the area of process capability analysis; see for example Mathew et al. [77], Kurian et al. [69], Hsu et al. [58], Meng et al. [80], Meng and Yang [79], Guo et al. [48], Guo et al. [49] and others.

Bootstrapping is a newer, non-parametric, but computer-intensive statistical technique that plays an important role in modern statistical analysis and applications. The bootstrap method was introduced by Efron [35]. According to Efron and Tibshirani [38], bootstrapping is a computer-based resampling approach for estimating the standard error of summary statistics. No theoretical calculations are required for estimating bootstrap standard error. This technique is helpful in situations where the estimation of the actual confidence interval of parameters becomes intractable. The bootstrap approach has been widely applied in estimating CIs of various PCIs, see Franklin and Gary [42], Franklin and Wasserman [43], Rao et al. [94], Saha et al. [98], Dey et al. [33], Dey and Saha [31], Dey and Saha [32].

1.6 Contributions of the Thesis

The main contributions of this thesis are the following:

1. We have analyzed the combined effect of autocorrelation and measurement errors on the PCI C_p, C_{pm} and the incapability index C''_{pp}. Approximate expressions of the mean and the variance are derived. The bias and mean square error values are reported under different levels of autocorrelation, measurement errors, and sample sizes.

2. One of the important contributions of this thesis is the derivation of the fiducial generalized confidence interval of the PCI $C_{N_{pk}}$ and C_{py} under the location-scale distributions. Four important distributions, namely normal, lognormal, Weibull and gamma are considered. The performance of the proposed method is compared with bootstrap confidence intervals in terms of their coverage probability and average width.

3. Applications of PCIs in supplier selection are discussed. Testing of hypothesis is performed to select a superior supplier among two competitive suppliers based on process capability index by using fiducial generalized confidence interval.

4. The lower confidence bound of the lifetime performance index C_L is derived under type II censored samples by using fiducial approach. We have considered the location-scale distributions. Three important lifetime distributions, namely, two parameter exponential, Weibull and two parameter Rayleigh, are discussed. Applications in purchase decisions are considered.

1.7 Summary of the Thesis Work

Process capability analysis is a beneficial tool for enhancing the level of performance in industrial processes, provided it is applied in a statistically correct way. So far, most of the statistical properties of PCIs have been studied under the simplifying assumption of normally distributed quality characteristics, independence of observations and absence of measurement errors. But, these assumptions are not always valid. Many quality

Table 1.2: A brief discussion about what has been done in our research

Chapter	Research done earlier	Research done in our thesis
2	Statistical properties of the PCI C_p in presence of autocorrelation and measurement was first attempted by Scagliarini [101]. But there was some serious calculation errors.	We have reattempted the problem and corrected the errors.
3	The effect of autocorrelation and measurement errors on the estimator of PCI C_{pm} was studied separately.	We have studied some statistical properties of the estimator of C_{pm} under the combined effect of autocorrelation and measurement errors.
4	A Study on statistical properties of process incapability index C''_{pp} was done in the presence of measurement error by assuming the process observations are identical and independently normal.	We investigate some properties of the estimator of C''_{pp} in the presence of measurement error when the process observations are autocorrelated.
5	Most of the existing research on PCIs assumes that the quality characteristics are iid normal. Confidence intervals of PCIs are derived by using bootstrap methods and asymptotic methods.	We have considered the quantile-based PCI $C_{N_{pk}}$ and derived a confidence interval by the fiducial approach under some important location-scale distributions. We also compare the performance of two suppliers by estimating the fiducial confidence interval of the difference and ratio of their PCIs.
6	Confidence interval of the PCI C_{py} was derived by using bootstrap confidence intervals and in most cases, it was assumeed that the quality characteristics are iid normal.	We have derived a fiducial generalized confidence interval of C_{py} under some location-scale distribution and have compared the performance of two processes by deriving the fiducial interval of the difference of their PCIs.
7	Statistical properties of the lifetime performance index C_L was derived under different censoring schemes by using different techniques.	We have derived the lower confidence bound of C_L by using the fiducial approach under some location-scale distributions when the sample is type-II censored.

characteristics follow non-normal distributions. Also, serial autocorrelation is inherent in some processes. Despite the use of modern measuring devices in industrial applications, gauge measurement errors are regarded as an essential component of measurements. The overall aim of the research work presented in this thesis is to address these issues of autocorrelation, measurement error and non-normality on some of the existing PCIs regarding their statistical properties and confidence intervals. Chapter 1 gives an introduction to the topic of study, some notations and terminologies used in the thesis, a brief review of literature, the contribution of the thesis, a summary of the work done, and a list of papers.

Chapter 2 discusses some statistical properties of C_p considering the combined effects of autocorrelation and measurement errors (refer Anis and Bera [3]). Zhang [131] studied some statistical properties and derived confidence intervals for PCIs C_p and C_{pk} when the process data exhibit autocorrelation. The impact of gauge measurement errors on the estimator of C_p when the underlying data are autocorrelated was attempted by Scagliarini [101]. Some of his results, however, were incorrect. We study the problem and correct the errors that were made. We have derived the approximate expectation, variance and mean square error (MSE) of the estimator. Our result indicates that the bias is always positive in the presence of autocorrelation, whereas bias can be positive, negative and even zero in the presence of both autocorrelation and measurement errors. Some properties of bias and MSE are derived. Method of estimation of parameters are discussed and an application of the proposed results is given.

Chapter 3 deals with some statistical properties of PCI C_{pm}, considering both autocorrelation and measurement errors (refer Bera and Anis [8]). We have derived approximate mathematical expressions for the expectation, variance and MSE of the estimator of the index in the presence of both autocorrelation and measurement error. Also, we have estimated an approximate lower confidence bound (LCB) of the index. The effect of autocorrelation and measurement error on the estimator is analyzed in detail. This study significantly contributes to the understanding of the estimator of C_{pm} in situations where process observations exhibit autocorrelation and measurement errors are inevitable.

In Chapter 4, we analyze some statistical properties of the process incapability index C''_{pp} when process observations are autocorrelated and measurement error is present (refer Bera and Anis [10]). We have derived the precise expressions for both the expectation and the variance of the estimator. The impact of autocorrelation and measurement error on the estimator is discussed in detail. This study helps to understand the behavior of the estimator of C''_{pp} when process observations are autocorrelated and measurement error is present.

In Chapter 5, we consider the PCI $C_{N_{pk}}$, which was derived to measure the capability of non-normal quality characteristics (refer Bera and Anis [9]). We have derived the fiducial generalized confidence interval (FGCI) of the PCI under some location-scale distribution. We have also derived the FGCI for the difference and ratio of the PCI between the two suppliers to identify the better supplier among them. Four important distributions are considered, namely normal, lognormal, Weibull and gamma distribution. The performance of the proposed FGCI is compared to some nonparametric bootstrap confidence intervals (BCI). Simulation results indicate that the proposed FGCI exhibits superior performance regarding coverage probability (CP) and average width (AW). Examples are given to demonstrate how the proposed method can be applied effectively.

In Chapter 6, we have studied some properties of PCI C_{py} in terms of fiducial inference. We have derived FGCI of C_{py} and the difference $\delta_{py} = C_{py1} - C_{py2}$ of PCIs between two suppliers. We have considered four important location-scale (log location-scale) distributions, namely normal, lognormal, Weibull and gamma distribution. The performance of the proposed FGCI is compared with some nonparametric BCIs in terms of CP. Simulation study shows that our proposed FGCI produces satisfactory results and can be used effectively. Examples are given to demonstrate the applicability of the proposed method.

Next, we examine the scenario where the samples are censored instead of being complete. Such a situation often arises when the quality characters are the lifetime of a specific component. Chapter 7 discusses the properties of the index C_L, known as the lifetime performance index under the type-II censoring scheme. The FGCI are derived under three important lifetime distributions, namely, two-parameter exponential, Weibull, and two-parameter Rayleigh. Simulation performance indicates that the coverage probability of the proposed interval is very close to the nominal confidence level, even when the sample size is small. This approach helps to minimize both the costs and time involved in sampling.

Finally, Chapter 8, concludes the thesis. We also talked about several significant research gaps that could be addressed in the future.

1.8 List of Papers Included in the Thesis

The following papers have been included in this thesis:

1. Bera, K. & Anis, M. Z. (2024). On some statistical properties of a stationary Gaussian process in the presence of measurement errors. *Statistics in Transition new series*, Vol. 25, No. 4, pp. 137–155 [Chapter 2].

2. Anis, M. Z., & Bera, K. (2022). Process Capability C_p Assessment for Auto-Correlated Data in the Presence of Measurement Errors. *International Journal of Reliability, Quality and Safety Engineering*, 29(06), 2250010 [Chapter 2].

3. Bera, K., & Anis, M. Z. (2023). Estimation of Cpm for autocorrelated data in the presence of random measurement errors. *Communications in Statistics-Simulation and Computation*, 54(5), 1255-1282, [Chapter 3].

4. Bera, K., & Anis, M. Z. (2024). Process incapability index for autocorrelated data in the presence of measurement errors. *Communications in Statistics-Theory and Methods*, 53(15), 5439-5459, [Chapter 4].

5. Bera, K., & Anis, M. Z. (2024). Comparison of two processes based on quantile-based process capability indices by using fiducial generalized confidence interval. *Journal of Statistical Computation and Simulation*, 95(5), 1071-1090, [Chapter 5].

6. Bera, K., & Anis, M. Z. (2024). Fiducial generalized confidence interval of quantile-based process capability indices under some location-scale distributions. Communicated, [Chapter 5].

7. Bera, K., & Anis, M. Z. (2024). Fiducial generalized confidence interval of C_{py} under some location-scale distributions. *Journal of the Indian Society for Probability and Statistics*. DOI: 10.1007/s41096-025-00244-w, Published online, [Chapter 6].

8. Bera, K., & Anis, M. Z. (2024). Fiducial inference on lifetime performance index based on type-II censored samples. Communicated, [Chapter 7].

Chapter 2

Assessment of C_p for Autocorrelated Data in the Presence of Measurement Errors

2.1 Introduction

In spite of its importance, the influence of autocorrelation has received relatively less attention in the PCI literature. To the best of our knowledge, attention to autocorrelation was first drawn by Kotz and Johnson [64], (p.76). Shore [103] demonstrated that neglecting process autocorrelations while evaluating PCIs can lead to incorrect interpretations. Zhang [131] established the variance of $\hat{C}_p$ as a function of C_p, the sample size n and the process autocorrelation ρ. Noorossana [85] postulates that autocorrelated data can lead to biased estimates of PCIs, resulting in incorrect decisions about process performance.

There have been attempts to look at the influence of measurement errors on PCIs. The causes of measurement errors include inadequate gauge calibration and various external factors affecting the measuring devices. McNeese and Klein [78] pointed out variability is inherent in the measurement system and hence affects process capability. However, it was Mittag [81] who theoretically discussed the effects of measurement errors on the performance of the four fundamental PCIs. Bordignon and Scagliarini [11] extended this analysis to the inferential properties of the estimators of C_p and C_{pk} when estimated from sample data.

However, so far the issues of autocorrelation and measurement errors have been examined separately in the context of PCIs. But, in real-world applications, process data from industries such as chemicals, pharmaceuticals, and food processing are often correlated. Additionally, even with advanced measuring instruments, it is impossible to completely eliminate measurement errors. Therefore, it is important to discuss PCIs for autocorrelated data when measurement errors are present, which is the focus of this work. Scagliarini [101] should be credited for attempting in this direction. However, some of his results were incorrect. In this chapter, we present a corrected version of the results of Scagliarini [101].

The chapter is organized as follows. Section 2.2 discusses the stationary Gaussian process and some of its properties. Section 2.3 derived some properties of a stationary Gaussian process in the presence of measurement error. Section 2.4 investigates the statistical features of C_p for autocorrelation data, while Section 2.5 analyzes the impact of gauge measurement errors on C_p when the underlying data are autocorrelated. Estimation of parameters and two applications are considered in Section 2.6. Section 2.7 offers some concluding remarks.

2.2 Some Properties of Stationary Gaussian Process

A process is considered stationary if it can be regarded as in a state of statistical equilibrium. This indicates that the distribution of quality characteristics used to monitor particular processes remains constant throughout time. As a result, shifting observations ahead or backward by any number of time units has no effect on the process mean and standard deviation estimates. Another consequence of a process being stationary is that the pattern of autocorrelations between observations separated by k time units $(k = 1, 2, \dots)$ remains stable.

The correlation in a process can be captured using time series models. This approach can offer a framework for implementing statistical control while monitoring autocorrelated processes. A time series $\{X_t\}$ is said to be a Gaussian process if the joint distribution of any finite number of random variables from the process follows a multivariate normal distribution. A process that is simultaneously stationary and Gaussian is said to be a stationary Gaussian process, refer to Brockwell and Davis [15].

We assume that the relevant quality characteristic 'X' is described by a discrete-time stationary Gaussian process $\{X_t,\ t \in \mathbb{Z}\}$. Under this assumption, the process has a constant mean $E(X_t) = \mu$ and auto-covariances are given by $\gamma(k) = E[(X_t - \mu)(X_{t+k} - \mu)]$, where $\gamma(k)$ $(k = 0, \pm 1, \pm 2, \dots)$ is a function of the lag k only. Under the assumption of stationarity, the standard deviation of the process is a constant, denoted by $\sigma_X = \sqrt{\gamma(0)}$. In our discussion, we focus on a particularly important stationary Gaussian process, specifically, a first-order stationary autoregressive process (AR(1)), which has the following structure:

$$X_t - \mu = \phi(X_{t-1} - \mu) + a_t \tag{2.1}$$

where, X_t is the value of observation at time t, μ is the process mean, a_t is white noise where $a_t \sim N(0, \sigma_a^2)$ and $|\phi| < 1$. Note that for AR(1) model, the autocorrelation coefficient between X_t and X_{t+k} is $\rho_k = \phi^k,\ k = 1, 2, \dots$. Furthermore the variance of X_t is given by $\sigma_X^2 = \sigma_a^2/(1 - \phi^2)$.

Let $\{X_1, X_2, \dots, X_n\}$ be a random sample of n consecutive observations from a stationary Gaussian process $\{X_t\}$. Then the sample mean and sample variance of the process are, respectively, given by $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S^2 = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2/(n-1)$. Zhang [131] derived the expected value and variance of $\bar{X}$ and S^2. We shall use the same notation as Zhang [131]. However, for completeness, we shall define the functions that will be used subsequently. The results are given below,

$$\begin{aligned} E\left(\bar{X}\right) &= \mu, \qquad Var\left(\bar{X}\right) = \frac{\sigma_X^2}{n} g(n,\rho_i), \\ E\left(S^2\right) &= \sigma_X^2 f(n,\rho_i), \qquad Var\left(S^2\right) = \frac{2\sigma_X^4}{(n-1)^2} F(n,\rho_i) \end{aligned} \tag{2.2}$$

where, $\rho_i = \rho_X(i)$ for $i = 1, 2, 3, \ldots, n$ is the autocorrelation of X_t at lag i,

$$f(n,\rho_i) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} (n-i)\,\rho_i, \tag{2.3}$$

$$F(n,\rho_i) = n + 2\sum_{i=1}^{n-1} (n-i)\,{\rho_i}^2 + \frac{1}{n^2}\left[n + 2\sum_{i=1}^{n-1}(n-i)\rho_i\right]^2 - \frac{2}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-i} (n-i-j)\,\rho_i\rho_j \tag{2.4}$$

and

$$g(n,\rho_i) = 1 + \frac{2}{n}\sum_{i=1}^{n-1}(n-i)\rho_i. \tag{2.5}$$

Observe that when the process $\{X_t\}$ is identically and independently normally distributed, then $\rho_i = 0$ for $i \geq 1$. In this case $f(n,\rho_i) = 1$, $g(n,\rho_i) = 1$ and $F(n,\rho_i) = (n-1)$.

For an AR(1) process defined by equation (2.1), $\rho_i = \phi^i$. So, the expected value and variance of the sample mean and sample variance are as follows:

$$\begin{aligned} E\left(\bar{X}\right) &= \mu, \qquad Var\left(\bar{X}\right) = \frac{\sigma_X^2}{n} g(n,\phi), \\ E\left(S^2\right) &= \sigma_X^2 f(n,\phi), \qquad Var\left(S^2\right) = \frac{2\sigma_X^4}{(n-1)^2} F(n,\phi) \end{aligned} \tag{2.6}$$

where

$$f(n,\phi) = 1 - \frac{2}{n(n-1)} \frac{\phi\left[n - 1 - n\phi + \phi^n\right]}{(1-\phi)^2},$$

$$F(n,\phi) = n + 2\sum_{i=1}^{n-1} (n-i)\,\phi^{2i} + \frac{1}{n^2}\left[n + 2\sum_{i=1}^{n-1}(n-i)\phi^i\right]^2 - \frac{2}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-i} (n-i-j)\,\phi^{i+j},$$

and

$$g(n,\phi) = 1 + \frac{2}{n}\frac{\phi\left[n - 1 - n\phi + \phi^n\right]}{(1-\phi)^2}.$$

2.3 Some Properties of Stationary Gaussian Process in Presence of Measurement Errors

We will now consider the situation where the observations are affected by random measurement errors. We suppose that the measurement error can be described by a Gaussian white noise process E_t with constant variance, that is, $E_t \sim N(0, {\sigma_E}^2)$. Additionally, it is assumed that X_t and E_t are linked additively and are stochastically independent of each other. The observable quality characteristic Y_t, is thus a Gaussian process $\{Y_t,\ t = 1, 2, \ldots\}$ given by

$$Y_t = X_t + E_t. \tag{2.7}$$

The sample mean and sample variance are given by,

$$\bar{Y} = \sum_{i=1}^{n} Y_i/n, \qquad S_Y^2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/(n-1). \tag{2.8}$$

We derive the expectation and the variance of $\bar{Y}$ and S_Y^2. Note that $\bar{X} \sim N(\mu, \sigma_X^2 g(n, \rho_i)/n)$ and $\bar{E} \sim N(0, \sigma_E^2/n)$, where $\bar{E} = \sum_{i=1}^{n} E_i/n$. Therefore, $E(\bar{X}) = \mu$ and $E(\bar{E}) = 0$.

$$\begin{aligned} E\left(\bar{Y}\right) &= E\left(\sum_{i=1}^{n} Y_i/n\right) \\ &= E\left(\sum_{i=1}^{n}(X_i + E_i)/n\right) \\ &= E\left(\bar{X}\right) + E\left(\bar{E}\right) \\ &= \mu. \end{aligned} \tag{2.9}$$

Since, X_t and E_t are independent, we have $Cov\left(\bar{X}, \bar{E}\right) = 0$.

$$\begin{aligned} Var\left(\bar{Y}\right) &= Var\left(\bar{X} + \bar{E}\right) \\ &= Var\left(\bar{X}\right) + Var\left(\bar{E}\right) + 2Cov\left(\bar{X}, \bar{E}\right) \\ &= \frac{\sigma_X^2 g(n, \rho_i)}{n} + \frac{\sigma_E^2}{n}. \end{aligned} \tag{2.10}$$

$$\begin{aligned} E\left(S_Y^2\right) &= E\left(S^2 + \frac{2}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(E_i - \bar{E}) + S_E^2\right) \\ &= E\left(S^2\right) + E\left(S_E^2\right) \\ &= \sigma_X^2 f(n, \rho_i) + \sigma_E^2 \end{aligned} \tag{2.11}$$

where $S_E^2 = \sum_{i=1}^{n}(E_i - \bar{E})^2/(n-1)$. Now, after simplification, we get,

$$Var\left(S_Y^2\right) = Var(S^2) + \frac{4}{(n-1)^2} Var\left(\sum_{i=1}^{n}(X_i - \bar{X})(E_i - \bar{E})\right) + Var(S_E^2). \tag{2.12}$$

Note that other covariance terms in Equation (2.12) will become zero as X_t, E_t are independent. From Zhang [131] we get,

$$Var(S^2) = \frac{2\sigma_X^4}{(n-1)^2} F(n, \rho_i). \tag{2.13}$$

Also, we have

$$Var(S_E^2) = \frac{2\sigma_E^4}{(n-1)}. \tag{2.14}$$

Now,

$$\begin{aligned}
& Var\left(\sum_{i=1}^{n}(X_i-\bar{X})(E_i-\bar{E})\right) \\
&= E\left(\sum_{i=1}^{n}(X_i-\bar{X})(E_i-\bar{E})\right)^2 \\
&= E\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-\bar{X})(X_j-\bar{X})(E_i-\bar{E})(E_j-\bar{E})\right) \\
&= \sum_{i=1}^{n}E(X_i-\bar{X})^2E(E_i-\bar{E})^2+\frac{\sigma_E^2}{n}\sum_{i=1}^{n}E(X_i-\bar{X})^2 \\
&\qquad -\frac{\sigma_E^2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left((X_i-\bar{X})(X_j-\bar{X})\right) \\
&= \sigma_E^2\sum_{i=1}^{n}E(X_i-\bar{X})^2 \\
&= (n-1)\sigma_E^2\sigma_X^2 f(n,\rho_i). \qquad (2.15)
\end{aligned}$$

Substituting Equations (2.13), (2.14) and (2.15) in Equation (2.12) we finally get,

$$Var(S_Y^2)=\frac{2\sigma_X^4}{(n-1)^2}\left\{F(n,\phi)+(n-1)\frac{\sigma_E^4}{\sigma_X^4}+2(n-1)f(n,\rho_i)\frac{\sigma_E^2}{\sigma_X^2}\right\}, \qquad (2.16)$$

In particular, if the process is an AR(1) process, then

$$Var\left(\bar{Y}\right) = \frac{\sigma_X^2}{n}\left[g(n,\phi)+(1-\phi^2){\tau_a}^2\right], \qquad (2.17)$$

$$E(S_Y^2)=\sigma_X^2\left[f(n,\phi)+(1-\phi^2)\tau_a^2\right], \qquad (2.18)$$

$$Var(S_Y^2)=\frac{2\sigma_X^4}{(n-1)^2}\left\{F(n,\phi)+(n-1)(1-\phi^2)^2\tau_a^4+2(n-1)(1-\phi^2)\tau_a^2 f(n,\phi)\right\} \qquad (2.19)$$

where,

$$\tau_a=\frac{\sigma_E}{\sigma_a} \qquad (2.20)$$

is the ratio between the standard deviations of the measurement error and the standard deviation of the white noise term a_t of AR(1) process.

2.4 The Effect of Autocorrelation on C_p

The sample estimator of $C_{p;a}$ (where the subscript 'a' denotes that we are considering autocorrelated data) is

$$\hat{C}_{p;a}=\frac{U-L}{6S}. \qquad (2.21)$$

The approximate expectation and variance of $\hat{C}_{p;a}$ are given in the following theorem.

Theorem 2.1. *Assume that the quality characteristic* $\{X_t\}$ *is described by a discrete-time stationary Gaussian process. Then the approximate expectation and the variance of* $\hat{C}_{p;a}$ *are*

$$E(\hat{C}_{p;a}) \approx C_{p;a} \frac{1}{[f(n,\rho_i)]^{\frac{1}{2}}} \left[1 + \frac{3F(n,\rho_i)}{4(n-1)^2 \left[f(n,\rho_i)\right]^2} \right] \tag{2.22}$$

and

$$Var(\hat{C}_{p;a}) \approx (C_{p;a})^2 \left[\frac{F(n,\rho_i)}{2(n-1)^2 \left[f(n,\rho_i)\right]^3} \right]. \tag{2.23}$$

Proof. Here we use the statistical differential technique to derive the approximate expression of expectation and variance. First we briefly discuss how to derive the approximate mean and variance of a statistic by the statistical differential technique. Let the random variable be X and $E(X) = \alpha$. Let $g(X)$ be a differentiable function of X. Using Taylor series expansion

$$\begin{aligned} g(X) &= g(\alpha + X - \alpha) = g(\alpha) + (X-\alpha)g^{'}(\alpha) + \frac{1}{2}(X-\alpha)^2 g^{''}(\alpha) + \ldots \\ &\approx g(\alpha) + (X-\alpha)g^{'}(\alpha) + \frac{1}{2}(X-\alpha)^2 g^{''}(\alpha). \end{aligned}$$

Taking the expectation on both sides

$$E\left[g(X)\right] \approx g(\alpha) + \frac{1}{2} g^{''}(\alpha) Var(X).$$

Similarly, to derive the approximate variance, we write the Taylor series expression as,

$$g(X) \approx g(\alpha) + (X-\alpha)g^{'}(\alpha).$$

Hence, approximate variance

$$Var\left[g(X)\right] \approx \left\{g^{'}(\alpha)\right\}^2 Var(X).$$

Now, using the above results, we can write

$$E\left(\frac{1}{S}\right) = E\left[\frac{1}{(S^2)^{1/2}}\right] \approx \left[E(S^2)\right]^{-1/2} \left[1 + \frac{3}{8}\frac{Var(S^2)}{[E(S^2)]^2}\right].$$

Similarly,

$$Var\left(\frac{1}{S}\right) \approx \frac{Var(S^2)}{4[E(S^2)]^3}.$$

Taking expectation on both sides of equation (2.21) and using the methods of statistical differential, we get

$$\begin{aligned} E\left(\hat{C}_{p;a}\right) &= E\left(\frac{U-L}{6S}\right) \\ &= \left(\frac{U-L}{6}\right) E\left(\frac{1}{S}\right) \\ &\approx \left(\frac{U-L}{6}\right) \left[E(S^2)\right]^{-1/2} \left[1 + \frac{3}{8}\frac{Var(S^2)}{[E(S^2)]^2}\right]. \end{aligned} \tag{2.24}$$

Substituting $E(S^2)$ and $Var(S^2)$ given by equation (2.2) into equation (2.24) we get,

$$E(\hat{C}_{p;a}) \approx C_{p;a}\frac{1}{[f(n,\rho_i)]^{\frac{1}{2}}}\left[1+\frac{3F(n,\rho_i)}{4(n-1)^2\,[f(n,\rho_i)]^2}\right].$$

Similarly, by taking variance on both sides of the equation (2.21) and using the methods of statistical differential, we get

$$\begin{aligned} Var\left(\hat{C}_{p;a}\right) &= Var\left(\frac{U-L}{6S}\right) \\ &= \left(\frac{U-L}{6}\right)^2 Var\left(\frac{1}{S}\right) \\ &\approx \left(\frac{U-L}{6}\right)^2 \frac{Var(S^2)}{4[E(S^2)]^3}. \end{aligned} \tag{2.25}$$

Substituting $E(S^2)$ and $Var(S^2)$ into equation (2.25) we get,

$$Var(\hat{C}_{p;a}) \approx (C_{p;a})^2\left[\frac{F(n,\rho_i)}{2(n-1)^2\,[f(n,\rho_i)]^3}\right].$$

□

For an AR(1) process, the expected value of $\hat{C}_{p;a}$ can be approximated by

$$E(\hat{C}_{p;a}) \approx C_{p;a}\ \mathcal{B}(n,\phi) \tag{2.26}$$

where,

$$\mathcal{B}(n,\phi) = \frac{1}{\sqrt{f(n,\phi)}}\left[1+\frac{3F(n,\phi)}{4(n-1)^2[f(n,\phi)]^2}\right]. \tag{2.27}$$

An approximate bias of $\hat{C}_{p;a}$ is given by

$$Bias(\hat{C}_{p;a}) \approx C_{p;a}\ \left[\mathcal{B}(n,\phi)-1\right]. \tag{2.28}$$

The variance and the means square error (MSE) of $\hat{C}_{p;a}$ are

$$Var(\hat{C}_{p;a}) \approx (C_{p;a})^2\left[\frac{F(n,\phi)}{2(n-1)^2\,[f(n,\phi)]^3}\right] \tag{2.29}$$

and

$$\begin{aligned} MSE(\hat{C}_{p;a}) &= Var(\hat{C}_{p;a}) + [Bias(\hat{C}_{p;a})]^2 \\ &\approx (C_{p;a})^2\left\{\frac{F(n,\phi)}{2(n-1)^2\,[f(n,\phi)]^3} + [\mathcal{B}(n,\phi)-1]^2\right\}. \end{aligned} \tag{2.30}$$

Numerical values of $Bias(\hat{C}_{p;a})$ for various n and ϕ are presented in Table 2.1. This analysis reveals the behavior of the bias of $\hat{C}_{p;a}$ when an AR(1) process describes the quality characteristic. It is observed that the bias of $\hat{C}_{p;a}$ is positive, indicating that the index is overestimated in the presence of autocorrelation. For a given value of ϕ, the bias decreases as the sample size increases. For a fixed sample size, the bias increases as the value of $|\phi|$ increases. Some numerical values of MSE are presented in Table 2.2, which exhibits properties similar to those of bias.

Table 2.1: Some Numerical Values of $Bias(\hat{C}_{p;a})$ for $C_{p;a} = 1$

ϕ \ n	10	20	25	50	100	150	200
−0.75	0.1758	0.0999	0.0819	0.0430	0.0220	0.0148	0.0111
−0.50	0.0844	0.0440	0.0355	0.0180	0.0091	0.0061	0.0046
−0.25	0.0661	0.0327	0.0261	0.0130	0.0065	0.0043	0.0033
0.25	0.1470	0.0658	0.0515	0.0247	0.0121	0.0080	0.0060
0.50	0.3466	0.1392	0.1066	0.0489	0.0235	0.0154	0.0115
0.75	1.6180	0.4872	0.3494	0.1403	0.0630	0.0406	0.0299

Table 2.2: Some Numerical Values of $MSE\left(\hat{C}_{p;a}\right)$ for $C_{p;a} = 1$

ϕ \ n	10	20	25	50	100	150	200
−0.75	0.1701	0.0890	0.0715	0.0359	0.0179	0.0119	0.0089
−0.50	0.0828	0.0416	0.0333	0.0167	0.0083	0.0056	0.0042
−0.25	0.0605	0.0292	0.0232	0.0115	0.0057	0.0038	0.0028
0.25	0.0997	0.0376	0.0284	0.0127	0.0060	0.0039	0.0029
0.50	0.2993	0.0811	0.0571	0.0219	0.0096	0.0061	0.0045
0.75	3.7601	0.4922	0.2903	0.0754	0.0264	0.0155	0.0109

2.5 The Effect of Measurement Errors on $C_{p;a}$

We denote the index in the presence of measurement error as,

$$C^e_{p;a} = \frac{(U-L)}{6\sigma_Y}. \tag{2.31}$$

When $\{X_t\}$ is an AR(1) process the variance of Y_t becomes

$$\sigma_Y^2 = \sigma_X^2 + {\sigma_E}^2 = {\sigma_X}^2\left[1 + (1-\phi^2){\tau_a}^2\right].$$

The impact of measurement errors on the index can be theoretically analyzed by examining the ratio between the observable index $C^e_{p;a}$ and the true (but unknown) index $C_{p;a}$ given by,

$$\frac{C^e_{p;a}}{C_{p;a}} = \frac{\sqrt{\frac{{\sigma_a}^2}{1-\phi^2}}}{\sqrt{\frac{{\sigma_a}^2}{1-\phi^2} + {\sigma_E}^2}} = \sqrt{\frac{{\sigma_a}^2}{{\sigma_a}^2 + (1-\phi^2){\sigma_E}^2}} = \frac{1}{\sqrt{1+(1-\phi^2){\tau_a}^2}}. \tag{2.32}$$

Clearly, the ratio (2.32) is a decreasing function of τ_a.

2.5.1 Statistical Analysis of $Bias\left(\hat{C}^e_{p;a}\right)$

The estimator of the index $C^e_{p;a}$ is,

$$\hat{C}^e_{p;a} = \frac{U-L}{6S_Y}. \tag{2.33}$$

Theorem 2.2. *Let the quality characteristic $\{X_t\}$ follow an AR(1) process defined by equation (2.1) and the measurement errors $E_t \sim \mathrm{N}(0, \sigma_E^2)$. Also, assume that X_t and E_t are additively linked and stochastically independent. Let, $Y_t = X_t + E_t$ be the observable process. Then the expected value of the estimator $\hat{C}^e_{p;a}$ can be approximated as,*

$$E\left(\hat{C}^e_{p;a}\right) \approx C_{p;a}\ \mathcal{B}^e(n,\phi,\tau_a) \tag{2.34}$$

where,

$$\mathcal{B}^e(n,\phi,\tau_a) = \frac{1}{[f(n,\phi)+(1-\phi^2)\tau_a^2]^{\frac{1}{2}}} \times \left\{1 + \frac{3\left[F(n,\phi)+(n-1)(1-\phi^2)^2\tau_a^4 + 2(n-1)(1-\phi^2)\tau_a^2 f(n,\phi)\right]}{4(n-1)^2[f(n,\phi)+(1-\phi^2)\tau_a^2]^2}\right\}.$$

Proof. From equation (2.33), we have

$$\begin{aligned} E\left(\hat{C}^e_{p;a}\right) &= E\left(\frac{U-L}{6S_Y}\right) \\ &= \frac{U-L}{6} E\left(\frac{1}{S_Y}\right). \end{aligned} \tag{2.35}$$

Using statistical differentials, it can be shown that

$$\begin{aligned} E\left(\frac{1}{S_Y}\right) &\approx \left[E(S_Y^2)\right]^{-\frac{1}{2}} \left\{1 + \frac{3Var(S_Y^2)}{8[E(S_Y^2)]^2}\right\} \\ &= \frac{1}{\sigma_X[f(n,\phi)+(1-\phi^2)\tau_a^2]^{\frac{1}{2}}} \times \\ &\quad \left\{1 + \frac{3\left[F(n,\phi)+(n-1)(1-\phi^2)^2\tau_a^4 + 2(n-1)(1-\phi^2)\tau_a^2 f(n,\phi)\right]}{4(n-1)^2[f(n,\phi)+(1-\phi^2)\tau_a^2]^2}\right\}. \end{aligned} \tag{2.36}$$

By substituting the result from equation (2.36) into equation (2.35), we obtain the approximate expectation as shown in equation (2.34). □

The bias of the estimator is

$$Bias\left(\hat{C}^e_{p;a}\right) = E\left(\hat{C}^e_{p;a}\right) - C_{p;a} \approx C_{p;a}\left[\mathcal{B}^e(n,\phi,\tau_a) - 1\right]. \tag{2.37}$$

Table 2.3: Some Numerical Values of $Bias\left(\hat{C}^e_{p;a}\right)$ for $C_{p;a} = 1$

ϕ \ n	10	20	25	50	100	150	200
$\tau_a = 0.2$							
−0.75	0.1622	0.0881	0.0706	0.0329	0.0126	0.0056	0.0021
−0.50	0.0676	0.0279	0.0197	0.0028	−0.0059	−0.0088	−0.0103
−0.25	0.0471	0.0140	0.0075	−0.0054	−0.0118	−0.0140	−0.0150
0.25	0.1227	0.0451	0.0314	0.0056	−0.0066	−0.0105	−0.0125
0.50	0.3124	0.1179	0.0870	0.0322	0.0078	0.0001	−0.0037
0.75	1.5119	0.4574	0.3267	0.1269	0.0524	0.0307	0.0204
$\tau_a = 0.4$							
−0.75	0.1241	0.0551	0.0390	0.0044	−0.0140	−0.0203	−0.0235
−0.50	0.0216	−0.0159	−0.0235	−0.0392	−0.0471	−0.0497	−0.0511
−0.25	−0.0042	−0.0365	−0.0428	−0.0552	−0.0614	−0.0634	−0.0644
0.25	0.0583	−0.0104	−0.0227	−0.0459	−0.0569	−0.0605	−0.0622
0.50	0.2241	0.0608	0.0342	−0.0136	−0.0351	−0.0419	−0.0453
0.75	1.2444	0.3775	0.2650	0.0894	0.0224	0.0027	−0.0067

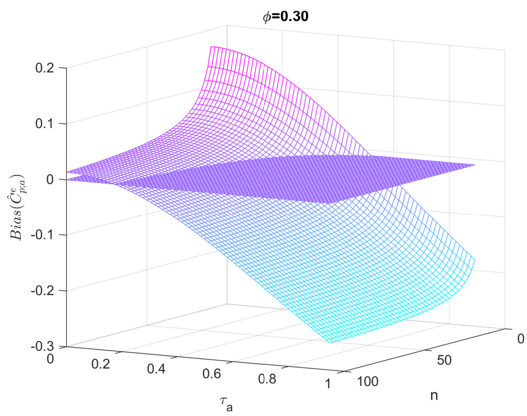

Figure 2.1: $Bias\left(\hat{C}^e_{p;a}\right)$ as a function of n (x axis) and τ_a (y axis) for $\phi = 0.30$ and $C_{p;a} = 1$.

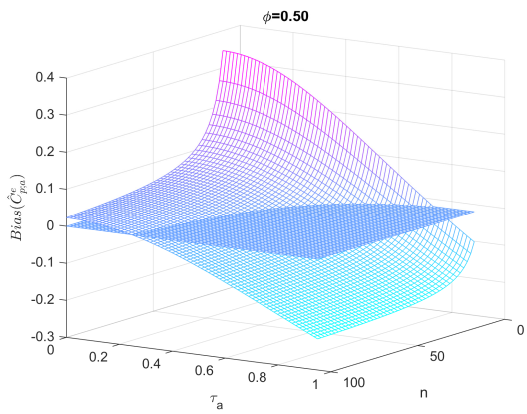

Figure 2.2: $Bias\left(\hat{C}^e_{p;a}\right)$ as a function of n (x axis) and τ_a (y axis) for $\phi = 0.50$ and $C_{p;a} = 1$.

Table 2.4: Some numerical values of τ_1

n \ ϕ	0.75	0.50	0.25	−0.25	−0.50	−0.75
10	1.44401	0.79534	0.53880	0.38650	0.47161	0.81056
20	1.08902	0.55890	0.36880	0.26726	0.33862	0.62354
30	0.90659	0.45426	0.29759	0.21695	0.27883	0.52839
40	0.79188	0.39222	0.25615	0.18740	0.24267	0.46745
50	0.71155	0.35009	0.22825	0.16737	0.21776	0.42395
60	0.65139	0.31911	0.20783	0.15265	0.19924	0.39083
70	0.60420	0.29511	0.19206	0.14123	0.18477	0.36448
80	0.56592	0.27581	0.17941	0.13205	0.17307	0.34288
90	0.53407	0.25985	0.16897	0.12445	0.16334	0.32473
100	0.50704	0.24637	0.16016	0.11803	0.15508	0.30920

To analyze the effect of measurement error on $Bias\left(\hat{C}^e_{p;a}\right)$, we present numerical values in Table 2.3 for $C_{p;a} = 1$ with $\tau_a = 0.2$ and $\tau_a = 0.4$, respectively. Table 2.3 clearly indicates that the bias can be positive, negative, or zero when both autocorrelation and measurement are present. Note that, for $\tau_a = 0$, $\mathcal{B}^e(n, \phi, \tau_a) = \mathcal{B}(n, \phi)$.

We investigate the nature of $Bias\left(\hat{C}^e_{p;a}\right)$ depending on τ_a for fixed values of n and ϕ. Since we cannot perform an analytical comparison, we shall compare the values of $Bias\left(\hat{C}^e_{p;a}\right)$ numerically. We have

$$Bias\left(\hat{C}^e_{p;a}\right) \begin{cases} > 0 & \text{for } 0 \leqslant \tau_a < \tau_1, \\ = 0 & \text{for } \tau_a = \tau_1, \\ < 0 & \text{for } \tau_a > \tau_1. \end{cases} \tag{2.38}$$

Numerical values of $\tau_1 > 0$ for various combinations of n and ϕ are provided in Table 2.4. Finding an analytical expression for τ_1 as a function of n and ϕ is impossible due to the complicated nature of the expression for $Bias\left(\hat{C}^e_{p;a}\right)$. To obtain values of τ_1 for specific n and ϕ, one can numerically solve the equation $Bias(\hat{C}^e_{p;a}) = 0$ for τ_a, focusing on the positive real root using computer-based equation-solving methods. As shown in Table 2.4, for a fixed value of ϕ, as n increases, τ_1 decreases. Also, for a fixed value of n as $|\phi|$ increases, the value of τ_1 increases. In order to understand the nature of $Bias\left(\hat{C}^e_{p;a}\right)$ properly, we present a surface plot of $Bias\left(\hat{C}^e_{p;a}\right)$ intersecting with the plane $z = 0$ in Figures 2.1 and 2.2. The points above the plane indicate a positive bias, while those below the plane represent a negative bias.

To complete our analysis of the behavior of $Bias\left(\hat{C}^e_{p;a}\right)$, we provide a numerical comparison between the absolute values of $Bias\left(\hat{C}^e_{p;a}\right)$ and the bias without measurement error, i.e., $Bias\left(\hat{C}_{p;a}\right)$. From some numerical computation and graphical presentations,

Table 2.5: Some numerical values of τ_2

n \ ϕ	0.75	0.50	0.25	−0.25	−0.50	−0.75
10	–	1.62759	0.86576	0.57609	0.71596	1.34791
20	2.79792	0.89322	0.54982	0.38767	0.49613	0.96158
30	1.68029	0.69027	0.43503	0.31198	0.40368	0.79220
40	1.32938	0.58350	0.37109	0.26834	0.34926	0.69074
50	1.13980	0.51482	0.32897	0.23906	0.31230	0.62096
60	1.01550	0.46585	0.29855	0.21766	0.28507	0.56907
70	0.92551	0.42866	0.27525	0.20115	0.26392	0.52849
80	0.85629	0.39916	0.25667	0.18790	0.24689	0.49558
90	0.80081	0.37503	0.24141	0.17697	0.23278	0.46820
100	0.75501	0.35480	0.22858	0.16775	0.22085	0.44494

it can be shown that the condition of equation (2.39) holds

$$Bias\left(\hat{C}_{p;a}\right)\begin{cases} > |Bias\left(\hat{C}^e_{p;a}\right)| & \text{for } 0 < \tau_a < \tau_2, \\ = |Bias\left(\hat{C}^e_{p;a}\right)| & \text{for } \tau_a = \tau_2, \\ < |Bias\left(\hat{C}^e_{p;a}\right)| & \text{for } \tau_a > \tau_2. \end{cases} \tag{2.39}$$

Numerical values of $\tau_2 > 0$ for various combinations of n and ϕ are presented in Table 2.5. The expression for τ_2 as a function of n and ϕ cannot be determined analytically. To obtain numerical values of τ_2 for specified values of n and ϕ, one can solve the equation $Bias\left(\hat{C}_{p;a}\right) - |Bias\left(\hat{C}^e_{p;a}\right)| = 0$ for τ_a using computer-based methods to find positive real solutions. The combination of n and ϕ for which τ_2 does not exist is such that $Bias\left(\hat{C}_{p;a}\right)$ is always greater than $|Bias\left(\hat{C}^e_{p;a}\right)|$ for any real positive value of τ_a. It is interesting to observe from Table 2.5 that for each fixed value of ϕ, as n increases, the values of τ_2 decrease. For each fixed value of n, as $|\phi|$ increases, the values of τ_2 also increase. To properly visualize this situation, we present a surface plot of $Bias\left(\hat{C}_{p;a}\right) - |Bias\left(\hat{C}^e_{p;a}\right)|$ with an intersection at the plane $z = 0$ in Figure 2.3. The points on the surface above the plane $z = 0$ correspond to $Bias\left(\hat{C}_{p;a}\right) > |Bias\left(\hat{C}^e_{p;a}\right)|$, while the points below the plane correspond to $Bias\left(\hat{C}_{p;a}\right) < |Bias\left(\hat{C}^e_{p;a}\right)|$.

2.5.2 Statistical Analysis of $MSE\left(\hat{C}^e_{p;a}\right)$

The variance of $\hat{C}^e_{p;a}$ is given in the following theorem.

Theorem 2.3. *Under the assumption of Theoren 2.2, the variance of $\hat{C}^e_{p;a}$ can be approximated as*

$$Var\left(\hat{C}^e_{p;a}\right) = (C_{p;a})^2\mathcal{M}^e(n, \phi, \tau_a) \tag{2.40}$$

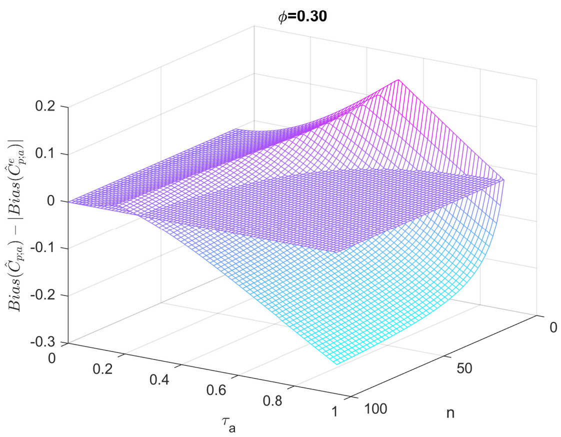

Figure 2.3: $Bias\left(\hat{C}_{p;a}\right) - |Bias\left(\hat{C}^e_{p;a}\right)|$ as a function of n (x-axis) and τ_a (y-axis) for $\phi = 0.30$ and $C_{p;a} = 1$.

where,

$$\mathcal{M}^e(n, \phi, \tau_a) = \frac{1}{2(n-1)^2}\left\{\frac{F(n,\phi) + (n-1)(1-\phi^2)^2{\tau_a}^4 + 2(n-1)(1-\phi^2){\tau_a}^2 f(n,\phi)}{\left[f(n,\phi) + (1-\phi^2){\tau_a}^2\right]^3}\right\}.$$

Proof. From equation (2.33), we have

$$\begin{aligned} Var\left(\hat{C}^e_{p;a}\right) &= Var\left(\frac{U-L}{6S_Y}\right) \\ &= \left(\frac{U-L}{6}\right)^2 Var\left(\frac{1}{S_Y}\right). \end{aligned} \tag{2.41}$$

Using statistical differentials, it follows that

$$\begin{aligned} Var\left(\frac{1}{S_Y}\right) &\approx \frac{Var(S_Y^2)}{4[E(S_Y^2)]^3} \\ &= \frac{F(n,\phi) + (n-1)(1-\phi^2)^2\tau_a^4 + 2(n-1)(1-\phi^2)\tau_a^2 f(n,\phi)}{2\sigma_X^2(n-1)^2\left[f(n,\phi) + (1-\phi^2)\tau_a^2\right]^3}. \end{aligned} \tag{2.42}$$

By substituting the relation from equation (2.42) into equation (2.41), we obtain the approximate variance expressed in equation (2.40). □

The expression for the MSE of the estimator is

$$MSE\left(\hat{C}^e_{p;a}\right) = Var\left(\hat{C}^e_{p;a}\right) + \left\{Bias\left(\hat{C}^e_{p;a}\right)\right\}^2 = \left(C_{p;a}\right)^2 \mathcal{H}^e(n,\phi,\tau_a) \tag{2.43}$$

where,

$$\mathcal{H}^e(n,\phi,\tau_a) = \mathcal{M}^e(n,\phi,\tau_a) + \left[\mathcal{B}^e(n,\phi,\tau_a) - 1\right]^2.$$

Since the estimator $\hat{C}^e_{p;a}$ is not unbiased, it is preferable to analyze the MSE instead of the variance. We now present numerical values of $MSE\left(\hat{C}^e_{p;a}\right)$ for $\tau_a = 0.2$ and $\tau_a = 0.4$

Table 2.6: Some Numerical Values of $MSE\left(\hat{C}^e_{p;a}\right)$ for $C_{p;a} = 1$

ϕ \ n	10	20	25	50	100	150	200
$\tau_a = 0.2$							
-0.75	0.1606	0.0838	0.0673	0.0337	0.0169	0.0113	0.0085
-0.50	0.0769	0.0386	0.0309	0.0155	0.0079	0.0053	0.0040
-0.25	0.0562	0.0272	0.0217	0.0108	0.0056	0.0038	0.0029
0.25	0.0887	0.0336	0.0255	0.0115	0.0056	0.0038	0.0029
0.50	0.2630	0.0717	0.0505	0.0195	0.0086	0.0055	0.0041
0.75	3.3367	0.4484	0.2654	0.0691	0.0241	0.0142	0.0100
$\tau_a = 0.4$							
-0.75	0.1365	0.0709	0.0571	0.0291	0.0150	0.0103	0.0080
-0.50	0.0644	0.0333	0.0272	0.0150	0.0090	0.0070	0.0060
-0.25	0.0485	0.0254	0.0211	0.0126	0.0086	0.0072	0.0065
0.25	0.0665	0.0276	0.0219	0.0122	0.0082	0.0069	0.0063
0.50	0.1833	0.0519	0.0372	0.0159	0.0086	0.0065	0.0056
0.75	2.3794	0.3425	0.2044	0.0539	0.0192	0.0116	0.0085

in Table 2.6, considering $C_{p;a} = 1$, and for various combinations of sample size n and autocorrelation parameter ϕ.

To analyze the impact of measurement errors on the estimator $\hat{C}^e_{p;a}$, we compare the values of $MSE\left(\hat{C}^e_{p;a}\right)$ and $MSE\left(\hat{C}_{p;a}\right)$. Numerical calculations and graphical presentations demonstrate the relationship between $MSE\left(\hat{C}^e_{p;a}\right)$ and $MSE\left(\hat{C}_{p;a}\right)$ as expressed in equation (2.44).

$$MSE\left(\hat{C}_{p;a}\right) \begin{cases} > MSE\left(\hat{C}^e_{p;a}\right) & \text{for } 0 < \tau_a < \tau_3, \\ = MSE\left(\hat{C}^e_{p;a}\right) & \text{for } \tau_a = \tau_3, \\ < MSE\left(\hat{C}^e_{p;a}\right) & \text{for } \tau_a > \tau_3. \end{cases} \tag{2.44}$$

Numerical values of $\tau_3 > 0$ for various combinations of n and ϕ are presented in Table 2.7. It is not possible to explicitly express τ_3 as a function of n and ϕ. Numerical values of τ_3 for given values of n and ϕ can be determined by solving the equation $MSE\left(\hat{C}_{p;a}\right) - MSE\left(\hat{C}^e_{p;a}\right) = 0$ for τ_a. This involves finding positive real solutions using computer-based equation-solving methods. The combination of n and ϕ for which τ_3 is not available indicates that $MSE\left(\hat{C}_{p;a}\right)$ is always greater than $MSE\left(\hat{C}^e_{p;a}\right)$ for any positive real value of τ_a. From Table 2.7, it is clear that for each fixed ϕ, the variable τ_3 decreases as n increases. Additionally, for each fixed n, as $|\phi|$ increases, the values of τ_3 also increase. To properly illustrate this situation, we present a surface plot of $MSE\left(\hat{C}_{p;a}\right) - MSE\left(\hat{C}^e_{p;a}\right)$ in Figure 2.4, showing the intersection with the plane $z = 0$. The points on the surface of Figure 2.4 above the plane $z = 0$ correspond to $MSE\left(\hat{C}_{p;a}\right) > MSE\left(\hat{C}^e_{p;a}\right)$, while the points below the plane correspond to $MSE\left(\hat{C}_{p;a}\right) < MSE\left(\hat{C}^e_{p;a}\right)$.

Table 2.7: Some numerical values of τ_3

n \ ϕ	0.75	0.50	0.25	-0.25	-0.50	-0.75
10	–	2.50195	1.17302	0.82837	1.11533	2.20354
20	5.07690	1.19618	0.71534	0.55497	0.77514	1.53529
30	2.40275	0.90650	0.55997	0.44689	0.63461	1.26602
40	1.81499	0.76107	0.47524	0.38470	0.55172	1.10884
50	1.52932	0.66931	0.42004	0.34296	0.49514	1.00166
60	1.35140	0.60454	0.38043	0.31243	0.45325	0.92210
70	1.22629	0.55563	0.35025	0.28885	0.42058	0.85976
80	1.13178	0.51700	0.32626	0.26993	0.39415	0.80909
90	1.05695	0.48547	0.30660	0.25430	0.37218	0.76677
100	0.99569	0.45911	0.29012	0.24110	0.35355	0.73068

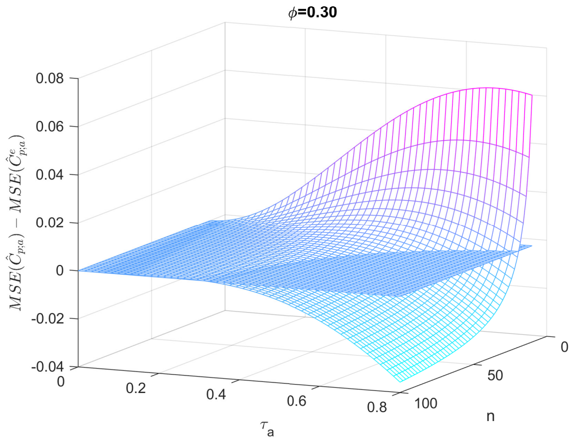

Figure 2.4: $MSE\left(\hat{C}_{p;a}\right) - MSE\left(\hat{C}^e_{p;a}\right)$ as a function of n (x-axis) and τ_a (y-axis) for $\phi = 0.30$ and $C_{p;a} = 1$.

2.6 Parameters Estimation and Implementation of the Results

Our results clearly indicate that τ_a is crucial in analyzing the statistical properties of the estimator for C_p. Although there is a significant amount of literature on gauge capability studies, estimating the unknown variance σ_a^2 of a_t and the measurement error variance σ_E^2 is quite complicated in this case. This complexity arises because the set $\{X_t\}$ is not independent but is auto-correlated and unobservable. Nonetheless, this issue can be addressed by utilizing the state space representation of the model. Basseville and Nikiforov [7] noted that state-space models are highly useful for modeling dynamic systems that describe manufacturing processes and for monitoring measurement devices. However, since X_t is not observable, what we essentially have is

$$Y_t = X_t + E_t.$$

In this setup, the unobservable variable X_t represents the state equation, while Y_t, the observed quality characteristics, represents the observation equation. The errors a_t and E_t follow a Gaussian distribution, as mentioned earlier. For a given initial state $X_0 \sim N(\mu_0, \sigma_0^2)$, the innovation sequence

$$\epsilon_t = Y_t - E\left(Y_t | Y_{t-1}, \cdots, Y_1\right)$$

may be obtained using the Kalman filter recursions procedure. The parameters σ_a^2, σ_E^2, and ϕ can be estimated using the maximum likelihood approach, which takes the prediction error decomposition of the likelihood function, refer Shumway et al. [104]. Noomene [84] examined the efficiency of various parameter estimation procedures for an AR(1) model with measurement error, using optimization methods.

Liu and He [74] have considered the same problem and obtained the estimates of the parameters. Let $\gamma_Y(k)$ be the autocovariance function of the process $\{Y_t\}$ at lag k. Then, estimates of the parameters are

$$\hat{\phi} = \frac{\hat{\gamma}_Y(2)}{\hat{\gamma}_Y(1)},$$

$$\hat{\sigma}_E^2 = \hat{\gamma}_Y(0) - \frac{\hat{\gamma}_Y^2(1)}{\hat{\gamma}_Y(2)}$$

and

$$\hat{\sigma}_a^2 = \frac{\hat{\gamma}_Y^2(1)}{\hat{\gamma}_Y(2)} - \hat{\gamma}_Y(2).$$

See Liu and He [74] for details. These results provide a reliable estimate, provided that the sample size is sufficiently large. They have also demonstrated that these estimates are consistent and follow an asymptotic normal distribution.

To analyze the estimated value of the capability index C_p, it is necessary to compare the value of τ_a with τ_1, τ_2, and τ_3. However, these values depend on the process parameters ϕ, σ_a^2, and σ_E^2. In many processes, the values of these parameters are unknown but can be estimated from an initial set of in-control data. After estimating these parameters, we can analyze the estimated value of C_p using a sample from the in-control process. We shall discuss the implementation of our results in two scenarios: one based on simulated data and the other from real industrial data.

2.6.1 Simulated data

To estimate the parameters of the model $\{Y_t\}$, we simulate a random data of size 500 of an AR(1) model with $\mu = 0$, $\phi = 0.50$, $\sigma_a^2 = 4$ and $\sigma_E^2 = 1$ (i.e., $\tau_a = 0.5$). Using the maximum likelihood parameter estimation procedure based on the Kalman filter recursion process described in Section 4, we obtain $\hat{\phi} = 0.46$, $\hat{\sigma}_a = 1.95$, and $\hat{\sigma}_E = 0.99$. Therefore, the estimated value of τ_a, i.e., $\hat{\tau}_a = 0.51$. These are the invariant parameter values of the model.

Suppose at some time, capability index C_p is estimated based on a sample size of 100 from the in-control process. To obtain some approximate knowledge about this estimated value, we have to calculate the values of τ_1, τ_2, τ_3 for $\phi = 0.46$ and $n = 100$ and have to compare these values with $\hat{\tau}_a = 0.51$. In this case we obtain $\tau_1 = 0.2270$, $\tau_2 = 0.3261$, and $\tau_3 = 0.4205$.

Since $\hat{\tau}_a > \tau_1$, we can conclude from equation (2.38) that the estimate will, on an average, underestimate in this case. Here, since $\hat{\tau}_a > \tau_2$, we can conclude from the results of equation (2.39) that the absolute values of the bias of the estimator will be greater than the bias in the actual case, that is, in the error-free scenario. Since $\hat{\tau}_a > \tau_3$, from equation (2.44) we can conclude that in this case, the MSE is likely to be larger than the actual MSE.

2.6.2 Industrial Data

Here, we consider the example of the 125 g yogurt cup filling process given in Costa and Castagliola [27]. A long-term study (Phase I) utilizing a large database of yogurt cup weights has shown that this data follows an AR(1) process and some measurement error is present. The findings indicate that the parameter ϕ is 0.38, the mean μ is 124.9, and the standard deviation of the process σ_X is 0.76. The standard deviation of the measurement error is $\sigma_E = 0.24$. From these values, we get the standard deviation of the white noise term a_t is $\sigma_a = 0.7030$, therefore $\hat{\tau}_a = 0.34$. If we estimate the capability index C_p for a process for given specific upper and lower limits, we can analyze the estimated value by comparing $\hat{\tau}_a$ with τ_1, τ_2, and τ_3. Suppose C_p is estimated based on a sample of size $n = 100$, then $\tau_1 = 0.1958$, $\tau_2 = 0.2804$, $\tau_3 = 0.3590$.

In this case, $\hat{\tau}_a > \tau_1$, hence from equation (2.38), we can say that the estimate will be on the average underestimate in this case. Here $\hat{\tau}_a > \tau_2$, using the results from equation (2.39), we can conclude that the absolute values of the bias of the estimator will be larger than the bias in the actual case, i.e., the error-free case. Since $\hat{\tau}_a < \tau_3$, we can state from equation (2.44) that in this case, the MSE will be less than the actual MSE.

2.7 Concluding Remarks

In this Chapter, we discuss the statistical properties of the process capability index C_p estimator for autocorrelated data, considering random measurement errors, a scenario that frequently arises in industrial applications. The results of our analysis suggest that the presence of measurement errors in the observations causes the behavior of the estimator to vary based on the level of variability in those errors.

Our results indicate that measurement errors in autocorrelated observations cause the estimator of the index C_p to be generally biased, with the nature of this bias differing significantly from that in error-free scenarios. Bias can be positive, negative, or zero, while in error-free cases, it is always positive.

Our analysis indicates that, for appropriate combinations of n and the autoregressive parameters ϕ, there exist values of τ_a such that $|Bias(\hat{C}^e_{p;a})|$ does not exceed $Bias(\hat{C}_{p;a})$. This suggests that a small value of τ_a does not significantly impact the estimation of the capability index C_p.

Our analysis of MSE of the estimator reveals that the value of $MSE(\hat{C}^e_{p;a})$ cannot exceed $MSE(\hat{C}_{p;a})$, provided that the value of τ_a does not surpass the corresponding value of τ_3 depending on the values of n and ϕ. This indicates that small measurement errors in observations are not a significant issue when estimating the process capability index C_p.

Chapter 3

Statistical Analysis of C_{pm} in Presence of Autocorrelation and Measurement Errors

3.1 Introduction

In today's competitive business, an ideal value of quality characteristics, or target value, is considered alongside the specification limits to assess process performance. The capability index C_{pm}, defined in equation (1.4), is important as it accounts for the deviation of quality characteristics from the target value. PCIs are metrics derived from sample observations of a process. Some uncertainty exists in the estimated capability index due to sampling error. It is essential to understand the behavior of the estimator before evaluating the effectiveness of the process based on the estimated value of the index.

Most of the research on C_{pm} assumes that the quality characteristics follow a normal distribution and are independent and identically distributed. The assumption that observations are independent is not applicable to certain processes. Certain quality characteristics show a natural correlation with themselves over time. Guevara and Vargas [47] derived an approximate variance for C_{pm} and C_{pmk} when the quality characteristics being studied follow a stationary Gaussian process. In their study, Sun et al. [108] examined various properties of C_{pm} in the context of an autoregressive model of order one (AR(1)). Recently, Srivastava et al. [106] estimated the exact expression of moments for the estimator of C_{pm} and analyzed the statistical properties based on these moments.

One issue that much of the existing research has overlooked is the error associated with gauge measurements. The existence of measurement error in the measuring items is an inherent characteristic of the measurement process and cannot be avoided, even with advanced modern measuring instruments. Bordignon and Scagliarini [12] investigated the statistical characteristics of the estimator for C_{pm} considering the presence of random measurement errors. For information regarding the impact of measurement errors on various statistical process monitoring schemes, we refer to Maleki et al. [76].

So far, the issues of autocorrelation and measurement error have been addressed separately in relation to C_{pm}. To the best of our knowledge, no study has examined C_{pm} while considering both autocorrelation and measurement errors. In Chapter 2, we studied the behavior of C_p considering both autocorrelation and measurement errors. In this chapter, we explore the statistical properties of C_{pm} under the same assumptions. This chapter is organized as follows. In Section 3.2, we examine the impact of autocorrelation on C_{pm} and provide an estimate for its lower confidence bound. In Section 3.3, we analyze the combined effects of autocorrelation and measurement error on the bias and MSE of the estimator for C_{pm}. An application is provided in Section 3.4, and conclusions are drawn in Section 3.5.

3.2 Assessment of C_{pm} for Autocorrelated Data

For autocorrelated data, we use the notation $C_{pm;a}$ ('a' to emphasize that the data are autocorrelated) and is defined as follows

$$C_{pm;a} = \frac{U-L}{6\sqrt{E(X_t-T)^2}} = \frac{U-L}{6\sqrt{\sigma_X^2+(\mu-T)^2}} = \frac{C_{p;a}}{\sqrt{1+\xi^2}} \tag{3.1}$$

where,

$$C_{p;a} = \frac{U-L}{6\sigma_X} \quad \text{and} \quad \xi = \frac{(\mu-T)}{\sigma_X}. \tag{3.2}$$

Let $\{X_1, X_2, X_3, \ldots, X_n\}$ be the n consecutive random observations from a stationary Gaussian process $\{X_t\}$. The sample estimator of $C_{pm;a}$ is

$$\hat{C}_{pm;a} = \frac{U-L}{6\widetilde{S}} \tag{3.3}$$

where,

$$\widetilde{S}^2 = \frac{\sum_{i=1}^n (X_i-T)^2}{n}.$$

Note that $\widetilde{S}^2$ is an unbiased estimator of $E(X_t-T)^2$. Under the assumption that $\{X_t\}$ is a stationary Gaussian process, we derive the expectation and variance of $\widetilde{S}^2$ in the following:

$$\begin{aligned}\widetilde{S}^2 &= \sigma_X^2+(\mu-T)^2 = \sigma_X^2\left[1+\xi^2\right] \\ &= \frac{1}{n}\left\{\sum_{i=1}^n (X_i-\mu)^2 + 2(\mu-T)\sum_{i=1}^n (X_i-\mu) + \sum_{i=1}^n (\mu-T)^2\right\}.\end{aligned} \tag{3.4}$$

Taking expectation on both sides of the equation (3.4), we get

$$E\left(\widetilde{S}^2\right) = \sigma_X^2+(\mu-T)^2. \tag{3.5}$$

Similarly, taking variance on both sides of the equation (3.4) we get

$$\begin{aligned}Var(\widetilde{S}^2) &= \frac{1}{n^2}Var\left[\sum_{i=1}^n (X_i-\mu)^2 + 2(\mu-T)\sum_{i=1}^n (X_i-\mu) + \sum_{i=1}^n (\mu-T)^2\right] \\ &= \frac{1}{n^2}Var\left[\sum_{i=1}^n (X_i-\mu)^2\right] + \frac{4(\mu-T)^2}{n^2}Var\left[\sum_{i=1}^n (X_i-\mu)\right].\end{aligned} \tag{3.6}$$

Now,

$$Var\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] = \sum_{i=1}^{n}Var(X_i-\mu)^2+\sum_{i=1}^{n}\sum_{(i\neq j)j=1}^{n}Cov\left[(X_i-\mu)^2,(X_j-\mu)^2\right]$$
$$= 2n\sigma_X^4+4\sigma_X^4\sum_{i=1}^{n-1}(n-i)\rho_i^2. \quad (3.7)$$

$$Var\left[\sum_{i=1}^{n}(X_i-\mu)\right] = \sum_{i=1}^{n}Var(X_i-\mu)+\sum_{i=1}^{n}\sum_{(i\neq j)j=1}^{n}Cov\left[(X_i-\mu),(X_j-\mu)\right]$$
$$= n\sigma_X^2+2\sigma_X^2\sum_{i=1}^{n}(n-i)\rho_i. \quad (3.8)$$

Now substituting the results (3.7) and (3.8) into equation (3.6) we get

$$Var(\widetilde{S}^2)=\frac{2\sigma_X^4}{n^2}G(n,\rho_i,\xi) \quad (3.9)$$

where,

$$G(n,\rho_i,\xi)=n+2\sum_{i=1}^{n-1}(n-i)\rho_i^2+2\xi^2\left[n+2\sum_{i=1}^{n-1}(n-i)\rho_i\right]. \quad (3.10)$$

We have the following theorem.

Theorem 3.1. *Assume that the quality characteristic $\{X_t\}$ is described by a discrete-time stationary Gaussian process. Then the approximate expressions of expectation and variance of the estimator $\hat{C}_{pm;a}$ are*

$$E(\hat{C}_{pm;a})\approx\frac{C_{p;a}}{\sqrt{1+\xi^2}}\left[1+\frac{3G(n,\rho_i,\xi)}{4n^2[1+\xi^2]^2}\right] \quad (3.11)$$

and

$$Var(\hat{C}_{pm;a})\approx\frac{C_{p;a}^2}{[1+\xi^2]^3}\frac{G(n,\rho_i,\xi)}{2n^2}, \quad (3.12)$$

where $\rho_i=\frac{\gamma(i)}{\gamma(0)}$ for $i=1,2,\dots,n$ is the autocorrelation of X_t at lag i, $\xi=\frac{\mu-T}{\sigma_X}$ and $G(n,\rho_i,\xi)$ is given by equation (3.10).

Proof. By taking the expectation of both sides of equation (3.3), we obtain:

$$E(\hat{C}_{pm;a})=\frac{U-L}{6}E\left(\frac{1}{\widetilde{S}}\right). \quad (3.13)$$

The expression for $E\left(\frac{1}{\widetilde{S}}\right)$ is difficult to calculate exactly, however approximations can be obtained by using the method of statistical differential. Thus utilizing equations (3.5) and (3.9) we get,

$$E(\frac{1}{\widetilde{S}})\approx[E(\widetilde{S}^2)]^{-\frac{1}{2}}\left[1+\frac{3Var(\widetilde{S}^2)}{8[E(\widetilde{S}^2)]^2}\right]$$
$$=\frac{1}{\sigma_X\sqrt{1+\xi^2}}\left[1+\frac{3G(n,\rho_i,\xi)}{4n^2(1+\xi^2)^2}\right]. \quad (3.14)$$

By substituting the result from equation (3.14) into equation (3.13), we obtain the approximate expectation expressed in equation (3.11).

Similarly,

$$Var(\hat{C}_{pm;a}) = \left(\frac{U-L}{6}\right)^2 Var\left(\frac{1}{\widetilde{S}}\right). \tag{3.15}$$

By applying the methods of statistical differentials and using equations (3.5) and (3.9), we obtain the following results:

$$\begin{aligned} Var\left(\frac{1}{\widetilde{S}}\right) &\approx \frac{Var(\widetilde{S}^2)}{4[E(\widetilde{S}^2)]^3} \\ &= \frac{G(n,\rho_i,\xi)}{2n^2\sigma_X^2(1+\xi^2)^3}. \end{aligned} \tag{3.16}$$

By substituting equation (3.16) into equation (3.15), we can derive the approximate variance given in equation (3.12). □

To check how good the approximations are in equations (3.14) and (3.16), we will perform a simulation based on an AR(1) process to compare their exact and estimated values. Here we simulate a model from a stationary AR(1) process defined by equation (2.1) with $\mu = 5$, $a_t \sim N(0,1)$. Let the target value be $T = 4.7$. The exact and estimated values for different values of ϕ and n are reported in Table (3.1). Here, the exact value is derived from the approximate expression, and the estimated values are the average of $10,000$ simulated values. From this table, we can observe that the exact and estimated values are very close to each other.

Table 3.1: Exact and estimated values of $E(\frac{1}{\widetilde{S}})$ and $Var(\frac{1}{\widetilde{S}})$.

ϕ	n	$E(\frac{1}{\widetilde{S}})$	$\hat{E}(\frac{1}{\widetilde{S}})$	$Var(\frac{1}{\widetilde{S}})$	$\hat{Var}(\frac{1}{\widetilde{S}})$
0.25	25	0.9702	0.9625	0.0250	0.0240
	50	0.9509	0.9458	0.0121	0.0113
	75	0.9421	0.9397	0.0078	0.0074
	100	0.9406	0.9368	0.0057	0.0054
0.50	25	0.8857	0.8805	0.0268	0.0285
	50	0.8633	0.8583	0.0140	0.0136
	75	0.8546	0.8520	0.0092	0.0089
	100	0.8505	0.8481	0.0068	0.0066

We discuss a special case of stationary Gaussian processes, namely an AR(1) process as defined in equation (2.1). For an AR(1) process, the expression in (3.10) simplifies to:

$$G(n,\phi,\xi) = n + 2\sum_{i=1}^{n-1}(n-i)\phi^{2i} + 2\xi^2\left[n + 2\sum_{i=1}^{n-1}(n-i)\phi^i\right].$$

In this case, the expected value of $\hat{C}_{pm;a}$ can be expressed as follows:

$$E(\hat{C}_{pm;a}) \approx \mathcal{B}_1(n,\phi,\xi)C_{p;a} \tag{3.17}$$

where,

$$\mathcal{B}_1(n,\phi,\xi) = \frac{1}{\sqrt{1+\xi^2}}\left[1 + \frac{3G(n,\phi,\xi)}{4n^2[1+\xi^2]^2}\right].$$

Hence, the bias of the estimator $\hat{C}_{pm;a}$ is

$$\begin{aligned} Bias(\hat{C}_{pm;a}) &= E(\hat{C}_{pm;a}) - C_{pm;a} \\ &\approx C_{p;a}\left[\mathcal{B}_1(n,\phi,\xi) - \frac{1}{\sqrt{1+\xi^2}}\right]. \end{aligned} \quad (3.18)$$

Similarly using formula (3.12), the variance of $\hat{C}_{pm;a}$ can be expressed as

$$Var(\hat{C}_{pm;a}) \approx \mathcal{M}_1(n,\phi,\xi)C_{p;a}^2 \quad (3.19)$$

where,

$$\mathcal{M}_1(n,\phi,\xi) = \frac{1}{[1+\xi^2]^3}\frac{G(n,\phi,\xi)}{2n^2}.$$

The mean square error (MSE) of the estimator is

$$\begin{aligned} MSE(\hat{C}_{pm;a}) &= Var(\hat{C}_{pm;a}) + \{Bias(\hat{C}_{pm;a})\}^2 \\ &\approx C_{p;a}^2\left[\mathcal{M}_1(n,\phi,\xi) + \left\{\mathcal{B}_1(n,\phi,\xi) - \frac{1}{\sqrt{1+\xi^2}}\right\}^2\right] \\ &= \mathcal{H}_1(n,\phi,\xi)C_{p;a}^2 \end{aligned} \quad (3.20)$$

where,

$$\mathcal{H}_1(n,\phi,\xi) = \left[\mathcal{M}_1(n,\phi,\xi) + \left\{\mathcal{B}_1(n,\phi,\xi) - \frac{1}{\sqrt{1+\xi^2}}\right\}^2\right].$$

Some numerical values of $Bias(\hat{C}_{pm;a})$ is reported in Table 3.2 for various combinations of n, ϕ and ξ. It can be observed from the table that bias decreases as the sample size n increases, while bias increases as $|\phi|$ increases. Here, for $C_{p;a} > 0$, bias is always positive.

3.2.1 Estimation of Lower Confidence Bound for $C_{pm;a}$

Since the process parameters are estimated from sample values, the estimator of the index is influenced by sampling error. Therefore, the point estimator is not a reliable method for estimating the index. Thus, we need to calculate a confidence interval. For PCIs, estimating a lower confidence bound is sufficient, as the focus is on the minimum capability of interest. The lower confidence bound of a PCI indicates its minimum possible value. In this section, we estimate a lower confidence bound for $C_{pm;a}$ when the sample observations follow an AR(1) model. Since the sample exhibits autocorrelation, the quantity $n\widetilde{S}^2/\sigma_X^2$ does not follow a chi-squared distribution. Due to the resemblance noted by Wallgren [118], it is reasonable to approximate this as a scaled χ^2 distribution of the form $c\chi_v^2$, as indicated by Boyles [14]. The parameters c and v can be determined by equating the mean and variance of the two distributions and solving the equations as follows:

$$cv = n(1+\xi^2), \qquad 2c^2v = 2G(n,\phi,\xi). \quad (3.21)$$

Table 3.2: Some numerical values of $Bias(\hat{C}_{pm;a})$ for $C_{p;a} = 1$

n \ ϕ	-0.75	-0.50	-0.25	0.25	0.50	0.75
$\xi = 0$						
25	0.1001	0.0489	0.0338	0.0338	0.0489	0.1001
50	0.0518	0.0247	0.0170	0.0170	0.0247	0.0518
75	0.0349	0.0165	0.0113	0.0113	0.0165	0.0349
100	0.0263	0.0124	0.0085	0.0085	0.0124	0.0263
125	0.0211	0.0100	0.0068	0.0068	0.0100	0.0211
150	0.0177	0.0083	0.0057	0.0057	0.0083	0.0177
$\xi = 0.5$						
25	0.0587	0.0310	0.0246	0.0334	0.0524	0.1092
50	0.0303	0.0156	0.0123	0.0168	0.0267	0.0576
75	0.0204	0.0104	0.0082	0.0112	0.0179	0.0391
100	0.0154	0.0078	0.0062	0.0084	0.0135	0.0296
125	0.0124	0.0063	0.0049	0.0067	0.0108	0.0238
150	0.0103	0.0052	0.0041	0.0056	0.0090	0.0199

As noted by Boyles [14], an approximate $100(1-\alpha)\%$ lower confidence bound for $C_{pm;a}$ is given by:

$$\hat{C}_{pm;a}(\chi^2_\alpha(\hat{v})/\hat{v})^{1/2} \tag{3.22}$$

where χ^2_α is the lower $100\alpha\%$ percentile of the χ^2 distribution and $\hat{v}$ is the estimated degree of freedom obtained by solving equation (3.21) as

$$\hat{v} = \frac{n^2(1+\hat{\xi}^2)^2}{G(n,\hat{\phi},\hat{\xi})}. \tag{3.23}$$

It is important to note that the same result was utilized by Wallgren [118], but he employed the large sample approximation of $G(n,\phi,\xi)$ instead of the exact form as:

$$G(n,\phi,\xi) \approx n\left(\frac{1+\phi^2}{1-\phi^2} + 2\xi^2\frac{1+\phi}{1-\phi}\right). \tag{3.24}$$

For large samples, a chi-square distribution can be approximated using a normal distribution. Hence lower confidence bound (3.22) becomes

$$\hat{C}_{pm;a}\left(1 - Z_{1-\alpha}(1/2\hat{v})^{1/2}\right) \tag{3.25}$$

where Z_α is the $100\alpha\%$ lower percentile for the standard normal distribution.

We now examine the performance of the lower confidence bound provided in (3.22) in terms of its coverage probability (CP) and average value (AV). This will be accomplished using Monte Carlo simulation methods. Without loss of generality, we assume that $U = 3$, $L = -3$, $T = 0$, and we consider the cases $\mu = 0.5, 0.9$, $C_{pm;a} = 0.90, 1.10$, $\phi = 0.25, 0.50, 0.75$, $n = 25, 50, 100, 150$. For the selected values of the parameters μ

Table 3.3: Coverage Probability (CP) and Average Value (AV) of 95% Lower Confidence Bound for $C_{pm;a}$ for Different Combinations of Parameters

			Ig.-Auto		$\phi = 0.25$		$\phi = 0.50$		$\phi = 0.75$	
$C_{pm;a}$	μ	n	CP	AV	CP	AV	CP	AV	CP	AV
0.90	0.5	25	0.8802	0.7290	0.9427	0.9429	0.9165	0.6621	0.8678	0.6365
		50	0.8766	0.7727	0.9468	0.7437	0.9254	0.7167	0.8979	0.6682
		100	0.8726	0.8073	0.9484	0.7880	0.9349	0.7642	0.9143	0.7160
		150	0.8699	0.8234	0.9454	0.8078	0.9349	0.7872	0.9196	0.7438
	0.9	25	0.8539	0.7725	0.9397	0.7265	0.9213	0.6948	0.8734	0.6744
		50	0.8505	0.8035	0.9502	0.7704	0.9382	0.7395	0.9126	0.6925
		100	0.8421	0.8298	0.9516	0.8068	0.9438	0.7810	0.9339	0.7341
		150	0.8403	0.8419	0.9471	0.8232	0.9438	0.8012	0.9342	0.7593
1.10	0.5	25	0.8724	0.9006	0.9413	0.8455	0.9135	0.8126	0.8649	0.7859
		50	0.8675	0.9505	0.9472	0.9111	0.9279	0.8760	0.8963	0.8181
		100	0.8598	0.9910	0.9491	0.9642	0.9352	0.9333	0.9163	0.8738
		150	0.8601	1.0098	0.9437	0.9881	0.9364	0.9612	0.9219	0.9072
	0.9	25	0.8297	1.0543	0.9296	1.0402	0.9139	1.0259	0.8634	1.0137
		50	0.8343	1.0662	0.9431	1.0557	0.9317	1.0432	0.9041	1.0235
		100	0.8286	1.0759	0.9466	1.0684	0.9396	1.0585	0.9287	1.0405
		150	0.8252	1.0803	0.9461	1.0741	0.9432	1.0658	0.9338	1.0500

and ϕ, the white noise variance (σ_a^2) is determined to achieve the desired value of the index $C_{pm;a}$. Table 3.3 presents the coverage probability and the average value for the 95% lower confidence bound. We also examine the case where autocorrelation is present but is ignored when estimating a lower confidence bound. The values of CP and the average of the lower confidence bound, while ignoring autocorrelation with $\phi = 0.50$, are presented in the column labeled Ig.-Auto (meaning ignoring autocorrelation). When ignoring autocorrelation, the corresponding lower confidence bound can be derived using equation (36) from Boyles [14], with sample mean and sample variance estimated from the autocorrelated data.

Table 3.3 shows that ignoring autocorrelation in the sample significantly reduces the coverage probability. Figure 3.1 shows the plot of the CP for the 95% lower confidence bound with sample sizes $n = 25, 50, 100, 150$ over the range $\phi \in [-9, 9]$. It can be observed from the figure that when $|\phi| < 0.4$, the estimated coverage probability closely aligns with the nominal coverage probability of 0.95. For $|\phi| > 0.4$, the estimated coverage probability diverges from the nominal coverage probability, and this divergence is more pronounced when the sample size n is small. Figure 3.2 shows the plot of the average values of the 95% lower confidence bounds of $C_{pm;a}$ for $C_{pm;a} = 1$ and sample sizes of $n = 25, 50, 100,$ and 150 respectively when $\phi \in [-9, 9]$. As the sample size n increases, the estimated lower confidence bound also increases and approaches the true value of the index.

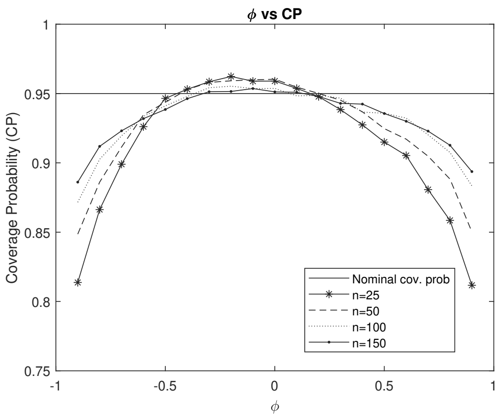

Figure 3.1: Coverage Probability for 95% lower confidence bound..

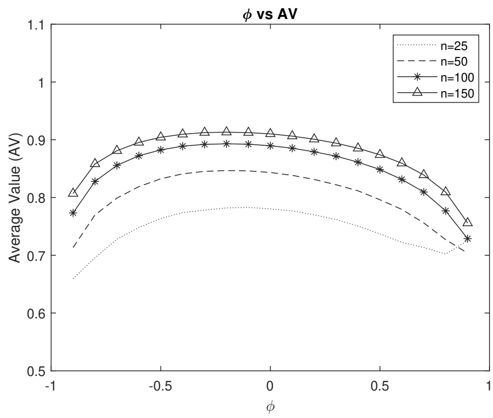

Figure 3.2: Average Value of 95% lower confidence bound.

3.3 Measuring $C_{pm;a}$ in the Presence of Measurement Errors

Now, we consider the case where the observations are subject to random measurement errors. All the assumptions are the same as Section 2.5 of Chapter 2. The observable quality characteristic $\{Y_t\}$ is given by equation (2.7). We observe the process $\{Y_t, t = 1, 2, \dots\}$ instead of true process $\{X_t, t = 1, 2, \dots\}$. When $\{X_t\}$ is an $AR(1)$ process defined by equation (2.1),

$$\begin{aligned} E(Y_t - T)^2 &= \sigma_X^2 + (\mu - T)^2 + \sigma_E^2 \\ &= \sigma_X^2[1 + \xi^2 + (1 - \phi^2)\tau_a^2] \end{aligned} \tag{3.26}$$

where τ_a is defined by equation (2.20).

The impact of measurement errors on the theoretical index can be analyzed by examining the ratio between the observable index $C^e_{pm;a}$ and the true but unobservable index $C_{pm;a}$. The ratio is given by

$$\frac{C^e_{pm;a}}{C_{pm;a}} = \frac{\sqrt{\sigma_X^2\,[1 + \xi^2]}}{\sqrt{\sigma_X^2\,[1 + \xi^2 + (1 - \phi^2){\tau_a}^2]}} = \frac{\sqrt{1 + \xi^2}}{\sqrt{1 + \xi^2 + (1 - \phi^2){\tau_a}^2}}. \tag{3.27}$$

It is evident from ratio (3.27) that the index $C_{pm;a}$ is underestimated due to measurement errors.

Let $\{Y_i, i = 1, 2, \dots, n\}$ be the n consecutive observations from the observable process $\{Y_t\}$. We use these observations to estimate the statistical properties of the estimator

$$\hat{C}^e_{pm;a} = \frac{U - L}{6\widetilde{S}_Y} \tag{3.28}$$

where $\widetilde{S}_Y^2 = \sum_{i=1}^{n}(Y_i - T)^2/n$ is an unbiased estimator of $E(Y_t - T)^2$.

First, we derive the expectation and variance of $\widetilde{S}_Y^2$.

$$\begin{aligned}\widetilde{S}_Y^2 &= \frac{1}{n}\sum_{i=1}^{n}(Y_i - T)^2 \\ &= \widetilde{S}_X^2 + \frac{2}{n}\sum_{i=1}^{n}(X_i - T)E_i + \frac{1}{n}\sum_{i=1}^{n}E_i^2. \\ E\left(\widetilde{S}_Y^2\right) &= E(\widetilde{S}_X^2) + \sigma_E^2 \\ &= \sigma_X^2 + (\mu - T)^2 + \sigma_E^2. \end{aligned} \tag{3.29}$$

Similarly, using the condition that X_t and E_t are stochastically independent, we get

$$\begin{aligned} Var\left(\widetilde{S}_Y^2\right) &= Var\left(\widetilde{S}_X^2\right) + \frac{4}{n^2}Var\left(\sum_{i=1}^{n}(X_i - T)E_i\right) + \frac{1}{n^2}Var\left(\sum_{i=1}^{n}E_i^2\right) \\ &= \frac{2\sigma_X^4}{n^2}G(n,\rho_i,\xi) + \frac{4\sigma_E^2\left[\sigma_X^2 + (\mu - T)^2\right]}{n} + \frac{2\sigma_E^4}{n} \\ &= \frac{2\sigma_X^4}{n^2}\left\{G(n,\rho_i,\xi) + 2n(1+\xi^2)\frac{\sigma_E^2}{\sigma_X^2} + n\frac{\sigma_E^4}{\sigma_X^4}\right\}. \end{aligned} \tag{3.30}$$

Theorem 3.2. *Let the quality characteristic $\{X_t\}$ be described by a discrete-time stationary Gaussian process and the measurement errors E_t be described by a normal distribution. Also, assume that X_t and E_t are additively linked and stochastically independent. Let $Y_t = X_t + E_t$ be the observable process. Then the expected value and variance of the sample estimator $\hat{C}^e_{pm;a}$ can be approximated as*

$$E(\hat{C}^e_{pm;a}) \approx \frac{C_{p;a}}{\sqrt{1+\xi^2+\frac{\sigma_E^2}{\sigma_X^2}}}\left[1 + \frac{3\left[G(n,\rho_i,\xi) + 2n(1+\xi^2)\frac{\sigma_E^2}{\sigma_X^2} + n\frac{\sigma_E^4}{\sigma_X^4}\right]}{4n^2\left[1+\xi^2+\frac{\sigma_E^2}{\sigma_X^2}\right]^2}\right] \tag{3.31}$$

and

$$Var(\hat{C}^e_{pm;a}) \approx C_{p;a}^2\left[\frac{G(n,\rho_i,\xi) + 2n(1+\xi^2)\frac{\sigma_E^2}{\sigma_X^2} + n\frac{\sigma_E^4}{\sigma_X^4})}{2n^2\left[1+\xi^2+\frac{\sigma_E^2}{\sigma_X^2}\right]^3}\right]. \tag{3.32}$$

Proof. By taking the expectations on both sides of equation (3.28), we obtain:

$$E(\hat{C}_{pm;a}) = \frac{U-L}{6}E\left(\frac{1}{\widetilde{S}_Y}\right). \tag{3.33}$$

The exact expression of $E\left(\frac{1}{\widetilde{S}_Y}\right)$ is difficult to calculate, but an approximate expression can be obtained using statistical differential methods as follows:

$$\begin{aligned} E(\frac{1}{\widetilde{S}_Y}) &\approx [E(\widetilde{S}_Y^2)]^{-\frac{1}{2}}\left[1 + \frac{3}{8}\frac{Var(\widetilde{S}_Y^2)}{[E(\widetilde{S}_Y^2)]^2}\right] \\ &= \frac{1}{\sigma_X\sqrt{1+\xi^2+\frac{\sigma_E^2}{\sigma_X^2}}}\left[1 + \frac{3\left[G(n,\rho_i,\xi) + 2n(1+\xi^2)\frac{\sigma_E^2}{\sigma_X^2} + n\frac{\sigma_E^4}{\sigma_X^4}\right]}{4n^2\left[1+\xi^2+\frac{\sigma_E^2}{\sigma_X^2}\right]^2}\right]. \end{aligned} \tag{3.34}$$

By substituting equation (3.34) into equation (3.33), we obtain the approximate expectation presented in equation (3.31).

Similarly,

$$Var(\hat{C}_{pm;a}) = \left(\frac{U-L}{6}\right)^2 Var\left(\frac{1}{\widetilde{S}_Y}\right). \tag{3.35}$$

Again, using the statistical differential technique, we get,

$$\begin{aligned} Var(\frac{1}{\widetilde{S}_Y}) &\approx \frac{Var(\widetilde{S}_Y^2)}{4[E(\widetilde{S}_Y^2)]^3} \\ &= \left[\frac{G(n,\rho_i,\xi) + 2n(1+\xi^2)\frac{\sigma_E^2}{\sigma_X^2} + n\frac{\sigma_E^4}{\sigma_X^4})}{2n^2\sigma_X^2\left[1+\xi^2+\frac{\sigma_E^2}{\sigma_X^2}\right]^3}\right]. \end{aligned} \tag{3.36}$$

By substituting equation (3.36) into equation (3.35), we can easily derive expression (3.32). □

In the case where $\{X_t\}$ is a stationary AR(1) process, then we have the following:

$$E(\hat{C}^e_{pm;a}) \approx \mathcal{B}^e_1\left(n,\phi,\xi,\tau_a\right) C_{p;a} \tag{3.37}$$

where,

$$\begin{aligned} \mathcal{B}^e_1\left(n,\phi,\xi,\tau_a\right) = & \frac{1}{\sqrt{1+\xi^2+(1-\phi^2)\tau_a^2}} \times \\ & \left[1+\frac{3[G(n,\phi,\xi)+2n(1+\xi^2)(1-\phi^2)\tau_a^2+n(1-\phi^2)^2\tau_a^4]}{4n^2\left[1+\xi^2+(1-\phi^2)\tau_a^2\right]^2}\right]; \end{aligned} \tag{3.38}$$

and

$$Var(\hat{C}^e_{pm;a}) \approx \mathcal{M}^e_1(n,\phi,\xi,\tau_a) C^2_{p;a} \tag{3.39}$$

where,

$$\mathcal{M}^e_1(n,\phi,\xi,\tau_a) = \left[\frac{G(n,\phi,\xi)+2n(1+\xi^2)(1-\phi^2)\tau_a^2+n(1-\phi^2)^2\tau_a^4}{2n^2\left[1+\xi^2+(1-\phi^2)\tau_a^2\right]^3}\right]$$

3.3.1 Estimation of Bias in the Presence of Measurement Errors

Centered process:

In this case $\mu = T$ i.e., $\xi = 0$. The bias of the estimated index is

$$\begin{aligned} Bias(\hat{C}^e_{pm;a}) &= E(\hat{C}^e_{pm;a}) - C_{pm;a} \\ &\approx \left[\mathcal{B}^e_1(n,\phi,\tau_a) - 1\right] C_{p;a} \end{aligned} \tag{3.40}$$

Table 3.4: Some numerical values of $Bias(\hat{C}^e_{pm;a})$ for $\xi = 0$ and $C_{p;a} = 1$

ϕ \ n	25	50	75	100	125	150
$\tau_a = 0.2$						
-0.75	0.0882	0.0415	0.0251	0.0168	0.0118	0.0084
-0.50	0.0325	0.0091	0.0013	-0.0027	-0.0051	-0.0067
-0.25	0.0147	-0.0017	-0.0072	-0.0100	-0.0116	-0.0127
0.25	0.0147	-0.0017	-0.0072	-0.0100	-0.0116	-0.0127
0.50	0.0325	0.0091	0.0013	-0.0027	-0.0051	-0.0067
0.75	0.0882	0.0415	0.0251	0.0168	0.0118	0.0084
$\tau_a = 0.4$						
-0.75	0.0549	0.0123	-0.0025	-0.0101	-0.0147	-0.0177
-0.50	-0.0125	-0.0336	-0.0407	-0.0443	-0.0464	-0.0479
-0.25	-0.0368	-0.0521	-0.0572	-0.0598	-0.0613	-0.0624
0.25	-0.0368	-0.0521	-0.0572	-0.0598	-0.0613	-0.0624
0.50	-0.0125	-0.0336	-0.0407	-0.0443	-0.0464	-0.0479
0.75	0.0549	0.0123	-0.0025	-0.0101	-0.0147	-0.0177

where

$$\mathcal{B}^e_1(n, \phi, \tau_a) \approx \frac{1}{\sqrt{1 + (1-\phi^2)\tau_a^2}} \times \left[1 + \frac{3\left[n + 2\sum_{i=1}^{n-1}(n-i)\phi^{2i} + 2n(1-\phi^2)\tau_a^2 + n(1-\phi^2)^2\tau_a^4\right]}{4n^2\left[1 + (1-\phi^2)\tau_a^2\right]^2}\right]. \quad (3.41)$$

We present numerical values of $Bias(\hat{C}^e_{pm;a})$ for various combinations of n, ϕ, and τ_a in Table 3.4. Note that in this case, the behavior of bias is quite complex.

It is clear that in cases of measurement errors, the bias $Bias(\hat{C}^e_{pm;a})$ can be positive, negative, or zero; however, in error-free cases, it is always positive. So, measurement error has a significant impact on the estimation of C_{pm}. Analysis of the surface plots of $Bias(\hat{C}^e_{pm;a})$ in Figures 3.3 and 3.4, along with numerical values, indicates that bias decreases as τ_a increases for any fixed values of n and ϕ. The equation (3.42) holds for bias. Here, $\tau_1^{'}$ is a function of n and ϕ, and its explicit expression cannot be determined. In Table 3.5, we present numerical values of $\tau_1^{'}$ for various combinations of n and ϕ. Table 3.5 shows that as n increases, the value of $\tau_1^{'}$ decreases. Additionally, as the value of ϕ increases, the value of $\tau_1^{'}$ also increases.

$$Bias(\hat{C}^e_{pm;a}) \begin{cases} > 0 & \text{if } 0 \leq \tau_a < \tau_1^{'}, \\ = 0 & \text{if } \tau_a = \tau_1^{'}, \\ < 0 & \text{if } \tau_a > \tau_1^{'}. \end{cases} \quad (3.42)$$

Next, we compare the absolute values of bias with and without measurement errors. An analytical comparison is not feasible but numerical calculations and surface plots

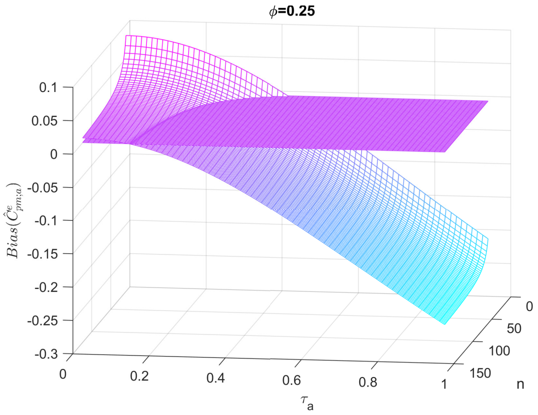

Figure 3.3: Surface plot of $Bias(\hat{C}^{e}_{pm;a})$ as function of n (x-axis) and τ_a (y-axis) for $\phi = 0.25$ and $C_{p;a} = 1$.

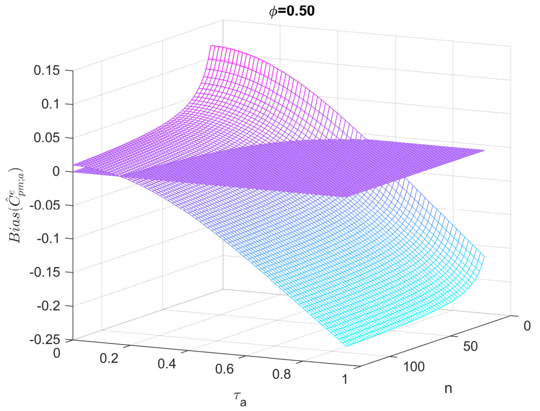

Figure 3.4: Surface plot of $Bias(\hat{C}^{e}_{pm;a})$ as function of n (x-axis) and τ_a (y-axis) for $\phi = 0.50$ and $C_{p;a} = 1$.

Table 3.5: Some numerical values of $\tau_1^{'}$

ϕ \ n	25	50	75	100	125	150
0.20	0.2607	0.1841	0.1503	0.1301	0.1164	0.1062
0.30	0.2795	0.1981	0.1619	0.1403	0.1255	0.1146
0.40	0.3090	0.2201	0.1802	0.1563	0.1399	0.1278
0.50	0.3535	0.2537	0.2083	0.1809	0.1621	0.1481
0.60	0.4222	0.3061	0.2523	0.2196	0.1970	0.1802
0.70	0.5348	0.3937	0.3265	0.2851	0.2564	0.2349

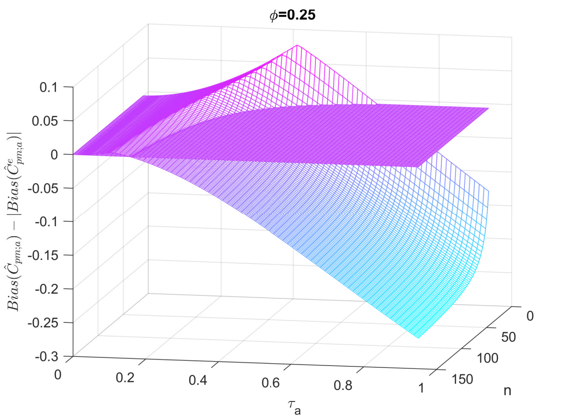

Figure 3.5: Surface plot of $Bias(\hat{C}_{pm;a}) - |Bias(\hat{C}^e_{pm;a})|$ as function of n (x-axis) and τ_a (y-axis) for $\phi = 0.25$ and $C_{p;a} = 1$.

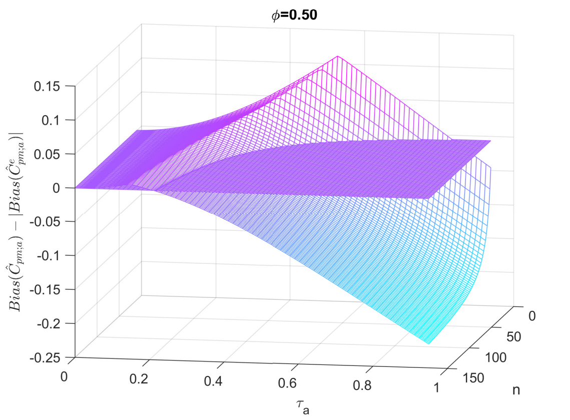

Figure 3.6: Surface plot of $Bias(\hat{C}_{pm;a}) - |Bias(\hat{C}^e_{pm;a})|$ as function of n (x-axis) and τ_a (y-axis) for $\phi = 0.50$ and $C_{p;a} = 1$.

Table 3.6: Some numerical values of $\tau_2^{'}$

ϕ \ n	25	50	75	100	125	150
0.20	0.3780	0.2636	0.2143	0.1852	0.1654	0.1509
0.25	0.3905	0.2726	0.2216	0.1916	0.1711	0.1561
0.30	0.4065	0.2840	0.2311	0.1997	0.1784	0.1628
0.40	0.4513	0.3163	0.2576	0.2228	0.1991	0.1817
0.50	0.5201	0.3658	0.2983	0.2583	0.2309	0.2108
0.75	0.9588	0.6827	0.5605	0.4872	0.4368	0.3995

(Figures 3.5 and 3.6) provide evidence that equation (3.43) holds true. Here, $\tau_2^{'}$ is a function of n and ϕ; however, its explicit expression cannot be determined.

$$Bias(\hat{C}_{pm;a}) \begin{cases} > |Bias(\hat{C}^e_{pm;a})| & \text{if } 0 < \tau_a < \tau_2^{'}\,, \\ = |Bias(\hat{C}^e_{pm;a})| & \text{if } \tau_a = \tau_2^{'}, \\ < |Bias(\hat{C}^e_{pm;a})| & \text{if } \tau_a > \tau_2^{'}. \end{cases} \tag{3.43}$$

Numerical values of $\tau_2^{'}$ for various combinations of n and ϕ are presented in Table 3.6. As n increases, the value of $\tau_2^{'}$ decreases, while when ϕ increases, the value of $\tau_2^{'}$ also increases.

Table 3.7: Some numerical values of $Bias(\hat{C}^e_{pm;a})_{nc}$ for $C_{p;a} = 1$

ϕ \ n	25	50	75	100	125	150
$\tau_a = 0.2,\ \xi = 0.5$						
−0.75	0.0512	0.0234	0.0138	0.0089	0.0059	0.0039
−0.50	0.0199	0.0048	−0.0003	−0.0028	−0.0044	−0.0054
−0.25	0.0113	−0.0009	−0.0050	−0.0070	−0.0082	−0.0091
0.25	0.0194	0.0032	−0.0022	−0.0049	−0.0066	−0.0077
0.50	0.0401	0.0152	0.0067	0.0025	−0.0001	−0.0018
0.75	0.1000	0.0498	0.0318	0.0226	0.0169	0.0131
$\tau_a = 0.4,\ \xi = 1$						
−0.75	0.0071	−0.0022	−0.0054	−0.0071	−0.0081	−0.0087
−0.50	−0.0073	−0.0138	−0.0160	−0.0171	−0.0177	−0.0181
−0.25	−0.0119	−0.0185	−0.0207	−0.0218	−0.0225	−0.0229
0.25	−0.0029	−0.0139	−0.0176	−0.0195	−0.0206	−0.0214
0.50	0.0155	−0.0020	−0.0080	−0.0111	−0.0129	−0.0141
0.75	0.0644	0.0288	0.0157	0.0090	0.0049	0.0021

Non-centered process

In this case $\mu \neq T$ i.e., $\xi \neq 0$. The bias of the estimated index for true capability is

$$\begin{aligned} Bias(\hat{C}^e_{pm;a})_{nc} &= E(\hat{C}^e_{pm;a}) - C_{pm;a} \\ &\approx \left[\mathcal{B}^e_1(n,\phi,\xi,\tau_a) - \frac{1}{\sqrt{1+\xi^2}}\right] C_{p;a} \end{aligned} \tag{3.44}$$

where $\mathcal{B}^e_1(n,\phi,\xi,\tau_a)$ is expressed by equation (3.41). We present numerical values of bias for various combinations of n, ϕ, ξ, and τ_a in Table 3.7.

Based on numerical calculations and surface plots (Figure 3.7 and Figure 3.8), it is shown that $Bias(\hat{C}^e_{pm;a})_{nc}$ decreases as the value of τ_a increases, and equation (3.46) holds for $Bias(\hat{C}^e_{pm;a})_{nc}$. Figure 3.7 displays a surface plot of $Bias(\hat{C}^e_{pm;a})_{nc}$, intersected by the plane $z = 0$. Points above the plane $z = 0$ correspond to $Bias(\hat{C}^e_{pm;a})_{nc} > 0$, while points below the plane correspond to $Bias(\hat{C}^e_{pm;a})_{nc} < 0$. In this context, $\tau_3^{'}$ is a function of n, ϕ, and ξ, for which the explicit expression cannot be determined. However, its numerical value can be obtained by solving the equation $Bias(\hat{C}^e_{pm;a})_{nc} = 0$ using computer-based equation-solving methods. Some numerical values of $\tau_3^{'}$ are reported in Table 3.8. We utilize MATLAB software for our numerical calculations.

$$Bias(\hat{C}^e_{pm;a})_{nc} \begin{cases} > 0 & \text{if } 0 \leq \tau_a < \tau_3^{'}, \\ = 0 & \text{if } \tau_a = \tau_3^{'}, \\ < 0 & \text{if } \tau_a > \tau_3^{'}. \end{cases} \tag{3.45}$$

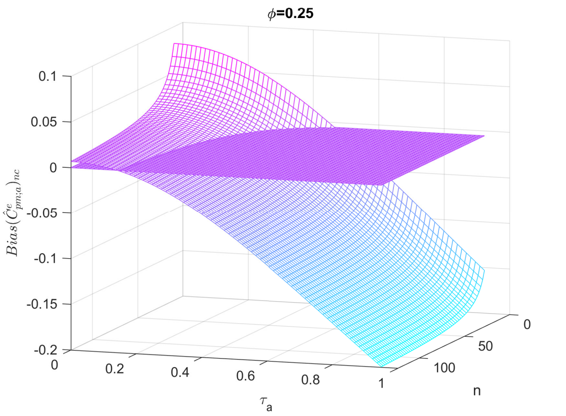

Figure 3.7: Surface plot of $Bias(\hat{C}^e_{pm;a})_{nc}$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.25$, $\xi = 0.5$.

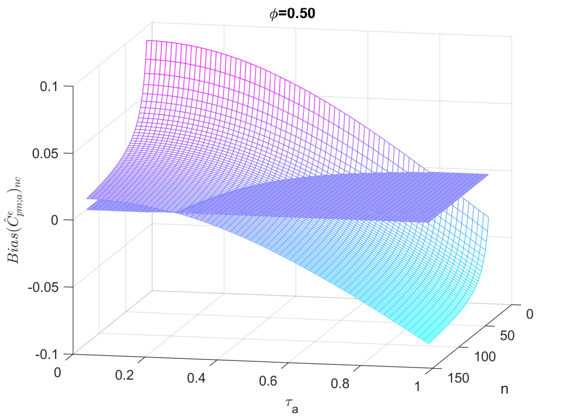

Figure 3.8: Surface plot of $Bias(\hat{C}^e_{pm;a})_{nc}$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.50$, $\xi = 1$.

Table 3.8: Some Numerical Values of $\tau_3^{'}$

ϕ \ n	25	50	75	100	125	150
$\xi = 0.5$						
0.25	0.3141	0.2231	0.1825	0.1582	0.1416	0.1293
0.40	0.3695	0.2651	0.2177	0.1890	0.1694	0.1548
0.50	0.4268	0.3092	0.2547	0.2216	0.1988	0.1819
0.75	0.7508	0.5679	0.4763	0.4186	0.3779	0.3472
$\xi = 1$						
0.25	0.3758	0.2664	0.2178	0.1887	0.1688	0.1541
0.40	0.4507	0.3232	0.2653	0.2304	0.2064	0.1886
0.50	0.5249	0.3804	0.3134	0.2727	0.2446	0.2237
0.75	0.9302	0.7059	0.5927	0.5211	0.4705	0.4324

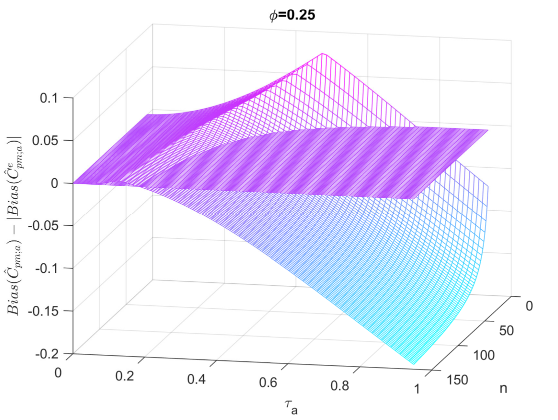

Figure 3.9: Surface plot of $Bias(\hat{C}_{pm;a})_{nc} - |Bias(\hat{C}^e_{pm;a})_{nc}|$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.25$, $\xi = 0.5$.

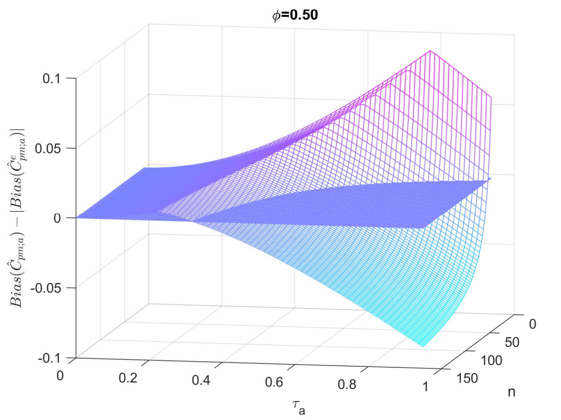

Figure 3.10: Surface plot of $Bias(\hat{C}_{pm;a})_{nc} - |Bias(\hat{C}^e_{pm;a})_{nc}|$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.50$, $\xi = 1$.

We finalize our analysis of bias by comparing the absolute values of the error case with those of the error-free case. Analytical analysis is not possible here. Numerical calculations and surface plots (Figures 3.9 and 3.10) demonstrate that equation (3.46) holds. Figure 3.9 is the surface plot of $Bias(\hat{C}_{pm;a})_{nc} - |Bias(\hat{C}^e_{pm;a})_{nc}|$ with intersection by the plane $z = 0$. Points on the surface above the plane $z = 0$ corresponds to $Bias(\hat{C}_{pm;a})_{nc} > |Bias(\hat{C}^e_{pm;a})_{nc}|$ and below the plane corresponds to $Bias(\hat{C}_{pm;a})_{nc} < |Bias(\hat{C}^e_{pm;a})_{nc}|$ and $Bias(\hat{C}_{pm;a})_{nc} = |Bias(\hat{C}^e_{pm;a})_{nc}|$ for the points along which the plane $z = 0$ intersects the surface. Here $\tau_4^{'}$ is a function of n, ϕ and ξ. As before, explicit expression of $\tau_4^{'}$ as a function of n, ϕ and ξ cannot be determined but numerical values of $\tau_4^{'}$ can be evaluate for given values of n, ϕ and ξ. Some numerical values of $\tau_4^{'}$ is reported in the Table 3.9. Table 3.9 indicates that as n increases, the value of $\tau_4^{'}$ decreases. As the value of ϕ increases, $\tau_4^{'}$ also increases; similarly, as ξ increases, $\tau_4^{'}$ increases.

$$Bias(\hat{C}_{pm;a})_{nc} \begin{cases} > |Bias(\hat{C}^e_{pm;a})_{nc}| & \text{if } 0 < \tau_a < \tau_4^{'}, \\ = |Bias(\hat{C}^e_{pm;a})_{nc}| & \text{if } \tau_a = \tau_4^{'}, \\ < |Bias(\hat{C}^e_{pm;a})_{nc}| & \text{if } \tau_a > \tau_4^{'}. \end{cases} \tag{3.46}$$

3.3.2 Estimation of MSE in the Presence of Measurement Errors

Centered process

In this $\xi = 0$. Therefore, the expression of MSE is

$$\begin{aligned} MSE(\hat{C}^e_{pm;a}) &= Var(\hat{C}^e_{pm;a}) + \left\{Bias(\hat{C}^e_{pm;a})\right\}^2 \\ &\approx C^2_{p;a}\left[\mathcal{M}^e_1(n,\phi,\tau_a) + \{\mathcal{B}^e_1(n,\phi,\tau_a) - 1\}^2\right] \\ &= C^2_{p;a}\mathcal{H}^e_1(n,\phi,\tau_a) \end{aligned} \tag{3.47}$$

Table 3.9: Some Numerical Values of $\tau_4^{'}$

ϕ \ n	25	50	75	100	125	150
$\xi = 0.5$						
0.20	0.4378	0.3060	0.2489	0.2152	0.1923	0.1754
0.25	0.4574	0.3201	0.2606	0.2253	0.2014	0.1837
0.30	0.4809	0.3371	0.2746	0.2375	0.2123	0.1937
0.40	0.5429	0.3821	0.3117	0.2698	0.2413	0.2203
0.50	0.6332	0.4476	0.3659	0.3171	0.2838	0.2592
0.75	1.1826	0.8476	0.6981	0.6079	0.5459	0.4997
$\xi = 1$						
0.20	0.5173	0.3614	0.2939	0.2540	0.2269	0.2070
0.25	0.5452	0.3816	0.3106	0.2685	0.2399	0.2189
0.30	0.5776	0.4052	0.3300	0.2855	0.2551	0.2328
0.40	0.6601	0.4651	0.3795	0.3286	0.2939	0.2683
0.50	0.7762	0.5499	0.4497	0.3899	0.3490	0.3187
0.75	1.4560	1.0511	0.8674	0.7560	0.6792	0.6219

where

$$\mathcal{M}_1^e(n, \phi, \tau_a) = \left[\frac{n + 2\sum_{i=1}^{n-1}(n-i)\phi^{2i} + 2n(1-\phi^2)\tau_a^2 + n(1-\phi^2)^2\tau_a^4}{2n^2\left[1 + (1-\phi^2)\tau_a^2\right]^3}\right].$$

$\mathcal{B}_1^e(n, \phi, \tau_a)$ is given by expression (3.41) and

$$\mathcal{H}_1^e(n, \phi, \tau_a) = \left[\mathcal{M}_1^e(n, \phi, \tau_a) + \{\mathcal{B}_1^e(n, \phi, \tau_a) - 1\}^2\right]. \tag{3.48}$$

We compare the MSE with and without measurement error. Through numerical calculations and graphical presentations (see Figure 3.11), it can be demonstrated that the relation in equation (3.49) is valid. Figure 3.11 and Figure 3.12 presents surface plot of $MSE(\hat{C}_{pm;a}) - MSE(\hat{C}^e_{pm;a})$ with intersection by the plane $z = 0$.

$$MSE(\hat{C}_{pm;a}) \begin{cases} > MSE(\hat{C}^e_{pm;a}) & \text{if } 0 < \tau_a < \tau_5^{'}, \\ = MSE(\hat{C}^e_{pm;a}) & \text{if } \tau_a = \tau_5^{'}, \\ < MSE(\hat{C}^e_{pm;a}) & \text{if } \tau_a > \tau_5^{'}. \end{cases} \tag{3.49}$$

Here $\tau_5^{'}$ is a function of n and ϕ. Numerical values of $\tau_5^{'}$ are reported for various combinations of n and ϕ in Table 3.10. As n increases, the value of $\tau_5^{'}$ decreases, while an increase in the value of ϕ leads to an increase in the value of $\tau_5^{'}$.

Table 3.10: Some Numerical Values of $\tau_5^{'}$

ϕ \ n	25	50	75	100	125	150
0.15	0.4893	0.3392	0.2752	0.2375	0.2121	0.1933
0.25	0.5291	0.3684	0.2994	0.2587	0.2311	0.2108
0.40	0.6356	0.4469	0.3648	0.3160	0.2827	0.2581
0.50	0.7512	0.5322	0.4361	0.3786	0.3393	0.3101
0.75	1.4565	1.0450	0.8660	0.7580	0.6833	0.6275

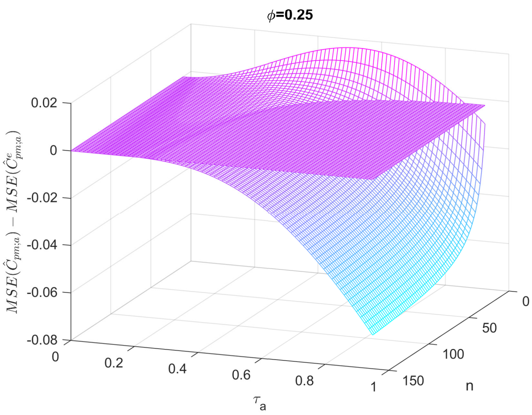

Figure 3.11: Surface plot of $MSE(\hat{C}_{pm;a}) - MSE(\hat{C}^e_{pm;a})$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.25$.

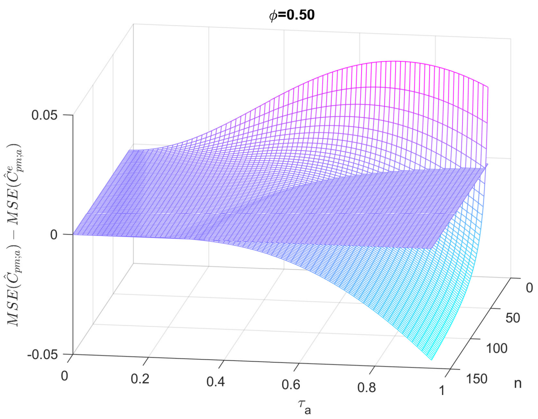

Figure 3.12: Surface plot of $MSE(\hat{C}_{pm;a}) - MSE(\hat{C}^e_{pm;a})$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.50$.

Non-centered process

In this case $\xi \neq 0$. The mean square error of the estimated index is

$$\begin{aligned} MSE(\hat{C}^e_{pm;a})_{nc} &= Var(\hat{C}^e_{pm;a}) + \left\{Bias(\hat{C}^e_{pm;a})_{nc}\right\}^2 \\ &\approx C^2_{p;a}\left[\mathcal{M}^e_1(n,\phi,\xi,\tau_a) + \left\{\mathcal{B}^e_1(n,\phi,\xi,\tau_a) - \frac{1}{\sqrt{1+\xi^2}}\right\}^2\right] \\ &= C^2_{p;a}\mathcal{H}^e_1(n,\phi,\xi,\tau_a) \end{aligned} \tag{3.50}$$

where

$$\mathcal{H}^e_1(n,\phi,\xi,\tau_a) = \left[\mathcal{M}^e_1(n,\phi,\xi,\tau_a) + \left\{\mathcal{B}^e_1(n,\phi,\xi,\tau_a) - \frac{1}{\sqrt{1+\xi^2}}\right\}^2\right]. \tag{3.51}$$

We conclude our analysis by comparing the MSE in cases where there is measurement error against cases that are free of measurement error when the process is non-centered. Numerical calculations and graphical representation in Figure 3.13 demonstrate that equation (3.52) holds. To visualize this situation we present a surface plot of $MSE(\hat{C}_{pm;a})_{nc} - MSE(\hat{C}^e_{pm;a})_{nc}$ with intersection by the plane $z = 0$. The points on the surface above the plane $z = 0$ corresponds to $MSE(\hat{C}_{pm;a})_{nc} > MSE(\hat{C}^e_{pm;a})_{nc}$ and below the plane $z = 0$ corresponds to $MSE(\hat{C}_{pm;a})_{nc} < MSE(\hat{C}^e_{pm;a})_{nc}$.

$$MSE(\hat{C}_{pm;a})_{nc} \begin{cases} > MSE(\hat{C}^e_{pm;a})_{nc} & \text{if } 0 < \tau_a < \tau_6^{'}, \\ = MSE(\hat{C}^e_{pm;a})_{nc} & \text{if } \tau_a = \tau_6^{'}, \\ < MSE(\hat{C}^e_{pm;a})_{nc} & \text{if } \tau_a > \tau_6^{'}. \end{cases} \tag{3.52}$$

We report some numerical values of $\tau_6^{'}$ in the Table 3.11. Note that $\tau_6^{'}$ is a function of n, ϕ and ξ. The analytical expression for $\tau_6^{'}$ cannot be determined; however, the numerical value of $\tau_6^{'}$ can be obtained for specific values of n, ϕ, and ξ by solving the equation $MSE(\hat{C}_{pm;a})_{nc} - MSE(\hat{C}^e_{pm;a})_{nc} = 0$. This involves finding the real, positive root of τ_a using computer-based equation-solving methods.

3.4 An Application

Here we consider the case of a 125 g yogurt cup filling process mentioned by Costa and Castagliola [27] where the quality characteristic is the weight of each yogurt cup. Based on a long-term study (Phase I) of the large data set, it was observed that the quality characteristics fit an AR(1) plus measurement error model with parameters $\phi = 0.38$, $\sigma_E = 0.24$, $\sigma_a = 0.703$, $\tau_a = 0.341$. Also for this process target value $T = 125$, mean of the process $\mu = 124.9$ and $\sigma_X = 0.76$ and hence $\xi = \frac{\mu - T}{\sigma_X} = -0.1316$. Let for given upper and lower specification range of the weight of yogurt cup, the capability of the process is estimated based upon the process capability index C_{pm} from a sample of size n. Then, we can approximately assess the estimated value of the index by using our obtained results.

If the process capability is estimated from a sample size of $n = 100$, then $\tau_3^{'} = 0.1554$, $\tau_4^{'} = 0.2215$, and $\tau_6^{'} = 0.3135$. In this case, since $\tau_3^{'} < \tau_a$, we can conclude from equation

Table 3.11: Some Numerical Values of $\tau_6^{'}$

ϕ \ n	25	50	75	100	125	150
$\xi = 0.5$						
0.15	0.5674	0.3947	0.3207	0.2771	0.2475	0.2257
0.25	0.6334	0.4438	0.3616	0.3129	0.2798	0.2554
0.40	0.7818	0.5542	0.4541	0.3942	0.3532	0.3228
0.50	0.9313	0.6650	0.5472	0.4763	0.4276	0.3913
0.75	1.8140	1.3038	1.0834	0.9506	0.8586	0.7897
$\xi = 1$						
0.15	0.6296	0.4356	0.3531	0.3047	0.2720	0.2479
0.25	0.7336	0.5127	0.4172	0.3607	0.3224	0.2941
0.40	0.9399	0.6666	0.5460	0.4739	0.4245	0.3879
0.50	1.1348	0.8122	0.6684	0.5818	0.5222	0.4779
0.75	2.2273	1.6156	1.3454	1.1815	1.0676	0.9822

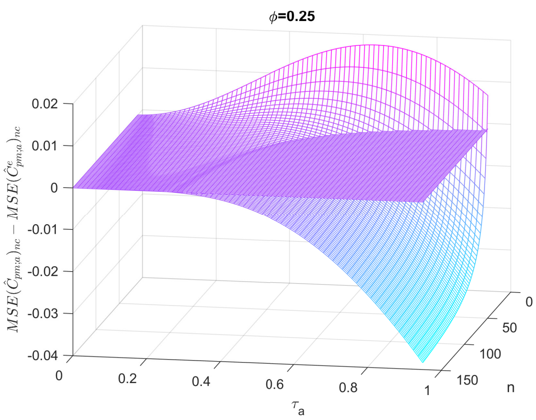

Figure 3.13: Surface plot of $MSE(\hat{C}_{pm;a})_{nc} - MSE(\hat{C}^e_{pm;a})_{nc}$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.25$, $\xi = 0.5$.

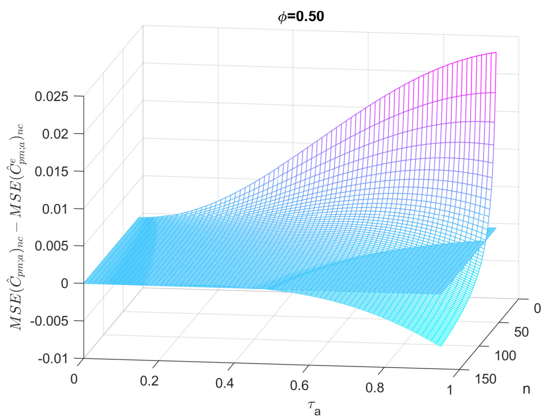

Figure 3.14: Surface plot of $MSE(\hat{C}_{pm;a})_{nc} - MSE(\hat{C}^e_{pm;a})_{nc}$ as function of n (x-axis) and τ_a (y-axis) for $C_{p;a} = 1$ and $\phi = 0.50$, $\xi = 1$.

(3.45) that the most probable estimate will likely yield an underestimated value. Here $\tau_4^{'} < \tau_a$, we can conclude from relation (3.46) that the absolute value of the estimator's bias will be greater than the actual bias. Since $\tau_6^{'} < \tau_a$, we can conclude from relation (3.52) that in this case, the MSE of the estimator is greater than the actual MSE.

3.5 Concluding Remarks

In this chapter, we discuss the statistical properties of the estimator of C_{pm} for autocorrelated data when observations are influenced by measurement errors. This situation frequently occurs in various industries, particularly in the chemical and process industries. Our analysis shows that measurement errors significantly impact the estimator of the capability index C_{pm} when observations are non-independent.

Our discussion shows that the behavior of the estimator is complex in this situation. The estimator can be biased, with bias being negative, positive, or zero depending on the variability of the measurement error. Here we have considered the case of an AR(1) process. Our discussion shows that there is a specific range of τ_a that depends on the sample size n, the autoregressive parameter ϕ, and the value of ξ, for which the absolute value of the bias, i.e., $|Bias(\hat{C}^e_{pm;a})|$, is less than the actual bias $Bias(\hat{C}_{pm;a})$.

Our discussion on MSE reveals a range for τ_a that depends on the values of n, ϕ, and ξ, for which $MSE(\hat{C}^e_{pm;a})$ is less than the actual MSE of the estimator, i.e., $MSE(\hat{C}_{pm;a})$. Based on our results, we can conclude that a small amount of measurement error in the observed data does not affect the estimated capability index value. Nonetheless, a significant measurement error could negatively impact the estimated value.

Chapter 4

Effect of Autocorrelation and Measurement Errors on Process Incapability Index C''_{pp}

4.1 Introduction

THE process incapability index is a valuable tool for assessing process performance and is often utilized by various industries to evaluate product quality. Process incapability represents a clear distinction between process accuracy and process precision. The process incapability index, denoted as C_{pp}, was introduced by Greenwich and Jahr-Schaffrath [45] and is defined by equation (1.9). It measures both process performance and the deviation of the mean from the target value, along with the variance. The generalized process incapability index C''_{pp}, as defined in equation (1.10), was introduced by Chen [19] to assess the performance of a process with asymmetric tolerance limits. The main difference between the process incapability index and other PCIs is that the process incapability index with a lower value is better, while for PCIs, a higher value is desirable.

Most studies on the process incapability index assume that quality characteristics follow an identically independent normal distribution. To the best of our knowledge, there are no studies on the process incapability index under autocorrelated observations, despite the fact that serial autocorrelation is a common phenomenon in many manufacturing and process industries. Another reason for variation is the presence of measurement errors in the sample observations. This work is inspired by the recent paper of Sadeghpour Gildeh and Abbasi Ganji [97] that examined the impact of measurement error on the process incapability index C''_{pp} when the sample observations are independent and identically distributed normally. In this chapter, we examine some statistical properties of C''_{pp} considering the combined influences of autocorrelation and measurement errors.

The chapter is organized as follows. In Section 4.2 we analyze the effect of autocorrelation on C''_{pp}. The effects of autocorrelation and measurement on the bias and MSE of the estimator of C''_{pp} are addressed in Section 4.3. Some conclusions are mentioned in

Section 4.4.

4.2 The Effect of Autocorrelation on C''_{pp}

In this section, we maintain the same assumptions as in Section 2.4, where the required quality characteristics $\{X_t\}$ are modeled as a discrete-time stationary Gaussian process. Let $\{X_1, X_2, \ldots, X_n\}$ be a sequence of n consecutive observable items from a stationary Gaussian process. Define $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, which are the maximum likelihood estimates (MLE) of the process mean and process variance, respectively. The MLE of the index $C''_{pp;a}$ ('a' indicates observations are autocorrelated) is

$$\hat{C}''_{pp;a} = \frac{\hat{A}^2}{D^2} + \frac{S_n^2}{D^2} \tag{4.1}$$

where,

$$\hat{A} = \max\left\{\frac{(\bar{X}-T)d}{D_u}, \frac{(T-\bar{X})d}{D_l}\right\}.$$

Here $D_u = U - T,\ D_l = T - L$ and $D = \min\{D_u, D_l\}/3$. Based on the findings of Zhang [131], it can be demonstrated that,

$$\begin{aligned} E\left(\bar{X}\right) &= \mu, \qquad Var\left(\bar{X}\right) = \frac{\sigma_X^2}{n} g(n,\rho_i) \\ E\left(S_n^2\right) &= \frac{(n-1)\sigma_X^2}{n} f(n,\rho_i), \qquad Var\left(S_n^2\right) = \frac{2\sigma_X^4}{n^2} F(n,\rho_i) \end{aligned} \tag{4.2}$$

where μ and σ_X^2 are mean and variance of the process $\{X_t\}$ and $f(n,\rho_i)$, $F(n,\rho_i)$, $g(n,\rho_i)$ are defined in equations (2.3), (2.4) and (2.5) respectively.

4.2.1 Bias of the Estimator for Autocorrelated Data

We have the following theorem.

Theorem 4.1. *Let $\{X_t\}$ be a discrete-time stationary Gaussian process. The expected value of the estimator is as follows:*

$$E\left(\hat{C}''_{pp;a}\right) = \begin{cases} C''_{pp;a} + \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\rho_i)}{nD_u^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right]; & \mu > T, \\ C''_{pp;a} + \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\rho_i)}{nD_l^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right]; & \mu \le T. \end{cases} \tag{4.3}$$

Proof. First, we consider the scenario when the mean of the process exceeds the target value, i.e., $\mu > T$. In this case $A = (\mu - T)d/D_u$ and hence

$$C''_{pp;a} = \frac{(\mu-T)^2 d^2}{D^2 D_u^2} + \frac{\sigma_X^2}{D^2}.$$

Therefore, the estimator of the index is

$$\hat{C}''_{pp;a} = \frac{(\bar{X}-T)^2 d^2}{D^2 D_u^2} + \frac{S_n^2}{D^2}. \tag{4.4}$$

The expected value of $\hat{C}''_{pp;a}$ is as follows:

$$\begin{aligned} E\left(\hat{C}''_{pp;a}\right) &= \frac{E(\bar{X}-T)^2)d^2}{D^2 D_u^2} + \frac{E(S_n^2)}{D^2} \\ &= \frac{d^2\left[\frac{\sigma_X^2}{n} g(n,\rho_i) + (\mu-T)^2\right]}{D^2 D_u^2} + \frac{(n-1)\sigma_X^2 f(n,\rho_i)}{nD^2} \\ &= \frac{(\mu-T)^2 d^2}{D^2 D_u^2} + \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\rho_i)}{nD_u^2} + \frac{(n-1)f(n,\rho_i)}{n}\right] \\ &= C''_{pp;a} + \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\rho_i)}{nD_u^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right]. \end{aligned}$$

Similarly, for $\mu \leq T$ we obtain

$$E\left(\hat{C}''_{pp;a}\right) = C''_{pp;a} + \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\rho_i)}{nD_l^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right].$$

□

The bias of the estimator $\hat{C}''_{pp;a}$ is given by the following:

$$Bias\left(\hat{C}''_{pp;a}\right) = \begin{cases} \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\rho_i)}{nD_u^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right]; & \mu > T, \\ \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\rho_i)}{nD_l^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right]; & \mu \leq T. \end{cases} \tag{4.5}$$

For an AR(1) process defined by equation (2.1),

$$E\left(\hat{C}''_{pp;a}\right) = \begin{cases} C''_{pp;a} + \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\phi)}{nD_u^2} + \frac{(n-1)f(n,\phi)}{n} - 1\right]; & \mu > T, \\ C''_{pp;a} + \frac{\sigma_X^2}{D^2}\left[\frac{d^2 g(n,\phi)}{nD_l^2} + \frac{(n-1)f(n,\phi)}{n} - 1\right]; & \mu \leq T. \end{cases} \tag{4.6}$$

4.2.2 Mean Square Error of the Estimator for Autocorrelated Data

We have the following result.

Theorem 4.2. *Let $\{X_t\}$ be a discrete-time stationary Gaussian process. Then the variance of the estimator $\hat{C}''_{pp;a}$ is*

$$Var\left(\hat{C}''_{pp;a}\right) = \begin{cases} \frac{d^4}{D^4 D_u^4}\left[\frac{2\sigma_X^4 g^2(n,\rho_i)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\rho_i)}{n}\right] + \frac{2\sigma_X^4 F(n,\rho_i)}{n^2 D^4} \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\rho_i\rho_j - n^2 g^2(n,\rho_i)\right]}{n^4 D^4 D_u^2}; & \mu > T, \\ \frac{d^4}{D^4 D_l^4}\left[\frac{2\sigma_X^4 g^2(n,\rho_i)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\rho_i)}{n}\right] + \frac{2\sigma_X^4 F(n,\rho_i)}{n^2 D^4} \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\rho_i\rho_j - n^2 g^2(n,\rho_i)\right]}{n^4 D^4 D_l^2}; & \mu \leq T \end{cases} \tag{4.7}$$

Proof. First, we examine the scenario where the process mean exceeds the target value, i.e., $\mu > T$. In this case

$$C''_{pp;a} = \frac{(\mu - T)^2 d^2}{D^2 D_u^2} + \frac{\sigma_X^2}{D^2}.$$

The estimator for the index is:

$$\hat{C}''_{pp;a} = \frac{(\bar{X} - T)^2 d^2}{D^2 D_u^2} + \frac{S_n^2}{D^2}. \tag{4.8}$$

We have,

$$Var\left(\hat{C}''_{pp;a}\right) = \frac{d^4}{D^4 D_u^4} Var\left(\bar{X} - T\right)^2 + \frac{Var\left(S_n^2\right)}{D^4} + \frac{2d^2}{D^4 D_u^2} Cov\left[(\bar{X} - T)^2, S_n^2\right]. \tag{4.9}$$

Now,

$$\begin{aligned} & Var(\bar{X} - T)^2 \\ = \; & Var(\bar{X} - \mu)^2 + 4(\mu - T)^2 Var(\bar{X} - \mu) + 2(\mu - T) Cov\left[(\bar{X} - \mu)^2, (\bar{X} - \mu)\right] \\ = \; & \frac{2\sigma_X^4 g^2(n, \rho_i)}{n^2} + \frac{4\sigma_X^2 (\mu - T)^2 g(n, \rho_i)}{n}. \end{aligned} \tag{4.10}$$

It is important to note that $Cov[(\bar{X} - \mu)^2, (\bar{X} - \mu)] = 0$. Furthermore, Zhang [131] has demonstrated that for a stationary Gaussian process, $Cov[(\bar{X} - \mu), S_n^2] = 0$. Hence, we have

$$\begin{aligned} Cov\left[(\bar{X} - T)^2, S_n^2\right] &= Cov[(\bar{X} - \mu)^2, S_n^2] + 2(\mu - T) Cov[(\bar{X} - \mu), S_n^2] \\ &= Cov[(\bar{X} - \mu)^2, S_n^2] \\ &= \frac{2\sigma_X^4 \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} (n - i - j)\rho_i \rho_j + 4n^2 \mu^2 \sigma_X^2 g(n, \rho_i)}{n^3} \\ &\quad - \frac{2n^2 \sigma_X^4 g^2(n, \rho_i) + 4n^3 \mu^2 \sigma_X^2 g(n, \rho_i)}{n^4} \\ &= \frac{2\sigma_X^4 \left[n \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} (n - i - j)\rho_i \rho_j - n^2 g^2(n, \rho_i)\right]}{n^4}. \end{aligned} \tag{4.11}$$

By substituting the results from equations (4.10), (4.11), and (4.2) into equation (4.9), we obtain:

$$\begin{aligned} Var\left(\hat{C}''_{pp;a}\right) = \; & \frac{d^4}{D^4 D_u^4}\left[\frac{2\sigma_X^4 g^2(n, \rho_i)}{n^2} + \frac{4(\mu - T)^2 \sigma_X^2 g(n, \rho_i)}{n}\right] + \frac{2\sigma_X^4 F(n, \rho_i)}{n^2 D^4} \\ & + \frac{4d^2 \sigma_X^4 \left[n \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} (n - i - j)\rho_i \rho_j - n^2 g^2(n, \rho_i)\right]}{n^4 D^4 D_u^2}. \end{aligned}$$

The proof for the case $\mu \leq T$ is similar. □

In an AR(1) process, the variance is expressed in the following manner:

$$Var\left(\hat{C}''_{pp;a}\right) = \begin{cases} \frac{d^4}{D^4 D_u^4}\left[\frac{2\sigma_X^4 g^2(n,\phi)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\phi)}{n}\right] + \frac{2\sigma_X^4 F(n,\phi)}{n^2 D^4} \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\phi^{i+j} - n^2 g^2(n,\phi)\right]}{n^4 D^4 D_u^2}; & \mu > T \\ \frac{d^4}{D^4 D_l^4}\left[\frac{2\sigma_X^4 g^2(n,\phi)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\phi)}{n}\right] + \frac{2\sigma_X^4 F(n,\phi)}{n^2 D^4} \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\phi^{i+j} - n^2 g^2(n,\phi)\right]}{n^4 D^4 D_l^2}; & \mu \leq T. \end{cases} \tag{4.12}$$

Since the estimator is biased, it is important to estimate the MSE of the estimator. The MSE of the estimator $\hat{C}''_{pp;a}$ is derived in the following theorem.

Theorem 4.3. *Let $\{X_t\}$ be a discrete-time stationary Gaussian process. Then the MSE of the estimator $\hat{C}''_{pp;a}$ is*

$$MSE\left(\hat{C}''_{pp;a}\right) = \begin{cases} \frac{d^4}{D^4 D_u^4}\left[\frac{2\sigma_X^4 g^2(n,\rho_i)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\rho_i)}{n}\right] + \frac{\sigma_X^4}{D^4}\left[\frac{d^2 g(n,\rho_i)}{nD_u^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right]^2 \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\rho_i\rho_j - n^2 g^2(n,\rho_i)\right]}{n^4 D^4 D_u^2} + \frac{2\sigma_X^4 F(n,\rho_i)}{n^2 D^4}; \qquad \mu > T \\ \frac{d^4}{D^4 D_l^4}\left[\frac{2\sigma_X^4 g^2(n,\rho_i)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\rho_i)}{n}\right] + \frac{\sigma_X^4}{D^4}\left[\frac{d^2 g(n,\rho_i)}{nD_u^2} + \frac{(n-1)f(n,\rho_i)}{n} - 1\right]^2 \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\rho_i\rho_j - n^2 g^2(n,\rho_i)\right]}{n^4 D^4 D_l^2} + \frac{2\sigma_X^4 F(n,\rho_i)}{n^2 D^4}; \qquad \mu \le T. \end{cases} \tag{4.13}$$

Proof. We know that,

$$MSE\left(\hat{C}''_{pp;a}\right) = Var\left(\hat{C}''_{pp;a}\right) + \left\{Bias\left(\hat{C}''_{pp;a}\right)\right\}^2.$$

Using the results from equations (4.7) and (4.5), we can derive the result shown in equation (4.13). □

For an AR(1) process, the mean squared error (MSE) of the estimator $\hat{C}''_{pp;a}$ is given as follows:

$$MSE\left(\hat{C}''_{pp;a}\right) = \begin{cases} \frac{d^4}{D^4 D_u^4}\left[\frac{2\sigma_X^4 g^2(n,\phi)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\phi)}{n}\right] + \frac{\sigma_X^4}{D^4}\left[\frac{d^2 g(n,\phi)}{nD_u^2} + \frac{(n-1)f(n,\phi)}{n} - 1\right]^2 \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\phi^{i+j} - n^2 g^2(n,\phi)\right]}{n^4 D^4 D_u^2} + \frac{2\sigma_X^4 F(n,\phi)}{n^2 D^4}; \qquad \mu > T \\ \frac{d^4}{D^4 D_l^4}\left[\frac{2\sigma_X^4 g^2(n,\phi)}{n^2} + \frac{4(\mu-T)^2\sigma_X^2 g(n,\phi)}{n}\right] + \frac{\sigma_X^4}{D^4}\left[\frac{d^2 g(n,\phi)}{nD_u^2} + \frac{(n-1)f(n,\phi)}{n} - 1\right]^2 \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\phi^{i+j} - n^2 g^2(n,\phi)\right]}{n^4 D^4 D_l^2} + \frac{2\sigma_X^4 F(n,\phi)}{n^2 D^4}; \qquad \mu \le T. \end{cases} \tag{4.14}$$

4.3 Incapability Index for Autocorrelated Data in Presence of Measurement Errors

This section examines the scenario in which measurement errors affect the process observations. The assumption is the same as that outlined in Section 2.5 of Chapter 2, that is, the observable process is $Y_t = X_t + E_t$, where X_t is a stationary Gaussian process and E_t be the measurement error. Then, Y_t is also a stationary Gaussian process with $\mu_Y = E(Y_t)$ and $\sigma_Y^2 = Var(Y_t)$. Note that,

$$\mu_Y = \mu, \qquad \sigma_Y^2 = \sigma_X^2 + \sigma_E^2.$$

The incapability index for autocorrelated data in the presence of measurement error can be expressed as:

$$C''^{e}_{pp;a} = C''_{pp;a} + \frac{\sigma_E^2}{D^2} = C''_{pp;a} + C_{ip;a} \times \left(\frac{\sigma_E^2}{\sigma_X^2}\right).$$

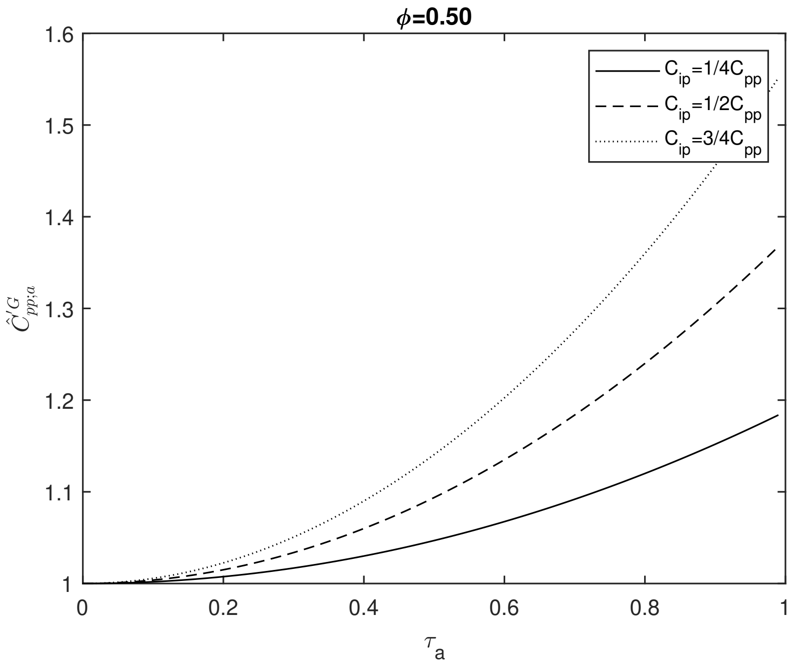

Figure 4.1: Plot of $C''^{e}_{pp;a}$ for τ_a in [0,0.99], $C''_{pp;a} = 1$, $C_{ip;a} = 1/4C''_{pp;a}, 1/2C''_{pp;a}, 3/4C''_{pp;a}$ and $\phi = 0.50$.

We use the superscript 'e' to denote gauge measurement error. When the underlying process $\{X_t\}$ is a stationary AR(1) process, then,

$$C''^{e}_{pp;a} = C''_{pp;a} + C_{ip;a} \times (1 - \phi^2){\tau_a}^2 \tag{4.15}$$

where τ_a is defined in equation (2.20). Equation (4.15) clearly shows that the true incapability index increases due to measurement error. Figure 4.1 shows the plot of $C''^{e}_{pp;a}$ for τ_a in the range $[0, 0.99]$. Here, $C''_{pp;a} = 1$, $C_{ip;a} = 1/4C''_{pp}$, $1/2C''_{pp}$, $3/4C''_{pp}$, and $\phi = 0.50$. The figure shows that when the imprecision index is large, indicating a high variance of the process compared to the specification tolerance, the impact of measurement error is greater than when the imprecision index is small.

Let $\{Y_1, Y_2, \ldots, Y_n\}$ be n consecutive observations from the observable process $\{Y_t\}$. The sample mean and sample variance of the process are $\bar{Y} = \sum_{i=1}^{n} Y_i/n$ and $S^2_{n;Y} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/n$. Now we get,

$$E\left(\bar{Y}\right) = \mu, \qquad Var\left(\bar{Y}\right) = \frac{\sigma_X^2 g(n,\rho_i)}{n} + \frac{\sigma_E^2}{n},$$

and

$$E\left(S^2_{n;Y}\right) = \frac{(n-1)\sigma_X^2 f(n,\rho_i)}{n} + \frac{(n-1)\sigma_E^2}{n},$$

$$Var\left(S^2_{n;Y}\right) = \frac{2\sigma_X^4}{n^2}\left[F(n,\rho_i) + \frac{(n-1)\sigma_E^4}{\sigma_X^4} + \frac{2(n-1)\sigma_E^2 f(n,\rho_i)}{\sigma_X^2}\right]. \tag{4.16}$$

In this context, the estimation of the index is as follows:

$$\hat{C}''^{e}_{pp;a} = \left(\frac{\hat{A}_Y^2}{D^2}\right) + \left(\frac{S^2_{n;Y}}{D^2}\right) \tag{4.17}$$

where,

$$\hat{A}_Y = \max\left\{\frac{(\bar{Y}-T)d}{D_u}, \frac{(T-\bar{Y})d}{D_l}\right\}.$$

4.3.1 Bias of the Estimator $\hat{C}''^{e}_{pp;a}$

We have the following theorem.

Theorem 4.4. *Let $\hat{C}''^{e}_{pp;a}$ be as defined in equation (4.17). Then the expectation of $\hat{C}''^{e}_{pp;a}$ is :*

$$E\left(\hat{C}''^{e}_{pp;a}\right) = \begin{cases} C''_{pp;a} + \frac{d^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{nD^2D_u^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\rho_i)+\sigma_E^2\right]-n\sigma_X^2}{nD^2}; & \mu > T \\ C''_{pp;a} + \frac{d^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{nD^2D_l^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\rho_i)+\sigma_E^2\right]-n\sigma_X^2}{nD^2}; & \mu \le T. \end{cases} \tag{4.18}$$

Proof. First, let us examine the scenario where $\mu > T$. In this case

$$\hat{C}''^{e}_{pp;a} = \frac{(\bar{Y}-T)^2 d^2}{D_u^2 D^2} + \frac{S^2_{n;Y}}{D^2}.$$

Therefore, the expectation of $\hat{C}''^{e}_{pp;a}$ is given by:

$$\begin{aligned} E\left(\hat{C}''^{e}_{pp;a}\right) &= \frac{d^2 E(\bar{Y}-T)^2}{D_u^2 D^2} + \frac{E(S^2_{n;Y})}{D^2} \\ &= \frac{d^2\left[\frac{\sigma_X^2 g(n,\rho_i)+\sigma_E^2}{n} + (\mu-T)^2\right]}{D^2 D_u^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\rho_i)+\sigma_E^2\right]}{nD^2} \\ &= C''_{pp;a} + \frac{d^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{nD^2D_u^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\rho_i)+\sigma_E^2\right]-n\sigma_X^2}{nD^2} \end{aligned}$$

Similarly, the results can be obtained for the case $\mu \le T$. □

Therefore, bias of the estimator $\hat{C}''^{e}_{pp;a}$ is

$$Bias\left(\hat{C}''^{e}_{pp;a}\right) = \begin{cases} \frac{d^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{nD^2D_u^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\rho_i)+\sigma_E^2\right]-n\sigma_X^2}{nD^2}, & \mu > T \\ \frac{d^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{nD^2D_l^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\rho_i)+\sigma_E^2\right]-n\sigma_X^2}{nD^2}, & \mu \le T. \end{cases} \tag{4.19}$$

When the actual process is an AR(1) process, the bias of the estimator $\hat{C}''^{e}_{pp;a}$ is,

$$Bias\left(\hat{C}''^{e}_{pp;a}\right) = \begin{cases} \frac{d^2\left[\sigma_X^2 g(n,\phi)+\sigma_E^2\right]}{nD^2D_u^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\phi)+\sigma_E^2\right]-n\sigma_X^2}{nD^2}; & \mu > T \\ \frac{d^2\left[\sigma_X^2 g(n,\phi)+\sigma_E^2\right]}{nD^2D_l^2} + \frac{(n-1)\left[\sigma_X^2 f(n,\phi)+\sigma_E^2\right]-n\sigma_X^2}{nD^2}; & \mu \le T. \end{cases} \tag{4.20}$$

Figures 4.2 and 4.3 represent the surface plot of $Bias(\hat{C}''^{e}_{pp;a})$ against sample size n and degree of error contamination τ_a for $C''_{pp;a} = 1$, $C_{ip;a} = 3/4C''_{pp;a}$, and $\phi = 0.25, 0.50$ respectively, where $LSL = -3$, $USL = 3$, $T = 0.75$, and $\mu > T$. The values of the parameters μ and σ_X can be obtained using the provided values of $C_{ip;a}$ and $C''_{ia;a}$. In this context, we have utilized the relationship $\sigma_E^2/\sigma_X^2 = \tau_a^2(1-\phi^2)$ for a stationary AR(1) process. It is evident from these figures that bias increases as the values of τ_a and ϕ increases.

Table 4.1: Some numerical values of $Bias(\hat{C}''^{e}_{pp;a})$ for $C''_{pp;a} = 1$

			τ_a						
$C_{ip;a}$	ϕ	n	0	0.1	0.2	0.3	0.4	0.5	0.6
$1/4C''_{pp;a}$	0.00	25	0.0078	0.0104	0.0181	0.0310	0.0490	0.0722	0.1006
		50	0.0039	0.0064	0.0140	0.0267	0.0445	0.0674	0.0953
		75	0.0026	0.0051	0.0127	0.0253	0.0430	0.0657	0.0935
		100	0.0019	0.0045	0.0120	0.0246	0.0423	0.0649	0.0926
		150	0.0013	0.0038	0.0113	0.0239	0.0415	0.0641	0.0918
	0.25	25	0.0127	0.0151	0.0224	0.0344	0.0514	0.0731	0.0997
		50	0.0064	0.0088	0.0159	0.0278	0.0445	0.0659	0.0921
		75	0.0043	0.0067	0.0138	0.0256	0.0422	0.0635	0.0895
		100	0.0032	0.0056	0.0127	0.0245	0.0410	0.0623	0.0883
		150	0.0022	0.0045	0.0116	0.0234	0.0398	0.0611	0.0870
	0.75	25	0.0470	0.0481	0.0515	0.0571	0.0650	0.0752	0.0876
		50	0.0254	0.0265	0.0298	0.0354	0.0431	0.0531	0.0653
		75	0.0173	0.0184	0.0217	0.0273	0.0350	0.0449	0.0571
		100	0.0131	0.0142	0.0176	0.0231	0.0308	0.0407	0.0528
		150	0.0089	0.0100	0.0133	0.0188	0.0265	0.0364	0.0484
$3/4C''_{pp;a}$	0.00	25	0.0233	0.0311	0.0543	0.0929	0.1471	0.2167	0.3017
		50	0.0117	0.0193	0.0421	0.0802	0.1335	0.2021	0.2859
		75	0.0078	0.0154	0.0381	0.0760	0.1290	0.1972	0.2806
		100	0.0058	0.0134	0.0361	0.0739	0.1268	0.1948	0.2779
		150	0.0039	0.0114	0.0340	0.0717	0.1245	0.1924	0.2753
	0.25	25	0.0381	0.0453	0.0671	0.1033	0.1541	0.2193	0.2991
		50	0.0192	0.0264	0.0478	0.0835	0.1335	0.1978	0.2763
		75	0.0129	0.0200	0.0413	0.0768	0.1265	0.1905	0.2686
		100	0.0097	0.0168	0.0380	0.0734	0.1230	0.1868	0.2648
		150	0.0065	0.0135	0.0347	0.0701	0.1195	0.1832	0.2609
	0.75	25	0.1410	0.1443	0.1545	0.1714	0.1951	0.2255	0.2628
		50	0.0761	0.0794	0.0894	0.1061	0.1294	0.1594	0.1960
		75	0.0520	0.0553	0.0652	0.0818	0.1050	0.1348	0.1713
		100	0.0394	0.0427	0.0527	0.0692	0.0923	0.1221	0.1585
		150	0.0266	0.0299	0.0398	0.0563	0.0794	0.1091	0.1453

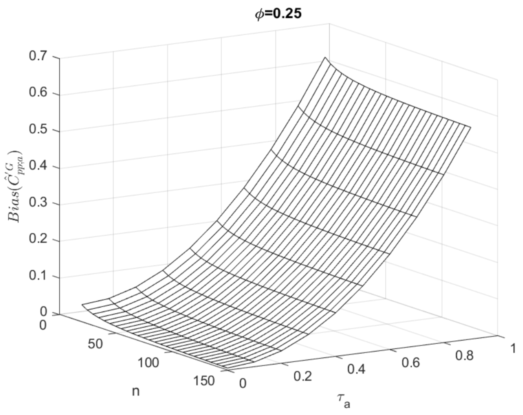

Figure 4.2: Plot of $Bias(C''^{e}_{pp;a})$ for τ_a in [0,0.99], $C_{ip;a} = 1/4C''_{pp;a}$,$1/2C''_{pp;a}$,$3/4C''_{pp;a}$, $C''_{pp;a} = 1$ and $\phi = 0.25$.

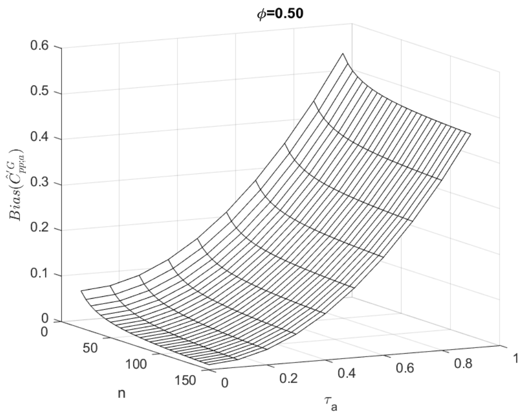

Figure 4.3: Plot of $Bias(C''^{e}_{pp;a})$ for τ_a in [0,0.99], $C_{ip;a} = 1/4C''_{pp;a}$, $1/2C''_{pp;a}$, $3/4C''_{pp;a}$, $C''_{pp;a} = 1$ and $\phi = 0.50$.

4.3.2 Mean Square Error of the Estimator $\hat{C}''^{e}_{pp;a}$

Now, we derive the MSE of the estimator. We have the following theorem.

Theorem 4.5. *Let the estimator $\hat{C}''^{e}_{pp;a}$ be as equation (4.17). Then the variance of the estimator is*

$$Var\left(\hat{C}''^{e}_{pp;a}\right) = \begin{cases} \frac{d^4}{D^4 D_u^4}\left\{\frac{\left[2\sigma_X^4 g^2(n,\rho_i)+4\sigma_X^2\sigma_E^2 g(n,\rho_i)+2\sigma_E^4\right]}{n^2} + \frac{4(\mu-T)^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{n}\right\} + \frac{2\sigma_X^4\left[F(n,\rho_i)+\frac{(n-1)\sigma_E^4}{\sigma_X^4}+\frac{2(n-1)\sigma_E^2 f(n,\rho_i)}{\sigma_X^2}\right]}{n^2 D^4} \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\rho_i\rho_j - n^2 g^2(n,\rho_i)\right]}{n^4 D^4 D_u^2}; \qquad \mu > T \\ \frac{d^4}{D^4 D_l^4}\left\{\frac{\left[2\sigma_X^4 g^2(n,\rho_i)+4\sigma_X^2\sigma_E^2 g(n,\rho_i)+2\sigma_E^4\right]}{n^2} + \frac{4(\mu-T)^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{n}\right\} + \frac{2\sigma_X^4\left[F(n,\rho_i)+\frac{(n-1)\sigma_E^4}{\sigma_X^4}+\frac{2(n-1)\sigma_E^2 f(n,\rho_i)}{\sigma_X^2}\right]}{n^2 D^4} \\ \quad + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\rho_i\rho_j - n^2 g^2(n,\rho_i)\right]}{n^4 D^4 D_l^2}; \qquad \mu \le T. \end{cases} \tag{4.21}$$

Proof. First, we consider the case $\mu > T$. In this case,

$$\hat{C}''^{e}_{pp;a} = \frac{(\bar{Y}-T)^2 d^2}{D_u^2 D^2} + \frac{S^2_{n;Y}}{D^2}.$$

Hence,

$$Var\left(\hat{C}''^{e}_{pp;a}\right) = \frac{d^4}{D^4 D_u^4} Var\left(\bar{Y}-T\right)^2 + \frac{Var(S^2_{n;Y})}{D^4} + \frac{2d^2}{D^4 D_u^2} Cov\left[(\bar{Y}-T)^2, S^2_{n;Y}\right]. \tag{4.22}$$

Now,

$$\begin{aligned}
& Var(\bar{Y}-T)^2 \\
= \ & Var(\bar{Y}-\mu)^2 + 4(\mu-T)^2 Var(\bar{Y}-\mu) + 4(\mu-T)Cov[(\bar{Y}-\mu)^2, (\bar{Y}-\mu)] \\
= \ & \frac{[2\sigma_X^4 g^2(n,\rho_i) + 4\sigma_X^2\sigma_E^2 g(n,\rho_i) + 2\sigma_E^4]}{n^2} + \frac{4(\mu-T)^2\left[\sigma_X^2 g(n,\rho_i)+\sigma_E^2\right]}{n}
\end{aligned} \tag{4.23}$$

and

$$Cov\left[(\bar{Y}-T)^2, S_{n;Y}^2\right] = \frac{2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\rho_i\rho_j - n^2g^2(n,\rho_i)\right]}{n^4}. \tag{4.24}$$

The calculations for equations (4.23) and (4.24) can be found in Appendix A of Bera and Anis [10]. By substituting the results obtained from equations (4.16), (4.23), and (4.24) into equation (4.22), we get the results presented in equation (4.21). Results can be obtained in a similar manner for the case when $\mu \leq T$. □

When the underlying process is a stationary AR(1) process, then

$$\begin{aligned}
& Var\left(\hat{C}''^{e}_{pp;a}\right) \\
= \ & \frac{d^4}{D^4D_u^4}\left\{\frac{[2\sigma_X^4 g^2(n,\phi) + 4\sigma_X^2\sigma_E^2 g(n,\phi) + 2\sigma_E^4]}{n^2} + \frac{4(\mu-T)^2\left[\sigma_X^2 g(n,\phi)+\sigma_E^2\right]}{n}\right\} \\
& + \frac{2\sigma_X^4\left[F(n,\phi) + \frac{(n-1)\sigma_E^4}{\sigma_X^4} + \frac{2(n-1)\sigma_E^2 f(n,\phi)}{\sigma_X^2}\right]}{n^2D^4} \\
& + \frac{4d^2\sigma_X^4\left[n\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(n-i-j)\phi^{i+j} - n^2g^2(n,\phi)\right]}{n^4D^4D_u^2}.
\end{aligned}$$

The MSE of the estimator can be calculated using the results from equations (4.19) and (4.21), along with the formula.

$$MSE\left(\hat{C}''^{e}_{pp;a}\right) = Var\left(\hat{C}''^{e}_{pp;a}\right) + \left\{Bias(\hat{C}''^{e}_{pp;a})\right\}^2. \tag{4.25}$$

Figures 4.4 and 4.5 illustrate the surface plot of $MSE(\hat{C}''^{e}_{pp;a})$, with n on the x-axis and τ_a on the y-axis. It is clear from this figure that the MSE of the estimator $\hat{C}''^{e}_{pp;a}$ increases as the values of τ_a and ϕ increase.

When the observations are free from measurement errors, $\sigma_E^2 = 0$. In this case, the expressions for expectation, variance, and MSE resemble the results obtained in Section 4.2. When observations are independent, we have $\rho_0 = 1$ and $\rho_i = 0$ for $i \geq 1$. In this special case, $g(n,\rho_i) = 1$, $f(n,\rho_i) = 1$, and $F(n,\rho_i) = n-1$. When the observations are independent, then the expression of expectation in equation (4.18) takes the form

$$E\left(\hat{C}''^{e}_{pp}\right) = \begin{cases} C''^{e}_{pp} + \frac{(\sigma_X^2+\sigma_E^2)}{nD^2}\left[\frac{d^2}{D_u^2}-1\right]; & \mu > T \\ C''^{e}_{pp} + \frac{(\sigma_X^2+\sigma_E^2)}{nD^2}\left[\frac{d^2}{D_l^2}-1\right]; & \mu \leq T. \end{cases} \tag{4.26}$$

Table 4.2: Some numerical values of $MSE(\hat{C}''^{e}_{pp;a})$ for $C''_{pp;a} = 1$

			τ_a						
$C_{ip;a}$	ϕ	n	0	0.1	0.2	0.3	0.4	0.5	0.6
$1/4C''_{pp;a}$	0.00	25	0.0588	0.0595	0.0617	0.0655	0.0716	0.0804	0.0927
		50	0.0293	0.0296	0.0308	0.0329	0.0364	0.0419	0.0502
		75	0.0195	0.0197	0.0205	0.0221	0.0248	0.0292	0.0361
		100	0.0146	0.0148	0.0154	0.0167	0.0190	0.0229	0.0291
		150	0.0097	0.0099	0.0103	0.0113	0.0132	0.0165	0.0221
	0.25	25	0.0937	0.0943	0.0964	0.1002	0.1060	0.1143	0.1259
		50	0.0471	0.0474	0.0485	0.0505	0.0539	0.0590	0.0666
		75	0.0314	0.0316	0.0324	0.0339	0.0364	0.0405	0.0468
		100	0.0236	0.0238	0.0243	0.0255	0.0277	0.0313	0.0369
		150	0.0157	0.0159	0.0163	0.0172	0.0189	0.0220	0.0270
	0.75	25	0.3493	0.3497	0.3509	0.3530	0.3561	0.3603	0.3657
		50	0.1855	0.1857	0.1863	0.1874	0.1891	0.1914	0.1944
		75	0.1260	0.1261	0.1265	0.1273	0.1284	0.1301	0.1324
		100	0.0953	0.0954	0.0958	0.0963	0.0972	0.0986	0.1005
		150	0.0641	0.0642	0.0644	0.0648	0.0655	0.0665	0.0681
$3/4C''_{pp;a}$	0.00	25	0.1028	0.1047	0.1113	0.1249	0.1493	0.1900	0.2540
		50	0.0503	0.0512	0.0549	0.0634	0.0803	0.1108	0.1614
		75	0.0333	0.0339	0.0366	0.0435	0.0580	0.0852	0.1314
		100	0.0249	0.0254	0.0276	0.0336	0.0470	0.0726	0.1166
		150	0.0165	0.0169	0.0186	0.0239	0.0361	0.0600	0.1019
	0.25	25	0.1469	0.1490	0.1559	0.1697	0.1937	0.2328	0.2931
		50	0.0717	0.0728	0.0765	0.0848	0.1010	0.1293	0.1757
		75	0.0474	0.0481	0.0508	0.0574	0.0710	0.0959	0.1378
		100	0.0354	0.0360	0.0382	0.0439	0.0562	0.0795	0.1191
		150	0.0235	0.0239	0.0256	0.0305	0.0416	0.0631	0.1006
	0.75	25	0.5660	0.5678	0.5733	0.5831	0.5977	0.6184	0.6463
		50	0.2788	0.2797	0.2825	0.2876	0.2958	0.3079	0.3253
		75	0.1838	0.1844	0.1863	0.1900	0.1960	0.2054	0.2193
		100	0.1369	0.1374	0.1389	0.1417	0.1467	0.1547	0.1670
		150	0.0906	0.0909	0.0919	0.0941	0.0980	0.1047	0.1152

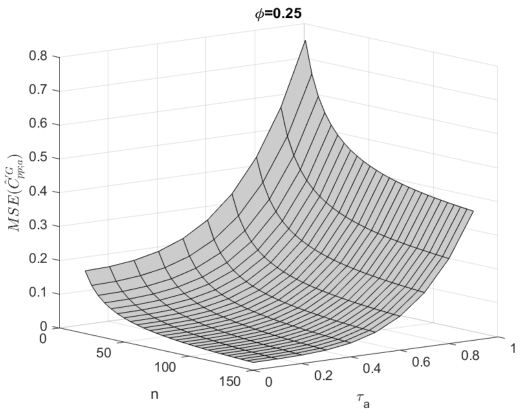

Figure 4.4: Plot of $MSE(\hat{C}''^{e}_{pp;a})$ as a function of n (x-axis), τ_a (y-axis) for $C''_{pp;a} = 1$, $C_{ip;a} = 3/4C''_{pp;a}$ and $\phi = 0.25$.

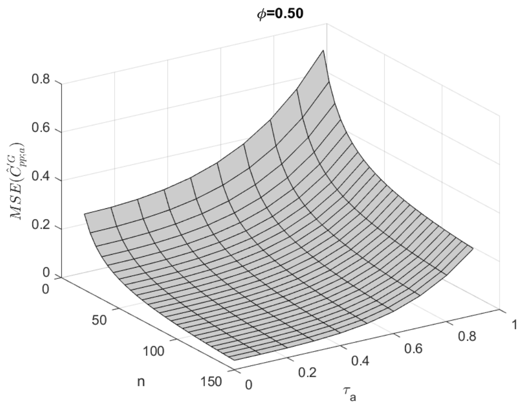

Figure 4.5: Plot of $MSE(\hat{C}''^{e}_{pp;a})$ as a function of n (x-axis), τ_a (y-axis) for $C''_{pp;a} = 1$, $C_{ip;a} = 3/4C''_{pp;a}$ and $\phi = 0.50$.

The equation (4.26) is the corrected version of Theorem 6.2 from Sadeghpour Gildeh and Abbasi Ganji [97], where some terms were missing. For independent observations, the variance expression in equation (4.21) can be rewritten as follows:

$$Var\left(\hat{C}''^{e}_{pp}\right) = \begin{cases} \frac{d^4}{D^4 D_u^4}\left\{\frac{2(\sigma_X^2+\sigma_E^2)^2}{n^2} + \frac{4(\mu-T)^2(\sigma_X^2+\sigma_E^2)}{n}\right\} + \frac{2(n-1)\left(\sigma_X^2+\sigma_E^2\right)^2}{n^2 D^4}; & \mu > T \\ \frac{d^4}{D^4 D_l^4}\left\{\frac{2(\sigma_X^2+\sigma_E^2)^2}{n^2} + \frac{4(\mu-T)^2(\sigma_X^2+\sigma_E^2)}{n}\right\} + \frac{2(n-1)\left(\sigma_X^2+\sigma_E^2\right)^2}{n^2 D^4}; & \mu \leq T. \end{cases} \tag{4.27}$$

The equation (4.27) is a revised and corrected version of Theorem 6.4 from Sadeghpour Gildeh and Abbasi Ganji [97]. Likewise, the MSE $(\hat{C}''^{e}_{pp})$ for independent process observations can be derived using the results from equations (4.26) and (4.27). This will yield a corrected version of Theorem 6.6 of Sadeghpour Gildeh and Abbasi Ganji [97].

Some values of bias and MSE are presented for an AR(1) process to investigate the behavior of the estimator $\hat{C}''^{e}_{pp;a}$. For simplicity, we assume that $LSL = -3$, $USL = 3$, $T = 0.75$, and $\mu > T$. We consider cases with $C''_{pp;a} = 0.44, 1.00$, $C_{ip;a} = 1/4C''_{pp;a}, 3/4C''_{pp;a}$, $\phi = 0, 0.25, 0.50, 0.75$, $\tau_a = 0(0.1)0.6$, and $n = 25, 50, 75, 100, 150$. Values of other parameters such as μ and σ_X can be obtained using the given values of $C_{ip;a}$, $C''_{ia;a}$, USL, LSL, and T. The value of $C''_{ia;a}$ can be derived from the relation $C''_{ia;a} = C''_{pp;a} - C_{ip;a}$. We have utilized the relation $\sigma_E^2/\sigma_X^2 = \tau_a^2(1-\phi^2)$ for a stationary AR(1) process. We have examined both smaller and larger values of the index $C''_{pp;a}$ along with the corresponding index $C_{ip;a}$. We have also examined small, moderate, and large values of ϕ, τ_a, and n. In the case where measurement error is absent, we have $\tau_a = 0$, and when the data are independent, $\phi = 0$. From Tables 4.1 and 4.2, we can infer that:

1. For fixed values of ϕ, bias increases with higher values of τ_a, and the bias is always positive. This means the estimator consistently overestimates the true value of the incapability index.

2. When the value of ϕ increases, bias increases if $\tau_a < 0.3$, but the bias value decreases when $\tau_a \geq 0.3$ and the sample size is moderate (≥ 50).

3. Bias value increases as the imprecision index $C_{ip;a}$ increases, meaning the variance of the process relative to the specification tolerance also increases.

4. The behavior of the mean squared error (MSE) resembles that of bias.

4.4 Concluding Remarks

The process incapability index is a widely used measure of production performance across various industries. One advantage of using the process incapability index over traditional capability indices is that it provides separate information regarding process inaccuracy and process imprecision. Process incapability is often estimated assuming observations are independent; however, for certain industries like chemicals, food, and pharmaceuticals, process observations may be auto-correlated. In this chapter, we discuss the statistical properties of the estimator for the process incapability index C''_{pp} when the process observations follow a stationary Gaussian process. We have also discussed the situation in which measurement errors affect the observations. Our analysis demonstrates that measurement errors increase the bias, variance, and MSE of the estimator. Additionally, our research has addressed and corrected some of the results given in Sadeghpour Gildeh and Abbasi Ganji [97].

Chapter 5

Statistical Inference and Supplier Selection Based on $C_{N_{pk}}$ by Using Fiducial Generalized Confidence Interval

5.1 Introduction

Many quality characteristics in the manufacturing process, such as roundness, taper, ovality, concentricity along with compositional data from chemical processes like zinc plating, follow non-normal distribution. It is not appropriate to evaluate the performance of non-normal processes using normal distribution-based classical PCIs. For the assessment of the capability of non-normal processes, Chen and Pearn [20] introduced a percentile-based PCI $C_{N_p}(u, v)$, which is a generalization of the index $C_p(u, v)$ for non-normal distributions, as defined by equation (1.11). Fiducial inference often provides a reliable confidence interval when traditional methods are difficult to apply. Recently, Meng et al. [80] discussed hypothesis testing of C_{pk} under some non-normal distributions based on generalized fiducial inference while Meng and Yang [79] examined hypothesis testing of the Taguchi index C_{pm} under similar non-normal distribution assumptions. Guo et al. [48] examined the estimation of the generalized confidence interval (GCI) for percentile-based process capability indices under the Birnbaum-Saunders distribution, while Guo et al. [49] estimated the GCI when the quality characteristics follow an inverse Gaussian distribution. In this chapter, we estimate the confidence interval of the PCI $C_{N_{pk}}$ using the fiducial approach when the quality characteristics of interest follow a location-scale (log location-scale) distribution.

PCIs have also been used to compare performance between processes and to select the best supplier from the available options. Two common methods are used to compare the PCIs of two suppliers:

(I) A 100% test is conducted to estimate the PCIs of each supplier individually, followed by a comparison of the two suppliers based on their true PCI values.

(II) A sample is collected, and statistical tests are conducted to evaluate the process capabilities of two suppliers.

Method (I) is both time-consuming and costly. Method (II) is challenging to implement since a suitable test statistic may not always be available for all PCIs. For these reasons, confidence intervals (CIs) for the differences and ratios of the two PCIs are estimated to evaluate the performance of the processes. Chen and Chen [18] examined four different methods for constructing the confidence interval of the ratio C_{pm1}/C_{pm2} to determine the better option between two processes or suppliers. Tong and Chen [112] implemented bootstrap approach to construct confidence interval of the difference $(C_{pk1} - C_{pk2})$. Perakis [91] applied bootstrap re-sampling approach to construct confidence interval of $(C_{pm1} - C_{pm2})$ and $(C_{pmk1} - C_{pmk2})$. Saha et al. [100] use the bootstrap re-sampling method to estimate the confidence interval of the difference between two generalized PCIs for selecting the better process or supplier. Kanichukattu and Luke [63] applied the generalized confidence interval (GCI) method to construct the confidence interval for the difference between two PCIs in order to evaluate the performance of two processes or suppliers. Wu and Huang [122] also used the Generalized Confidence Interval (GCI) approach to analyze the differences and ratios of PCIs. Wu et al. [123] developed an efficient method for comparing two process yields. As far as we know, no comparison has been made between the two processes using percentile-based PCIs under general location-scale distributions.

In this chapter, we propose the estimation of the fiducial generalized confidence interval (FGCI) of $C_{N_{pk}}$, as well as the difference and ratio of $C_{N_{pk}}$ for two processes or suppliers, under some important location-scale (or log location-scale) distributions. This chapter is structured as follows. Section 5.2 addresses the fiducial pivotal quantity (FPQ) and the fiducial generalized confidence interval (FGCI). The estimation procedure for FPQ and FGCI of $C_{N_{pk}}$ under a general location-scale distribution is discussed in Section 5.3. The estimation of the FPQ of percentile functions for important location-scale (log location-scale) distributions is discussed in Section 5.4. Section 5.5 discusses comparing two processes or suppliers by estimating FGCI differences and the ratios of their $C_{N_{pk}}$. The process for estimating three key bootstrap confidence intervals (BCIs) is outlined in Section 5.6. In Section 5.7, we discuss the performance comparison of the proposed FGCI of $C_{N_{pk}}$, analyzing their differences and ratios with three BCIs through Monte Carlo simulation, focusing on coverage probability (CP) and average width (AW). Two applications of the proposed method are discussed in Section 5.8. The conclusions are discussed in Section 5.9.

5.2 Fiducial Pivotal Quantities (FPQ) and Fiducial Generalized Confidence Intervals (FGCI)

At first, we give the definition of fiducial generalized pivotal quantity (FGPQ) for completeness and outline the procedure for estimating FGCIs. Let 'X' be a random variable with cumulative distribution function (cdf) $F(x; \psi, \zeta)$, where x is the observed value of X, ψ is the parameter of interest, and ζ is a nuisance parameter. The FGPQs and FGCIs, as outlined by Hannig et al. [52], are defined as follows:

Definition 1. *Let $\mathcal{R}_\psi = \mathcal{R}_\psi(X, x; \psi, \zeta)$ be a function of random variable X, observed value x, parameters ψ and ζ. Then $\mathcal{R}_\psi$ is said to be a FGPQ if*

(a) The distribution of $\mathcal{R}_\psi$ is free from any unknown parameters for a fixed x.

(b) The observed value of $\mathcal{R}_\psi$ is the parameter of interest, i.e., $\mathcal{R}_\psi(x, x; \psi, \zeta) = \psi$.

Definition 2. *Let $\mathcal{R}_{\psi;\gamma}$ be the γ-th percentile of the distribution $\mathcal{R}_\psi(X, x; \psi, \zeta)$. Then a $100(1-\alpha)\%$ two-sided equal-tailed FGCI of ψ is given by $\mathcal{R}_{\psi;\alpha/2} \leq \psi \leq \mathcal{R}_{\psi;(1-\alpha/2)}$.*

5.3 Location-scale Distribution and Fiducial Generalized Confidence Interval

The location-scale distribution is an important class of distribution in parametric statistics. Several significant distributions are part of this family. The probability density function (pdf) of a location-scale distribution can be expressed as:

$$f(x; \mu, \sigma) = \frac{1}{\sigma} f_0\left(\frac{x-\mu}{\sigma}\right) \tag{5.1}$$

where f_0 is a specified probability density function, and the parameter μ $(-\infty < \mu < \infty)$ is known as the location parameter, while σ $(\sigma > 0)$ is referred to as the scale parameter. Some well-known location-scale distributions include the normal distribution, logistic distribution, two-parameter exponential distribution, Laplace (double exponential) distribution, and generalized extreme value distribution. Additionally, some distributions that do not belong to the location-scale family can be transformed into a location-scale distribution through the appropriate variable transformation technique. The logarithms of Weibull and lognormal variables follow the extreme value and normal distributions, respectively. Such distributions are known as log location-scale distributions. The cube root of the gamma variable approximately follows a normal distribution.

If a random variable X follows a location-scale distribution, then the standardized variable $Z = (X-\mu)/\sigma$ follows a location-scale distribution with location value 0 and scale value 1. This is known as the standard location-scale distribution. The percentile of a location-scale distribution can be defined as follows:

$$Q_p(\mu, \sigma) = \mu + Q_p(0, 1)\,\sigma \tag{5.2}$$

where $Q_p(0, 1)$ denotes the p-th percentile of the standardised distribution. Therefore the PCIs $C_{N_{pk}}$ for location-scale distribution can be written as:

$$C_{N_{pk}}(\mu, \sigma) = \frac{2\left(d - |Q_{p_2} - M|\right)}{[Q_{p_3} - Q_{p_1}]}. \tag{5.3}$$

Let $\hat{\mu}$ and $\hat{\sigma}$ be equivariant estimators of μ and σ, respectively, based on a random sample $X_1, X_2, \ldots, X_n$ of size n. The quantities $(\hat{\mu}-\mu)/\sigma$ and $\hat{\sigma}/\sigma$ are both pivotal, as noted in Theorem E2 of Lawless [71]. As a consequence,

$$\frac{\hat{\mu}-\mu}{\sigma} \sim \hat{\mu}^*, \qquad \frac{\hat{\sigma}}{\sigma} \sim \hat{\sigma}^* \tag{5.4}$$

where '$U \sim V$' means U and V are identically distributed, and $\hat{\mu}^*$, $\hat{\sigma}^*$ are equivariant estimators based on a random sample of size n from the standardized location-scale distribution, i.e., with location parameter $\mu = 0$ and scale parameter $\sigma = 1$.

Let $(\hat{\mu}_0, \hat{\sigma}_0)$ be an observed value of $(\hat{\mu}, \hat{\sigma})$. By solving equation (5.4) for μ and σ, and substituting $\hat{\mu}$ with $\hat{\mu}_0$ and $\hat{\sigma}$ with $\hat{\sigma}_0$, we derive the FGPQ of μ and σ as follows:

$$\mathcal{R}_\mu = \hat{\mu}_0 - \frac{\hat{\mu}^*}{\hat{\sigma}^*}\hat{\sigma}_0 \quad \text{and} \quad \mathcal{R}_\sigma = \frac{\hat{\sigma}_0}{\hat{\sigma}^*}. \tag{5.5}$$

By substituting $\mathcal{R}_\mu$ for μ and $\mathcal{R}_\sigma$ for σ in equation (5.2), we obtain the FGPQ of the percentile function $Q_p(\mu, \sigma)$ as follows:

$$\mathcal{R}_{Q_p} = \mathcal{R}_\mu + Q_p\left(0,1\right)\mathcal{R}_\sigma = \hat{\mu}_0 + Q_p^*\hat{\sigma}_0 \tag{5.6}$$

where $Q_p^* = \left(Q_p(0,1) - \hat{\mu}^*\right)/\hat{\sigma}^*$ is also a pivotal quantity. By substituting FGPQ for percentiles in equation (5.6), we obtain FGPQ for the PCIs as follows:

$$\mathcal{R}_{C_{N_{pk}}} = \frac{2\left(d - |\mathcal{R}_{Q_{p2}} - M|\right)}{\left[\mathcal{R}_{Q_{p3}} - \mathcal{R}_{Q_{p1}}\right]}. \tag{5.7}$$

The FGCI of $C_{N_{pk}}$ can be derived by estimating the percentiles of the distribution of their FGPQ, which is known as the *fiducial distribution*. These percentiles can be estimated through Monte Carlo simulation, as the fiducial distributions do not contain unknown parameters. We present a general algorithm for deriving the FGCI of PCIs related to location-scale (or log location-scale) distributions. Additionally, we shall discuss some important distributions that are either location-scale distributions (or log location-scale distributions) or can be transformed into a location-scale distribution. The maximum likelihood estimator (MLE) is an equivariant estimator.

Algorithm for estimating FGCI

1. Given a random sample of size n from a location-scale distribution.
2. Derive FGPQs of location and scale parameters, say $\mathcal{R}_\mu$ and $\mathcal{R}_\sigma$ based on the random sample.
3. Compute FGPQs of the percentile function $\mathcal{R}_{Q_p} = \mathcal{R}_\mu + Q_p(0,1)\mathcal{R}_\sigma$, where $Q_p(0,1)$ denotes the p-th percentile of standard location-scale distribution.
4. Compute FGPQ of PCIs by using FGPQs of percentile function.
5. Repeat steps 2-4 a large number of times, say N times.
6. Then the $100\alpha/2$ th and the $100(1-\alpha/2)$ th percentiles of these N generated FGPQs will provide a two-sided $100(1-\alpha)\%$ fiducial generalized confidence interval for the PCIs.

5.4 Some Important Distributions

Now, we discuss some important distributions that are location-scale or log location-scale distributions. We discuss estimating the FGPQs of percentile functions for normal, lognormal, Weibull, and gamma distributions. The gamma distribution is not a location-scale or log location-scale distribution; however, the cube root of a gamma variate approximates a normal distribution. The primary task in estimating the FGPQs of differences and ratios of two percentile-based PCIs is to assess the FGPQs of the percentile functions.

5.4.1 Normal distribution

Let 'X' be a random variable following normal distribution with mean μ and variance σ^2, i.e., $X \sim N(\mu, \sigma^2)$. The pdf of the normal distribution is,

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \; -\infty < x < \infty, -\infty < \mu < \infty, \; \sigma > 0. \tag{5.8}$$

Normal distribution is a location-scale distribution and equivariant estimator of μ and σ^2 are given by $\hat{\mu} = \bar{X}$, $\hat{\sigma}^2 = S^2$ where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S^2 = \sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2/(n-1)$ are the sample mean and the sample variance respectively. Let $\hat{\mu}^*$, $\hat{\sigma}^{*2}$ be the sample mean and sample variance based on a random sample of size n from the standard normal distribution, then $\hat{\mu}^* \sim N(0, 1/n)$ and $\hat{\sigma}^{*2} \sim \chi^2_{n-1}/(n-1)$. Let Z be the standard normal random variable, V be the chi-squared random variable with $n-1$ degrees of freedom and $(\hat{\mu}_0, \hat{\sigma}_0^2) = (\bar{x}, s^2)$ be an observed value of $(\hat{\mu}, \hat{\sigma}^2)$. Using equation (5.6), we can express the FGPQ of the percentile function as follows:

$$\mathcal{R}^N_{Q_p} = \bar{x} + \frac{Q_p(0,1)\sqrt{n} - Z}{\sqrt{V/(n-1)}}\frac{s}{\sqrt{n}}. \tag{5.9}$$

Here $Q_p(0,1) = \Phi^{-1}(p)$ is the p-th percentile of the standard normal distribution, Φ being the cdf of the standard normal distribution.

5.4.2 Lognormal distribution

A positive random variable 'X' is said to follow a lognormal distribution if the random variable $Y = \ln(X)$ follows a normal distribution. Let, Y follows a normal distribution with mean μ and variance σ^2 i.e., $Y \sim N(\mu, \sigma^2)$, then we denote this lognormal distribution by $X \sim \mathrm{LN}(\mu, \sigma^2)$ and pdf of X is

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}x\sigma}\exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right], \; x > 0, \; -\infty < \mu < \infty, \; \sigma > 0. \tag{5.10}$$

Note that lognormal distribution is a log location-scale distribution. From the definition, it follows that the percentile of the lognormal distribution is $\exp(\mu + Q_p(0,1)\sigma)$, where $Q_p(0,1)$ is the p-th percentile of the standard normal distribution. We derive the FGPQ of the percentile function of the lognormal distribution based on the percentile of the normal distribution. Let, $Y_i = \ln(X_i), i = 1, 2, \ldots, n$ be a random sample of size n from the normal distribution Y. The equivariant estimator of the parameters are $\hat{\mu} = \bar{Y}$ and $\hat{\sigma}^2 = S_Y^2$ where $\bar{Y} = \sum_{i=1}^n Y_i/n$, $S_Y^2 = \sum_{i=1}^n \left(Y_i - \bar{Y}\right)^2/(n-1)$ are the sample mean and the sample variance respectively. Let, $(\hat{\mu}_0, \hat{\sigma}_0^2) = (\bar{y}, s_y^2)$ be an observed value of $(\hat{\mu}, \hat{\sigma}^2)$. Similar to the normal distribution, the FGPQ of percentiles of the lognormal distribution is

$$\mathcal{R}_{Q_p}^{LN} = \exp\left[\bar{y} + \frac{Q_p(0,1)\sqrt{n} - Z}{\sqrt{V/(n-1)}}\frac{s_y}{\sqrt{n}}\right]. \tag{5.11}$$

5.4.3 Weibull distribution

Let 'X' be a positive random variable following the Weibull distribution with shape parameter β and scale parameter α. We denote this distribution by $X \sim Weib(\alpha, \beta)$ and its pdf is given by,

$$f(x; \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], \; x > 0, \; \alpha > 0, \; \beta > 0. \tag{5.12}$$

Let, $Y = \ln(X)$. Then the pdf of Y is,

$$f(y; \mu, \sigma) = \frac{1}{\sigma}\exp\left[\frac{y-\mu}{\sigma} - \exp\left(\frac{y-\mu}{\sigma}\right)\right], \; -\infty < y < \infty \tag{5.13}$$

where $\sigma = 1/\beta > 0$ and $\mu = \ln(\alpha)$ $(-\infty < \mu < \infty)$. It is important to note that the distribution of Y is a location-scale distribution, specifically known as the extreme value distribution. We will refer to this extreme value distribution as $Y \sim \mathrm{EV}(\mu, \sigma)$. Therefore, Weibull distribution is a log location-scale distribution. First, we derive the FGPQ of the location parameter μ and the scale parameter σ of the extreme value distribution based on the FGPQ of the parameters α and β of the Weibull distribution. Krishnamoorthy et al. [67] derived the FGPQ of the parameter α and β as follows,

$$\mathcal{R}_\alpha = \hat{\alpha}_0\left(\frac{1}{\hat{\alpha}^*}\right)^{\hat{\beta}^*/\hat{\beta}_0} \quad \text{and} \quad \mathcal{R}_\beta = \hat{\beta}_0/\hat{\beta}^* \tag{5.14}$$

where $(\hat{\alpha}_0, \hat{\beta}_0)$ is the MLE of (α, β) based on a random sample of size n from a $Weib(\alpha, \beta)$ distribution and $(\hat{\alpha}^*, \hat{\beta}^*)$ is the MLE of (α, β) based on a random sample of size n from a $Weib(1, 1)$ distribution. See Cohen [26] to estimate the MLE of Weibull distribution. Then, the FGPQ of μ and σ are

$$\mathcal{R}_\mu = \ln(\hat{\alpha}_0) - \left(\frac{\hat{\beta}^*}{\hat{\beta}_0}\right)\ln(\hat{\alpha}^*) \quad \text{and} \quad \mathcal{R}_\sigma = \hat{\beta}^*/\hat{\beta}_0.$$

From the definition and after some simplification, it follows that the FGPQ of the percentiles of the Weibull distribution is

$$\mathcal{R}_{Q_p}^{W} = \hat{\alpha}_0 \exp\left[\frac{\left(Q_p(0,1) - \ln(\hat{\alpha}^*)\right)\hat{\beta}^*}{\hat{\beta}_0}\right] \tag{5.15}$$

where $Q_p(0,1) = \ln[-\ln(1-p)]$ is the p-th percentile of the standard extreme value distribution.

5.4.4 Gamma distribution

Let 'X' be a random variable following a gamma distribution with shape parameter α and rate parameter β. We denote this distribution as $X \sim \text{Gamma}(\alpha, \beta)$ and pdf of this distribution is given by

$$f(x; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \ x > 0, \ \alpha > 0, \ \beta > 0. \tag{5.16}$$

Note that the gamma distribution is neither a location-scale distribution nor a log-location-scale distribution. We derive the FGPQs of the parameters based on the cube root transformed sample. Let $Y = X^{1/3}$, based on the Wilson and Hilferty [120] approximation, Y approximately follow a normal distribution with mean μ and variance σ^2, see Krishnamoorthy and Wang [68] for details. For various methods of estimating the fiducial quantity for gamma parameters, refer to Gao and Tian [44]. Let, $Y_i = X_i^{1/3}, i = 1, 2 \ldots, n$ be a random sample of size n. The equivariant estimator of μ, σ^2 are $\hat{\mu} = \bar{Y}$ and $\hat{\sigma}^2 = S_Y^2$ where $\bar{Y} = \sum_{i=1}^{n} Y_i/n$, $S_Y^2 = \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)^2 / (n-1)$ are the sample mean and the sample variance respectively. Let, $(\hat{\mu}_0, \hat{\sigma}_0^2) = \left(\bar{y}, s_y^2\right)$ be an observed value of $(\hat{\mu}, \hat{\sigma}^2)$. Then FGPQ of the p-th percentile function $Q_p(\mu, \sigma)$ of Y can be obtained similarly as equation (5.9) as

$$\mathcal{R}_{Q_p} = \left[\bar{y} + \frac{Q_p(0,1)\sqrt{n} - Z}{\sqrt{V/(n-1)}} \frac{s_y}{\sqrt{n}}\right] \tag{5.17}$$

where $Q_p(0, 1)$ is the p-th percentile of the standard normal distribution, Z is the standard normal variable and V is the chi-squared random variable with $n-1$ degrees of freedom. Then it follows that an approximate FGPQ of the p-th percentile function of the gamma distribution is

$$\mathcal{R}_{Q_p}^{G} = \left[\bar{y} + \frac{Q_p(0,1)\sqrt{n} - Z}{\sqrt{V/(n-1)}} \frac{s_y}{\sqrt{n}}\right]^3. \tag{5.18}$$

5.5 Comparison Between Two Processes

Comparisons of performance between two processes using process capability indices are discussed below.

5.5.1 FGCI Method for the Difference of Two PCIs

To compare the capabilities of two processes using a difference test, we first denote

$$\delta_{pk} = C_{N_{pk1}} - C_{N_{pk2}}.$$

The hypothesis to be tested can be expressed as follows:

$$\text{H}_0 : C_{N_{pk1}} = C_{N_{pk2}} \ \text{ vs. } \ \text{H}_1 : C_{N_{pk1}} \neq C_{N_{pk2}}. \tag{5.19}$$

The FGPQ of the difference is defined as

$$\mathcal{R}_{\delta_{pk}} = \mathcal{R}_{C_{N_{pk1}}} - \mathcal{R}_{C_{N_{pk2}}}. \tag{5.20}$$

Since the distribution of $\mathcal{R}_{\delta_{pk}}$ is free of any unknown parameters, the FGCI of δ_{pk} can be obtained by estimating the percentiles of the FGPQ $\mathcal{R}_{\delta_{pk}}$ using Monte Carlo simulations.

5.5.2 FGCI Method for the Ratio of Two PCIs

To compare the capabilities of two processes based on a ratio test, we first denote

$$\xi_{pk} = \mathrm{C}_{N_{pk1}}/\mathrm{C}_{N_{pk2}}.$$

The hypothesis to be tested is similar as equation (5.19). The FGPQ of the ratios are

$$\mathcal{R}_{\xi_{pk}} = \frac{\mathcal{R}_{C_{N_{pk1}}}}{\mathcal{R}_{C_{N_{pk2}}}}. \tag{5.21}$$

FGCI of ξ_{pk} can be obtained by estimating the percentiles of the distribution of their FGPQ $\mathcal{R}_{\xi_{pk}}$, known as fiducial distribution. These percentiles can be obtained by using Monte Carlo simulation because the fiducial distributions are free of unknown parameters.

Algorithm for estimating FGCI and testing hypothesis

We will illustrate the procedure for estimating the FGCI of the ratio and difference between two PCIs under a location-scale distribution to compare the performance of two processes or suppliers.

1. Construct the null and the alternative hypothesis: $\mathrm{H}_0 : C_{N_{pk1}} = C_{N_{pk2}}$ vs. $\mathrm{H}_1 : C_{N_{pk1}} \neq C_{N_{pk2}}$.
2. Collect random observations $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ of size n_1 and n_2 respectively, from the two processes under study.
3. Derive FGPQs of the location and scale parameters, say $\mathcal{R}_{\mu_i}$ and $\mathcal{R}_{\sigma_i}$, $(i = 1, 2)$ for the two processes based on the random sample.
4. Compute FGPQs of the percentile function for the two processes $\mathcal{R}_{Q_p;i} = \mathcal{R}_{\mu_i} + Q_p(0,1)\mathcal{R}_{\sigma_i}$; $i = 1, 2$; $Q_p(0, 1)$ denotes the p-th percentile of standard location-scale distribution.
5. Compute the FGPQ of the ratio and difference of PCIs by using the FGPQ of percentile functions.
6. Repeat steps 3-5 a large number of times say N times.
7. Then the $100\alpha/2$ th and the $100(1-\alpha/2)$ th percentiles of these N generated FGPQ will provide a two-sided $100(1-\alpha)\%$ fiducial generalized confidence interval for the ratio and difference of the PCIs. The FGCI of the difference is, $\left(\mathcal{R}_{\delta_{pk};\alpha/2}, \mathcal{R}_{\delta_{pk};1-\alpha/2}\right)$ and FGCI for the ratio is $\left(\mathcal{R}_{\xi_{pk};\alpha/2}, \mathcal{R}_{\xi_{pk};1-\alpha/2}\right)$.

8. Decisions can be taken based on the differences test as follows:

   (i) Reject the null hypothesis if the estimated CI does not contain zero. Choose supplier I if both the confidence limits are positive and choose supplier II if both the confidence limits are negative.

   (ii) No significant difference exists between the two suppliers if the estimated CI contains zero.

9. Decisions can be taken based on the ratio test as follows:

   (i) Reject the null hypothesis if the estimated CI does not contain one. Choose supplier I if both the confidence limits are greater than one and choose supplier II if both the confidence limits are less than one.

   (ii) No significant difference exists between the two suppliers if the estimated CI contains one.

5.6 Bootstrap Confidence Interval

The general procedure for estimating non-parametric bootstrap confidence intervals (BCIs) for independent and identically distributed random variables is outlined below. We outline the procedure for estimating the bootstrap confidence interval for the difference and ratio of $C_{N_{pk}}$.

Algorithm for estimating BCIs of the differences and ratios

(i) Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ be two random samples of size n_1 and n_2 respectively, drawn from two processes under investigation. Bootstrap samples $X^*_{11}, X^*_{12}, \ldots, X^*_{1n_1}$ and $X^*_{21}, X^*_{22}, \ldots, X^*_{2n_2}$ of size n_1 and n_2 are drawn respectively from the two original samples with replacement.

(ii) Based on these bootstrap samples compute the MLE of the parameters of the corresponding distributions as well as the corresponding MLE of $C_{N_{pk1}}$ and $C_{N_{pk2}}$, denoted by $\hat{C}^*_{N_{pk1}}$ and $\hat{C}^*_{N_{pk2}}$.

(iii) There are total $n_1^{n_1}$ possible such re-samples for the first sample and $n_2^{n_2}$ possible re-samples for the second sample. Among these re-samples, we choose B bootstrap samples in each case and calculate bootstrap estimators $\hat{C}^*_{N_{pk1}}$ and $\hat{C}^*_{N_{pk2}}$. Then we obtain B bootstrap estimator $\hat{\delta}^*_{pk} = (\hat{C}^*_{N_{pk1}} - \hat{C}^*_{N_{pk2}})$ and $\hat{\xi}^*_{pk} = \hat{C}^*_{N_{pk1}}/\hat{C}^*_{N_{pk2}}$ and ordered them from smallest to largest value as $\hat{\delta}^{*(1)}_{pk} \leq \hat{\delta}^{*(2)}_{pk} \leq \ldots, \leq \hat{\delta}^{*(B)}_{pk}$ and $\hat{\xi}^{*(1)}_{pk} \leq \hat{\xi}^{*(2)}_{pk} \leq \ldots, \leq \hat{\xi}^{*(B)}_{pk}$ respectively. Let denote these ordered sets as $\mathcal{S}_1 = \left\{\hat{\delta}^{*(j)}_{pk}; j = 1, 2, \ldots, B\right\}$ and $\mathcal{S}_2 = \left\{\hat{\xi}^{*(j)}_{pk}; j = 1, 2, \ldots, B\right\}$. Efron and Tibshirani [37] indicate that a minimum $B = 1000$ bootstrap resamples are sufficient to estimate a reliable CI.

5.6.1 Standard bootstrap

Let, $\bar{\hat{\delta}}^*_{pk}$ and $S_{\hat{\delta}^*_{pk}}$ represents the sample mean and sample standard deviation of the set $\mathcal{S}_1 = \left\{\hat{\delta}^{*(j)}_{pk}; j = 1, 2, \ldots, B\right\}$, and $\bar{\hat{\xi}}^*_{pk}$ and $S_{\hat{\xi}^*_{pk}}$ be the sample mean and sample standard deviation of $\mathcal{S}_2 = \left\{\hat{\xi}^{*(j)}_{pk}; j = 1, 2, \ldots, B\right\}$, i.e., $\bar{\hat{\delta}}^*_{pk} = (1/B)\sum_{i=1}^{B}\hat{\delta}^{*(i)}_{pk}$; $S_{\hat{\delta}^*_{pk}} = \sqrt{\frac{1}{(B-1)}\sum_{i=1}^{B}\left(\hat{\delta}^{*(i)}_{pk} - \bar{\hat{\delta}}^*_{pk}\right)^2}$ and $\bar{\hat{\xi}}^*_{pk} = (1/B)\sum_{i=1}^{B}\hat{\xi}^{*(i)}_{pk}$; $S_{\hat{\xi}^*_{pk}} = \sqrt{\frac{1}{(B-1)}\sum_{i=1}^{B}\left(\hat{\xi}^{*(i)}_{pk} - \bar{\hat{\xi}}^*_{pk}\right)^2}$. To obtain a $100(1-\alpha)\%$ SB confidence interval for $\hat{\delta}_{pk}$ and $\hat{\xi}_{pk}$, we can use the formulas: $\left(\bar{\hat{\delta}}^*_{pk} - z_{(1-\alpha/2)} \cdot S_{\hat{\delta}^*_{pk}}, \bar{\hat{\delta}}^*_{pk} + z_{(1-\alpha/2)} \cdot S_{\hat{\delta}^*_{pk}}\right)$ and $\left(\bar{\hat{\xi}}^*_{pk} - z_{(1-\alpha/2)} \cdot S_{\hat{\xi}^*_{pk}}, \bar{\hat{\xi}}^*_{pk} + z_{(1-\alpha/2)} \cdot S_{\hat{\xi}^*_{pk}}\right)$, respectively. Here, $z_{(1-\alpha/2)}$ represents the $(1-\alpha/2)$-th percentile of the standard normal distribution.

5.6.2 Percentile bootstrap

Let, $\hat{\delta}^{*(\gamma)}_{pk}$ and $\hat{\xi}^{*(\gamma)}_{pk}$ denote the γ-th percentile of $\mathcal{S}_1 = \left\{\hat{\delta}^{*(j)}_{pk}; j = 1, 2, \ldots, B\right\}$ and $\mathcal{S}_2 = \left\{\hat{\xi}^{*(j)}_{pk}; j = 1, 2, \ldots, B\right\}$ respectively. Then a $100(1-\alpha)\%$ PB confidence interval of δ_{pk} and ξ_{pk} are, $\left(\hat{\delta}^{*(B.(\alpha/2))}_{pk}, \hat{\delta}^{*(B.(1-\alpha/2))}_{pk}\right)$ and $\left(\hat{\xi}^{*(B.(\alpha/2))}_{pk}, \hat{\xi}^{*(B.(1-\alpha/2))}_{pk}\right)$ respectively.

5.6.3 Bias corrected percentile bootstrap

To obtain the BCPB confidence interval of δ_{pk} and ξ_{pk}, first we have to obtain their bias-corrected factor Z_1 and Z_2 respectively. Based on ordered bootstrap $\mathcal{S}_1$ and $\mathcal{S}_2$, first we calculate $P_1 = 1/B\sum_{j=1}^{B} I\left(\hat{\delta}^{*(j)}_{pk} \leq \hat{\delta}_{pk}\right)$ and $P_2 = 1/B\sum_{j=1}^{B} I\left(\hat{\xi}^{*(j)}_{pk} \leq \hat{\xi}_{pk}\right)$, where $I(.)$ is the indicator function and $\hat{\delta}_{pk}$, $\hat{\xi}_{pk}$ are the estimators based on original samples. Then we calculate $Z_1 = \Phi^{-1}(P_1)$ and $Z_2 = \Phi^{-1}(P_2)$, where $\Phi(.)$ is the cdf of standard normal distribution. Subsequently, we calculate the probabilities $P_{L_1} = \Phi\left(2Z_1 - Z_{(1-\alpha/2)}\right)$, $P_{U_1} = \Phi\left(2Z_1 + Z_{(1-\alpha/2)}\right)$ and $P_{L_2} = \Phi\left(2Z_2 - Z_{(1-\alpha/2)}\right)$, $P_{U_2} = \Phi\left(2Z_2 + Z_{(1-\alpha/2)}\right)$. Then a $100(1-\alpha)\%$ BCPB confidence interval of δ_{pk} and ξ_{pk} are, $\left(\hat{\delta}^{*(B.P_{L_1})}_{pk}, \hat{\delta}^{*(B.P_{U_1})}_{pk}\right)$ and $\left(\hat{\xi}^{*(B.P_{L_2})}_{pk}, \hat{\xi}^{*(B.P_{U_2})}_{pk}\right)$.

5.7 Performance Analysis of Different Methods via Monte Carlo Simulation

In this section, we estimate the coverage probability (CP) and average width (AW) of the proposed fiducial generalized confidence interval using Monte Carlo simulation. We also compare the performance of the proposed FGCI method with the BCIs obtained through the SB, PB, and BCPB methods. For a sample of size n, we generate $10,000$ FGPQ of the PCIs to estimate the 95% FGCI. To estimate the coverage probability (CP) and average width (AW) of the estimated FGCIs, the simulation is replicated $10,000$ times. To estimate 95% bootstrap confidence intervals (BCIs), we draw $B = 1000$ bootstrap samples of size n from each original sample. To estimate the coverage probability (CP) and average width (AW) of the estimated BCIs, the simulation is replicated $N = 5000$ times. The estimated CP is the proportion of the estimated CIs that include the true value of the PCIs. Here we consider sample size $n = 20, 30, 40, 50$. Various combinations of parameters, along with specification limits and target values, are outlined in the corresponding table. The simulated values of CPs and AWs for FGCI and BCIs, under the normal, lognormal, Weibull, and gamma distributions for $C_{N_{pk}}$, are reported in Tables 5.1, 5.2, 5.3, and 5.4 respectively. Our study demonstrates that the proposed FGCI outperforms BCIs in every instance regarding CP. The CP of the estimated FGCIs is close to the nominal confidence level, even with small sample sizes, and is not significantly affected by sample size. The CP of the estimated BCIs are affected by the sample size. The CP of the estimated BCIs converges to the nominal confidence level as the sample size increases. In every case, the AWs of the CIs decrease as the sample size increases. The performance of the CIs based on CP can be arranged as FGCI $>$ SB $>$ BCPB $>$ PB for all the cases discussed here. The AWs of the BCPB method are the smallest in every case and are very close to the AWs of the FGCI. The performance of the CIs in terms of AW can be arranged as BCPB $>$ FCI $>$ PB $>$ SB. Therefore, we can conclude that the overall performance of our proposed FGCI is superior to that of BCIs.

Also, we estimate the coverage probability (CP) and average width (AW) of the 95% confidence intervals (CIs) for δ_{pk} and ξ_{pk} under normal, lognormal, Weibull, and gamma distributions. The estimated values for CP and AW are presented in Tables 5.5, 5.6, 5.7, and 5.8. For every distribution discussed here, we consider $(n_1, n_2) = (20, 20), (25, 20), (25, 25), (30, 35), (50, 50)$ which includes equal as well as unequal sample sizes. The values of the parameters (μ_1, μ_2) and (σ_1, σ_2) of the two processes, where μ_1 is the localization parameter of the first process, μ_2 is the localization parameter of the second process, σ_1 is the scale parameter of the first process and σ_2 is the scale parameter of the second process for each distribution as well as L, U, T values are provided in the corresponding table. The first row in the column of CP and AW under each value of parameter σ_2 (β_2 for Weibull and gamma distribution) in each table corresponds to the difference, and the second row corresponds to the ratio. For estimating FGCI, we generate $10,000$ samples of size n_1 and n_2 from two processes corresponding to each combination of parameters. We used $10,000$ simulation runs for each pair of samples to estimate the CIs. The CP of the estimated FGCI is the proportion of $10,000$ CIs that include the true values of the parameter. The AW of FGCI is the average length of $10,000$ estimated CIs. For estimating BCIs, $B = 1000$ bootstrap resampling is drawn from both processes. The simulation is replicated for $N = 5,000$ to estimate CP and AW.

The performance of an estimated CI can also be verified in another way. The frequency of a $100(1-\alpha)\%$ CI containing the true parameter value follows a binomial distribution with parameter N and $p = (1-\alpha)$, where N is the number of simulations. For FGCI, $\mathrm{N} = 10,000$ and for BCIs $\mathrm{N} = 5,000$. Then it follows that a 99% confidence interval of the observed CP for FGCI is given by $0.95 \pm 2.57\sqrt{(0.95 \times 0.05)/10000} = (0.944, 0.956)$ and for BCIs is given by $0.95 \pm 2.57\sqrt{(0.95 \times 0.05)/5000} = (0.942, 0.958)$.

Table 5.1: Coverage probability (CP) and Average width (AW) of estimated 95% CIs of $C_{N_{pk}}$ under normal distribution, $L = 5$, $U = 15$, $T = 10.5$

		CP				AW			
(μ, σ)	Methods	$n = 20$	$n = 30$	$n = 40$	$n = 50$	$n = 20$	$n = 30$	$n = 40$	$n = 50$
(10.8, 1.0)	FGCI	0.947	0.952	0.946	0.949	0.971	0.774	0.664	0.590
	SB	0.927	0.929	0.928	0.933	1.104	0.834	0.695	0.611
	PB	0.832	0.865	0.888	0.898	1.079	0.823	0.687	0.606
	BCPB	0.869	0.893	0.918	0.915	0.936	0.746	0.641	0.572
(10.2, 1.8)	FGCI	0.943	0.946	0.942	0.947	0.622	0.500	0.431	0.385
	SB	0.937	0.941	0.939	0.941	0.667	0.517	0.437	0.388
	PB	0.892	0.915	0.922	0.926	0.656	0.513	0.434	0.386
	BCPB	0.897	0.915	0.927	0.920	0.627	0.500	0.428	0.382

Based on the results obtained, we can draw the following conclusions:

(i) The confidence intervals (CIs) for both the differences and the ratios perform similarly.

(ii) The sample size has a significant impact on the AWs of confidence intervals (CIs) for all methods. As the sample size increases, the AW of all confidence intervals decreases.

(iii) FGCI delivers optimal results regarding the differences in terms of CP and AW. In this case, the CP is very close to the nominal coverage rate i.e., 0.95 and in most cases, it belongs to the 99% CI $(0.944, 0.956)$. Among the BCIs, SB provides the best result both in terms of CP and AW.

(iv) In terms of the ratio, FGCI delivers the best results regarding CP. Among the BCIs, SB provides the best result in terms of CP, while the BCPB provides the smallest AW.

(v) The sample size (n_1, n_2) does not affect the CP of FGCI. Even with small sample sizes, the actual CP of FGCIs is very close to the nominal confidence level. On the other hand, the sample size (n_1, n_2) affects the performance of BCIs. As the sample size increases, the CP of BCIs gradually tends to the nominal coverage rate. Therefore, the FGCI method is superior to the BCI method.

(vi) The performance of the confidence intervals for differences and ratios is not significantly different. Therefore, either the difference test or the ratio test can be used to compare the performance of two processes or suppliers.

Table 5.2: Coverage probability (CP) and Average width (AW) of estimated 95% CIs of $C_{N_{pk}}$ under lognormal distribution, $L = 20$, $U = 130$, $T = 65$

		CP				AW			
(μ, σ)	Methods	$n = 20$	$n = 30$	$n = 40$	$n = 50$	$n = 20$	$n = 30$	$n = 40$	$n = 50$
$(4.1, 0.18)$	FGCI	0.947	0.951	0.948	0.952	0.858	0.683	0.585	0.521
	SB	0.922	0.927	0.926	0.929	0.971	0.733	0.609	0.538
	PB	0.832	0.867	0.884	0.896	0.948	0.723	0.603	0.534
	BCPB	0.871	0.896	0.905	0.913	0.815	0.656	0.561	0.504
$(3.9, 0.22)$	FGCI	0.948	0.950	0.948	0.953	0.641	0.511	0.438	0.390
	SB	0.921	0.926	0.925	0.930	0.722	0.546	0.455	0.402
	PB	0.832	0.870	0.885	0.896	0.705	0.539	0.450	0.399
	BCPB	0.873	0.900	0.903	0.915	0.610	0.491	0.420	0.378

Table 5.3: Coverage probability (CP) and Average width (AW) of estimated 95% CIs of $C_{N_{pk}}$ under Weibull distribution, $L = 10$, $U = 80$, $T = 50$

		CP				AW			
(α, β)	Methods	$n = 20$	$n = 30$	$n = 40$	$n = 50$	$n = 20$	$n = 30$	$n = 40$	$n = 50$
$(50.25, 6.5)$	FGCI	0.943	0.942	0.941	0.948	0.724	0.572	0.487	0.430
	SB	0.925	0.923	0.932	0.930	0.801	0.601	0.496	0.437
	PB	0.842	0.873	0.884	0.901	0.786	0.595	0.492	0.434
	BCPB	0.879	0.896	0.902	0.911	0.715	0.550	0.468	0.413
$(45.50, 3.8)$	FGCI	0.941	0.947	0.946	0.951	0.608	0.492	0.425	0.379
	SB	0.932	0.933	0.939	0.935	0.650	0.509	0.432	0.384
	PB	0.876	0.899	0.917	0.918	0.643	0.506	0.430	0.383
	BCPB	0.881	0.901	0.912	0.914	0.609	0.482	0.414	0.373

Table 5.4: Coverage probability (CP) and Average width (AW) of estimated 95% CIs of $C_{N_{pk}}$ under gamma distribution, $L = 30$, $U = 120$, $T = 70$

		CP				AW			
(α, β)	Methods	$n = 20$	$n = 30$	$n = 40$	$n = 50$	$n = 20$	$n = 30$	$n = 40$	$n = 50$
$(50.3, 0.70)$	FGCI	0.949	0.949	0.952	0.952	0.925	0.738	0.634	0.563
	SB	0.935	0.932	0.929	0.936	1.030	0.786	0.656	0.578
	PB	0.848	0.875	0.885	0.901	1.008	0.776	0.650	0.573
	BCPB	0.879	0.901	0.905	0.916	0.894	0.715	0.609	0.543
$(53.2, 0.61)$	FGCI	0.952	0.949	0.947	0.950	0.718	0.577	0.496	0.442
	SB	0.931	0.927	0.937	0.938	0.809	0.616	0.517	0.456
	PB	0.855	0.877	0.900	0.908	0.794	0.609	0.513	0.453
	BCPB	0.888	0.901	0.919	0.926	0.704	0.563	0.485	0.433

Table 5.5: Coverage probability (CP) and Average width (AW) of estimated 95% CIs of difference and ratio of $C_{N_{pk}}$ under normal distribution, $L = 20$, $U = 40$, $T = 30$, $(\mu_1, \mu_2) = (29.7, 30.4)$, $\sigma_1 = 3.0$

		CP				AW			
(n_1, n_2)	σ_2	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
(20,20)	2.5	0.950	0.945	0.923	0.933	1.136	1.272	1.287	1.274
		0.952	0.931	0.924	0.932	0.894	0.831	0.823	0.830
	3.0	0.953	0.946	0.925	0.935	1.043	1.152	1.167	1.165
		0.953	0.935	0.926	0.932	1.114	1.023	1.012	1.008
	3.5	0.954	0.944	0.923	0.936	0.977	1.067	1.091	1.077
		0.953	0.939	0.923	0.933	1.342	1.224	1.209	1.201
	4.0	0.955	0.944	0.919	0.930	0.930	1.016	1.026	1.017
		0.953	0.943	0.923	0.933	1.586	1.443	1.435	1.402
(25,20)	2.5	0.953	0.948	0.921	0.932	1.086	1.197	1.212	1.194
		0.953	0.924	0.927	0.933	0.853	0.754	0.749	0.760
	3.0	0.953	0.951	0.928	0.932	0.986	1.071	1.076	1.067
		0.953	0.930	0.929	0.932	1.062	0.928	0.921	0.927
	3.5	0.954	0.951	0.929	0.933	0.915	0.988	0.998	0.993
		0.955	0.936	0.930	0.932	1.277	1.112	1.105	1.102
	4.0	0.955	0.950	0.930	0.932	0.866	0.930	0.939	0.933
		0.955	0.940	0.930	0.931	1.532	1.311	1.298	1.292
(25,25)	2.5	0.954	0.943	0.924	0.928	1.013	1.076	1.087	1.084
		0.954	0.926	0.924	0.926	0.776	0.721	0.716	0.722
	3.0	0.956	0.944	0.924	0.931	0.926	0.985	0.994	0.990
		0.956	0.932	0.923	0.927	0.957	0.884	0.877	0.878
	3.5	0.956	0.944	0.925	0.931	0.866	0.919	0.927	0.922
		0.957	0.935	0.926	0.927	1.155	1.062	1.053	1.044
	4.0	0.956	0.944	0.924	0.927	0.826	0.874	0.880	0.873
		0.957	0.940	0.925	0.928	1.357	1.242	1.235	1.214
(30,35)	2.5	0.954	0.942	0.929	0.933	0.880	0.918	0.924	0.921
		0.952	0.934	0.928	0.930	0.662	0.631	0.626	0.630
	3.0	0.955	0.942	0.929	0.930	0.808	0.841	0.846	0.843
		0.955	0.938	0.930	0.931	0.812	0.771	0.765	0.764
	3.5	0.955	0.945	0.927	0.930	0.761	0.790	0.794	0.790
		0.955	0.943	0.929	0.931	0.972	0.919	0.912	0.903
	4.0	0.955	0.942	0.924	0.926	0.729	0.755	0.759	0.752
		0.956	0.945	0.930	0.931	1.139	1.068	1.072	1.054
(50,50)	2.5	0.955	0.946	0.937	0.938	0.705	0.719	0.721	0.718
		0.956	0.938	0.937	0.940	0.521	0.494	0.491	0.495
	3.0	0.956	0.948	0.939	0.940	0.644	0.654	0.656	0.655
		0.957	0.941	0.938	0.941	0.638	0.603	0.599	0.601
	3.5	0.956	0.947	0.940	0.941	0.604	0.612	0.614	0.612
		0.958	0.946	0.938	0.940	0.761	0.717	0.712	0.710
	4.0	0.957	0.948	0.941	0.938	0.576	0.583	0.584	0.581
		0.956	0.949	0.941	0.938	0.891	0.836	00.831	0.824

Table 5.6: Coverage probability (CP) and average width (AW) of estimated 95% CIs of difference and ratio of $C_{N_{pk}}$ under lognormal distribution, $L = 10$, $U = 130$, $T = 70$, $(\mu_1, \mu_2) = (4.0, 4.2)$, $\sigma_1 = 0.20$

		CP				AW			
(n_1, n_2)	σ_2	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
(20,20)	0.16	0.946	0.949	0.925	0.937	1.549	1.757	1.781	1.768
		0.948	0.942	0.928	0.928	0.866	0.806	0.798	0.784
	0.20	0.947	0.947	0.927	0.938	1.377	1.556	1.581	1.567
		0.947	0.942	0.926	0.928	1.171	1.057	1.053	1.024
	0.24	0.949	0.945	0.922	0.936	1.276	1.445	1.463	1.439
		0.946	0.943	0.925	0.929	1.535	1.362	1.345	1.303
	0.28	0.949	0.942	0.913	0.935	1.208	1.357	1.383	1.346
		0.946	0.946	0.925	0.929	2.012	1.724	1.686	1.635
(25,20)	0.16	0.948	0.948	0.923	0.932	1.475	1.661	1.673	1.654
		0.948	0.936	0.932	0.932	0.815	0.723	0.717	0.716
	0.20	0.948	0.950	0.928	0.942	1.311	1.453	1.457	1.456
		0.949	0.937	0.930	0.933	1.113	0.956	0.948	0.940
	0.24	0.949	0.950	0.931	0.942	1.204	1.322	1.331	1.329
		0.949	0.939	0.928	0.932	1.467	1.233	1.221	1.197
	0.28	0.948	0.950	0.929	0.944	1.124	1.226	1.253	1.246
		0.948	0.941	0.929	0.933	1.918	1.556	1.543	1.512
(25,25)	0.16	0.950	0.947	0.921	0.935	1.367	1.486	1.513	1.548
		0.950	0.933	0.925	0.927	0.739	0.695	0.690	0.682
	0.20	0.950	0.946	0.925	0.939	1.221	1.317	1.354	1.336
		0.951	0.935	0.926	0.929	0.991	0.914	0.908	0.891
	0.24	0.950	0.946	0.925	0.936	1.131	1.224	1.245	1.227
		0.949	0.939	0.925	0.928	1.287	1.168	1.167	1.134
	0.28	0.951	0.944	0.922	0.933	1.071	1.162	1.168	1.159
		0.949	0.940	0.925	0.928	1.667	1.472	1.463	1.422
(30,35)	0.16	0.948	0.941	0.927	0.937	1.171	1.247	1.256	1.255
		0.947	0.938	0.925	0.928	0.627	0.605	0.602	0.593
	0.20	0.948	0.940	0.925	0.935	1.051	1.124	1.136	1.134
		0.947	0.940	0.923	0.928	0.832	0.793	0.788	0.772
	0.24	0.948	0.939	0.922	0.932	0.984	1.037	1.046	1.042
		0.948	0.941	0.924	0.929	1.067	1.011	1.016	0.977
	0.28	0.949	0.937	0.919	0.929	0.936	0.991	0.999	0.985
		0.948	0.945	0.924	0.929	1.359	1.257	1.248	1.212
(50,50)	0.16	0.949	0.945	0.936	0.940	0.938	0.969	0.976	0.975
		0.950	0.941	0.938	0.936	0.489	0.468	0.466	0.464
	0.20	0.949	0.946	0.937	0.941	0.836	0.861	0.868	0.867
		0.950	0.942	0.937	0.937	0.645	0.612	0.610	0.605
	0.24	0.952	0.944	0.937	0.942	0.775	0.796	0.802	0.800
		0.950	0.943	0.937	0.937	0.828	0.777	0.774	0.765
	0.28	0.951	0.944	0.934	0.940	0.734	0.754	0.758	0.754
		0.950	0.945	0.937	0.938	1.052	0.971	0.967	0.951

Table 5.7: Coverage probability (CP) and Average width (AW) of estimated 95% CIs of differences and ratio of $C_{N_{pk}}$ under Weibull distribution, $L = 10$, $U = 80$, $T = 45$, $(\alpha_1, \alpha_2) = (53.0, 50.0)$, $\beta_1 = 6.0$

		CP				AW			
(n_1, n_2)	β_2	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
(20,20)	5.0	0.945	0.945	0.925	0.928	0.876	0.958	0.975	0.986
		0.944	0.949	0.924	0.923	0.892	0.816	0.816	0.794
	5.5	0.949	0.946	0.931	0.932	0.901	0.991	1.013	1.018
		0.952	0.948	0.929	0.923	0.810	0.749	0.748	0.735
	6.0	0.951	0.947	0.932	0.934	0.927	1.032	1.038	1.058
		0.950	0.949	0.930	0.928	0.746	0.694	0.694	0.687
	6.5	0.954	0.949	0.934	0.936	0.955	1.058	1.075	1.088
		0.951	0.949	0.932	0.928	0.694	0.649	0.650	0.648
(25,20)	5.0	0.949	0.943	0.926	0.937	0.824	0.899	0.913	0.917
		0.950	0.937	0.927	0.935	0.844	0.745	0.745	0.725
	5.5	0.947	0.947	0.925	0.938	0.850	0.935	0.950	0.951
		0.949	0.932	0.923	0.934	0.766	0.682	0.683	0.670
	6.0	0.946	0.948	0.923	0.939	0.879	0.973	0.989	0.986
		0.953	0.929	0.925	0.933	0.703	0.632	0.633	0.625
	6.5	0.949	0.947	0.921	0.937	0.908	1.011	1.035	1.022
		0.954	0.927	0.925	0.933	0.653	0.591	0.593	0.588
(25,25)	5.0	0.946	0.939	0.919	0.926	0.768	0.817	0.831	0.834
		0.946	0.939	0.920	0.923	0.758	0.699	0.699	0.683
	5.5	0.943	0.942	0.925	0.929	0.790	0.844	0.858	0.863
		0.948	0.935	0.924	0.926	0.692	0.642	0.642	0.632
	6.0	0.950	0.943	0.928	0.929	0.812	0.873	0.888	0.892
		0.950	0.934	0.928	0.925	0.638	0.596	0.596	0.591
	6.5	0.948	0.942	0.930	0.927	0.836	0.903	0.919	0.920
		0.950	0.931	0.928	0.926	0.595	0.558	0.559	0.557
(30,35)	5.0	0.938	0.939	0.928	0.936	0.666	0.692	0.701	0.696
		0.939	0.942	0.926	0.934	0.627	0.603	0.603	0.581
	5.5	0.945	0.937	0.929	0.940	0.682	0.711	0.720	0.718
		0.944	0.938	0.928	0.936	0.575	0.555	0.555	0.540
	6.0	0.947	0.940	0.928	0.944	0.699	0.731	0.741	0.740
		0.945	0.937	0.928	0.939	0.533	0.516	0.516	0.506
	6.5	0.945	0.941	0.928	0.945	0.717	0.752	0.763	0.762
		0.944	0.936	0.927	0.940	0.498	0.484	0.484	0.478
(50,50)	5.0	0.945	0.940	0.938	0.932	0.526	0.531	0.535	0.537
		0.946	0.944	0.938	0.930	0.487	0.462	0.462	0.452
	5.5	0.942	0.940	0.937	0.933	0.539	0.545	0.550	0.552
		0.944	0.944	0.937	0.931	0.447	0.426	0.426	0.419
	6.0	0.943	0.942	0.936	0.933	0.553	0.561	0.566	0.568
		0.944	0.942	0.935	0.931	0.414	0.396	0.396	0.393
	6.5	0.944	0.942	0.937	0.934	0.568	0.578	0.583	0.585
		0.944	0.941	0.937	0.933	0.388	0.372	0.372	0.370

Table 5.8: Coverage probability (CP) of estimated 95% CIs of difference and ratio of $C_{N_{pk}}$ under gamma distribution, $L = 40$, $U = 140$, $T = 90$, $(\alpha_1, \alpha_2) = (48, 50)$, $\beta_1 = 0.55$

		CP				AW			
(n_1, n_2)	β_2	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
(20,20)	0.50	0.951	0.946	0.924	0.932	1.134	1.281	1.304	1.302
		0.950	0.923	0.928	0.926	1.477	1.372	1.336	1.367
	0.55	0.952	0.946	0.922	0.934	1.221	1.367	1.389	1.389
		0.953	0.942	0.923	0.928	1.089	1.012	0.986	0.985
	0.60	0.950	0.946	0.921	0.934	1.189	1.356	1.378	1.367
		0.950	0.926	0.923	0.926	1.067	1.012	0.998	1.014
	0.65	0.951	0.944	0.922	0.934	1.147	1.322	1.345	1.336
		0.950	0.929	0.920	0.926	1.156	1.101	1.089	1.112
(25,20)	0.50	0.953	0.940	0.922	0.934	1.051	1.167	1.189	1.178
		0.954	0.917	0.917	0.926	1.421	1.256	1.253	1.278
	0.55	0.951	0.943	0.924	0.935	1.153	1.256	1.278	1.273
		0.952	0.934	0.924	0.931	1.044	0.917	0.913	0.908
	0.60	0.953	0.942	0.921	0.934	1.123	1.245	1.267	1.253
		0.952	0.915	0.922	0.928	1.013	0.906	0.903	0.923
	0.65	0.952	0.940	0.922	0.935	1.078	1.202	1.221	1.214
		0.951	0.913	0.922	0.928	1.089	0.987	0.983	1.012
(25,25)	0.50	0.949	0.949	0.930	0.941	0.994	1.078	1.102	1.101
		0.951	0.930	0.931	0.934	1.278	1.189	1.178	1.202
	0.55	0.951	0.948	0.929	0.936	1.083	1.164	1.178	1.176
		0.951	0.944	0.927	0.932	0.939	0.869	0.865	0.851
	0.60	0.950	0.947	0.930	0.939	1.052	1.155	1.167	1.164
		0.951	0.932	0.929	0.933	0.921	0.870	0.866	0.878
	0.65	0.950	0.948	0.927	0.940	1.011	1.113	1.132	1.118
		0.951	0.933	0.930	0.933	1.002	0.947	0.943	0.957
(30,35)	0.50	0.953	0.943	0.928	0.937	0.872	0.923	0.933	0.930
		0.951	0.941	0.935	0.934	1.082	1.034	1.032	1.032
	0.55	0.952	0.940	0.928	0.939	0.938	0.983	0.995	0.991
		0.953	0.949	0.925	0.937	0.792	0.753	0.749	0.733
	0.60	0.953	0.944	0.932	0.940	0.913	0.968	0.980	0.978
		0.953	0.936	0.932	0.934	0.788	0.760	0.757	0.762
	0.65	0.953	0.945	0.933	0.939	0.884	0.937	0.949	0.947
		0.953	0.937	0.933	0.934	0.858	0.827	0.824	0.830
(50,50)	0.50	0.954	0.948	0.940	0.943	0.688	0.714	0.719	0.718
		0.952	0.938	0.935	0.938	0.852	0.822	0.819	0.822
	0.55	0.951	0.940	0.932	0.941	0.747	0.768	0.773	0.774
		0.951	0.946	0.932	0.938	0.618	0.593	0.591	0.582
	0.60	0.952	0.945	0.935	0.944	0.724	0.751	0.757	0.756
		0.953	0.941	0.936	0.939	0.621	0.602	0.600	0.604
	0.65	0.952	0.945	0.936	0.943	0.699	0.725	0.731	0.730
		0.953	0.939	0.935	0.938	0.676	0.656	0.654	0.658

5.8 Some Illustrative Examples

Two applications of the proposed method are discussed below.

5.8.1 Example 1: Drilling machine manufacture data

In this example, we consider lifetime data from a 1.88 mm drill manufacturing process. This data set is referenced in Piao and Zhi-Sheng [93], and it is presented in Table 5.9. Drills are a crucial part of a drilling machine. The factory set specification limits (in minutes) are set at $(U, L) = (60, 180)$ with a target value of $T = 120$. Using a model discrimination method, the authors indicated that the data actually follows a gamma distribution. We assess the reliability of the drills in terms of estimated CIs. We plot the histogram and Q-Q plot of the data in Figure 5.1 and Figure 5.2, respectively. Also, we test the data set based on K-S test. The p-value of the K-S test is 0.6933, which is greater than 0.05, providing strong evidence that the data follows a gamma distribution. The MLE of the parameters are $(\hat{\alpha}, \hat{\beta}) = (72.364, 0.628)$. The estimated CIs and corresponding widths under different methods are provided in Table 5.10.

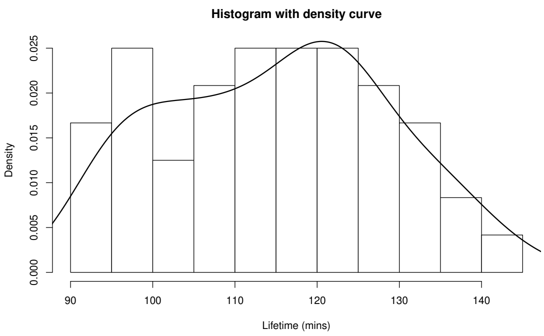

Figure 5.1: Histogram of the sample

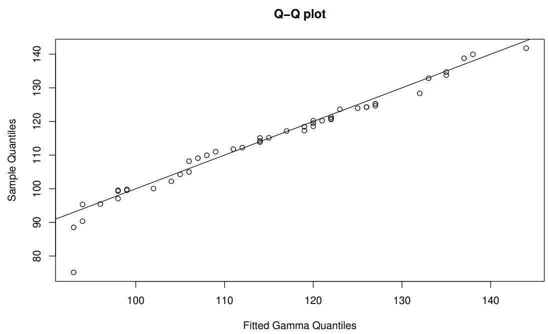

Figure 5.2: Q-Q plot of the sample

Table 5.9: Lifetime data of 48 observations of 1.88 mm drills (mins)

135	98	114	137	138	144	99	93	115	106
132	122	94	98	127	122	102	133	114	120
93	126	119	104	119	114	125	107	98	117
111	106	108	127	126	135	112	94	127	99
120	120	121	122	96	109	123	105		

5.8.2 Example 2: Bond finger width data

In this case study, the process data of bond fingers of IC substrate were extracted from a famous printed circuit board (PCB) manufacturer in Taiwan. This data set was originally reported by Tong et al. [113]. The quality characteristic being examined is the width of

Table 5.10: Estimated CIs and widths of $C_{N_{pk}}$ under different methods

Methods	CI	Width
FGCI	$(1.093, 1.567)$	0.474
SB	$(1.133, 1.591)$	0.458
PB	$(1.173, 1.641)$	0.468
BCPB	$(1.160, 1.571)$	0.411

the bond fingers. The width data of 60 bond fingers were collected from an in-control process before and after quality improvement to assess whether the process improvement plan was effective. The data sets are reported in Table 5.11. The specifications for the bond finger widths are, $T = 3.05$ mm, $U = 3.25$ mm, $L = 2.85$ mm. The histogram of these two samples is shown in Figure 5.3 and Figure 5.4 respectively. The Shapiro-Wilk normality test is performed to check the normality of sample I and sample II. The result of the Shapiro-Wilk test for the two samples are, sample I: $W = 0.985,\ p = 0.668$; sample II: $W = 0.971,\ p = 0.159$, indicates that both samples are normally distributed. For sample I, sample size, sample mean and sample variance are $n_1 = 60$, $\bar{x}_1 = 2.865$, $s_1^2 = 0.0076$ and for sample II, $n_2 = 60$, $\bar{x}_2 = 2.996$, $s_2^2 = 0.0092$. We estimate the confidence intervals (CIs) for both the difference and the ratio of the two indices. The estimated CIs under different methods are reported in Table 5.12. Both the upper and lower confidence limits of the difference for both PCIs are negative under each of the FGCI, SB, PB, and BCPB methods. This indicates that the process performance has improved due to the corrective action. Similar results can be achieved using the ratio test across all methods. It is notable that both the upper and lower confidence limits of the ratio of the two PCIs are less than one using all methods, indicating an improvement in process performance.

Table 5.11: Sample Data of Bond Finger Width Before and After Quality Improvement (Unit: mm)

The sample data before improvement (sample I)	2.94	2.76	2.65	2.86	2.75	2.75	2.83	2.86	2.91	2.92
	3.12	2.84	2.86	2.95	3.02	2.90	2.82	2.98	3.00	2.95
	2.99	2.72	2.83	2.80	2.91	2.91	2.91	2.86	2.93	2.79
	2.93	2.70	2.98	2.88	2.93	2.87	2.80	2.88	2.88	2.98
	2.76	2.84	2.89	2.85	2.81	2.86	2.81	2.80	2.88	2.85
	2.85	2.68	2.91	2.80	2.72	2.91	2.89	2.90	2.89	2.86
The sample data after improvement (sample II)	2.97	2.93	2.88	3.09	2.97	2.83	2.91	3.02	2.92	3.01
	3.03	2.96	2.86	3.25	3.15	2.84	2.93	3.03	3.04	3.06
	2.89	3.03	2.96	3.19	3.15	2.94	2.99	3.02	3.00	2.97
	3.15	3.11	2.97	3.21	3.03	3.09	2.91	2.95	2.98	3.00
	3.05	2.99	3.03	3.01	2.97	2.93	2.83	2.96	3.06	3.16
	3.00	2.91	2.89	3.08	2.89	2.90	2.96	2.88	3.10	2.94

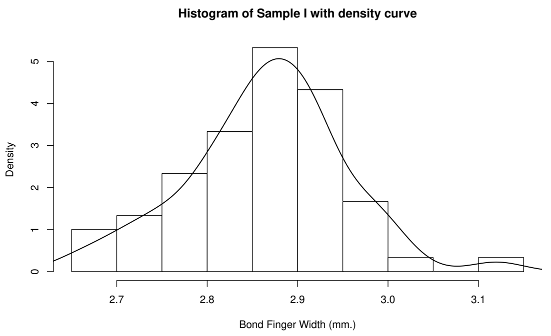

Figure 5.3: Histogram of the sample 1

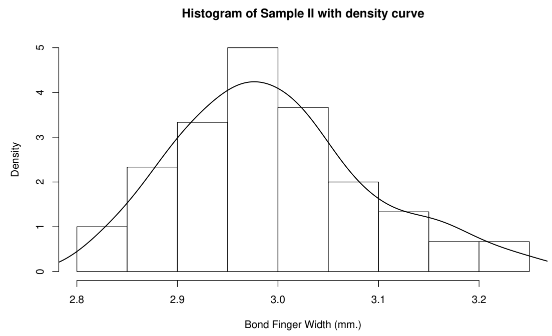

Figure 5.4: Histogram of the sample 2

Table 5.12: 95% CI of the differences and ratios of $C_{N_{pk}}$ under different methods

Methods	δ_{pk}	ξ_{pk}
FGCI	$(-0.5979, -0.2967)$	$(-0.0528, 0.2989)$
SB	$(-0.5937, -0.3300)$	$(-0.0522, 0.2829)$
PB	$(-0.5961, -0.3358)$	$(-0.0523, 0.2926)$
BCPB	$(-0.5912, -0.3202)$	$(-0.0553, 0.2936)$

5.9 Concluding Remarks

The fiducial inference has been applied in many practical applications. In this chapter, we propose FGCI of $C_{N_{pk}}$ and discuss the difference and ratio of $C_{N_{pk}}$ between two processes under location-scale (log location-scale) distributions. We consider four important distributions, namely, normal, lognormal, Weibull, and gamma distributions. The performance of the proposed FGCI is evaluated against three non-parametric bootstrap confidence intervals through Monte Carlo simulations. Our simulation studies indicated that the CP of the proposed FGCI closely aligns with the nominal confidence level, even with a small sample size. The CP of the BCIs is affected by the sample size. The confidence intervals (CIs) for both the difference and the ratio exhibit similar performance. Thus, either the difference or ratio test can be used to evaluate the performance of two processes or suppliers.

Chapter 6

Fiducial Generalized Confidence Interval of C_{py} Under Some Location-scale Distributions

6.1 Introduction

The majority of PCIs in the literature are only associated with processes whose characteristics are described by continuous distributions. Most often it is assumed that the characteristics under study may be well described by the normal distribution. However, some PCIs have been introduced to measure the performance of a process when the quality characteristic is not adequately modelled by the normal distribution; see for example, Chen and Pearn [20], Clements [25]. Most of the existing PCIs deal with continuous distributions. There are only a few PCIs that can be used for continuous as well as for discrete distributions; see, for example, Yeh and Bhattcharya [126], Perakis and Xekalaki [92]. Maiti et al. [75] proposed a generalized PCI C_{py} as the ratio of process yield to the proportion of desired conformance, defined by equation (1.13). The index C_{py} is easy to use for both normal and non-normal distributions, as well as for continuous and discrete distributions.

The index C_{py} is the ratio of two proportions of conformance, making it challenging to estimate the exact confidence interval. In this chapter, we discuss the estimation of the fiducial generalized confidence interval (FGCI) of C_{py} under some location-scale or log location-scale distributions. To the best of our knowledge, no attempts have been made to estimate C_{py} using a fiducial approach. We also compare the performance of two processes or two suppliers using FGCI, based on the difference $\delta_{py} = C_{py1} - C_{py2}$. The performance of the proposed FGCI is compared with three key BCIs, namely, SB, PB, and BCPB, in terms of CP. Our investigation shows that the proposed FGCI outperforms the comparison, and the CP of FGCI closely aligns with the nominal confidence level, even with small sample sizes.

This chapter is organized as follows. In Section 6.2, we discuss the estimation of the fiducial quantity (FQ) and fiducial generalized confidence interval (FGCI) of PCIs under location-scale distributions. The procedure of supplier selection and estimation of FGCI of the difference between two PCIs are discussed in Section 6.3. Some non-parametric bootstrap confidence interval procedures are presented in Section 6.4. Performance comparison of the proposed FGCI with the BCIs is discussed in Section 6.5. Two applications of the proposed FGCI are reported in Section 6.6. Finally, some concluding remarks are added in Section 6.7.

6.2 Fiducial Generalized Confidence Interval of C_{py} Under Some Location-scale Distributions

Let $F(x;\mu,\sigma)$ be the cdf of a location-scale distribution discussed in Section 5.3. Also, let $F(x;0,1) = F^*(x)$ be the cdf of standard location-scale distribution. We can write

$$F(x;\mu,\sigma) = F^*\left(\frac{x-\mu}{\sigma}\right). \tag{6.1}$$

Replacing (μ,σ) in relation (6.1) by their FGPQ $(\mathcal{R}_\mu, \mathcal{R}_\sigma)$ derived in equation (5.5), we obtain FGPQ of the cdf as follows:

$$\mathcal{R}_{F(x)} = F^*\left(\frac{x-\mathcal{R}_\mu}{\mathcal{R}_\sigma}\right) = F^*\left(\hat{\sigma}^*\left(\frac{x-\hat{\mu}_0}{\hat{\sigma}_0}\right)+\hat{\mu}^*\right). \tag{6.2}$$

Therefore, FGPQ of C_{py} is

$$\mathcal{R}_{C_{py}} = \frac{\mathcal{R}_{F(U)} - \mathcal{R}_{F(L)}}{p_0}. \tag{6.3}$$

Then, the FGCI of C_{py} can be obtained by estimating the percentiles of the distribution of their FGPQ, known as fiducial distribution. These percentiles can be obtained using Monte Carlo simulation, as the fiducial distributions are free of unknown parameters. We present a general algorithm for deriving the FGCI of PCIs for location-scale (and log location-scale) distributions and then discuss some key distributions that are either location-scale distributions (or can be converted to a location-scale distribution). It should be noted that the maximum likelihood estimator (MLE) is an equivariant estimator.

Algorithm for Estimating FGCI

1. Generate a random sample of size n from a location-scale distribution.
2. Derive FGPQs of location and scale parameters, say $\mathcal{R}_\mu$ and $\mathcal{R}_\sigma$ based on the random sample.
3. Compute FGPQ of the cdf as $\mathcal{R}_{F(x)} = F^*\left(\hat{\sigma}^*\left(\frac{x-\hat{\mu}_0}{\hat{\sigma}_0}\right)+\hat{\mu}^*\right)$.
4. Compute FGPQ of C_{py}, say $\mathcal{R}_{C_{py}}$ by using FGPQs of cdf.
5. Repeat steps 2-4 a large number of times, say N times.
6. Then the $100\alpha/2$ th and the $100(1-\alpha/2)$ th percentiles of these N generated FGPQs will provide a two-sided $100(1-\alpha)\%$ fiducial generalised confidence interval for C_{py}.

6.2.1 Normal distribution

Let 'X' be a random variable following normal distribution with mean μ and variance σ^2, i.e., $X \sim N(\mu, \sigma^2)$. The pdf of the normal distribution is defined in equation (5.8). The equivariant estimator of μ and σ^2 are given by $\hat{\mu} = \bar{X}$, $\hat{\sigma}^2 = S^2$ where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S^2 = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 /(n-1)$ are the sample mean and the sample variance respectively. Let, $\hat{\mu}^*$, $\hat{\sigma}^{*2}$ be the sample mean and the sample variance based on a random sample of size n from the standard normal distribution, then $\hat{\mu}^* \sim N(0, 1/n)$ and $\hat{\sigma}^{*2} \sim \chi^2_{n-1}/(n-1)$. Let Z be a standard normal random variable, V be a chi-squared random variable with $n-1$ degrees of freedom and $(\hat{\mu}_0, \hat{\sigma}_0^2) = (\bar{x}, s^2)$ be an observed value of $(\hat{\mu}, \hat{\sigma}^2)$. Then from equation (6.2) we can write the FGPQ of the cdf as

$$\mathcal{R}^N_{F(x)} = \Phi\left(\frac{V}{\sqrt{(n-1)}}\frac{(x-\bar{x})}{s} + \frac{Z}{\sqrt{n}}\right) \tag{6.4}$$

where $\Phi(.)$ is the cdf of standard normal distribution. Substituting this FGPQ in equation (6.3) we get FGPQ quantity of C_{py}.

6.2.2 Lognormal distribution

A positive random variable 'X' is said to follow a lognormal distribution if the random variable $Y = \ln(X)$ follows a normal distribution. Let Y follow a normal distribution with mean μ and variance σ^2 i.e., $Y \sim N(\mu, \sigma^2)$, then we denote this lognormal distribution by $X \sim \text{LN}(\mu, \sigma^2)$ and pdf of X is given in equation (5.10). Note that lognormal distribution is a log-location-scale distribution. From the definition it follows that $F_X(x) = \text{P}(X \leq x) = \text{P}(\ln(X) \leq \ln(x)) = F_Y(\ln(x))$. We derive the FGPQ of the cdf of the lognormal distribution based on the cdf of normal distribution. Let, $Y_i = \ln(X_i), i = 1, 2, \ldots, n$ be a random sample of size n from the normal distribution Y. The equivariant estimator of the parameters are $\hat{\mu} = \bar{Y}$ and $\hat{\sigma}^2 = S_Y^2$ where $\bar{Y} = \sum_{i=1}^{n} Y_i/n$, $S_Y^2 = \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)^2 /(n-1)$ are the sample mean and the sample variance respectively. Let, $(\hat{\mu}_0, \hat{\sigma}_0^2) = (\bar{y}, s_y^2)$ be an observed value of $(\hat{\mu}, \hat{\sigma}^2)$. Following a similar way as in the case of a normal distribution, we get the FGPQ of cdf of the lognormal distribution,

$$\mathcal{R}^{LN}_{F(x)} = \Phi\left(\frac{V}{\sqrt{(n-1)}}\frac{(\ln(x)-\bar{y})}{s_y} + \frac{Z}{\sqrt{n}}\right) \tag{6.5}$$

where $\Phi(.)$ is the cdf of the standard normal distribution. Substituting this FGPQ in equation (6.3) we get the FGPQ quantity of C_{py}.

6.2.3 Gamma distribution

Let 'X' be a random variable following a gamma distribution with shape parameter α and rate parameter β. We denote this distribution as $X \sim \text{Gamma}(\alpha, \beta)$, and the pdf of this distribution is given by equation (5.16). Note that gamma distribution is neither a location-scale nor a log-location-scale distribution. We derive the FGPQs of the parameters based on the cube root transformed sample. Let $Y = X^{1/3}$, then based on the Wilson and Hilferty [120] approximation, Y approximately follows a normal distribution with mean μ and variance σ^2, see [68]. For different methods of estimating fiducial quantity for gamma parameters see, Gao and Tian [44]. Then $F_X(x) = \mathrm{P}(X \le x) = \mathrm{P}\left(X^{1/3} \le x^{1/3}\right) = F_Y(x^{1/3})$. Let, $Y_i = X_i^{1/3}, i = 1, 2 \ldots, n$ be a random sample of size n. The MLE of μ, σ are $\hat{\mu} = \bar{Y}$ and $\hat{\sigma} = S_Y^2$ where $\bar{Y} = \sum_{i=1}^{n} Y_i/n$, $S_Y^2 = \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2/(n-1)$ are sample mean and sample variance respectively. Let, $(\hat{\mu}_0, \hat{\sigma}_0) = \left(\bar{y}, s_y^2\right)$ be an observed value of $(\hat{\mu}, \hat{\sigma})$. Then an approximate FGPQ of the cdf of gamma distribution can be obtained similarly as equation (6.5) as

$$\mathcal{R}_{F(x)}^{G} = \Phi\left(\frac{V}{\sqrt{(n-1)}}\frac{\left(x^{1/3} - \bar{y}\right)}{s_y} + \frac{Z}{\sqrt{n}}\right) \tag{6.6}$$

where $\Phi(.)$ is the cdf of standard normal distribution. Substituting this FGPQ in equation (6.3), we get the FGPQ quantity of C_{py}.

6.2.4 Weibull distribution

Let 'X' be a positive random variable following the Weibull distribution with shape parameter β and scale parameter α. We denote this distribution by $X \sim Weib(\alpha, \beta)$ and its pdf is given by equation (5.12) and cdf are given by,

$$F(x; \alpha, \beta) = 1 - \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]. \tag{6.7}$$

Krishnamoorthy et al. [67] derived the FGPQ of the parameters α and β given in equation (5.14). Next, the FGPQ of the cumulative distribution function (cdf) is obtained by substituting the parameters α and β with their FGPQ in equation (6.7) as follows:

$$\mathcal{R}_{F(x)}^{W} = 1 - \exp\left[-\hat{\alpha}^{*}\left(\frac{x}{\hat{\alpha}_0}\right)^{\hat{\beta}_0/\hat{\beta}^{*}}\right]. \tag{6.8}$$

Substituting this FGPQ in equation (6.3), we get the FGPQ quantity of C_{py}.

6.3 Selection of Supplier Between Two Suppliers

Suppose a manufacturing company wants to select the supplier based on the generalized PCI C_{py}. The supplier with the higher value of C_{py} will be chosen. Let us define the difference between the PCIs of the two suppliers as

$$\delta_{py} = C_{py1} - C_{py2},$$

where C_{py1} is the generalised PCI of supplier I and C_{py2} for supplier II. Testing of the hypothesis based on C_{py} is performed to select the proper purchase decision. The hypothesis testing problem can be written as,

$$\mathrm{H}_0 : C_{py1} = C_{py2} \quad \text{vs.} \quad \mathrm{H}_1 : C_{py1} \neq C_{py2}.$$

The FGPQ of δ_{py} is defined as,

$$\mathcal{R}_{\delta_{py}} = \mathcal{R}_{C_{py1}} - \mathcal{R}_{C_{py2}}.$$

Let, $\mathcal{R}_{\delta_{py};\gamma}$ denote the γ-th quantile of the distribution $\mathcal{R}_{\delta_{py}}$, known as fiducial distribution. Then a $100(1-\alpha)\%$ FGCI of δ_{py} is $\left(\mathcal{R}_{\delta_{py};\alpha/2}, \mathcal{R}_{\delta_{py};1-\alpha/2}\right)$. These quantiles can be obtained by using Monte Carlo simulation because fiducial distributions are free of unknown parameters.

Algorithm for Estimating FGCI of the Difference Between Two PCIs and Test of the Hypothesis

1. Construct the null and the alternative hypothesises: $\mathrm{H}_0 : C_{py1} = C_{py2}$ *vs.* $\mathrm{H}_1 : C_{py1} \neq C_{py2}$.
2. Collect random observations $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ of size n_1 and n_2 respectively, from the two processes under comparison having the same distribution.
3. Derive FGPQ of the location and scale parameters for the two distributions, say $\mathcal{R}_{\mu_i}$ and $\mathcal{R}_{\sigma_i}$ for $i = 1, 2$ based on the random sample.
4. By substituting the FGPQs of the parameters in the cdf, obtain FGPQs of the cdf of two processes as $\mathcal{R}_{F(x);i} = F^*\left(\hat{\sigma}_i^*\left(\frac{x-\hat{\mu}_{0;i}}{\hat{\sigma}_{0;i}}\right) + \hat{\mu}_i^*\right)$, $i = 1, 2$.
5. Compute the FGPQ of the two PCIs, $\mathcal{R}_{C_{py1}}$, $\mathcal{R}_{C_{py2}}$ by substituting FGPQ of the parameters in the cdf. Then compute FGPQ of the difference as $\mathcal{R}_{\delta_{py}} = \mathcal{R}_{C_{py1}} - \mathcal{R}_{C_{py2}}$.
6. Repeat steps 2-5 a large number of times, say N times.
7. Then the $100\alpha/2$-th and the $100(1-\alpha/2)$-th percentiles of these N generated FGPQs will provide a two-sided $100(1-\alpha)\%$ fiducial generalized confidence interval of δ_{py} denoted by, $\left(\mathcal{R}_{\delta_{py};\alpha/2}, \mathcal{R}_{\delta_{py};1-\alpha/2}\right)$.
8. Decisions are to be taken as follows:
   (i) Reject the null hypothesis if the estimated FGCI does not contain zero. Choose supplier I if both the confidence limits are positive and choose supplier II if both the confidence limits are negative.
   (ii) No significant difference exists between the two suppliers if the estimated FGCI contains zero.

6.4 Bootstrap Confidence Interval

Several types of bootstrap methods have been developed for constructing confidence intervals, e.g., standard bootstrap (SB), percentile bootstrap (PB), student's t bootstrap (STB), bias-corrected percentile bootstrap (BCPB). The general procedure of estimating non-parametric bootstrap CIs of C_{py} for iid random variables is explained below.

Algorithm for estimating BCIs

(i) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from a distribution of interest. A bootstrap sample of size n, denoted by $X_1^*, X_2^*, \ldots, X_n^*$ is drawn from this original sample with replacement.

(ii) Based on the bootstrap sample, compute the MLE of the parameters as well as the MLE of C_{py} denoted by $\hat{C}_{py}^*$.

(iii) There are total n^n possible such re-samples. Among these re-samples, we choose B bootstrap samples and calculate bootstrap estimators $\hat{C}_{py}^*$. Thus we obtain B bootstrap estimator of C_{py} and order them from smallest to largest value as $\hat{C}_{py}^{*(1)} \leq \hat{C}_{py}^{*(2)} \leq \ldots, \leq \hat{C}_{py}^{*(B)}$. We denote this ordered set as $\mathcal{S} = \left\{\hat{C}_{py}^{*(j)}; j = 1, 2, \ldots, B\right\}$. Efron and Tibshirani [37] indicated that a minimum $B = 1000$ bootstrap samples are sufficient to estimate a CI.

6.4.1 Standard bootstrap (SB) confidence interval

Let $\bar{\hat{C}}_{py}^*$ and $S_{\hat{C}_{py}^*}$ represent the sample mean and sample standard deviation, respectively, of the set $\mathcal{S} = \left\{\hat{C}_{py}^{*(j)}; j = 1, 2, \ldots, B\right\}$. This means that $\bar{\hat{C}}_{py}^* = \frac{1}{B}\sum_{i=1}^{B} \hat{C}_{py}^{*(i)}$ and $S_{\hat{C}_{py}^*} = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}\left(\hat{C}_{py}^{*(i)} - \bar{\hat{C}}_{py}^*\right)^2}$. Then a $100(1-\alpha)\%$ SB confidence interval of C_{py} is, $\left(\bar{\hat{C}}_{py}^* - z_{(1-\alpha/2)}.S_{\hat{C}_{py}^*}, \bar{\hat{C}}_{py}^* + z_{(1-\alpha/2)}.S_{\hat{C}_{py}^*}\right)$, where $z_{(1-\alpha/2)}$ is the $(1-\alpha/2)$-th quantile of the standard normal distribution.

6.4.2 Percentile bootstrap (PB) confidence interval

Let, $\hat{C}_{py}^{*(\gamma)}$ be the γ-th percentile of $\mathcal{S} = \left\{\hat{C}_{py}^{*(j)}; j = 1, 2, \ldots, B\right\}$. Then a $100(1-\alpha)\%$ PB confidence interval of C_{py} is, $\left(\hat{C}_{py}^{*(B.(\alpha/2))}, \hat{C}_{py}^{*(B.(1-\alpha/2))}\right)$.

6.4.3 Bias corrected percentile bootstrap (BCPB) confidence interval

This approach has been introduced to correct the potential bias. To obtain the BCPB confidence interval, we have to obtain the bias-corrected factor Z_0. Based on ordered bootstrap $\mathcal{S} = \left\{\hat{C}_{py}^{*(j)}; j = 1, 2, \ldots, B\right\}$, first we calculate $P_0 = 1/B\sum_{j=1}^{B} I\left(\hat{C}_{py}^{*(j)} \leq \hat{C}_{py}\right)$, where $I(.)$ is the indicator function and $\hat{C}_{py}$ is the estimator of C_{py} based on original sample. Then we calculate $Z_0 = \Phi^{-1}(P_0)$, where $\Phi(.)$ is the cdf of standard normal distribution. Subsequently, we calculate the probabilities $P_L = \Phi\left(2Z_0 - Z_{(1-\alpha/2)}\right)$ and

$P_U = \Phi\left(2Z_0 + Z_{(1-\alpha/2)}\right)$. Then a $100(1-\alpha)\%$ BCPB confidence interval of C_{py} is, $\left(\hat{C}_{py}^{\prime*(B.P_L)}, \hat{C}_{py}^{\prime*(B.P_U)}\right)$.

6.5 Comparison of Performance Between FGCI and BCIs

We now compare the performance of our proposed fiducial generalized confidence interval with three non-parametric bootstrap confidence intervals in terms of their coverage probability using Monte Carlo simulations. We estimate CP of 90% and 95% CI of C_{py} and δ_{py} under normal, lognormal, gamma and Weibull distributions. The estimated CPs are reported in Table 6.1-Table 6.8. For estimating CP of C_{py} we consider the sample sizes $n = 20, 30, 40, 50$ and for estimating CP of δ_{py} we consider $n_1 = n_2 = n = 20, 30, 40, 50$. The upper and lower specification limits, the desired proportion of conformance (p_0) and different combinations of parameters for different distributions are given in the corresponding tables. For estimating FGCI, $10,000$ FGPQs are generated for each sample of size n and for estimating CP the simulation is run $10,000$ times. For estimating BCIs $B = 1000$ bootstrap re-sample of size n is drawn with replacement from the original sample of size n and the simulation is run $5,000$ times to estimate CP. The estimated CP is the proportion of the estimated CIs that includes the true value of the PCIs. The simulation study shows that the CP of the FGCI is very close to the nominal confidence level even for small sample sizes and is not substantially affected by the sample size. The CP of BCIs are not satisfactory and highly deviates from the nominal confidence level. The sample size affects the performance of the BCIs. The CP of BCIs increases as the sample size increases and tends to the nominal confidence level. Thus we conclude that the performance of the FGCI is superior to the BCIs. Hence FGCI can be effectively used to estimate a reliable confidence interval or to choose a better supplier among two suppliers.

6.6 Illustrative Examples

In this section, we consider two examples to show the applications of the proposed method. The first example estimates the FGCI of C_{py} and the second example selects the better supplier between the two competing suppliers.

6.6.1 Example 1

The drill bit is one of the important components of drill machines. Drill bits of different sizes are needed in a drilling machine production process. The quality characteristic is the lifetimes of 1.88 mm drill bit. A sample of size $n = 48$ of the lifetime of 1.88 mm drill bit is reported in Table 6.9. The factory set specifications (in minutes) are $(L, U) = (80, 180)$ and the desire proportion of conformance is $p_0 = 0.97$. Based on the model discrimination method, Piao and Zhi-Sheng [93] showed that gamma distribution fits well for this sample. The ML estimate of parameters are $(\hat{\alpha}, \hat{\beta}) = (72.364, 0.629)$. The estimated 90% FGCI of C_{py} is $(1.0197, 1.0305)$. Therefore, 95% lower confidence

Table 6.1: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of C_{py} under normal distribution, $L = 20$, $U = 30$

				95%				90%			
p_0	(μ, σ)	C_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.9973	(25,1.5)	1.002	20	0.944	0.847	0.865	0.889	0.891	0.828	0.814	0.833
			30	0.948	0.876	0.895	0.908	0.895	0.857	0.845	0.856
			40	0.942	0.892	0.914	0.923	0.890	0.873	0.858	0.865
			50	0.946	0.897	0.915	0.926	0.893	0.878	0.866	0.873
	(25,2)	0.990	20	0.943	0.837	0.871	0.888	0.891	0.805	0.819	0.836
			30	0.948	0.867	0.897	0.909	0.894	0.835	0.847	0.856
			40	0.943	0.886	0.915	0.925	0.890	0.849	0.859	0.866
			50	0.946	0.893	0.918	0.923	0.893	0.860	0.870	0.874
0.9900	(26,1.5)	1.006	20	0.949	0.815	0.848	0.878	0.895	0.794	0.791	0.829
			30	0.952	0.844	0.874	0.902	0.900	0.817	0.825	0.846
			40	0.944	0.861	0.902	0.925	0.894	0.836	0.846	0.866
			50	0.948	0.871	0.908	0.919	0.899	0.842	0.853	0.874
	(26,2)	0.986	20	0.948	0.827	0.862	0.886	0.895	0.792	0.808	0.835
			30	0.951	0.855	0.883	0.907	0.899	0.818	0.836	0.850
			40	0.944	0.873	0.906	0.920	0.891	0.833	0.853	0.867
			50	0.947	0.882	0.915	0.925	0.897	0.836	0.861	0.875

Table 6.2: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of δ_{py} under normal distribution, $L = 20$, $U = 40$ and $(\mu_1, \sigma_1) = (30, 3)$

				95%				90%			
p_0	(μ_2, σ_2)	δ_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.9973	(30,4)	0.012	20	0.958	0.872	0.868	0.876	0.912	0.823	0.817	0.828
			30	0.957	0.874	0.893	0.891	0.910	0.829	0.839	0.841
			40	0.956	0.882	0.903	0.917	0.909	0.847	0.852	0.871
			50	0.955	0.892	0.912	0.923	0.905	0.863	0.860	0.875
	(28,3.5)	0.011	20	0.958	0.846	0.849	0.863	0.908	0.804	0.798	0.811
			30	0.957	0.851	0.883	0.877	0.906	0.810	0.823	0.832
			40	0.953	0.872	0.894	0.912	0.905	0.834	0.836	0.864
			50	0.953	0.881	0.906	0.920	0.904	0.845	0.848	0.870
0.9900	(31,4)	0.015	20	0.958	0.853	0.863	0.874	0.912	0.811	0.810	0.827
			30	0.955	0.859	0.887	0.899	0.911	0.817	0.836	0.847
			40	0.956	0.876	0.901	0.912	0.906	0.837	0.850	0.865
			50	0.954	0.894	0.912	0.924	0.902	0.855	0.856	0.879
	(28,2)	-0.001	20	0.956	0.893	0.861	0.842	0.912	0.872	0.799	0.796
			30	0.956	0.899	0.883	0.879	0.907	0.880	0.834	0.832
			40	0.957	0.901	0.906	0.896	0.905	0.882	0.855	0.847
			50	0.954	0.905	0.910	0.907	0.903	0.880	0.856	0.857

Table 6.3: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of C_{py} under lognormal distribution, $L = 10$, $U = 160$

				95%				90%			
p_0	(μ, σ)	C_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.970	(4,0.50)	1.014	20	0.951	0.821	0.859	0.884	0.894	0.790	0.807	0.832
			30	0.951	0.849	0.886	0.901	0.900	0.813	0.833	0.850
			40	0.945	0.867	0.900	0.917	0.892	0.830	0.847	0.864
			50	0.948	0.878	0.906	0.919	0.898	0.835	0.848	0.866
	(4,0.60)	0.991	20	0.948	0.832	0.870	0.890	0.892	0.794	0.816	0.837
			30	0.950	0.862	0.891	0.905	0.898	0.820	0.839	0.851
			40	0.944	0.876	0.906	0.919	0.890	0.834	0.850	0.866
			50	0.946	0.885	0.911	0.921	0.896	0.838	0.854	0.867
0.950	(4.1,0.50)	1.026	20	0.951	0.821	0.862	0.887	0.896	0.787	0.809	0.836
			30	0.952	0.848	0.886	0.903	0.901	0.811	0.832	0.852
			40	0.945	0.867	0.902	0.920	0.893	0.827	0.845	0.866
			50	0.947	0.878	0.907	0.919	0.900	0.833	0.849	0.868
	(4.1,0.60)	0.996	20	0.950	0.836	0.872	0.892	0.893	0.795	0.822	0.841
			30	0.950	0.864	0.893	0.906	0.899	0.815	0.838	0.855
			40	0.944	0.882	0.908	0.922	0.891	0.834	0.854	0.871
			50	0.947	0.888	0.909	0.922	0.898	0.839	0.857	0.869

Table 6.4: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of δ_{py} under lognormal distribution, $L = 5$, $U = 150$ and $(\mu_1, \sigma_1) = (4, 0.5)$

				95%				90%			
p_0	(μ_2, σ_2)	δ_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.970	(4,0.6)	0.025	20	0.952	0.926	0.890	0.854	0.903	0.869	0.836	0.792
			30	0.953	0.930	0.912	0.880	0.906	0.875	0.855	0.817
			40	0.953	0.938	0.926	0.905	0.904	0.882	0.871	0.846
			50	0.952	0.940	0.924	0.910	0.907	0.886	0.879	0.854
	(4.1,0.4)	-0.011	20	0.952	0.952	0.884	0.826	0.902	0.896	0.828	0.759
			30	0.952	0.961	0.908	0.861	0.904	0.910	0.852	0.796
			40	0.952	0.958	0.920	0.885	0.905	0.903	0.869	0.817
			50	0.951	0.959	0.929	0.897	0.902	0.906	0.875	0.833
0.950	(4.2,0.45)	0.015	20	0.952	0.955	0.890	0.847	0.902	0.903	0.846	0.779
			30	0.953	0.957	0.916	0.876	0.906	0.904	0.863	0.804
			40	0.953	0.957	0.932	0.897	0.904	0.905	0.878	0.836
			50	0.951	0.954	0.928	0.903	0.905	0.904	0.880	0.839
	(4.1,0.55)	0.029	20	0.953	0.920	0.889	0.856	0.902	0.863	0.834	0.793
			30	0.953	0.926	0.910	0.882	0.906	0.871	0.854	0.821
			40	0.952	0.935	0.923	0.905	0.904	0.880	0.869	0.849
			50	0.953	0.936	0.923	0.910	0.906	0.884	0.877	0.856

Table 6.5: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of C_{py} under gamma distribution, $L = 70$, $U = 130$

				95%				90%			
p_0	(α, β)	C_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.970	(48,0.50)	0.997	20	0.940	0.838	0.886	0.906	0.889	0.805	0.835	0.849
			30	0.944	0.872	0.894	0.903	0.894	0.825	0.847	0.850
			40	0.945	0.883	0.916	0.921	0.894	0.840	0.860	0.874
			50	0.950	0.895	0.918	0.921	0.898	0.853	0.868	0.870
	(48,0.55)	0.950	20	0.946	0.838	0.875	0.899	0.899	0.801	0.828	0.851
			30	0.948	0.873	0.887	0.912	0.898	0.818	0.834	0.853
			40	0.948	0.887	0.901	0.915	0.900	0.835	0.852	0.866
			50	0.949	0.896	0.918	0.930	0.901	0.852	0.869	0.883
0.950	(50,0.50)	1.018	20	0.944	0.852	0.887	0.905	0.888	0.805	0.835	0.856
			30	0.943	0.865	0.901	0.906	0.889	0.825	0.842	0.853
			40	0.947	0.884	0.913	0.918	0.893	0.843	0.854	0.863
			50	0.949	0.889	0.908	0.914	0.898	0.851	0.863	0.866
	(50,0.55)	1.005	20	0.945	0.839	0.876	0.899	0.896	0.801	0.827	0.851
			30	0.945	0.861	0.892	0.908	0.895	0.820	0.844	0.856
			40	0.948	0.881	0.908	0.923	0.900	0.835	0.854	0.870
			50	0.946	0.891	0.911	0.919	0.899	0.844	0.863	0.871

Table 6.6: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of δ_{py} under gamma distribution, $L = 70$, $U = 150$, $(\alpha_1, \beta_1) = (50, 0.5)$

				95%				90%			
p_0	(α_2, β_2)	δ_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.970	(50,0.45)	0.002	20	0.954	0.912	0.928	0.870	0.906	0.856	0.870	0.783
			30	0.950	0.927	0.927	0.877	0.904	0.871	0.875	0.793
			40	0.953	0.932	0.934	0.893	0.904	0.881	0.879	0.813
			50	0.952	0.940	0.942	0.905	0.904	0.879	0.890	0.829
	(48,0.50)	0.011	20	0.953	0.873	0.893	0.858	0.903	0.819	0.839	0.790
			30	0.953	0.902	0.906	0.878	0.903	0.849	0.857	0.812
			40	0.952	0.913	0.910	0.875	0.904	0.863	0.846	0.802
			50	0.949	0.922	0.922	0.898	0.901	0.867	0.873	0.838
0.950	(50,0.55)	0.033	20	0.951	0.861	0.875	0.879	0.903	0.797	0.823	0.818
			30	0.950	0.887	0.901	0.898	0.900	0.835	0.851	0.852
			40	0.955	0.902	0.898	0.898	0.903	0.841	0.841	0.838
			50	0.952	0.912	0.916	0.919	0.902	0.857	0.862	0.868
	(48,0.45)	-0.002	20	0.954	0.903	0.913	0.862	0.905	0.847	0.854	0.780
			30	0.952	0.919	0.920	0.877	0.903	0.860	0.861	0.794
			40	0.949	0.930	0.921	0.889	0.898	0.873	0.875	0.815
			50	0.954	0.941	0.936	0.904	0.902	0.879	0.880	0.824

Table 6.7: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of C_{py} under Weibull distribution, $L = 20$, $U = 60$

				95%				90%			
p_0	(α, β)	C_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.970	(40,4.0)	0.962	20	0.946	0.851	0.893	0.915	0.888	0.811	0.837	0.864
			30	0.944	0.886	0.914	0.928	0.894	0.846	0.863	0.871
			40	0.947	0.902	0.921	0.931	0.891	0.852	0.868	0.880
			50	0.950	0.905	0.929	0.942	0.896	0.860	0.873	0.888
	(40,4.5)	0.984	20	0.944	0.846	0.888	0.915	0.885	0.805	0.830	0.866
			30	0.945	0.883	0.912	0.926	0.892	0.840	0.858	0.877
			40	0.942	0.899	0.919	0.933	0.895	0.854	0.870	0.882
			50	0.953	0.899	0.928	0.944	0.896	0.862	0.872	0.888
0.950	(42,4.0)	0.984	20	0.945	0.851	0.888	0.908	0.890	0.810	0.836	0.860
			30	0.944	0.884	0.917	0.926	0.892	0.843	0.860	0.864
			40	0.946	0.893	0.915	0.922	0.892	0.845	0.862	0.872
			50	0.946	0.903	0.926	0.933	0.896	0.854	0.870	0.882
	(42,4.5)	1.009	20	0.946	0.855	0.889	0.911	0.882	0.809	0.837	0.862
			30	0.945	0.884	0.916	0.928	0.891	0.842	0.862	0.870
			40	0.944	0.892	0.915	0.926	0.893	0.849	0.865	0.878
			50	0.947	0.904	0.928	0.938	0.898	0.855	0.870	0.883

Table 6.8: Coverage probability (CP) of estimated 95% and 90% FGCI and BCIs of δ_{py} under Weibull distribution, $L = 20$, $U = 70$, $(\alpha_1, \beta_1) = (50, 4.5)$

				95%				90%			
p_0	(α_2, β_2)	δ_{py}	n	FGCI	SB	PB	BCPB	FGCI	SB	PB	BCPB
0.970	(45,4.0)	0.015	20	0.953	0.913	0.908	0.878	0.910	0.862	0.848	0.808
			30	0.955	0.938	0.926	0.904	0.911	0.877	0.878	0.840
			40	0.948	0.940	0.934	0.907	0.908	0.889	0.882	0.836
			50	0.952	0.944	0.936	0.916	0.904	0.891	0.883	0.846
	(48,5.0)	-0.013	20	0.958	0.905	0.898	0.858	0.905	0.849	0.841	0.795
			30	0.956	0.906	0.918	0.880	0.907	0.847	0.865	0.825
			40	0.948	0.923	0.934	0.892	0.905	0.863	0.867	0.836
			50	0.949	0.924	0.932	0.904	0.905	0.870	0.878	0.849
0.950	(55,5.0)	0.016	20	0.950	0.890	0.902	0.871	0.904	0.832	0.855	0.806
			30	0.958	0.908	0.910	0.881	0.904	0.859	0.851	0.816
			40	0.949	0.915	0.914	0.893	0.898	0.867	0.867	0.838
			50	0.953	0.923	0.922	0.910	0.902	0.863	0.872	0.839
	(50,4.0)	0.021	20	0.952	0.912	0.912	0.882	0.908	0.854	0.859	0.817
			30	0.954	0.927	0.920	0.899	0.906	0.866	0.866	0.834
			40	0.951	0.930	0.925	0.907	0.905	0.870	0.873	0.845
			50	0.954	0.932	0.929	0.912	0.903	0.881	0.877	0.849

bound of C_{py} is 1.0197, which indicates that the process yield is greater than 97%.

Table 6.9: Lifetimes (in minutes) of 1.88 millimetre drill bit

135	98	114	137	138	144	99	93	115	106
132	122	94	98	127	122	102	133	114	120
93	126	119	104	119	114	125	107	98	117
111	106	108	127	126	135	112	94	127	99
120	120	121	122	96	109	123	105		

6.6.2 Example 2

This example is taken from Tong and Chen [112]. In this case study, we select the better supplier between two competing suppliers who provided aluminium foil materials to an electronic company in Taiwan. Aluminium foil is a key component that governs the quality of capacitors and the voltage is an important quality characteristic of aluminium foil. The product specifications (in voltage) are $(L, U) = (510, 530)$ and the desire of conformance is $p_0 = 0.99$. If the voltage falls outside this interval, the aluminium foil will break and thus be rejected. Samples of size $n = 50$ of voltage data from both supplier I and supplier II are collected by a quality inspector and are reported in Table 6.10. The Shapiro-Wilk normality test is performed to check the normality of both samples. The result of the Shapiro-Wilk test for the two samples are supplier I: $W = 0.9852$, $p = 0.7795$; supplier II: $W = 0.9901$, $p = 0.9493$, indicates that both samples are normally distributed. For supplier I, sample size $n_1 = 50$, sample mean $\bar{x}_1 = 519.748$, sample variance $s_1^2 = 3.168$ and for supplier II, $n_2 = 50$, $\bar{x}_2 = 522.172$, $s_2^2 = 8.847$. The estimated 95% FGCI of δ_{py} is $(0.0007, 0.0213)$. Since both the confidence limits are positive, we can conclude that the product quality of supplier I is better than supplier II.

Table 6.10: Voltages of aluminium foils from two suppliers

Supplier I	519.9	519.5	520.1	517.0	521.1	517.1	518.7	520.1	521.2	521.7
	520.4	517.9	522.9	517.7	517.2	520.7	521.0	519.1	518.4	518.9
	517.9	518.4	520.4	519.3	520.6	516.6	519.0	520.6	517.9	519.6
	519.6	522.6	518.3	522.1	523.1	519.9	519.8	520.7	516.5	521.5
	519.2	521.2	518.9	517.8	521.3	521.3	517.4	519.5	522.0	523.8
Supplier II	521.7	521.3	523.5	524.4	522.5	523.3	527.1	524.9	522.9	524.2
	523.9	523.5	527.5	517.3	518.7	518.7	521.9	519.7	520.4	520.4
	523.7	526.8	517.7	528.1	517.5	523.8	514.7	522.6	518.5	526.3
	523.2	524.4	522.7	519.6	520.4	520.6	525.2	524.1	519.3	522.2
	520.1	521.9	516.7	520.9	525.2	522.6	523.1	521.7	520.9	526.3

6.7 Concluding Remarks

The fiducial approach is a useful inferential tool and has been applied in many practical applications. In this chapter, we construct the FGCI of the generalized PCI C_{py} and analyze the difference between two generalized PCIs under some location-scale distributions. The performance of the proposed FGCI is compared with three non-parametric bootstrap confidence intervals: SB, PB, and BCPB, in terms of coverage probability using Monte Carlo simulations. The simulation result shows that the FGCI performs better than the BCIs. The coverage probability of FGCI closely matches the nominal confidence level and remains unaffected by the sample size. The proposed fiducial method is highly effective for constructing confidence intervals.

Chapter 7

Fiducial Inference on Lifetime Performance Index Based on Type-II Censored Samples

7.1 Introduction

Various types of PCIs are proposed for different purposes. Among the PCIs C_p, C_{pk}, C_{pm} and C_{pmk} are widely used in the industry. These indices measure nominal-the-better type quality characteristics with bilateral specification limits. To measure the performance of a process with unilateral specifications, Kane [62] proposed two one-sided PCIs C_{PL} and C_{PU}. The index C_{PL} measures 'larger-the-better' type quality characteristics such as lifetime, tensile strength, durability, hardness, etc., while C_{PU} measures 'smaller-the-better' type quality characteristics such as noise, pollution, degree of radiation, etc. To assess the performance of a process based on the lifetime of products, Montgomery [82] proposed the index C_L, known as the lifetime performance index, defined in equation (1.8). The lifetime random variable can often be modelled as Weibull, gamma, exponential, Rayleigh distribution, etc.

In life testing experiments, there are time constraints and other limitations such as material resources, financial constraints, and mechanical or experimental difficulties in data collection. Due to these restrictions, the experimenter may not always observe the lifetime of all the products in a test. As a result, censored samples are often collected instead of complete samples. Several types of censoring schemes exist in survival analysis, with type-II censoring being one of the most common (refer Balakrishnan and Aggarwala [5], Lawless [71]). In this chapter, we consider type-II censored data. In type-II censoring, a group of n products is put on a lifetime test and among them, only the first r lifetimes $X_{(1)} < X_{(2)} < \cdots < X_{(r)}$ have been observed and the remaining $(n-r)$ lifetimes are unobserved or missing. Several works have been done on the statistical inference of C_L under type-II censoring, see for example, Hong et al. [56], Hong et al. [57], Wu et al. [125], Lee et al. [72].

In this chapter, using a fiducial approach, we estimate the lower confidence bound (LCB) of the lifetime performance index C_L under type-II censored samples. We consider three important lifetime distributions: two-parametric exponential, Weibull distribution, and two-parametric Rayleigh. This chapter is organized as follows: In Section 7.2, we estimate the fiducial quantity of the location and scale parameters and estimate the lower confidence bound of three distributions using a fiducial approach and assess their performance by estimating coverage probability and average value. A real-life application of the proposed method is given in Section 7.3. Finally, some conclusions are added in Section 7.4.

7.2 Estimation of LCB Under Type-II Censored Samples By Fiducial Approaches

Let $\hat{\mu}$ and $\hat{\sigma}$ be equivariant estimators of μ and σ respectively, based on a type-II censored sample in which lifetime of the first r products are observed and the lifetime of the remaining $(n-r)$ products are censored. Then $(\hat{\mu}-\mu)/\sigma$ and $\hat{\sigma}/\sigma$ are two pivotal quantities, (refer Lawless [71], Theorem E2) and follow the relations (5.4) where $\hat{\mu}^*$, $\hat{\sigma}^*$ are equivariant estimators based on a type-II censored sample from the location-scale distribution with location parameter $\mu = 0$ and scale parameter $\sigma = 1$. Then, FGPQ of μ and σ can be obtained by the equation (5.5).

The procedure for estimating LCB by using a fiducial approach is given below.

Algorithm for Estimating LCB

1. Given a type-II random sample of a lifetime distribution follows a location-scale distribution.

2. Estimate the MLE of the location and scale parameters.

3. Derive the fiducial quantity of the parameters μ and σ, say $\mathcal{R}_\mu$ and $\mathcal{R}_\sigma$.

4. Derive the fiducial quantity of the lifetime performance index $\mathcal{R}_{C_L}$ by substituting the fiducial quantity of the parameters.

5. Repeat steps $2-4$ a large number of times, say N times.

6. Then the 100α-th percentiles of these N generated fiducial quantity will provide a $100(1-\alpha)\%$ lower confidence bound of the lifetime performance index.

Now, we discuss the estimation of the fiducial quantity of the parameters and the index C_L under three important lifetime distributions. The fiducial quantities are estimated using a maximum likelihood estimator (MLE). The likelihood functions and estimation of MLE under the type-II censoring scheme are discussed under corresponding distributions.

7.2.1 Two-parameter exponential distribution

Let the lifetime 'X' of a product follow the two-parameter exponential distribution, i.e., $X \sim \exp(\mu, \sigma)$. The probability density function (pdf) and cumulative distribution function (cdf) of the distribution are given by,

$$f(x) = \frac{1}{\sigma}\exp\left[-\frac{x-\mu}{\sigma}\right]; \quad x > \mu, \quad \sigma > 0$$

and

$$F(x) = 1 - \exp\left[-\frac{x-\mu}{\sigma}\right]$$

where μ is the location parameter and σ is the scale parameter. The mean and standard deviation of the distribution are $\mu_X = \mu + \sigma$ and $\sigma_X = \sigma$. Therefore, the lifetime index reduces to

$$C_L = 1 + \frac{\mu - L}{\sigma}. \tag{7.1}$$

Let $x_{(1)} < x_{(2)} < \cdots < x_{(r)}$ be an ordered set of uncensored observations from a type-II censored sample of size n from a two-parameter exponential distribution. Let, $x_i^* = x_{(i)}$ for $i = 1, 2, \ldots, r$ and $x_i^* = x_{(r)}$ for $i = (r+1), (r+2), \ldots, n$. The log-likelihood function is given by,

$$\ell(\mu, \theta) = -r \log(\sigma) - \sum_{i=1}^{n} \frac{(x_i^* - \mu)}{\sigma}.$$

By using the condition that $\mu < x_{(1)}$, the MLE of the parameters are

$$\hat{\mu} = x_{(1)} \quad \text{and} \quad \hat{\sigma} = \frac{1}{r}\left[\sum_{i=1}^{r}(x_{(i)} - x_{(1)}) + (n-r)(x_{(r)} - x_{(1)})\right]. \tag{7.2}$$

Here $\hat{\mu}$, $\hat{\sigma}$ are independent with $\frac{2n(\hat{\mu}-\mu)}{\sigma} \sim \chi^2_{(2)}$ and $\frac{2r\hat{\sigma}}{\sigma} \sim \chi^2_{(2r-2)}$, see, Theorem 4.5.1 of Lawless [71]. Let, $V \sim \chi^2_{(2)}$, $W \sim \chi^2_{(2r-2)}$ and $(\hat{\mu}_0, \hat{\sigma}_0)$ be an observed value of MLE $(\hat{\mu}, \hat{\sigma})$. Therefore, by using relations (5.4), the fiducial quantity of the parameters is obtained as

$$\mathcal{R}_\mu = \hat{\mu}_0 - \frac{r\hat{\sigma}_0 V}{nW}, \quad \text{and} \quad \mathcal{R}_\sigma = \frac{2r\hat{\sigma}_0}{W}. \tag{7.3}$$

Hence, the fiducial quantity of C_L is obtained by substituting the fiducial quantity of the parameters as

$$\mathcal{R}^E_{C_L} = 1 + \frac{\mathcal{R}_\mu - L}{\mathcal{R}_\sigma}. \tag{7.4}$$

7.2.2 Weibull distribution

Let the lifetime of a product 'X' follow the Weibull distribution with shape parameter β and scale parameter α. We denote this distribution by $X \sim Weib(\alpha, \beta)$ and its pdf and cdf are given by equation (5.12) and equation (6.7) respectively. The mean and standard deviation of the process are given by, $\mu_X = \alpha\ \Gamma(1+1/\beta)$ and $\sigma_X = \alpha\sqrt{\Gamma(1+2/\beta)-(\Gamma(1+1/\beta))^2}$. Therefore, the lifetime performance index is reduced to,

$$C_L = \frac{1}{\sqrt{\Gamma(1+\frac{2}{\beta})-(\Gamma(1+\frac{1}{\beta}))^2}}\left[\Gamma\left(1+\frac{1}{\beta}\right)-\frac{L}{\alpha}\right]. \tag{7.5}$$

Let $x_{(1)} < x_{(2)} < \cdots < x_{(r)}$ be an ordered set of uncensored observations from a type-II censored sample of size n from a Weibull distribution. Let, $x_i^* = x_{(i)}$ for $i = 1, 2, \ldots, r$ and $x_i^* = x_{(r)}$ for $i = (r+1), (r+2), \ldots, n$. Cohen [26] estimated the MLEs of the parameters of the Weibull distribution for the complete and censored samples. The MLE of β under the type-II censoring scheme can be obtained by solving,

$$\frac{1}{\hat{\beta}} - \frac{\sum_{i=1}^{n}(x_i^*)^{\hat{\beta}}\ \ln(x_i^*)}{\sum_{i=1}^{n}(x_i^*)^{\hat{\beta}}} + \frac{1}{r}\sum_{i=1}^{r}\ln(x_i^*) = 0$$

and the MLE of α can be obtained as,

$$\hat{\alpha} = \left(\sum_{i=1}^{n}(x_i^*)^{\hat{\beta}}/r\right)^{1/\hat{\beta}}.$$

Krishnamoorthy et al. [67] derived the fiducial quantity of the parameters as,

$$\mathcal{R}_\alpha = \hat{\alpha}_0\left(\frac{1}{\hat{\alpha}^*}\right)^{\hat{\beta}^*/\hat{\beta}_0} \quad \text{and} \quad \mathcal{R}_\beta = \hat{\beta}_0/\hat{\beta}^* \tag{7.6}$$

where $(\hat{\alpha}_0, \hat{\beta}_0)$ is the MLE of (α, β) based on a type-II censored sample from a $Weib\,(\alpha, \beta)$ distribution and $(\hat{\alpha}^*, \hat{\beta}^*)$ is the MLE of (α, β) based on a type-II censored sample from a $Weib\,(1, 1)$ distribution where $\hat{\beta}^*$ can be obtained by solving

$$\frac{1}{\hat{\beta}^*} - \frac{\sum_{i=1}^{n}(z_i^*)^{\hat{\beta}^*}\ \ln(z_i^*)}{\sum_{i=1}^{n}(z_i^*)^{\hat{\beta}^*}} + \frac{1}{r}\sum_{i=1}^{r}\ln(z_i^*) = 0$$

and $\hat{\alpha}^* = \left(\sum_{i=1}^{n}(x_i^*)^{\hat{\beta}^*}/r\right)^{1/\hat{\beta}^*}$. Here, $z_i^* = z_{(i)},\ i = 1, 2, \ldots, r$ and $z_i^* = z_{(r)}\ , i = (r+1), \ldots, n$ where $z_1, z_2, \ldots, z_r$ are random sample from $Weib(1, 1)$ distribution. Therefore, the fiducial quantity of C_L reduces to

$$\mathcal{R}_{C_L}^{W} = \frac{1}{\sqrt{\Gamma(1+\frac{2}{\mathcal{R}_\beta})-(\Gamma(1+\frac{1}{\mathcal{R}_\beta}))^2}}\left[\Gamma\left(1+\frac{1}{\mathcal{R}_\beta}\right)-\frac{L}{\mathcal{R}_\alpha}\right]. \tag{7.7}$$

7.2.3 Two-parameter Rayleigh distribution

Let the lifetime of a product 'X' follow a two-parameter Rayleigh distribution. Different estimation methods of the parameters of two-parameter Rayleigh distribution are discussed by Dey et al. [30]. The pdf of the distribution is,

$$f(x) = \frac{(x-\mu)}{\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad x > \mu \quad \sigma > 0$$

where μ is the location parameter and σ is the scale parameter. The mean and standard deviation of the distribution are $\mu_X = \mu + \sigma\sqrt{\pi/2}$ and $\sigma_X = \sigma\sqrt{(4-\pi)/2}$. Therefore, the lifetime performance index becomes

$$C_L = \frac{1}{\sqrt{(4-\pi)/2}} \left[\sqrt{\frac{\pi}{2}} + \frac{\mu - L}{\sigma}\right]. \tag{7.8}$$

We follow exactly the same procedure as discussed by Hoang-Nguyen-Thuy and Krishnamoorthy [53] to derived the MLE of the parameters. Let $x_{(1)} < x_{(2)} < \cdots < x_{(r)}$ be an ordered set of uncensored observations from a type-II censored sample of size n from a two-parameter Rayleigh distribution. Let, $x_i^* = x_{(i)}$ for $i = 1, 2, \ldots, r$ and $x_i^* = x_{(r)}$ for $i = (r+1), (r+2), \ldots, n$. Now, letting $\theta = \sigma^2$, the log-likelihood function be written as,

$$\ell(\mu, \theta) = -r \ln \theta + \sum_{i=1}^{r} \ln\left(x_i^* - \mu\right) - \frac{1}{2\theta} \sum_{i=1}^{n} \left(x_i^* - \mu\right)^2.$$

The MLE of the parameters μ and σ can be obtained by solving the system of equations,

$$\frac{\partial \ell(\mu, \theta)}{\partial \mu} = -\sum_{i=1}^{r} \frac{1}{(x_i^* - \mu)} + \frac{1}{\theta} \sum_{i=1}^{n} (x_i^* - \mu) = 0, \tag{7.9}$$

$$\frac{\partial \ell(\mu, \sigma)}{\partial \theta} = -\frac{r}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} (x_i^* - \mu)^2 = 0. \tag{7.10}$$

By solving equation (7.10), we get $\theta = \sum_{i=}^{n} (x_i^* - \mu)^2/(2r)$. By substituting this value of θ in equation (7.9) we get,

$$\frac{2r \sum_{i=1}^{n} (x_i^* - \mu)}{\sum_{i=1}^{n} (x_i^* - \mu)^2} - \frac{1}{\sum_{i=1}^{r} (x_i^* - \mu)} = 0. \tag{7.11}$$

Then, the MLE $\hat{\mu}$ of μ can be obtained as a root of the equation (7.11) by using the interval $\left(x_{(1)} - 12\tilde{\sigma}_r/\sqrt{r}, x_{(1)}\right)$ as root bracketing interval, where $\tilde{\sigma}_r = 2s_r^2/(4-\pi)$ and $s_r^2 = \sum_{i=1}^{r} (x_i^* - \bar{x}^*)^2/(r-1)$ is the variance of the uncensored observations. Hence, the MLE of σ is

$$\hat{\sigma} = \sqrt{\sum_{i=}^{n} (x_i^* - \hat{\mu})^2/(2r)}. \tag{7.12}$$

Let, $(\hat{\mu}_0, \hat{\sigma}_0)$ be an observed value of MLE of the parameters of a Rayleigh (μ, σ) distribution based on a type-II censored sample and $(\hat{\mu}^*, \hat{\sigma}^*)$ be the MLE of Rayleigh $(0, 1)$ based on a type-II censored sample. Then the fiducial quantity of the parameters is,

$$\mathcal{R}_\mu = \hat{\mu}_0 - \frac{\hat{\mu}^*}{\hat{\sigma}^*}\hat{\sigma}_0 \quad \text{and} \quad \mathcal{R}_\sigma = \frac{\hat{\sigma}_0}{\hat{\sigma}^*}. \tag{7.13}$$

Therefore, the fiducial quantity of C_L is,

$$\mathcal{R}_{C_L}^{R} = \frac{1}{\sqrt{(4-\pi)/2}} \left[\sqrt{\frac{\pi}{2}} + \frac{\mathcal{R}_\mu - L}{\mathcal{R}_\sigma}\right]. \tag{7.14}$$

Table 7.1: Coverage probability (CP) and average value (AV) of the 90% and 95% lower confidence bound of C_L for two-parameter exponential distribution, $\mu = 10$, $L = 9.5$

				90%		95%	
σ	C_L	n	r	CP	AV	CP	AV
3	1.17	20	10	0.8959	1.0787	0.9491	1.0422
			14	0.8985	1.0835	0.9487	1.0480
			18	0.8999	1.0870	0.9511	1.0519
		30	15	0.9042	1.1001	0.9535	1.0753
			22	0.8957	1.1063	0.9484	1.0824
			27	0.8968	1.1090	0.9511	1.0854
		40	25	0.8997	1.1153	0.9511	1.0968
			30	0.8990	1.1185	0.9481	1.1003
			37	0.9006	1.1205	0.9498	1.1026
5	1.10	20	10	0.8975	1.0263	0.9479	0.9910
			14	0.8987	1.0278	0.9476	0.9929
			18	0.9007	1.0292	0.9502	0.9944
		30	15	0.9051	1.0466	0.9548	1.0232
			22	0.8965	1.0499	0.9475	1.0267
			27	0.8986	1.0510	0.9514	1.0278
		40	25	0.8955	1.0596	0.9506	1.0421
			30	0.8979	1.0613	0.9481	1.0439
			37	0.9024	1.0620	0.9494	1.0446

7.3 Performance Analysis and Illustrative Example

The performance of the proposed lower confidence interval is discussed in terms of their coverage probability and average value based on Monte Carlo simulations. The simulated results are reported in Table 7.1-Table 7.3 for two-parameter exponential, Weibull, and two-parameter Rayleigh distributions, respectively. We considered sample size $n = 20,\ 30,\ 40$ and different values of number of complete observations r. From the simulated results, we can observe that the coverage probability is not affected by sample size but the average value of the lower confidence bound increases as sample size increases and approaches to the true value of the index value. Also, we can observe that as the number of uncensored observations r increases, the coverage probability and average value improve under a fixed sample size. Therefore, our proposed method can be used to estimate a reliable lower confidence bound of the lifetime performance index. Two examples are given to illustrate the applications of our proposed method.

Table 7.2: Coverage probability (CP) and average value (AV) of the 90% and 95% lower confidence bound of C_L for Weibull distribution, $L = 25$, $\alpha = 45$

				90%		95%	
β	C_L	n	r	CP	AV	CP	AV
3.0	1.04	20	10	0.8976	0.7380	0.9444	0.6491
			14	0.8968	0.7425	0.9468	0.6549
			18	0.8928	0.7457	0.9464	0.6583
		30	15	0.9112	0.7840	0.9584	0.7115
			22	0.9040	0.7869	0.9540	0.7149
			27	0.9024	0.7898	0.9516	0.7187
		40	25	0.9160	0.8116	0.9632	0.7486
			30	0.9152	0.8135	0.9596	0.7505
			37	0.9060	0.8189	0.9556	0.7568
3.5	1.21	20	10	0.8964	0.8737	0.9448	0.7752
			14	0.8964	0.8782	0.9460	0.7810
			18	0.8928	0.8850	0.9476	0.7894
		30	15	0.9104	0.9222	0.9568	0.8410
			22	0.9028	0.9270	0.9560	0.8470
			27	0.9024	0.9332	0.9500	0.8550
		40	25	0.9168	0.9517	0.9648	0.8814
			30	0.9176	0.9559	0.9608	0.8860
			37	0.9116	0.9652	0.9532	0.8971

Table 7.3: Coverage probability (CP) and average value (AV) of the 90% and 95% lower confidence bound of C_L for two-parameter Rayleigh distribution, $L = 12$, $\mu = 10$

				90%		95%	
σ	C_L	n	r	CP	AV	CP	AV
3.5	1.04	20	10	0.8976	0.7885	0.9452	0.7034
			14	0.8980	0.7953	0.9488	0.7105
			18	0.8980	0.7970	0.9492	0.7126
		30	15	0.9040	0.8408	0.9504	0.7754
			22	0.9056	0.8488	0.9520	0.7834
			27	0.9036	0.8506	0.9508	0.7855
		40	25	0.8984	0.8768	0.9528	0.8220
			30	0.8988	0.8786	0.9515	0.8239
			37	0.8980	0.8798	0.9508	0.8250
4.0	1.15	20	10	0.8936	0.9006	0.9476	0.8123
			14	0.8956	0.9030	0.9500	0.8157
			18	0.8924	0.9048	0.9472	0.8172
		30	15	0.8904	0.9518	0.9420	0.8835
			22	0.8900	0.9566	0.9440	0.8883
			27	0.8908	0.9574	0.9444	0.8894
		40	25	0.8992	0.9848	0.9476	0.9282
			30	0.8996	0.9859	0.9448	0.9295
			37	0.8992	0.9864	0.9476	0.9301

7.3.1 Example

The data reported in Table 7.4 represents the number of millions of revolutions of 23 ball bearings before failure is taken from Thoman et al. [111]. Lawless [71] observed that the Weibull distribution is appropriate for this data. To illustrate our proposed method for type-II censored data, we assume the test is terminated after the 17th failure, that is $n = 23$ and $r = 17$. The MLEs based on this type-II censored data are $\hat{\alpha} = 79.438$ and $\hat{\beta} = 2.293$. Assuming $L = 25$, the estimated 95% lower confidence bound of C_L is 0.9246. Let us consider, the hypothesis testing $H_0 : C_L \leq 0.9$ vs. $H_1 : C_L > 0.9$. Since the estimated 95% lower confidence bound of C_L is $0.9246 > 0.9$, therefore we reject the null hypothesis at 5% significant level and conclude that $C_L > 0.9$.

Table 7.4: Number of millions of revolutions of 23 ball bearings before failure

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84
51.96	54.12	55.56	67.80	68.64	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

7.4 Concluding Remarks

The fiducial inference has been applied in many practical applications. In this chapter, we considered using the fiducial approach to estimate $100(1-\alpha)\%$ one-sided confidence interval of the lifetime performance index C_L under a type-II censored sample. We considered three important lifetime distributions. The simulation performance showed that the proposed one-sided confidence interval is very effective and can be used to assess whether the process performance meets the customer's expectations.

Chapter 8

Conclusions and Future Work

PROCESS capability analysis is undoubtedly an important and evolving field within the realm of statistical quality control. Although there is a large body of literature on PCI, significantly less attention has been paid to issues such as autocorrelation, measurement error, non-normality of quality characteristics, and censored samples. The overall aim of this thesis is to address some of these issues. We have discussed the combined effect of autocorrelation and measurement error on the estimator of PCIs. One of the objectives of this research was to propose a more reliable and effective method for computing confidence intervals for PCIs using a fiducial approach. The concept of fiducial inference is relatively new and is effectively utilized in various areas of statistical inference. We have examined most existing methods for computing the confidence interval of PCIs and found some inadequacies among them. This inspired us to find a more efficient method. Our findings will benefit both theorists and practitioners. However, there is still room for further study, and some areas for future exploration are listed below.

1. We have discussed how autocorrelation and measurement errors affect PCIs. We have assumed that the process follows an AR(1) model and that the measurement error is distributed normally. Although autocorrelation and measurement error significantly impact results, the focus should be on reducing these effects. Vännman and Kulahci [117] and Grimshaw et al. [46] discussed the iterative skipping strategy to reduce the effect of autocorrelation on PCIs. More research is necessary on this topic.

2. The selection of suppliers and monitoring the supplier's performance is a major challenging task for the manufacturers. Manufacturers should produce high-quality products to increase customer loyalty and to be competitive. Most manufacturing firms purchase a significant percentage of components and parts from suppliers and perform other related in-house operations to produce finished products. Therefore, the quality of the finished products is highly dependent on the quality of the parts purchased from the suppliers. Improving quality is the responsibility of both manufacturers and suppliers. Evaluating a group of suppliers and choosing one or more of them are difficult decisions since many criteria need to be taken into account.

Among these criteria, quality is undoubtedly the most important. Hence, PCIs are often used to assess the performance of suppliers. Most of the existing literature on supplier selection based on PCIs considers two suppliers. The literature on the selection of suppliers involving multiple suppliers by using PCIs is very limited and most of them have considered that the quality characteristics follow normal distribution. More investigation is needed for supplier selection based on PCIs involving multiple suppliers, especially when the quality characteristics are autocorrelated or non-normal. We will focus on this in our future research.

3. Acceptance sampling plans are important practical tools for quality control applications. In the acceptance sampling plan, the producers provide the consumers with a predetermined quantity of things, called *lot*. The customers collect a sample from the lot to decide whether to accept or reject the lot based on the quality of the products. In the acceptance sampling plan, there is a risk of incorrectly accepting a bad product lot, called the consumer's risk, or incorrectly rejecting a good product lot, called the producer's risk. The acceptance sampling plan determines the sample size that is to be taken from the product lot and the criteria that will be used to determine whether the product lot is to be accepted or rejected. PCIs are used to design a reliable acceptance sampling plan. Several variable acceptance sampling plans have been proposed based on different PCIs. Still, there are some research gaps, and this could be an interesting subject of future research.

4. Before performing the capability analysis of a process, it is recommended to check the stability of the process. The stability of a process is typically assessed using control charts for mean and variance, which utilize retrospective samples. This is called Phase I analysis. If the process is stable, then PCIs are estimated to measure the capability of a process. But a single estimated PCI value is not reliable enough to justify the performance of the whole process. For the continuous assessment of the process performance, PCI-based control charts were proposed. In this chart, the process capability indices are monitored over time instead of sample statistics. A PCI-based control chart can assess the process capability over the lifetime of the process. This is called Phase II monitoring. Several control charts have been proposed to monitor the process capability of a process based on different PCIs in different situations. More investigation is needed on this topic, and we will consider these problems in the future.

5. Most studies on PCIs have utilized a parametric approach, assuming that the distribution of quality characteristics follows a known distribution. Identifying the distribution of quality characteristics can sometimes be challenging. In such cases, a nonparametric approach is utilized. Chen et al. [21] discussed the nonparametric estimation of a yield-based index using the kernel estimate of the cdf. This is another challenging area that could be explored in future.

6. Generally, the statistical properties of PCIs are analyzed using the complete sample. But, lifetime data of products often comes as censored rather than a complete sample. Statistical analysis of classical PCIs using censored data has not been much explored, except for the lifetime performance index. Akdoğan [1] has studied some properties of C_{pm} under progressive type-II censored sample. Recently, Saha

and Dutta [99] have studied the classical and the Bayesian estimation of S'_{pk} under type-II progressive right censored samples. This could be a promising area for future research.

7. Our discussion has been limited to univariate process capability indices (PCIs), focusing solely on one quality characteristic at a time. But in most processes, the product involves multiple quality characteristics. Multivariate process capability indices (MPCIs) were proposed for assessing the performance of processes with multiple quality characteristics. The research on MPCIs are still very limited in comparison to univariate PCIs. It is more difficult to obtain related statistical properties needed for the detailed inference on MPCIs. Furthermore, there is lack of consistency in methodology for the assessment of process capability through MPCIs. More investigation is needed in this field, and future research can be focused on this topic.

Bibliography

[1] Akdoğan, Y. (2022). On the confidence intervals of process capability index Cpm based on a progressive type-II censored sample. *Quality and Reliability Engineering International*, 38(5):2845–2861.

[2] Anis, M. Z. (2008). Basic process capability indices: An expository review. *International Statistical Review*, 76(3):347–367.

[3] Anis, M. Z. and Bera, K. (2022). Process Capability Cp Assessment for Auto-Correlated Data in the Presence of Measurement Errors. *International Journal of Reliability, Quality and Safety Engineering*, 29(06):2250010.

[4] Antony, J., Knowles, G., and Roberts, P. (1999). Gauge capability analysis: classical versus ANOVA. *Quality Assurance*, 6(3):173–181.

[5] Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: Theory, Methods, and Applications*. Springer Science & Business Media.

[6] Bartlett, M. S. (1946). On the theoretical specification and sampling properties of autocorrelated time-series. *Supplement to the Journal of the Royal Statistical Society*, 8(1):27–41.

[7] Basseville, M. and Nikiforov, I. V. (1993). *Detection of abrupt changes: theory and application*, volume 104. Prentice Hall Englewood Cliffs.

[8] Bera, K. and Anis, M. Z. (2023). Estimation of Cpm for autocorrelated data in the presence of random measurement errors. *Communications in Statistics-Simulation and Computation*. DOI:10.1080/03610918.2023.2278025.

[9] Bera, K. and Anis, M. Z. (2024a). Comparison of two processes based on quantile-based process capability indices by using fiducial generalized confidence interval. *Journal of Statistical Computation and Simulation*. DOI:10.1080/00949655.2024.2446381.

[10] Bera, K. and Anis, M. Z. (2024b). Process incapability index for autocorrelated data in the presence of measurement errors. *Communications in Statistics-Theory and Methods*, 53(15):5439–5459.

[11] Bordignon, S. and Scagliarini, M. (2002). Statistical analysis of process capability indices with measurement errors. *Quality and Reliability Engineering International*, 18(4):321–332.

[12] Bordignon, S. and Scagliarini, M. (2006). Estimation of Cpm when measurement error is present. *Quality and Reliability Engineering International*, 22(7):787–801.

[13] Box, G. E., Jenkins, G. M., Reinsel, G. C., and Ljung, G. M. (2015). *Time series analysis: forecasting and control*. John Wiley & Sons.

[14] Boyles, R. A. (1991). The Taguchi capability index. *Journal of Quality Technology*, 23(1):17–26.

[15] Brockwell, P. J. and Davis, R. A. (2002). *Introduction to Time Series and Forecasting*. Springer.

[16] Burdick, R. K., Borror, C. M., and Montgomery, D. C. (2003). A review of methods for measurement systems capability analysis. *Journal of Quality Technology*, 35(4):342–354.

[17] Chan, L. K., Cheng, S. W., and Spiring, F. A. (1988). A new measure of process capability: C_{pm}. *Journal of Quality Technology*, 20(3):162–175.

[18] Chen, J. P. and Chen, K. (2004). Comparing the capability of two processes using Cpm. *Journal of Quality Technology*, 36(3):329–335.

[19] Chen, K. (1998). Incapability index with asymmetric tolerances. *Statistica Sinica*, 8(1):253–262.

[20] Chen, K. and Pearn, W. (1997). An application of non-normal process capability indices. *Quality and Reliability Engineering International*, 13(6):355–360.

[21] Chen, P., Wang, B. X., and Ye, Z. S. (2019). Yield-based process capability indices for nonnormal continuous data. *Journal of Quality Technology*, 51(2):171–180.

[22] Chen, S. M. and Hsu, Y. S. (2004). Uniformly most powerful test for process capability index Cpk. *Quality Technology & Quantitative Management*, 1(2):257–269.

[23] Chou, Y. M., Owen, D., Salvador, A., and Borrego, A. (1990). Lower confidence limits on process capability indices. *Journal of Quality Technology*, 22(3):223–229.

[24] Chou, Y.-m. and Owen, D. B. (1989). On the distributions of the estimated process capability indices. *Communications in Statistics-Theory and Methods*, 18(12):4549–4560.

[25] Clements, J. A. (1989). Process capability calculations for non-normal distributions. *Quality Progress*, 22:95–100.

[26] Cohen, A. C. (1965). Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. *Technometrics*, 7(4):579–588.

[27] Costa, A. F. and Castagliola, P. (2011). Effect of measurement error and autocorrelation on the $\bar{X}$ chart. *Journal of Applied Statistics*, 38(4):661–673.

[28] de Felipe, D. and Benedito, E. (2017). A review of univariate and multivariate process capability indices. *The International Journal of Advanced Manufacturing Technology*, 92:1687–1705.

[29] Deleryd, M. (1999). A pragmatic view on process capability studies. *International Journal of Production Economics*, 58(3):319–330.

[30] Dey, S., Dey, T., and Kundu, D. (2014). Two-parameter Rayleigh distribution: different methods of estimation. *American Journal of Mathematical and Management Sciences*, 33(1):55–74.

[31] Dey, S. and Saha, M. (2019). Bootstrap confidence intervals of generalized process capability index Cpyk using different methods of estimation. *Journal of Applied Statistics*, 46(10):1843–1869.

[32] Dey, S. and Saha, M. (2020). Bootstrap confidence intervals of process capability index Spmk using different methods of estimation. *Journal of Statistical Computation and Simulation*, 90(1):28–50.

[33] Dey, S., Saha, M., Maiti, S. S., and Jun, C.-H. (2018). Bootstrap confidence intervals of generalized process capability index Cpyk for Lindley and power Lindley distributions. *Communications in Statistics-Simulation and Computation*, 47(1):249–262.

[34] Durbin, J. and Watson, G. S. (1971). Testing for serial correlation in least squares regression. III. *Biometrika*, 58(1):1–19.

[35] Efron, B. (1982). *The Jackknife, the Bootstrap and Other Resampling Plans*. SIAM.

[36] Efron, B. (1998). RA Fisher in the 21st century. *Statistical Science*, pages 95–114.

[37] Efron, B. and Tibshirani, R. (1986). Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Statistical Science*, 1(1):54–75.

[38] Efron, B. and Tibshirani, R. J. (1994). *An Introduction to the Bootstrap*. CRC press.

[39] Feigenbaum, A. V. (1991). *Total quality control*. New York.

[40] Fisher, R. A. (1930). Inverse probability. *Mathematical Proceedings of the Cambridge Philosophical Society*, 26(4):528–535.

[41] Fisher, R. A. (1935). The fiducial argument in statistical inference. *Annals of Eugenics*, 6(4):391–398.

[42] Franklin, L. A. and Gary, W. (1991). Bootstrap confidence interval estimates of Cpk: an introduction. *Communications in Statistics-Simulation and Computation*, 20(1):231–242.

[43] Franklin, L. A. and Wasserman, G. S. (1992). Bootstrap lower confidence limits for capability indices. *Journal of Quality Technology*, 24(4):196–210.

[44] Gao, Y. and Tian, L. (2022). Confidence interval estimation for the difference and ratio of the means of two gamma distributions. *Communications in Statistics-Simulation and Computation*. DOI:10.1080/03610918.2022.2116646.

[45] Greenwich, M. and Jahr-Schaffrath, B. L. (1995). A process incapability index. *International Journal of Quality & Reliability Management*, 12(4):58–71.

[46] Grimshaw, S. D., Guo, Z., and Duke, T. (2024). Unbiased process capability estimation for autocorrelated data using exhaustive systematic sampling. *Quality and Reliability Engineering International*. DOI:10.1002/qre.3617.

[47] Guevara, R. D. and Vargas, J. A. (2007). Comparison of process capability indices under autocorrelated data. *Revista Colombiana de Estadística*, 30(2):301–316.

[48] Guo, B., He, X., Xia, Q., Sun, Y., and Xuan, J. (2022). Generalized interval estimation of process capability indices for the Birnbaum–Saunders distribution. *Quality and Reliability Engineering International*, 38(8):4015–4032.

[49] Guo, B., Xia, Q., Sun, Y., and Aslam, M. (2023). Generalized confidence intervals of quantile-based process capability indices for inverse Gaussian distribution. *Quality Technology & Quantitative Management*, 20(3):405–417.

[50] Hannig, J. (2009). On generalized fiducial inference. *Statistica Sinica*, 19:491–544.

[51] Hannig, J., Iyer, H., Lai, R. C., and Lee, T. C. (2016). Generalized fiducial inference: A review and new results. *Journal of the American Statistical Association*, 111(515):1346–1361.

[52] Hannig, J., Iyer, H., and Patterson, P. (2006). Fiducial generalized confidence intervals. *Journal of the American Statistical Association*, 101(473):254–269.

[53] Hoang-Nguyen-Thuy, N. and Krishnamoorthy, K. (2021). A method for computing tolerance intervals for a location-scale family of distributions. *Computational Statistics*, 36(2):1065–1092.

[54] Hoffman, L. L. (1993). A general approach for testing the process capability index. *Quality and Reliability Engineering International*, 9(5):445–449.

[55] Hoffman, L. L. (2001). Obtaining confidence intervals for Cpk using percentiles of the distribution of Cp. *Quality and Reliability Engineering International*, 17(2):113–118.

[56] Hong, C. W., Wu, J. W., and Cheng, C. H. (2007). Computational procedure of performance assessment of lifetime index of businesses for the Pareto lifetime model with the right type II censored sample. *Applied Mathematics and Computation*, 184(2):336–350.

[57] Hong, C. W., Wu, J. W., and Cheng, C. H. (2008). Computational procedure of performance assessment of lifetime index of Pareto lifetime businesses based on confidence interval. *Applied Soft Computing*, 8(1):698–705.

[58] Hsu, B. M., Wu, C. W., and Shu, M. H. (2008). Generalized confidence intervals for the process capability index C_{pm}. *Metrika*, 68:65–82.

[59] Juran, J. (1951). *Quality Control Handbook, McGraw-Hill Book Company*. Inc., New York.

[60] Juran, J. (1962). *Quality Control Handbook McGraw-Hill Book Comp.* Inc., New York.

[61] Juran, J. and Gryna, F. (1980). *Quality Planning and Analysis Control Handbook*. McGrawHill, New York.

[62] Kane, V. E. (1986). Process capability indices. *Journal of Quality Technology*, 18(1):41–52.

[63] Kanichukattu, J. K. and Luke, J. A. (2013). Comparison between two process capability indices using generalized confidence intervals. *The International Journal of Advanced Manufacturing Technology*, 69:2793–2798.

[64] Kotz, S. and Johnson, N. L. (1993). *Process Capability Indices*. CRC Press.

[65] Kotz, S. and Johnson, N. L. (2002). Process capability indices—a review, 1992–2000. *Journal of Quality Technology*, 34(1):2–19.

[66] Krishnamoorthi, K. (1990). Capability indices for processes subject to unilateral and positional tolerances. *Quality Engineering*, 2(4):461–471.

[67] Krishnamoorthy, K., Lin, Y., and Xia, Y. (2009). Confidence limits and prediction limits for a Weibull distribution based on the generalized variable approach. *Journal of Statistical Planning and Inference*, 139(8):2675–2684.

[68] Krishnamoorthy, K. and Wang, X. (2016). Fiducial confidence limits and prediction limits for a gamma distribution: Censored and uncensored cases. *Environmetrics*, 27(8):479–493.

[69] Kurian, K., Mathew, T., and Sebastian, G. (2008). Generalized confidence intervals for process capability indices in the one-way random model. *Metrika*, 67:83–92.

[70] Kushler, R. H. and Hurley, P. (1992). Confidence bounds for capability indices. *Journal of Quality Technology*, 24(4):188–195.

[71] Lawless, J. F. (2011). *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons.

[72] Lee, W. C., Wu, J. W., and Lei, C. L. (2010). Evaluating the lifetime performance index for the exponential lifetime products. *Applied Mathematical Modelling*, 34(5):1217–1224.

[73] Lin, P. and Pearn, W. (2005). Testing manufacturing performance based on capability index Cpm. *The International Journal of Advanced Manufacturing Technology*, 27:351–358.

[74] Liu, J. and He, S. (2011). Autoregressive process with measurement errors. *Communications in Statistics-Theory and Methods*, 40(6):1041–1048.

[75] Maiti, S. S., Saha, M., and Nanda, A. K. (2010). On generalizing process capability indices. *Quality Technology & Quantitative Management*, 7(3):279–300.

[76] Maleki, M. R., Amiri, A., and Castagliola, P. (2017). Measurement errors in statistical process monitoring: A literature review. *Computers & Industrial Engineering*, 103:316–329.

[77] Mathew, T., Sebastian, G., and Kurian, K. (2007). Generalized confidence intervals for process capability indices. *Quality and Reliability Engineering International*, 23(4):471–481.

[78] McNeese, W. H. and Klein, R. A. (1991). Measurement systems, sampling, and process capability. *Quality Engineering*, 4(1):21–39.

[79] Meng, F. and Yang, J. (2023). Process capability analysis of Taguchi index Cpm based on generalized p-value. *Quality and Reliability Engineering International*, 39(6):2311–2329.

[80] Meng, F., Yang, J., and Huang, S. (2021). Hypothesis testing of process capability index Cpk from the perspective of generalized fiducial inference. *Quality and Reliability Engineering International*, 37(4):1578–1598.

[81] Mittag, H.-J. (1997). Measurement error effects on the performance of process capability indices. *Frontiers in Statistical Quality Control*, pages 195–206.

[82] Montgomery, D. C. (2007). *Introduction to Statistical Quality Control*. John Wiley & Sons.

[83] Montgomery, D. C. and Runger, G. C. (1993). Gauge capability and designed experiments. Part I: basic methods. *Quality Engineering*, 6(1):115–135.

[84] Noomene, R. (2007). *Procedures of Parameters' estimation of AR (1) models into lineal state-space models*. World Congress on Engineering.

[85] Noorossana, R. (2002). Process capability analysis in the presence of autocorrelation. *Quality and Reliability Engineering International*, 18(1):75–77.

[86] Palmer, K. and Tsui, K.-L. (1999). A review and interpretations of process capability indices. *Annals of Operations Research*, 87:31–47.

[87] Pearn, W. and Chen, K. (2002). One-sided capability indices C_{PU} and C_{PL}: decision making with sample information. *International Journal of Quality & Reliability Management*, 19(3):221–245.

[88] Pearn, W. L. and Kotz, S. (2006). *Encyclopedia and handbook of process capability indices: a comprehensive exposition of quality control measures*, volume 12. World Scientific.

[89] Pearn, W. L., Kotz, S., and Johnson, N. L. (1992). Distributional and inferential properties of process capability indices. *Journal of Quality Technology*, 24(4):216–231.

[90] Pearn, W. L. and Lin, P. (2004). Testing process performance based on capability index Cpk with critical values. *Computers & Industrial Engineering*, 47(4):351–369.

[91] Perakis, M. (2010). Estimation of differences between process capability indices Cpm or Cpmk for two processes. *Journal of Statistical Computation and Simulation*, 80(3):315–334.

[92] Perakis, M. and Xekalaki, E. (2002). A process capability index that is based on the proportion of conformance. *Journal of Statistical Computation and Simulation*, 72(9):707–718.

[93] Piao, C. and Zhi-Sheng, Y. (2018). A systematic look at the gamma process capability indices. *European Journal of Operational Research*, 265(2):589–597.

[94] Rao, G. S., Aslam, M., and Kantam, R. (2016). Bootstrap confidence intervals of $C_{N_{pk}}$ for inverse Rayleigh and log-logistic distributions. *Journal of Statistical Computation and Simulation*, 86(5):862–873.

[95] Rodriguez, R. N. (1992). Recent developments in process capability analysis. *Journal of Quality Technology*, 24(4):176–187.

[96] Runger, G. C. and Willemain, T. R. (1995). Model-based and model-free control of autocorrelated processes. *Journal of Quality Technology*, 27(4):283–292.

[97] Sadeghpour Gildeh, B. and Abbasi Ganji, Z. (2020). The effect of measurement error on the process incapability index. *Communications in Statistics-Theory and Methods*, 49(3):552–566.

[98] Saha, M., Dey, S., and Maiti, S. S. (2018). Parametric and non-parametric bootstrap confidence intervals of $C_{N_{pk}}$ for exponential power distribution. *Journal of Industrial and Production Engineering*, 35(3):160–169.

[99] Saha, M. and Dutta, S. (2025). Bayesian and non-Bayesian inference of the process capacity index Spk' under progressive type-II right censored scheme. *Quality and Reliability Engineering International*, 41(1):449–463.

[100] Saha, M., Kumar, S., and Sahu, R. (2020). Comparison of two generalized process capability indices by using bootstrap confidence intervals. *International Journal of Statistics and Reliability Engineering*, 7(1):187–195.

[101] Scagliarini, M. (2002). Estimation of Cp for autocorrelated data and measurement errors. *Communications in Statistics-Theory and Methods*, 31(9):1647–1664.

[102] Shewhart, W. (1931). *Economic Control of Quality of Manufactured Product*. New York, Van Nostrand Co.

[103] Shore, H. (1997). Process capability analysis when data are autocorrelated. *Quality Engineering*, 9(4):615–626.

[104] Shumway, R. H., Stoffer, D. S., and Stoffer, D. S. (2000). *Time Series Analysis and Its Applications*, volume 3. Springer.

[105] Somerville, S. E. and Montgomery, D. C. (1996). Process capability indices and non-normal distributions. *Quality Engineering*, 9(2):305–316.

[106] Srivastava, A., Chaturvedi, A., and Kumar, N. (2023). Finite sample performance of an estimator of process capability index Cpm for the autocorrelated data. *Communications in Statistics-Simulation and Computation*, 52(9):4348–4360.

[107] Sullivan, L. P. (1984). Reducing variability-A new approach to quality. *Quality Progress*, 17(7):15–21.

[108] Sun, J., Wang, S., and Fu, Z. (2009). The effect of autocorrelated data on Taguchi process capability index Cpm based on ar (1) model. *2009 International Conference on Management and Service Science*, pages 1–4.

[109] Taguchi, G. (1985). A Tutorial on Quality Control and Assurance-the Thaguchi Methods. *1985 ASA Annual Meeting, Las Vegas.*

[110] Tang, L. C. and Than, S. E. (1999). Computing process capability indices for non-normal data: a review and comparative study. *Quality and Reliability Engineering International*, 15(5):339–353.

[111] Thoman, D. R., Bain, L. J., and Antle, C. E. (1969). Inferences on the parameters of the Weibull distribution. *Technometrics*, 11(3):445–460.

[112] Tong, L. and Chen, J. (2003). Bootstrap confidence interval of the difference between two process capability indices. *The International Journal of Advanced Manufacturing Technology*, 21:249–256.

[113] Tong, L. I., Chen, H. T., and Tai, Y. F. (2008). Constructing BCa Bootstrap Confidence Interval for the Difference between Two Non-normal Process Capability Indices $C_{N_{pmk}}$. *Quality Engineering*, 20(2):209–220.

[114] Tsui, K. W. and Weerahandi, S. (1989). Generalized p-values in significance testing of hypotheses in the presence of nuisance parameters. *Journal of the American Statistical Association*, 84(406):602–607.

[115] Vännman, K. (1995). A unified approach to capability indices. *Statistica Sinica*, 5(2):805–820.

[116] Vännman, K. and Kotz, S. (1995). A superstructure of capability indices: Distributional properties and implications. *Scandinavian Journal of Statistics*, 22(4):477–491.

[117] Vännman, K. and Kulahci, M. (2008). A model-free approach to eliminate autocorrelation when testing for process capability. *Quality and Reliability Engineering International*, 24(2):213–228.

[118] Wallgren, E. (1998). *Properties of the Taguchi capability index for Markov dependent quality characteristics*. Springer.

[119] Weerahandi, S. (1993). Generalized Confidence Intervals. *Journal of the American Statistical Association*, 88:899–905.

[120] Wilson, E. B. and Hilferty, M. M. (1931). The distribution of chi-square. *Proceedings of the National Academy of Sciences*, 17(12):684–688.

[121] Wright, P. A. (1995). A process capability index sensitive to skewness. *Journal of Statistical Computation and Simulation*, 52(3):195–203.

[122] Wu, C. W. and Huang, P. (2010). Generalized confidence intervals for comparing the capability of two processes. *Communications in Statistics—Theory and Methods*, 39(13):2351–2364.

[123] Wu, C. W., Liao, M. Y., and Yang, T. T. (2013). Efficient methods for comparing two process yields-strategies on supplier selection. *International Journal of Production Research*, 51(5):1587–1602.

[124] Wu, C. W., Pearn, W., and Kotz, S. (2009). An overview of theory and practice on process capability indices for quality assurance. *International Journal of Production Economics*, 117(2):338–359.

[125] Wu, J. W., Lee, H. M., and Lei, C. L. (2007). Computational testing algorithmic procedure of assessment for lifetime performance index of products with two-parameter exponential distribution. *Applied Mathematics and Computation*, 190(1):116–125.

[126] Yeh, A. B. and Bhattcharya, S. (1998). A robust process capability index. *Communications in Statistics-Simulation and Computation*, 27(2):565–589.

[127] Yum, B.-J. (2023a). A bibliography of the literature on process capability indices (PCIs): 2010–2021, Part I: Books, review/overview papers, and univariate PCI-related papers. *Quality and Reliability Engineering International*, 39(4):1413–1438.

[128] Yum, B.-J. (2023b). A bibliography of the literature on process capability indices (PCIs): 2010–2021, Part II: Multivariate PCI-and functional PCI-related papers, special applications, software packages, and omitted papers. *Quality and Reliability Engineering International*, 39(4):1439–1464.

[129] Yum, B.-J. and Kim, K.-W. (2011). A bibliography of the literature on process capability indices: 2000–2009. *Quality and Reliability Engineering International*, 27(3):251–268.

[130] Zappa, D. and Deldossi, L. (2009). Misclassification rates, critical values and size of the design in measurement systems capability studies. *Applied Stochastic Models in Business and Industry*, 25(5):601–611.

[131] Zhang, N. F. (1998). Estimating process capability indexes for autocorrelated data. *Journal of Applied Statistics*, 25(4):559–574.

[132] Zimmer, L., Hubele, N., and Zimmer, W. (2001). Confidence intervals and sample size determination for Cpm. *Quality and Reliability Engineering International*, 17(1):51–68.