ESSAYS ON DEPRIVATION, POVERTY AND WELL-BEING

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Preface

Issues like inequity, deprivation and poverty have been a focus of interest for social scientists for a long time. The present thesis explores a few aspects of the above issues. It embodies the fruit of my intellectual perambulation in the Economic Research Unit of Indian Statistical Institute (ISI) during the years 1995 - 97. The text is divided into six chapters. The first chapter gives a general overview of different aspects of measuring inequality, deprivation and poverty. The second, third and fourth chapters take up the issue of deprivation measurement in detail. They contain discussions on deprivation ordering, indices and deprivation reducing tax functions. Chapter 5 characterizes a family of subgroup and factor decomposable measures of multidimensional poverty and chapter 6 studies a family of multidimensional additive measures of improvement in well-being.

I cannot express in words my sense of gratitude towards my thesis supervisor Professor Satya R. Chakravarty. I had the privilege of occupying a great deal of his precious time during which I have learnt a lot from his way of thinking and his approach to a problem. I must thank Professor Chakravarty and Professor Ravindra R. Ranade of Kagawa University, Japan for allowing me to include our joint work in this thesis.

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Chapter 1

General Introduction

1.1 Income Inequality

In the theory of income distribution we often attempt to rank alternative income profiles on the basis of some value judgements. One of the common value judgements is the "social preference for a more equitable profile" or "equity preference".

To make our discussion precise, we introduce some notation first. For a population of size n, the set of income distributions is denoted by D^n, the non-negative orthant R^n_+ of R^n with the origin deleted, where R^n is the Euclidean n-space and the positive integer n ≥ 2 is arbitrary. The set of all possible income distributions is $D = \bigcup_{n \in N} D^n$, where N is the set of positive integers. For any $n \in N$, any $\mathbf{x} \in D^n$, we write $\lambda(\mathbf{x})$ for the mean of $\mathbf{x}$. An n-coordinated vector of ones will be denoted by 1^n. For any $n \in N$, $\mathbf{x} \in D^n$, let $\bar{\mathbf{x}}$ be the illfare ranked permutation of $\mathbf{x}$, that is, $\bar{\mathbf{x}}$ is that permutation of $\mathbf{x}$ such that $\bar{x}_1 \leq \bar{x}_2 \leq .. \leq \bar{x}_n$. For all $n \in N$, $\mathbf{x} \in D^n$, the normalized distribution $\frac{\mathbf{x}}{\lambda(\mathbf{x})}$ and the mean centered distribution $\mathbf{x}$ - $\lambda(\mathbf{x})1^n$ will be denoted by $\widehat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ respectively. For any function $f : D \rightarrow R^1$, f^n will stand for the restriction of f on D^n. Similarly we will use $R_+ = \bigcup_{n \in N} R^n_+$ as the set of all possible income distributions.

An important criterion for ranking alternative income profiles on the basis of equity preference is Lorenz domination. To define this, let us first consider the Lorenz curve (LC). For any $\mathbf{p} \in D^n$, LC associated with $\mathbf{p}$ is obtained by joining the points $\{(0,0), \{\frac{k}{n}, L(\frac{k}{n};\mathbf{p})\}^n_{k=1}\}$ by straight lines, where

$$L(\frac{k}{n};\mathbf{p}) = \frac{1}{n}\sum_{j=1}^{k} \widehat{\bar{p}}_j \tag{1.1}$$

for k = 1,..,n. Given arbitrary $\mathbf{x}$, $\mathbf{y} \in D^n$, we say that $\mathbf{x}$ Lorenz dominates $\mathbf{y}$, $\mathbf{x} \geq_L \mathbf{y}$ for short, if

$$L(\frac{k}{n};\mathbf{x}) \geq L(\frac{k}{n};\mathbf{y}) \tag{1.2}$$

for all k=1,..,n. If the inequality is strict for some k < n in (1.2), then we say that $\mathbf{x}$ strictly lorenz dominates $\mathbf{y}$.

Note that Lorenz domination only works with the normalized distribution of incomes. We can similarly consider a dominance relation in terms of mean centered distributions. This is defined in terms of the Absolute lorenz curve (ALC). ALC is obtained by joining the points $\{\frac{k}{n}, LA(\frac{k}{n}; \mathbf{p})\}_{k=1}^{n}$ by straight lines, where

$$LA(\frac{k}{n}; \mathbf{p}) = \frac{1}{n}\sum_{j=1}^{k}\tilde{\bar{p}}_j \tag{1.3}$$

for k = 1,..,n and $\mathbf{p} \in D^n$ is arbitrary. For $\mathbf{x}$, $\mathbf{y} \in D^n$, $\mathbf{x}$ absolute Lorenz dominates $\mathbf{y}$, $\mathbf{x} \geq_{AL} \mathbf{y}$ for short, if

$$LA(\frac{k}{n}; \mathbf{x}) \geq LA(\frac{k}{n}; \mathbf{y}) \tag{1.4}$$

for all k = 1,...,n. If the inequality is strict for some k < n in (1.4), then we say that $\mathbf{x}$ strictly absolute lorenz dominates $\mathbf{y}$.

An index of inequality is a summary statistic of the spread of incomes. Let us now introduce inequality indices satisfying some standard postulates and see how ranking of distributions by such indices coincides with the ranking suggested by the Lorenz (absolute Lorenz) criterion. An inequality index I is a *real valued function* defined on D. An inequality index I is said to be a *relative inequality index* or a *rightist inequality index* if for all n ≥ 2, $\mathbf{x} \in D^n$

$$I^n(c\mathbf{x}) = I^n(\mathbf{x}) \tag{1.5}$$

where c > 0 is any scaler. In contrast, an inequality index is said to be an *absolute inequality index* or *leftist inequality index* if for all n ≥ 2, $\mathbf{x} \in D^n$

$$I^n(\mathbf{x} + c1^n) = I^n(\mathbf{x}) \tag{1.6}$$

where c is any scaler such that $\mathbf{x} + c1^n \in D^n$.

The problem of choosing between absolute and relative inequality is essentially a matter of value judgment. **Bossert and Pfingstein (1986, 90)** proposed an intermediate position by defining I to be an *intermediate inequality index* if for all n ≥ 2, $\mathbf{x} \in D^n$

$$I^n(\mathbf{x} + c(\mu\mathbf{x} + (1-\mu)1^n)) = I^n(\mathbf{x}) \tag{1.7}$$

where $0 \leq \mu \leq 1$ and c is a scaler such that $\mathbf{x}+c(\mu\mathbf{x}+(1-\mu)1^n) \in D^n$. Here μ is the compromise parameter. Equation 1.7 becomes 1.5 (1.6) if $\mu = 1$ (0).

Several postulates have been suggested in the literature for an arbitrary inequality index. For discussion on such postulates see **Dalton (1920), Kolm(1969,76,76a), Atkinson(1970), Kats (1972), Sheshinski (1972), Sen (1973), Champernowne (1974), Blackorby and Donaldson (1978,80,84), Fields and Fei (1978), Bourguignon (1979), Cowell (1980,94), Donaldson and Weymark (1980,83), Kakwani (1980,81,85), Shorrocks (1980, 84), Cowell and Kuga (1981, 81a), Weymark (1981,95), Thon (1982), Foster (1983,85), Russell (1985), Ebert(1987,88,88a,88b), Chakravarty (1988,1990), Foster** and **Shorrocks (1988)**, **Yaari (1988)**, **Bossert (1990)**, **Lambert (1993)**, **Ben Porath and Gilboa (1994)**, **Chakravarty and Ghosh (1995)**.

A few of the postulates that have been suggested for an arbitrary I are:

Normalization (NM) : For all n ≥ 2, $I^n(c1^n) = 0$, where c > 0 is any scaler.

Symmetry (SM) : For all n ≥ 2, $\mathbf{x} \in D^n$, $I^n(P\mathbf{x}) = I^n(\mathbf{x})$ where P is any n x n permutation matrix.[1]

Pigou-Dalton Transfer Principle (PD) : Suppose $\mathbf{y}$ is obtained from $\mathbf{x} \in D^n$ by a regressive income transfer from person j to person i, that is, (i) $x_j \leq x_i$, (ii) $y_i - x_i = x_j - y_j > 0$ and (iii) $y_k = x_k$ for all $k \neq i, j$. Then $I^n(\mathbf{y}) > I^n(\mathbf{x})$.

Dalton Population Principle (PP) : For all n ≥ 2, $\mathbf{x} \in D^n$, $I^{nk}(\mathbf{y}) = I^n(\mathbf{x})$ where $\mathbf{y}$ is a k-replication of $\mathbf{x}$, that is, $\mathbf{y} = (x^1, .., x^k)$, where each $x^i = \mathbf{x}$.

Normalization means that inequality is zero when the distribution is egalitarian. Symmetry says that the individuals are not distinguished by anything other than income. Thus the names of the individuals are irrelevant to the question of inequality. PD requires that a regressive transfer increases inequality. Analogously, a progressive transfer (that is, a rich to poor transfer) should reduce inequality. PP demands that the inequality of a given distribution is same as that of it's replicated version. In other words, PP leads us to view inequality in average terms. Clearly, PP is helpful in cross-population comparison of inequality.

The following proposition of **Foster (1985)** links the Lorenz ordering with a particular class of inequality indices.

Proposition 1 (Foster, 1985) : Let $\mathbf{x}, \mathbf{y} \in D$ be arbitrary. Then the following statements are equivalent.

(i) $I(\mathbf{x}) > I(\mathbf{y})$ for all relative inequality indices that satisfy SM, PD and PP.

(ii) $\mathbf{x} \geq_L \mathbf{y}$ in the strict sense.

Foster also has obtained particular cases of the above proposition for fixed (variable) mean and fixed population size.

Moyes (1987) showed an analogous result linking absolute inequality indices and the absolute Lorenz ordering. For intermediate inequality indices, a consistency result of this type was developed in **Chakravarty (1989)**.

We now provide some examples of different types of inequality indices. A general example of a relative (rightist) index satisfying NM, SM, PD, PP is the **Shorrocks (1980)** *generalized entropy family* (see also **Cowell (1980)**).

$$\begin{aligned} S_c^n(\mathbf{x}) &= \frac{1}{nc(c-1)} \sum_{i=1}^{n} \left(\left(\frac{x_i}{\lambda(\mathbf{x})} \right)^c - 1 \right), \quad c \neq 0, 1 \\ S_0^n(\mathbf{x}) &= \frac{1}{n} \sum_{i=1}^{n} ln \frac{\lambda(\mathbf{x})}{x_i} \end{aligned} \tag{1.8}$$

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[1] A n x n matrix with entries 0 and 1 is called a permutation matrix if each of its rows and columns sum to one.

$$S_1^n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{\lambda(\mathbf{x})} ln \frac{x_i}{\lambda(\mathbf{x})}$$

where $\mathbf{x} \in D^n$. (If $x_i = 0$ for some i, then continuity requires that $c > 0$.) The parameter c reflects different perception of inequality. As c decreases, S_c becomes more sensitive to transfers lower down the scale. S_0^n is called the mean logarithmic deviation. S_1^n is the **Theil (1967)** entropy index and $2S_2^n$ is the squared coefficient of variation. The **Atkinson (1970)** index, which requires positivity of all incomes, is given by

$$\begin{aligned} A_c^n(\mathbf{x}) &= 1 - [c(c-1)S_c^n(\mathbf{x}) + 1]^{1/c}, \quad for\ c \neq 0,\ c < 1 \\ A_c^n(\mathbf{x}) &= 1 - e^{-S_0^n(\mathbf{x})} \end{aligned} \tag{1.9}$$

or

$$\begin{aligned} A_c^n(\mathbf{x}) &= 1 - \frac{1}{\lambda(\mathbf{x})}\left[\frac{1}{n}\sum_{i=1}^{n} x_i^c\right]^{1/c}, \quad for\ c \neq 0,\ c < 1 \\ A_0^n(\mathbf{x}) &= 1 - \frac{1}{\lambda(\mathbf{x})}(\Pi_{i=1}^n x_i)^{1/n}. \end{aligned} \tag{1.10}$$

As $c \to -\infty$, $A_c^n(\mathbf{x}) \to 1 - \frac{1}{\lambda(\mathbf{x})} min_i\{x_i\}$, the *relative maximin index* of inequality.

Another relative index that satisfies all the above postulates is the well known *Gini coefficient*,

$$G^n(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j|/2n^2\lambda(\mathbf{x}). \tag{1.11}$$

The standard examples of absolute indices satisfying the stated properties are the *variance* $2(\lambda(\mathbf{x}))^2 S_2^n(\mathbf{x})$, the *absolute Gini index* $\lambda(\mathbf{x})G^n(\mathbf{x})$ and the **Kolm (1976)** index

$$K_\alpha^n(\mathbf{x}) = \frac{1}{\alpha} ln \frac{1}{n} \sum_{i=1}^{n} e^{\alpha(\lambda(\mathbf{x}) - x_i)} \tag{1.12}$$

where $\mathbf{x} \in D^n$, and $\alpha > 0$. As $\alpha \to \infty$, $K_\alpha^n \to \lambda(\mathbf{x})$ - $min\{x_i\}$, the absolute maximin index.

Axiomatic characterizations of alternative inequality indices consistent with the Lorenz ordering have been carried out by different authors. See for example **Bourguignon (1979), Cowell (1980), Shorrocks (1980,83), Cowell and Kuga (1981,81a), Foster (1985), Ebert (1988a,88b), Chakravarty (1997), Chakravarty and Ghosh (1995)**. Such an axiomatic characterization of an index states the necessary and sufficient conditions for identifying the index uniquely in some specific framework.

So far we have been talking about ranking of distributions via inequality indices. We now present some results which use *social welfare function* (SWF) defined on income distributions for the purpose of ranking.

Proposition 2 (Kolm, 1969 and Atkinson, 1970) : Suppose that the SWF is utilitarian symmetric[2] in incomes, so $W^n(\mathbf{x}) = \sum_{i=1}^{n} U(x_i)$, where $U(x_i)$ is the utility of person having income x_i. Then for $\mathbf{x}, \mathbf{y} \in D^n$ where $\lambda(\mathbf{x}) = \lambda(\mathbf{y})$, the following conditions are equivalent.

(i) $\mathbf{x} \geq_L \mathbf{y}$ in the strict sense.

(ii) $W^n(\mathbf{x}) > W^n(\mathbf{y})$ for any strictly concave real valued function U.

Dasgupta, Sen and Starett (1973) and **Rothschild and Stiglitz (1973)** generalized the above result by weakening the concavity and the additivity assumptions (see also **Kolm (1969)**).

Proposition 3 (Kolm 1969 and Dasgupta, Sen and Starett, 1973) : Let $\mathbf{x}, \mathbf{y} \in D^n$ where $\lambda(\mathbf{x}) = \lambda(\mathbf{y})$. Then the following statements are equivalent.

(i) $\mathbf{x} \geq_L \mathbf{y}$ in the strict sense.

(ii) $W^n(\mathbf{x}) > W^n(\mathbf{y})$ for all strictly S-concave[3] SWFs $W^n : D^n \to R$.

Now for propositions 2 and 3, the consideration of differing mean income is absent. Hence, the efficiency considerations are ruled out. For this, **Shorrocks (1983)** defined an extension of the LC which is known as the Generalized Lorenz curve (GLC). The GLC is constructed by joining the points $\{(0,0), \{\frac{k}{n}, LG(\frac{k}{n}; \mathbf{z})\}_{k=1}^{n}\}$ by straight lines, where

$$LG(\frac{k}{n}; \mathbf{z}) = \frac{1}{n}\sum_{j=1}^{k} \bar{z}_j \tag{1.13}$$

for k = 1,..,n and $\mathbf{z} \in D^n$ arbitrary. For $\mathbf{x}, \mathbf{y} \in D^n$, $\mathbf{x}$ generalized Lorenz dominates $\mathbf{y}$, $\mathbf{x} \geq_{GL} \mathbf{y}$ for short, if

$$LG(\frac{k}{n}; \mathbf{x}) \geq LG(\frac{k}{n}; \mathbf{y}) \tag{1.14}$$

for all k=1,..,n.

The following proposition of **Shorrocks (1983)** shows how GLC can be used for comparing distributions with different means (see also **Kakwani (1984)**).

Proposition 4 (Shorrocks (1983), Kakwani (1984)): Let $\mathbf{x}, \mathbf{y} \in D^n$ be arbitrary. Then the following statements are equivalent:

(a) $\mathbf{x} \geq_{GL} \mathbf{y}$.

(b) W^n $(\mathbf{x}) > W^n$ $(\mathbf{y})$ for all increasing, strictly S-concave SWFs $W^n : D^n \to R^1$.

Interesting examples of increasing, strictly S-concave SWFs is the **Gini** SWF:

$$W_G^n(\mathbf{x}) = \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)\bar{x}_i,$$

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[2] A function $f^n : D^n \to R$ is said to be symmetric if $f^n(P\mathbf{x}) = f^n(\mathbf{x})$ for all $\mathbf{x} \in D^n$, where P is any n x n permutation matrix.

[3] A function $f^n : D^n \to R$ is S-concave if $f^n(B\mathbf{x}) \geq f^n(\mathbf{x})$ for all $\mathbf{x} \in D^n$, where B is any n x n bistochastic matrix. A square matrix is said to be bistochastic it all its entries are non-negative and each of its rows and columns sums to one. For strict S-concavity of f^n, the weak inequality is to be replaced by a strict inequality whenever Bx is not a permutation of $\mathbf{x}$.

the **symmetric mean of order** r $(r < 1)$:

$$W_r^n(\mathrm{x}) = (\frac{1}{n}\sum_{i=1}^{n} x_i^r)^{1/r}, \quad r \neq 0,\ r < 1,$$
$$= \Pi_{i=1}^n x_i^{1/n}, \quad r = 0,$$

where $x_i > 0$ for all i; and the **Kolm-Pollack** SWF:

$$W_\alpha^n(\mathrm{x}) = -\sum_{i=1}^{n} e^{-\alpha x_i}$$

where $\alpha > 0$. The parameters r and α in W_r^n and W_α^n determine the curvature of the social indifference surfaces. As r $\rightarrow -\infty$ $(\alpha \rightarrow \infty)$ W_r^n $(W_\alpha^n) \rightarrow min\{x_i\}$, the **Rawlsian (1970)** maximin criterion.

Proposition 2, 3 and 4 also hold for varying population sizes given that the SWF is population replication invariant.

1.2 Concept of Deprivation

Deprivation can roughly be defined as the utility foregone because of not possessing the economic variable under consideration, here income. A person's feeling of deprivation in a society arises out of the comparison of his position with those of better off persons. **Runciman (1966)** used the example of promotion to describe an individual's feeling of deprivation and argued that the extent of deprivation felt by an individual for not being promoted is an increasing function of the number of persons who have been promoted. 'The more people a man sees promoted when he is not promoted himself, the more people he may compare himself within a situation where the comparison will make him relatively deprived' (op.cit. p 19). Yitzhaki (1979) considered relative deprivation in term of income and quantified a particular case and demonstrated that one plausible index of deprivation in the society is the absolute Gini index for the society.

Hey and Lambert (1980) provided an alternative characterization of the **Yitzhaki (1979)** result. Their characterization is based on **Runciman's (1966)** remark 'the magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it' (op. cit. p 10). Alternatives and variations of **Yitzhaki's** index have been suggested by many authors including **Chakravarty and Chakraborty (1984), Kakwani (1984), Berrebi and Silber (1985), Chakravarty (1990)** and **Paul (1991)**.

An interesting interpretation of the Gini index in the deprivation framework was given by **Sen(1973)**. In any pairwise comparison, according to Sen, the individual with lower income may suffer from depression upon discovering that his income is lower. The average of all such depressions in all pairwise comparisons becomes the Gini index if the extent of depression is proportional to the differences in the incomes. **Kakwani (1980)** showed that if the individual's depression is proportional to the square of the income difference, we get the coefficient of variation as the average deprivation index.

Now, in view of Runciman's remark, following **Hey and Lambert (1980)**, the deprivation d_{ij} felt by an individual with income x_i relative to j^{th} person's income x_j, where $x_j \geq x_i$, can be considered to be their income share differential (see also **Kakwani (1984)**). That is

$$\begin{aligned} d_{ij} &= \frac{\widehat{x_j} - \widehat{x}_i}{n} \quad if\ x_j \geq x_i \\ &= 0 \quad otherwise \end{aligned} \tag{1.15}$$

Viewing deprivation in terms of income differentials is formally equivalent to the **Temkin (1986)** approach to the inequality measurement. According to **Temkin**, a person has a complaint if he has income less than others and inequality can be formulated in terms of such complaints.

The function d_{ij} is increasing in $\bar{x}_j$ and decreasing in $\bar{x}_i$. Now an individual with income $\bar{x}_i$ is deprived of incomes $\bar{x}_{i+1}, .., \bar{x}_n$. Therefore the total relative deprivation felt by this person is

$$\begin{aligned} d_i(\mathbf{x}) &= \sum_{j=i+1}^{n} \frac{\widehat{\bar{x}_j} - \widehat{\bar{x}_i}}{n} \\ &= \sum_{j=i+1}^{n} \frac{\widehat{\bar{x}_j}}{n} - \frac{(n-i)\widehat{\bar{x}_i}}{n} \end{aligned} \tag{1.16}$$

This shows the i^{th} person's aggregate complaint or deprivation.

The individual deprivation function d_i possesses many interesting properties.

(i) d_i is decreasing in $\bar{x}_i$.

(ii) d_i is independent of $\widehat{\bar{x}_1}, .., \widehat{\bar{x}_{i-1}}$.

(iii) d_i decreases under a rank preserving income transfer from some one with income higher than $\bar{x}_i$ to some one with income smaller than $\bar{x}_i$.

(iv) An increase in any income higher than $\bar{x}_i$ increases d_i.

(v) d_i remains unaffected when income transfer takes place among persons, who are all richer than individual i, assuming that the donor's income does not fall below $\bar{x}_i$ as a result.

(vi) An equiproportionate variation in all incomes does not alter d_i.

(vii) d_i is population replication invariant. If $\mathbf{x}$ is k- replicated then d_i remains unaltered.

(viii) d_i is continuous in it's argument.

The **relative deprivation curve (RDC)** associated with the distribution $\mathbf{x}$ is defined as the plot of $d_i(\mathbf{x})$ against $\frac{i}{n}$, i=0,..,n ; where $d_0(\mathbf{x})$ =1 (see **Kakwani (1984)**). **Kakwani (1984)** showed that the area under RDC is the Gini index for the society.

We can rewrite $d_i(\mathbf{x})$ in (1.16) as

$$d_i(\mathbf{x}) = 1 - L(\frac{i}{n}, \mathbf{x}) - \frac{(n-i)\widehat{\bar{x}_i}}{n} \tag{1.17}$$

Given $\mathbf{x}, \mathbf{y} \in D^n$ we say that $\mathbf{x}$ dominates $\mathbf{y}$ by the *relative deprivation criterion*, $\mathbf{x} \geq_{RDC} \mathbf{y}$ in short, if

$$d_i(\mathbf{x}) \geq d_i(\mathbf{y}) \tag{1.18}$$

for all i=1,..,n.

Yitzhaki (1979), Hey and Lambert (1980) and **Stark and Yitzhaki (1988)** referred to the function $L(\frac{i}{n}, \mathbf{x}) + \frac{(n-i)\widehat{\bar{x}}_i}{n}$ as the *relative satisfaction function* $s_i(\mathbf{x})$ of the i^{th} person. That is

$$s_i(\mathbf{x}) = \sum_{j=1}^{i} \frac{\widehat{\bar{x}}_j}{n} + \frac{(n-i)\widehat{\bar{x}}_i}{n} \tag{1.19}$$

The function $s_i(\mathbf{x})$ can be interpreted as follows. Since person i is not deprived of incomes $\bar{x}_1, \bar{x}_2, ..., \bar{x}_{i-1}$, he may be regarded as being satisfied if he compares these incomes with his own income $\bar{x}_i$. Therefore, the first term of the RHS of (1.19) depends on $\widehat{\bar{x}}_1, ..., \widehat{\bar{x}}_i$. Next, since this person is deprived of the incomes $\bar{x}_{i+1}, ..., \bar{x}_n$, to eliminate his feeling of deprivation about these higher incomes we censor each of them at $\bar{x}_i$. Given that, in addition to person i, there are now (n-i) persons with income $\bar{x}_i$ and since all these persons are treated symmetrically, the second term on the RHS of (1.19) depends on (n-i) $\widehat{\bar{x}}_i$. It follows from equations (1.17) and (1.19) that y $\geq_{RDC}$ x if and only if x dominates y by the *relative satisfaction criterion*, that is $s_i(\mathbf{x}) \geq s_i(\mathbf{y})$ for all i=1,..,n, x $\geq_{RSC}$ y for short. (See **Chakravarty (1997a)** for further discussions on this.) Like $d_i(\mathbf{x})$, $s_i(\mathbf{x})$ also possesses properties analogous to (i)-(viii).

So far we have considered income shares, but nominal income differences may also become a source of envy. In such cases individual's deprivation depends on the absolute income differences only. We therefore assume now that the individual deprivation functions are of the type

$$d_i^a(\mathbf{x}) = \sum_{j=i+1}^{n} \frac{\bar{x}_j - \bar{x}_i}{n} \tag{1.20}$$

We say that y dominates x by the absolute deprivation criterion, y $\geq_{ADC}$ x for short, if

$$d_i^a(\mathbf{x}) \geq d_i^a(\mathbf{y}) \tag{1.21}$$

for all i=1,.., n. Defining an individual's absolute satisfaction function $s_i^a(\mathbf{x})$ by $s_i^a(\mathbf{x}) = \lambda(\mathbf{x})$ - $d_i^a(\mathbf{x})$ we can also talk about ordering in terms of absolute satisfaction.

It may be interesting to note that the idea of aggregate complaint differs from the conventional approach to inequality measurement in at least one important way. Here 'we may not be able to deal with just a set of pure income transfers as in the conventional approach' (see **Amiel and Cowell (1994)**).[4] The following result makes the difference more precise.

Proposition 5 (Chakravarty, Chattopadhyay and Majumder, 1995) Let x, y $\in D^n$ be arbitrary. Then x $\geq_{RDC}$ y $\Rightarrow$ y $\geq_L$ x. But the converse is not true.

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[4] An interesting application of the concept and measurement of relative deprivation is in the study of the prevalence or absence of equity in a theoretical model of society or an economy. Some social psychologists like **Homans (1961), Adams** (1965) and **Runciman** (1966) examines human behaviour in a broad multidimensional framework which postulates that social behaviour is motivated more by 'envy' than by 'hunger' (**Maslow** (1970)). **Ehud Satt** (1996), in an interesting article, explored the equity characteristics of the ideal *Kibbutz* economy. He showed that, in this model, equity is supported within the usual one-dimensional income distribution framework but in the multidimensional and relative approach, equity is no longer upheld.

To see that the converse is not true, let $\mathbf{x} = (2,4,7,9)$ and $\mathbf{y} = (2,5,6,9)$. Here $\mathbf{y} \geq_L \mathbf{x}$ but *not*$[\mathbf{x} \geq_{RDC} \mathbf{y}]$. The reasoning behind this is that **y** is obtained from **x** by a transfer of income from the second richest person to the third richest person. This transfer shifts the Lorenz curve upwards. But the transfer while increasing the relative deprivation of the donor decreases that of the recipient and the net effect becomes ambiguous.

It may also be noted that $\mathbf{y} \geq_{ADC} \mathbf{x} \Rightarrow \mathbf{x} \geq_{AL} \mathbf{y}$ but the converse does not follow. This can be demonstrated using Proposition 5.

The above result therefore motivates us to study deprivation from a somewhat different perspective than inequality indices. In view of this, in **chapter 2**, we consider rightist or relative and leftist or absolute measures of deprivation using social satisfaction function which are defined on individual absolute satisfaction functions. The rightist (leftist) measure gives us the amount by which social satisfaction can be increased in proportional (absolute) terms by redistributing incomes equally. We demonstrate the existence of a relationship between summary indices of deprivation (including the Gini Coefficient, the maximin index, the coefficient of variation and their leftist counterparts) and social satisfaction. A numerical illustration of the RDC and ADC ranking is also provided using Indian consumer expenditure data. Finally, we try to examine the influence of contaminated data on deprivation.

1.3 Tax Progression and Inequality

The principle of taxation adopted by any government is an important economic decision which affects the people in many ways. So looking at the degree of progression of the tax structure is an important job. We call a tax system to be **progressive** if the average rate of taxation increases with income. That is, a tax function with increasing average rate shifts the income distribution towards Lorenz criterion to decide whether a tax system is egalitarian.

A taxation scheme is a function f:$D^1 \rightarrow D^1$ that associates a pre-tax income u to a post tax income f(u). Therefore T(u) = u- f(u) is the tax liability. The person with income level u will be called a tax payer, unaffected or subsidized according as T(u) is positive, zero or negative. For any $u \in D^n$ we write $f^{(n)}(u)$ for $(f(u_1), f(u_2), .., f(u_n))$. We assume that the function f is continuously differentiable. Continuous differentiability assumption is just a technical condition to ensure simpler analysis. We now state some properties of income taxation in terms of the function f defined above.

(a) Incentive Preservation (IP): Post-tax income f is non-decreasing in pre-tax incomes, that is, for all $u > 0$, $f'(u) \geq 0$.

(b) Minimal Progression (MP): Tax liability is non-decreasing in pre-tax incomes, that is, for all $u > 0$, $f'(u) \leq 1$.

(c) Average Progression (AP): Average tax rate in non-decreasing in pre-tax incomes, that is, for all $u > 0$, f(u)/u is non-increasing.

Properties **IP** and **MP** were introduced by **Fei (1981)**. Note that if tax liabilities are positive then **AP** $\Rightarrow$ **MP**. Strong versions of **IP, MP** and **AP** can be defined by replacing the weak inequalities in the above definitions by strict inequalities.

Kakwani showed that average progression will make the post-tax income distribution at least as equitable as the pre-tax income distribution according to the Lorenz criterion. **Jakobsson (1976)** and later **Eichhorn, Funke and Richter (1984)** and **Thon (1987)** showed that the implication is an equivalence. Below we formally state the **Eichhorn, Funke and Richter (1984)** result which we will use in chapters 3 and 4.

Proposition 6 (Eichhorn, Funke and Richter, 1984) : The following statements are equivalent

(i) For all $n \geq 2$, for all $\mathbf{x} \in D^n$, $f^{(n)}(\mathbf{x}) \geq_L \mathbf{x}$.

(ii) f(x) satisfies IP and AP.

The absolute counterpart to this result was demonstrated by **Moyes (1988)**.

Proposition 7 (Moyes, 1988) : The following statements are equivalent

(i) For all $n \geq 2$, for all $\mathbf{x} \in D^n$, $f^{(n)}(\mathbf{x}) \geq_{AL} \mathbf{x}$.

(ii) f(x) satisfies IP and MP.

Tax functions satisfying **IP** and **AP (MP)** are called Lorenz (absolute Lorenz) - consistent.

Although the relation between Lorenz and absolute Lorenz dominance criteria and tax progressivity has been extensively studied in the literature, there are relatively few references that explore relationships between tax progression and other dominance criteria, in particular the deprivation criteria. In **chapter 3** we consider the two deprivation criteria and also some other orderings which are consistent with the Lorenz ordering following **Marshall and Olkin (1979)**. We then identify the classes of tax functions that make the pre-tax distribution not less deprived than the post-tax distribution according to the two notions of deprivation ordering.

In **chapter 4** we examine another ranking relation suggested by **Hey and Lambert (1980)**. Of two income distribution **x** and **y**, over a given population size, we say that the former dominates the latter by the **utilitarian deprivation rule** if, for any person the aggregate utility shortfall from richer persons under **x** is at least as large as that under **y**. In this chapter we show that if the relative risk aversion associated with the utility function does not exceed one, then any Lorenz-consistent income tax function will make the post-tax distribution not more deprived than the pre-tax distribution according to the utilitarian deprivation rule. The converse of this proposition holds if the risk aversion measure is not less than one. It then follows that the only utility function for which the consistency between the two criteria holds is the logarithmic utility function. Finally, we relate our results to the **equal sacrifice principle**.

1.4 Poverty

Poverty is in the news every year. To understand the threat that the problem of poverty poses, it is necessary to know it's dimensions and the process through which it is aggravated. In his pioneering article on the measurement of poverty, **Sen (1976)** argued that an index of poverty should satisfy two desirable properties: the *monotonocity* axiom, which requires poverty to increase if the income of a poor person decreases; and the *transfer* axiom, which demands that poverty should increase under a transfer of income from a poor person to anyone who is richer. **Sen** also noted that the **head-count ratio** H, the proportion of poor people in the economy, violates these axioms and the **income gap ratio** I, the proportional average income shortfall from the poverty line $z > 0$, violates the transfer axiom. He then axiomatically derived a sophisticated poverty index.

Alternatives and variations of the Sen index have been suggested by many researchers including **Anand (1977, 83)**, **Hamada and Takayama (1977)**, **Kakwani (1977, 80, 80a)**, **Takayama (1979)**, **Thon (1979, 83)**, **Blackorby and Donaldson (1980)**, **Clark, Hemming and Ulph (1981)**, **Chakravarty (1983, 83a, 83b, 90, 97b)**, **Foster, Greer and Thorbecke (1984)**, **Hagenaars (1987)**, **Pyatt (1987)**, **Vaughan (1987)**, **Lewis and Ulph (1988)**, **Pattanaik and Sengupta (1995)** and **Shorrocks (1995)**. Several studies concentrated on problems of poverty ordering and provided very useful ordering conditions. See, for example, **Atkinson (1987, 92)**, **Foster and Shorrocks (1988, 88a, 88b)**, **Spencer and Fisher (1992)**, **Jenkins and Lambert (1993)**, **Foster and Jin (1994)** and **Mitra and Ok (1996)**. There are now some surveys on the subject: **Sen (1979)**, **Foster (1984)**, **Seidl (1988)**, **Chakravarty (1990)**, **Borooah (1991)**, **Callan and Nollan (1991)** and **Zheng (1997)**.

We now state two definitions of the poor as put forward by **Donaldson and Weymark (1986)**. Let $z > 0$ be the poverty line, that is, z is the income level required for maintaining subsistence standard of living.

Strong definition of the poor : For all $\mathbf{x} \in R_+$, the set of poor people are

$$\Pi(\mathbf{x}) = \{i \mid x_i \leq z\},$$

Weak definition of the poor : For all $\mathbf{x} \in R_+$, the set of poor people are

$$\Pi^W(\mathbf{x}) = \{i \mid x_i < z\}.$$

Throughout this section we will use the strong definition of the poor which is more commonly adopted in the literature on construction of poverty indices.

An index of poverty is a function $P: R_+ \times D^1 \rightarrow R^1$. For any $\mathbf{x} \in R_+$ and $z \in D^1$, $P(\mathbf{x}, z)$ indicates the level of poverty associated with the income distribution $\mathbf{x}$, given the poverty line z. A poverty index is relative (absolute) if it is unchanged when all incomes and the poverty line itself are changed equiproportionately (augmented by equal amounts).[5] We now state some properties for an index of poverty.

Focus (F) : $P(\mathbf{x}, z) = P(\mathbf{y}, z)$ whenever $\mathbf{x} \in R_+$ is obtained from $\mathbf{y} \in R_+$ by an increment to a non-poor person.

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[5] For further discussions, see **Zheng (1994)**.

Weak Monotonocity (WM) : For all x, y $\in R_+$, if $x_j = y_j$ for all $j \neq i$, $i \in \Pi(\mathbf{x})$ and $x_i > y_i$, then P(x, z) < P(y, z).

Symmetry (SM) : P(x, z) = P(y, z) whenever x = Py for some permutation matrix P.

Replication Invariance (RI) : P(x, z) = P(y, z) whenever x is obtained from y by a k-replication.

Continuity (CON) : P(x, z) is a continuous function of x $\in R_+$, for any given z.

Transfer (TA) : P(x, z) > P(y, z) whenever x $\in R_+$ is obtained from y $\in R_+$ by a regressive transfer with the donor being poor.

Non-poverty Growth (NPG) : P(x, z) < P(y, z) whenever x $\in R_+$ is obtained from y $\in R_+$ by adding a non-poor person to the population.

Increasing Poverty Line (IP) : P(x, z) < P(x, z′) whenever z < z′.

Non-poverty Normalization (N) : P(x, z) = 0 if Π is an empty set.

Subgroup Decomposability (SD) : Let $x^i \in D^{n_i}$, i = 1,..,k. Then

$$P(\mathbf{x}, z) = \sum_{i=1}^{k} \frac{n_i}{n} P(\mathbf{x}^i, z) \qquad (1.22)$$

where n = $\sum_{i=1}^{k} n_i$ and x = (x^1,..., x^k).

F (**Sen (1976)**) means that the incomes of the non-poor are irrelevant to the measurement of poverty. **WM**, which was suggested by **Sen (1976)**, says that if one poor person's income decreases, then the index of poverty must increase. We can analogously define strong monotonocity axiom, which requires poverty to decrease when the income of a poor person goes up. Thus, this version of the monotonocity axiom includes the possibility that the beneficiary becomes rich as a result of income increase. **SM** requires that any characteristic of the poor individuals, other than their incomes, are irrelevant to poverty measurement. **RI** leads us to look at poverty in per capita or average terms. **CON**, proposed by **Watts (1968)** and **Donaldson and Weymark (1986)**, demands that the poverty measure should change by small amount if there are small changes in the income values. **TA** generalizes **WM** in the sense that if a poor person donates some of his income to a rich person, the poverty value for the society must increase. **TA** is weaker than the transfer axiom considered by Sen. For formulation and discussion on other versions of the transfer principle, see **Sen (1981, 82)**, **Donaldson and Weymark (1986)**, **Seidl (1988)**, **Chakravarty (1990)** and **Zheng (1997)**. **NPG**, considered by **Kundu and Smith (1983)**, says that if a rich person is added to the population, poverty must decrease.[6] **IP** was proposed by **Clark, Hemming and Ulph (1981)** and **Chakravrty (1983)**. It says that if the poverty line increases, the level of poverty, given other things, must increase. **N** similarly requires the poverty index to have a value zero whenever there are no poor people (**Foster and Shorrocks (1981)**). **SD** provides the prescription for decomposition of overall poverty into subgroup-wise poverty values. Poverty indices satisfying SD are helpful in calculating the

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[6] Kundu and Smith also defined a poverty growth axiom which says that poverty must increase if a poor person is added to the population. But this axiom is not compatible with TA.

percentage contributions of individual subgroups to overall poverty and hence in formulating poverty alleviation policies. This was introduced in **Foster, Greer and Thorbecke (1984)**.

We now state some of the poverty measures that have been proposed in the literature and used for empirical purposes.

Head count ratio : $H(\mathbf{x}, z) = \frac{q(\mathbf{x},z)}{n}$, where $q(\mathbf{x}, z)$, or simply q, is the number of poor people in the economy, when $\mathbf{x} \in R^n_+$.

Income gap ratio : $I(\mathbf{x}, z) = 1 - \frac{\lambda_p(\mathbf{x},z)}{z}$, where λ_p is the mean income of the poor, $\mathbf{x} \in R^n_+$.

The **Sen (1976) index :**

$$S(\mathbf{x}, z) = \frac{2}{(q+1)nz}\sum_{i=1}^{q}(z-\bar{x}_i)(q+1-i) \tag{1.23}$$

where $\mathbf{x} \in R^n_+$. For a large number of poor, $S(\mathbf{x}, z)$ can be approximated by

$$S'(\mathbf{x}, z) = H[I + (1-I)G_p] \tag{1.24}$$

where G_p is the Gini coefficient of the poor.

Blackorby and Donaldson offered an alternative interpretation and generalization of S'. To see this, let W^p be the social welfare function (SWF) of the poor. W^p is assumed to be continuous, increasing, strictly S-concave and homothetic.[7] The representative income ξ_p of the poor is then defined by

$$W^p(\xi_p 1^q) = W^p(\bar{x}_1, .., \bar{x}_q), \tag{1.25}$$

that is, ξ_p is that level of income which, if given to each poor person, will make the existing distribution ethically indifferent. Then **Blackorby and Donaldson (1980)** general poverty index is given by

$$BD(\mathbf{x}, z, W^p) = H(\mathbf{x}, z)[1 - \frac{\xi_p}{z}]. \tag{1.26}$$

If we take W^p to be the Gini SWF then $BD(\mathbf{x}, z, W^p)$ becomes S'.

If ξ_p in (1.26) is given by

$$\xi_p = \sum_{i=1}^{q}\bar{x}_i(q+1-i)^k / \sum_{i=1}^{q} i^k,$$

we obtain the **Kakwani (1980) index :**

$$K(\mathbf{x}, z, k) = \frac{q}{nz\sum_{i=1}^{q} i^k}\sum_{i=1}^{q}(z-x_i)(q+1-i)^k, \quad k \geq 0. \tag{1.27}$$

where $\mathbf{x} \in R^n_+$.

Unfortunately the general index BD violates continuity and transfer principle (**Chakravarty 1983, 90)**).

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[7] Homotheticity means that W^p can be expressed as an ordinal transform of a linearly homogeneous function of the incomes of the poor.

Takayama (1979) defined the censored income x_i^* associated with x_i by $x_i^* = min(x_i, z)$. That is, the censored income distribution $\mathbf{x}^*$ corresponding to $\mathbf{x}$ replaces all non-poor incomes by the poverty line income. **Takayama** then suggested the Gini index of the censored income distribution as a measure of poverty. Other indices of inequality based on censored income distributions were suggested as poverty indices in **Hamada and Takayama (1977)**. However, all such axioms violate **WM**.

Let W be the continuous, increasing, strictly S-concave, homothetic SWF defined on censored income distributions. We define the representative income ξ^* corresponding to $\mathbf{x}^*$ by

$$W(\xi^* 1^n) = W(\mathbf{x}^*). \tag{1.28}$$

The general poverty index introduced by **Chakravarty (1983)** is then defined by

$$Ch(\mathbf{x}^*, z, W) = 1 - \frac{\xi^*}{z}. \tag{1.29}$$

If we use the Gini SWF in (1.28), then obtain the **Shorrocks modification of the Sen index**

$$Sh(\mathbf{x}^*, z) = \frac{1}{n^2 z} \sum_{i=1}^{q} (z - \bar{x}_i^*)(2(n-i)+1). \tag{1.30}$$

The use of symmetric mean of order α ($\alpha \leq 1$) in (1.28) leads us to the **Clark, Hemming and Ulph (1981)** measure of poverty

$$\begin{aligned} C(\mathbf{x}^*, z, \alpha) &= 1 - \frac{1}{z}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i^{*\alpha}\Big)^{\frac{1}{\alpha}}, \quad \alpha > 0 && (1.31) \\ &= 1 - \frac{1}{z}[\Pi_{i=1}^{n} x_i^*]^{\frac{1}{n}}, \quad \alpha = 0. \end{aligned}$$

Next, if ξ^* in (1.29) is of the form $\sum_{i=1}^{n} x_i^*(n+1-i)/n^2$, then $Ch(\mathbf{x}^*, z, W)$ becomes the **Thon (1979)** index:

$$T(\mathbf{x}^*, z) = \frac{2}{(n+1)nz} \sum_{i=1}^{n} (z - x_i^*)(n+1-i). \tag{1.32}$$

Unfortunately, *Ch* violates RI and SD.

Finally we give three examples of poverty measures that satisfy SD. These are the **Chakravarty** (**1983**) measure

$$C_h(\mathbf{x}, z, e) = \frac{1}{n} \sum_{i=1}^{q} [1 - (\frac{x_i}{z})^e], \quad 0 < e < 1, \tag{1.33}$$

the **Foster, Greer and Thorbecke (1984)** measure

$$F(\mathbf{x}, z, \alpha) = \frac{1}{n} \sum_{i=1}^{q} (1 - \frac{x_i}{z})^\alpha, \quad \alpha \geq 0, \tag{1.34}$$

and the **Watts (1968)** measure

$$Wa(\mathbf{x}, z) = \frac{1}{n} \sum_{i=1}^{n} (lnz - lnx_i^*). \tag{1.35}$$

C_h is increasing in e. It satisfies all the properties considered above. For $e = 1$, C_h becomes I. F becomes H and I for $\alpha = 0$ and $\alpha = 1$ respectively. For WM to hold we require positivity of α. On the other hand, for TA to hold, $\alpha > 1$ is needed. (For policy applications of F, see **Besley and Kanbur (1988)**, **Ravallion and Chao (1989)**, **Besley (1990)**, **Keen (1992)** and **Kanbur, Keen and Tuomala (1994)**.) It is important to note that Wa is defined only for positive incomes. In fact C_h, F and Wa are members of the class of subgroup consistent poverty indices. Subgroup consistency, which has been introduced by **Foster and Shorrocks (1991)**, requires that overall poverty falls if there is a reduction in the poverty of a subgroup of the population assuming that the poverty in the rest of the subgroups remains fixed. An additively decomposable poverty index 'must be subgroup consistent' (**Foster and Shorrocks (1991)**, p.692).

The concept of poverty, however, is essentially multi-dimensional, income being only one factor among many. If we define poverty as 'the failure of basic capabilities to reach minimally acceptable levels' (**Sen (1992)**, p.139) and if this capacity depends on income, health, housing, clothing etc., then poverty should be measured in more than one dimension. (See also **Ravallion (1996)**.) In fact, according to **Sen (1981)**, capability method is 'superior to the income method' and 'the income method is at most a second best'. Though research on income distribution based poverty has flourished in the last two decades, the multi-dimensional measurement of poverty remains a rather neglected area.

In **chapter 5** we characterize the class of subgroup and factor decomposable measures of multidimensional poverty where, as stated earlier, according to subgroup decomposability, overall poverty is a weighted average of subgroup poverty levels, with the weights being the population share of the subgroups. Factor decomposability, on the other hand, demands that total poverty is a simple average of poverty levels for individual basic needs or factors. These two postulates are quite helpful in formulating antipoverty policies. The chapter goes on to provide an empirical illustration of such decomposition of poverty using data on some factors from five districts in West Bengal, India.

1.5 Improvement in Well-being

In recent years, researchers have often focussed on the measurement of well-being in an economy. Any measure of well-being should be a function of some quality of life indicators, for example, life expectancy, educational attainment, health indicators, housing conditions etc. Such indices are frequently used for international comparisons in well-being. **Sen (1985, 87)** defined the concept of standard of living in terms of (i) *functioning* which indicates achievement of different attributes and (ii) *capacity*, which is the ability to achieve. **Kakwani (1993)** suggested a class of achievement indices to represent the actual standard of living and a class of improvement indices to represent the improvement in the standard of living of a country. An improvement in well-being indicates an increase in the attainment level of some attribute of well-being from, say,

w_1 to w_2. An improvement index is a real valued function $Q(w_1, w_2)$ summarizing this improvement. Alternatively, we can view improvement as a reduction in deprivation. Deprivation is measured by the shortfall of some attribute of well-being from its upper bound, say M. Then the improvement index will be $Q(M - w_1, M - w_2)$. See **Sen (1981)**, **Kakwani (1993)**, **UNDP (1990)**. Thus, it is worthwhile to explore the axiomatic foundations of improvement indices from both the shortfall and attainment perspectives. Let m stand for the lower bound of the attribute. Therefore, $w \in$ [m, M].

Kakwani (1993) suggested some interesting axioms for an improvement index:

Range Subdivision (RS) : Q$(M - w_1, M - w_2)$ is $<$, $=$ or $>$ 0 according as w_2 is $<$, $=$ or $> w_1$.

Normalization (NM) : Q$(M - w_1, M - w_2) = 1$ if $w_1 =$ m and $w_2 =$ M.

Monotonocity (MO) : Given other things, Q$(M - w_1, M - w_2)$ is decreasing in w_1 and increasing in w_2.

Period Consistency (PC) : For all $w_1, w_2, w_3 \in$ [m, M],

$$Q(M - w_1, M - w_3) = Q(M - w_1, M - w_2) + Q(M - w_2, M - w_3).$$

Increasing Difficulty of Improvement (ID) : Q$(M - w_1, M - (w_1 + \delta)) >$ Q$(M - v_1, M - (v_1 + \delta))$ for all $\delta > 0$ and $w_1 > v_1$.

RS subdivides the range of Q depending on the relationship between w_1 and w_2. **NM** states that Q should take on the highest value when the improvement is maximum. **MO** ensures that Q increases with w_2 and decreases with w_1. **PC** says that if there is an improvement from w_1 to w_2 and then further to w_3, the improvement index for the improvement from w_1 to w_3 should equal the sum of improvements from w_1 to w_2 and from w_2 to w_3. **ID** states that improvement is harder for higher initial attainment. For instance, it is harder to improve life expectancy from 80 to 85 years than to increase it from 60 to 65 years.

Kakwani also suggested a **full comparability axiom (FC)** which says that it is always possible for a country with achievement level v_1 to attain some achievement level v_2 so that the degree of improvement becomes equal to that of another country with achievement levels w_1 and w_2. Kakwani noted some problems with FC. **Majumder and Chakravarty (1996)** developed a sufficient condition under which FC can be fulfilled. Given the problems associated with FC, we will give up FC and not impose it as a postulate for Q.

In addition to the Kakwani axioms, **Majumder and Chakravarty (1996)** considered two more axioms for an improvement index. These are

Continuity (CN) : Q is a continuous function.

Dimensionality (DM) : Q$(M - w_1, M - w_2) =$ Q$(\alpha M - \alpha w_1, \alpha M - \alpha w_2)$ for all $\alpha > 0$.

CN ensures that small changes in one or more arguments of Q will generate small changes in its functional value. **DM** demands insensitivity of an improvement index to changes in the unit of measurement of the attribute.

Kakwani's (1993) improvement index is given by

$$Q(M-w_1, M-w_2) = \frac{(M-w_1)^r-(M-w_2)^r}{(M-m)^r}, \quad 0<r<1 \tag{1.36}$$

$$= \frac{ln(M-w_1)-ln(M-w_2)}{ln(M-m)}$$

where w_t is the achievement level of the attribute in question in year t, t=1,2; M (m) is the upper (lower) bound of w_t. But this is in one dimension only and in reality the well-being of a population depends on a bundle of attributes. At the same time people do not isolate different aspects of their lives. Instead, they speak about an overall level of well-being. It thus appears that the construction of a composite index of well-being is a worthwhile exercise. In fact, building on ideas put forward by **Sen**, the **UNDP** recently started publishing **Human Development Reports** which show ranking of countries by human development index (HDI) that depends on several attributes of well-being. This motivated **Tsui (1996)** to extend **Kakwani's** set-up to a multidimensional framework and suggest a multidimensional improvement index which aggregates the changes in achievements, that is, the improvement levels of a group of attributes of well-being. If there is only one attribute of well-being, the **Tsui** index turns out to be the **Kakwani** index of improvement.

In **chapter 6** we suggest a family of multidimensional indices of improvement in well-being which, in addition to satisfying all the desirable properties for a multidimensional improvement index suggested by **Tsui (1996)**, fulfills an additivity criterion. An additive index can be helpful in calculating the percentage contributions made by individual attributes of well-being to overall improvement level and hence in identifying the attributes whose contributions are poor or negative and formulating appropriate policies for improving the level of well-being. In the single-dimensional case a member of this family coincides with the **Kakwani (1993)** index and the single-dimensional **Tsui (1996)** index. We also provide a numerical illustration using UNDP data for 6 countries.

Before concluding, a few words about the organization of the material presented in the following chapters. Each chapter has its own introduction which motivates the investigation carried out thereafter. The next section introduces the additional notation and definitions. The results are then presented in the subsequent sections along with numerical illustrations (in chapters 2, 5 and 6).

Finally, we provide a bibliography where we cite the relevant references.

Chapter 2

Measurement of Deprivation

2.1 Introduction

As noted in chapter 1, most of the existing deprivation indices have been proposed on an ad hoc basis. The purpose of this chapter is to develop a unified approach, which relies on the social satisfaction function, to the measurement of deprivation. We suggest indices of both relative and absolute variety. Needless to say, the general deprivation indices proposed here are not meant to supplement the existing indices. Rather, we show how the existing indices (including the Gini coefficient, the maximin index, the coefficient of variation and their absolute counterparts) can be interpreted in our framework and hence can be related to social satisfaction functions in a negative monotonic way.

In section 2.2 we discuss social satisfaction functions. The relative and absolute indices are proposed in sections 2.3 and 2.4 respectively. Section 2.5 gives a numerical illustration of the relative and the absolute deprivation orderings. In section 2.6 we examine the effect of using erroneous data on deprivation orderings. Finally, section 2.7 concludes.

Social Satisfaction Functions

For any income distribution $\mathbf{x} \in D^n$, the absolute satisfaction function $s_i^a(\mathbf{x})$ of person i is given by

$$
\begin{aligned}
s_i^a(\mathbf{x}) &= \lambda(\mathbf{x}) - d_i^a(\mathbf{x}) \\
&= \sum_{j=1}^{i} \frac{\bar{x}_j}{n} + \frac{n-i}{n}\bar{x}_i
\end{aligned}
\tag{2.1}
$$

(see **Yitzhaki (1979), Hey and Lambert (1980)** and **Stark and Yitzhaki (1988)**). Clearly, $s_i^a(\mathbf{x})$ can be interpreted in the way we have interpreted $s_i(\mathbf{x})$ in section 1.2.

The individual satisfaction function s_i^a possesses many interesting properties:

(i) s_i^a is increasing in $\bar{x}_i$.

(ii) s_i^a is independent of incomes higher than $\bar{x}_i$.

(iii) A rank preserving increase in any income smaller than $\bar{x}_i$ increases s_i^a.

(iv) A rank preserving transfer of income between any two persons with income smaller than $\bar{x}_i$ does not change s_i^a.

(v) s_i^a is unit-translatable - an equal absolute change in all incomes changes s_i^a by the same absolute amount. More precisely, $s_i^a(\mathbf{x} + c1^n) = s_i^a(\mathbf{x}) + c$, where c is any scaler such that $\mathbf{x} + c1^n \in D^n$.

(vi) s_i^a is linearly homogeneous - $s_i^a(c\mathbf{x}) = cs_i^a(\mathbf{x})$ for all $c > 0$.

(vii) s_i^a's are illfare ranked. That is, for any $\mathbf{x} \in D^n$, $\bar{x}_1 = s_1^a(\mathbf{x}) \leq s_2^a(\mathbf{x}) \leq .. \leq s_n^a(\mathbf{x}) = \lambda(\mathbf{x})$. For any $\mathbf{x} \in D^n$, $s(\mathbf{x}) = (s_1^a(\mathbf{x}), .., s_n^a(\mathbf{x})) \in D^n$. Further, if x_i's are equally distributed, that is, if $x_i = c > 0$ for all i, then $s_i^a = c$ for all i, $1 \leq i \leq n$.

(viii) s_i^a is population replication invariant - an income by income replication of the population leaves s_i^a unchanged.

(ix) s_i^a is a continuous function.

Clearly, other specifications of satisfaction functions are also possible. For instance, we can express satisfaction in terms of income squares. However, we prefer to work with the form (2.1) because of its simplicity and interesting interpretation.

We now assume that social satisfaction function S^n is a real valued function of individual satisfaction levels, where S^n is ordinally significant and the superscript n of S^n indicates the dependence of the function on the population size n. That is, S^n, which given $\mathbf{x} \in D^n$, associates to the corresponding satisfaction vector $s(\mathbf{x}) = (s_1^a(\mathbf{x}), .., s_n^a(\mathbf{x}))$, a value $S^n(s(\mathbf{x}))$ indicating the level of social satisfaction. This is analogous to the requirement that the social welfare function denotes social welfare as a general function of individual utilities. It is supposed that S^n is (i) continuous, (ii) increasing along ray of equality (minimal increasingness), (iii) level surfaces of S^n cut the ray of equality, (iv) symmetric and (v) quasi-concave. The continuity assumption ensures that minor changes in incomes (hence in satisfactions) will generate minor change in S^n. Therefore a continuous satisfaction function will not be over-sensitive to minor observational errors in incomes. According to minimal increasingness, if all individuals enjoy the same level of absolute satisfaction, more satisfaction is preferred to less. (The term minimal increasingness is due to **Blackorby and Donaldson (1984a).**) Condition (iii) shows that each satisfaction profile $(s_1^a, .., s_n^a)$ is indifferent to some perfectly equal distribution of satisfaction. According to symmetry, social satisfaction does not alter under any permutation of individual satisfactions. Thus, any information other than individual satisfaction levels (for instance, the names of the individuals) are irrelevant to the measurement of social satisfaction. An implication of symmetry is that S^n can be defined directly on ordered satisfaction vectors (as we have done). Finally, quasi-concavity is the requirement that satisfaction contours are convex to the origin. A satisfaction function S^n satisfying conditions (i)-(v) will be called **regular.**

Let us now define the representative level of satisfaction $s_e(\mathbf{x})$ associated with $s(\mathbf{x})$ as that level of satisfaction which, if enjoyed by everybody, will make the existing distribution s(x) socially indifferent. More precisely,

$$S^n(s_e(\mathbf{x})1^n) = S^n(s(\mathbf{x})). \tag{2.2}$$

Since S^n satisfies (ii) and (iii), we can solve (2.2) for the unique representative satisfaction

$$s_e(\mathbf{x}) = E^n(s(\mathbf{x})). \tag{2.3}$$

By continuity of S^n, $E^n(s(\mathbf{x}))$ is continuous. Furthermore, $E^n(s(\mathbf{x}))$ is a particular numerical representation of S^n, that is, for any $\mathbf{x},\mathbf{y} \in D^n$,

$$S^n(s(\mathbf{x})) \geq S^n(s(\mathbf{y})) \Leftrightarrow E^n(s(\mathbf{x})) \geq E^n(s(\mathbf{y})). \tag{2.4}$$

Thus, one satisfaction profile is preferred to another with the same population size if and only if its representative satisfaction is higher. The indifference surfaces of E^n are numbered so that for all c > 0,

$$E^n(s(c1^n)) = c. \tag{2.5}$$

By symmetry and quasi-concavity of S^n, for any $\mathbf{x} \in D^n$, $s_e(\mathbf{x}) \leq \sum_{i=1}^n s_i^a(\mathbf{x})/n \leq \lambda(\mathbf{x}) = s_n^a(\mathbf{x})$.

2.3 Relative Measures of Deprivation

As a general measure of deprivation we suggest the use of $J^n(\mathbf{x})$, the proportionate gap between the representative satisfaction $E^n(s(\mathbf{x}))$ and its maximum attainable value $\lambda(\mathbf{x})$, where the income distribution $\mathbf{x}$ is arbitrary. More precisely,

$$J^n(\mathbf{x}) = 1 - \frac{E^n(s(\mathbf{x}))}{\lambda(\mathbf{x})}. \tag{2.6}$$

For a regular satisfaction function, $J^n(\mathbf{x})$ is continuous, symmetric in incomes and bounded between zero and one, where the lower bound is achieved whenever income (hence satisfaction) levels are equal. When efficiency considerations are absent (that is, mean income is fixed), an increase in J^n is equivalent to a reduction in satisfaction and vice-versa. From policy point of view J^n is a measure of the amount (in proportional terms) by which social satisfaction could be increased if incomes were redistributed equally. Given a functional form for J^n, we can recover E^n as

$$E^n(s(\mathbf{x})) = \lambda(\mathbf{x})(1 - J^n(\mathbf{x})). \tag{2.7}$$

Next, using (2.7), we can retrieve S^n with the help of (2.3) and (2.2). (In fact, S^n is an ordinal transform of E^n, that is, $S^n(s(\mathbf{x})) = f(E^n(s(\mathbf{x}))) = f(\lambda(\mathbf{x})(1 - J^n(\mathbf{x})))$, where f is increasing in its argument.)

In general, J^n is not a relative index. Since $\lambda(\mathbf{x})$ is linearly homogeneous, from (2.6) it follows that J^n will be a relative index, that is, it remains invariant under equiproportionate changes in all incomes, whenever $E^n(s(\mathbf{x}))$ is linearly homogeneous in incomes. This is equivalent to the requirement that S^n is homothetic. The converse is also true, that is, given homotheticity of S^n, J^n becomes a relative index.

To illustrate the general formula in (2.6) let us assume that S^n is the **symmetric mean of order r**, that is,

$$\begin{aligned} E_r^n(s(\mathbf{x})) &= \left(\frac{1}{n}\sum_{i=1}^{n}(s_i^a(\mathbf{x}))^r\right)^{\frac{1}{r}},\ r \leq 1, r \neq 0, \\ &= \prod_{i=1}^{n}(s_i^a(\mathbf{x}))^{\frac{1}{n}}, \quad r = 0. \end{aligned} \tag{2.8}$$

The corresponding deprivation index becomes

$$\begin{aligned} J_r^n(\mathbf{x}) &= 1 - \frac{1}{\lambda(\mathbf{x})}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\frac{\bar{x}_j}{n} + \frac{n-i}{n}\bar{x}_i\right)^r\right)^{\frac{1}{r}},\ r \leq 1, r \neq 0, \\ &= 1 - \frac{1}{\lambda(\mathbf{x})}\prod_{i=1}^{n}\left(\sum_{j=1}^{i}\frac{\bar{x}_j}{n} + \frac{n-i}{n}\bar{x}_i\right)^{\frac{1}{n}}, \quad r = 0. \end{aligned} \tag{2.9}$$

The parameter r in (2.8) determines the curvature of the social satisfaction contour. As r decreases $E_r^n(J_r^n)$ in (2.8)((2.9)) becomes more sensitive to the satisfaction (deprivation) of the poorer persons. For r=1, E_r^n becomes the Gini satisfaction function

$$E_1^n(s(\mathbf{x})) = \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)\bar{x}_i, \tag{2.10}$$

whose associated index in (2.9) is the relative Gini deprivation index $G(\mathbf{x}) = 1 - \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)\bar{x}_i/\lambda(\mathbf{x})$. On the other hand, as r → $-\infty$, $E_r^n \rightarrow min_i s_i^a(\mathbf{x}) = \bar{x}_1$, the **Rawlsian(1971)** maximin satisfaction function and the corresponding index becomes the relative maximin deprivation index $1 - \bar{x}_1/\lambda(\mathbf{x})$.

We now wish to illustrate how one can identify the satisfaction functions associated with a given numerical measure of deprivation. As the first example let us consider the deprivation index $C^n(\mathbf{x})$ suggested in **Chakravarty and Chakraborty(1984)**, where

$$\begin{aligned} C^n(\mathbf{x}) &= \sqrt{}\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}(\bar{x}_j - \bar{x}_i)^2/(n^2\lambda^2(\mathbf{x}))\right) \\ &= \sqrt{}\left(\frac{n-1}{n}((v^n(\mathbf{x}))^2+1) - 2\sum_{i=1}^{n}\bar{x}_i\sum_{j=i+1}^{n}\bar{x}_j/(n^2\lambda^2(\mathbf{x}))\right). \end{aligned} \tag{2.11}$$

where $v^n(\mathbf{x}) = \sqrt{}(\frac{1}{n}\sum(\bar{x}_i/\lambda(\mathbf{x}))^2 - 1)$ is the coefficient of variation. Given other things, an increase in the coefficient of variation increases C^n and vice-versa. Now, C^n is bounded above by $\sqrt{}(n-1)$. This bound is achieved when the entire income is monopolized by the richest person and all the other persons receive zero income. Therefore we consider the transformed index

$C^n(\mathbf{x})/\sqrt{(n-1)}$ which is normalized over [0,1]. Then the satisfaction function corresponding to this normalized index becomes

$$E_c^n(s(\mathbf{x})) = \lambda(\mathbf{x})(1 - C^n(\mathbf{x})/(\sqrt{(n-1)})). \tag{2.12}$$

Thus, given any homothetic satisfaction function we can generate the corresponding deprivation index. Conversely, we can associate a satisfaction function to any deprivation index in a negative monotonic way.

2.4 Absolute Measures of Deprivation

As an alternative to J^n in (2.6) we propose the use of A^n, the absolute shortfall of the representative satisfaction from its maximum attainable value as a general index of deprivation. More precisely,

$$A^n(\mathbf{x}) = \lambda(\mathbf{x}) - E^n(s(\mathbf{x})). \tag{2.13}$$

Given regularity of S^n, A^n is continuous and symmetric in incomes, and has zero as its greatest lower bound, which is achieved whenever incomes are equal. Given a form of A^n, we can recover E^n (hence S^n) with the help of (2.13),(2.3) and (2.2).

Using arguments similar to that employed in section 2.3 we can show that A^n is an absolute index, that is, it remains invariant under equal absolute changes in all incomes, if and only if E^n is unit-translatable (equivalently, S^n is translatable). From policy point of view the absolute index A^n tells us by how much social satisfaction could be increased (in absolute terms) if incomes were redistributed equally. It also gives us the amount of money that must be given to each person to achieve a distribution that generates the same level of social satisfaction as the distribution where each individual receives the present mean income. Thus, the absolute index calculates the per capita cost of deprivation. This is another policy interpretation of A^n.

To illustrate the general formula in (2.13), let us suppose that

$$E_\theta^n(s(\mathbf{x})) = -\frac{1}{\theta} log \frac{\sum_{i=1}^n e^{-\theta s_i^a}}{n}, \tag{2.14}$$

whose associated index becomes

$$A_\theta^n(\mathbf{x}) = \frac{1}{\theta} log \frac{\sum_{i=1}^n e^{\theta(\lambda(\mathbf{x}) - s_i^a)}}{n}, \tag{2.15}$$

where the parameter θ, which determines the curvature of satisfaction contours, is non-negative. As $\theta \to 0$, $E_\theta^n(s(\mathbf{x})) \to$ Gini satisfaction function and the corresponding deprivation index becomes the Gini absolute index of deprivation $\lambda(\mathbf{x}) - \frac{1}{n^2}\sum_{i=1}^n (2(n-i)+1)\bar{x}_i$, the index proposed by Yitzhaki. On the other hand as $\theta \to \infty$, $A_\theta^n(\mathbf{x}) \to \lambda(\mathbf{x}) - \bar{x}_1$, the absolute maximin deprivation index. Note that both the Gini and the maximin relative indices satisfy a compromise property - when multiplied by the mean income they become absolute indices. The associated satisfaction functions are both homothetic and translatable. The compromise property is also satisfied by the index considered in (2.11). In this case the absolute index $\lambda(\mathbf{x})C^n(\mathbf{x})$ becomes monotonically related to the standard deviation $\lambda(\mathbf{x})v^n(\mathbf{x})$, the absolute counterpart to the coefficient of variation.

Thus, we can determine a unique absolute measure of deprivation from a given translatable social satisfaction function. Given a deprivation index we can relate it to a satisfaction function in a negative monotonic way. Choice of a particular satisfaction function is essentially a matter of value judgment.

2.5 A Numerical Illustration of Deprivation Orderings

The purpose of this section is to illustrate the deprivation orderings using statewise per capita expenditure distribution for rural and urban India thrown up by the Indian National Sample Survey Organization for the accounting period January - December, 1992. The states considered are: (1) Andhra Pradesh, (2) Bihar, (3) Karnataka, (4) Madhya Pradesh, (5) Maharashtra, (6) Punjab and (7) Uttar Pradesh.

Table 2.1: Ranking of expenditure distributions in Urban India by deprivation orderings

State	1	2	3	4	5	6	7	
1		x	x	x	<	<	<	
2	x		<	<	<	<	<	
3	x	x		x	<	<	x	A
4	x	x	x		<	<	x	D
5	>	>	>	>		>	>	C
6	x	x	x	x	<		>	
7	x	>	x	x	<	x		
	RDC							

Table 2.2: Ranking of expenditure distributions in Rural India by deprivation orderings

State	1	2	3	4	5	6	7	
1		>	x	x	<	<	<	
2	x		x	<	<	<	<	
3	x	x		<	<	<	<	A
4	>	x	>		<	<	<	D
5	>	>	>	x		<	>	C
6	>	>	>	x	x		>	
7	>	>	>	x	x	<		
	RDC							

The first row and first column of table 2.1 indicate the names of the states considered. In superdiagonal part of the table we present the ranking generated by the ADC criterion between states i and j for urban expenditure distribution, where $i \neq j = 1,2,..,7$. On the other hand, the subdiagonal part shows the ranking results for the RDC criterion using the same data set. We use the following notation: For any column state i and row state j, > means that i dominates j, < indicates that i is dominated by j and x means that the curves for the states under comparison cross. For instance, in table 2.1, Uttar Pradesh dominates Bihar but is dominated by Maharashtra by the ADC criterion.

Certain interesting features emerge from table 2.1. Out of ${}^7C_2 = 21$ comparable situations, the ADC criterion generates unambiguous ranking in 15 cases in contrast to the 7 unambiguous rankings shown by the RDC criterion. Thus, many ambiguous comparisons in the RDC case become Unambiguous in the ADC case. The reasoning behind this improved performance of the ADC criterion is that the multiplication of the RDC by the mean income can sufficiently offset the lower part of the RDC so that the distributions with intersecting RDCs may have non-intersecting ADCs. This particularly holds when we consider the ranking of Punjab with other states - in the situation of RDC comparison out of 7 only 2 cases are unambiguous, whereas the ADC criterion shows that 6 cases are unambiguous. We note that Maharashtra dominates all the other six states by both ADC and RDC criterion. Though mean income for Maharashtra is higher than that for any other state, this dominance is a consequence of the concentration of incomes in Maharashtra in high income brackets. On the other extreme we have Bihar which is dominated by maximum number of states by either of the criteria. Inspite of the fact that Bihar has minimum average income among the states considered, its undomination follows from low income dispersion. Therefore, the equity-efficiency trade-off involved in the context of ranking income distributions shows a clear bias towards equity if the ranking relation used is the ordering $\geq_{ADC}$.

Table 2.2, whose format is similar to that of table 2.1, presents the ranking for rural expenditure data. Here the ADC criterion generates unambiguous ranking in 18 out of 21 cases and the RDC criterion shows 12 unambiguous rankings. Thus, the number of unambiguous rankings by both criteria is higher in the rural sector than in the urban sector. Here it turns out that Punjab dominates all other states by the ADC principle and four states by the RDC principle. This again is a consequence of concentration of incomes in high income brackets. On the other extreme Bihar and Karnataka is dominated by the maximum number of states by the ADC criterion and the RDC criterion respectively.

2.6 Deprivation Judgement in the Presence of Contaminated Data

Formal propositions on income distributions can be satisfactorily formulated for empirical purposes only if the data are reasonable representation of the underlying income distributions. It is well known that in practice there is contamination of data by recording errors, measurement errors etc. and if removal of those is not possible, we should at least be clear about their possible influence on our conclusions. Practitioners are often confronted with such problems. It is our aim, in this section, to examine one possible consequence of using contaminated data.

Here we consider income distribution $\mathbf{x} = (x_1, x_2, .., x_n) \in D^n$ for some n fixed. Assume that the true distribution of income is F. The empirical distribution is then

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_x(x_i)$$

where $I_x(x_i) = 1$ if $x_i \leq x$ and 0 otherwise.

The two deprivation orderings would then be a function of F. With the sample observations, the deprivation values $d_i(\mathbf{x})$ are defined as in equation (1.20)

$$d_i^a(\mathbf{x}) = \frac{1}{n} \sum_{j=i+1}^{n} (\bar{x}_j - \bar{x}_i).$$

This can be rewritten in terms of F_n as

$$d_i^F(\mathbf{x}) = \int (x - \bar{x}_i) I_x(\bar{x}_i) dF_n(x). \tag{2.16}$$

In order to represent the impact of contamination on an income distribution, we need a specific model of the contamination. Consider the distribution $H^{(z)}$ which is given by

$$H^{(z)}(x) = I_x(z). \tag{2.17}$$

We use this to model an elementary form of contamination by introducing a mixture distribution

$$F_\epsilon^{(z)}(x) = (1 - \epsilon)F(x) + \epsilon H^{(z)}(x). \tag{2.18}$$

The parameter ϵ $(0 < \epsilon < 1)$ captures the importance of the contamination relative to the true distribution. Hence, an observation drawn from $F_\epsilon^{(z)}$ has probability (1-ϵ) of being generated by F (the true distribution) and probability ϵ of being equal to z. We define d_i^a corresponding to the distribution F_ϵ as

$$\begin{aligned} d_i^{F_\epsilon}(\mathrm{x}) &= \int (x - \bar{x}_i) I_x(\bar{x}_i) dF_{n\epsilon}(x) \\ &= \int (x - \bar{x}_i) I_x(\bar{x}_i) d[(1-\epsilon)F_n(x) + \epsilon H^{(z)}(x)] \qquad (2.19) \\ &= (1-\epsilon) \int (x - \bar{x}_i) I_x(\bar{x}_i) dF_n(x) + \epsilon (z - \bar{x}_i) I_z(\bar{x}_i). \end{aligned}$$

We now introduce a measure of this importance by the following function which is called the *influence function* IF, which has been introduced in the statistical literature by **Huber (1964), Hampel (1968, 74)** in the framework of robust statistics. It has been further studied in **Hampel, Ronchetti, Rousseauw and Stahel (1986), Heritier and Ronchetti (1994), Victoria-Feser (1995)** among others. IF is defined by

$$IF(F_\epsilon^{(z)}) = lim_{\epsilon \to 0} \frac{d_i^{F_\epsilon}(\mathrm{x}) - d_i^F(\mathrm{x})}{\epsilon}. \qquad (2.20)$$

This function measures the impact of an infinitesimal amount of contamination at z upon the estimate. The impact of such contamination upon inequality measures, poverty measures and estimation of income distributions have been studied in **Cowell and Victoria-Feser (1994, 96, 96a)** and **Victoria-Feser (1993).**

Now we substitute $d_i^a(x)$ from (1.20) in $IF(F_\epsilon^{(z)})$ to obtain

$$IF(F_\epsilon^{(z)}) = lim_{\epsilon \to 0} \frac{(1-\epsilon) \int (x - \bar{x}_i) I_x(\bar{x}_i) dF_n(x) + \epsilon (z - \bar{x}_i) I_z(\bar{x}_i) - \int (x - \bar{x}_i) I_x(\bar{x}_i) dF_n(x)}{\epsilon}. \qquad (2.21)$$

Hence (2.21) simplifies to,

$$\begin{aligned} IF(F_\epsilon^{(z)}) &= lim_{\epsilon \to 0} (z - \bar{x}_i) I_z(\bar{x}_i) - d_i^F(x) \qquad (2.22) \\ &= (z - \bar{x}_i) I_z(\bar{x}_i) - d_i^F(x), \qquad (2.23) \end{aligned}$$

as (2.22) is independent of ϵ.

Now the central issue is that, if the contamination value is large relative to the true data, will the calculated deprivation values be very misleading? This can be examined if we take the limit of (2.23) as $z \to \infty$ (that is, the contamination is arbitrarily large).

Now, if $z \to \infty$ then eventually $I_z(\bar{x}_i) = 1$ (as $z > \bar{x}_i$ occurs). Therefore $IF(F_\epsilon^{(z)})$ becomes

$$IF(F_\epsilon^{(z)}) = z - \bar{x}_i - d_i^F(x). \qquad (2.24)$$

Now as $z \to \infty$, LHS of (2.24) $\to \infty$. Hence the IF is unbounded in this case. This is a caution towards the use of contaminated data in deprivation measurement as the calculated deprivation values become very misleading. It can be shown that even if we use the normalized deprivation values as defined in (1.16),

$$d_i(\mathbf{x}) = \frac{1}{n} \sum_{j=i+1}^{n} \frac{\bar{x}_j - \bar{x}_i}{\lambda(x)},$$

the IF still remains unbounded. In this context, it might be of interest to note that the most popular measure of inequality, the Gini inequality index, which is also a measure of aggregate deprivation, has an unbounded influence function (**Cowell and Victoria-Feser (1996a)**). *Lorenz curve* and the *absolute Lorenz curve* both has unbounded IF, so does most of the second order stochastic dominance criteria excepting for some very restricted cases. (See **Cowell and Victoria-Feser (1996a)**).

2.7 Conclusion

In this chapter we have proposed general indices of relative (absolute) notions of deprivation using social satisfaction functions. We have shown that to every homothetic (translatable) social satisfaction function, there corresponds a unique relative (absolute) index of deprivation. Conversely, for each deprivation index a social satisfaction function can be found that imply the index. We have analyzed a number of indices of deprivation along this line. In particular, we take the Gini coefficient, the maximin index, the coefficient of variation and their absolute versions. Which social satisfaction function will be adopted becomes an issue of value judgment. An empirical illustration of the two deprivation orderings has also been provided in the chapter. The performance of the absolute deprivation ordering is found to be better than the relative deprivation ordering. Finally, we examine the impact of using contaminated data on deprivation orderings.

Chapter 3

Deprivation Reducing Income Tax Functions[1]

3.1 Introduction

A great deal has been written by economists on the relationship between tax progressivity and income inequality when the pre-tax income distribution remains fixed. As stated in section 1.3, **Jakobsson (1976)**, **Kakwani (1984)**, **Eichhorn et.al. (1984)** and **Thon (1987)**, among others, have demonstrated that under certain circumstances non-decreasing average tax rate will make the post-tax income distribution at least as equitable as the pre-tax income distribution according to the Lorenz criterion. The absolute version of the above mentioned progressivity result was demonstrated by **Moyes (1988)**.

A recent paper by **Moyes (1994)** makes some further investigations along this line in a quite general set-up. In particular, Moyes analyses the redistributive effect of transformations of income when the same transformation is applied to all incomes. More precisely, he considered the absolute and relative approaches to the measurement of inequality and within each framework he chose three nested criteria. He then shows that within each set-up all the three criteria generate exactly the same class of transformations so that the transformed distribution becomes at least as equitable as the original one.

In this chapter we consider the deprivation orderings introduced in section 1.2. Building on the paper by **Moyes (1994)** we then look for the tax functions that will make the post-tax income distribution at least as satisfied as (no more deprived than) the pre-tax distribution according to these two criteria.

The chapter is organized as follows: we begin section 3.2 by specifying some results of Moyes concerning inequality reducing transformations. These results become helpful in determining the deprivation reducing tax functions. Section 3.3 presents the main results of the chapter and

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[1] This chapter is based on **Mukherjee (1997)**.

finally section 3.4 concludes.

3.2 Notation, Definitions and Preliminaries

With the notation introduced in chapter 1, following **Marshall and Olkin (1979)**,we consider two orderings that are consistent with the Lorenz ordering. According to the first one the distribution **x** dominates distribution **y** in the *relative differentials* , $\mathbf{x} \geq_{RD} \mathbf{y}$ for short, if

$$\frac{\bar{x}_i}{\bar{y}_i} \geq \frac{\bar{x}_{i+1}}{\bar{y}_{i+1}} \tag{3.1}$$

for all i=1,..,n-1. That is, under distribution **y** the ratio of any two consecutive incomes taken in non-decreasing order does not exceed the corresponding ratio under distribution **x**.

Similarly the second ordering says that the distribution **x** dominates distribution **y** in the *absolute differentials* , $\mathbf{x} \geq_{AD} \mathbf{y}$ for short,if

$$\bar{x}_i - \bar{y}_i \geq \bar{x}_{i+1} - \bar{y}_{i+1} \quad . \tag{3.2}$$

for all i = 1,..,n-1. Here we require that under distribution **y** the differences between any two consecutive incomes arranged in non-decreasing order should not be greater than the corresponding difference under distribution **x**.

Let us now say that the distribution **x** dominates distribution **y** according to the *single-crossing condition*, $\mathbf{x} \geq_S \mathbf{y}$ for short,if there exists $k\,(1 \leq k \leq n)$ such that

$$\bar{x}_i - \bar{y}_i \geq 0 \; for \; all \; 1 \leq i < k \tag{3.3}$$

and

$$\bar{x}_i - \bar{y}_i \leq 0 \; for \; all \; k \leq i \leq n. \tag{3.4}$$

This condition says that we can partition the population into two groups such that in the first group any individual's income in **x** is at least as large as that in **y**, while the reverse situation arises for any individual in the second group.

We say that the distribution **x** dominates distribution **y** according to the *absolute single-crossing condition*, $\mathbf{x} \geq_{AS} \mathbf{y}$ for short,if $\tilde{\mathbf{x}} \geq_S \tilde{\mathbf{y}}$. Similarly, distribution **x** dominates distribution **y** according to the *relative single-crossing condition*, $\mathbf{x} \geq_{RS} \mathbf{y}$ for short,if $\widehat{\mathbf{x}} \geq_S \widehat{\mathbf{y}}$. (**Thistle (1989)** provides further discussions on single crossing condition.)

The following remark made by Moyes concerning relationship between alternative dominance criteria will be useful for our results to be presented in the next section.

Remark 1 (Moyes, 1994):

$$\geq_{RD} \Rightarrow \geq_L \; and \geq_{AD} \Rightarrow \geq_{AL} \quad for \; n \geq 3.$$

For a tax function f, as defined in section 1.3, we have

Proposition 1 (Moyes, 1994): Suppose r = RD,RS,L. Then $f(u)/u$ is non-increasing in u *if and only if* for all $\mathbf{y} \in D^n$, $f^{(n)}(\mathbf{y}) \geq_r \mathbf{y}$.

Proposition 2 (Moyes, 1994): Suppose r = AD,AS,AL. Then $f(u) - u$ is non-increasing in u *if and only if* for all $\mathbf{y} \in D^n$, $f^{(n)}(\mathbf{y}) \geq_r \mathbf{y}$.

Note that Proposition 2 becomes the **Moyes (1988)** minimal progressivity result (Proposition 7 of Chapter 1) if r = AL. Similarly Proposition 1 corresponds to the average progressivity Proposition if r = L.

3.3 The Results

To facilitate presentation of our subsequent more general results we first prove a comparatively restricted result. We state the result as follows.

Lemma 1: Let $\mathbf{x}, \mathbf{y} \in D^n$ be arbitrary. Then $\mathbf{x} \geq_{AD} \mathbf{y}$ implies that $\mathbf{y} \geq_{ADC} \mathbf{x}$ but the converse is not true.

Proof:

$$\begin{aligned} & \mathbf{x} \geq_{AD} \mathbf{y} \\ \Rightarrow\ & (\bar{x}_i - \bar{y}_i) \ is\ non-increasing\ in\ i, \quad i = 1,..,n \\ \Rightarrow\ & \sum_{j=i+1}^{n} (\bar{x}_j - \bar{y}_j) \leq (n-i)(\bar{x}_i - \bar{y}_i) \quad for\ all\ i = 1,..n. \\ \Leftrightarrow\ & \frac{\sum_{j=i+1}^{n}(\bar{x}_j - \bar{x}_i)}{n} \leq \frac{\sum_{j=i+1}^{n}(\bar{y}_j - \bar{y}_i)}{n} \quad for\ all\ i = 1,..n \\ \Leftrightarrow\ & \mathbf{y} \geq_{ADC} \mathbf{x} \end{aligned}$$

To prove the converse we consider the example $\mathbf{x} = (2,5,7,10)$ and $\mathbf{y} = (2,7,7,8)$. Then $\mathbf{y} - \mathbf{x} = (0,2,0,-2)$ and it is easy to see that $\mathbf{x} \geq_{ADC} \mathbf{y}$ but not $[\mathbf{y} \geq_{AD} \mathbf{x}]$. ■

Remark 2: Note that if $\lambda(\mathbf{x}) = \lambda(\mathbf{y})$, then $\mathbf{y} \geq_{RDC} \mathbf{x}$ is same as $\mathbf{y} \geq_{ADC} \mathbf{x}$. Therefore, we can restate **Lemma 1** by replacing ADC by RDC.

In the following proposition we identify the class of tax functions that makes the post-tax distribution not more deprived than the pre-tax distribution according to the ADC criterion.

Proposition 3: The following statements are equivalent:

(a) f(x) - x is non-increasing in x.

(b) For all $\mathbf{x} \in D^n$, $f^{(n)}(\mathbf{x}) \leq_{ADC} \mathbf{x}$.

Proof:

(a) ⇒ (b): Statement (a) implies that $f^{(n)}(\mathbf{x}) \geq_{AD} \mathbf{x}$, for all $\mathbf{x} \in D^n$. Appealing to Lemma 1, we obtain (b).

(b) ⇒ (a): If (b) holds then we have

$$\sum_{j=i+1}^{n} [f(\bar{x}_j) - f(\bar{x}_i)] \leq \sum_{j=i+1}^{n} [\bar{x}_j - \bar{x}_i], \tag{3.5}$$

for all i=1,2,..,(n-1), and all $\mathbf{x} \in D^n$. Choosing $\mathbf{x}$ = (u,v,..,v) $\in D^n$ we have for i=1, (n-1)[f(u)-u] ≥ (n-1)[f(v)-v]. Since this is true for all u < v, we have proved that (a) must hold. ■

Proposition 3 says that the pre-tax distribution **x** dominates the corresponding post-tax distribution according to the ADC rule in (1.21) if and only if taxes are non-decreasing in incomes, a requirement that parallels the **Moyes (1988)** minimal progressivity result.

In the next proposition we determine the tax functions that agree with the relative deprivation criterion. Before stating the proposition we need to impose a further condition on the post tax income functions, which is

$$Lim_{x \to \infty} f'(x) = Sup_{x \in D^1}\{f'(x)\}. \tag{3.6}$$

We refer to the function f' as the marginal post-tax income function. It indicates how much is left to the taxpayer on the margin if post-tax income increases by one unit. The assumption (3.6) then means that the *least upper bound* of the marginal post-tax income function, which may be one or a positive number less than one, is attained as pre-tax income increases asymptotically. We now have,

Proposition 4: The following statements are equivalent:

(a) $Inf_{x \in D^1}\{\frac{f(x)}{x}\} \geq Sup_{x \in D^1}\{f'(x)\}$.

(b) For all $\mathbf{x} \in D^n$, $f^{(n)}(\mathbf{x}) \leq_{RDC} \mathbf{x}$.

To interpret condition (a), note that we can rewrite this condition in terms of the tax function T(x) = x - f(x) as

$$Sup_{x \in D^1}\{\frac{T(x)}{x}\} \leq Inf_{x \in D^1}\{T'(x)\}. \tag{3.7}$$

That is, the ***greatest lower bound*** of the marginal tax function is at least as large as the ***least upper bound*** of the average tax function. Now, progressivity in the average sense requires that marginal tax function is nowhere smaller than average tax function. Condition (a), in addition to ensuring this, imposes some further egalitarian restrictions on the tax function in the sense that in this case we can always separate the average tax rate from the marginal tax rate by some constant value. However in the **average progressivity** case this may not be possible. The following are two examples of this type of tax functions.

Example 1: f(x) = a + bx where b ∈ (0,1) and a > 0.

Example 2: f(x) = a + bx + $\frac{c}{x+1}$ where a > 0 and 0 < c ≤ b < 1.

Proof of the proposition:

(a) ⇒ (b): We need to show that for any n ≥ 2 and for any $\mathbf{x} \in D^n$,

$$\frac{\sum_{j=i+1}^{n}[f(\bar{x}_j) - f(\bar{x}_i)]}{\sum_{i=1}^{n} f(\bar{x}_i)} \leq \frac{\sum_{j=i+1}^{n}[\bar{x}_j - \bar{x}_i]}{\sum_{i=1}^{n} \bar{x}_i}, \tag{3.8}$$

for i=1,..,n-1 with < for some i. This implies and is implied by the condition

$$\frac{\sum_{j=i+1}^{n}[f(\bar{x}_j)-f(\bar{x}_i)]}{\sum_{j=i+1}^{n}[\bar{x}_j-\bar{x}_i]} \leq \frac{\sum_{i=1}^{n} f(\bar{x}_i)}{\sum_{i=1}^{n} \bar{x}_i}, \tag{3.9}$$

for i=1,..,n-1 with < for some i.

Now, using the *Mean Value Theorem (MVT)* we have $f(\bar{x}_j)-f(\bar{x}_i)=(\bar{x}_j-\bar{x}_i)f'(\theta_j)$ for some $\bar{x}_i < \theta_j < \bar{x}_j$. Hence we can rewrite (3.9) as,

$$\frac{\sum_{j=i+1}^{n}[\bar{x}_j-\bar{x}_i]f'(\theta_j)}{\sum_{i=1}^{n}[\bar{x}_j-\bar{x}_i]} \leq \frac{\sum_{i=1}^{n} f(\bar{x}_i)}{\sum_{i=1}^{n} \bar{x}_i}, \tag{3.10}$$

for i=1,..,n-1 with < for some i.

It is evident that,

$$\frac{\sum_{j=i+1}^{n}[\bar{x}_j-\bar{x}_i]f'(\theta_j)}{\sum_{i=1}^{n}[\bar{x}_j-\bar{x}_i]} \leq Max_{j=i+1}^{n}\{f'(\theta_j)\} \leq Sup_{x\in D^1}\{f'(x)\} \tag{3.11}$$

$$and\ \frac{\sum_{i=1}^{n} f(\bar{x}_i)}{\sum_{i=1}^{n} \bar{x}_i} \geq Min_{j=i+1}^{n}\{\frac{f(\bar{x}_i)}{\bar{x}_i}\} \geq Inf_{x\in D^1}\{\frac{f(x)}{x}\}. \tag{3.12}$$

Therefore, given condition (a), inequalities (3.11) and (3.12) together imply (3.10). But (3.10) is equivalent to condition (b). Hence (a) ⇒ (b).

(b) ⇒ (a): To show (b) ⇒ (a), let us suppose that (a) does not hold. Then it is enough to show that (b) does not hold.

Since (a) does not hold, we have $Inf_{x\in D^1}\{\frac{f(x)}{x}\} < Sup_{x\in D^1}\{f'(x)\}$ = L (say). Then there exists y such that $\frac{f(y)}{y}$ < L and this implies that there exists x,y such that $\frac{f(y)}{y} < f'(x)$. Now, one of the following cases must hold.

Case (i): x = y, that is, there exists x such that $\frac{f(x)}{x} < f'(x)$.

Then we can find u ∈ $(x-\epsilon, x+\epsilon)^n \subset D^n$ for some ϵ small (> 0) such that $\frac{\sum_{j=i+1}^{n}[f(\bar{u}_j)-f(\bar{u}_i)]}{\sum_{j=i+1}^{n}[\bar{u}_j-\bar{u}_i]}$ can be made > $\frac{\sum f(\bar{u}_i)}{\sum \bar{u}_i}$ for some i=1,..,(n-1), where $(x-\epsilon, x+\epsilon)^n$ is the n-fold cartesian product of $(x-\epsilon, x+\epsilon)$. This in turn implies that $\frac{\sum_{j=i+1}^{n}[f(\bar{u}_j)-f(\bar{u}_i)]}{\sum f(\bar{u}_i)} > \frac{\sum_{j=i+1}^{n}[\bar{u}_j-\bar{u}_i]}{\sum \bar{u}_i}$ for some i. And this implies not[$f^{(n)}$(**x**) $\geq_{RDC}$ **x**]. (Essentially we are choosing u such that the rate of increase of the numerator is > that of the denominator in the above expression.)

Case (ii): $\frac{f(y)}{y} < f'(x)$ for some x > y.

Now, given x, there are two possibilities, either $f'(x) > \frac{f(x)}{x}$ (which is case (i) and hence already considered), or $f'(x) \leq \frac{f(x)}{x}$. Therefore, we consider only the situation $\frac{f(x)}{x} \geq f'(x) > \frac{f(y)}{y}$. Hence, again by MVT with the function $\frac{f(u)}{u}$, there exists $z \in (x,y)$ such that $(\frac{f(z)}{z})' > 0$ or $f'(z) > \frac{f(z)}{z}$ which is case (i) with x = y = z. This contradicts (b).

Case (iii): $\frac{f(y)}{y} < f'(x)$ for some x < y.

In view of condition (3.6), which says that $f'(u)$ tends to it's maximum value as $u \to \infty$, we have $Lim_{p\to\infty}f'(p) \geq f'(x) > \frac{f(y)}{y}$. So there exists z > y such that $f'(z) > \frac{f(y)}{y}$. This means that we now go back to case (ii). Hence, in this case also, (b) does not hold.

Thus we have shown that if (a) does not hold then neither does (b). Thus (b) $\Rightarrow$ (a). ■

Remark 3: In view of its equivalence with condition (a) of proposition 4, which implies non-increasingness of $\frac{f(x)}{x}$, RDC-domination implies Lorenz-domination. But the converse is not true.

Remark 4: For a pair of transformations $f^{(n)}$ and $g^{(n)}$ $(:D^n \rightarrow D^n)$ on the income distributions, we have,

(a) $f^{(n)}(\mathbf{x}) \leq_{ADC} g^{(n)}(\mathbf{x})$ for all $\mathbf{x} \in D^n \Leftrightarrow$ f(u) - g(u) is non-increasing in u.

(b) $f^{(n)}(\mathbf{x}) \leq_{RDC} g^{(n)}(\mathbf{x})$ for all $\mathbf{x} \in D^n \Leftrightarrow Inf_{x \in D^1}\{\frac{f(x)}{g(x)}\} \geq Sup_{x \in D^1}\{\frac{f'(x)}{g'(x)}\}$ (under the condition that $Lim_{x \rightarrow \infty}\frac{f'(x)}{g'(x)} = Sup_{x \in D^1}\{\frac{f'(x)}{g'(x)}\}$).

3.4 Conclusions

Jakobsson (1976), Kakwani (1984) and **Eichhorn et.al. (1984)** indicated that an average progressive taxation scheme is uniformly equalizing according to the Lorenz criterion. Later **Moyes (1988, 1994)** showed that a minimally progressive tax function makes the post-tax income distribution at least as equitable as the pre-tax income distribution according to the absolute Lorenz criterion. In this chapter we consider two concepts of deprivation domination principle following **Runciman (1966), Kakwani (1984)** and **Temkin (1986)**. We then try to determine conditions to be imposed on a tax function for the resulting income distribution to be less deprived than the original one. While in the former case the tax function turns out to be non-decreasing in incomes, in the later case a stronger condition than **average progressivity** is required, which states that the marginal retention rate is not greater than the average retention rate at all possible income levels.

Chapter 4

Lorenz Domination, Utilitarian Deprivation Rule and Equal Sacrifice Principle[1]

4.1 Introduction

As stated earlier, the relationship between income discrepancy and tax progressivity has been investigated by many authors. In section 1.3 we have discussed about the equivalence between the **Lorenz** dominance criterion and tax progressivity (Proposition 6 of Chapter 1). But the Lorenz criterion gives us only one notion of distributive justice, an alternative basis for the tax policy is the principle of **equal absolute (proportional) sacrifice**, which states that all taxpayers should suffer the same absolute (proportional) utility loss due to taxation. This traditional justification for progressive taxation is one of the earliest solutions proposed for the problem of vertical equity in income taxation. However, it was widely believed that this principle involves interpersonal comparison of welfare. To apply the equal sacrifice approach in a real set-up, a particular form of utility function would have to be specified and in practice the same form would have to hold for all taxpayers. **Young(1988)** has shown that the equal sacrifice approach can be justified on entirely different grounds than interpersonal comparison of welfare. More precisely, he showed that a set of reasonable and non-utilitarian axioms imply the existence of a utility function relative to which everyone suffers the same loss of welfare. **Ok(1995)** demonstrated that under certain minor restrictions, a progressive tax function equalizes the level of absolute or proportional sacrifice according to a continuous, increasing and concave utility function.

We can also think of a third type of distributive justice by which a tax policy can be formulated. It is based on a ranking relation suggested and characterized by **Hey and Lambert**

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[1] This chapter is based on **Chakravarty and Mukherjee (1997)**.

(1980), which, itself, builds on an idea of **Runciman(1966)** and **Yitzhaki (1979)**. Following **Hey and Lambert (1980)**, we measure the aggregate deprivation felt by a person as the sum of his income utility shortfalls from richer individuals. We will say that of two income distributions x and y over a given population size, x dominates y by the **utilitarian deprivation criterion** if, for any person, the sum of utility shortfalls under y does not exceed the corresponding shortfall sum under x. The distributive justice we consider then is that the pre-tax income distribution should dominate the post-tax distribution according to the utilitarian deprivation rule.

The purpose of this chapter is to establish some interrelationship between the three types of distributive justice mentioned above. More precisely, we show that, if the relative risk aversion associated with the income utility function does not exceed one, then any weakly incentive preserving and weakly average progressive tax function will make the pre-tax distribution at least as deprived as the post-tax distribution according to the utilitarian deprivation rule. The converse holds if the risk aversion index is not smaller than one. These two results indicate that the deprivation rule becomes consistent with the Lorenz criterion if relative risk aversion is one, that is, the utility function is of logarithmic type. Next, it is shown that the only tax function which represents equal sacrifice with respect to the logarithmic utility function is the flat tax function. Thus, the logarithmic utility function and the flat tax function are the vehicles which connect the three types of distributive justice considered here.

The chapter is organized as follows. In the next section we present notation, some definitions and preliminaries. **Section 4.3** presents the main findings. Finally, **section 4.4** concludes.

4.2 Notation, Definitions and Preliminaries

Throughout this chapter we will assume that the post-tax income function f, as defined in section 1.3, satisfies **IP**. Before demonstrating our results, we need to impose one further condition on f. It is well-known that for sacrifice to make sense, we need positivity of taxes. Since we will be relating our results to equal sacrifice principle, in this chapter, we will assume that all individuals are taxpayers, that is $T(u) = u - f(u) > 0$ for all u > 0.

Following **Hey and Lambert (1980)**, we can say that, the deprivation felt by person i due to shortfall of his income $\bar{x}_i$ from the j^{th} person's income $\bar{x}_j$ ($\bar{x}_i \leq \bar{x}_j$) is given by the utility differential $U(\bar{x}_j) - U(\bar{x}_i)$, where U is the identical utility function of the individuals in the society. U is assumed to be increasing, concave and twice differentiable. If we assume $U(z) = z$, then the utility differential becomes the simple income differential ($\bar{x}_j - \bar{x}_i$). Since in the ordered distribution **x**, the person i is deprived of all incomes $\bar{x}_{i+1}, .., \bar{x}_n$, the total deprivation felt by this person is

$$d_i^u(\mathbf{x}) = \sum_{j=i+1}^{n} (U(\bar{x}_j) - U(\bar{x}_i)). \tag{4.1}$$

We say that for **x**,**y** $\in D^n$, **x** dominates **y** by the **utilitarian deprivation criterion**,

$\mathbf{x} \geq_{D(U)} \mathbf{y}$ for short,if for all i = 1,..,n-1,

$$\sum_{j=i+1}^{n} [U(\bar{x}_j) - U(\bar{x}_i)] \geq \sum_{j=i+1}^{n} [U(\bar{y}_j) - U(\bar{y}_i)]. \tag{4.2}$$

That is, for any person, the aggregate deprivation in terms of utility shortfall is not larger under **y** than under **x**.

If U is *affine*, that is, $U(z) = a + bz$ where b > 0, then the ordering (4.2) becomes the absolute deprivation ordering defined in (1.21).

The relation $\geq_{D(U)}$ remains invariant under *affine* transformations of U. That is, for any scalars a,b where b > 0, $\mathbf{x} \geq_{D(U)} \mathbf{y}$ coincides with $\mathbf{x} \geq_{D(a+bU)} \mathbf{y}$.

We define the **Arrow-Pratt relative risk aversion measure** r(U) for a utility function U by

$$r(U) = -zU''(z)/U'(z). \tag{4.3}$$

Evidently, $r(U) = r(a + bU)$, where a and b are scalars, b > 0.

As stated earlier, one of the fundamental concepts of distributive justice is that of equal absolute (proportional) sacrifice. Equal absolute (proportional) sacrifice states that everyone foregoes the same amount (proportion) of utility in paying taxes.

We say that t is an equal absolute sacrifice tax function with respect to a utility function U with properties as above if there exists a constant c > 0 such that

$$U(x) - U(x - t(x)) = c \quad \textit{for all } x > 0. \tag{4.4}$$

Analogously, t satisfies equal proportional sacrifice principle if there exists a c > 1 such that for some U (assuming that U is positive),

$$U(x)/U(x - t(x)) = c \quad \textit{for all } x > 0. \tag{4.5}$$

Clearly, if (4.4) holds for some U, then (4.5) holds for e^U and conversely if (4.5) holds for some U, then (4.4) holds for $logU$. In view of this equivalence, from now on we will deal with equal absolute sacrifice principle only. This way of defining equal absolute (proportional) sacrifice can be interpreted by considering the utility function U as representing the preferences of a representative person of the society, and thereby acting as a *social norm* (**Musgrave(1959) and Young(1990)**). Interpreted this way, the equal sacrifice principles do not subsume interpersonal utility comparisons.

4.3 The Results

In this section we first identify a sufficient condition on utility functions so that D(U)-dominance is implied by Lorenz-dominance. This condition is imposed through the measure r(U) as defined in (4.3).

Proposition 1: If the utility function U(x) is such that $r(U) \leq 1$, then for all $n \geq 2$, for all $\mathbf{x} \in D^n$,

$$f^{(n)}(\mathbf{x}) \geq_L \mathbf{x} \textit{ implies } f^{(n)}(\mathbf{x}) \leq_{D(U)} \mathbf{x}. \tag{4.6}$$

Proof: For $f^{(n)}(\mathbf{x}) \leq_{D(U)} \mathbf{x}$ to hold we need,

$$\sum_{j=i+1}^{n} [U(\bar{x}_j) - U(\bar{x}_i)] \geq \sum_{j=i+1}^{n} [U(f(\bar{x}_j)) - U(f(\bar{x}_i))] \tag{4.7}$$

for i = 1,..,n-1, for all $n \geq 2$, for all $\mathbf{x} \in D^n$. So in particular, taking n = 2 and pre-tax income distribution (y,x), we see that the requirement (4.7) is equivalent to the requirement

$$U(x) - U(y) \geq U(f(x)) - U(f(y)), \tag{4.8}$$

for all $x \geq y > 0$. We may write (4.8) as

$$\frac{U(x) - U(y)}{x - y} \geq \frac{U(f(x)) - U(f(y))}{f(x) - f(y)} \frac{f(x) - f(y)}{x - y}, \tag{4.9}$$

for all $x \geq y > 0$, as f satisfies IP (assumed). Now as x, y are arbitrary we may take $x \to y$ and then taking limit and using the continuity of f, we see that the requirement (4.7) implies,

$$U'(x) \geq U'(f(x))f'(x), \tag{4.10}$$

for all x > 0. To show the reverse implication, let us take definite integrals of both sides of (4.10) from lower limit y to upper limit x to obtain

$$\int_y^x U'(t)dt \geq \int_y^x U'(f(t))f'(t)dt,$$

which is equation (4.8).

Now define U'(x) = V(x) for all x > 0. Increasingness and concavity of U implies that V is positive and non-increasing for all x > 0. Then (4.10) may be rewritten as

$$V(x) \geq V(f(x))f'(x), \tag{4.11}$$

for all x > 0.

Now for any f we define W(x) = x V(x). Then (4.11) becomes,

$$\frac{W(x)}{x} \geq \frac{W(f(x))}{f(x)} f'(x). \tag{4.12}$$

This is same as

$$W(x)\frac{f(x)}{x} \geq W(f(x))f'(x). \tag{4.13}$$

Firstly we see that $\frac{f(x)}{x} \geq f'(x)$, as f satisfies AP (Proposition 6 of chapter 1). Now notice that if $r(U) \leq 1$ then W is a non-decreasing function of x and hence, as x > f(x), we have $W(x) \geq W(f(x))$. So (4.11) holds. ■

The above proposition shows that if we assume the tax function to satisfy IP and AP then under the condition of $r(U) \leq 1$, we have D(U)-dominance.

In the next proposition we isolate the utility functions for which the converse of proposition 1 holds.

Proposition 2: If the utility function U(x) is such that $r(U) \geq 1$, then for all $n \geq 2$, for all $\mathbf{x} \in D^n$,

$$f^{(n)}(\mathbf{x}) \leq_{D(U)} \mathbf{x} \textit{ implies } f^{(n)}(\mathbf{x}) \geq_L \mathbf{x}. \tag{4.14}$$

Proof : From the proof of Proposition 1 we can say that $f^{(n)}(\mathbf{x}) \leq_{D(U)} \mathbf{x}$ is equivalent to $W(x)\frac{f(x)}{x} \geq W(f(x))f'(x)$ (equation (4.13)). Now $r(U(x)) \geq 1$ implies that $W(x)$ is non-increasing. Since $x > f(x)$ for all x > 0, we must have

$$W(x) \leq W(f(x)). \tag{4.15}$$

Hence, from (4.13) and (4.15), $\frac{f(x)}{x} \geq f'(x)$ for all x > 0, which means that **AP** holds. Thus, from Proposition 6 of chapter 1, $f^{(n)}(\mathbf{x}) \geq_L \mathbf{x}$ (since **IP** has been assumed). ■

In proving propositions 1 and 2 we have made specific assumptions about the Arrow-Pratt measure of relative risk aversion. The following proposition shows that propositions 1 and 2 may not hold if these assumptions are not fulfilled.

Proposition 3: (a) if r(U) > 1, then proposition 1 need not hold. (b) If r(U) < 1, then proposition 2 need not hold.

Proof : The proof consists of two examples. As the first example let us consider the tax function $f(x) = 0.95x$. We can easily check that this tax function satisfies **MP, IP** and **AP.** So for this tax function Lorenz dominance holds. Take the utility function $U(x) = 1 - e^{-x}$. It can be shown that, for this utility function, $r(U) = x$, so $r(U) > 1$ if $x > 1$.

Now, for D(U)-dominance to hold for all $n \geq 2$ and for all $x \in D^n$, we need $W(x)\frac{f(x)}{x} \geq W(f(x))f'(x)$ for all $\mathbf{x} > 0$ (equation (4.13)). Here $W(x) = xe^{-x}$. So, $W(x)\frac{f(x)}{x} = 0.95xe^{-x}$ and $W(f(x))f'(x) = (0.95)^2xe^{-0.95x}$. Then $W(x)\frac{f(x)}{x} < W(f(x))f'(x)$ is equivalent to $x > 1.05$. So (4.13) does not hold for all $\mathbf{x} > 0$. Hence D(U)-dominance does not hold.

Next, let us consider the tax function

$$\begin{aligned} &= 0.99x, 0 < x \leq 9.99 \\ f(x) &= 8.9 + 0.1x, \quad 10 < x \leq 10.99 \\ &= -0.35 + 0.95x, \quad 11 \leq x \end{aligned}$$

and connected in (9.99, 10) and (10.99, 11) by continuously differentiable curves for continuation. This f satisfies IP and MP but not AP (for x ≥ 11).

Now consider the utility function

$$U(x) = x^{0.95}/0.95. \tag{4.16}$$

Here $r(U) = 0.05 < 1$. Now one can check that this f satisfies (4.7) or the D(U)-dominance condition but obviously it does not satisfy Lorenz-dominance. ■

Propositions 1 and 2 also show that, if $U(x) = \log(x)$ then utilitarian deprivation dominance coincides with Lorenz dominance.

Corollary 1: For all $n \geq 2$ and for all $\mathbf{x} \in D^n$, $f^{(n)}(\mathbf{x}) \leq_{D(U)} \mathbf{x}$ implies and is implied by $f^{(n)}(\mathbf{x}) \geq_L \mathbf{x}$ if $U(x) = log(x)$ (that is, $r(U) = 1$).

Given that the logarithmic utility function makes the deprivation dominance consistent with the Lorenz domination, it will be interesting to look at the nature of the tax function that generates equal sacrifice with respect to this utility function. This is demonstrated in the following remark.

Remark 1: The only tax function which represents equal sacrifice with respect to the logarithmic utility function is the flat tax function.

Proof: Consider the flat tax function $t(x) = ax$ ($0 < a < 1$). Then equal sacrifice with respect to $t(x) = ax$ means that

$$U(x) - U(x - ax) = c \tag{4.17}$$

for all $x > 0$ and for some $c > 0$. We rewrite equation (4.17) as $U(kx) = U(x) - A(k)$ where $k = 1 - a$ and $A(k) = -c$ (a constant). This is a *linear affine functional equation* whose only solution is

$$U(x) = alog(x) + b, \quad a \neq 0$$

(see **Aczél (1987)**, p.25-26). This therefore establishes the sufficiency part of the proposition.

To demonstrate the necessity part, we start with the logarithmic utility function $U(x) = alog(x) + b$ and see that the equal sacrifice tax function under this specification of U is given by $t(x) = [1 - e^{-c/a}]x$, which is the flat tax function. ■

The following result drops out as an interesting corollary to remark 1.

Corollary 2: For any $\mathbf{x} \in D^n$, any two of the following conditions implies the third.

(i) Tax function is the flat tax function $t(x) = ax$.

(ii) Utility function is the logarithmic utility function $U(x) = log(x)$.

(iii) The equal sacrifice principle holds, that is $U(x) - U(x - t(x)) = c$, for $c > 0$, constant.

Remark 1 provides an axiomatic characterization of the logarithmic utility function using equal sacrifice. **Young(1987)** characterized the logarithmic and power function type utilities with the help of a scale invariance condition. Scale invariance condition requires that the sacrifice is independent of the money unit in which the incomes are measured. In other words, for all r

> 0,

$$U(x) - U(y) = U(x') - U(y') \Leftrightarrow U(rx) - U(ry) = U(rx') - U(ry').$$

Young shows that, equal sacrifice with respect to continuous, non-decreasing, non-constant utility function U is scale invariant *if and only if* U is a positive linear transform of log(x) or $\frac{1}{p}x^p$, $p \neq 0$, $p \leq 1$. Since **Young** considers a general tax function rather than the flat tax function as we have done, the utility function with respect to which equal sacrifice holds in his characterization turns out to be more general than ours.

Corollary 1 shows consistency between the logarithmic utility function based deprivation dominance and the Lorenz criterion (hence AP, since IP has been assumed). From corollary 2 it follows that, for the flat tax function, the logarithmic utility function gives rise to equal sacrifice. Thus, for the logarithmic utility function and the flat tax function (which satisfies AP) all three notions of distributive justice turn out to be nonconflicting. More precisely, they all are progressive, that is, they all make any post-tax income distribution at least as equitable as the corresponding pre-tax distribution in the respective sense. Therefore, if all individuals in a society have the logarithmic utility function and if a policy maker recommends the proportional or flat taxation as a tool of fiscal policy, then he is free to choose any one of the three notions of distributive justice considered here as an egalitarian principle.

As stated earlier, Proposition 6 of chapter 1 is based on the relative or rightist approach to inequality measurement. We know that another possibility is to assume that inequality indices are of absolute or leftist type. For ranking distributions by *leftist* inequality indices the domination in terms of absolute-Lorenz curve has been proved to be necessary and sufficient (**Moyes(1988)**, Proposition 7 of chapter 1). It is then easy to see that, for any increasing, concave U, D(U)-dominance implies the absolute Lorenz dominance. However, in the absence of **AP**, the converse may not hold without further assumptions. This should be clear from the equivalence between equation (4.13) and D(U)-dominance.

4.4 Conclusion

In this chapter we have considered three notions of distributive justice, namely, **average tax progressivity**, **equal sacrifice principle** and **utilitarian deprivation rule** and tried to identify the tax function that will interlink the three types of distributive justice. The identified tax function turns out to be the proportional or **flat tax function**. The utility function which relates average progressivity with utilitarian deprivation rule and sacrifice principle is the **logarithmic utility function**.

Chapter 5

On the Family of Subgroup and Factor Decomposable Measures of Multidimensional Poverty[1]

5.1 Introduction

In **section 1.4** we have argued about the need for viewing poverty measurement from a multidimensional perspective. Determination of poverty values for a distribution of basic need attributes in a population is however a first step in the analysis of poverty. For deeper analysis we should enquire into the causations of poverty.

The identification of a subgroup in a population which is particularly susceptible to poverty allows policymakers to design antipoverty policies, where the partitioning of the population into subgroups can be done with respect to characteristics like age, sex, region, race, religion etc. Clearly, a subgroup decomposable poverty measure, that is, one which satisfies the requirement that the overall poverty index is a weighted average of subgroup poverty indices, where the weights are the population shares of the subgroups, can be employed for calculating the different subgroup's contributions to overall poverty. This in turn enables us to identify the subgroups whose contributions are major and hence to formulate antipoverty policies. An alternative decomposability formula for a poverty index can be factor decomposability which requires overall poverty to be expressed as an weighted average of poverty levels for individual basic needs. It is obvious that an index satisfying this property can also be helpful in formulating antipoverty policies.

If an index exhibits both the subgroup and factor decomposabilities then we can have a two-way poverty breakdown and it is then possible to calculate the contributions of different subgroups to the total poverty with respect to different attributes or basic needs. Consequently,

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[1] **This chapter is based on Chakravarty, Mukherjee and Ranade (1997).**

we can identify the subgroup-attribute combinations that are most affected by poverty. Evidently this kind of poverty breakdown is finer than the one done for subgroups or factors. Identification of most afflicted subgroup-attribute combinations become particularly important for designing antipoverty policies when a society's limited financial ability/resources does not enable it to eliminate poverty from an entire subgroup or for a specific attribute.

The purpose of this chapter is to identify the class of multidimensional poverty indices that satisfy both subgroup and factor decomposability criteria. In particular, the class contains simple multidimensional extensions of single dimensional subgroup decomposable indices suggested by **Chakravarty (1983)** and **Foster, Greer and Thorbecke (1984)** and also of the **Watts (1968)** index (under certain minor restrictions) as particular members. In the next section we suggest properties for a multidimensional poverty index. **Section 5.3** presents the family of indices and discusses its properties. A numerical illustration of the family is provided in **section 5.4** using Indian data. Finally **section 5.5** concludes.

5.2 Properties for a Multidimensional Poverty Index

The purpose of this section is to lay down the postulates for a multidimensional poverty index. For a set of n persons, the ith person possesses a k-vector $x_i \in R^k_+$ of attributes (basic needs).[2] Let M^n be the set of all n x k matrices whose entries are non-negative real numbers. A typical X $\in M^n$ shows an arrangement of values of k basic needs possessed by n persons in a matrix form. The ith row of X is $x_i \in R^k_+$. On the other hand the jth column $x_{.j}$ of X shows a distribution of basic need j among the n persons in the society. The (i,j)th entry of X is x_{ij}, the quantity of the jth basic need possessed by the person i. Let M = $\bigcup_{n \in N} M^n$, where N is the set of natural numbers.

Instead of setting a poverty line income, a threshold or subsistence level is determined for each basic need. These levels determine the minimally acceptable levels of different basic needs. Let z = (z$_1$,z$_2$,...,z$_k$) be a vector of thresholds for the k basic needs such that z $\in$ Z where Z may be R^k_{++}, the strictly positive part of R^k, or some nondegenerate subset thereof.

A person i is said to be poor with respect to attribute j if the amount of basic need j possessed by person i does not exceed the corresponding subsistence level, i.e. $x_{ij} \leq z_j$. We will say that a person i is poor if $x_{ij} \leq z_j$ for at least one j. If $x_{ij} \leq z_j$ for all j, then person i is certainly poor. However, person i may be poor with respect to certain basic needs, say h, but the quantities of the remaining (k-h) basic needs may exceed the respective threshold levels. If we regard such a person as rich and assume that the poverty index is independent of the amounts of different attributes possessed by a rich person, then we simply ignore the deprivations of person i for these h attributes from their respective thresholds in the calculation of the poverty index. We will therefore refer to a person of this type as poor since he is not rich with respect to all basic

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[2] See **Streeten (1981)** for an extensive discussion on the basic needs approach to development.

needs. For further discussions along this line see **Tsui (1994)**.

Let us now introduce some more notation. For any X $\in$ M the corresponding population size is n(X) (or n); the set of poor persons with respect to the basic need j is $S_j(X)$ (or S_j) $= \{ 1 \le i \le \text{n}: x_{ij} \le z_j \}$; the cardinality of S_j, that is, the number of poor with respect to the basic need j is $q_j(\text{x})$ (or q_j); the average amount of the attribute j possessed by the people in S_j is $\mu_j(\text{X})$ or (μ_j); the set of poor people is S(X) (or S) $= \{ 1 \le i \le \text{n}: x_{ij} \le z_j$ for some j $\}$ and the number of poor people is q(X) or q.

Following **Watts (1968)** and **Tsui (1994)** we say that a multidimensional poverty index is a function P:M x Z $\rightarrow R^1$. For any X $\in$ M and **z** $\in$ Z, **P(X; z)** indicates the poverty level associated with the basic need matrix X and the minimally acceptable levels of basic needs given by **z**. The index P is assumed to satisfy certain postulates.

Symmetry (SYM): For all (X ; z) $\in$ M x Z, P(X ; z) = P(ΠX ; z) where Π is any permutation matrix of appropriate order.

Focus (FOC): For (X ; z)$\in$ M x Z and for any individual i and attribute j such that $x_{ij} \ge z_j$, P(X; z) does not change for an increase in x_{ij}.

Monotonicity (MON): For (X ; z) $\in$ M x Z and for any person i and attribute j such that $x_{ij} \le z_j$, an increase in x_{ij} does not increase the poverty value P(X; z).

Principle of Population (POP): For any (X ; z) $\in$ M x Z, P(X ; z) = P(X^m; z) where X^m is the m-fold replication of X.

Continuity (CON): For any z $\in$ Z, P is continuous on M.

Non-Poverty Growth (NPG): For any (X ; z) $\in$ M x Z, if Y is obtained from X by adding a rich person to the population then P(Y ; z) $\le$ P(X; z).

Nondecreasingness in Subsistence Levels of Basic Needs (NSB): For a given X $\in$ M, *P(X ; z) does not decrease if* z_j *increases for any j.*

Scale Invariance (SCI): P(X^1; z^1) = P(X^2; z^2) whenever (X^2; z^2) $\in$ M x Z is obtained from (X^1; z^1) $\in$ M x Z by a scale transformation, that is, $X^2 = X^1 \Lambda$ and $z^2 = z^1 \Lambda$ where $\Lambda =$ diag $(\lambda_1, ..., \lambda_k)$, $\lambda_i > 0$ for all i.

Normalization (NOM): For any z $\in$ Z, P(X; z) = 1 whenever $x_{ij} = 0$ for all i and j.

Subgroup Decomposability (SUD): For any X^1,X^2,...,X^m $\in$ M and z $\in$ Z,

$$P(\mathbf{X^1}, \mathbf{X^2}, ..., \mathbf{X^m}; \mathbf{z}) = \sum_{i=1}^{m} \frac{n_i}{n} P(\mathbf{X^i}; \mathbf{z}) \tag{5.1}$$

where n_i is the population size corresponding to X^i, i = 1,2,...,m and n $(= \sum_{i=1}^{m} n_i)$ is the total population.

Factor Decomposability (FAD): For any (X ; z) $\in$ M x Z,

$$P(\mathbf{X}; \mathbf{z}) = \sum_{j=1}^{k} a_j P(\mathbf{x_{.j}}; z_j), \tag{5.2}$$

where $\mathbf{x_{.j}}$ is the j^{th} column of X, $a_j > 0$ is the weight attached to attribute j, j = 1,..,k, such that $\sum_{j=1}^{k} a_j = 1$.

Before we introduce two final postulates, let us discuss the ones considered above. Excepting FAD and NOM, these are essentially generalizations of the axioms suggested for a unidimensional index. The properties SYM, FOC, MON, POP, CON and SCI were considered by **Tsui (1994)**. SYM demands anonymity. FOC states that if a person is not poor with respect to an attribute then giving him more of that attribute does not change the level of poverty even if this person is poor with respect to some other attribute(s). A poverty index satisfying FOC will be called a *focussed* index. MON requires that the index does not increase if the condition of the poor improves. Under POP, if a particular basic need matrix X is pooled several times, then the poverty index does not change for the pooled population. Given the difficulties in getting accurate data on basic needs quantities, it is reasonable to require the index to vary continuously with basic needs levels. The nonpoverty growth axiom implies that the poverty index is a nonincreasing function of the population size of the rich. The intuition behind NSB is quite reasonable. Between two identical communities, the one with higher subsistence levels of one or more basic needs should not have a lower poverty. In our definition of the poverty index different attributes may have different scales of measurement. According to SCI, if we post multiply the matrix X and the vector **z** by the same diagonal matrix Λ, that is, if we change the scales of measurement of the attributes then the index does not vary. NOM states that the poverty index takes on the value unity when the quantities of different attributes possessed by all individuals are zero.

Subgroup decomposability shows that if a population is divided into several subgroups, say m, defined along ethnic, geographical or other lines, then the overall poverty is the population share weighted average of subgroup poverty levels. In our formulation (5.1), $\frac{n_i}{n}$P(X^i ; z) is the contribution of subgroup i to the overall poverty. This is the amount by which the community poverty will be reduced if the subgroup poverty is eliminated. $\frac{n_i P(X^i;z)}{nP(X;z)}$ × 100 is the percentage contribution of subgroup i to total poverty, where $X = (X^1, ..., X^m)$. Each of these statistics is useful to the policymakers.

Factor decomposability means that an weighted average of the poverty levels for individual attributes gives the overall poverty. The weight a_j may be assumed to reflect the importance that we attach in our aggregation to attribute j. It may also be assumed to reflect the importance that the government attaches for alleviating poverty for attribute j. The percentage contribution of attribute j to the total poverty is $\frac{a_j P(x_{.j};z_j)}{P(X;z)}$ × 100. The elimination of poverty for attribute j will lower community poverty by the amount a_j P($x_{.j}$; z_j).

We can use the two decomposability postulates to construct a two-way poverty profile and to calculate each subgroup's poverty for each attribute. To illustrate this, let us assume that there are two attributes and the population has been divided into two subgroups of sizes n_1 and n_2. Assume further for simplicity that $a_1 = a_2 = \frac{1}{2}$. The basic need matrices of the subgroups are X^1 and X^2. Then by SUD,

$$P(\mathbf{X^1},\mathbf{X^2};\mathbf{z}) = \frac{n_1}{n}P(\mathbf{X^1};\mathbf{z}) + \frac{n_2}{n}P(\mathbf{X^2};\mathbf{z}) \tag{5.3}$$

which in view of FAD can be written as:

$$\frac{n_1}{n}\Big\{\frac{1}{2}P(x^1_{\cdot 1};z_1) + \frac{1}{2}P(x^1_{\cdot 2};z_2)\Big\} + \frac{n_2}{n}\Big\{\frac{1}{2}P(x^2_{\cdot 1};z_1) + \frac{1}{2}P(x^2_{\cdot 2};z_2)\Big\} \tag{5.4}$$

where $x^j_{\cdot i}$ is the i^{th} column of matrix X^j.

Denoting $P(x^j_{\cdot i};\, z_i)$ by P_{ij} we can present the poverty levels in the following tabular form.

Table 5.1: A two-way poverty profile

Subgroup → Basic need ↓	1	2	Average poverty
1	P_{11}	P_{12}	$P_{1\cdot}$
2	P_{21}	P_{22}	$P_{2\cdot}$
Average poverty	$P_{\cdot 1}$	$P_{\cdot 2}$	P

Note that $P_{\cdot i} = P(X^i; z)$ is subgroup i's average poverty for attributes 1 and 2. On the other hand, $P_{i\cdot}$ is the population share weighted average of subgroup poverty levels for the attribute i. Overall poverty level P can now be obtained by taking the simple average of $P_{1\cdot}$ and $P_{2\cdot}$ or by taking population share weighted average of $P_{\cdot 1}$ and $P_{\cdot 2}$. Clearly, this type of micro level breakdown of poverty will help us to identify simultaneously the population subgroup(s) as well as attribute(s) for which poverty levels are severe. In terms of table 5.1, this means that we can identify the row and column of the table for which the poverty values are high. For instance, if P_{21} is the maximum P_{ij} in the above table then subgroup 1's poverty for attribute 2 is the maximum contributor to overall poverty and the subgroup-attribute combination (1,2) of the population needs maximum attention from antipoverty perspective. Elimination of poverty for this pair will reduce P by $\frac{n_1 P_{21}}{2n}$. This type of decomposability becomes especially important when the society's limited resource may not be sufficient for poverty elimination from one subgroup or for one attribute.

The next property that we wish to consider, following **Tsui (1994)**, is about redistribution of attributes. As argued earlier, **Sen (1976)** suggested that an index should be sensitive to the inequality of income among the poor (the transfer axiom). This idea can be generalized to the multidimensional case. It is well known that in the unidimensional context a rank-preserving transfer of income from the rich to the poor decreases inequality if and only if the post-transfer distribution v is obtained by multiplying the pre-transfer distribution u by a bistochastic matrix A where Au is not a permutation of u (see **Kolm (1969)**, **Dasgupta, Sen and Starrett (1973)** and **Chakravarty (1990)**). In the multidimensional context an analogous property suggested by **Kolm (1977)** says that the distribution of a set of attributes summarized by some matrix Y is more equal than another such matrix X (whose rows are not identical) if and only if Y = BX, where B is some bistochastic matrix and Y cannot be derived by permutations of the rows of X. Intuitively, multiplication of X by B makes the resulting distribution less concentrated. Now, to have a property for multidimensional poverty indices which is analogous to the transfer axiom, let X_p be the bundle of attributes possessed by the poor corresponding to the basic need matrix X. The i^{th} row of X_p gives the quantities of different attributes owned by person i where i ∈ S. The transfer axiom is then stated as:

Transfer Axiom (TRA): For any (X ; z) ∈ M x Z, if Y is obtained from X by multiplying X_p by a bistochastic matrix B and BX_p is not a permutation of the rows of X_p then P(Y ; z) ≤ P(X ; z), given that the bundles of attributes of the rich remain unaltered.

Tsui (1994, 1996) argues that TRA ignores an important aspect of multidimensional inequality, viz., the correlation among attributes. The property we wish to introduce now, which is a generalization of an egalitarian principle considered by **Atkinson and Bourguignon (1982)**, takes care of this problem. The Atkinson-Bourguignon (AB) study focuses on the ranking of matrices using the additively separable social welfare function:

$$W(X) = U(X_1) + U(X_2) + ... + U(X_n)$$

where, as stated earlier, X_i is the i^{th} row of the basic need matrix X. Let us explain the AB issue by assuming that there are three poor persons and two attributes. The original distribution of the attributes between the persons is given by $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 7 & 2 \end{bmatrix}$. If we switch the amounts of the second attribute between the second and the third person then the distribution of the attributes is given by $G = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 7 & 5 \end{bmatrix}$. After the transfer the third person gets more of the two attributes than the second person. The movement from A to G increases the correlation of the attributes, that is, a person who has more of one attribute has more of the other too. Note that the first row of A coincides with the first row of G, but g_2 = (min {3,7}, min{2,5}) = (3,2) and g_3 = (max{3,7}, max{2,5}) = (7,5), where g_2 (g_3) is the second (third) row of G.

AB explored the restriction on U for W(A) ≥ W(G), or in the more general case W(X) ≥ W(Y) to hold, where Y is obtained from X by a correlation increasing switch. Assuming that U is twice differentiable, the restriction turns out to be $U_{ij} \leq 0$ where U_{ij} is the $(i,j)^{th}$ cross partial derivative of U (**Marshall and Olkin (1979)**, p.146). Note that the concavity of U is not sufficient to guarantee this.

In the context of poverty measurement it seems intuitively reasonable to argue that the poverty level P(A ; z) should not be higher than the level P(G ; z). This is because the third person who was originally richer for the first attribute and poorer for the second attribute than the second person has become richer for both the attributes under the switch that takes us from A to G. The following property suggested by **Borland and Proschan (1988)** and **Tsui**

(1994), which captures this view, is a generalization of the AB condition.

Nondecreasing Poverty under Correlation Increasing Rearrangement (NPC): Let X and Y be two n x k basic needs matrices such that $x_{11} \leq x_{21} \leq ... \leq x_{n1}$ and $y_{11} \leq y_{21} \leq ... \leq y_{n1}$. Then for any symmetric index P, P(X ; z) $\leq$ P(Y ; z) whenever Y is obtained from X by a sequence of matrices $X^t = (x^t_{ij})$ showing correlation increasing rearrangements, where $0 \leq t \leq m$ and t and m are nonnegative integers such that $X^0 = X$ and $X^m = Y$. For any persons i and r, with i>r, $x^t_{ij} = \max(x^{t-1}_{ij}, x^{t-1}_{rj})$ and $x^t_{rj} = \min(x^{t-1}_{ij}, x^{t-1}_{rj})$ for all j and t $\geq$ 1.

5.3 The Family of Poverty Indices

In this section we present our general family of multidimensional poverty indices. We first show that a subgroup of the set of postulates proposed in section 5.2 identifies the class of indices. The class is then shown to meet the remaining axioms as well. We formally state the first result as:

Proposition 1: The only nonconstant focussed poverty index P:M x Z $\rightarrow R^1$ that satisfies SUD, FAD, SCI, MON, TRA, CON and NOM is:

$$P(\mathbf{X}; \mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} a_j f(\frac{x_{ij}}{z_j}),$$

where f:[0, ∞] $\rightarrow R^1$ is continuous, nonincreasing, convex, f(0)=1 and f(t) = c for all t $\geq$ 1, where c < 1 is a constant. Also, $a_j > 0$ are constants such that $\sum_{j=1}^{k} a_j = 1$.

Proof : : For any (X ; z) $\in$ M x Z, by SUD we have,

$$P(\mathbf{X}; \mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n} P(\mathbf{x_i}; \mathbf{z}). \tag{5.5}$$

Applying FAD to (5.5), we get

$$P(\mathbf{X}; \mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} a_j P(x_{ij}; z_j), \tag{5.6}$$

where $a_j > 0$ and $\sum a_j = 1$. By SCI, $P(x_{ij}; z_j) = P(\frac{x_{ij}}{z_j}; 1)$ for all (X; z) $\in$ M x Z. Setting $f(\frac{x_{ij}}{z_j}) = P(\frac{x_{ij}}{z_j}; 1)$, we have

$$P(\mathbf{X}; \mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} a_j f(\frac{x_{ij}}{z_j}) \tag{5.7}$$

where by CON f:[0, ∞] $\rightarrow R^1$ is continuous. For TRA to hold we require convexity of f (see **Marshall and Olkin (1979)**, p.433).

Consider the extreme situation where x_{ij} are all identically 0 for all i,j. In this case P(X ; z) in (5.7) becomes f(0). But by the normalization axiom we require the poverty index to take on the value 1 when x_{ij} = 0 for all i and j. Hence f(0) = 1. In addition, MON and FOC along with nonconstancy of P imply that f is nonincreasing and $f(t) = c$ for all $t \geq 1$, where $c < 1$ is a constant. This completes the necessity part.

The sufficiency can be verified by checking that P(X; z) in (5.7) fulfills FOC, SUD, FAD, SCI, MON, TRA, CON and NOM. ■

Sometimes it becomes convenient to normalize the poverty index over the interval [0, 1]. In view of this we define the function $g(t) = \frac{f(t)-c}{1-c}$. Clearly, the real valued function g defined on $[0, \infty]$ is continuous, nonincreasing, convex and $g(0) = 1$ and $g(t) = 0$ for all $t \geq 1$. Therefore we can redefine the poverty index P as

$$P(\mathbf{X}; \mathbf{z}) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{k} a_j g(\frac{x_{ij}}{z_j}). \tag{5.8}$$

Since P in (5.8) is a rescaled version of (5.7), from now on we deal with P as defined in (5.8).

Next, in the following proposition we show that the general measure P in (5.8) also satisfies the properties not considered in proposition 1.

Proposition 2: The poverty measure P given by (5.8) satisfies SYM, POP, NPG, NSB. Assume that g is twice differentiable on (0, 1). Then P satisfies NPC also.

Proof : : It is obvious that P in (5.8) meets SYM and POP. Nonincreasingness of g ensures that P meets NSB. Since the index does not depend on the quantities of different attributes possessed by the rich and is inversely related to the population size of the rich, it fulfills NPG.

Since $g(t) = 0$ for all $t \geq 1$, we need to verify NPC only on (0, 1). Now, note that for any poverty index which is additive across individuals the conditions NPC and AB are equivalent. (See **Borland and Proschan (1988)** and **Tsui (1996)**.) Therefore it is sufficient to verify whether the AB condition is satisfied, that is, to check whether $g_{ij} \geq 0$ holds on (0, 1), where g_{ij} is the $(i,j)^{th}$ cross derivative of g. Since in the poverty index above, the function g is single coordinated, its cross derivatives are always zero, which means that P meets the AB condition. ■

The index given by (5.8) achieves the maximum value 1 in the extreme case described in NOM. On the other hand, the minimum value 0 is attained when the quantities of different attributes possessed by all the persons are never less than the corresponding subsistence levels. Therefore, the poverty index is always bounded between 0 and 1.

The function g associated with the poverty index P can be regarded as a deprivation function. Thus, $g(\frac{x_{ij}}{z_j})$ gives the deprivation felt by person i when the quantity of attribute j possessed by him is less than or equal to the corresponding subsistence level. $g(0) = 1$ means that the deprivation is maximum if the person has absolutely nothing. On the other hand, $g(t) = 0$, for $t \geq 1$ means that a person has no feeling of deprivation if the quantity of his possession of an attribute is at least as high as its subsistence level. Nonincreasingness shows that deprivation does not increase if a person has more of an attribute and convexity means that the rate of decrease of deprivation is nonnegative. We will say that a deprivation function $g:[0, \infty] \rightarrow R^1$

is regular if it meets the aforementioned restrictions, that is, if $g(0) = 1$, $g(t) = 0$ for $t \geq 1$, g is nonincreasing, convex and twice differentiable on (0,1).

Given any regular g, there will be a corresponding poverty index that meets all the postulates laid down in section 5.2. These indices will differ only in the manner in which we use the function g to aggregate the deprivations of different persons for different attributes into an overall indicator. For instance, one can derive poverty indices associated with the following functional forms:

$$
\begin{array}{rrcl}
(i) & f_1(t) & = & 1 - t^e, \quad for\ 0 \leq t \leq 1,\ where\ 0 \leq e \leq 1 \\
 & & = & 0, \quad for\ t > 1; \\
(ii) & f_2(t) & = & (1-t)^{\alpha}, \quad for\ 0 \leq t \leq 1,\ where\ \alpha \geq 1 \\
 & & = & 0, \quad for\ t > 1; \\
(iii) & f_3(t) & = & \dfrac{1-t}{1+t}, \quad for\ 0 \leq t \leq 1 \\
 & & = & 0, \quad for\ t > 1; \\
(iv) & f_4(t) & = & \dfrac{(e^{1/2} - e^{t/(1+t)})}{e^{1/2} - 1}, \quad for\ 0 \leq t \leq 1 \\
 & & = & 0, \quad for\ t > 1.
\end{array}
$$

If k, the number of attributes, is 1 then P in (5.8) is a member of the class of subgroup consistent indices that was identified by **Foster and Shorrocks (1991)**.

To illustrate the general formula in (5.8), let us consider the poverty index associated with the regular function f_1. In this case the index is:

$$P_e(\mathbf{X}; \mathbf{z}) = \frac{1}{n} \sum_{j=1}^{k} \sum_{i \in S_j} a_j [1 - (\frac{x_{ij}}{z_j})^e]. \tag{5.9}$$

This index is a multidimensional extension of (1.33). The parameter e reflects different perceptions of poverty. For a given X, as e increases, P_e increases. As $e \to 0$, $P_e \to 0$. For $e = 1$, P_e becomes:

$$P_1(\mathbf{X}; \mathbf{z}) = \frac{1}{n} \sum_{j=1}^{k} \sum_{i \in S_j} a_j [\frac{z_j - x_{ij}}{z_j}] = \sum_{j=1}^{k} a_j H_j I_j, \tag{5.10}$$

where $H_j = q_j/n$ is the head-count ratio for attribute j and the poverty gap ratio I_j for attribute j is given by $\sum_{i \in S_j} [\frac{z_j - x_{ij}}{q_j z_j}]$. Thus, for a given H_j, an increase in I_j increases the index.

The following alternative of interest arises from the specification f_2:

$$P_{\alpha}(\mathbf{X}; \mathbf{z}) = \frac{1}{n} \sum_{j=1}^{k} \sum_{i \in S_j} a_j [1 - (\frac{x_{ij}}{z_j})]^{\alpha}. \tag{5.11}$$

This is a multidimensional generalization of (1.34). For $\alpha = 1$, the index coincides with the particular case of $e = 1$ discussed above. If $\alpha = 2$, P_α can be written as:

$$P_2(\mathbf{X};\mathbf{z}) = \sum_{j=1}^{k} a_j H_j \{ I_j^2 + (1 - I_j)^2 C_j^2 \}, \tag{5.12}$$

where $\mathrm{C}_j^2 = \sum_{i \in S_j} \{(x_{ij} - \mu_j)^2 / q_j \mu_j^2\}$ is the squared coefficient of variation for attribute j. Now, C_j^2 (or more commonly, its positive square root) is a measure of inequality of the distribution for attribute j among the poor. Clearly, given I_j and H_j, P_2 increases as C_j increases. Thus, the decomposition in (5.12) shows that the poverty measure is related in a positive monotonic way with the inequality levels of the poor for different attributes.

It might be of interest to see how the poverty indices look like when the weights, a_j's, are all equal ($= \frac{1}{k}$). For instance, when $a_1 = a_2 = ... = a_k = \frac{1}{k}$, P_α becomes

$$P_\alpha(\mathbf{X};\mathbf{z}) = \frac{1}{nk} \sum_{j=1}^{k} \sum_{i \in S_j} [1 - (\frac{x_{ij}}{z_j})]^\alpha. \tag{5.13}$$

The other indices are similarly modified.

Finally, we briefly discuss two alternative multidimensional poverty indices axiomatically derived by **Tsui (1994)**. They are:

$$T_1(\mathbf{X};\mathbf{z}) = \frac{b}{n} \sum_{i=1}^{n} \{1 - \prod_{j=1}^{k} [\frac{\widehat{x}_{ij}}{z_j}]^{r_j}\}, \quad br_j \geq 0, \tag{5.14}$$

$$T_2(\mathbf{X};\mathbf{z}) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{k} d_j log(\frac{z_j}{\widehat{x}_{ij}}), \quad d_j \geq 0. \tag{5.15}$$

where $\widehat{x}_{ij} = \min \{ x_{ij}, z_j \}$. The first is an alternative multidimensional generalization of (1.33). The log-linear expression in T_2 is a multidimensional extension of the Watts index. It is important to note that for T_2 to be well defined we require positivity of all attribute levels. (See **Zheng (1993)** and **Tsui (1996a)** for discussions on the single dimensional Watts index.) Both T_1 and T_2 satisfy SUD but neither satisfies FAD. However, a special case of T_2, where d_j's are strictly positive and sum to 1, meets FAD under the assumption of positivity of all attribute levels. In fact under these assumptions (5.15) becomes a member of the general family in (5.8).

5.4 A Numerical Illustration

The illustration provided in this section is based on survey data on basic needs collected by **Rudra, Chakrabarti, Mazumdar and Bhattacharya (1995)** from five districts of West Bengal, India. This survey was conducted during December 1990 - May 1991 with a view to evolving a simple criteria for identifying poor households in rural areas. The 5 districts covered are Darjeeling (DAJ), Jalpaiguri (JAL), Midnapore (MNP), Birbhum (BBM) and Bardhaman

(BMN). The total number of households surveyed was 2598. A schedule on fulfillment of basic needs covering a wide range of items relating to basic needs of food, clothing, shelter, health care etc. was canvassed for every sample household. Out of the 17 poverty indicators chosen by these authors, most indicators were of qualitative type. For instance, one attribute was whether the dwelling was adequate for protection against rainstorms. If the answer turned out to be 'no', then the household was identified as poor according to this attribute, rich otherwise. Consequently, exact quantitative information for such attributes were not available. We therefore chose 3 indicators for which quantitative information were available. These are (i) number of 'saris' per adult female, where a sari is a chief garment for an Indian woman; (ii) the roof height and (iii) the number of months for which the household members had two square meals a day throughout the last year (365 days). Let us denote the three indicators by SR, RH and FU respectively. The subsistence levels for these 3 indicators chosen by the surveyors are 1.75, 1.68 meters and 10 months respectively. Since the number of saris per adult female was determined by dividing the total number of saris the household possessed by the number of female members, it may not be an integer. If the roof height was found to be less than or equal to 1.68 meters, then the house was considered as a shanty and household was regarded as poor for the attribute. The total number of households that were found to be poor with respect to these 3 indicators is 605. For simplicity we assume that all the attributes are equally important, so that $a_{SR} = a_{RH} = a_{FU} = \frac{1}{3}$.

District wise, attribute wise and overall poverty levels were computed to create a profile of poverty for identifying those districts and attributes which are most afflicted by poverty. Numerical estimates of poverty are presented in tables 5.2 and 5.3 the formats of which are identical. In table 5.2 we present the poverty estimates for the index P_e where $e = 0.5$. The first row of the table gives the partitioning of the population with respect to districts. The figures in the parentheses in this row are the number of sample households in each district. The first column gives the basic needs considered. The index values inside the table are the poverty levels for different districts and different attributes. For instance, the level of poverty in the BBM district for attribute SR is 0.0154. The average figures presented in the fifth row are the poverty values for different districts, whereas the average figures in the seventh column are poverty values for alternative attributes. The district poverty levels, given by the fifth row, are now weighted by the corresponding population shares to determine the contributions of different districts to total poverty which are given as percentages of total poverty in the last row. The percentage contributions of different attributes to total poverty are shown in the last column. The overall poverty value .0139 is shown in the south-east corner of the table. Table 5.3 presents similar calculations for P_α when α is 2.

Table 5.2: Two-way breakdown of the poverty measure P_e ($e = .5$) for rural India (West Bengal) by districts and by basic needs

District →	MNP	BBM	BDN	DJL	JAL	Average	Percentage
Basic need ↓	(436)	(478)	(1161)	(154)	(369)	poverty	contribution
RH	.0167	.0007	.0019	.0150	.0042	.0053	12.63
SR	.0195	.0154	.0080	.0618	.0380	.0187	45.02
FU	.0246	.0045	.0010	.1320	.0311	.0176	42.35
Average poverty	.0202	.0069	.0036	.0696	.0244	.0139	
Percentage contribution	24.48	09.13	11.66	29.74	24.99		

Table 5.3: Two-way breakdown of the poverty measure P_α ($\alpha = 2$) for rural India (West Bengal) by districts and by basic needs

District →	MNP	BBM	BDN	DJL	JAL	Average	Percentage
Basic need ↓	(436)	(478)	(1161)	(154)	(369)	poverty	contribution
RH	.0086	.0002	.0010	.0141	.0042	.0034	09.91
SR	.0168	.0139	.0070	.0538	.0344	.0166	48.60
FU	.0154	.0043	.0009	.1266	.0202	.0141	41.49
Average poverty	.0136	.0061	.0030	.0648	.0196	.0114	
Percentage contribution	20.07	09.92	11.64	33.86	24.50		

Certain interesting features emerge from these tables. The ranking of different districts and basic needs by P_α ($\alpha = 2$) coincide with that by P_e for $e = 0.5$. However, at the micro level the rankings are not always the same. Both the tables show that the district DJL is the poorest with respect to the attribute FU, whereas minimum poverty was observed in the district BBM for the attribute RH. From antipoverty policy point of view, the pair of district and attribute that needs maximum attention is (DJL, FU). If we look at the contributions to total poverty by different districts, we note that the maximum contribution comes from DJL. Complete elimination of poverty from DJL would lower aggregate poverty by the percentage by which it contributes to total poverty. The highest contribution of this district according to both the indices are due to the high poverty levels for attributes SR and FU. On the other hand, the most susceptible basic need of poverty is SR. Its high contribution is a consequence of high poverty levels in districts DJL and JAL for the attribute.

5.5 Conclusions

This chapter has identified the class of multidimensional poverty indices which along with other properties, satisfies subgroup and factor decomposability postulates. While subgroup decomposability ensures that overall poverty is the population share weighted average of poverty levels within subgroups, factor decomposability expresses overall poverty as an weighted average of poverty levels for altenative factors. These two postulates enable us to isolate the subgroup, factor pairs which are most affected by poverty. This type of isolation becomes important from antipoverty point of view. The results developed in the chapter heve been analyzed using rural India data.

Chapter 6

A Family of Additive Indices of Improvement in Well-being

6.1 Introduction

A multidimensional improvement index gives us a national average of improvement in well-being with respect to different attributes. The use of an index of this type hides the extents to which different attributes contribute individually to the overall improvement. Therefore, for deeper analysis we should look at the percentage contributions made by different attributes to the overall improvement. This will enable a policy maker to identify the attributes whose contributions are rather low or negative and recommend policies under which more resources can be allocated for improving the levels of these attributes of well-being. Clearly, a multidimensional improvement index which satisfies additivity, that is, which can be expressed as the average of improvement indices for individual attributes of well-being will help us to carry out this type of analysis.

It may be important to note that the additivity assumption ignores the cross effects among the variables. For instance, with an additive improvement index, we cannot determine the impact of an extra unit of income on life expectancy. However, additivity assumption can be justified by arguments put forward by UNDP. According to **UNDP (1991)** 'Human development cannot take place without human life and health; people do not just want to be alive; they want to know their way around in life. They want to be knowledgeable; and they certainly may want a decent life, one that is not constantly undermined by extreme poverty and constant worry about sheer physical survival. All three of the HDI (Human Development Index) components thus deserve equal weight. And that is why the HDI proposes an unweighted average of a country's rank on the life expectancy, literacy and income scale' (op.cit. p.88). Since improvement refers to gain in achievement which reflects human development, following UNDP, we can also adopt the unweighted averaging principle, that is, the additivity assumption.

We may mention that according to the above notion of policy prescription, an assessment of overall progress or of a particular policy may be conditional on the implicit valuations of our aggregate index. This notion can be contrasted with the alternative view where a policy maker may begin with the valuation and then judge what to do. However, it may be regarded as one of the policy issues that may arise in the current context. It may be useful to do this for two reasons. First, following **Sen (1985)**, the nonwelfarist approach to policy analysis is becoming quite popular. Second, often policy is evaluated by the use of such indices. We also note that the additivity assumption parallels the factor decomposability postulate suggested in the case of poverty measurement (see section 5.2). Therefore, any policy recommendation that seeks to reduce poverty associated with a particular factor is similar to the above notion of policy perception.

The purpose of this chapter is to suggest a general family of additive indices of improvement, discuss its properties and illustrate it numerically.

6.2 Properties for a Measure of Improvement in Well-being

Suppose that there are n attributes of well-being. An improvement in well-being with respect to the i^{th} attribute can be conceptualized as an increase in the attainment level of the attribute from one value to another. Let w_{it} stand for the value of attribute i in period t. Then $\mathbf{w_t} = (w_{1t}, w_{2t}, ..., w_{nt})$ is the vector of attributes in period t. Suppose that we wish to determine the improvement in well-being between periods 1 and 2. Then an improvement index showing the extent of improvement in people's welfare between these two periods should be a real valued function of $\mathbf{w_1}$ and $\mathbf{w_2}$.

Let M_i be the upper bound of the i^{th} attribute and m_i be its lower bound. (See **Morris (1979)**, **Sen (1981)**, **Dasgupta (1993)** and **Dasgupta and Weale (1992)** for discussion on such bounds.) Thus, $w_{it} \in [m_i, M_i]$ for i = 1,2,..,n and t = 1,2. We assume that $m_i < M_i$. (This assumption is implicit in **Kakwani (1993)**, **Tsui (1996b)** and **Majumder and Chakravarty (1996, 1996a)**.) This assumption ensures that the open set (m_i, M_i) is nonempty. Let $\mathbf{M} = (M_1, M_2, ..., M_n)$ and $\mathbf{m} = (m_1, m_2, ..., m_n)$. The shortfall of the value of the attribute from its maximum attainable value $M_i - w_{it}$ is the deprivation with respect to this attribute. The smaller is this shortfall the better off the society is with respect to the attribute under consideration. If we view improvement in terms of such deprivations, then the improvement index becomes a real valued function of $(M_1 - w_{11}, M_2 - w_{21}, ..., M_n - w_{n1})$ and $(M_1 - w_{12}, M_2 - w_{22}, ..., M_n - w_{n2})$. We can as well regard improvement index as a real valued function of $(d_{11}, ..., d_{n1})$ and $(d_{12}, ..., d_{n2})$ where d_{it}, more precisely, $d_{it}(M_i, m_i, w_{it}) = (M_i - w_{it})/(M_i - m_i)$ for all i = 1,2,..,n and t = 1,2. Since $w_{it} \geq m_i$ for all i and t, d_{it} is the deprivation with respect to attribute i in period t, expressed as a proportion of its maximal attainable value. Clearly, d_{it} is normalized over the set $[0, 1]$, that is, $d_{it} \in [0, 1]$. Let $\mathbf{d_t} = (d_{1t}, ..., d_{nt})$, where t = 1,2. Since human development indicators (eg. UNDP (1993)) and the **Kakwani** and **Tsui** measures of improvement are based on normalized deprivations, we will also define the improvement index directly on normalized deprivations. That is, an improvement index is a real valued function G, which associates to any vector of normalized deprivations $\mathbf{d_1}$ and $\mathbf{d_2}$ in periods 1 and 2, a value $G(\mathbf{d_1};\mathbf{d_2})$ indicating the level of improvement that actually takes place when normalized deprivation changes from $\mathbf{d_1}$ to $\mathbf{d_2}$. We will show later that improvement indices defined this way satisfy a desirable property.

We now suggest some postulates for an arbitrary G. The first property is regarding the domain of G. Since $d_{it} \in [0,1]$, $\mathbf{d_t} \in [0,1]^n$, where $[0,1]^n$ is the n-fold cartesian product of $[0,1]$. Thus, we have :

Domain Restriction (DR) : G is a real valued function defined on $[0,1]^n\mathbf{x}[0,1]^n$. More precisely, $G : [0,1]^n\mathbf{x}[0,1]^n \rightarrow R^1$, where R^1 is the real line.
We may point out that **Tsui (1996)** adopted the absolute shortfall domain $\prod_{i=1}^{n} D^i\mathbf{x}\prod_{i=1}^{n} D^i$, where $D^i = (0, M_i - m_i]$, instead of our normalized domain. However, the final form of the **Tsui** index, which is derived axiomatically, is shown to depend on normalized deprivations which are elements of our normalized domain.

The next four axioms, which have been suggested by **Tsui**, are essentially generalizations of the corresponding Kakwani axioms. These are:

Monotonocity (MN) : G is an increasing function of $\mathbf{d_1}$ and a decreasing function of $\mathbf{d_2}$.

Period Consistency (PC) : For any $\mathbf{w_t} \in \prod_{i=1}^{n}[m_i, M_i]$, $t = 1,2,3$; $G(\mathbf{d_1};\mathbf{d_3}) = G(\mathbf{d_1};\mathbf{d_2}) + G(\mathbf{d_2};\mathbf{d_3})$.

Normalization (NR) : $G(\mathbf{d_1};\mathbf{d_2}) = 1$ when $\mathbf{d_2} = 01^n$ and $\mathbf{d_1} = 1^n$, where 1^n is n-coordinated vector of ones.

Increasing Difficulty of Improvement (ID) : For any $\mathbf{w_1}, \mathbf{u_1}$ such that $w_{j1} = u_{j1}$ for all $j \neq i$ and $w_{i1} > u_{i1}$,

$$G(\mathbf{d^{w_1}};\mathbf{d^{w_1+ce_i}}) > G(\mathbf{d^{u_1}};\mathbf{d^{u_1+ce_i}})$$

where $\mathbf{d^{w_1}} = (\frac{M_1 - w_{11}}{M_1 - m_1}, ..., \frac{M_n - w_{n1}}{M_n - m_n})$, and $\mathbf{d^{w_1+ce_i}}$ is the normalized deprivation vector all of whose components, except the i^{th}, are same as those of $\mathbf{d^{w_1}}$ and the i^{th} component of $\mathbf{d^{w_1+ce_i}}$ is $(M_i - w_{i1} - c)/(M_i - m_i)$, with $c > 0$ being any arbitrary constant such that $w_{i1} + c \leq M_i$. The vectors $\mathbf{d^{u_1}}$ and $\mathbf{d^{u_1+ce_i}}$ are defined analogously.

In addition to the above, Tsui also suggested a continuity property, which is formally stated as

Continuity (CN) : G is a continuous function.

The seventh property is the **dimensionality (DM)** postulate.

Dimensionality (DM) : For any $\mathbf{w_t} \in \prod_{i=1}^{n}[m_i, M_i]$, $t = 1,2$,

$$\begin{aligned}&G(d_{11}(\alpha_1 M_1, \alpha_1 m_1, \alpha_1 w_{11}), d_{21}(\alpha_2 M_2, \alpha_2 m_2, \alpha_2 w_{21}), ..., d_{n1}(\alpha_n M_n, \alpha_n m_n, \alpha_n w_{n1});\\ &d_{12}(\alpha_1 M_1, \alpha_1 m_1, \alpha_1 w_{12}), d_{22}(\alpha_2 M_2, \alpha_2 m_2, \alpha_2 w_{22}), ..., d_{n2}(\alpha_n M_n, \alpha_n m_n, \alpha_n w_{n2})) =\\ &\qquad G(d_{11}(M_1, m_1, w_{11}), d_{21}(M_2, m_2, w_{21}), ..., d_{n1}(M_n, m_n, w_{n1});\\ &\qquad d_{12}(M_1, m_1, w_{12}), d_{22}(M_2, m_2, w_{22}), ..., d_{n2}(M_n, m_n, w_{n2})),\end{aligned}$$

where $\alpha_i > 0$ is any scaler.
Since each d_{it} is homogeneous of degree zero in its arguments, that is, $d_{it}(\alpha_i M_i, \alpha_i m_i, \alpha_i w_{it}) = d_{it}(M_i, m_i, w_{it})$ for all positive α_i, DM is always satisfied by an improvement index defined on normalized deprivation levels. **Tsui (1996b)** adopted an analogous property, which he called homotheticity (HM), for pinning down a specific class of indices. According to HM, the ranking of a pair of shortfalls $(M - w_1, M - w_2)$ and $(M - v_1, M - v_2)$ remains unchanged if, for each i, $(M_i - w_{it})$ and $(M_i - v_{it})$ are multiplied by some positive scaler.

These properties can be interpreted in an analogous manner in which we have interpreted their single-dimensional counterparts.

The final property we wish to suggest is additivity.

Additivity (AD) : For any $w_t \in \prod_{i=1}^{n}[m_i, M_i]$, $t = 1,2$;

$$G(\mathbf{d_1}; \mathbf{d_2}) = \frac{1}{n}\sum_{i=1}^{n} G^i\left(\frac{M_i - w_{i1}}{M_i - m_i}, \frac{M_i - w_{i2}}{M_i - m_i}\right)$$

where $G^i : [0,1]\mathrm{x}[0,1] \to R^1$ is the improvement index based on attribute i only. This property says that the overall improvement level is simply the arithmetic average of improvement indices based on individual attributes. This subdivision allows qualitative as well as quantitative assessment of attribute - wise improvement. The quantity $T_i = G^i(d_{i1}, d_{i2})/n$ may be interpreted as the total contribution for attribute i to overall improvement G, while $100T_i/G$ is the percentage contribution of attribute i. Therefore, this type of breakdown will help us to isolate the attributes which are less susceptible to aggregate level of improvement.

6.3 A family of Additive Improvement Indices

To suggest a general family of improvement measures let us consider the class $\mathbf{F}$ of all real valued functions defined on $[0,1]$ which are increasing, strictly concave, differentiable on $(0,1)$, continuous at end points, and take on the values zero and one at zero and one respectively. More precisely, $f : [0,1] \to R^1$ is a member of $\mathbf{F}$ if f is increasing, strictly concave, differentiable on $(0,1)$, continuous at 0 and 1, and $f(0) = 0$ and $f(1) = 1$. Strict concavity of f ensures that it is differentiable at each point of $(0,1)$, except on a countable set (**Royden (1968)**, p.109). To rule out nondifferentiability of f on a countable subset of $(0,1)$ we assume at the outset that f is differentiable on $(0,1)$.

Examples of functions which are members of $\mathbf{F}$ are

$$\begin{aligned} (i)\ f_1(t) &= t^r, \quad 0 < r < 1. \\ (ii)\ f_2(t) &= \frac{1 - e^{-t}}{1 - e^{-1}}. \\ (iii) f_3(t) &= \frac{2t}{1+t}. \end{aligned}$$

In the proposition we prove in this section we show that under the assumptions made about members of **F**, the general measure suggested below satisfies all of the properties discussed in the previous section.

The general measure of improvement we suggest is

$$\begin{aligned} G(\mathbf{d_1};\mathbf{d_2}) &= \frac{1}{n}\sum_{i=1}^{n} G^i(d_{i1}, d_{i2}) \\ &= \frac{1}{n}\sum_{i=1}^{n}[f_i(d_{i1}) - f_i(d_{i2})] \\ &= \frac{1}{n}\sum_{i=1}^{n}[f_i(\frac{M_i - w_{i1}}{M_i - m_i}) - f_i(\frac{M_i - w_{i2}}{M_i - m_i})] \qquad (6.1) \end{aligned}$$

where $f_i \in \mathbf{F}$, i = 1,..,n; and $w_t \in \prod_{i=1}^{n}[m_i, M_i]$, t = 1,2 are arbitrary. Since the function f_i is defined on normalized deprivation levels, its values represent transformed normalized deprivations and we interpret $f_i(d_{i1}) - f_i(d_{i2})$ as the achievement index $G^i(d_{i1}, d_{i2})$ for attribute i.

Proposition 1 : Let $f_i \in \mathbf{F}$. Then the general measure G suggested in (6.1) satisfies DR, CN, MN, PC, NR, ID, DM, AD.

Proof : It follows from the definition of G that it satisfies DR. Next, using strict concavity of f_i, we show that G satisfies ID. For this, choose $\mathbf{w_1}, \mathbf{u_1} \in \prod_{i=1}^{n}[m_i, M_i]$ such that $w_{j1} = u_{j1}$ for $j \neq i$ and $w_{i1} > u_{i1}$. Then $G(\mathbf{d}^{\mathbf{w_1}}; \mathbf{d}^{\mathbf{w_1}+c\mathbf{e}_i}) = f_i(\frac{M_i - w_{i1}}{M_i - m_i}) - f_i(\frac{M_i - w_{i1} - c}{M_i - m_i})$. Similarly, $G(\mathbf{d}^{\mathbf{u_1}}; \mathbf{d}^{\mathbf{u_1}+c\mathbf{e}_i}) = f_i(\frac{M_i - u_{i1}}{M_i - m_i}) - f_i(\frac{M_i - u_{i1} - c}{M_i - m_i})$. Therefore, $G(\mathbf{d}^{\mathbf{w_1}}; \mathbf{d}^{\mathbf{w_1}+c\mathbf{e}_i}) > G(\mathbf{d}^{\mathbf{u_1}}; \mathbf{d}^{\mathbf{u_1}+c\mathbf{e}_i})$, means that

$$f_i(\frac{M_i - w_{i1}}{M_i - m_i}) - f_i(\frac{M_i - w_{i1} - c}{M_i - m_i}) > f_i(\frac{M_i - u_{i1}}{M_i - m_i}) - f_i(\frac{M_i - u_{i1} - c}{M_i - m_i}). \qquad (6.2)$$

Dividing inequality (6.2) by c and letting c tend to zero, the left and right hand sides become $-f_i'(\frac{M_i - w_{i1}}{M_i - m_i})$ and $-f_i'(\frac{M_i - u_{i1}}{M_i - m_i})$ respectively, where f_i' is the derivative of f_i. Since f_i is strictly concave, f_i' is decreasing and since $w_{i1} > u_{i1}$, we must have $-f_i'(\frac{M_i - w_{i1}}{M_i - m_i}) > -f_i'(\frac{M_i - u_{i1}}{M_i - m_i})$, which shows that G meets ID.

Given that f_i is differentiable on $(0,1)$, it is continuous on $(0,1)$. This along with continuity of f_i at zero and one establishes that f_i is continuous on $[0,1]$. Hence G^i is continuous on $[0,1]$x$[0,1]$ from which continuity of G on $[0,1]^n$x$[0,1]^n$ follows (**Apostol (1971)**, p.132).

Increasingness of f_i shows that as w_{i2} rises G^i rises. Similarly, if w_{i1} rises G^i falls for an increasing f_i. Hence postulate MN follows. By construction, G satisfies DM and AD. It is very easy to check that G satisfies PC. Since $f_i(0) = 0$ and $f_i(1) = 1$, $f_i(\frac{M_i - w_{i1}}{M_i - m_i})$ and $f_i(\frac{M_i - w_{i2}}{M_i - m_i})$ take on the values one and zero respectively when w_{i1} coincides with m_i and w_{i2} coincides with M_i, that is, G^i becomes one in this case. Hence G satisfies NR. This therefore completes the proof of the proposition. ■

The proof of the proposition shows that strict concavity of f_i can be regarded as a sufficient condition for G to satisfy ID. Another interesting point that may be noted here is that G is uniformly continuous. Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point. In general, uniform continuity implies but is not implied by continuity. However, since on a compact (that is, closed and bounded) set continuity and uniform continuity are equivalent (**Apostol (1971)**, p.75), continuity of f_i on $[0,1]$ implies its uniform continuity on $[0,1]$, from which uniform continuity of G follows.

To illustrate the general formula G, let us suppose for simplicity that f_i's are identical, that is $f_i = f$ for all i and the function f is of the type $f(t) = t^r$, $0 < r < 1$. Then G becomes

$$\begin{aligned} G_r &= \frac{1}{n}\sum_{i=1}^{n} G_r(d_{i1}, d_{i2}) \\ &= \frac{1}{n}\sum_{i=1}^{n}[(\frac{M_i - w_{i1}}{M_i - m_i})^r - (\frac{M_i - w_{i2}}{M_i - m_i})^r] \end{aligned} \tag{6.3}$$

The parameter r reflects different perceptions of improvement. If $w_{i1} < w_{i2}$ for all i, that is, if improvement takes place with respect to all attributes, then G_r increases as r increases. In this case a higher value of r gives greater emphasis to the attributes whose contributions are low to the overall improvement. However, if the inequality $w_{i1} < w_{i2}$ is violated for some i, then nothing can be concluded unambiguously about the monotonocity of G_r with respect to r. On the other hand, if $w_{i1} > w_{i2}$ for all i, that is, if there has not been improvement with respect to any attribute, then G_r decreases as r increases.

For $r = 0$, $G_r = 0$. In contrast, for $r = 1$, $G_r = \frac{1}{n}\sum_{i=1}^{n}(\frac{w_{i2} - w_{i1}}{M_i - m_i})$, the average of increase in the attainment levels of the attributes, expressed as fraction of maximum increase in attainment levels. However, for $r = 1$ the postulate (ID) is violated.

If there is only one attribute, $G_r = (\frac{M_1 - w_{11}}{M_1 - m_1})^r - (\frac{M_1 - w_{12}}{M_1 - m_1})^r$, the **Kakwani** index of improvement which is also the **Tsui** index for the single attribute case.[1] For $r = 1$, in the single attribute case, G_r is related to the **Sen (1981)** index S by $G_r = S\,\frac{M_1 - m_1}{M_1 - w_{11}}$.

It is clear that, given any $f \in$ F, there exists a corresponding improvement index G_f. These indices will differ only in the manner how we transform the deprivation levels $(\frac{M_i - w_{i1}}{M_i - m_i})$ and $(\frac{M_i - w_{i2}}{M_i - m_i})$ into values $f(\frac{M_i - w_{i1}}{M_i - m_i})$ and $f(\frac{M_i - w_{i2}}{M_i - m_i})$ using the transformation f. However, for any $f \in$ F, the index G_f will meet all the desirable properties of an improvement index.

We may mention here that the **Tsui** multidimensional index, which is given by

$$U = \frac{\prod_{i=1}^{n}(M_i - w_{i1})^{r_i} - \prod_{i=1}^{n}(M_i - w_{i2})^{r_i}}{\prod_{i=1}^{n}(M_i - m_i)^{r_i}}, \tag{6.4}$$

$0 < r_i < 1$, will meet all the postulates except additivity. However, **Tsui's** major objective was to generalize the **Kakwani** index to a multidimensional framework, which he did quite successfully.

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[1] As stated in section 1.5, **Kakwani (1993)** also argues that $log(\frac{M_1 - w_{11}}{M_1 - m_1}) - log(\frac{M_1 - w_{12}}{M_1 - m_1})$ can be regarded as an improvement index. But because of some shortcoming pointed out by **Tsui** for this index, we will not go for further analysis of this form.

6.4 Numerical Illustration

In this section the improvement index G_r is applied to UNDP data on human development to illustrate the usefulness of additivity. It may be important to note that the UNDP reports presents numerical values of the Human Development Index (HDI) for more than 170 countries. The HDI in year t is defined as

$$H = \frac{1}{k}\sum_{i=1}^{k}\frac{w_{it} - m_i}{M_i - m_i}.$$

The indicators used by UNDP for calculating H are: life expectancy at birth, e (in years); educational attainment as measured by a combination of adult literacy, l (in %) and combined primary, secondary and tertiary enrolment ratios c (in %), where the weights are $\frac{2}{3}$ and $\frac{1}{3}$ respectively; and real GDP per capita, g (in PPP$). For the construction of the index, fixed minimum and maximum values have been established for each of these indicators. Depending on the value of HDI, the UNDP reports subdivide the countries into three groups : high human development (HHD), medium human development (MHD) and low human development (LHD).

For our analysis we choose six countries. The countries chosen are Canada, France and Romania (HHD), Saudi-Arabia (MHD), and India and Rwanda (LHD). The periods over which we look at change in well-being is the interval 1987 - 1993. In fact, given the UNDP data, this is the longest period that can be covered. The attributes of well-being we take for our analysis are e, l and g. Although here we are looking at changes in well-being between the periods 1987 and 1993, we can consider any intermediate period t between these two years. By period consistency, the sum of improvements from 1987 to t and then from t to 1993 is same as that from 1987 to 1993. As noted above, UNDP calculates enrolment ratio using enrolment figures from the three levels of education with the help of some weighting scheme and then this enrolment ratio is combined with the literacy rate to arrive at the educational attainment indicator. Thus, some double scaling is involved in the calculation of the educational attainment indicator. In order to avoid this, we use l as an indicator of well-being.

In recent studies **UNDP** has found that there exists high correlation among these three attributes of well-being. The correlations, based on 1992 data, are: 0.848 between i and e, 0.871 between e and educational attainment (a combination of adult literacy and combined primary, secondary and tertiary enrollment ratios) and 0.729 between i and educational attainment. **Endow (1996)** has also reported such high values (0.67 between l and e and 0.46 between e and i) based on 1993 data. Endow reports similar findings with 1988-92 data from 15 major states in India. The correlation between l and e is 0.67 though other correlations are smaller.

The above findings seems to vindicate the choice of these 3 indicators simultaneously (due to possible redundancy in explanatory power). But Principal Component analysis carried out by UNDP shows that the eigenvector corresponding to the leading eigenvalue (which explains 88% of the total variance in the data) puts virtually equal weight on the three variables. 'Thus, it does not advocate omitting or downgrading a variable' (**UNDP (1993)**, p.119).

For each attribute the maximum value is taken as the maximum observed value of the attribute over countries and over two time periods considered. More precisely, let w_{it}^{c} be the observed value of the attribute i for country c at time t where i = e, l, g and t = 1987, 1993. Then $M_i = max_{c,t} w_{it}^{c}$. We choose the minimum values in a similar way. That is, $m_i = min_{c,t} w_{it}^{c}$. The maximum and minimum values of different attributes and the corresponding countries and year for which they are observed are $M_e = 79.6$ (Japan, 1993), $M_l = 99.0$ (many countries including Canada, France, USA belonging to the HHD, 1993 and also many countries in HHD group, 1987), $M_g = 25390$ (Luxembourg, 1993), $m_e = 39.2$ (Sierra Leone, 1993), $m_l = 12$ (Somalia, 1987), $m_g = 400$ (Chad, 1987).

The relationship between well-being improvement and specific factors of improvement may be analyzed with the aid of a collection of tables called an improvement profile. *Tables 6.1 - 6.4* show such a profile, describing improvement in well-being generated by different sources. In *table 6.1*, the first column gives the names of the countries for which the analysis is carried out. In columns 2-4 we present, for each country, the level of improvement for the three sources of well-being assuming that r = .25. These improvement levels are then averaged to determine the overall improvement $G_{.25}$ which is presented in column 5. Finally, columns 6-8 show, for each country, the percentage contributions of improvement with respect to the alternative attributes of well-being to overall improvement.

From *table 6.1* several interesting features emerge. Although for each country, the overall level of improvement turns out to be positive, the picture appears to be dismal for some specific attributes for some countries. For instance, for Rwanda, life expectancy at birth has decreased significantly during the period 1987 - 1993. The percentage contribution to overall well-being improvement with respect to improvement in this factor is significantly negative (-34.4 %). A very high adult literacy improvement makes the overall improvement index positive. (The contribution of the third factor is rather low as compared to these two factors.) On the other hand, for India we see that all the sources contribute positively to overall improvement, though the contribution of adult literacy is higher compared to that of the other two attributes. For Romania a very high negative contribution comes from life expectancy at birth. This significant negative contribution outweighs the extremely high positive effect of literacy rate and makes the overall index lower than that for India. It may be interesting to note that at the base period (1987) Romania belonged to the group HHD but low improvement during the period 1987 - 1993 relegated it to the group MHD in 1993. However, the groups of attachment of all other countries remained unaltered. We also note that, for Saudi Arabia, the variation among the attributes in relation to their percentage contributions to overall improvement is minimum among the countries considered. In fact, among the countries considered, the maximum improvement during the reference period has been observed for this country only. For Canada and France adult literacy remained at the maximal level in both 1987 and 1993, but real GDP is found to contribute significantly to overall improvement. There has been some improvement in life expectancy at birth.

Tables 6.2, 6.3 and 6.4 present similar figures for r = .5, .75 and 1. We can analyze these tables in a manner in which we analyzed *table 6.1*. *Tables 6.1 - 6.4* show that the index values as well as percentage contributions are sensitive to the value of r. We have pointed out earlier that if positive improvement takes place with respect to all attributes, then G_r increases as r increases. This is confirmed by index values calculated for India and Saudi Arabia.

6.5 Concluding Remarks

Kakwani (1993) constructed a class of improvement indices of well-being satisfying certain desirable properties. **Tsui (1996b)** interestingly extended **Kakwani's** analysis to a multidimensional set-up. In this chapter we have suggested a family of multidimensional improvement indices which can be expressed as the average of improvement levels for different attributes of well-being. An index showing this type of breakdown becomes quite important from policy point of view - sources for which improvement is negative or low can be identified. In the single-dimensional case one of the members of this family coincides with the **Kakwani** class and unidimensional **Tsui** class of indices.

Table 6.1: Subdivision of Improvement $G_{.25}$ in Well-being by Sources of Improvement

Country	Improvement based on				Percentage contribution to $G_{.25}$ based on		
	e	l	g	$G_{.25}$	e	l	g
CANADA	.0087	0	.0419	0.0506	17.2	0	82.8
FRANCE	.0142	0	.0384	0.0526	27.0	0	73.0
SAUDI-ARABIA	.0294	.0107	.0211	0.0612	48.1	17.4	34.5
ROMANIA	-.0069	.0123	.0027	0.0080	-86.4	153.1	33.3
INDIA	.0060	.0107	.0006	0.0173	34.6	61.7	3.7
RWANDA	-.0045	.0169	.0006	0.0130	-34.4	130.1	4.4

Table 6.2: Subdivision of Improvement $G_{.5}$ in Well-being by Sources of Improvement

Country	Improvement based on				Percentage contribution to $G_{.5}$ based on		
	e	l	g	$G_{.5}$	e	l	g
CANADA	.0086	0	.0597	0.0683	12.5	0	87.5
FRANCE	.0149	0	.0587	0.0737	20.3	0	79.7
SAUDI-ARABIA	.0438	.0176	.0370	0.0985	44.5	17.9	37.6
ROMANIA	-.0095	.0101	.0052	0.0057	-166.3	176.3	90.0
INDIA	.0100	.0188	.0013	0.0301	33.3	62.5	4.2
RWANDA	-.0084	.0289	.0011	0.0216	-38.9	133.7	5.2

Table 6.3: Subdivision of Improvement $G_{.75}$ in Well-being by Sources of Improvement

Country	Improvement based on				Percentage contribution to $G_{.75}$ based on		
	e	l	g	$G_{.75}$	e	l	g
CANADA	.0063	0	.0639	0.0702	9.0	0	91.0
FRANCE	.0118	0	.0675	0.0793	14.9	0	85.1
SAUDI-ARABIA	.0489	.0219	.0488	0.1196	40.9	18.3	40.8
ROMANIA	-.0099	.0063	.0075	0.0039	-253.2	160.6	192.6
INDIA	.0126	.0248	.0019	0.0393	32.0	63.2	4.8
RWANDA	-.0119	.0370	.0017	0.0268	-44.2	137.9	6.3

Table 6.4: Subdivision of Improvement G_1 in Well-being by Sources of Improvement

Country	Improvement based on				Percentage contribution to G_1 based on		
	e	l	g	G_1	e	l	g
CANADA	.0041	0	.0610	0.0651	6.3	0	93.7
FRANCE	.0083	0	.0691	0.0773	10.7	0	89.3
SAUDI-ARABIA	.0487	.0241	.0571	0.1299	37.5	18.6	43.9
ROMANIA	-.0091	.0034	.0097	0.0041	-223.0	84.7	238.3
INDIA	.0140	.0291	.0025	0.0456	30.7	63.8	5.5
RWANDA	-.0149	.0421	.0023	0.0295	-50.2	142.6	7.6

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