C^*-Extreme Quantum Instruments; Completion and Disintegration of Completely Positive Maps

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Dedicated

to

my beloved family and teachers

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November, 2025 *Arghya Chongdar*

Contents

List of Symbols

$\varnothing$	Empty set
$\mathbb{N}$	Set of all natural numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{M}_n(\mathbb{C})$	$n \times n$ matrix algebra over $\mathbb{C}$
$\mathbb{M}_n(\mathcal{A})$	Set of all $n \times n$ matrices over a C^*-algebra $\mathcal{A}$
$\mathcal{B}(\mathcal{H})$	Algebra of bounded operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}(\mathcal{H}, \mathcal{K})$	Algebra of bounded operators between the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$
$CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$	Set of all completely positive maps from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$
$\mathcal{O}(X)$	σ-algebra of subsets of a set X
$Ins_{\mathcal{H}}(X, \mathcal{A})$	Set of all CP instruments on $(X, \mathcal{O}(X))$ with values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$
$I_{\mathcal{H}}(X, \mathcal{A})$	Set of all unital instruments on $(X, \mathcal{O}(X))$ taking values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$
$\mathcal{R}$-$Ins_{\mathcal{H}}(X, \mathcal{A})$	Set of all regular instruments on $(X, \mathcal{O}(X))$ taking values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$
$\mathcal{R}$-$I_{\mathcal{H}}(X, \mathcal{A})$	Set of all regular instruments on $(X, \mathcal{O}(X))$ taking values in $UCP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$
$[S]$	Closure of the linear span of a subset S of $\mathcal{B}(\mathcal{H})$
$\mathcal{M}'$	Commutant of a subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$
Tr	Trace of a trace class operator
1_A	Indicator function of a set A

Introduction

The Hilbert space formulation of quantum mechanics, pioneered by von Neumann, established the rigorous mathematical foundation that later evolved into non-commutative functional analysis. As this framework matured, it became necessary to reinterpret and extend classical probabilistic notions to the non-commutative setting, giving rise to quantum probability theory. This reformulation was motivated not only by the desire to understand the formal structure of quantum mechanics more deeply but also by the intrinsic mathematical richness of the resulting framework. A decisive turning point occurred with the pioneering work of Murray and von Neumann in the late 1930s and early 1940s, which laid the foundations for non-commutative integration and probability through replacing classical functions by operators. The subsequent development of this non-commutative approach was marked by several key milestones, notably including the Gelfand–Naimark–Segal construction for C^*-algebras.

The profound influence of classical convexity in analysis naturally inspired the search for its non-commutative counterparts. This effort has led to several operator-theoretic generalizations, each capturing a distinct aspect of convexity in the operator-algebraic setting. Effros and Winkler introduced matrix convexity [EW97], providing operator analogues of the bipolar and Hahn–Banach theorems. Loebl and Paulsen [LP81], and later Farenick and Morenz [FM93], developed C^*-convexity, which extends convexity notions to sets of C^*-algebra elements. Fujimoto [Fuj93] formulated CP-convexity, emphasizing completely positive maps as the non-commutative analogue of classical convex combinations. More recently, Davidson and Kennedy [DK19] explored non-commutative convexity in the broader operator system context, providing a unified framework that generalizes many of the previous approaches. Together, these developments illustrate the rich interplay between convexity and operator-algebraic structures in the non-commutative setting. In this thesis, we focus primarily on the notion of C^*-convexity, in which positive scalars adding to unity in a classical convex combination are replaced by not-necessarily commuting operators with similar property.

The classical Riesz representation theorem, which establishes a correspondence between regular Borel measures on compact Hausdorff spaces and positive linear functionals, provides a natural bridge to extending measure-theoretic ideas into the realm of quantum probability, bypassing the limitations of standard measure-theoretic frameworks. Within this setting, completely positive (CP) maps on C^* -algebras, introduced by Stinespring [Sti55], emerge as a fundamental concept, often serving as operator-theoretic analogues of positive linear functionals. Building on this foundation, Arveson [Arv69] developed a comprehensive structural theory of CP maps, establishing a systematic framework for their study. Over time, CP maps have become indispensable tools across operator algebras, with specific classes such as Markov maps and trace-preserving CP maps playing a pivotal role in quantum probability and quantum information theory, respectively.

This framework of CP maps naturally extends to the mathematical modeling of quantum measurements, where the need arises to describe not only outcomes but also the corresponding changes in quantum states. In this context, Davies and Lewis [DL70] introduced a mathematical framework for quantum measurements. Their key idea was to define an *"instrument"* to capture the complexities of quantum measurement processes. According to them, an instrument generalizes both the concept of an observable and that of an operation. It is particularly relevant for situations involving continuous observables and repeated measurements, and can be interpreted in different ways [Sri80]. A decisive advance in this theory was achieved by Ozawa [Oza84], who showed that the state changes determined by measuring processes naturally correspond to completely positive instruments and vice versa. In simple terms, quantum instruments or CP instruments are completely positive (CP) map–valued measures, generalizing both positive operator valued measures (POVMs) and CP maps. From a mathematical point of view, Ozawa developed a dilation theory for CP instruments, which became a cornerstone for subsequent progress. In particular, Holevo [Hol98] established a Radon–Nikodym type theorem for CP instruments, offering deeper structural insights. This line of investigation was further extended in [CDP09], and more recently, the marginal problems for CP instruments have attracted significant attention, as studied in [DPS11] and by Juha-Pekka Pellonpää et al. [Pel13a; Pel13b; HHP14; HP11].

Just as CP instruments provide a non-commutative analogue of joint probability measures, the classical notion of disintegration [Fre06; Kal02] admits a quantum counterpart within the framework of CP maps on C^*-algebras. Heuristically, disintegration can be viewed as an inverse operation of the product measure construction using conditional probability. Several approaches to non-commutative disintegration have appeared in the literature (see, for instance, [BFL11; Fri20; CJ19]), each addressing different aspects of this generalization. Among these, the formulation by Parzygnat and Russo [PR23] provides a natural and rigorous framework by interpreting CP maps as quantum Markov maps and introducing non-commutative disintegration in a categorical setting. This approach unifies several concepts—including perfect quantum error-correcting codes, regular conditional probabilities, conditional expectations, and classical disintegration consistent with measure-preserving maps—within a single scheme, and establishes existence results in finite-dimensional settings, particularly for $*$-homomorphisms.

In this work, we focus on the operator-algebraic aspects of the topics discussed above, emphasizing their structural and conceptual implications.

Organization of the thesis:

The rest of the thesis is structured as follows.

- In Chapter 1 we start with fundamental concepts from the theory of C^*-algebras and von Neumann algebras. The chapter then introduces completely positive (CP) maps, positive operator-valued measures (POVMs), and CP instruments, along with their associated dilation theory and foundational results such as the bi-dilation theorem and a Radon–Nikodym-type theorem. Key contributions include the introduction of pure instruments (Definition 1.4.18) and their characterization via minimal bi-dilations, as well as the study of sub-minimal dilations and their relationship to the canonical bi-dilation of an instrument. We also revisit the notion of decomposable instruments (Definition 1.4.29) and provide a characterization in terms of minimal bi-dilations (Theorem 1.4.30). Notably, the chapter presents the realization of CP instruments as bivariate maps (Theorem 1.4.4), providing a rigorous framework for extending classical joint measures to the quantum setting.
- In Chapter 2, we commence with a review of standard characterization results concern-

ing classical extreme points of instruments, emphasizing their connection with the classical Choi–Kraus representation. In addition, we present a characterization of extreme instruments formulated via a natural partial order (Theorem 2.1.6). The principal result of this chapter, Theorem 2.1.37, provides a structural characterization of C^*-extreme unital completely positive (UCP) instruments on finite-dimensional Hilbert spaces. The proof of this result draws upon methods from the theory of nest algebras, highlighting the deep interplay between operator-algebraic structure and convexity. Subsequently, we investigate the relationship between an instrument and its marginals within the framework of C^*-convexity. In particular, it is shown that for both classically extreme and C^*-extreme instruments whose ranges are commutative, the corresponding POVM marginal is necessarily spectral. The chapter concludes with Theorem 2.2.12, which establishes that the C^*-extremality of the marginals entails the C^*-extremality of the instrument itself, thereby revealing further structural consequences. We also revisit a result due to Pellonpää *et al.* concerning the determination of an instrument from its marginals and establish that the C^*-extremity of any one marginal suffices to uniquely determine the entire instrument. Moreover, in such cases, the instrument assumes a decomposable form, mirroring the classical property of a joint probability measure being expressible as the product of its marginals.

- Chapter 3 develops a systematic framework for integration with respect to completely positive (CP) instruments, drawing inspiration from Bartle's theory of vector integration. Beginning with simple functions and extending the construction via instrument-measurable approximations, we obtain a coherent and internally consistent integration theory that captures the operational and operator-algebraic aspects of quantum measurement. A principal outcome of this development is the CP–instrument correspondence (Theorem 3.1.27), which extends the classical CP–POVM correspondence to the setting of regular CP instruments on compact Hausdorff spaces. The chapter also presents an operator-algebraic formulation of integration based on tensor-product extensions of instruments to CP maps, offering a complementary structural perspective that will be utilized in later chapters.

  The Krein–Milman theorem, a foundational result in classical functional analysis, asserts that every compact convex subset of a locally convex topological vector space is the closed convex hull of its extreme points. Motivated by this paradigm, it is natural to investigate an analogue within the framework of C^*-convexity, particularly for classes of completely positive maps and instruments. Several such analogues are known: compact C^*-convex subsets of $M_n(\mathbb{C})$ satisfying Krein–Milman type results [Mor94]; BW-compact C^*-convex sets $CP(\mathcal{A}, \mathcal{H})$ of unital CP maps for certain C^*-algebras $\mathcal{A}$ and separable Hilbert spaces $\mathcal{H}$ [BK22]; as well as examples of C^*-convex weak-$*$ compact sets that do not admit any C^*-extreme points [Mag18]. In contrast, no corresponding result had previously been established for CP instruments. As an application of the integration theory developed in this chapter, we prove a Krein–Milman type theorem for instruments on separable C^*-algebras $\mathcal{A}$ (Theorem 3.1.33), thereby filling this gap in the existing literature.

- Chapter 4 is devoted to extending the classical positive completion problem for matrices to the setting of completely positive (CP) maps, a problem we refer to as the CP completion problem—namely, the completion of linear maps to CP maps on C^*-algebras. The Minimal Completion Theorem (Theorem 4.2.4) establishes that whenever a linear map admits a CP completion, there exists a unique minimal CP completion. The chapter also introduces and develops the concept of quasi-pure maps in detail. As a final outcome, we extend the key result of Parzygnat and Russo (Theorem 2.48 in [PR23]) to the most abstract setting—showing, on matrix algebras, if a CP map is equal to the identity map almost everywhere, then it must in fact coincide with it. This result is generalized in full generality for von-Neumann algebras in Corollary 4.3.6.

- The study of non-commutative disintegration forms the core theme of Chapter 5. We begin by revisiting the classical notion of disintegration to highlight the intuition behind its operator-algebraic extension. In the non-commutative setting, this problem translates, for matrix algebras, into the question of existence of a left inverse. Motivated by this connection, we analyze left-invertible completely positive maps in a general framework (Theorem 5.2.4) and obtain a structural description of left-invertible normal CP maps on $\mathcal{B}(\mathcal{H})$ (Corollary 5.2.6). The chapter concludes by extending the disintegration theory to infinite-dimensional Hilbert spaces, establishing both existence and uniqueness results (Theorem 5.2.7).

Chapter 1

Preliminaries

We begin by briefly recalling some foundational results from the literature on operator algebras that form the basis of this thesis. This will also serve to establish the notations and terminologies that will be consistently used throughout.

Notations

Unless stated otherwise, we adhere to the conventions listed below.

- Throughout the thesis, all Hilbert spaces are assumed to be separable and over $\mathbb{C}$.
- Whenever used, inner products are assumed to be linear in the second variable and conjugate-linear in the first.
- $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$.
- For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we write $\mathcal{B}(\mathcal{H},\mathcal{K}) = \{T : \mathcal{H} \to \mathcal{K} | \ T \textit{ is bounded and linear}\}$.
- Unless specified otherwise, every projection is taken to be self-adjoint, idempotent, and all subspaces are understood to be closed.
- If S is a non-empty subset of $\mathcal{H}$, $[S]$ will denote the closure of the linear span of S.
- $M_n(\mathbb{C})$ and $M_{k\times l}(\mathbb{C})$, $n, k, l \in \mathbb{N}$, denotes the algebra of $n \times n$ and $k \times l$ matrices over $\mathbb{C}$, respectively.

Additional notations can be found in the 'List of Symbols' appended to the thesis.

1.1 Overview on Operator Algebras

Francis Murray and John von Neumann initiated the study of specific classes of subalgebras of bounded linear operators on Hilbert spaces. This line of inquiry led to the development of operator algebras, particularly von Neumann algebras (also known as W^*-algebras), which exhibit rich structural properties and play a central role in the mathematical formulation of quantum theory.

Subsequently, in a landmark paper in 1943, Gelfand and Naimark introduced the abstract definition of C^*-algebras as Banach $*$-algebras satisfying the C^*-identity and showed that every such algebra can be faithfully represented as a norm-closed $*$-subalgebra of bounded operators on a Hilbert space. Their work unified the concrete and abstract perspectives, and laid the groundwork for the modern theory of C^*-algebras. These two classes of operator algebras now serve as foundational tools in noncommutative functional analysis, which, in turn, provides the mathematical backbone of contemporary quantum physics, quantum information theory, and quantum field theory. In this section, we briefly review some of the fundamental concepts and ideas in the theory of operator algebras, with a primary focus on C^*-algebras and von Neumann algebras.

Comprehensive introductions to the theory of operator algebras can be found in the classical texts by Arveson [Arv76], Conway [Con90], Kadison–Ringrose [KR83], Murphy[Mur90], and Takesaki [Tak02].

1.1.1 C^*-algebras

As noted previously, the abstract framework of C^*-algebras was profoundly shaped by the seminal work of Gelfand and Naimark, who formulated them as Banach algebras endowed with a $*$-involution and satisfying the C^*-identity, thus establishing a deep interplay between algebraic structure and analytic norm properties.

A complex algebra $\mathcal{A}$ is called a Banach algebra if it is a Banach space equipped with a norm $\|\cdot\|$ satisfying

$$\|a_1 a_2\| \leq \|a_1\|\|a_2\| \quad \text{for all } a_1, a_2 \in \mathcal{A}.$$

Any Banach algebra $\mathcal{A}$ equipped with an involution $* : \mathcal{A} \to \mathcal{A}$ is called a Banach $*$-algebra.

Definition 1.1.1. (C^*-algebra) A complex Banach algebra $\mathcal{A}$ becomes a *C^*-algebra* when it admits an involution $* : \mathcal{A} \to \mathcal{A}$, such that the norm and the involution are compatible in the sense that : $\|a^*a\| = \|a\|^2, \ \forall\, a \in \mathcal{A}$.

Remark 1.1.2. The identity above is known as the C^*-identity.

A C^*-algebra $\mathcal{A}$ is called unital if it contains a multiplicative identity, denoted by $1_{\mathcal{A}}$. Given unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, a linear map $\psi : \mathcal{A} \to \mathcal{B}$ is called unital whenever it sends the unit of $\mathcal{A}$ to the unit of $\mathcal{B}$, i.e. $\psi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

Definition 1.1.3 ($*$-homomorphisms)**.** Let $\mathcal{A}, \mathcal{B}$ be two C^*-algebras. A linear map $\psi : \mathcal{A} \to \mathcal{B}$ is called a $*$-homomorphism when it preserves both the algebraic product and the involution, that is,

$$\psi(ab) = \psi(a)\psi(b), \quad \psi(a^*) = \psi(a)^* \quad \forall\, a, b \in \mathcal{A}.$$

A C^*-isomorphism is a $*$-homomorphism between C^*-algebras that is both injective and surjective. Two C^*-algebras are regarded as isomorphic when they are related by such a structure-preserving map.

Remark 1.1.4. Unless otherwise specified, the term isomorphism between two C^*-algebras will refer to a C^*-isomorphism.

Remark 1.1.5. Every $*$-homomorphism between C^*-algebras is contractive, meaning that for all $a \in \mathcal{A}$,

$$||\psi(a)|| \leq ||a|| \,.$$

In particular, this implies that all $*$-homomorphisms are continuous. Moreover, it is well known that if a $*$-homomorphism between two C^*-algebras is injective, then it is isometric. Notably, there exists a unique norm on a Banach algebra with involution that turns it into a C^*-algebra. This rigidity is a defining characteristic that sets the category of C^*-algebras apart from general Banach $*$-algebras.

Examples:

1. Let $\mathcal{H}$ be a Hilbert space. The collection $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, equipped with the operator norm and the adjoint operation, forms a unital C^*-algebra.

2. Further, any norm-closed $*$-subalgebra i.e. closed under the usual adjoint of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra equppied with the inherited operator norm.

3. Let X be a compact Hausdorff space, and denote by $C(X)$ the set of all complex-valued continuous functions on X. With pointwise operations, $C(X)$ forms a unital C^*-algebra. More precisely, for $g, h \in C(X)$ and $x \in X$,

$$(gh)(x) = g(x)h(x), \qquad g^*(x) = \overline{g(x)}.$$

The natural norm on this algebra is the supremum (uniform) norm,

$$\|g\|_\infty = \sup_{x \in X} |g(x)|,$$

and equipped with this norm and involution, $C(X)$ satisfies the C^*-identity. Hence $C(X)$ is a unital C^*-algebra.

A C^*-algebra $\mathcal{A}$ is said to be *abelian* (or *commutative*) if every pair of elements in $\mathcal{A}$ commute; that is, $a_1a_2 = a_2a_1$ for all $a_1, a_2 \in \mathcal{A}$.

In the next theorem, due to Gelfand, we will see that the realization given in Example 3 completely characterizes all unital commutative C^*-algebras, up to isomorphisms.

Theorem 1.1.6. (Gelfand Theorem for commutative unital C^*-algebras) Let $\mathcal{A}$ be a unital commutative C^*-algebra. Then there exists a compact, Hausdorff space X for which $\mathcal{A}$ is C^*-isomorphic to the algebra $C(X)$ of continuous complex-valued functions on X.

Remark 1.1.7. This result establishes an equivalence between the category of commutative C^*-algebras and that of compact Hausdorff spaces. It forms the conceptual foundation of non-commutative geometry.

In order to bridge the abstract theory of C^*-algebras with their concrete operator-theoretic manifestations, we now recall the classical Gelfand–Naimark–Segal (GNS) Theorem. This foundational theorem guarantees that any C^*-algebra admits a concrete representation as a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, in accordance with Example 2.

Theorem 1.1.8 (Gelfand–Naimark–Segal (GNS) Representation Theorem)**.** For every C^*-algebra $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}$ and an injective $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\pi(\mathcal{A})$ is a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. In particular, every C^*-algebra can be faithfully represented as a concrete operator algebra acting on a Hilbert space.

Definition 1.1.9. Let $\mathcal{A}$ be a unital C^*-algebra. For an element $a \in \mathcal{A}$ we use the following terminology:

1. a is called *self-adjoint (Hermitian)* if $a = a^*$;

2. a is called *normal* if it commutes with its adjoint, i.e. $a^*a = aa^*$;

3. a is called a *projection* if it is idempotent and self-adjoint, namely $a^2 = a$ and $a = a^*$;

4. a is called an *isometry* if $a^*a = 1_{\mathcal{A}}$;

5. a is called a *co-isometry* if $aa^* = 1_{\mathcal{A}}$;

6. a is called *unitary* if it is both an isometry and a co-isometry, equivalently $a^*a = aa^* = 1_{\mathcal{A}}$.

1.1.2 von Neumann algebra

von Neumann algebras, also known as W^*-algebras, form a distinguished subclass of C^*-algebras. They were originally introduced by and Francis Murray and John von Neumann in a seminal series of papers titled "Rings of Operators" during the 1930s and 1940s. Their development was motivated by foundational questions in ergodic theory and the mathematical formulation of quantum mechanics. For a foundational development of the theory, the reader may consult standard references such as Conway [Con90], Kadison–Ringrose [KR83], and Takesaki[Tak02].

von Neumann algebras play a central role in mathematical physics, particularly in the contexts of quantum statistical mechanics, quantum field theory, and the operator-theoretic formulation of quantum mechanics. Their rich structural properties make them indispensable in capturing both algebraic and topological aspects of quantum systems.

We introduce three important locally convex topologies on the operator algebra $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space.

The *weak operator topology (WOT)* is defined as the topology generated by the scalar maps

$$T \longmapsto \langle u, Tv \rangle,$$

with $u, v \in \mathcal{H}$. Equivalently, it is the weakest topology for which every seminorm

$$q_{u,v}(T) = |\langle u, Tv \rangle|$$

is continuous.

The *strong operator topology (SOT)* is determined by pointwise norm convergence on vectors, that is, by the seminorms

$$q_u(T) = \|Tu\|, \qquad u \in \mathcal{H}.$$

Whenever a subset of $\mathcal{B}(\mathcal{H})$ is convex, its WOT and SOT closures coincide.

Another important topology is the σ-*weak* (or *ultraweak*) topology. This is the topology induced by the seminorms

$$q_F(T) = |\operatorname{Tr}(TF)|,$$

where F ranges over all trace-class operators on $\mathcal{H}$.

Definition 1.1.10 (von Neumann algebra)**.** A unital $*$-subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra* if it is closed under the weak operator topology.

For any family $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H})$, its *commutant* is defined by

$$\mathcal{T}' = \{R \in \mathcal{B}(\mathcal{H}) : RT = TR \text{ for all } T \in \mathcal{T}\},$$

and its *double commutant* is given by $\mathcal{T}'' = (\mathcal{T}')'$.

The deep relation between these algebraic commutants and the operator topologies is captured by the classical result below.

Theorem 1.1 (Double Commutant Theorem)**.** *Let $\mathcal{N}$ be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The following conditions are equivalent:*

1. *$\mathcal{N}$ is a von Neumann algebra;*
2. *$\mathcal{N}$ is SOT-closed;*
3. *$\mathcal{N}$ is σ-weakly closed;*
4. *$\mathcal{N}'' = \mathcal{N}$.*

Examples:

1. The algebra $\mathcal{B}(\mathcal{H})$ is von Neumann algebra.
2. Consider a measure space (X, μ). The space $L^\infty(X, \mu)$, consisting of all (equivalence classes of) essentially bounded measurable functions on X, forms a commutative von Neumann algebra.

   This algebra admits a canonical representation on the Hilbert space $L^2(X, \mu)$ through multiplication operators. For $g \in L^\infty(X, \mu)$, define the bounded operator

   $$T_g : L^2(X, \mu) \to L^2(X, \mu)$$

   by

   $$(T_g\varphi)(x) = g(x)\,\varphi(x) \qquad \text{for } \mu\text{-a.e. } x \in X,$$

   where $\varphi \in L^2(X, \mu)$.

   Under this identification, $L^\infty(X, \mu)$ appears as a concrete, abelian von Neumann algebra inside $\mathcal{B}(L^2(X, \mu))$: it contains the identity operator, and it is closed with respect to the weak operator topology.

1.2 Completely positive maps

The concept of completely positive (CP) maps on C^*-algebras—and the dilation framework associated with them—originates from the pioneering work of Stinespring [Sti55]. His celebrated result, now referred to as the Stinespring dilation theorem, provides a representation-theoretic description of CP maps and serves as the cornerstone of dilation theory. A major advance in the structural understanding and applications of CP maps was made by Arveson in his seminal paper [Arv69], where he introduced key ideas that influenced the development of noncommutative dynamics. Over the decades, the theory of CP maps has attracted considerable attention, particularly in the realms of quantum probability and quantum information theory, where they serve as the natural mathematical formulation of quantum operations and channels.

For a comprehensive treatment of CP maps and their connections to operator space theory and noncommutative analysis, the reader is referred to Paulsen [Pau02].

Let $\mathcal{A}$ be a unital C^*-algebra. By the GNS construction, $\mathcal{A}$ admits a faithful representation on a Hilbert space; in particular, we may regard $\mathcal{A}$ as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

For each $n \in \mathbb{N}$, consider the algebra $M_n(\mathcal{A})$ of $n \times n$ matrices with entries from $\mathcal{A}$. Equipped with the usual matrix operations (entrywise scalar multiplication and addition, matrix multiplication, and the involution

$$[a_{ij}]^* = [a_{ji}^*],$$

this becomes a unital $*$-algebra. Since elements of $\mathcal{A}$ act on $\mathcal{H}$ as bounded operators, the space $M_n(\mathcal{A})$ acts naturally on

$$\mathcal{H}^{(n)} := \mathcal{H} \oplus \cdots \oplus \mathcal{H}$$

via the standard block-matrix action. Consequently, $M_n(\mathcal{A})$ embeds into $\mathcal{B}(\mathcal{H}^{(n)})$, from which it inherits the operator norm. It is routine to verify that $M_n(\mathcal{A})$ is norm-closed in this representation, and therefore is a unital C^*-algebra.

Now let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Given a linear map $\psi : \mathcal{A} \to \mathcal{B}$ and a fixed $n \geq 1$, the *ampliation* of ψ is the map

$$\psi_n([a_{ij}]) = [\,\psi(a_{ij})\,], \qquad [a_{ij}] \in M_n(\mathcal{A}).$$

Definition 1.2.1. A linear map $\psi : \mathcal{A} \to \mathcal{B}$ is called *positive* if $\psi(a) \geq 0$ whenever $a \geq 0$ in $\mathcal{A}$. It is said to be *completely positive (CP)* if each ampliation ψ_n is positive for every $n \in \mathbb{N}$.

Remark 1.2.2. In what follows, we shall frequently use the abbreviations *CP* and *UCP* to refer to completely positive and unital completely positive maps, respectively.

Notation. We denote by $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ the set of all completely positive maps from the C^*-algebra $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$, and by $UCP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ the subset consisting of all unital completely positive maps.

A basic but important fact is that positive linear maps between unital C^*-algebras are automatically bounded. More precisely (see [Pau02, Proposition 3.6]):

Proposition 1.2.3. For positive map ψ between unital C^*-algebras, $\|\psi\| = \|\psi(1)\|$. Furthermore, if ψ is CP, then $\sup_{n\geq 1} \|\psi_n\| = \|\psi(1)\|$. Conversely, any linear map ψ satisfying $\psi(1) = 1$ and $\|\psi\| \leq 1$ must be positive.

Example 1.2.4. Let $\mathcal{A}$ be a unital C^*-algebra. Every unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, i.e. every *representation*, is a UCP map. More generally, if $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a representation and $V : \mathcal{H} \to \mathcal{K}$ is a bounded operator, then

$$\psi(a) = V^*\pi(a)V, \qquad a \in \mathcal{A},$$

defines a CP map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Moreover, ψ is unital exactly when V is an isometry.

In his seminal paper [Sti55] Stinespring established the converse of the construction above: every completely positive map admits such a factorization. His proof parallels the philosophy of the GNS construction (see [Pau02, Theorem 4.1]).

Theorem 1.2 (Stinespring dilation theorem)**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map. Then there exist a Hilbert space $\mathcal{K}$, a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ such that*

$$\psi(a) = V^*\pi(a)V, \qquad a \in \mathcal{A}.$$

This triple $(\mathcal{K}, \pi, V)$ may be taken to satisfy the minimality condition

$$\mathcal{K} = [\,\pi(\mathcal{A})V\mathcal{H}\,].$$

Such a triple is called the minimal Stinespring dilation*. It is unique up to unitary equivalence: if $(\mathcal{K}_1, \pi_1, V_1)$ and $(\mathcal{K}_2, \pi_2, V_2)$ are two minimal dilations, then there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that*

$$UV_1 = V_2, \qquad U\pi_1(a) = \pi_2(a)U \quad \textit{for all } a \in \mathcal{A}.$$

Remark 1.2.5. The dilation space $\mathcal{K}$ associated with a minimal Stinespring triple $(\mathcal{K}, \pi, V)$ is not guaranteed to be separable, even under the assumption that the input Hilbert space $\mathcal{H}$ is separable. However, an inspection of the construction reveals that separability of both the C^*-algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ ensures that the resulting Stinespring space $\mathcal{K}$ is separable.

As often occurs in the theory of C^*-algebras, commutativity brings additional structural regularity. In particular, Stinespring [Sti55] showed that when the *domain* of a positive map is commutative, the map is automatically completely positive. Similarly, Arveson [Arv69] proved that the same conclusion holds when the *codomain* is commutative. The following result restates this fact in the case of commutative domains (see also [Pau02, Theorem 4.11]).

Proposition 1.2.6. Let X be a compact Hausdorff space and $\mathcal{A}$ a unital C^*-algebra. Then any positive linear map $\psi : C(X) \to \mathcal{A}$ is automatically completely positive.

Radon-Nikodym type Theorem

Some fundamental results on completely positive maps, due to Arveson [Arv69], will play an important role in the subsequent development. Before stating these results, we recall a few standard notions.

Definition 1.2.7. Let $\psi_1, \psi_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely positive maps. We say that ψ_1 is *dominated* by ψ_2 (denoted $\psi_1 \leq \psi_2$) whenever the map $\psi_2 - \psi_1$ is itself completely positive.

Motivated by the classical Radon–Nikodym theorem from measure theory, Arveson established an operator-algebraic analogue for CP maps (Theorem 1.4.2 in [Arv69]), providing a precise criterion for comparing CP maps.

Theorem 1.2.8 (Radon–Nikodym type theorem)**.** Let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map, and let $(\mathcal{K}, \pi, V)$ denote its minimal Stinespring representation. A CP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies the domination relation $\phi \leq \psi$ exactly when there exists an operator T in the commutant $\pi(\mathcal{A})'$ which is positive and contractive, for which

$$\phi(a) = V^* T \, \pi(a) \, V, \qquad \text{for all } a \in \mathcal{A}.$$

Let us recall the notion of pure completely positive maps, beginning with a standard definition from representation theory.

Definition 1.2.9. A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be *irreducible* when it admits no non-trivial closed subspace of $\mathcal{H}$ that is invariant under $\pi(\mathcal{A})$. Equivalently, the commutant of $\pi(\mathcal{A})$ is as small as possible, namely $\pi(\mathcal{A})' = \mathbb{C}\, I_{\mathcal{H}}$.

Definition 1.2.10. A CP map ψ is said to be *pure* if the only completely positive maps dominated by it are its scalar multiples. More precisely, whenever a CP map ϕ satisfies $\phi \leq \psi$, there exists a scalar $\lambda \in [0, 1]$ with $\phi = \lambda \, \psi$.

The next result (see Corollary 1.4.3 in [Arv69]) provides a characterization of pure CP maps in terms of their Stinespring dilations and is an immediate consequence of Theorem 1.2.8.

Proposition 1.2.11. Let ψ be a CP map and let $(\mathcal{K}, \pi, V)$ denote its minimal Stinespring representation. Then ψ is pure precisely when the associated representation π is irreducible; that is, $\pi(\mathcal{A})' = \mathbb{C} I_{\mathcal{K}}$.

Bounded weak (BW) topology

In what follows, we endow the space $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ of CP maps with a topology analogous in spirit to the weak*–topology on the state space. This topology, formulated in terms of net convergence in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, was first introduced by Arveson in [Arv69](see also [Pau02, Lemma 7.2]).

Definition 1.2.12. Let $\{\psi_i\}$ be a bounded net in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$. We say that ψ_i *converges* to ψ in the **bounded weak** (BW) topology if $\psi_i(a) \to \psi(a)$ in the weak operator topology (WOT) for every $a \in \mathcal{A}$.

Fix a map $\psi' \in CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$. Neighbourhoods of ψ' in the BW topology are generated by sets of the form

$$\Big\{\psi \in CP(\mathcal{A}, \mathcal{B}(\mathcal{H})) : \big|\langle(\psi(a_j) - \psi'(a_j))h_j,\, k_j\rangle\big| < \varepsilon,\ 1 \leq j \leq n\Big\},$$

where $a_j \in \mathcal{A}$, $h_j, k_j \in \mathcal{H}$, and $\varepsilon > 0$.

In close analogy with the weak*-compactness of the classical state space, the space $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ is compact when equipped with the BW topology. The following theorem (see Theorem 7.4 in [Pau02]) establishes this fact.

Theorem 1.2.13. Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{H}$ be a Hilbert space. The space $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, consisting of all unital completely positive maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$, is compact with respect to the BW topology.

Normal UCP maps

We now recall the structural characterization of normal unital completely positive (UCP) maps on algebras of bounded operators over separable Hilbert spaces.

Definition 1.2.14 (Normal positive map)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A positive linear map $\psi : \mathcal{M} \to \mathcal{N}$ is called *normal* if, for every increasing net $\{X_i\}$ of self-adjoint operators in $\mathcal{M}$ converging strongly to X (in the strong operator topology, SOT), we have

$$\psi(X_i) \to \psi(X) \quad \text{in SOT}.$$

The following theorem characterizes the structure of normal unital completely positive (UCP) maps on $\mathcal{B}(\mathcal{H})$, with $\mathcal{H}$ separable, via the Stinespring dilation (see [Pis20, Theorem 1.41]).

Theorem 1.2.15 (Stinespring dilation for normal UCP maps)**.** Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, and let $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a normal UCP map. Then we can present ψ as

$$\psi(X) = V^*(X \otimes I_{\mathcal{P}})V, \qquad X \in \mathcal{B}(\mathcal{H}),$$

for a Hilbert space $\mathcal{P}$ and an isometry $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$ such that

$$\mathcal{H} \otimes \mathcal{P} = \overline{\text{span}}\{(X \otimes I_{\mathcal{P}})Vk : X \in \mathcal{B}(\mathcal{H}),\ k \in \mathcal{K}\},$$

the condition of minimality is satisfied.

Identifying $\mathcal{H} \otimes \ell^2(\mathbb{N})$ with a countable direct sum of copies of $\mathcal{H}$, one obtains the familiar Choi–Kraus decomposition for normal UCP maps (see [Dav76, Theorem 2.3]). Choi, in his seminal paper [Cho75], established this representation for completely positive maps on matrix algebras, and the same structure extends naturally to normal UCP maps on $\mathcal{B}(\mathcal{H})$.

Corollary 1.2.16. Let $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a normal UCP map. Then there exists a countable (finite or infinite) family of operators $\{V_n\}_{n\geq 1} \subseteq \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\psi(X) = \sum_{n\geq 1} V_n^* X V_n, \quad \text{with SOT convergence},$$

for all $X \in \mathcal{B}(\mathcal{H})$.

Remark 1.2.17. The commutant of the set $\{X \otimes I_{\mathcal{P}} : X \in \mathcal{B}(\mathcal{H})\}$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{P})$ is precisely the algebra $\{I_{\mathcal{H}} \otimes T : T \in \mathcal{B}(\mathcal{P})\}$. As a consequence, a normal UCP map ψ on $\mathcal{B}(\mathcal{H})$ is *pure* if and only if $\dim \mathcal{P} = 1$. In this case, ψ admits the simplified form

$$\psi(X) = V^* X V,$$

for some isometry $V : \mathcal{K} \to \mathcal{H}$.

1.3 Positive operator valued measures

Here we introduce the concept of *positive operator valued measures (POVMs)*. The notion of POVMs originates from the theory of measurements in quantum mechanics and has since become indispensable in areas such as quantum information theory, quantum computing, and quantum field theory. POVMs generalize the classical notion of projection-valued measurements, allowing a more flexible description of quantum observables, particularly in scenarios involving unsharp measurements or open quantum systems. For further study, the reader may consult [Dav76; Hol01; Pau02; Han+14] and the references therein.

Throughout this thesis, unless stated otherwise, X denotes a non-empty set and $\mathcal{O}(X)$ a σ-algebra of subsets of X. The tuple $(X, \mathcal{O}(X))$ will be referred to as a *measurable space*, and the elements of $\mathcal{O}(X)$ are called *measurable sets*.

Definition 1.3.1. Let $\mathcal{H}$ be a Hilbert space and $(X, \mathcal{O}(X))$ a measurable space. A map $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is said to be a *positive operator valued measure (POVM)* on X with values in $\mathcal{B}(\mathcal{H})$ if it satisfies the following conditions:

- $\mu(A) \geq 0$ for all $A \in \mathcal{O}(X)$, and
- for every $h, k \in \mathcal{H}$, the function $\mu_{h,k} : \mathcal{O}(X) \to \mathbb{C}$ defined by

$$\mu_{h,k}(A) = \langle h, \mu(A)k \rangle, \qquad A \in \mathcal{O}(X), \tag{1.3.1}$$

  is a complex measure.

Moreover, a POVM μ is called

1. *normalized* if $\mu(X) = I_{\mathcal{H}}$;
2. a *projection valued measure (PVM)* if $\mu(A)$ is a projection for every $A \in \mathcal{O}(X)$;
3. a *spectral measure* if it is a normalized PVM.

Naimark's dilation theorem

Naimark [Neu43] observed that every normalized POVM can be obtained as a compression of a spectral measure. He originally established this dilation theorem in the context of finitely additive POVMs. In the present discussion, we focus on the countably additive case. It is noteworthy that this result serves as a foundational milestone in dilation theory, as many subsequent developments in the field build upon the ideas introduced by Naimark's theorem. The proof of this dilation theorem follows the standard GNS construction; relevant references include [Hol01, Theorem 2.1.2] and [Sch96, Theorem II.11.F]. An alternative proof based on Stinespring's theorem for completely positive maps is also well known (see [Pau02, Theorem 4.6]); however, in that approach, the POVMs under consideration are typically required to be regular on the Borel σ-algebra of a locally compact Hausdorff space.

Theorem 1.3.2 (Naimark Dilation Theorem)**.** Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a positive operator valued measure (POVM). Then there exists a Hilbert space $\mathcal{K}$, a spectral measure $\pi : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K})$, and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$\mu(A) = V^*\pi(A)V, \qquad \forall\, A \in \mathcal{O}(X),$$

and the minimality condition

$$\mathcal{K} = \overline{\text{span}}\{\pi(A)Vh : A \in \mathcal{O}(X),\, h \in \mathcal{H}\}$$

is satisfied. Moreover, this dilation is unique up to unitary equivalence: if $(\mathcal{K}_1, \pi_1, V_1)$ is another such dilation, there exists a unitary $U : \mathcal{K} \to \mathcal{K}_1$ such that

$$UV = V_1 \qquad \text{and} \qquad U\pi(A) = \pi_1(A)U, \quad \forall\, A \in \mathcal{O}(X).$$

1.4 Quantum instruments

In this section, we review the definition of CP instruments within the framework of general C^*-algebras and note some of their fundamental properties. See [DL70], [Hol98], [Oza84], [Pel13a], [Pel13b], and [Sri80] for general literature on the topic of quantum instruments.

To date, the existing theory has focused on CP instruments in von Neumann algebra setting, where the value space consists of normal CP maps. Here, perhaps for the first time, we are considering CP instruments in the broader context of general C^*-algebras. A suggestion that for physical reasons we should be looking at finitely additive measures, which amounts going beyond normal maps, was made already in [Sri80]. Let us begin with some definitions.

Definition 1.4.1 (Quantum instruments)**.** Let $(X, \mathcal{O}(X))$ be a measurable space, $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a Hilbert space. A *CP instrument* on $(X, \mathcal{O}(X))$ with values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ is a map $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ satisfies:

1. $\mathcal{I}(A) \in CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, for all $A \in \mathcal{O}(X)$;
2. for every $h, k \in \mathcal{H}$ and any $a \in \mathcal{A}$, the map $\mathcal{I}_{a,h,k} : \mathcal{O}(X) \to \mathbb{C}$ defined by
$$\mathcal{I}_{a,h,k}(A) = \langle h, \mathcal{I}(A)(a)k\rangle \text{ for all } A \in \mathcal{O}(X), \tag{1.4.1}$$
is a complex measure.

Moreover, a quantum instrument $\mathcal{I}$ is said to be

- *normalized* or *unital* if $\mathcal{I}(X)(1_{\mathcal{A}}) = 1_{\mathcal{H}}$ i.e. if the completely positive map $\mathcal{I}(X)$ is *unital*.
- *normal* if $\mathcal{A}$ is a von Neumann algebra and $\mathcal{I}(A)$'s are normal completely positive map for all $A \in \mathcal{O}(X)$.
- *spectral* if $\mathcal{I}(A)$ is a $*$-homomorphism for all $A \in \mathcal{O}(X)$ and is unital.

It follows from the definition of CP instrument that, for any increasing (or decreasing) sequence $\{A_n\}$ of measurable subsets converging to A i.e. $A_n \subseteq A_{n+1}$ and $\cup_n A_n = A$ (or $A_n \supseteq A_{n+1}$ and $\cap A_n = A$), $\mathcal{I}(A_n)(a) \to \mathcal{I}(A)(a)$ in weak operator topology (WOT) in $\mathcal{B}(\mathcal{H})$, for all $a \in \mathcal{A}$. Since a bounded net, $\{\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})\}_{i\in\mathcal{I}}$ of completely positive maps converges to a completely positive map, $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ in the *bounded weak* (BW) topology if $\phi_i(a) \to \phi(a)$ in WOT, $\forall\ a\ \in \mathcal{A}$. Consequently, for a countably additive CP instrument $\mathcal{I}$, we have

$$\sum_{n=1}^{\infty} \mathcal{I}(B_n) \;\longrightarrow\; \mathcal{I}\Big(\bigcup_{n=1}^{\infty} B_n\Big) \quad \text{in the BW topology of CP maps,}$$

where $\{B_n\}_{n\geq 1} \subset \mathcal{O}(X)$ are mutually disjoint, i.e., $B_n \cap B_m = \varnothing$ for $n \neq m$.

The following example shows that CP maps and POVMs can be regarded as special types of instruments.

Example 1.4.2. 1. Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. Consider $\mathcal{A} = \mathbb{C}$. Define a map $\mathcal{I}_\mu : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ by,

$$\mathcal{I}_\mu(A)(a) = a\mu(A), \ \forall A \in \mathcal{O}(X), \ a \in \mathbb{C}.$$

This defines an instrument, showing that every POVM can be considered as a CP instrument.

2. Similarly given a CP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, consider the trivial measurable space $\mathcal{O}(X) = \{\varnothing, X\}$ for some non-empty set X. Define the map $\mathcal{I}_\phi : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ by

$$\mathcal{I}_\phi(\varnothing)(a) = 0, \ \ \mathcal{I}_\phi(X)(a) = \phi(a), \forall \ a \in \mathcal{A}.$$

This shows that every CP map can be regarded as a CP instrument in a natural way.

Remark 1.4.3. For any instrument $\mathcal{I}$, we denote by $\mathcal{I}_{a,h,k}$ the complex measure defined in (1.4.1). It can be verified that a CP instrument $\mathcal{I}$ is completely determined by its associated family of complex measures $\{\mathcal{I}_{a,h,k} : a \in \mathcal{A}, \ h, k \in \mathcal{H}\}$.

Remark 1.4.4 (Bivariate realization)**.** Observe that a CP instrument $\mathcal{I}$ on $(X, \mathcal{O}(X))$ with values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ can be thought of as a bivariate map $\tilde{\mathcal{I}} : \mathcal{O}(X) \times \mathcal{A} \to \mathcal{B}(\mathcal{H})$ given by,

$$\tilde{\mathcal{I}}(A, a) = \mathcal{I}(A)(a), \ \forall a \in \mathcal{A}, \ A \in \mathcal{O}(X),$$

where,

- for each $A \in \mathcal{O}(X)$, the map, $\tilde{\mathcal{I}}(A, \cdot) : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by $\mathcal{I}(A, a) = \mathcal{I}(A)(a)$ gives a completely positive map and
- by fixing any positive element $a \in \mathcal{A}$, the map, $\tilde{\mathcal{I}}(\cdot, a) : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ given by $\mathcal{I}(A, a) = \mathcal{I}(A)(a)$ defines a positive operator valued measure on X.

Conversely, if there exists a bivariate map $\tilde{\mathcal{I}} : \mathcal{O}(X) \times \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that:

1. for each $A \in \mathcal{O}(X)$, the map, $\tilde{\mathcal{I}}(A, \cdot) : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely positive map,
2. for every $a \in \mathcal{A}_+$, ($\mathcal{A}_+$, the set of all positive elements in $\mathcal{A}$) the map, $\tilde{\mathcal{I}}(\cdot, a) : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ defines a positive operator valued measure on X,

then we can construct a map $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ such that $\mathcal{I}(A)a = \tilde{\mathcal{I}}(A, a)$ for all $A \in \mathcal{O}(X)$ and $a \in \mathcal{A}$ respectively. The fact that $\mathcal{I}$ is a CP instrument follows directly from the definition of $\tilde{\mathcal{I}}$.

Based on Remark 1.4.4, CP instruments can be described unambiguously either in terms of the 1.4.1 or equivalently as a bivariate map, as outlined in 1.4.4.

Remark 1.4.5. Bivariate realization of CP instruments implies that corresponding to every CP instrument $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, we have a POVM, $\mathcal{I}(\cdot, 1_{\mathcal{A}}) : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ and a CP map, $\mathcal{I}(X, \cdot) : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Furthermore, if the quantum instrument is unital then $\mathcal{I}(., 1_{\mathcal{A}})$ becomes a normalized POVM, and $\mathcal{I}(X, .)$, a UCP map. These are called associated POVM and CP map of the CP instrument $\mathcal{I}$.

Notation. We use the symbols $\mu_{\mathcal{I}}, \phi_{\mathcal{I}}$ to denote the associated POVM and CP map of a quantum instrument $\mathcal{I}$. We also call them POVM marginal and CP marginal of the instrument $\mathcal{I}$, collectively as *marginals*. Let $Ins_{\mathcal{H}}(X, \mathcal{A})$ denote the collection of all CP instruments on $\mathcal{O}(X)$ with values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ and let$I_{\mathcal{H}}(X, \mathcal{A})$ denote the collection of all unital elements in $Ins_{\mathcal{H}}(X, \mathcal{A})$.

Analogous to classical joint measures, in the non-commutative setting, the entire instrument vanishes if and only if either of its marginals vanishes.

Proposition 1.4.6. Let, $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ ba a CP instrument. Then, the following are equivalent (TFAE):

(i) $\mathcal{I} = 0$, (ii) $\mu_{\mathcal{I}} = 0$, (iii) $\phi_{\mathcal{I}} = 0$.

Proof. If the instrument $\mathcal{I} = 0$, then by definition, its marginals satisfy $\phi_{\mathcal{I}} = 0$ and $\mu_{\mathcal{I}} = 0$. Conversely, if $\mu_{\mathcal{I}} = 0$ i.e. $\mathcal{I}(A, 1_{\mathcal{A}}) = 0$, for any $A \in \mathcal{O}(X)$, in particular, $\mathcal{I}(X, 1_{\mathcal{A}}) = 0$, which implies that $\mathcal{I}(A) = 0$, for all $A \in \mathcal{O}(X)$. Similarly, we can verify that if the CP marginal $\phi_{\mathcal{I}} = 0$ then the entire instrument $\mathcal{I} = 0$. ■

Bi-dilation theorem

The classical dilation theorems of Naimark, Theorem 1.3.2 and Stinespring Theorem 1.2 are foundational results in the theory of operator algebras, establishing that POVMs and completely positive (CP) maps can be dilated to spectral measures and $*$-homomorphisms, respectively. Ozawa [Oza84] proved a dilation theorem for CP instruments, demonstrating that such instruments can be dilated to spectral instruments (as characterized in 1.4.10). Although his treatment focused on normal CP instruments on von Neumann algebras, the underlying arguments generalize seamlessly to quantum instruments on arbitrary C^*-algebras. However, this dilation in the C^*- algebraic framework has already been studied independently in Theorem 7.1 of [Bus+16], as well as in [HP17]. Here, we present the dilation theorem in the general C^*-algebra setting along with a complete proof.

Theorem 1.4.7. Let, $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be an instrument. Then there exists a quadruple $(\mathcal{K}, \pi, E, V)$, where $\mathcal{K}$ is a Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a unital $*$-homomorphism, $E : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K})$ is a spectral measure, and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that,

$$\mathcal{I}(A)(a) = V^*\pi(a)E(A)V \quad \text{and} \quad E(A)\pi(a) = \pi(a)E(A), \forall a \in \mathcal{A}, A \in \mathcal{O}(X) \tag{1.4.2}$$

and satisfies the minimality condition: $[\pi(\mathcal{A})E(\mathcal{O}(X))V\mathcal{H}] = \mathcal{K}$.

Therefore, one can check that such a dilation is unique up to unitary equivalence. We call the quadruple $(\mathcal{K}, \pi, E, V)$, the minimal bi-dilation quadruple for the instrument $\mathcal{I}$. Since π is a unital $*$-homomorphism and E is spectral, the equation 1.4.2 implies that V is an isometry if and only if $\mathcal{I}$ is a *normalized* instrument.

The proof of this theorem primarily involves bookkeeping. The proof involves the standard GNS construction. However, for clarity we provide a brief outline of the proof.

Proof. Consider the product space, $\mathcal{M} = \mathcal{O}(X) \times \mathcal{A} \times \mathcal{H}$ and define the map $K : \mathcal{M} \times \mathcal{M} \to \mathbb{C}$ by $K((A, a, h), (B, b, k)) = \langle h, \mathcal{I}(A \cap B)(a^*b)k\rangle$. For arbitrary $(A_1, a_1, h_1), \cdots, (A_n, a_n, h_n) \in \mathcal{M}$ and $c_1, \cdots, c_n \in \mathbb{C}$,

$$\sum_{i,j} \overline{c_i} c_j K((A_i, a_i, h_i), (A_j, a_j, h_j)) = \sum_{i,j} \overline{c_i} c_j \langle h_i, \mathcal{I}(A_i \cap A_j)(a_i^* a_j) h_j\rangle$$

Using standard measure theoretic arguments, note that each element from the collection $\{A_1, \cdots, A_n\}$ can be written as a disjoint union of a new collection of disjoint measurable subsets $\{B_1, \cdots, B_k\}$. Consequently,

$$\sum_{i,j} \overline{c_i} c_j \langle h_i, \mathcal{I}(A_i \cap A_j)(a_i^* a_j) h_j\rangle = \sum_{l} \sum_{i,j \in \{r : B_l \subseteq A_r\}} \overline{c_i} c_j \langle h_i, \mathcal{I}(B_l)(a_i^* a_j) h_j\rangle.$$

Due to the complete positivity of the CP instrument $\mathcal{I}$, the second summand in the above equation is positive. Hence K is a positive definite kernel. By GNS construction we have a Hilbert space $\mathcal{K}$ with a map $\lambda : \mathcal{M} \to \mathcal{K}$ such that,

$$\langle \lambda(A,a,h), \lambda(B,b,k) \rangle = \langle h, \mathcal{I}(A\cap B)(a^*b)k \rangle, \; for \; (A,a,h),(B,b,k) \in \mathcal{M},$$

and the set $\{\lambda(A,a,h) : (A,a,h) \in \mathcal{M}\}$ forms a total set in $\mathcal{K}$. For each unitary $u \in \mathcal{A}$, we define $\pi(u)$ by $\pi(u)(\lambda(A,a,h)) = \lambda(A,ua,h)$. Then it's easy to verify that $\pi(u)$ is a unitary operator on $\mathcal{K}$. By extending π linearly via the functional calculus, we obtain a unital $*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}(\mathcal{K})$. Similarly, define the map, $E : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K})$ by $E(B)(\lambda(A,a,h)) = \lambda(A \cap B, a, h)$, for each $B \in \mathcal{O}(X)$. One can verify that E is a spectral measure on the measurable space $(X, \mathcal{O}(X))$. It follows directly from the definitions of the maps π, E that $\pi(a)E(A) = E(A)\pi(a)$ for each $a \in \mathcal{A}$ and $A \in \mathcal{O}(X)$. Finally, define a linear map $V : \mathcal{H} \to \mathcal{K}$ by $Vh = \lambda(X, 1_{\mathcal{A}}, h)$. Routine verification shows that V is bounded, in fact $\|V\| = \|\mathcal{I}(X)(1_{\mathcal{A}})\|^{\frac{1}{2}}$, and the desired dilation is given by $\mathcal{I}(A)(a) = V^*E(A)\pi(a)V$. ■

As a consequence of Theorem 1.4.2, every CP instrument on a finite set, taking values in completely positive maps on matrix algebras, admits a Choi–Kraus representation analogous to that of completely positive maps Corollary 1.2.16. Further, we can characterize minimal *Choi-Kraus* decompositions in terms of linear independence. In the following, for notational simplicity, for any singleton $\{i\}$, we denote $\mathcal{I}(\{i\}, \cdot)$ by $\mathcal{I}(i, \cdot)$.

Proposition 1.4.8 (Choi-Kraus decomposition)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(M_d, M_k)$ be a CP instrument where $X = \{1, \cdots, n\}$ is a finite set. Then there exists a collection of matrices $\{V_j^i\}_{j=1}^{l_i} \subset M_{k\times d}$ for some $l_i \in \mathbb{N}$, for each $i \in \{1, \ldots, n\}$ such that,

$$\mathcal{I}(i,a) = \sum V_j^{i*} a V_j^i, \; \forall \; a \in M_d. \tag{1.4.3}$$

Moreover, $\{V_j^i\}_{j=1}^{l_i}$ can be chosen to be linearly independent for each $i \in \{1, \cdots, n\}$.

Remark 1.4.9. We call the representation in 1.4.3 as the *Choi-Kraus decomposition*. We refer the collection of matrices $\{V_j^i\}_{j=1}^{l_i}$ as the *Choi-Kraus* operators associated to the instrument $\mathcal{I}$. If they are linearly independent, the decomposition is said to be minimal and then for any other decomposition

$$\mathcal{I}(i,a) = \sum {W^i}_k^* a W_k^i, \quad \forall \; a \in M_d,$$

there exist a family of isometries $(\mu_{k,j}^i)$ such that,

$$W_k^i = \sum \mu_{k,j}^i V_j^i.$$

Another important implication of the dilation theorem is that spectral instruments admit a canonical factorization as the composition of a spectral measure and a unital $*$-homomorphism.

Theorem 1.4.10. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument. Then TFAE:

(i) $\mathcal{I}$ is a spectral instrument,

(ii) $\mathcal{I}(A,a) = \phi_{\mathcal{I}}(a)\mu_{\mathcal{I}}(A) = \mu_{\mathcal{I}}(A)\phi_{\mathcal{I}}(a)$ for all $a \in \mathcal{A}, \;\; A \in \mathcal{O}(X)$, where $\phi_{\mathcal{I}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a unital $*$-homomorphism and $\mu_{\mathcal{I}} : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a spectral measure.

Proof. The proof, *(ii)* $\implies$ *(i)* is obvious. For the converse, let $(\mathcal{K}, \pi, E, V)$ be the minimal bi-dilation tuple of $\mathcal{I}$. As $\mathcal{I}$ is unital, it follows that V is an isometry. Since, $\mathcal{I}$ is spectral, both the associated POVM, $\mu_{\mathcal{I}}$ and the UCP map, $\phi_{\mathcal{I}}$ are spectral, with $\mu_{\mathcal{I}}$ being a spectral measure and $\phi_{\mathcal{I}}$ a $*$-homomorphism respectively. Then for any $A \in \mathcal{O}(X)$ we have,

$$\begin{aligned}[V\mu_{\mathcal{I}}(A) - E(A)V]^* \cdot [V\mu_{\mathcal{I}}(A) - E(A)V] &= [\mu_{\mathcal{I}}(A)V^* - V^*E(A)] \cdot [V\mu_{\mathcal{I}}(A) - E(A)V] \\ &= \mu_{\mathcal{I}}(A)^2 - \mu_{\mathcal{I}}(A)^2 - \mu_{\mathcal{I}}(A)^2 + \mu_{\mathcal{I}}(A) \\ &= \mu_{\mathcal{I}}(A)^2 - \mu_{\mathcal{I}}(A) \\ &= 0.\end{aligned}$$

So we obtain $V\mu_{\mathcal{I}}(A) = E(A)V$. As a result, for all $a \in \mathcal{A}$ and $A \in \mathcal{O}(X)$, we have

$$\mathcal{I}(A, a) = V^*\pi(a)E(A)V = V^*\pi(a)V\mu_{\mathcal{I}}(A) = \phi_{\mathcal{I}}(a)\mu_{\mathcal{I}}(A).$$

■

Remark 1.4.11. We may sometimes use (ii) of Theorem 1.4.10 as the definition of a spectral instrument for convenience. Hereafter, we use the notation πE to denote a spectral instrument on its respective domain and range:

$$\pi E(A, a) = \pi(a)E(A), \ \ A \in \mathcal{O}(X), a \in \mathcal{A}.$$

Definition 1.4.12 (Irreducible instruments)**.** A spectral instrument $\pi E : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ is said to be *irreducible* if πE has no proper invariant subspace. This is equivalent to requiring $\{\pi E\}' := \{\pi(a)E(A) : a \in \mathcal{A}, A \in \mathcal{O}(X)\}' = \mathbb{C}I$.

Remark 1.4.13. For any spectral instrument $\pi E : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, one can check that

$$\{\pi(\mathcal{A})E(\mathcal{O}(X))\}' = \{\pi(\mathcal{A})\}' \cap \{E(\mathcal{O}(X))\}'.$$

Proposition 1.4.14. Let $\pi E : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a spectral instrument. Then TFAE,

(i) πE is irreducible,

(ii) $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an irreducible representation of the C^*-algebra $\mathcal{A}$.

Furthermore, in such a case, the associated spectral measure $E : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ satisfies $E(\mathcal{O}(X)) \subseteq \{0, I_{\mathcal{H}}\}$.

Proof. It follows from Remark 1.4.13, that (ii) implies (i). To prove (i) $\implies$ (ii), we need to show that $\pi(\mathcal{A})' = \mathbb{C}I_{\mathcal{H}}$. We will show that $E(\mathcal{O}(X)) \subseteq \{0, I_{\mathcal{H}}\}$. This combined with Remark 1.4.13, yields the desired result. Assume that there exists a proper projection $P \in E(\mathcal{O}(X))$. Since E is a spectral measure, every element in $E(\mathcal{O}(X))$ is a projection, which are mutually orthogonal and they commute with unital $*$-homomorphism π, i.e., $E(\mathcal{O}(X)) \subseteq E(\mathcal{O}(X))' \cap \pi(\mathcal{A})' = \{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$. However, by hypothesis, we have $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}' = \mathbb{C}I_{\mathcal{H}}$, which has only trivial projections. As a consequence $E(\mathcal{O}(X)) \subseteq \{0, I_{\mathcal{H}}\}$. ■

Definition 1.4.15. A CP instrument $\mathcal{I}$ is said to be *concentrated* on a set $S \in \mathcal{O}(X)$ if $\mathcal{I}(B) = \mathcal{I}(B \cap S)$, for all $B \in \mathcal{O}(X)$.

Proposition 1.4.16. Let, $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument with the minimal bi-dilation tuple $(\mathcal{K}, \pi, E, V)$. Then for any $B \in \mathcal{O}(X)$, $\mathcal{I}(B) = 0$ if and only if $(\pi E)(B) = 0$. In particular, $\mathcal{I}$ is concentrated on $S \in \mathcal{O}(X)$ if and only if πE is concentrated on S.

Proof. The second assertion follows from the first one. So, it is enough to prove the first one. Let $\mathcal{I}(B) = 0$. Then for any $a \in \mathcal{A}$, $A \in \mathcal{O}(X)$ and $h \in \mathcal{H}$, we get,

$$\begin{aligned}\langle \pi(a)E(A)Vh, (\pi E)(B)(1_{\mathcal{A}})(\pi(a)E(A)Vh)\rangle &= \langle h, V^*\pi(a^*)E(B \cap A)(\pi(a)E(A)Vh)\rangle \\ &= \langle h, \mathcal{I}(B \cap A)(a^*a)h\rangle \\ &\leq \langle h, \mathcal{I}(B)(a^*a)h\rangle = 0.\end{aligned}$$

Since $\{\pi(a)E(A)Vh : a \in \mathcal{A}, A \in \mathcal{O}(X), h \in \mathcal{H}\}$ is a total set in $\mathcal{K}$, by the minimality condition, we conclude that $(\pi E)(B)(1_{\mathcal{A}}) = 0$. Hence, we have $\pi E(B)(a) = 0, \ \forall\ a \in \mathcal{A}$. The converse is obvious. ∎

Radon-Nikodym type theorem

In classical measure theory, the Radon-Nikodym derivative of a (σ-finite) positive measure absolutely continuous with respect to another (σ-finite) positive measure, is a well-established result. In the case of POVMS ([BBK21, Theorem 2.8]) and CP maps ([Arv69, Theorem 1.4.2]) the notion of Radon-Nikodym derivative has been introduced using the partial order defined by comparison. An analogous Radon-Nikodym type of theorem exists for quantum instruments as well. For two quantum instruments $\mathcal{I}, \mathcal{J}$, we say $\mathcal{J}$ is *dominated* by $\mathcal{I}$ (denoted by $\mathcal{J} \leq \mathcal{I}$) if $\mathcal{I} - \mathcal{J}$ is a CP instrument.

Theorem 1.4.17 (Radon-Nikodym type theorem)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be an instrument with the minimal bi-dilation quadruple $(\mathcal{K}, \pi, E, V)$. Then for any instrument $\mathcal{J} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, $\mathcal{J} \leq \mathcal{I}$ if and only if there exists a positive contraction $D \in \{\pi(a)E(A) : a \in \mathcal{A}, A \in \mathcal{O}(X)\}'$ such that,

$$\mathcal{J}(A, a) = V^*D\pi(a)E(A)V, \forall A \in \mathcal{O}(X),\ a \in \mathcal{A}. \tag{1.4.4}$$

We will call the operator D of this theorem as the *Radon-Nikodym derivative* of $\mathcal{J}$ with respect to $\mathcal{I}$.

Motivated by the notion of pure completely positive (CP) maps for C^*-algebras, as introduced in [Arv69], we have the following definition for quantum instruments.

Definition 1.4.18 (Pure instruments)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument. $\mathcal{I}$ is called a pure instrument if for any instrument $\mathcal{J} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ dominated by $\mathcal{I}$ is of the form $\mathcal{J} = t\mathcal{I}$ for some $t \in [0, 1]$.

Here is a characterization of pure quantum instruments generalizing a similar result by Arveson for pure CP maps.

Proposition 1.4.19. Let, $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument with minimal bi-dilation quadruple $(\mathcal{K}, \pi, E, V)$. Then $\mathcal{I}$ is pure if and only if the spectral instrument $\pi E : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ is irreducible.

Proof. If the associated spectral instrument $\pi E : \mathcal{O}(X) \to Cp(\mathcal{A}, \mathcal{B}(\mathcal{K}))$ in the minimal bi-dilation of $\mathcal{I}$ is irreducible, then $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}' = \mathbb{C}I_{\mathcal{K}}$. As a consequence of Theorem 1.4.17, for any instrument $\mathcal{J} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, $\mathcal{J} \leq \mathcal{I}$ we have that $\mathcal{J} = t\mathcal{I}$ for some $t \in (0, 1)$, i.e. the instrument $\mathcal{I}$ is pure. Conversely, suppose the set $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ contains a proper projection P. Consider the instrument $\mathcal{I}' : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, defined by

$$\mathcal{I}'(A)(a) = V^*\pi(a)E(A)PV\ \forall\ a \in \mathcal{A},\ A \in \mathcal{O}(X),$$

It follows that $\mathcal{I}'$ is dominated by the instrument $\mathcal{I}$. Since the instrument is pure, we must have $\mathcal{I}' = t\mathcal{I}$ for some $t \in [0, 1]$. Therefore,

$$V^*\pi(a)E(A)PV = tV^*\pi(a)E(A)V$$

for all $a \in \mathcal{A}$, $A \in \mathcal{O}(X)$, which implies that $P \in \{0, I_{\mathcal{K}}\}$. Hence, the spectral measure πE is irreducible.

■

As a direct consequence of Proposition 1.4.19, we obtain the following result.

Corollary 1.4.20. POVM marginal of any pure instrument is trivial.

Remark 1.4.21. On a measure space $(X, \mathcal{O}(X))$, a POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is said to be trivial if $\mu(A) = \nu(A) I_{\mathcal{H}}$, $A \in \mathcal{O}(X)$, where ν is a positive measure on $(X, \mathcal{O}(X))$. Furthermore, μ is called a trivial spectral measure if $\nu(\mathcal{O}(X)) \subseteq \{0, 1\}$.

Definition 1.4.22 (Compression of instruments)**.** Let $\mathcal{I}_i : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}_i))$, $i = 1, 2$, be two CP instruments. We say that $\mathcal{I}_2$ is a *compression* of $\mathcal{I}_1$ if there exists an isometry $Z : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\mathcal{I}_2(A) = Z^* \mathcal{I}_1(A) Z, \qquad \text{for all } A \in \mathcal{O}(X)$.

Remark 1.4.23. Let $\mathcal{I}_1, \mathcal{I}_2$ be two quantum instruments, which are compressions of a common spectral instrument πE. Suppose $(\mathcal{K}, \pi, E, V_i)$ denote the minimal bi-dilation tuple of $\mathcal{I}_i, i = 1, 2$. Then one can verify that $\mathcal{I}_2$ is a compression of $\mathcal{I}_1$ iff $V_2 V_2^* \leq V_1 V_1^*$, which is equivalent to, $\text{range}(V_2) \subseteq \text{range}(V_1)$.

Sub-minimal dilations

Within the framework of the bivariate realization outlined in Remark 1.4.4, it becomes evident that, unlike completely positive (CP) maps or POVMs, quantum instruments admit multiple types of dilations. However, a closer investigation reveals that all such dilations ultimately reduce to the fundamental bi-dilation structure. Our next objective is to explore these variants, collectively referred to as *sub-minimal dilations*, a notion originally introduced in [HHP14].

While the authors in [HHP14] introduced the notion of sub-minimal dilations in the context of instruments arising from completely positive maps defined on the von Neumann tensor product of algebras into $\mathcal{B}(\mathcal{H})$, their focus was primarily on understanding the connection between the marginals and the joint CP map. The intrinsic relationships between two sub-minimal dilations and the minimal bi-dilation was not their focus. In this work, we reinterpret and generalize the notion of sub-minimal dilations in a more abstract setting, offering what we believe is a more systematic treatment of the topic.

Based on their construction methods, these dilations are classified into two types: CP sub-minimal dilations and POVM sub-minimal dilations. Briefly speaking, in the CP sub-minimal dilation, the CP marginal is dilated using the Stinespring representation, ignoring the POVM marginal. Similarly, in the POVM sub-minimal dilation the POVM marginal is dilated with help from the Naimark dilation, disregarding the other marginal. For clarity and convenience, we provide a concise overview of these constructions below.

Theorem 1.4.24 (CP Sub-minimal dilation)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument. Then there exists a Hilbert space $\mathcal{K}_1$, a unital $*$-homomorphism $\pi_1 : \mathcal{A} \to \mathcal{B}$, a POVM $\mu : \mathcal{O}(X) \to \pi_1(\mathcal{A})' \subset \mathcal{B}(\mathcal{K}_1)$ and a bounded linear map $V_1 : \mathcal{H} \to \mathcal{K}_1$ such that,

$$\mathcal{I}(A, a) = V_1^* \pi_1(a) \mu(A) V_1 \qquad \text{and} \qquad \mu(A) \pi_1(a) = \pi_1(a) \mu(A),$$

for all $a \in \mathcal{A}$, $A \in \mathcal{O}(X)$. Moreover, it satisfies the minimality condition: $[\pi_1(\mathcal{A}) V_1 \mathcal{H}] = \mathcal{K}_1$.

Proof. Let $(\mathcal{K}_1, \pi_1, V_1)$ be the minimal Stinespring dilation of the associated CP map $\phi_{\mathcal{I}}$ of the instrument $\mathcal{I}$. For every $a \in \mathcal{A}_+$ and $A \in \mathcal{O}(X)$ it holds that $\mathcal{I}(A, a) \leq \mathcal{I}(X, a)$. By the

Radon Nikodym theorem for CP maps, there exists $D \in \pi_1(\mathcal{A})'$ with $0 \leq D \leq I_{\mathcal{K}_1}$ such that

$$\mathcal{I}(A, a) = V_1^* \pi_1(a) D V_1, \ for\ all\ a \in \mathcal{A}.$$

This implies that every element of $\mathcal{O}(X)$ corresponds to a positive contraction of $\pi_1(\mathcal{A})'$. Let, $\mu : \mathcal{O}(X) \to \pi_1(\mathcal{A})'$ denote this correspondence. It follows directly from the definition of μ that it commutes with the $*$-homomorphism π_1. To establish that μ is a POVM, it remains to verify the following:

$$\mu(\cup_i A_i) = \sum_i \mu(A_i),$$

for any countable collection of disjoint subsets $\{A_i\} \subset \mathcal{O}(X)$, where the sum on the right-hand side converges in WOT. Since the collection of vectors of the form $\pi_1(a)V_1 h$ forms a total set for $\mathcal{K}_1$, it suffices to prove that,

$$\langle \pi_1(a)V_1 h, \mu(\cup A_i)\pi_1(b)V_1 k\rangle = \sum_i \langle \pi_1(a)V_1 h, \mu(A_i)\pi_1(b)V_1 k\rangle, \tag{1.4.5}$$

for all $a, b \in \mathcal{A}$ and $h, k \in \mathcal{H}$. Equation 1.4.5 is an immediate consequence of the fact that $\mathcal{I}$ is an instrument. Consequently, we have that $\mathcal{I}(A, a) = V_1^* \pi_1(a)\mu(A)V_1$,$\forall\ a \in \mathcal{A}, A \in \mathcal{O}(X)$. ∎

Remark 1.4.25. Furthermore, if we consider the minimal Naimark dilation of the POVM μ that appears in the subminimal dilation described above, we obtain,

$$\mathcal{I} = V_1^* \pi_1 W_1^* E_1 W_1 V_1,$$

where $(\widetilde{\mathcal{K}_1}, E_1, W_1)$ is the minimal Naimark tuple for μ with

$$\{E_1(A)W_1\pi_1(a)V_1 h : A \in \mathcal{O}(X),\ a \in \mathcal{A},\ h \in \mathcal{H}\}$$

forming a total set for $\widetilde{\mathcal{K}_1}$. Following an argument analogous to the construction of the bi-dilation, one sees that the tuple $(\widetilde{\mathcal{K}_1}, \widetilde{\pi_1}, E_1, W_1V_1)$ forms a minimal bi-dilation tuple of the instrument $\mathcal{I}$, where $\widetilde{\pi_1} : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{K}_1})$ is the unital $*$-homomorphism commutes with E_1 given by,

$$\widetilde{\pi_1}(b)(E_1(A)W_1\pi_1(a)V_1 h) = E_1(A)W_1\pi_1(ba)V_1 h,\ A \in \mathcal{O}(X),\ a, b \in \mathcal{A},\ h \in \mathcal{H}\}$$

Hence, this calculation clarifies that the notion of subminimal dilation does not introduce any additional structural insight beyond the framework of minimal bi-dilation.

We call the quadruple $(\mathcal{K}_1, \pi_1, \mu, V_1)$ in Theorem 1.4.24, the CP subminimal dilation quadruple of $\mathcal{I}$ and adopt this notation throughout the paper. Next we introduce another type of subminimal dilation. Instead of Stinespring dilation of the associated CP map, we start with the Naimark dilation of the associated POVM $\mu_{\mathcal{I}}$. The proof strategy mirrors that of Theorem 1.4.24 and hence we omit the proof. However we present the formal statement.

Theorem 1.4.26 (POVM Subminimal dilation)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument. Then there exists a Hilbert space $\mathcal{K}_2$,a spectral measure $E_2 : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K}_2)$, a CP map $\phi : \mathcal{A} \to E_2(\mathcal{O}(X))' \subset \mathcal{B}(\mathcal{K}_2)$, and a linear map $V_2 : \mathcal{H} \to \mathcal{K}_2$ such that,

$$\mathcal{I}(A, a) = V_2^* \phi(a) E_2(A) V_2 \qquad \text{and} \qquad E_2(A)\phi(a) = \phi(a)E_2(A),$$

for all $a \in \mathcal{A},\ A \in \mathcal{O}(X)$. Moreover, it satisfies the minimality condition: $[E_2(\mathcal{A})V_2\mathcal{H}] = \mathcal{K}_2$.

Remark 1.4.27. Similar to Remark 1.4.25, in this case as well, if we further take the minimal Stinespring dilation $(\widetilde{\mathcal{K}_2}, \pi_2, W_2)$ of the UCP map ϕ, we obtain a minimal bi-dilation of the instrument $\mathcal{I}$, given by, $(\widetilde{\mathcal{K}_2}, \pi_2, \widetilde{E_2}, W_2V_2)$, where, $\widetilde{\mathcal{K}_2} = \overline{span}\{\pi_2(a)W_2(E_2(A)V_2(h)) : a \in \mathcal{A},\ A \in \mathcal{O}(X),\ h \in \mathcal{H}\}$ and $\widetilde{E_2} : \mathcal{O}(X) \to \mathcal{B}(\widetilde{\mathcal{K}_2})$ is a spectral measure commutes with the $*$-homomorphism π_2 given by,

$$\widetilde{E_2}(B)(\pi_2(a)W_2(E_2(A)V_2(h))) = \pi_2(a)W_2(E_2(A \cap B)V_2(h)),\ a \in \mathcal{A},\ A, B \in \mathcal{O}(X),\ h \in \mathcal{H}.$$

We call the quadruple $(\mathcal{K}_2, \phi, E_2, V_2)$ in Theorem 1.4.26, the POVM subminimal dilation quadruple of $\mathcal{I}$ and we fix this notation throughout the paper.

It is not difficult to construct examples where all three dilations differ from one another. We present one such example below.

Example 1.4.28. Let $X = \{1, 2\}$ equipped with the dicrete σ-algebra $\mathcal{O}(X)$. Consider the instrument $\mathcal{I} : \mathcal{O}(X) \times M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by:

$$\mathcal{I}(i, T) = \alpha_i W_1^* T W_1 + \beta_i W_2^* T W_2 + \gamma_i W_3^* T W_3,\ i \in \{1, 2\},$$

where $\{W_j : j = 1, 2, 3\} \subset M_2(\mathbb{C})$ is a linearly independent set and satisfies the condition $\sum\limits_j W_j^* W_j \leq I_2$, and $\alpha_i, \beta_i, \gamma_i > 0$ are scalars satisfying $\alpha_1 + \alpha_2 = 1, \beta_1 + \beta_2 = 1, \gamma_1 + \gamma_2 = 1$. The two marginals of $\mathcal{I}$ are as follows. The CP marginal $\phi_{\mathcal{I}} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is given by $\phi_{\mathcal{I}}(T) = \sum\limits_j W_j^* T W_j$, and the POVM marginal $\mu_{\mathcal{I}} : \mathcal{O}(X) \to M_2(\mathbb{C})$ is given by, $\mu_{\mathcal{I}}(i) = \alpha_i W_1^* W_1 + \beta_i W_2^* W_2^* + \gamma_i W_3^* W_3,\ i \in \{1, 2\}$.

One can verify that the minimal bi-dilation space for the instrument $\mathcal{I}$ is $\mathbb{C}^{12}$, the CP sub-minimal dilation space is $\mathbb{C}^6$ and the POVM sub-minimal dilation space is $\mathbb{C}^4$. Since the minimal dilation spaces are mutually distinct, it follows immediately that the corresponding dilations are necessarily distinct.

Recovering sub-minimal dilations from minimal bi-dilations

From the previous section on sub-minimal dilation and Remark 1.4.25, it is clear that every sub-minimal dilation naturally leads to the minimal bi-dilation of an instrument. Conversely, suppose we begin with with the minimal dilation quadruple $(\mathcal{K}, \pi, E, V)$, of an instrument $\mathcal{I}$. Consider the orthogonal projections $P_1, P_2 \in \mathcal{B}(\mathcal{K})$ with ranges as the sub-spaces $\mathcal{K}_1, \mathcal{K}_2$ of the minimal dilation space $\mathcal{K}$, defined as

$$\mathcal{K}_1 = [\pi(\mathcal{A})V(\mathcal{H})] \text{ and } \mathcal{K}_2 = [E(\mathcal{O}(X))V(\mathcal{H})].$$

Since $\mathcal{K}_1$ is invariant under the representation π, the compression $P_1 \pi P_1$ induces a sub-representation of π. Then it is immediate from the definition of $\mathcal{K}_1$ that the tuple $(\mathcal{K}_1, P_1\pi P_1, V)$ forms a minimal Stinespring triple for the CP marginal

$$\phi_{\mathcal{I}}(a) = V^*\pi(a)V, \text{ for all } a \in \mathcal{A}.$$

Moreover, the POVM μ appearing in the CP sub-minimal dilation Theorem 1.4.24 can be identified as

$$\mu(A) = P_1 E(A) P_1,\ \forall\ A \in \mathcal{O}(X).$$

Thus, we conclude that starting from the minimal bi-dilation quadruple $(\mathcal{K}, \pi, E, V)$ of an instrument $\mathcal{I}$, we can explicitly identify the CP sub-minimal dilation as $(\mathcal{K}_1, P_1\pi P_1, P_1 E P_1, V)$. By a similar line of reasoning, the POVM subminimal dilation is $(\mathcal{K}_2, P_2\pi P_2, P_2 E P_2, V)$ where the minimal bi-dilation quadruple of the instrument $\mathcal{I}$ is $(\mathcal{K}, \pi, E, V)$.

Decomposable instruments

In classical measure theory, a product measure combines two measure spaces into a single joint space in a consistent and well-understood manner. Motivated by the discussion in Remark 1.4.4, we now consider the quantum analogue of this concept. In [Oza84], Ozawa introduced the notion of *decomposable* instruments, extending the idea of product measures to the non-commutative setting of operator algebras. Such instruments are those whose statistical structure can be factored along two components, paralleling the factorization of product measures into their marginals in the classical case. We now formally present the definition of decomposable instruments:

Definition 1.4.29 (Decomposable instruments)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument. Then $\mathcal{I}$ is said to be a decomposable instrument if

$$\mathcal{I}(A, a) = \phi_{\mathcal{I}}(a)\, \mu_{\mathcal{I}}(A), \quad \forall a \in \mathcal{A},\ A \in \mathcal{O}(X),$$

where $\phi_{\mathcal{I}}$ and $\mu_{\mathcal{I}}$ are the CP map and POVM associated with the instrument $\mathcal{I}$, respectively.

It is evident that, if $\mathcal{I} : \mathcal{O}(X) \times \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a decomposable instrument, then $\mathcal{I}(A, a) = \phi_{\mathcal{I}}(a)\mu_{\mathcal{I}}(A) = \mu_{\mathcal{I}}(A)\phi_{\mathcal{I}}(a)$, $\forall a \in \mathcal{A},\ A \in \mathcal{O}(X)$.

We will now present a characterization of decomposable instruments utilizing the concept of sub-minimal dilations of quantum instruments.

Theorem 1.4.30. Let, $\mathcal{I} : \mathcal{O}(X) \times \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP instrument with the minimal bi-dilation quadruple $(\mathcal{K}, \pi, E, V)$. Let P_1, P_2 be the orthogonal projections onto the sub-spaces $[\pi(\mathcal{A})V(\mathcal{H})]$ and $[E(\mathcal{O}(X))V(\mathcal{H})]$ of the dilation space $\mathcal{K}$. Then TFAE:

(i) $\mathcal{I}$ is decomposable,

(ii) $P_1E(A)P_1V = VV^*P_1E(A)P_1V$, for all $A \in \mathcal{O}(X)$,

(iii) $P_2\pi(a)P_2V = VV^*P_2\pi(a)P_2V$, for all $a \in \mathcal{A}$.

Proof. From the Definition 1.4.29 we have, $\mathcal{I}$ is decomposable if and only if

$$\mathcal{I}(A, a) = \phi_{\mathcal{I}}(a)\mu_{\mathcal{I}}(A), \quad \forall a \in \mathcal{A},\ A \in \mathcal{O}(X). \tag{1.4.6}$$

Using the CP sub-minimal dilation of $\mathcal{I}$, for all $A \in \mathcal{O}(X)$ and $a \in \mathcal{A}$, equation 1.4.6 is equivalent to:

$$\begin{aligned} & V^*P_1\pi(a)P_1E(A)P_1V = V^*P_1\pi(a)P_1VV^*P_1E(A)P_1V \\ \iff\ & V^*P_1E(A)P_1(1 - VV^*)P_1\pi(a)P_1V = 0 \\ \iff\ & P_1E(A)P_1V = VV^*P_1E(A)P_1V, \end{aligned}$$

where the last equivalence follows from the fact that the preceding equality holds for all $a \in \mathcal{A}$. This establishes the equivalence (i) $\iff$ (ii). Following a similar line of reasoning one can establish the equivalence (i) $\iff$ (iii). Combining these two results, we conclude the equivalence between (ii) and (iii). ■

Remark 1.4.31. In the preceding theorem, if the instrument $\mathcal{I}$ is unital, then we have the following additional equivalent criteria. Condition (ii) is equivalent to:

$$P_1E(A)P_1VV^* = VV^*P_1E(A)P_1VV^* = VV^*P_1E(A)P_1, \ \textit{for all}\ \ A \in \mathcal{O}(X),$$

and condition (iii) is equivalent to:

$$P_2\pi(a)P_2VV^* = VV^*P_2\pi(a)P_2VV^* = VV^*P_2\pi(a)P_2, \ \textit{for all}\ a \in \mathcal{A}.$$

As a consequence of Theorem 1.4.30, we obtain the following corollary, originally established by Ozawa ([Oza84, Proposition 4.3]). Making use of this it is possible to give an alternative proof of Theorem 1.4.10.

Corollary 1.4.32. A CP instrument $\mathcal{I} : \mathcal{O}(X) \times \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is decomposable if either $\phi_{\mathcal{I}}$ is a $*$-homomorphism or $\mu_{\mathcal{I}}$ is a spectral measure.

Atomic and non-atomic instruments

In classical measure theory atomic and non-atomic measures have been widely studied. To better understand the structure of POVMs Ramsey, Plosker, et al. [MPR20] extended these notions to the quantum setting of POVMs. More recently, the authors of [BBK21] utilized the decomposition of POVMs into atomic and non-atomic POVMs to analyze the C^*-extreme points. Here we extend these ideas to the framework of quantum instruments, aiming to recast and build upon some of these results.

Definition 1.4.33. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument. A set $A \in \mathcal{O}(X)$ is called an *atom* for $\mathcal{I}$ if $\mathcal{I}(A) \neq 0$ and for every $B \subseteq A$, $B \in \mathcal{O}(X)$, we have$\mathcal{I}(B) = 0 \quad \text{or} \quad \mathcal{I}(B) = \mathcal{I}(A)$. The instrument $\mathcal{I}$ is said to be *atomic* if every measurable set A with $\mathcal{I}(A) \neq 0$ contains at least one atom. Conversely, $\mathcal{I}$ is called *non-atomic* if it contains no atoms.

Remark 1.4.34. From the definition, it is straight forward to verify that A is an atom for the instrument $\mathcal{I}$ iff it is an atom for the associated POVM $\mu_{\mathcal{I}}$.

It is a well-known fact that every finite (and more generally, every σ-finite) positive measure decomposes uniquely into the sum of an atomic and a non-atomic positive measure (see [Joh70]). In an analogous manner, every POVM admits a unique decomposition as the sum of an atomic POVM and a non-atomic POVM, as established in [MPR20]. While the proof in [MPR20] was formulated for POVMs on locally compact Hausdorff spaces, it was later fully generalized to arbitrary measurable spaces in [BBK21]. We extend this result to the setting of instruments and state the corresponding decomposition theorem in this context. The proof closely follows the classical case as presented in [Joh70] and we skip it.

Theorem 1.4.35. Every Quantum instrument decomposes uniquely as a sum of an atomic CP instrument and a non-atomic CP instrument.

We conclude this discussion with an useful observation on atoms of instruments.

Proposition 1.4.36. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument with the minimal bi-dilation tuple $(\mathcal{K}, \pi, E, V)$. Then a subset $A \in \mathcal{O}(X)$ is an atom for $\mathcal{I}$ if and only if A is an atom for the spectral instrument πE. In particular, $\mathcal{I}$ is atomic (non-atomic) if and only if πE is atomic (non-atomic).

Proof. For any subset $A \in \mathcal{O}(X)$, A is an atom for $\mathcal{I}$ if and only if $\mathcal{I}(A) \neq 0$ and for each $A' \subseteq A$ in $\mathcal{O}(X)$, we have either $\mathcal{I}(A') = 0$ or $\mathcal{I}(A \setminus A') = 0$. Equivalently, $(\pi E)(A) \neq 0$ and by Proposition 1.4.16, we have either $(\pi E)(A') = 0$ or $(\pi E)(A \setminus A') = 0$. This is precisely the condition for A to be an atom for πE. The second assertion follows easily from the first. ∎

Chapter 2

Convexities, Instruments and Marginals

The set $I_{\mathcal{H}}(X, \mathcal{A})$ of all unital completely positive (UCP) instruments on a measurable space $(X, \mathcal{O}(X))$ with values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ forms a convex set. Various works have explored this convexity structure (see [Pel13a; Pel13b; HHP14; HP11; DPS11]). Though we use a somewhat different setup and do not use the technical framework of direct integrals of Hilbert spaces, many of the results here are motivated by these articles. The main goal of this chapter is to study the role of quantum convexity or C^*-convexity (See Definition 2.1.7) in understanding quantum instruments. A particularly interesting aspect arises from viewing CP instruments as quantizations of joint measures, as highlighted in their bivariate realization (Remark 1.4.4). There, the marginals consist of a CP map on a C^*-algebra and a POVM on a measurable space. One may, however, consider other quantizations—where both marginals are either CP maps (see, for example, [Kur18]) or POVMs. From the perspective of quantum measurement theory, the concept of CP instruments with one marginal being a POVM (which could be called as semi-classical) and another being a CP map (purely quantum) has physical relevance. Mathematically, this structure is interesting as we get to see different features in two marginals.

Classically, a joint probability measure on a product space is an extreme point in the convex set of all joint measures if and only if it is the product of its marginals and each marginal is itself extreme. In other words, the extremality of the joint distribution is equivalent to the extremality of the marginals together with the product structure. In particular, the extremality of even one marginal suffices to uniquely determine the entire joint distribution. Thus, in the classical setting, the convex structure of the marginals plays a fundamental role in characterizing the joint distribution.

From a mathematical standpoint, it becomes natural to investigate the structure of CP instruments through their marginals, particularly through their convexity theory. In the setting of completely positive (CP) instruments, the classical characterization of extremality holds only in one direction: if both marginals of a CP instrument are extreme, then the instrument itself is extreme. This result was established by Haapasalo, Heinosaari, and Pellonpää (see Theorem 4.1 in [HHP14]). Furthermore, they established that the extremality of even one of the marginals is sufficient to uniquely determine the instrument (Theorem 4.1, [HHP14]). However, a product representation of an instrument in terms of its extreme marginals generally does not hold. These findings provide a partial analogue of the classical case in the quantum setting, while simultaneously revealing key structural differences arising from the inherent noncommutativity of quantum observables. However, the converse does not hold—that is, a CP instrument may be extreme even if its marginals are not extreme. A concrete example illustrating this phenomenon is discussed in Example 2.1.4.

These observations naturally motivate a re-investigation of such results through the lens of C*-convexity. Here, we have characterized the set of C^*-extreme points within the space of unital completely positive (UCP) instruments. Furthermore, we have obtained a partial counterpart of the classical result: if the marginals of a CP instrument are C^*-extreme, then the instrument itself is decomposable (see Definition 1.4.29)—that is, it coincides with the product of its marginals. In particular, the instrument is uniquely determined by its C^*-extreme marginals. However, the converse does not hold in general: the C^*-extremality of an instrument does not necessarily imply the C^*-extremality of its marginals, as demonstrated by Example 2.2.5.

2.1 Classical and C^*-Convexity of Instruments

2.1.1 Classical convexity and extreme instruments

The classical convex structure of unital completely positive (UCP) maps and normalized POVMs, along with several of their subclasses, has attracted considerable attention in the literature. It was W. Arveson who first established an abstract characterization of extreme UCP maps, presented as Theorem 1.4.6 in [Arv69]. Later, M. D. Choi revisited this result in the setting of matrix algebras [Cho75], and a generalization of Choi's result can be found in [Tsu96]. For characterizations of extreme points of normalized POVMs, we refer to [Pel13a],[Par99],[DLPP05],[FPS11], and [HP11], especially in the cases where X is finite, countable, or a compact Hausdorff space, and the Hilbert space $\mathcal{H}$ is finite-dimensional.

These mathematical objects—namely UCP maps and POVMs—can be regarded as particular instances of completely positive (CP) instruments. Consequently, a thorough understanding of CP instruments becomes essential. In this context, classical convexity theory provides a natural and powerful framework for analyzing the convex set $\mathcal{I}_{\mathcal{H}}(X, \mathcal{A}) :=$ {all normalized CP instruments on X with values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$}. Inspired by Arveson's abstract characterization of extremal UCP maps [Arv69], one observes that extremal CP instruments admit an analogous abstract description. This has already been observed by several authors: in particular, Pellonpää [Pel13a] studied this using the framework of direct integrals, and D'Ariano et al. [DPS11] discussed it in the finite-dimensional setting. We state the result below in our general setting without proof, as its derivation follows similar lines of reasoning to those used in Theorem 1.4.6 of [Arv69].

Theorem 2.1.1 (Extreme point condition)**.** Suppose that $\mathcal{I} \in I_{\mathcal{H}}(X, \mathcal{A})$ has the minimal dilation tuple $(\mathcal{K}, \pi, E, V)$. Then a necessary and sufficient criterion for $\mathcal{I}$ to be extreme in $I_{\mathcal{H}}(X, \mathcal{A})$ is that the map $D \mapsto V^*DV$ from $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ to $\mathcal{B}(\mathcal{H})$ is injective.

As a direct consequence of Theorem 2.1.1, we obtain the following corollary.

Corollary 2.1.2. Every spectral instrument is an extreme point in $I_{\mathcal{H}}(X, \mathcal{A})$.

Next, we present a characterization of the extremity in terms of the Choi–Kraus decomposition (Proposition 1.4.8) for CP instruments defined on finite sets. This is a consequence of Theorem 2.1.1 and extends Choi's characterization of extreme UCP maps on matrix algebras ([Cho75, Theorem 5]).

Corollary 2.1.3. Let $\mathcal{I} : \mathcal{O}(X) \to CP(M_d(\mathbb{C}), M_k(\mathbb{C}))$ be a CP instrument, where $X = \{1, \cdots, n\}$ be a finite set. Suppose,

$$\mathcal{I}(i, T) = \sum_{j \in I_i} V_j^{i*} T V_j^i, \ \forall\, T \in M_d,$$

is a minimal Choi–Kraus decomposition of $\mathcal{I}$, where I_i denotes the index set of the minimal Choi-Kraus operators $\{V_j^i : j \in I_i\} \subset M_{d\times k}(\mathbb{C})$. Then $\mathcal{I}$ is extreme if and only if the set $\{V_j^{i*} V_k^i : j, k \in I_i, i \in X\}$ is linearly independent.

Corollary 2.1.3 enables the construction of numerous extreme instruments that are not spectral in nature. As an illustration, consider the following example of a unital completely positive (UCP) instrument that is extreme, yet not spectral.

Example 2.1.4. Fix $0 < t < \frac{1}{2}$. Let $\mu : \mathcal{O}(X) \to M_2(\mathbb{C})$ be a POVM on $X = \{1, 2\}$, defined by

$$\mu(1) = tE_{11} + (1-t)E_{22}, \quad \mu(2) = (1-t)E_{11} + tE_{22}.$$

where E_{ii} denote the matrix unit with 1 in the (i,i)-th entry and 0 elsewhere. Consider the instrument $\mathcal{I} : \mathcal{O}(X) \to CP(M_2(\mathbb{C}), M_2(\mathbb{C}))$ given by,

$$\mathcal{I}(i)(T) = \mu(i)^{\frac{1}{2}} T \mu(i)^{\frac{1}{2}}, \text{ for all } T \in M_2(\mathbb{C}).$$

It follows immediately from Corollary 2.1.3 that $\mathcal{I}$ is an extreme unital completely positive (UCP) instrument. Moreover, observe that the POVM marginal $\mu_{\mathcal{I}}$ of $\mathcal{I}$ coincides with the original POVM μ. Since μ is not spectral, it follows that $\mathcal{I}$ cannot be a spectral instrument either.

Remark 2.1.5. The instruments of the form described in Example 2.1.4 are referred to as generalized Lüders instruments. It can be verified that such an instrument is extreme if and only if the set $\{\mu(i) : i \in X\}$ is linearly independent for the associated POVM μ.

Here is an alternative abstract characterization of the extremity of instruments, in terms of the natural partial order on instruments. Recall that an instrument $\mathcal{J}$ is *dominated* by an instrument $\mathcal{I}$, if $\mathcal{I}(A) - \mathcal{J}(A)$ is CP for every A in $\mathcal{O}(X)$.

Theorem 2.1.6. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument. Then $\mathcal{I}$ is extreme iff for any two CP instruments $\mathcal{J}_1, \mathcal{J}_2$ dominated by $\mathcal{I}$ such that $\mathcal{J}_1(X) = \mathcal{J}_2(X)$ then $\mathcal{J}_1 = \mathcal{J}_2$.

Proof. To prove the 'if' part, we consider the UCP instruments, $\mathcal{I}_1 = \mathcal{I} - \mathcal{J}_1 + \mathcal{J}_2$ and $\mathcal{I}_2 = \mathcal{I} - \mathcal{J}_2 + \mathcal{J}_1$. It is clear that if $\mathcal{J}_1 \neq \mathcal{J}_2$, then we have a proper convex combination for the extreme instrument $\mathcal{I}$. Next we assume the converse part in the hypothesis. If $\mathcal{I}$ is not extreme then there exists a proper convex combination, $\mathcal{I} = \sum p_i \mathcal{I}_i$, for $\mathcal{I}$. Consider the instruments, $\mathcal{J}_1 = \mathcal{I} - (p_1 \wedge p_2)\mathcal{I}_1$ and $\mathcal{J}_2 = \mathcal{I} - (p_1 \wedge p_2)\mathcal{I}_2$, where $p_1 \wedge p_2 = \min\{p_1, p_2\}$. Then it is straight forward to verify that $\mathcal{J}_1(X) = \mathcal{J}_2(X)$ and both of them are dominated by $\mathcal{I}$. By the hypothesis, $\mathcal{J}_1 = \mathcal{J}_2$, which further implies that $\mathcal{I}_1 = \mathcal{I}_2$. Therefore, $\mathcal{I}$ must be extreme. ∎

2.1.2 C^*-convexity and C^*-extreme instruments

Inspired by the C^*-convexity of POVMs and unital completely positive (UCP) maps, we introduce the concepts of C^*-convexity and C^*-extreme points in the framework of instruments. In this context, we also explore abstract characterizations of C^*-extremity for instruments. To begin, we formally define C*-convexity for the collection of all normalized instruments $I_{\mathcal{H}}(X, \mathcal{A})$.

Definition 2.1.7 (C^*-convexity)**.** For any $\mathcal{I}_i \in I_{\mathcal{H}}(X, \mathcal{A})$ and $T_i \in \mathcal{B}(\mathcal{H})$, $1 \leq i \leq n$ with $\sum_{i=1}^n T_i^* T_i = I_{\mathcal{H}}$, a sum of the form

$$\mathcal{I}(\cdot) = \sum_{i=1}^{n} T_i^* \mathcal{I}_i(\cdot) T_i \tag{2.1.1}$$

is called a C^*-*convex combination* for $\mathcal{I}$. The operators T_i's here are called C^*-*coefficients*. When T_i's are invertible, the sum in (2.1.1) is called a *proper C^*-convex combination* for $\mathcal{I}$.

Observe that $I_{\mathcal{H}}(X,\mathcal{A})$ is a C^**-convex set* in the sense that it is closed under C^*-convex combinations i.e. $\sum_{i=1}^n {T_i}^*\mathcal{I}_i(\cdot)T_i \in I_{\mathcal{H}}(X,\mathcal{A})$, whenever $\mathcal{I}_i \in I_{\mathcal{H}}(X,\mathcal{A})$ and $T_i \in \mathcal{B}(\mathcal{H})$ satisfying $\sum_{i=1}^n {T_i}^*T_i = I_{\mathcal{H}}$.

Definition 2.1.8 (C^*-extreme point)**.** An instrument $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A},\mathcal{B}(\mathcal{H}))$ is called a C^**-extreme point* in $I_{\mathcal{H}}(X,\mathcal{A})$ if, whenever $\sum_{i=1}^n T_i^*\mathcal{I}_i(\cdot)T_i$ is a proper C^*-convex combination of $\mathcal{I}$, then each $\mathcal{I}_i$ is unitarily equivalent to $\mathcal{I}$ i.e. there are unitary operators $U_i \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{I}_i(\cdot) = U_i^*\mathcal{I}(\cdot)U_i$ for $1 \le i \le n$.

Remark 2.1.9. Throughout, a C^*-extreme instrument will refer to a C^*-extreme point of $\mathcal{I}_{\mathcal{H}}(X,\mathcal{A})$.

It is immediate from the definition that any UCP instrument unitarily equivalent to a C^*-extreme instrument is itself C^*-extreme.

Next, we present a few examples of C^*-extreme instruments, inspired by Proposition 1.2 of [FM97].

Example 2.1.10. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A},\mathcal{B}(\mathcal{H}))$ be a UCP instrument, and let $(\mathcal{K},\pi,E,V)$ denote its minimal bi-dilation. Then $\mathcal{I}$ is C^**-extreme* if it fulfills any of the following criteria:

(i) $\mathcal{I}$ is a decomposable instrument whose UCP marginal is the inflation of a pure state, i.e., $\phi_{\mathcal{I}}(a) = \xi(a)I_{\mathcal{H}}$ for all $a \in \mathcal{A}$, where ξ is a pure state on $\mathcal{A}$, and whose POVM marginal is trivial.

(ii) $\mathcal{I}$ is a pure instrument with a trivial POVM marginal $\mu_{\mathcal{I}}$.

(iii) $\mathcal{I}$ is a spectral instrument.

(iv) The subspace $V(\mathcal{H})$ is invariant under the commutant $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$.

Remark 2.1.11. Then, from (i) and the existence of pure states, it follows that the set of C^*-extreme points of $I_{\mathcal{H}}(X,\mathcal{A})$ is non-empty.

2.1.3 Abstract characterizations of C^*-extreme points

Farenick and Zhou [FZ98], taking cue from the Arveson's extreme point criteria for UCP maps, introduced an abstract characterization of C^*-extreme points for unital completely positive maps. Later, Bhat and Kumar [BK22], adapted this characterization to the setting of C^*-extreme points of normalized POVMs. In this work, we extend these ideas further by presenting an abstract characterization of C*-extreme instruments. Our proof follows an adaptation of the method used for the POVM case.

Theorem 2.1.12. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A},\mathcal{B}(\mathcal{H}))$ be a CP instrument and let $(\mathcal{K},\pi,E,V)$ be the minimal bi-dilation quadruple. Then $\mathcal{I}$ is a C^*-extreme if and only if for any positive operator $D \in \{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ with V^*DV being invertible, there exists a co-isometry $U \in \{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ (i.e. $UU^* = I_{\mathcal{H}}$) satisfying $U^*UD^{1/2} = D^{1/2}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VS$.

The following is an immediate corollary of Theorem 2.1.12.

Corollary 2.1.13. Spectral instruments are C^*-extreme points within the convex set $I_{\mathcal{H}}(X,\mathcal{A})$ of normalized CP instruments.

Proof. If $\mathcal{I}$ is a spectral instrument then the quadruple $(\pi,E,I_{\mathcal{H}},\mathcal{H})$ can be taken to be the minimal bi-dilation for $\mathcal{I}$. For $D \in \{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ with $D(= I_{\mathcal{H}}^*DI_{\mathcal{H}})$ positive and invertible, we can take $U = I_{\mathcal{H}}$ and $S = D^{1/2}$ to satisfy the criterion. ■

We now present an alternative abstract characterization of C^*-extreme points, adapted from a corollary originally formulated for completely positive maps by Bhat and Kumar in [BK22]. This result plays a crucial role in the analysis of C^*-extremity in the setting of instruments.

Corollary 2.1.14. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument, and let $(\mathcal{K}, \pi, E, V)$ denote its minimal bi-dilation. Then $\mathcal{I}$ is a C^*-extreme point of $I_{\mathcal{H}}(X, \mathcal{A})$ if and only if the following holds: for any positive operator $D \in \{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ such that V^*DV is invertible, there exists an operator $S \in \{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ satisfying $D = S^*S$, $SVV^* = VV^*SVV^*$, and V^*SV is invertible, i.e., S maps $V\mathcal{H}$ into itself and the restriction $S|_{V\mathcal{H}}$ is invertible.

Analogous to Theorem 2.1.6, which characterizes extreme instruments, we now present a result concerning C^*-extremity. This criterion, originally due to Farenick and Zhou, was established in the context of completely positive maps in [FZ98].

Corollary 2.1.15. Let $\mathcal{I} \in I_{\mathcal{H}}(X, \mathcal{A})$ be a CP instrument. Then $\mathcal{I}$ is a C^*-extreme point in $I_{\mathcal{H}}(X, \mathcal{A})$ if and only if the following condition holds: for any CP instrument $\mathcal{J} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ satisfying $\mathcal{J} \leq \mathcal{I}$ and such that $\mathcal{J}(X, 1_{\mathcal{A}})$ is invertible, there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\mathcal{J}(A, a) = S^* \mathcal{I}(A, a) S, \ \forall\, a \in \mathcal{A},\ A \in \mathcal{O}(X)$.

2.1.4 Direct sums of pure instruments

We now consider the scenario in which the direct sum of two C^*-extreme instruments continues to be C^*-extreme. This discussion is motivated by the necessary and sufficient condition established in [BK22] for the direct sum of unital completely positive maps to be C^*-extreme. We present a brief outline of the corresponding results for instruments. As a first step, we define the notion of the direct sum of instruments.

Definition 2.1.16 (Direct sum of instruments)**.** For any countable collection of CP instruments $\{\mathcal{I}_i : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}_i))\}_{i\in\Lambda}$, their direct sum $\oplus_{i\in\Lambda}\mathcal{I}_i$ is the instrument $\oplus_{i\in\Lambda}\mathcal{I}_i : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, defined by $(\oplus_{i\in\Lambda}\mathcal{I}_i)(A)(a) = \oplus_{i\in\Lambda}\mathcal{I}_i(A)(a)$, for all $A \in \mathcal{O}(X), a \in \mathcal{A}$, where $\mathcal{H} = \oplus_{i\in\Lambda}\mathcal{H}_i$.

Motivated by the concept of disjoint representations, we introduce the notion of disjointness in the context of instruments. We begin by defining when two spectral instruments are said to be mutually disjoint, and subsequently extend this notion to general instruments.

Definition 2.1.17 (Mutually disjoint spectral instruments)**.** Two spectral instruments $\pi_i E_i : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}_i)), i = 1, 2$ are said to be mutually disjoint if there does not exist any non-zero $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ satisfying $T\pi_1(a)E_1(A) = \pi_2(a)E_2(A)T$ for all $a \in \mathcal{A}, A \in \mathcal{O}(X)$ i.e., there is no non-trivial intertwiner between the two spectral instruments.

Now we are in a position to extend this definition for instruments.

Definition 2.1.18. Let $\mathcal{I}_i : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}_i)), i = 1, 2$ two CP instruments with respective minimal bi-dilation $(\mathcal{K}_i, \pi_i, E_i, V_i)$, we say $\mathcal{I}_1$ and $\mathcal{I}_2$ are disjoint if $\pi_1 E_1$ is disjoint to $\pi_2 E_2$.

The following proposition is easy to prove using the characterizations of C^*-extrmity proved above.

Proposition 2.1.19. Let $\{\mathcal{I}_i : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}_i))\}_{i\in\Lambda}$ be a countable collection of mutually disjoint instruments. Then $\mathcal{I} = \oplus_{i\in\Lambda}\mathcal{I}_i$ is C^*-extreme iff each $\mathcal{I}_i$ is C^*-extreme.

The following theorem is a direct adaptation of Theorem 3.7 in [BK22] in the context of instruments.

Theorem 2.1.20. Let $\mathcal{I}$ be a direct sum of pure UCP instruments, so that $\mathcal{I}$ is unitarily equivalent to $\bigoplus_{\alpha\in\Gamma}\bigoplus_{i\in\Lambda_\alpha}\mathcal{J}_\alpha^i(\cdot)\otimes I_{\mathcal{K}_\alpha^i}$, where $\mathcal{K}_\alpha^i$ is a Hilbert space and $\mathcal{J}_\alpha^i$ is a pure UCP instrument with minimal bi-dilation tuple $(\mathcal{H}_{\pi_\alpha E_\alpha},\pi_\alpha,E_\alpha,V_\alpha^i)$ such that $\mathcal{J}_\alpha^i$ is non-unitarily equivalent to $\mathcal{J}_\alpha^j$ for each $i\neq j$ in Λ_α, $\alpha\in\Gamma$, and $\pi_\alpha E_\alpha$ is disjoint to $\pi_\beta E_\beta$ for $\alpha\neq\beta$. Then $\mathcal{I}$ is C^*-extreme in $I_{\mathcal{H}}(X,\mathcal{A})$ if and only if the following holds for each $\alpha\in\Gamma$:

1. The family $\{\text{range}(V_\alpha^i)\}_{i\in\Lambda_\alpha}$ is a nest (see Definition 2.1.26) in $\mathcal{H}_{\pi_\alpha E_\alpha}$, which endows Λ_α with the structure of a totally ordered set.
2. Setting $\mathcal{L}_\alpha^i := \bigoplus_{j\leq i}\mathcal{K}_\alpha^j$ for each $i\in\Lambda_\alpha$, the completion of the nest $\{\mathcal{L}_\alpha^i\}_{i\in\Lambda_\alpha}$ in $\bigoplus_{i\in\Lambda_\alpha}\mathcal{K}_\alpha^i$ (that is, the closure of the nest with respect to arbitrary joins and meets of its elements) is countable.

2.1.5 Structure of C^*-extreme instruments in finite dimensions

In the context of completely positive (CP) maps and POVMs on finite-dimensional Hilbert spaces, every C^*-extreme point also qualifies as an extreme point in the classical convex sense. Following the approach presented in Proposition 2.1 of [FPS11], this correspondence can be established for instruments. However, it remains unclear whether the same holds when $\mathcal{H}$ is infinite-dimensional. Conversely, (see [FZ98], page 1470) there is an example of an extreme UCP map which is not C^*-extreme. Since CP maps are natural examples of instruments, this immediately provides an instrument that is extreme but not C^*-extreme.

Theorem 2.1.21. If dim $(\mathcal{H})<\infty$, then every C^*-extreme point in $I_{\mathcal{H}}(X,\mathcal{A})$ is an extreme point.

Proof. Let $\mathcal{I}$ ba a C^*-extreme UCP instrument. Suppose $\mathcal{I}=t\,\mathcal{I}_1+(1-t)\,\mathcal{I}_2$ with $0<t<1$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are UCP instruments. Any convex combination is a C^*-convex combination (using the coefficients $\sqrt{t}\,I$ and $\sqrt{1-t}\,I$), so C^*-extremality implies $\mathcal{I}_i=\text{Ad}_{U_i}\circ\mathcal{I}$ for some unitaries $U_i\in\mathcal{B}(\mathcal{H}), i=1,2$.

Fix $(A,a)\in\mathcal{O}(X)\times\mathcal{A}$ and set $T:=\mathcal{I}(A,a)$. Then

$$T=t\,U_1^*TU_1+(1-t)\,U_2^*TU_2,\qquad \|T\|_2=\|U_i^*TU_i\|_2\ (i=1,2),$$

where $\|\cdot\|_2$ is the Hilbert-Schimdt norm on the finite-dimensional Hilbert space $\mathcal{B}(\mathcal{H})$. However, the sphere of a Hilbert space contains no non-trivial line segments, we conclude $U_1^*TU_1=T=U_2^*TU_2$. Thus $\mathcal{I}_1=\mathcal{I}_2=\mathcal{I}$, proving extremality. ■

We begin by recalling several key concepts from the theory of log-modular and factorization algebras, as well as from nest algebras. The investigation of the factorization property for subalgebras of C^*-algebras has a long-standing history. A classical instance is provided by Cholesky's theorem, which demonstrates that the algebra of upper-triangular matrices in $M_n(\mathbb{C})$ possesses the factorization property. For further details and references, see [BK24]. These ideas will be essential in our subsequent analysis and characterization of C^*-extreme quantum instruments.

Definition 2.1.22. Let $\mathcal{M}$ be a subalgebra of a C^*-algebra $\mathcal{B}$. We say that $\mathcal{M}$ is *log-modular* in $\mathcal{B}$ if

$$\{a^*a : a\in\mathcal{M},\ a^{-1}\in\mathcal{M}\}$$

is norm dense in $\mathcal{B}_+^{-1}$, the set of all positive invertible elements of $\mathcal{B}$.

In particular, if

$$\mathcal{B}_+^{-1}=\{a^*a : a\in\mathcal{M},\ a^{-1}\in\mathcal{M}\},$$

then $\mathcal{M}$ is said to have the *factorization property* in $\mathcal{B}$. The factorization property is also referred to as *strong log-modularity.*

Remark 2.1.23. It is immediate from the definition that if a subalgebra $\mathcal{M}$ of a C^*-algebra $\mathcal{B}$ has the factorization property in $\mathcal{B}$, then $\mathcal{M}$ is automatically log-modular in $\mathcal{B}$.

We now restate a weaker form of Corollary 2.1.14 using the language of factorization.

Proposition 2.1.24. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{H})$ be a C^*-extreme instrument with minimal bi-dilation $(\mathcal{K}, \pi, E, V)$. Then the subalgebra

$$\mathcal{M} := \{\, S \in \{\pi(\mathcal{A})E(\mathcal{O}(X))\}' : SVV^* = VV^*SVV^* \,\}$$

has the factorization property in $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$.

For a collection $\{P_i\}_{i\in\Lambda}$ of projections in $\mathcal{B}(\mathcal{H})$, we denote by $\vee_{i\in\Lambda}P_i$ the projection onto the smallest closed subspace containing the ranges of all P_i, and by $\wedge_{i\in\Lambda}P_i$ the projection onto the intersection of the ranges of all P_i.

Definition 2.1.25 (Lattice)**.** A set $\mathcal{L}$ of projections in a von Neumann algebra $\mathcal{B}$ is called a *lattice* if, for any $P, Q \in \mathcal{L}$, both $P \wedge Q$ and $P \vee Q$ belong to $\mathcal{L}$.

Let $\mathcal{M}$ be a subalgebra of a von Neumann algebra $\mathcal{B}$. We define

$$Lat_{\mathcal{B}}\mathcal{M} := \{P \in \mathcal{B} : P = P^* = P^2 \text{ and } AP = PAP \ \forall A \in \mathcal{M}\},$$

which is the collection of all projections in $\mathcal{B}$ whose ranges are invariant under each operator in $\mathcal{M}$. When $\mathcal{B} = \mathcal{B}(\mathcal{H})$, we write $Lat\mathcal{M}$ for simplicity. Moreover, if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, then $Lat_{\mathcal{B}}\mathcal{M} = \mathcal{B} \cap Lat\mathcal{M}$. It is immediate that $0, 1 \in Lat_{\mathcal{B}}\mathcal{M}$, and this set is closed under the operations $\wedge$ and $\vee$ for arbitrary subcollections, as well as under the weak operator topology (WOT).

Dually, for any collection $\mathcal{S}$ of projections in $\mathcal{B}$ (not necessarily forming a lattice), we define

$$Alg_{\mathcal{B}}\mathcal{S} := \{X \in \mathcal{B} : XP = PXP \ \forall P \in \mathcal{S}\},$$

which consists of all operators in $\mathcal{B}$ leaving the range of each $P \in \mathcal{S}$ invariant. If $\mathcal{B} = \mathcal{B}(\mathcal{H})$, we write $Alg\mathcal{S}$. Clearly, $Alg_{\mathcal{B}}\mathcal{S} = \mathcal{B} \cap Alg\mathcal{S}$ and it forms a unital subalgebra of $\mathcal{B}$ that is WOT-closed.

Definition 2.1.26 (Nest)**.** A lattice $\mathcal{N}$ of projections in a von Neumann algebra $\mathcal{B}$ is called a *nest* if it is totally ordered with respect to the operator ordering; that is, for any $P, Q \in \mathcal{N}$, either $P \le Q$ or $Q \le P$. The nest $\mathcal{N}$ is said to be *complete*, denoted $\overline{\mathcal{N}}$, if $0, 1 \in \mathcal{N}$ and for any family $\{P_i\}_{i\in\Lambda} \subseteq \mathcal{N}$, both $\vee_{i\in\Lambda}P_i$ and $\wedge_{i\in\Lambda}P_i$ belong to $\mathcal{N}$.

Definition 2.1.27 (Atoms and Atomic Nests)**.** Let $\mathcal{N}$ be a complete nest of projections in a von Neumann algebra $\mathcal{B}$. A non-zero projection $r \in \mathcal{B}$ is called an *atom* of $\mathcal{N}$ if there exists $P \in \mathcal{N}$ such that

$$r = P - \vee_{Q<P}Q,$$

where the join is taken over all $Q \in \mathcal{N}$ strictly below P.

The nest $\mathcal{N}$ is said to be *atomic* if there exists a countable family of atoms $\{r_n\}$ in $\mathcal{N}$ satisfying

$$\sum_n r_n = 1_{\mathcal{B}} \quad \text{in the weak operator topology (WOT).}$$

Example 2.1.28 (Trivial Nest)**.** Let $\mathcal{H}$ be any Hilbert space. The collection

$$\mathcal{N} = \{\{0\}, \mathcal{H}\}$$

is a complete nest, called the *trivial nest*. It is the smallest complete nest on $\mathcal{H}$, and completeness is immediate since the only possible join and meet of $\{0\}$ and $\mathcal{H}$ remain within $\mathcal{N}$.

Example 2.1.29 (Finite Nest on $\mathbb{C}^n$)**.** Let $\mathcal{H} = \mathbb{C}^n$ with standard orthonormal basis $\{e_1, \dots, e_n\}$. Define

$$\mathcal{N} = \big\{\{0\}, \operatorname{span}\{e_1\}, \operatorname{span}\{e_1, e_2\}, \dots, \operatorname{span}\{e_1, \dots, e_n\}\big\}.$$

This is a complete nest of $n + 1$ elements. Completeness follows from the fact that every totally ordered chain of subspaces of a finite-dimensional Hilbert space is automatically closed under finite joins and meets.

Example 2.1.30 (Volterra Nest)**.** Let $\mathcal{H} = L^2[0, 1]$. For each $t \in [0, 1]$, define

$$\mathcal{H}_t := \big\{f \in L^2[0, 1] : f = 0 \text{ a.e. on } (t,\, 1]\big\}.$$

The collection $\mathcal{N} = \{\mathcal{H}_t : t \in [0, 1]\}$ is a complete nest, known as the *Volterra nest*. Completeness holds since

$$\bigvee_{s<t} \mathcal{H}_s = \mathcal{H}_t, \qquad \bigcap_{s>t} \mathcal{H}_s = \mathcal{H}_t,$$

with $\mathcal{H}_0 = \{0\}$ and $\mathcal{H}_1 = L^2[0, 1]$. This nest is the prototypical example in nest algebra theory, and is associated with the *Volterra integration operator* $V : L^2[0, 1] \to L^2[0, 1]$ defined by

$$(V f)(x) := \int_0^x f(t)\, dt.$$

Example 2.1.31 (Atomic Nest on $\ell^2(\mathbb{N})$)**.** Let $\mathcal{H} = \ell^2(\mathbb{N})$ with standard orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, set

$$\mathcal{H}_n := \operatorname{span}\{e_1, e_2, \dots, e_n\},$$

and define

$$\mathcal{N} = \{\{0\}\} \cup \{\mathcal{H}_n : n \in \mathbb{N}\} \cup \{\ell^2(\mathbb{N})\}.$$

This is a countable complete nest. Completeness requires the inclusion of the top element:

$$\bigvee_{n\in\mathbb{N}} \mathcal{H}_n = \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\} = \ell^2(\mathbb{N}) \in \mathcal{N}.$$

Each pair of consecutive elements $\mathcal{H}_{n-1} \subsetneq \mathcal{H}_n$ contributes an *atom* $\mathcal{H}_n \ominus \mathcal{H}_{n-1} = \operatorname{span}\{e_n\}$ of dimension one, making $\mathcal{N}$ a purely *atomic* nest.

Example 2.1.32 (Bilateral Nest on $\ell^2(\mathbb{Z})$)**.** Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with standard orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. For each $n \in \mathbb{Z}$, set

$$\mathcal{H}_n := \overline{\operatorname{span}}\{e_k : k \leq n\},$$

and define

$$\mathcal{N} = \{\{0\}\} \cup \{\mathcal{H}_n : n \in \mathbb{Z}\} \cup \{\ell^2(\mathbb{Z})\}.$$

This is a complete nest. Completeness follows since

$$\bigvee_{n\in\mathbb{Z}} \mathcal{H}_n = \ell^2(\mathbb{Z}) \in \mathcal{N}, \qquad \bigcap_{n\in\mathbb{Z}} \mathcal{H}_n = \{0\} \in \mathcal{N}.$$

As in Example 2.1.31, each consecutive pair contributes the atom $\operatorname{span}\{e_n\}$, so $\mathcal{N}$ is again purely atomic and countable.

Definition 2.1.33 (Nest Subalgebra)**.** Let $\mathcal{M}$ be a subalgebra of a von Neumann algebra $\mathcal{B}$. We say that $\mathcal{M}$ is a *nest subalgebra* of $\mathcal{B}$ (or a *nest algebra* if $\mathcal{B} = \mathcal{B}(\mathcal{H})$) if there exists a nest $\mathcal{N} \subseteq \mathcal{B}$ such that $\mathcal{M} = Alg_{\mathcal{B}}\mathcal{N}$. Furthermore, $\mathcal{M}$ is called $\mathcal{B}$*-reflexive* (or simply *reflexive* when $\mathcal{B} = \mathcal{B}(\mathcal{H})$) if $\mathcal{M} = Alg_{\mathcal{B}} Lat_{\mathcal{B}}\mathcal{M}$.

Next, we present a particular instance of Corollary 5.8 from [BK24], formulated here as Lemma 2.1.34. This result provides a characterization of log-modular subalgebras within finite-dimensional von Neumann algebras and will be instrumental in describing C^*-extreme instruments in finite-dimensional settings.

Lemma 2.1.34. Let $\mathcal{B}$ be a von Neumann algebra of finite dimension. A subalgebra $\mathcal{M} \subseteq \mathcal{B}$ is log-modular if and only if it is a nest subalgebra of $\mathcal{B}$. Furthermore, any such subalgebra $\mathcal{M}$ is $\mathcal{B}$-reflexive.

Remark 2.1.35. It is important to observe that in Lemma 2.1.34, if $\mathcal{M} = \mathrm{Alg}_{\mathcal{B}}(\mathcal{N})$ for a nest $\mathcal{N}$ in $\mathcal{B}$, then the completion of the nest, $\overline{\mathcal{N}}$, is contained in $\mathrm{Lat}_{\mathcal{B}}(\mathcal{N})$, which forms a finite, complete, and atomic nest.

Theorem 2.1.36. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a C^*-extreme UCP instrument,where $\mathcal{H}$ is a finite-dimensional Hilbert space. Then $\mathcal{I}$ is unitarily equivalent to a direct sum of pure UCP instruments.

Proof. Let $(\mathcal{K}, \pi, E, V)$ be the minimal bi-dilation tuple of the C^*-extreme UCP instrument $\mathcal{I}$. Since $\mathcal{H}$ is finite-dimensional, by Theorem 2.1.21 it follows that $\mathcal{I}$ is an extreme UCP instrument. Furthermore, by Theorem 2.1.1, the map $D \mapsto V^*DV$ from $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ to $\mathcal{B}(\mathcal{H})$ is injective. Consequently, $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ is finite-dimensional. Now, since $\mathcal{I}$ is C^*-extreme, Proposition 2.1.24 implies that the sub-algebra

$$\mathcal{M} := \{T : TV(H) \subseteq V(H) \ \&\ T \in \{\pi(\mathcal{A})E(\mathcal{O}(X))\}'\},$$

has factorization in $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$. By applying Lemma 2.1.34 together with Remark 2.1.35, we conclude that $\mathcal{M}$ is a reflexive sub-algebra of $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ i.e.

$$\mathcal{M} = Alg_{\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'} Lat_{\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'}\mathcal{M}$$

and that $Lat_{\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'}\mathcal{M}$ contains a finite, complete, atomic nest. Let $\mathscr{E}$ denote such a finite, complete, atomic nest contained in $Lat_{\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'}\mathcal{M}$, and let $\{P_i : i = 1, \cdots, n\}$ be the set of atoms of $\mathscr{E}$. By definition 2.1.27, each P_i is a non-zero projection of the form $P - P_-$ for some $P \in \mathscr{E}$, where $P_- = \vee_{\{Q \in \mathscr{E}; Q < P\}} Q$. It is immediate to note that each P_i, as well as any sub-projection of it, belongs to

$$Alg_{\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'}\mathscr{E} = Alg_{\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'} Lat_{\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'}\mathcal{M} = \mathcal{M}.$$

Since $\{\pi(\mathcal{A})E(\mathcal{O}(X))\}'$ is finite dimensional, each projection P_i, can be further decomposed into a finite set of minimal orthogonal sub-projections P_{ij}, i.e., $P_i = \sum_{j=1}^{n_i} P_{ij}$ with each $P_{ij} \in \mathcal{M}$. This collection $\{P_{ij}\}$ forms a resolution of the identity on $\mathcal{K}$, i.e., $\mathcal{K} = \oplus_{i,j}\mathcal{K}_{ij}$, where $\mathcal{K}_{ij} = \mathrm{range}(P_{ij})$ are mutually orthogonal sub-spaces. This induces a direct sum decomposition of the spectral instrument πE :

$$\pi E = \oplus_{i,j}\pi_{ij}E_{ij}, \text{ where } \pi_{ij}E_{ij} := P_{ij}\pi E.$$

The minimality of each P_{ij} ensures that the instruments $\pi_{ij}E_{ij}$ are irreducible. Since each $P_{ij} \in \mathcal{M}$, we also obtain a decomposition of the subspace $V(\mathcal{H})$: as $V(\mathcal{H}) = \oplus_{i,j}V_{ij}(\mathcal{H})$, leading to a decomposition of $\mathcal{H}$, itself: $\mathcal{H} = \oplus_{ij}\mathcal{H}_{ij}$, with $V_{ij} : \mathcal{H}_{ij} \to \mathcal{K}$ isometry. Therefore the instrument $\mathcal{I}$ decomposes a finite direct sum of irreducible instruments: $\mathcal{I} = \oplus_{ij}V_{ij}^*\pi_{ij}E_{ij}V_{ij}$. ∎

Combining Theorem 2.1.20, Theorem 2.1.19, and the earlier Theorem 2.1.36, we obtain the following complete characterization of C^*-extreme instruments in finite dimensions.

Theorem 2.1.37. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument, where $\mathcal{H}$ is finite-dimensional Hilbert space. Then $\mathcal{I}$ is C^*-extreme if and only if there exist finitely many mutually disjoint irreducible instruments $\{\pi_i E_i\}_{i=1}^m$ on X and nested sequence of compression $\mathcal{I}_j^i(1 \leq j \leq n_i)$ of each irreducible instruments $\pi_i E_i$ such that $\mathcal{I}$ is unitarily equivalent to $\oplus_{i=1}^m \oplus_{j=1}^{n_i} \mathcal{I}_j^i$.

Remark 2.1.38. This theorem recovers the characterizations of C^*-extreme unital completely positive (UCP) maps on finite-dimensional spaces established by Farenick and Zhou in [FZ98]. It also yields an analogous characterization for normalized positive operator-valued measures (POVMs).

We now point out an immediate consequence of the structural description of C^*-extreme instruments obtained in Theorem 2.1.37. In the finite-dimensional setting, this structural rigidity implies that the two natural dilations—CP sub-minimal and minimal bi dilation must agree.

Corollary 2.1.39. Let $\mathcal{I} : \mathcal{O}(X) \to \mathrm{CP}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a C*-extreme UCP instrument. If $\mathcal{H}$ is finite-dimensional, then the CP sub-minimal dilation coincides with the bi-dilation of the instrument.

However, it is interesting to note that the POVM subminimal dilation does not, in general, provide any information about the minimal bi-dilation. For instance, consider the instrument

$$\mathcal{I} : \mathcal{O}(\{1,2\}) \times M_4(\mathbb{C}) \longrightarrow M_2(\mathbb{C}), \qquad \mathcal{I}(1, X) = V^*XV, \;\; \mathcal{I}(2, X) = 0,$$

where $V \in M_{4\times 2}(\mathbb{C})$ is an isometry. This defines a pure instrument whose POVM marginal is spectral, and hence $\mathcal{I}$ is a C^*-extreme UCP instrument. Let us now compare the POVM sub-minimal dilation and the minimal bi-dilation of $\mathcal{I}$. $(\mathbb{C}^2, \phi_{\mathcal{I}}, \mu_{\mathcal{I}}, V)$ denotes the POVM sub-minimal dilation tuple, whereas the minimal bi-dilation is the tuple $(\mathbb{C}^4, \pi, E, V)$ where $\pi : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ is the unital $*$-homomorphism given by, $\pi(T) = T$, $T \in M_4(\mathbb{C})$ and $E : \mathcal{O}(X) \to M_4(\mathbb{C})$ the spectral measure has the form $E(1) = I_4, E(2) = 0$. Since the POVM sub-minimal dilation space is $\mathbb{C}^2$ while the minimal bi-dilation space is $\mathbb{C}^4$, the two dilations are clearly distinct.

2.2 Instrument and Its Marginals Through Different Notions of Convexity

In this section, we explore the relationship between an instrument and its marginals through the lens of classical and C^*-convexity structures of CP instruments introduced in Section 2.1.

2.2.1 Extreme instruments and their marginals

In general the extremity of an instrument does not imply the extremity of its marginals. That is, there exist extreme instruments whose marginals—both the POVM and the CP part—are not extreme. This phenomenon was first discussed in [DPS11], and is illustrated in Example 2.1.4. As previously noted, the instrument $\mathcal{I}$ in that example is an extreme instrument. However, the choice of the POVM μ clearly indicates that the POVM marginal is not extreme. Moreover, the non-extremity of the CP marginal follows from the commutativity of μ, together with Theorem 2.1.1 for CP maps.

Interestingly, we see below that under certain natural additional conditions, the extremality of the POVM marginal can be inferred from the extremality of the instrument itself.

The same can't be said about the CP marginal. For instance, if an extreme instrument is commutative, then its POVM marginal must be a spectral measure (see Theorem 2.2.1). However, there exist examples (see Example 2.2.5) where the corresponding CP marginal is not extreme.

Theorem 2.2.1. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be an extreme UCP instrument in the set $I_{\mathcal{H}}(X, \mathcal{A})$ with commutative range. Then the POVM marginal, $\mu_{\mathcal{I}} : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ of $\mathcal{I}$ is a spectral measure.

Proof. Fix $A \in \mathcal{O}(X)$ such that $\mu(A) \neq 0$. Define two POVMs

$$\nu_1(B) = \mu(A \cap B)\mu(A^{\complement}) \text{ and } \nu_2(B) = \mu(A^{\complement} \cap B)\mu(A), \ \forall\, B \in \mathcal{O}(X).$$

It is easy to verify that $\nu_i \leq \mu$ for $i = 1, 2$ and further, $\nu_1(X) = \mu(A)\mu(A^{\complement}) = \nu_2(X)$. Since μ is an extreme point, it follows from the previous theorem that $\nu_1 = \nu_2$. In particular, evaluating at $B = A$ we obtain, $\nu_1(A) = \nu_2(A) = \mu(A)\mu(A^{\complement}) = 0$. Now using the relation $\mu(A) + \mu(A^{\complement}) = 1$, we conclude that $\mu(A)$ is a projection. Since A was arbitrary, it follows that all $\mu(A)$'s are projections. Hence, μ is a spectral measure. ■

In the converse direction, it can be seen that the extremity of both marginals implies the extremity of the instrument. This result is due to E. Haapasalo, T. Heinosaari, and J.-P. Pellonpää (see Theorem 4.1 in [HHP14]). Although the original result was established for CP maps on tensor products of von Neumann algebras, the underlying technique adapts naturally to our framework. For the reader's convenience, we restate the result here in our setting.

Theorem 2.2.2. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument. If both the marginals $\phi_{\mathcal{I}}$ and $\mu_{\mathcal{I}}$ are extreme, then the instrument $\mathcal{I}$ is extreme.

Proof. Let $(\mathcal{K}_1, \pi_1, \mu, V_1)$ denote the CP subminimal dilation of the instrument $\mathcal{I}$. Suppose that

$$\mathcal{I} = t\mathcal{I}_1 + (1-t)\mathcal{I}_2, \qquad t \in (0, 1), \tag{2.2.1}$$

is a proper convex decomposition of $\mathcal{I}$. Applying Theorem 1.4.17 to the CP marginal $\phi_{\mathcal{I}}$, and following the same line of construction as in Theorem 1.4.24, we obtain CP subminimal dilation tuples for $\mathcal{I}_i$ of the form $(\mathcal{K}_1, \pi_1, \mu_i, V_1)$, $i = 1, 2$. Equation (2.2.1) then induces $V_1^*\mu V_1 = tV_1^*\mu_1 V_1 + (1-t)V_1^*\mu_2 V_1$ $t \in (0, 1)$. As $\mu_{\mathcal{I}} = V_1^*\mu V_1$ is an extreme POVM, we obtain $V_1^*\mu V_1 = V_1^*\mu_i V_1$ for $i = 1, 2$. Finally, invoking the injectivity of the map $D \mapsto V_1^* D V_1$ (as established in Theorem 2.1.1 for completely positive maps), we infer that $\mu = \mu_i$ for each i. It then follows that $\mathcal{I} = \mathcal{I}_i$, and therefore $\mathcal{I}$ is extreme. ■

Remark 2.2.3. It is interesting to note from the proof of Theorem 2.2.2 that if we have two CP instruments $\mathcal{I}_i$, $i = 1, 2$ sharing the same marginals $\phi_{\mathcal{I}_i} = \phi$, $i = 1, 2$ and POVM marginal $\mu_{\mathcal{I}_i} = \mu$, then the extremality of either marginal ensures that the two instruments coincide. For instance, if the CP marginal ϕ is extreme, then by invoking its extremality, we deduce that the corresponding CP subminimal dilations of both $\mathcal{I}_i$, $i = 1, 2$ are identical, and hence $\mathcal{I}_1 = \mathcal{I}_2$. A similar line of reasoning applies if we assume that the POVM marginal μ is extreme, which likewise ensures that the two instruments coincide

As noted in Remark 2.2.3, the extremity of a single marginal is sufficient to uniquely determine the instrument. However, extremality of the marginals alone does not gaurantee that the instrument is decomposable. This is a feature that sharply contrasts with the classical case. This distinction is highlighted in the following example, which uses the correspondence between regular POVMs on a compact Hausdorff space X and completely positive maps on $C(X)$ (see Chapter 4, [Pau02]).

Example 2.2.4. Let $X = \{1, 2, 3, 4\}$, $\mathcal{O}(X) = \mathcal{P}(X)$ and $\mathcal{H} = \mathbb{C}^2$. Set $\omega = e^{2\pi i/3}$ and define $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ by,

$$\mu(\{1\}) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mu(\{2\}) = \frac{1}{6}\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix},$$

$$\mu(\{3\}) = \frac{1}{6}\begin{bmatrix} 1 & \sqrt{2}\omega^2 \\ \sqrt{2}\omega & 2 \end{bmatrix}, \quad \mu(\{4\}) = \frac{1}{6}\begin{bmatrix} 1 & \sqrt{2}\omega \\ \sqrt{2}\omega^2 & 2 \end{bmatrix}.$$

Then μ is an extreme POVM. Let $(\mathcal{K}, E, V)$ be its minimal Naimark dilation. Consider the instrument $\mathcal{I} : \mathcal{O}(X) \to CP(C(X), \mathcal{B}(\mathcal{H}))$ given by

$$\mathcal{I}(A)(f) = V^* \pi_\mu(f) E(A) V,$$

where $\pi_\mu : C(X) \to \mathcal{B}(\mathcal{K})$ is the $*$-homomorphism corresponding to E defined by

$$E(f) = \sum_{i=1}^{4} f(i) E(\{i\}), \quad f \in C(X).$$

The extremality of μ ensures that $\mathcal{I}$ is an extreme UCP instrument. However, $\mathcal{I}$ is not decomposable.

2.2.2 C^*-extreme instruments and their marginals

In this section, we investigate the interplay between C^*-convexity properties of instruments and their marginals.

Example 2.1.4 revealed that, even in finite dimensions, the classical extremality of an instrument does not, in general, imply the extremality of its marginals. In contrast to the classical convex setting, the framework of C^*-extremity reveals a more intricate relationship. We will see in Theorem 2.2.6 below that the C^*-extremity of an instrument does ensure that its POVM marginal is spectral - hence, C^*-extreme at least in the finite-dimensional setting. However, no such conclusion can be drawn for the CP marginal. This asymmetry is illustrated in the following example and in Theorem 2.2.6.

Example 2.2.5. Consider the instrument $\mathcal{I} : \mathcal{O}(X) \to CP(M_2, M_2)$, on the set $X = \{1, 2\}$, defined by $\mathcal{I}(i)(T) = t_{ii} E_{ii}$, $i = 1, 2$, where $T = \sum_{i,j=1}^{2} t_{ij} E_{ij}$, and $E_{ij}, i, j = 1, 2$, are the standard matrix units in M_2. It is easy to verify that $\mathcal{I}$ is a C^*-extreme instrument with commutative range. However, the associated CP marginal $\phi_{\mathcal{I}} : M_2 \to M_2$, mapping a matrix to its diagonal part, is not extreme in the convex set of unital completely positive maps, and hence not C^*-extreme.

We now use the explicit decomposition of C^*-extreme instruments obtained in Theorem 2.1.37 to show that C^*-extremity of an instrument implies the C^*-extremity of its POVM marginal in the finite-dimensional setting.

Theorem 2.2.6. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a C^*-extreme UCP instrument with $\mathcal{H}$ finite-dimensional. Then its POVM marginal $\mu_{\mathcal{I}}$ is C^*-extreme.

Proof. By Theorem 2.1.37, any C^*-extreme instrument $\mathcal{I}$ admits a decomposition of the form:

$$\mathcal{I} = \oplus_{i=1}^{m} \oplus_{j=1}^{n_i} \mathcal{I}_j^i,$$

where $\mathcal{I}_j^i (1 \leq j \leq n_i)$ is a nested sequence of compressions of irreducible instruments $\pi_i E_i, 1 \leq i \leq m$. By Corollary 1.4.20, the POVM marginal of each $\mathcal{I}_j^i$ is trivial. Since the POVM marginal of $\mathcal{I}$ is the direct sum of the marginals of the $\mathcal{I}_j^i$, it follows that the POVM marginal of $\mathcal{I}$ is a direct sum of trivial measures, hence spectral, and therefore C^*-extreme. ■

In parallel with the case of extremality (see Theorem 2.2.1), we will see in Corollary 2.2.9 that commutativity of a C^*-extreme instrument guarantees that its POVM marginal is spectral. This result, however, appears as a consequence of the following more general theorem, which captures a broader structural phenomenon. The formulation and the underlying idea of this theorem are motivated by the main result (Theorem 3.8) of [BBK21] and so we omit the proof.

Theorem 2.2.7. Let $\mathcal{I}$ be a C^*-extreme instrument in $I_{\mathcal{H}}(X, \mathcal{A})$. Suppose there exists $E \in \mathcal{O}(X)$ such that $\mathcal{I}(A, a)\mathcal{I}(E, 1_{\mathcal{A}}) = \mathcal{I}(E, 1_{\mathcal{A}})\mathcal{I}(A, a)$ for all $A \subseteq E$ in $\mathcal{O}(X)$ and all $a \in \mathcal{A}$. Then the operator $\mathcal{I}(E, 1_{\mathcal{A}})$ is a projection. In particular, if $\mathcal{I}(E, 1_{\mathcal{A}})$ commutes with $\mathcal{I}(B, a)$ for every $B \in \mathcal{O}(X)$ and $a \in \mathcal{A}$, it follows that $\mathcal{I}(E, 1_{\mathcal{A}})$ is a projection.

The following two corollaries are immediate:

Corollary 2.2.8. For every atomic C^*-extreme instrument, the associated POVM marginal is a spectral measure.

Corollary 2.2.9. The associated POVM of every commutative C^*-extreme instrument is a spectral measure.

It was established in Theorem 4.1, [HHP14] that if both the marginals of an instrument are extreme, then the instrument itself must also be extreme. It is natural to ask whether an analogous result holds in the setting of C^*-convexity. In this section, we provide an affirmative answer to this question in the finite-dimensional case.

Before presenting the main result, we establish a few preparatory results that will play a crucial role in the proof. To this end, let us recall the relationship between sub-minimal dilations and minimal bi-dilations as discussed in Section 1.4. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument, and suppose that $(\mathcal{K}, \pi, E, V)$ is its minimal bi-dilation. Let P_1 be the orthogonal projection onto the subspace $\mathcal{K}_1 := [\pi(\mathcal{A})V(\mathcal{H})] \subseteq \mathcal{K}$. Then the minimal Stinespring dilation of the associated CP marginal $\phi_{\mathcal{I}}$ is given by the triple $(\mathcal{K}_1, P_1\pi P_1, V)$. Consequently, the CP sub-minimal dilation of the instrument $\mathcal{I}$ is described by the quadruple $(\mathcal{K}_1, P_1\pi P_1, P_1EP_1, V)$.

Theorem 2.2.10. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument with minimal bi-dilation quadruple $(\mathcal{K}, \pi, E, V)$, and let P_1 denote the orthogonal projection onto the subspace $\mathcal{K}_1 := [\pi(\mathcal{A})V(\mathcal{H})]$
$\subseteq \mathcal{K}$. Suppose that the associated UCP map $\phi_{\mathcal{I}}$ is extreme and the associated POVM marginal $\mu_{\mathcal{I}}$ is spectral. Then the compression P_1EP_1 is a spectral measure. Consequently, the CP sub-minimal dilation coincides with the minimal bi-dilation of $\mathcal{I}$.

Proof. Since the POVM marginal $\mu_{\mathcal{I}}$ is spectral, Corollary 1.4.32 together with Theorem 1.4.30 implies that

$$P_1E(A)P_1VV^* = VV^*P_1E(A)P_1VV^* = VV^*P_1E(A)P_1, \text{for all } A \in \mathcal{O}(X), \qquad (2.2.2)$$

Recall that $\mu_{\mathcal{I}} = V^*P_1EP_1V$. The spectrality of $\mu_{\mathcal{I}}$, in conjunction with (2.2.2), leads to the equivalence:

$$\begin{aligned} \mu_{\mathcal{I}}(A) = \mu_{\mathcal{I}}(A)^2 \quad &\iff \quad V^*P_1E(A)P_1V = V^*P_1E(A)P_1VV^*P_1E(A)P_1V \\ &\iff \quad V^*P_1E(A)P_1V = V^*(P_1E(A)P_1)^2V, \ \forall\ A \in \mathcal{O}(X). \end{aligned}$$

Since, the associated UCP map $\phi_{\mathcal{I}}$ is extreme, it follows that the map: $D \mapsto V^*P_1DP_1V$, is injective on the von Neumann algebra $\{P_1\pi(\mathcal{A})P_1\}'$. Therefore, the equality above implies that $P_1E(A)P_1$ is a projection for each $A \in \mathcal{O}(X)$, i.e., the POVM P_1EP_1 is spectral. Thus, the quadruple $(\mathcal{K}_1, P_1\pi P_1, P_1EP_1, V)$ is a bi-dilation of the instrument $\mathcal{I}$. Since this is already a sub-minimal dilation, it follows that it is also minimal as a bi-dilation. ■

Having established the preparatory results, we are in a position to present some principal contributions of this paper.

Theorem 2.2.11. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument such that its POVM marginal $\mu_{\mathcal{I}}$ is spectral and its CP marginal admits the decomposition

$$\phi_{\mathcal{I}} = \oplus_{i=1}^{\ell} \psi_i \otimes 1_{\mathbb{C}^{n_i}},$$

where $\psi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, \dots, \ell$, are pure unital completely positive maps with $\dim \mathcal{H}_i < \infty$. Assume that each ψ_i admits a minimal Stinespring dilation $(\mathcal{H}_\pi, \pi, V_i)$ with respect to a fixed irreducible representation π of $\mathcal{A}$, and that the family of subspaces $\{\text{range}(V_i)\}_{i=1}^{\ell}$ forms a nest in $\mathcal{H}_\pi$. Then $\mathcal{I}$ is C^*-extreme.

Proof. We begin by establishing the necessary notations. Let $\mathcal{K} = \oplus_{i=1}^{\ell}(\mathcal{H}_\pi \otimes \mathbb{C}^{n_i})$, and consider the representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, given by

$$\rho(a) = \oplus_{i=1}^{\ell}(\pi(a) \otimes 1_{\mathbb{C}^{n_i}}), \quad \forall a \in \mathcal{A},$$

where π is a fixed irreducible representation of $\mathcal{A}$. Define the isometry, $V : \mathcal{H} \to \mathcal{K}$ by $V = \oplus_{i=1}^{\ell}(V_i \otimes 1_{\mathbb{C}^{n_i}})$. It is evident that the triple $(\mathcal{K}, \rho, V)$ forms a minimal Stinespring dilation for the UCP map $\phi_{\mathcal{I}}$. By Theorem 2.1.37 for UCP maps we can conclude that $\phi_{\mathcal{I}}$ is C^*-extreme. Since $\mathcal{H}$ is finite-dimensional and $\phi_{\mathcal{I}}$ is C^*-extreme, it is also classically extreme. Since $\mathcal{I}$ satisfies the hypothesis of Theorem 2.2.10, we conclude that the CP subminimal dilation of $\mathcal{I}$ coincides with its minimal Naimark dilation, which is given by the quadruple $(\mathcal{K}, \rho, E, V)$, where $E : \mathcal{O}(X) \to \rho(\mathcal{A})' \subset \mathcal{B}(\mathcal{K})$ is a spectral measure. Since π is irreducible, we have $\pi(\mathcal{A})' = \mathbb{C} \cdot 1_{\mathcal{H}_\pi}$, and therefore,

$$\rho(\mathcal{A})' = \{1_{\mathcal{H}_\pi} \otimes A : A \in M_m(\mathbb{C}), \quad m = \sum_{i=1}^{\ell} n_i\}.$$

As E is a spectral measure and takes values in $\rho(\mathcal{A})'$, with out loss of generality it decomposes as $E = \oplus_{i=1}^{m} E_i$, where each E_i is a trivial spectral measure on X with values in $\mathcal{B}(\mathcal{H}_\pi)$. Accordingly, the instrument $\mathcal{I}$ can be expressed as,

$$\mathcal{I} = \oplus_{i=1}^{m} W_i^* \pi_i E_i W_i,$$

where each $\pi_i = \pi$, and the isometries W_i are drawn from $\{V_1, \dots, V_l\}$ with the common dilation space $\mathcal{K}$. It follows immediately that the instruments $\{\pi_i E_i\}_{i=1}^{m}$ are irreducible. Therefore, by invoking Theorem 2.1.37, we conclude that $\mathcal{I}$ is indeed C^*-extreme. ■

Combining Proposition 2.1.19 with Theorem 2.2.11, we conclude that in finite dimensions the C^*-convexity of the marginals of an instrument ensures that the instrument itself is C^*-extreme. Moreover, under additional assumptions, the converse implications can also be established.

Theorem 2.2.12. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a UCP instrument, where $\mathcal{H}$ is finite dimensional. Then the marginals $\mu_{\mathcal{I}}$ and $\phi_{\mathcal{I}}$ are C^*-extreme if and only if:

1. $\mathcal{I}$ is C^*-extreme and
2. if $\mathcal{I}_1, \mathcal{I}_2 : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ dominated by $\mathcal{I}$ are two pure instruments such that $\phi_{\mathcal{I}_1}$ and $\phi_{\mathcal{I}_2}$ dilate to a common irreducible representation then one of them is a compression of the other.

Proof. ($\Longrightarrow$) In the forward direction, the C^*-extremity of $\mathcal{I}$ follows from Proposition 2.1.19 along with the Theorem 2.2.11. To prove (2), let $\mathcal{J}' : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a pure instrument with minimal bi-dilation $(\mathcal{K}_{\mathcal{J}'}, \pi_{\mathcal{J}'}, E_{\mathcal{J}'}, V_{\mathcal{J}'})$, and let $\mathcal{J} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be the direct sum of pure instruments such that $\mathcal{J}' \le \mathcal{J}$, where $\mathcal{J} = \oplus_{i=1}^n V_i^* \pi_i E_i V_i$. Then necessarily $\mathcal{J}' = tV_\ell^* \pi_\ell E_\ell V_\ell$ for some $\ell \in \{1, \dots, n\}$ and $t \in [0, 1]$.

Since $\phi_\mathcal{I}$ is C^*-extreme, by Theorem 2.1.37 for UCP maps, there exists a family of disjoint irreducible representations $\{\pi_i\}_{i=1}^n$ together with a nested sequence of pure CP maps $\{V_{ij}^* \pi_i V_{ij}\}_{j=1}^{n_i}$ for each i, such that

$$\phi_\mathcal{I} \; = \; \oplus_i \oplus_j V_{ij}^* \pi_i V_{ij}.$$

Let $\mathcal{I}_1, \mathcal{I}_2 : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be pure instruments dominated by $\mathcal{I}$ such that $\phi_{\mathcal{I}_1}$ and $\phi_{\mathcal{I}_2}$ dilate to the same irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$. Then $\phi_{\mathcal{I}_i} \le \phi_\mathcal{I}$ for $i = 1, 2$. From the discussion in the beginning it follows that $\pi = \pi_m$ for some $m \in \{1, \dots, n\}$, and moreover

$$\phi_{\mathcal{I}_1} = t_1 V_{mj_1}^* \pi_m V_{mj_1}, \quad \phi_{\mathcal{I}_2} = t_2 V_{mj_2}^* \pi_m V_{mj_2}, \quad t_i \in [0, 1].$$

Since the family $\{V_{ij}^* V_{ij}\}$ is nested for each i, it follows that $\phi_{\mathcal{I}_1}$ and $\phi_{\mathcal{I}_2}$ are compressions of either one, establishing condition (2).

($\Longleftarrow$) For the converse, the C^*-extremity of the POVM marginal $\mu_\mathcal{I}$ follows from Theorem 2.2.6. Since $\mathcal{I}$ itself is C^*-extreme, by Theorem 2.1.37 we have $\mathcal{I} = \oplus_i \oplus_j V_{ij}^* \pi_i E_i V_{ij}$, where $\{\pi_i E_i\}_{i=1}^n$ are disjoint irreducible instruments, and $\{V_{ij}^* \pi_i E_i V_{ij}\}$ is a nested sequence of pure instruments for each i. Thus, $\phi_\mathcal{I} = \oplus_i \oplus_j V_{ij}^* \pi_i V_{ij}$.

To show that $\phi_\mathcal{I}$ is C^*-extreme, it suffices to prove that

$$\phi_\mathcal{I} \; = \; \oplus_{i'} \oplus_{j'} W_{i'j'}^* \pi_{i'} W_{i'j'},$$

where $\{\pi_{i'}\}_{i'=1}^m \subseteq \{\pi_i\}_{i=1}^n$ are disjoint irreducible representations and, for each i', the family $\{W_{i'j'} W_{i'j'}^*\}$ is nested, with $\{W_{i'j'}\} \subseteq \{V_{ij}\}$. The disjointness of $\{\pi_{i'}\}$ follows directly, and condition (2) guarantees the nesting property. Hence, $\phi_\mathcal{I}$ is C^*-extreme. ∎

Chapter 3

Integration with respect to Instruments

In this chapter, we develop a rigorous framework for integration with respect to quantum instruments, aiming to extend classical measure-theoretic ideas to the operator-algebraic setting relevant for quantum information and measurement theory.

The theory of integration is classical in nature and constitutes a fundamental pillar of modern analysis. Beginning in the 1930s, the geometric and isomorphic study of Banach spaces was significantly influenced by the development of vector measure theory. In this context, a systematic integration theory for scalar- and vector-valued functions with respect to both scalar and vector measures became essential for understanding the geometric structure of Banach spaces. The subject evolved through the foundational contributions of Bochner, Dunford, and Pettis, particularly (see [DU77]) in the integration of vector-valued functions with respect to scalar measures, as well as Dunford's work on the integration of scalar functions with respect to vector measures. These developments led to a rich theory with deep applications in the study of geometric properties of Banach spaces. At the same time, increasing interest in operators on spaces of vector-valued functions motivated representations in terms of vector integration, namely, the integration of vector-valued functions with respect to vector-valued measures.

The integration of vector-valued functions with respect to vector-valued measures appears in the work of Bartle [Bar56], where a general and flexible framework was introduced. In the present work, we adopt and tailor this framework to suit our setting. Over time, these integration theories have also played a significant role in the study of convexity in functional-analytic structures. Motivated by these developments, we investigate the geometry of the C^*-convex set of UCP instruments by introducing an appropriate notion of integration with respect to instruments.

We present two complementary formulations of this integration theory, each reflecting a distinct methodological perspective.

The first formulation extends the classical theory of vector-valued measures to a non-commutative setting. In this direction, we draw on the work of Farenick, Plosker, Ramsey, and MacLaren [MPR20],[FPS11] concerning the integration of vector-valued functions with respect to POVMs. Building on this line of development, and inspired by Bartle's framework [Bar56] for vector integration, we introduce suitable notions of measurability and integrability for $\mathcal{A}$-valued functions with respect to a given instrument. Starting from simple functions and proceeding via measurable approximations determined by the instrument, we develop a consistent integration theory that captures both the operational and algebraic aspects intrinsic to quantum measurement. A principal outcome of this approach is the CP–instrument correspondence (Theorem 3.1.27), which extends the classical CP–POVM correspondence to the setting of regular completely positive instruments on compact Hausdorff spaces. Furthermore, by equipping the space of unital CP instruments with a natural topology, we establish a Krein–Milman type theorem (Theorem 3.1.33) within the framework of C^*-convexity.

The second formulation approaches the integration problem from an operator-algebraic perspective, by realizing instruments as completely positive maps on suitable tensor product spaces. This viewpoint is rooted in the interpretation of instruments as noncommutative analogues of joint measures. In particular, the classical product structure of measure theory admits a natural extension to the tensor product of algebras of bounded measurable functions, suggesting that the appropriate framework for integration with respect to instruments is given by completely positive maps on tensor product spaces. Proceeding in this direction, integration is defined through the structural properties of operator systems and tensor products. This formulation accommodates a broader class of measurable functions and highlights the intrinsic connection between integration and complete positivity. To illustrate the scope of this approach, we present a concrete example (Example 3.2.1), demonstrating that the tensor-product formulation yields an integration theory applicable to a wider class of measurable maps.

Together, these two formulations provide a coherent and comprehensive account of integration with respect to quantum instruments, bridging classical measure theory with noncommutative analysis and laying the groundwork for subsequent investigations into the geometric and structural properties of quantum measurement processes.

3.1 Classical approach

To formally introduce the notion of integration with respect to an instrument, in this approach, we draw upon the theory of the Bartle integral (see [Bar56]). In order to establish the necessary theoretical foundation, we first recall the definition of vector-valued measures taking values in a Banach space and review some of their fundamental properties. For a comprehensive exposition see [Rya02].

Definition 3.1.1 (Vector valued measures)**.** A vector measure on a measurable space $(X, \mathcal{O}(X))$ is a countably additive function μ on $\mathcal{O}(X)$ with values in a Banach space $\mathcal{B}$. Thus if $\{E_n\}$ is a sequence of mutually disjoint measurable subsets of X with $\cup_n E_n = E$, then the series $\sum_n \mu(E_n)$ converges to $\mu(E)$ unconditionally.

Remark 3.1.2. In finite-dimensional Banach spaces, absolute convergence and unconditional convergence are equivalent. However, in the infinite-dimensional setting, unconditional convergence does not necessarily imply absolute convergence.

Analogous to the case of complex measures, the notions of semi-variation and total variation are equally applicable in this context.

Definition 3.1.3 (Semi-variation and Total variation)**.** Let μ be a vector measure on $(X, \mathcal{O}(X))$ with values in a Banach space $\mathcal{B}$. The semi-variation of μ is the extended non-negative function $\|\mu\|$ whose value on a set $A \in \mathcal{O}(X)$, denoted by $\|\mu\|(A)$ or $\|A\|$, is defined to be

$$\|A\| = \sup\{\| \sum_i \alpha_i \mu(E_i)\| : \{E_1, \cdots, E_n\} \text{ is a partition of } A \text{ and } |\alpha_i| \leq 1\}$$

and the total variation of μ, the extended non-negative function $|\mu|$, is defined to be

$$|\mu|(A) = \sup\{\sum_i \|\mu(E_i)\| : \{E_1, \cdots, E_n\} \text{ is a partition of } A\}.$$

Remark 3.1.4. Unlike complex measures, the total variation $|\mu|$ of a vector measure μ need not be finitely valued.

Definition 3.1.5 (Bounded variation)**.** μ is said to be of bounded variation if $|\mu|(X) < \infty$, i.e $|\mu|$ is a finite positive measure on $(X, \mathcal{O}(X))$.

As noted in Remark 1.4.3, the complex measures associated with an instrument can completely determine the instrument. However, it is also possible to associate a broader class of vector measures to an instrument, extending beyond the scope of complex measures. These vector measures offer a more general framework for understanding and analyzing the structure and behavior of instruments.

Proposition 3.1.6. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument and let ρ be a state (i.e., a positive trace-class operator with $Tr(\rho) = 1$)on the separable Hilbert space $\mathcal{H}$. Then the map $\mathcal{I}_\rho : \mathcal{O}(X) \to \mathcal{A}^*$ defines a $\mathcal{A}^*$-valued vector measure on $(X, \mathcal{O}(X))$ which are of bounded variation, given by $\mathcal{I}_\rho(A)(a) = Tr(\mathcal{I}(A, a)\rho)$, where Tr is the usual trace functional on $\mathcal{B}(\mathcal{H})$.

Proof. Observe that, the Equation (1.4.1) implies that for any $a \in \mathcal{A}$ and countable collection of disjoint measurable sets $\{A_i\}_{i=1}^\infty$ in $\mathcal{O}(X)$ the series $\sum_{i=1}^\infty \mathcal{I}(A_i, a)$ converges in WOT. Since WOT and σ-weak topology both agree on the bounded subsets of $\mathcal{B}(\mathcal{H})$, and for each $a \in \mathcal{A}$, $\{\mathcal{I}(\cup_{i=1}^k A_i, a)\}_{k=1}^\infty$ is bounded, it follows that, for any trace class operator $\rho \in \mathcal{B}(\mathcal{H})$,

$$\lim_{k\to\infty} Tr(\mathcal{I}(\cup_{i=1}^k A_i, a)\rho) = Tr(\mathcal{I}(\cup_{i=1}^\infty A_i, a)\rho) = Tr(\sum_{i=1}^\infty \mathcal{I}(A_i, a)\rho) = \sum_{i=1}^\infty Tr(\mathcal{I}(A_i, a)\rho). \tag{3.1.1}$$

It's clear from the definition of $\mathcal{I}_\rho$ that for every $A \in \mathcal{O}(X)$, $\mathcal{I}_\rho$ defines a positive linear functional on the C^*-algebra $\mathcal{A}$. Let us prove the countable additivity of $\mathcal{I}_\rho$. Let $\{A_i\}_{i=1}^\infty$ be a countable family of disjoint measurable subsets of X. We have to prove that,

$$\mathcal{I}_\rho(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mathcal{I}_\rho(A_i), \tag{3.1.2}$$

where the right-hand side of the Equation 3.1.2 converges unconditionally in the norm. Observe that,

$$\begin{aligned}
\sum_{i=1}^\infty \|\mathcal{I}_\rho(A_i)\| &= \sum_{i=1}^\infty \mathcal{I}_\rho(A_i)(1_\mathcal{A}) \quad \text{(since, $\mathcal{I}_\rho$ is a positive linear functional on $\mathcal{A}$)} \\
&= \sum_{i=1}^\infty Tr(\mathcal{I}(A_i, 1_\mathcal{A})\rho) \\
&= Tr(\mathcal{I}(A, 1_\mathcal{A})\rho) \quad \text{(follows from the Equation 3.1.1)} \\
&= \mathcal{I}_\rho(A)(1_\mathcal{A}) \\
&< \infty,
\end{aligned} \tag{3.1.3}$$

proving that the series in Equation 3.1.2 is a Cauchy series. Also for each $a \in \mathcal{A}$ we have,

$$\begin{aligned}
\mathcal{I}_\rho(\cup_{i=1}^\infty A_i)(a) &= Tr(\mathcal{I}(\cup_{i=1}^\infty A_i, a)\rho) \\
&= \sum_{i=1}^\infty Tr(\mathcal{I}(A_i, a)\rho) \\
&= \sum_{i=1}^\infty \mathcal{I}_\rho(A_i)(a)
\end{aligned} \tag{3.1.4}$$

establishing the point-wise convergence of the series in Equation 3.1.2. Combining Equation 3.1.3 and Equation 3.1.4, we can conclude that the series in Equation 3.1.2 converges unconditionally in the norm. It remains to prove that $\mathcal{I}_\rho$ is of bounded variation. We have,

$$\begin{aligned}
|\mathcal{I}_\rho|(X) &= \sup\{\sum_i \|\mathcal{I}_\rho(E_i)\| : \{E_1, \cdots, E_n\} \text{ is a partition of X}\} \\
&= \sup\{\sum_i \mathcal{I}_\rho(E_i)(1_\mathcal{A}) : \{E_1, \cdots, E_n\} \text{ is a partition of X}\} \\
&= \sup\{\sum_i Tr(\mathcal{I}(E_i, 1_\mathcal{A})\rho) : \{E_1, \cdots, E_n\} \text{ is a partition of X}\} \\
&= Tr(\mathcal{I}(X, 1_\mathcal{A})\rho) \\
&= \mathcal{I}_\rho(X)(1_\mathcal{A}) \\
&< \infty,
\end{aligned}$$

finishing the proof. ■

Remark 3.1.7. More generally, since equation (3.1.1) holds for every trace-class operator,$\tau \in \mathcal{B}(\mathcal{H})$, the map $\mathcal{I}_\tau : \mathcal{O}(X) \to \mathcal{A}^*$ defined as $\mathcal{I}_\tau(A)(a) = Tr(\mathcal{I}(A)(a)\tau)$ produces $\mathcal{A}^*$ valued measure of bonded variation. Since, for any $h, k \in \mathcal{H}, |x\rangle\langle y|$ is a rank one trace class operator, therefore, the set of $\mathcal{A}^*$-valued measures $\{\mathcal{I}_{h,k} : h, k \in \mathcal{H}\}$ on $(X, \mathcal{O}(X))$ defined as $\mathcal{I}_{h,k}(A)(a) = \langle h, \mathcal{I}(A, a)k\rangle$, determines the instrument $\mathcal{I}$, similar to complex measure scenario in Remark 1.4.3.Furthermore, these measures satisfy the bound $|\mathcal{I}_{h,k}|(X) \leq ||h||||k||||\mathcal{I}(X, 1_\mathcal{A})\|$.

Remark 3.1.8. To conclude that $\mathcal{I}_{h,k}$ is an $\mathcal{A}^*$ valued measure, for any $h, k \in \mathcal{H}$, the separability of $\mathcal{H}$ is not required.

To develop a theory of integration, we first need to establish a suitable notion of measurability—and, in particular, integrability—for $\mathcal{A}$-valued functions with respect to a given instrument $\mathcal{I}$. In general, three notions of measurability are commonly considered: strong, Borel, and weak measurability. In this context, our focus will be on the first and third, namely, strong and weak measurability. We begin by recalling their definitions, with particular attention to strong measurability, as it forms the foundation of our subsequent construction of integration. Before doing so, we introduce the notion of a simple function, which, as in the classical theory, serves as the basic building block for the development of integration.

Definition 3.1.9 (Simple functions)**.** A function $f : X \to \mathcal{A}$ is said to be simple if there exists $a_1, a_2, \cdots, a_n \in \mathcal{A}$ and $A_1, A_2, \cdots, A_n \in \mathcal{O}(X)$ such that $f = \sum_{i=1}^n a_i 1_{A_i}$, where $1_{A_i}(x) = 1$ if $x \in A_i$ and $1_{A_i}(x) = 0$ if $x \notin A_i$.

Remark 3.1.10. In his setting, Bartle employed the concept of semi-variation of a vector-valued measure to define various modes of convergence, including almost everywhere convergence, convergence in measure, and almost uniform convergence. In a similar spirit, we extend these notions to the vector-valued measures $\mathcal{I}_{h,k}$ induced by an instrument $\mathcal{I}$, for each $h, k \in \mathcal{H}$.

To introduce the notion of integrable functions, we need to extend the concept of almost everywhere convergence with respect to instruments.

Definition 3.1.11 (Almost everywhere convergence)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument and $\{f_n\}$ be a sequence of simple functions. We say that, $\{f_n\}$ converges to f point-wise $\mathcal{I}$-almost everywhere if $lim_n \|f_n(x) - f(x)\| = 0$, for $\|\mathcal{I}_{h,k}\|$-almost every $x \in X$ and for all $h, k \in \mathcal{H}$.

Definition 3.1.12 (Strongly $\mathcal{I}$-measurable functions)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument then a function $f : X \to \mathcal{A}$ is called strongly $\mathcal{I}$-measurable if for every $h, k \in \mathcal{H}$ there exists a sequence of simple functions ${f^{h,k}}_n : X \to \mathcal{A}$, such that $\{{f^{h,k}}_n\}$ converges to f point-wise $\|\mathcal{I}_{h,k}\|$-almost everywhere.

Next we introduce the weakly measurable functions.

Definition 3.1.13 (Weakly $\mathcal{I}$-measurable functions)**.** Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument then a function $f : X \to \mathcal{A}$ is called weakly $\mathcal{I}$-measurable if a^*f is strongly $\mathcal{I}$-measurable for every $a^* \in \mathcal{A}^*$.

The set of all strongly measurable functions is strictly contained within the set of weakly measurable functions. An explicit example illustrating this distinction is provided in Example II.2.5 of [DU77], where a function is constructed to be weakly measurable but not strongly measurable.

However, under certain natural conditions, Petti (see Theorem II.2,[DU77]) showed that weak measurability coincides with strong measurability in the context of vector-valued measures. Here we establish a corresponding result for instruments. The proof of this result follows a similar line of reasoning as that of the original theorem.

Theorem 3.1.14. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be an instrument. A function $f : X \to \mathcal{A}$ is strongly $\mathcal{I}$-measurable if and only if

1. f is $\mathcal{I}$-essentially separably valued i.e. for every $h, k \in \mathcal{H}$ there exists $A_{h,k} \in \mathcal{O}(X)$ with $\|\mathcal{I}_{h,k}\|(A_{h,k}) = 0$ and $f(X \setminus A_{h,k})$ is a separable subset of $\mathcal{A}$ and
2. f is weakly $\mathcal{I}$-measurable.

Proof. Assume that f is strongly $\mathcal{I}$-measurable. Then for every $h, k \in \mathcal{H}$ there exists a sequence of simple functions g_n and a measurable set $A_{h,k} \in \mathcal{O}(X)$ such that $\lim_{n\to\infty} \|g_n(x) - f(x)\| = 0$, for all $x \in X \setminus A_{h,k}$ with $\|\mathcal{I}_{h,k}\|(A_{h,k}) = 0$. Each g_n takes values in a finite-dimensional subspace $\mathcal{A}_n := span\ g_n(X)$. Define

$$\mathcal{A}' = \overline{span \bigcup \mathcal{A}_n}.$$

Then $\mathcal{A}'$ is a separable subspace of $\mathcal{A}$, and clearly $f(X \setminus A_{h,k}) \subset \mathcal{A}'$. Hence, the assertion of 1 follows. The proof of 2 follows immediately from the assumptions, thereby completing the proof of one direction.

To prove the converse, for $h, k \in \mathcal{H}$, let $A_{h,k} \in \mathcal{O}(X)$ be chosen such that $\|\mathcal{I}_{h,k}\|(A_{h,k}) = 0$ and $f(X \setminus A_{h,k})$ is separable. Let $\{a_n\}$ be a countable dense subset of $f(X \setminus A_{h,k})$. Let us define $g_n : \mathcal{A} \to \{a_1, a_2, \cdots, a_n\}$ such that $\forall a \in \mathcal{A}$, $g_n(a)$ is the one among $\{a_1, a_2, \cdots, a_n\}$ having the smallest index in $\{1, 2, \cdots, n\}$ for which

$$\|a - g_n(a)\| = min_{1\leq j\leq n}\|a - a_j\|.$$

Then $lim_n g_n(a) = a$ for all $a \in \mathcal{A}$. We define $f_n^{h,k} : X \setminus A_{h,k} \to \mathcal{A}$ by $f_n^{h,k}(x) = g_n(f(x))$ for all $x \in X \setminus A_{h,k}$. Clearly, $lim_n f_n^{h,k}(x) = f(x),\ \forall\ x \in X \setminus A_{h,k}$ i.e. $f_n^{h,k}$ converges to f point-wise. It remains to verify that $f_n^{h,k}$ are measurable i.e. $(f_n^{h,k})^{-1}(a_k) \in \mathcal{O}(X)$ for $1 \leq k \leq n$. Here $(f_n^{h,k})^{-1}(a_k) = \{x \in X \setminus A_{h,k} : g_n(f(x)) = a_k\} = \{x \in X \setminus A_{h,k} : \|f(x) - a_k\| = min_{1\leq j\leq n}\|f(x) - a_j\|\}$. By Hahn Banach theorem, we choose a sequence $\{a_n^*\} \subset \mathcal{A}^*$ such that $sup_n|a_n^*(a)| = \|a\|$. Since infimum and supremum of countably many measurable functions are measurable, the function

$$x \mapsto \|f(x) - a_j\| = min_{1\leq j\leq n} sup_m |a_m^*(f(x) - a_j)|$$

is measurable as well. Therefore, each set $(f_n^{h,k})^{-1}(a_k)$ is measurable.

■

Remark 3.1.15. It is evident from the previous theorem for separably valued functions, both notions of measurability coincide.

Remark 3.1.16. It is easy to verify from the Theorem 3.1.14 that the collection of strongly-measurable functions on X is a linear space which is closed under the operation of convergence in point-wise limit of sequences.

In order to develop the theory of integration with respect to an instrument, we employ the framework of integration for $\mathcal{A}$-valued, $\mathcal{I}$-integrable functions with respect to $\mathcal{A}^*$-valued measures, as introduced in Bartle's integration theory. In this setting, we take $X = \mathcal{A}$, $Y = \mathcal{A}^*$, and $Z = \mathbb{C}$. Rather than appealing to the full generality of his framework, we recall his construction in a more concrete setting suited to our context, which will be invoked in several places throughout this chapter.

In his formulation, Bartle defined the integral initially for all functions that can be approximated in measure by simple functions. He then showed that, under certain regularity conditions on the underlying vector measure, this class coincides with the set of functions that can be approximated by simple functions almost everywhere. He referred to the corresponding regularity condition as the $*$-property of the underlying vector measure μ, which is stated as follows: the measure μ has the $*$-property if there exists a finite non-negative measure ν, such that

$$\mu(A) = 0 \textit{ if and only if } \nu(A) = 0.$$

This property ensures that the measure μ induces a well-behaved integration theory analogous to the scalar case. Bartle provided several sufficient conditions under which a vector-valued measure μ satisfies the $*$-property, bounded variation being one of them. Thus, whenever μ possesses bounded variation, the theory simplifies. Moreover, since for any instrument $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$, each $\mathcal{A}^*$-valued measure $\mathcal{I}_{h,k}$ has bounded variation (as noted in Remark 3.1.7) for every $h, k \in \mathcal{H}$, we can develop a consistent theory of integration with respect to an instrument by appealing to Bartle's framework of vector-valued integration.

In the following, we recall the framework of Bartle integration adapted to our setting. Let $\mu : \mathcal{O}(X) \to \mathcal{A}^*$ be a vector-valued measure. For a simple function $f : X \to \mathcal{A}$ defined by $f(x) = \sum_i a_i 1_{A_i}(x)$, its Bartle integral with respect to μ is given by

$$\int_X f \, d\mu := \sum_i \mu(A_i)(a_i).$$

Definition 3.1.17 (Bartle integral)**.** Let $\mu : \mathcal{O}(X) \to \mathcal{A}^*$ be a vector-valued measure. A function $f : X \to \mathcal{A}$ is said to be *Bartle integrable* with respect to μ if there exists a sequence of simple functions $\{f_n\}$ on X such that:

(i) $f_n(x) \to f(x)$ pointwise $\|\mu\|$-almost everywhere; and

(ii) the sequence of indefinite integrals $\left\{\int_X f_n \, d\mu\right\}$ converges.

In this case, we define

$$\int_X f \, d\mu := \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Next, we recall one of the most crucial results from Bartle's paper ([Bar56, Theorem 10]) in the present setting, which will be used on several occasions throughout this chapter.

Theorem 3.1.18. Let $(X, \mathcal{O}(X))$ be a measurable space and $\mathcal{A}$ be a Banach algebra. Let $\mu : \mathcal{O}(X) \to \mathcal{A}^*$ be a vector-valued measure possessing the $*$-property. Suppose f_n is a sequence of μ-integrable functions such that:

(i) the sequence $\{f_n\}$ converges to f, $\|\mu\|$ almost everywhere;

(ii) given $\epsilon > 0$ there is a $\delta > 0$ such that if $E \in \mathcal{O}(X)$ and $\|\mu\|(E) < \delta$, then $|\int_E f_n d\mu| < \epsilon$, for all $n \in \mathbb{N}$.

Then f is integrable with respect to μ, and moreover, $\int f d\mu = lim_{n\to\infty} \int f_n d\mu$, for $E \in \mathcal{O}(X)$.

As outlined earlier, to develop a rigorous theory of integration, we begin by defining the integral on the class of simple functions and subsequently extend it to the space of strongly measurable functions. Let $f : X \to \mathcal{A}$ be a simple function of the form $f = \sum_{i=1}^n a_i 1_{A_i}$, as mentioned in Definition 3.1.9, then we define its integral over any set $A \in \mathcal{O}(X)$ as

$$\int_A f d\mathcal{I} = \sum_{i=1}^n \mathcal{I}(A \cap A_i, a_i).$$

This serves as the foundational definition of integration in our framework. Based on this construction, we proceed to extend the integral from simple functions to a broader class of measurable functions, thereby formulating a general integration theory associated with instruments.

Definition 3.1.19 ($\mathcal{I}$-integrability)**.** A function f on X is said to be $\mathcal{I}$-*integrable* if it satisfies the following conditions:

(i) f is strongly $\mathcal{I}$-measurable;

(ii) for every $h, k \in \mathcal{H}$, the function f is $\mathcal{I}_{h,k}$-integrable in the sense of Bartle (Definition 3.1.17).

In this case, the *integral of f with respect to $\mathcal{I}$* is defined by

$$\langle h, \Big(\int_X f\, d\mathcal{I}\Big) k\rangle = \int_X f\, d\mathcal{I}_{h,k}, \qquad \text{for all } h, k \in \mathcal{H}.$$

Remark 3.1.20. In close analogy with classical integration theory over $*$-algebras, integration with respect to an instrument is inherently tied to positivity—more precisely, to complete positivity, as discussed in Theorem 3.1.27. This correspondence aligns naturally with the expectations set by the classical Riesz representation framework.

It follows from the definition of integrability that:

Theorem 3.1.21. Every $\mathcal{I}$-essentially separable and bounded measurable function $f : X \to \mathcal{A}$ is $\mathcal{I}$-integrable and satisfies

$$|\int_E f\, d\mathcal{I}_{h,k}| \leq \|f\|_\infty \|\mathcal{I}_{h,k}\|(E),$$

for every $h, k \in \mathcal{H}$ and $E \in \mathcal{O}(X)$.

As a direct application of the previous theorem we have the following corollary.

Corollary 3.1.22. Let X be a compact, Hausdorff space with $\mathcal{O}(X)$ be the Borel σ-algebra of X. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a CP instrument then every $\mathcal{A}$-valued continuous function on X is $\mathcal{I}$-integrable.

3.1.1 Correspondence between instruments and CP maps

We now move our attention to the case when X is a topological space and $\mathcal{O}(X)$ is to be the Borel σ-algebra on X. In this setting, to establish the correspondence between regular instruments and completely positive maps, we first recall the notion of regular vector-valued measures, which naturally generalizes the classical concept of regularity from scalar to operator-valued contexts.

Definition 3.1.23 (Regular vector measures)**.** Let $\mathcal{B}$ be a Banach space, X be a topological space and $\mathcal{O}(X)$ be the σ-algebra of Borel subsets of X. A vector measure μ on $(X, \mathcal{O}(X))$ with values in $\mathcal{B}$ is said be regular if for every Borel set $E \in \mathcal{O}(X)$ and $\epsilon > 0$, there exists an open set O containing E and a compact set C contained in E such that $||\mu||(O \setminus C) < \epsilon$.

Remark 3.1.24. The collection of all regular vector measures of bounded variation on $(X, \mathcal{O}(X))$, where X is compact and the measures take values in a Banach space $\mathcal{B}$, forms a Banach space when equipped with the norm given by

$$||\mu|| = |\mu|(X)$$

and we denote this space by $M(X, \mathcal{B})$. For the convenience of the reader we have added the proof here.

Proof. First, we show that $M(X, \mathcal{B})$ is a vector space. Let $\mu, \nu \in M(X, \mathcal{B})$ and $\alpha, \beta \in \mathbb{C}$. Define the linear combination $\lambda = \alpha\mu + \beta\nu$ pointwise by $\lambda(E) = \alpha\mu(E) + \beta\nu(E)$ for all $E \in \mathcal{O}(X)$.

To verify that $\lambda \in M(X, \mathcal{B})$, we first check countable additivity. Let $\{A_k\}$ be pairwise disjoint sets in $\mathcal{O}(X)$ with $A = \bigcup_{k=1}^{\infty} A_k$. Then:

$$\lambda(A) = \alpha\mu(A) + \beta\nu(A) = \alpha \sum_{k=1}^{\infty} \mu(A_k) + \beta \sum_{k=1}^{\infty} \nu(A_k) = \sum_{k=1}^{\infty} (\alpha\mu(A_k) + \beta\nu(A_k)) = \sum_{k=1}^{\infty} \lambda(A_k)$$

where the interchange of scalars with the series is valid by the continuity of addition and scalar multiplication in the Banach space $\mathcal{B}$.

Next, we verify that λ has bounded variation. For any finite disjoint partition $\{E_i\}$ of X:

$$\sum_i \|\lambda(E_i)\| \leq |\alpha| \sum_i \|\mu(E_i)\| + |\beta| \sum_i \|\nu(E_i)\| \leq |\alpha||\mu|(X) + |\beta||\nu|(X) < \infty$$

Taking the supremum over all such partitions, we obtain $|\lambda|(X) \leq |\alpha||\mu|(X) + |\beta||\nu|(X) < \infty$. The regularity of λ follows similarly from the subadditivity of the semivariation.

The total variation $\|\mu\| = |\mu|(X)$ defines a norm on $M(X, \mathcal{B})$, which follows from the norm properties of $\mathcal{B}$ and the properties of the supremum.

Finally, we show that $M(X, \mathcal{B})$ is complete. Let $(\mu_n)_{n=1}^{\infty}$ be a Cauchy sequence in $M(X, \mathcal{B})$. For any $E \in \mathcal{O}(X)$, we have:

$$\|\mu_m(E) - \mu_n(E)\| \leq |\mu_m - \mu_n|(E) \leq |\mu_m - \mu_n|(X) \to 0$$

Since $\mathcal{B}$ is complete, the pointwise limit $\mu(E) := \lim_{n\to\infty} \mu_n(E)$ exists in $\mathcal{B}$ for each $E \in \mathcal{O}(X)$.

To show μ is countably additive, fix $\varepsilon > 0$ and choose N such that $|\mu_m - \mu_n|(X) < \varepsilon/3$ for all $m, n \geq N$. For a disjoint union $A = \cup A_k$:

$$\left\| \mu(A) - \sum_{k=1}^{K} \mu(A_k) \right\| \leq \|\mu(A) - \mu_N(A)\| + \left\| \mu_N(A) - \sum_{k=1}^{K} \mu_N(A_k) \right\| + \sum_{k=1}^{K} \|\mu_N(A_k) - \mu(A_k)\|$$

Taking $K \to \infty$ and N sufficiently large, each term is bounded by $\varepsilon/3$, proving countable additivity.

For regularity, since $\|\mu - \mu_N\|(X) \leq |\mu - \mu_N|(X) < \varepsilon/2$, and μ_N is regular (i.e., there exist closed F and open U with $F \subseteq E \subseteq U$ such that $\|\mu_N\|(U \setminus F) < \varepsilon/2$), the subadditivity of semivariation implies:

$$\|\mu\|(U \setminus F) \leq \|\mu - \mu_N\|(X) + \|\mu_N\|(U \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\mu \in M(X, \mathcal{B})$ and $\mu_n \to \mu$ in the variation norm. We conclude that $M(X, \mathcal{B})$ is a Banach space. ■

Next we introduce regularity for instruments in a similar manner as discussed in [BBK21, Definition 6.1] in the context of POVMs.

Definition 3.1.25 (Regular instruments)**.** Let X be a topological space and $\mathcal{O}(X)$ be the Borel σ-algebra of X. An instrument $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ is said to be a regular instrument if $\mathcal{I}_{h,k}$, as defined in 3.1.6, is a regular $\mathcal{A}^*$ valued measure on $(X, \mathcal{O}(X))$ for all $h, k \in \mathcal{H}$.

Remark 3.1.26. It is worth noting that $\mathcal{I}_{h,k} = \frac{1}{4}\sum_{l=0}^{3} i^{-l}\mathcal{I}_{h+i^{-l}k,h+i^{-l}k}$, where i denotes the imaginary unit. Consequently, by Remark 3.1.24, , it suffices to establish regularity for the measures $\mathcal{I}_{h,h}, h \in \mathcal{H}$.

Notation. We denote by $\mathcal{R}$-$Ins_{\mathcal{H}}(X, \mathcal{A})$ the collection of all regular instruments on $(X, \mathcal{O}(X))$ with values in $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$. Similarly, $\mathcal{R}$-$I_{\mathcal{H}}(X, \mathcal{A})$ denotes the set of all regular instruments on $(X, \mathcal{O}(X))$.

In the following theorem, we extend the classical correspondence between regular positive operator-valued measures (POVMs) on compact Hausdorff spaces and completely positive (CP) maps on commutative C^*-algebras to the broader framework of CP instruments. This result naturally generalizes both the Riesz–Markov theorem ([Con90, Theorem III.5.7]) and the CP–POVM correspondence ([Pau02, Theorem 4.5]) to the instrument setting.

Theorem 3.1.27. Let X be a compact Hausdorff space and $\mathcal{A}$ be a unital C^*-algebra. Then there is a one-to-one correspondence between regular CP instruments $\mathcal{I}$ on $(X, \mathcal{O}(X))$ and CP maps from $C(X, \mathcal{A})$ to $\mathcal{B}(\mathcal{H})$, where $C(X, \mathcal{A})$ denotes the C^*-algebra of $\mathcal{A}$-valued continuous functions on X.

Proof. Let $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ be a regular instrument. It follows from Corollary 3.1.22 that for $f \in C(X, \mathcal{A})$, the following map $\Phi_{\mathcal{I}} : C(X, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ defined by,

$$\Phi_{\mathcal{I}} = \int f \; d\mathcal{I},$$

is well defined.

We claim that the association $C(X, \mathcal{A}) \ni f \to \Phi_{\mathcal{I}}(f) \in \mathcal{B}(\mathcal{H})$ defines a CP map such that,

$$\langle h, \Phi_{\mathcal{I}}(f)k\rangle = \int_X f \; d\mathcal{I}_{h,k}\,.$$

To prove the claim, for any $\{f_1, \cdots f_n\} \subset C(X, \mathcal{A})$ and $\{h_1, \cdots, h_n\} \subset \mathcal{H}$, we need to verify that $\sum_{i,j}\langle h_i, \Phi_{\mathcal{I}}(f_i^* f_j)h_j\rangle \geq 0$. Observe that,

$$\sum_{i,j}\langle h_i, \Phi_{\mathcal{I}}(f_i^* f_j)h_j\rangle \quad = \quad \sum_{i,j}\int f_i^* f_j \; d\mathcal{I}_{h_i,h_j}$$

Hence, in order to prove the claim, it is enough to prove that for any $\{f_1, \cdots f_n\} \subset C(X, \mathcal{A})$ and $\{h_1, \cdots, h_n\} \subset \mathcal{H}$,

$$\sum_{i,j} \int f_i^* f_j \; d\mathcal{I}_{h_i,h_j} \geq 0.$$

First consider the indicator functions $g_i = a_i \cdot 1_{A_i} \in B(X, \mathcal{A})$ for $a_i \in \mathcal{A}$ and $A_i \in \mathcal{O}(X)$ where $B(X, \mathcal{A})$ denotes the collection of all bounded $\mathcal{A}$-valued measurable functions. Then we have,

$$\begin{aligned} \sum_{i,j} \int g_i^* g_j \; d\mathcal{I}_{h_i h_j} &= \sum_{i,j} \int a_i^* a_j 1_{A_i \cap A_j} \; d\mathcal{I}_{h_i h_j} \\ &= \sum_{i,j} \langle h_i, \mathcal{I}(A_i \cap A_j, a_i^* a_j) h_j \rangle \\ &\geq 0, \end{aligned} \tag{3.1.5}$$

where this concluding inequality is a direct consequence of the calculations performed during the proof of Theorem 1.4.7. Since $C(X, \mathcal{A})$ is the C^*-norm closed linear span of the functions $af,\ a \in \mathcal{A},\ f \in C(X)$. Hence any $f \in C(X, \mathcal{A})$ can be approximated uniformly by a sequence of simple functions in every regular $\mathcal{A}^*$ measures $\mathcal{I}_{h_i,h_j},\ h_i, h_j \in \mathcal{H}$. Let $\{g_{i,m}\}$ be a sequence of simple functions converging uniformly to f_i. Therefore,

$$\begin{aligned} \sum_{i,j} \int f_i^* f_j \; d\mathcal{I}_{h_i,h_j} &= \sum_{i,j} \int (\lim_m g_{i,m}^* g_{j,m}) \; d\mathcal{I}_{h_i,h_j} \quad \text{(by Theorem 3.1.18)} \\ &= \sum_{i,j} \lim_m \int (g_{i,m}^* g_{j,m}) \; d\mathcal{I}_{h_i,h_j} \\ &= \lim_m \sum_{i,j} \int g_{i,m}^* g_{j,m} \; d\mathcal{I}_{h_i,h_j} \\ &= \lim_m \sum_{i,j} \langle h_i, \Phi_{\mathcal{I}}(g_{i,m}^* g_{j,m}) h_j \rangle \\ &\geq 0 \qquad \text{(By Equation 3.1.5)}, \end{aligned}$$

proving the claim.

To establish the other way correspondence, note that the dual $C(X, \mathcal{A})^*$ can be identified with $M(X, \mathcal{A}^*)$, (see [Rya02], page 112), by the following correspondence:

$$M(X, \mathcal{A}^*) \ni \nu \longmapsto (f \to \int f \; d\nu) \in C(X, \mathcal{A})^*.$$

Now let $\Phi : C(X, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ be a UCP map. For each $h, k \in \mathcal{H}$, the map $C(X, \mathcal{A}) \ni f \longmapsto \langle h, \Phi(f) k \rangle$ defines a bounded linear functional on $C(X, \mathcal{A})$. Hence it corresponds to a regular $\mathcal{A}^*$ - valued measure $\nu_{h,k} \in M(X, \mathcal{A}^*)$ on $(X, \mathcal{O}(X))$ such that

$$\int f \; d\nu_{h,k} = \langle h, \Phi(f) k \rangle. \tag{3.1.6}$$

It is clear from Equation 3.1.6 that $\nu_{h,h}$ is a $(\mathcal{A}^*)_+$ - valued measure for all $h \in \mathcal{H}$. Now for $h, k \in \mathcal{H}$, the total variation of the $\mathcal{A}^*$ - valued measure $\nu_{h,k}$ is given by

$$|\nu_{h,k}|(A) = \sup\{\sum_i \|\nu_{h,k}(E_i)\| : \{E_1, \cdots, E_n\} \text{ is a partition of } A\}.$$

Since $\nu_{h,k}$ are of bounded variation, we have $|\nu_{h,k}|$ is a finite positive measure on $(X, \mathcal{O}(X))$. For a simple function $g = a.1_A$, with $a \in \mathcal{A}$ and $A \in \mathcal{O}(X)$ the map $(h,k) \to \int g \, d\nu_{h,k}$ defines a bounded sesquilinear form $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$. Hence by Riesz theorem, we get a unique bounded operator $\tilde{\Phi}(g) \in \mathcal{B}(\mathcal{H})$ satisfying,

$$\langle h, \tilde{\Phi}(g)k\rangle = \int g \, d\nu_{h,k}.$$

It is easily verified that $\tilde{\Phi}(g) \geq 0$ for every simple function $g \in B(X, \mathcal{A})$ satisfying $g(x) \in \mathcal{A}_+$ everywhere on X. For $A \in \mathcal{O}(X), a \in \mathcal{A}$, define $\mathcal{I}_\Phi(A, a) = \tilde{\Phi}(a.1_A)$ such that,

$$\langle h, \mathcal{I}_\Phi(A, a)k\rangle = \langle h, \tilde{\Phi}(a.1_A)k\rangle = \int a.1_A \, d\nu_{h,k} = \nu_{h,k}(A)(a). \tag{3.1.7}$$

We claim that $\mathcal{I}_\Phi : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ is an instrument. Let $a \in \mathcal{A}$ be fixed. We need to show that $\mathcal{I}_\Phi(\cdot, a)$ defines an operator valued measure and in particular if $a \in \mathcal{A}_+$, we need to prove that $\mathcal{I}_\Phi(\cdot, a)$ is a POVM. Note that for any countable collection $\{B_i\} \subset \mathcal{O}(X)$ of disjoint measurable subsets, we have the following:

$$\langle h, \mathcal{I}_\Phi(\cup_i B_i, a)k\rangle = \langle h, \tilde{\Phi}(a.1_{\cup_i B_i})k\rangle = \nu_{h,k}(\cup_i B_i)(a) = \sum_i \nu_{h,k}(B_i)(a) = \sum_i \langle h, \mathcal{I}_\Phi(B_i, a)k\rangle$$

for all $h, k \in \mathcal{H}$ proving that $\mathcal{I}_\Phi(\cdot, a)$ defines an operator valued measure. Now since for each $A \in \mathcal{O}(X)$, $\nu_{h,h}(A)$ is a positive linear functional on $\mathcal{A}$, it is immediate from Equation 3.1.7 that $\mathcal{I}_\Phi(\cdot, a)$ defines a POVM for all $a \in \mathcal{A}_+$.

To prove that $\mathcal{I}_\Phi : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ is an instrument, we also need to show that $\mathcal{I}_\Phi(A, \cdot)$ defines a CP map for every $A \in \mathcal{O}(X)$. Let $A \in \mathcal{O}(X)$ be fixed. We need to see that,

$$\sum_{i,j} \langle h_i, \mathcal{I}_\Phi(a_i^* a_j, A)h_j\rangle \geq 0 \quad \text{for } \{a_1, \cdots, a_n\} \subset \mathcal{A} \text{ and } \{h_1, \cdots, h_j\} \subset \mathcal{H}. \tag{3.1.8}$$

Since X is compact Hausdorff, we have $C(X) \subset L^\infty(X, \nu) \subset L^1(X, \nu)$ for any finite scalar measure ν on X. Now by the denseness of $C(X)$ in $L^1(X, \nu)$, there exists a sequence of positive continuous functions $\{f_n\} \subset C(X)$ such that f_n converges to 1_A in L^1 norm and so there exists a subsequence f_{n_k} converges point-wise ν almost everywhere to 1_A. In fact by Lusin's theorem we can take the sequence $\{f_{n_k}\}$ to be uniformly bounded.

To prove Equation 3.1.8 , let us fix $\{a_1, \cdots, a_n\} \subset \mathcal{A}$ and $\{h_1, \cdots, h_j\} \subset \mathcal{H}$. Consider the finite positive measure $\nu = \sum_{i,j} |\nu_{h_i,h_j}|$. Then it is clear that $|\nu_{h_i,h_j}| \ll \nu$ for $i, j = 1, \cdots, n$. We have from the above discussion that f_{n_k} converges point-wise ν almost everywhere to 1_A. In particular f_{n_k} converges point-wise $|\nu_{h_i,h_j}|$ almost everywhere to 1_A, for $i, j = 1, \cdots, n$. Hence $a_i^* a_j \cdot f_{n_k}$ converges point-wise ν_{h_i,h_j} almost everywhere to $a_i^* a_j \cdot 1_A$ for $i, j = 1, \cdots, n$. Since f_{n_k}'s are uniformly bounded, we can conclude that $a_i^* a_j \cdot f_{n_k}$'s are uniformly bounded and hence by the Theorem 3.1.18 we have,

$$\lim_k \int a_i^* a_j \cdot f_{n_k} d\nu_{h_i,h_j} = \int a_i^* a_j \cdot 1_A \, d\nu_{h_i,h_j}, \quad \text{for } i, j = 1, \cdots, n. \tag{3.1.9}$$

Now observe that,

$$\begin{aligned}
\sum_{i,j}\langle h_i, \mathcal{I}_\Phi(a_i^* a_j, A) h_j\rangle &= \sum_{i,j}\langle h_i, \tilde{\Phi}((a_i^* a_j)\cdot 1_A) h_j\rangle \\
&= \sum_{i,j}\int a_i^* a_j \cdot 1_A \, d\nu_{h_i,h_j} \\
&= \sum_{i,j}\lim_k \int a_i^* a_j \cdot f_{n_k} \, d\nu_{h_i,h_j} \quad \text{(By Equation 3.1.9)} \\
&= \lim_k \sum_{i,j}\int a_i^* a_j \cdot f_{n_k} \, d\nu_{h_i,h_j} \\
&= \lim_k \sum_{i,j}\langle h_i, \Phi((a_i^* a_j)\cdot f_{n_k}) h_j\rangle \\
&= \lim_k \sum_{i,j}\langle h_i, \Phi((\sqrt{f_{n_k}} a_i)^*(\sqrt{f_{n_k}} a_j)) h_j\rangle \\
&\geq 0 \quad \text{(since } \Phi \text{ is CP)},
\end{aligned}$$

finishing the proof of Equation 3.1.8.

An easy verification shows that:
(i) $\mathcal{I}_{\Phi_{\mathcal{I}}} = \mathcal{I}$;
(ii) $\Phi_{\mathcal{I}_\Phi} = \Phi$.
concluding the proof of the bijection between the sets $\mathcal{R}$-$Ins_{\mathcal{H}}(X, \mathcal{A})$ and $CP(C(X, \mathcal{A}), \mathcal{B}(\mathcal{H}))$. ■

Remark 3.1.28. To prove the converse direction, one may appeal to dilation theory, thereby avoiding the technical intricacies of integration theory. We outline this alternative approach below. Let $\Phi : C(X, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ be a completely positive map, and consider its minimal Stinespring dilation. Since $C(X, \mathcal{A}) \equiv C(X) \otimes \mathcal{A}$, $C(X)$ is nuclear, the tensor product is uniquely determined. Thus, we may write $\Phi(f \otimes a) = V^*\pi_E(f)\pi(a)V$. By the CP–POVM correspondence (see [Pau02, Theorem 4.5]), there exists a spectral measure $E : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ on the measurable space $(X, \mathcal{O}(X))$. Define the map $\mathcal{I}_\Phi : \mathcal{O}(X) \times \mathcal{A} \to \mathcal{B}(\mathcal{H})$ given by $\mathcal{I}_\Phi(A, a) = V^*E(A)\pi(a)V,\ a \in \mathcal{A},\ A\mathcal{O}(X)$. It is straightforward to check that $\mathcal{I}_\Phi$ defines a CP instrument. Moreover, a routine argument shows that $\Phi = \Phi_{\mathcal{I}_\Phi}$.

Remark 3.1.29. It is immediate to verify that the correspondence described above respects the C^*-convexity as well as the C^*-extreme point structure.

3.1.2 BW Topology on Instruments

Let X be a topological space, and let $I_{\mathcal{H}}(X, \mathcal{A})$ denote the set of all unital instruments from $\mathcal{O}(X)$ to $CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$. In this context, we introduce a topology on $I_{\mathcal{H}}(X, \mathcal{A})$, inspired by the work of Bhat and Kumar [BBK21], which extends the bounded weak (BW) topology previously developed for CP maps (Definition 1.2.12).

Let $C_b(X, \mathcal{A})$ denote the space of all bounded, continuous, separably valued functions on X taking values in the C^*-algebra $\mathcal{A}$, with the additional requirement that for every $f \in C_b(X, \mathcal{A})$, there exists a sequence of simple functions converging uniformly to f. Recall that, for any instrument $\mathcal{I}$, the corresponding $\mathcal{A}^*$-valued measure $\mathcal{I}_{h,k}$ is defined as in Remark 3.1.6. We now introduce a topology on $I_{\mathcal{H}}(X, \mathcal{A})$ by specifying the convergence of nets.

Definition 3.1.30. Let $\mathcal{I}^j$ be a net in $I_{\mathcal{H}}(X, \mathcal{A})$ and $\mathcal{I} \in I_{\mathcal{H}}(X, \mathcal{A})$. We say that $\mathcal{I}^j \to \mathcal{I}$ in the *bounded weak (BW)* topology if

$$\int_X f d\mathcal{I}^j_{h,k} \to \int_X f d\mathcal{I}_{h,k}$$

for all $f \in C_b(X, \mathcal{A})$ and $h, k \in \mathcal{H}$.

It is worth noting that the topology defined above on $I_{\mathcal{H}}(X, \mathcal{A})$ is the weakest topology that renders all maps of the form: $\mathcal{I} \mapsto \int_X f d\mathcal{I}_{h,k}$ from $I_{\mathcal{H}}(X, \mathcal{A})$ to $\mathbb{C}$, continuous for all $f \in C_b(X, \mathcal{A})$ and $h, k \in \mathcal{H}$. Consequently, for any fixed $\mathcal{I} \in I_{\mathcal{H}}(X, \mathcal{A})$, a neighborhood basis at $\mathcal{I}$ is given by the sets

$$O = \left\{ \mathcal{J} \in I_{\mathcal{H}}(X, \mathcal{A}) : \left| \int_X f_i d\mathcal{J}_{h_i,k_i} - \int_X f_i d\mathcal{I}_{h_i,k_i} \right| < \epsilon, 1 \leq i \leq n \right\},$$

where $f_i \in C_b(X, \mathcal{A})$, $h_i, k_i \in \mathcal{H}$ for $1 \leq i \leq n$, $\epsilon > 0$. The structure of this topology is analogous to the weak topology commonly employed in classical probability theory, thereby generalizing the notion to the setting of operator-valued instruments. Moreover, following the correspondence established in Theorem 3.1.27, it is evident that the bounded weak topology introduced above is naturally linked to the BW topology on the space of completely positive maps $CP(C(X, \mathcal{A}), \mathcal{B}(\mathcal{H}))$, where X is a compact, Hausdorff space and $\mathcal{A}$ a C^*-algebra. For a net $\mathcal{I}^i$ and $\mathcal{I} \in \mathcal{R}\text{-}Ins_{\mathcal{H}}(X, \mathcal{A})$, recall that the associated completely positive map is defined as

$$\Phi_{\mathcal{I}}(f) = \int_X f d\mathcal{I} \textit{ for all } f \in C(X, \mathcal{A}),$$

It then follows that the convergence $\mathcal{I}^i \to \mathcal{I}$ in $\mathcal{R}\text{-}I_{\mathcal{H}}(X, \mathcal{A})$ is equivalent to the convergence $\Phi_{\mathcal{I}^i}(f) \to \Phi_{\mathcal{I}}(f)$ in WOT for all $f \in C(X, \mathcal{A})$. The following proposition formalizes this observation and, in essence, states that the spaces $\mathcal{R} - I_{\mathcal{H}}(X, \mathcal{A})$ and $UCP(C(X, \mathcal{A}), \mathcal{B}(\mathcal{H}))$ are topologically homeomorphic with respect to their respective bounded weak topologies.

Proposition 3.1.31. Let $\mathcal{I}^i$ be a net in $\mathcal{R} - I_{\mathcal{H}}(X, \mathcal{A})$ and $\mathcal{I} \in \mathcal{R} - I_{\mathcal{H}}(X, \mathcal{A})$. Then the following are equivalent:

1. $\mathcal{I}^i \to \mathcal{I}$ in $\mathcal{R} - I_{\mathcal{H}}(X, \mathcal{A})$ (and, in $I_{\mathcal{H}}(X, \mathcal{A})$) in BW topology.
2. $\Phi_{\mathcal{I}^i} \to \Phi_{\mathcal{I}}$ in BW topology in $UCP(C(X, \mathcal{A}), \mathcal{B}(\mathcal{H}))$.

3.1.3 Krein-Milman type theorem

The Krein–Milman theorem stands as one of the fundamental results in classical functional analysis. It asserts that, in any locally convex topological vector space, every compact convex subset is the closure of the convex hull of its extreme points. Motivated by this classical result, it is natural to seek an analogue of the Krein–Milman theorem in the framework of C^*-convexity, particularly within the space of CP instruments. To this end, we first establish the following proposition, whose proof closely follows the classical line of argument.

Proposition 3.1.32. Let X be a topological space and let $\mathcal{H}$ be a Hilbert space. We regard singleton sets as measurable elements of $\mathcal{O}(X)$. Then the collection of all normalized instruments on $(X, \mathcal{O}(X))$ that are concentrated on finite subsets of X is dense in $I_{\mathcal{H}}(X, \mathcal{A})$.

Proof. Let $\mathcal{I} \in I_{\mathcal{H}}(X, \mathcal{A})$, and let E be a typical open neighbourhood of $\mathcal{I}$ in $I_{\mathcal{H}}(X, \mathcal{A})$ of the form

$$E = \left\{ \mathcal{J} \in I_{\mathcal{H}}(X, \mathcal{A}) : \left| \int_X f_i \, d\mathcal{J}_{h_i,k_i} - \int_X f_i \, d\mathcal{I}_{h_i,k_i} \right| < \epsilon, \ 1 \leq i \leq n \right\},$$

for some fixed $f_i \in C_b(X, \mathcal{A})$, $h_i, k_i \in \mathcal{H}$ $(1 \leq i \leq n)$, and $\epsilon > 0$. We shall construct an element $\mathcal{J} \in E$ that is concentrated on a finite subset of X, which will yield the desired density result.

For each $i \in \{1, \dots, n\}$, there exists a simple function g_i on X such that

$$\sup_{x \in X} \|f_i(x) - g_i(x)\| < \frac{\epsilon}{2M},$$

where M is a positive constant satisfying $M > \sup_i \|h_i\| \, \|k_i\|$. Since each g_i is a simple function, we may consider a common finite refinement (partition), $\{A_{ij}\}$ of X along with corresponding elements $\{a_{ij}\} \subseteq \mathcal{A}$, where j ranges over some finite index set Λ_i (depending on i), such that

$$g_i = \sum_{j \in \Lambda_i} a_{ij} 1_{A_{ij}}$$

for each $i = 1, \dots, n$. Now, for each i and j, choose a point $x_{ij} \in A_{ij}$ and define

$$\mathcal{J} := \sum_{i=1}^{n} \sum_{j \in \Lambda_i} \delta_{x_{ij}}(\cdot) \mathcal{I}(A_{ij})(\cdot).$$

Clearly, $\mathcal{J}$ is an instrument concentrated on the finite subset $\{x_{ij}\} \subseteq X$. Moreover,

$$\mathcal{J}(X)(1_{\mathcal{A}}) = \sum_{i=1}^{n} \sum_{j \in \Lambda_i} \mathcal{I}(A_{ij})(1_{\mathcal{A}}) = \mathcal{I}(X)(1_{\mathcal{A}}) = I_{\mathcal{H}},$$

hence $\mathcal{J}$ is normalized. We now claim that $\mathcal{J} \in E$. For each $m \in \{1, \dots, n\}$, we have

$$\int_X g_m \, d\mathcal{J} = \sum_{i=1}^{n} \sum_{j \in \Lambda_i} \mathcal{I}(A_{ij})(g_m(x_{ij})) = \sum_{j \in \Lambda_m} \mathcal{I}(A_{mj})(a_{mj}) = \int_X g_m \, d\mathcal{I}.$$

Then, for each $i = 1, \dots, n$, we estimate:

$$\begin{aligned}
\left| \int_X f_i \, d\mathcal{J}_{h_i,k_i} - \int_X f_i \, d\mathcal{I}_{h_i,k_i} \right| &\leq \left| \int_X f_i d\mathcal{J}_{h_i,k_i} - \int_X g_i d\mathcal{J}_{h_i,k_i} \right| + \left| \int_X g_i d\mathcal{J}_{h_i,k_i} - \int_X g_i d\mathcal{I}_{h_i,k_i} \right| \\
&\quad + \left| \int_X g_i d\mathcal{I}_{h_i,k_i} - \int_X f_i d\mathcal{I}_{h_i,k_i} \right| \\
&\leq \left(\sup_{x \in X} \|f_i(x) - g_i(x)\| \right) (\|\mathcal{J}_{h_i,k_i}\|(X) + \|\mathcal{I}_{h_i,k_i}\|(X)) \\
&\leq \frac{\epsilon}{2M} \cdot 2\|h_i\| \, \|k_i\| < \epsilon.
\end{aligned}$$

We have used the fact that $\|\mathcal{J}_{h_i,k_i}\|(X), \|\mathcal{I}_{h_i,k_i}\|(X) \leq \|h_i\| \, \|k_i\|$, which follows directly from the definitions of $\mathcal{I}_{h_i,k_i}, \mathcal{J}_{h_i,k_i}$. Hence $\mathcal{J} \in E$, completing the proof. ■

Below, we present a Krein–Milman type theorem for the space of normalized instruments endowed with the BW topology. It is important to emphasize that, unlike the classical setting, compactness of $I_{\mathcal{H}}(X, \mathcal{A})$ is not required for the result.

Theorem 3.1.33. Let X be a Hausdorff topological space and let $\mathcal{A}$ and $\mathcal{H}$ be separable C^*-algebra and separable Hilbert space, respectively. Then the C^*-convex hull of C^*-extreme points is dense in $I_{\mathcal{H}}(X, \mathcal{A})$ with respect to the BW topology.

Proof. Fix an instrument $\mathcal{I} \in I_{\mathcal{H}}(X, \mathcal{A})$. By Proposition 3.1.32, there exists a net $\mathcal{I}_i \in I_{\mathcal{H}}(X, \mathcal{A})$ such that $\mathcal{I}_i \to \mathcal{I}$ in $I_{\mathcal{H}}(X, \mathcal{A})$ in BW topology and each $\mathcal{I}_i$ is concentrated on a finite subset of X. Hence, it suffices to show that any instrument concentrated on a finite subset belongs to the C^*-convex hull. Without loss of generality, assume that $\mathcal{I}$ is concentrated on a finite subset, say $\{x_1, \dots, x_n\}$. We may therefore identify X with this finite set. By the correspondence between completely positive maps and instruments (see Theorem 3.1.27), the instrument $\mathcal{I}$ corresponds to a unital completely positive (UCP) map $\Phi_{\mathcal{I}} : \oplus_{i=1}^{n}\mathcal{A} \to \mathcal{B}(\mathcal{H})$. According to the Krein–Milman type theorem for UCP maps [BK22, Theorem 5.3], there exists a net Φ_j of C^* extreme UCP maps such that $\Phi_j \to \Phi_{\mathcal{I}}$. The corresponding instruments $\mathcal{I}_{\Phi_j}$ are then C^*-extreme points of $I_{\mathcal{H}}(X, \mathcal{A})$, and as discussed in Proposition 3.1.31, the convergence $\Phi_j \to \Phi_{\mathcal{I}}$ implies $\mathcal{I}_{\Phi_j} \to \mathcal{I}$ in BW toplogy. ■

3.2 Tensor product approach

Classical joint measures are inherently defined by their action on bivariate scalar functions. In transitioning to a noncommutative framework, tensor product algebras naturally assume the role of these bivariate structures, dictating that the bivariate realization of instruments requires a corresponding integration theory on tensor products. Rather than a traditional progression, this section advances an inverted methodology: instruments are first extended to tensor product spaces as completely positive maps, providing the structural scaffolding required to develop an integration framework. To ensure compatibility with the theory of bounded measurable functions—which favors von Neumann algebras over C^*-algebras—our investigation is anchored in the von Neumann context, with specific attention given to algebras characterized by separable preduals.

Consider $\mathcal{H}$ to be a separable Hilbert space. Let $(X, \mathcal{O}(X))$ be a measurable space and $\mathcal{A}$ be a C^*-algebra. For any σ-finite measure ν on $(X, \mathcal{O}(X))$ define the set

$$Ins_{\mathcal{H}}(X, \mathcal{A})_{\nu} := \{\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H})) : \mu_{\mathcal{I}_\rho} \ll \nu, \quad \forall \rho \in \mathcal{S}(\mathcal{H})\},$$

where $\mathcal{S}(\mathcal{H})$ is the state space of $\mathcal{H}$ and $\mu_{\mathcal{I}_\rho}$ is the complex measure $\mu_{\mathcal{I}_\rho}(A) = \mathrm{Tr}(\rho\mu_{\mathcal{I}}(A))$, $A \in \mathcal{O}(X)$. This set corresponds to the collection of CP maps from $L^\infty(X, \nu) \otimes_{max} \mathcal{A}$ to $\mathcal{H}$. It's easy to check that the set $Ins_{\mathcal{H}}(X, \mathcal{A})_\nu$ is always non-empty for any σ-finite measure ν on $(X, \mathcal{O}(X))$. Moreover, the set of instruments can be expressed as

$$Ins_{\mathcal{H}}(X, \mathcal{A}) = \cup_\nu Ins_{\mathcal{H}}(X, \mathcal{A})_\nu,$$

where the union is taken over all σ-finite measures on $(X, \mathcal{O}(X))$. Let, $\mathcal{I} \in Ins_{\mathcal{H}}(X, \mathcal{A})_\nu$ with minimal bi-dilation $(\mathcal{K}, \pi, E, V)$ i.e. $\mathcal{I}(A, a) = V^*\pi(a)E(A)V$. Since $E : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K})$ is a spectral measure then it extends to a CP map $\psi_E : L^\infty(X, \nu) \to \mathcal{B}(\mathcal{K})$ defined by,

$$\psi_E(f) = \int f dE, \ \forall\ f \in L^\infty(X, \nu).$$

Then it is staright forward to verify that $\pi(\mathcal{A}) \subseteq (\psi_E(L^\infty(X, \nu)))'$, implying that we can define a CP map on the maximal tensor product of C^*-algebras,

$$\Phi : L^\infty(X, \nu) \otimes_{max} \mathcal{A} \to \mathcal{B}(\mathcal{K}) \text{ where } \Phi(f \otimes a) = \pi(a)\psi_E(f).$$

Consequently, we obtain a CP map $\Phi_{\mathcal{I}} : L^\infty(X, \nu) \otimes_{max} \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\Phi_{\mathcal{I}} = V^*\Phi V$, arising naturally from the instrument $\mathcal{I} : \mathcal{O}(X) \to CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$.

In particular, if we restrict our attention to normal instruments, assume that $\mathcal{A}$ is a von Neumann algebra with a separable predual, and consider the von Neumann algebra tensor product $\overline{\otimes}$ instead of the maximal C^*-tensor product, then whenever the associated map

$$\Phi_{\mathcal{I}} : L^\infty(X, \nu)\overline{\otimes}\mathcal{A} \to \mathcal{B}(\mathcal{H})$$

is normal, we can define the integration of a bounded ν-measurable function $f : X \to \mathcal{A}$ with respect to the normal instrument $\mathcal{I}$ by

$$\int f d\mathcal{I} := \Phi_{\mathcal{I}}(f),$$

where we identify $L^\infty(X,\nu)\overline{\otimes}\mathcal{A}$ with $L^\infty(X,\mathcal{A},\nu)$ (see Theorem 1.22.13 in [Sak12]).

In what follows, we present a concrete example of an instrument that naturally arises from a completely positive map. Although this example was not originally framed in the language of instruments, it fits seamlessly within the framework introduced by Farenick, Plosker, Ramsey, and MacLaren [MPR20; FPS11], who developed a theory of integration with respect to POVM. Let $(X,\mathcal{O}(X))$ be a measurable space equipped with a POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots\}$. For any full-rank trace-class operator $\rho \in \mathcal{B}(\mathcal{H})$, one defines the associated finite complex measure μ_ρ on $\mathcal{O}(X)$ by $\mu_\rho(A) = \mathrm{Tr}(\rho\mu(A)),\ A \in \mathcal{O}(X)$. The Radon–Nikodym derivative of μ with respect to μ_ρ is then defined by

$$\langle e_i, \frac{d\mu}{d\mu_\rho}(x)(e_j)\rangle = \frac{d\mu_{ij}}{d\mu_\rho}(x),\ x \in X,$$

where μ_{ij} denotes the complex measure $\mu_{ij}(A) = \langle e_i, \mu(A)e_j\rangle,\ A \in \mathcal{O}(X)$ and $\frac{d\mu_{ij}}{d\mu_\rho}$ is the classical Radon-Nikodym derivative. It is known that $\frac{d\mu}{d\mu_\rho}$ exists as a Borel measurable operator-valued map whenever $\mathcal{H}$ is finite-dimensional. However, as shown in [MPR20, Example 2.4], the derivative need not exist as a bounded operator-valued function in general. Nevertheless, whenever $\frac{d\mu}{d\mu_\rho}$ does exist, it defines a positive operator-valued map that is independent of both the choice of orthonormal basis of $\mathcal{H}$ and the full-rank trace-class operator ρ (see [MPR20, Theorem 2.12]).

In the following example, we illustrate how a normal completely positive (CP) map naturally induces an instrument.

Example 3.2.1. Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM, and let ρ be a full-rank trace-class operator on $\mathcal{H}$ with $\frac{d\mu}{d\mu_\rho}$ exists. Define the map $\Phi_\mu : L^\infty(X,\mathcal{B}(\mathcal{H}),\mu_\rho) \to \mathcal{B}(\mathcal{H})$ by,

$$\langle h, \Phi_\mu(f)k\rangle = \int \mathrm{Tr}(|h\rangle\langle k|\sqrt{(\frac{d\mu}{d\mu_\rho})}f\sqrt{(\frac{d\mu}{d\mu_\rho})})d\mu_\rho,\ h,k \in \mathcal{H},$$

for every $f \in L^\infty(X,\mathcal{B}(\mathcal{H}),\mu_\rho)$. It follows from [PR19, Theorem 2.8] that Φ_μ defines a normal completely positive map on the von Neumann algebra $L^\infty(X,\mathcal{B}(\mathcal{H}),\mu_\rho)$.

Consequently, the map Φ_μ induces a normal instrument $\mathcal{I}_\mu : \mathcal{O}(X) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with POVM marginal μ, given by

$$\mathcal{I}_\mu(A,a) = \Phi_\mu(a \cdot 1_A) \textit{ for all } A \in \mathcal{O}(X),\ a \in \mathcal{B}(\mathcal{H}).$$

From the above example, it is evident that the normal map Φ_μ provides a natural framework for defining integration with respect to the instrument $\mathcal{I}_\mu$. In particular, for any essentially bounded measurable function $f \in L^\infty(X,\mathcal{B}(\mathcal{H}),\mu_\rho)$, one may define its integral with respect to $\mathcal{I}_\mu$ via the relation

$$\int_X f d\mathcal{I}_\mu := \Phi_\mu(f).$$

An important observation here is that an element $f \in L^\infty(X,\mathcal{B}(\mathcal{H}),\mu_\rho)$ need not be separably valued. This stands in contrast to the classical approach, where Theorem 3.1.14 ensures that all integrable functions are $\mathcal{I}_{h,k}$-almost everywhere separably valued. In this sense, the tensor product approach extends the theory of integration to a broader class of operator-valued functions, offering a more general and flexible formulation within the framework of normal instruments.

Chapter 4

CP Completion

Motivated by the notion of almost everywhere equality for measurable functions Parzygnat and Russo [PR23] came up with a definition of almost everywhere equivalence (with respect to a state) for completely positive maps. The definition requires the familiar concept of null ideal and support projection for states and completely positive maps and we recall it here.

Let $\mathcal{B}, \mathcal{C}$ be unital C^*-algebras and let $\xi : \mathcal{B} \to \mathcal{C}$ be a completely positive (CP) map. Then

$$\mathcal{N}_\xi := \{X \in \mathcal{B} : \xi(X^*X) = 0\}$$

is a left ideal of $\mathcal{B}$. It is called the *null ideal* of ξ. If $\mathcal{B}, \mathcal{C}$ are von Neumann algebras and ξ is normal then

$$\mathcal{N}_\xi = \mathcal{B}(I - P) := \{X(I - P) : X \in \mathcal{B}\} \tag{4.0.1}$$

for a unique projection P in $\mathcal{B}$. The projection P is is known as the *support projection* of ξ (See [Sak12]) and is the smallest projection P such that $\xi(P) = \xi(I)$ and satisfies $\xi(X) = \xi(XP), \forall X \in \mathcal{B}$. Now the definition by Parzygnat and Russo [PR23] reads as follows.

Definition 4.0.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be unital C^*-algebras and let $\xi : \mathcal{B} \to \mathcal{C}$ be a unital completely positive map. Then two linear maps ϕ, ψ from $\mathcal{A}$ to $\mathcal{B}$ are said to be *equal almost everywhere (a.e.)* with respect to ξ if $\phi(X) - \psi(X) \in \mathcal{N}_\xi$ for every X in $\mathcal{A}$. This is denoted by $\phi \underset{\xi}{=} \psi$.

These authors were mainly interested in the case where ξ is a state. One of the surprising results of theirs is that on the C^*-algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices, given any state ξ, if a unital completely positive map ϕ is equal almost everywhere to the identity map with respect to ξ, then ϕ must be the identity map (see Theorem 2.48 of [PR23]). The result was obtained through detailed computations involving Choi-Kraus coefficients of the given completely positive map. Here we try to have a more conceptual understanding of their result by putting the problem in a more abstract setting.

There is extensive literature on completing partially specified matrices to positive matrices (see [Gro+84], [Smi08] and Chapter 12 of [Bap14]). It appears in a variety of different contexts and has a number of real life applications. Furuta [Fur94] has investigated a completion problem of partial matrices whose entries are completely bounded maps on a C*-algebra. We have a somewhat different setting. This we call as the CP completion problem. The minimal completion theorem in Section 3 (See Theorem 4.2.4) shows that if a linear map admits CP completion then there is a unique minimal CP completion. A direct application of this yields a result much more general than Theorem 2.48 of [PR23]. It is valid for a large class of maps called quasi-pure maps between arbitrary C^*-algebras (See Theorem 4.3.3 and Corollary 4.3.6). Moreover the proof becomes simpler and more algebraic. The notion of quasi-pure maps is introduced and studied in Section 2. The usefulness of it will be clear in subsequent sections.

Generalizing the concept of positive completability from matrices to maps on C^*-algebras we have the following definition.

Definition 4.0.2. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space. Fix $R \in \mathcal{B}(\mathcal{H})$. Then a linear map $\beta : \mathcal{A} \to \mathcal{B}.R := \{YR : Y \in \mathcal{B}\}$, is said to be CP completable (with respect to R) if there exists a CP map $\phi : \mathcal{A} \to \mathcal{B}$ such that

$$\beta(X) = \phi(X)R, \quad \forall X \in \mathcal{A}.$$

In such a case, ϕ is called a CP completion of β.

We are mostly interested in the case when R is a projection. It is a natural problem to characterize linear maps which can be CP completed. We give some necessary conditions for CP completability in Theorem 4.2.3. More interestingly, in the main theorem (Theorem 4.2.4) it is shown that every CP completable linear map admits a unique 'minimal' CP completion. The notion of minimality will be made clear below. Arguably the setting of this Definition 4.0.2 is somewhat peculiar. It is motivated by the theory we are able to develop in Section 3 and is justified by the applications it has in Section 4.

We now introduce the notion of minimality in the context of CP completions in a rigorous manner.

Definition 4.0.3. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Fix $R \in \mathcal{B}(\mathcal{H})$. Let $\beta : \mathcal{A} \to \mathcal{B}R$ be a linear map. Then a CP completion $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a minimal CP completion of β if $\psi : \mathcal{A} \to \mathcal{B}$ is any CP completion of β then $\phi \le \psi$, that is, ϕ is dominated by ψ.

4.1 Quasi-pure CP maps

In this section, we introduce a new class of completely positive maps, called quasi-pure CP maps, which serve as a key tool in generalizing Theorem 2.48 from [PR23]. We will also present some fundamental characterizations of these maps that highlight their structural and operational significance.

From Proposition 1.2.11, we know that a completely positive map ϕ is pure if and only if every non-zero vector in the dilation space $\mathcal{K}$ is cyclic for the corresponding Stinespring representation. Making this condition significantly weaker we have the following definition.

Definition 4.1.1. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{H}$ be a Hilbert space. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map with minimal Stinespring triple $(\mathcal{K}, \pi, V)$. Then ϕ is said to be *quasi-pure* if every non-zero vector in the range of V is cyclic for the representation π, that is,

$$\mathcal{K} = \overline{\text{span}}\{\pi(X)g : X \in \mathcal{A}\},$$

for every $0 \neq g \in V(\mathcal{H})$.

MD Choi in his seminal paper [Cho75] showed that any non-zero CP map, say, $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ is of the form, $\phi(X) = \sum_{j=1}^{k} L_j^* X L_j, \quad X \in M_{d_1}(\mathbb{C})$ such that $k \in \mathbb{N}$ and the set of $d_1 \times d_2$ matrices $\{L_1, L_2, \ldots, L_k\}$ forms a linearly independent set. This can be put in Stinespring's representation form: $\phi(X) = V^*\pi(X)V$, where

$$\pi(X) = \begin{bmatrix} X & 0 & \ldots & 0 \\ 0 & X & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & X \end{bmatrix}_{k\times k}, \quad V = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_k \end{bmatrix}_{k\times 1}.$$

In particular, if ϕ is pure, as the representation π has to be irreducible, we must have $k = 1$. Consequently non-zero pure CP maps have the form

$$X \mapsto L^* X L,$$

for some $d_1 \times d_2$ non-zero matrix L and conversely any such CP map is pure. Now it is easy to provide examples of quasi-pure maps which are not pure.

Example 4.1.2. Let $d_1, d_2 \in \mathbb{N}$. Let ρ be a positive matrix in $M_{d_1}(\mathbb{C})$ and let v be a unit vector in $\mathbb{C}^{d_2}$. Define $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ by

$$\phi(X) = \text{trace } (\rho X)|v\rangle\langle v|, \quad X \in M_{d_1}(\mathbb{C}), \tag{4.1.1}$$

where $|v\rangle\langle v|$ is the projection onto the one dimensional subspace spanned by v. Clearly ϕ is completely positive. By spectral theorem, there exist orthonormal vectors $u_1, u_2, \ldots, u_k$, and positive scalars $p_1, p_2, \ldots, p_k$, such that

$$\rho = \sum_{j=1}^{k} p_j |u_j\rangle\langle u_j|,$$

where k is the rank of ρ. Then ϕ has the minimal Stinespring representation, $(\mathcal{K}, \pi, V)$ where $\mathcal{K} = \mathbb{C}^{d_1} \otimes \mathbb{C}^k$, $\pi(X) = X \otimes I_{\mathbb{C}^k}$ and $V : \mathbb{C}^{d_2} \to \mathcal{K}$, is defined by $Vh = \sum_{j=1}^{k} \langle v, h\rangle(\sqrt{p_j} u_j) \otimes e_j$, where $e_1, e_2, \ldots, e_k$ is an orthonormal basis of $\mathbb{C}^k$. Now it is not hard to see that ϕ is quasi-pure. It is not pure if $k > 1$.

Here is a characterization of quasi-pure CP maps without referring to their Stinespring dilations and just using kernels of some maps.

Theorem 4.1.3. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{H}$ be a Hilbert space. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map. Then ϕ is quasi-pure if and only if whenever a non-zero CP map α is dominated by ϕ, $\ker(\alpha(1)) = \ker(\phi(1))$.

Proof. Let $(\mathcal{K}, \pi, V)$ be a minimal Stinespring triple of ϕ.

First assume that $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a quasi-pure CP map. Let $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a non-zero CP map such that $\alpha \leq \phi$. As $\alpha(1) \leq \phi(1)$, $ker(\phi(1)) \subseteq ker(\alpha(1))$. This reverse inclusion of the kernels follows from a standard property of positive operators on a Hilbert space $\mathcal{H}$: if $0 \leq T_1 \leq T_2$, then $T_2 h = 0$ forces $\langle T_2 h, h\rangle = 0$, which in turn implies $\langle T_1 h, h\rangle = 0$ and thus $T_1 h = 0$ for any $h \in \mathcal{H}$. By Radon-Nikodym type theorem (Theorem 1.2), there exists a positive contraction $D \in \pi(\mathcal{A})'$ such that

$$\alpha(X) = V^* D\pi(X)V, \text{ for all } X \in \mathcal{A}.$$

If we assume that $ker(\alpha(1)) \subsetneqq ker(\phi(1))$, there exists a non-zero $h_0 \in \mathcal{H}$ such that $DVh_0 = 0$ but $Vh_0 \neq 0$. This implies, $D\pi(X)Vh_0 = 0, \forall\ X \in \mathcal{A}$. Recalling that $(\mathcal{K}, \pi, V)$ is a minimal Stinespring triple for the quasi-pure CP map ϕ, $\mathcal{K} = \overline{\text{span}}\{\pi(X)Vh : X \in \mathcal{A}, h \in \mathcal{H}\} = \overline{\text{span}}\{\pi(X)Vh_0 : X \in \mathcal{A}\}$. Thus $D = 0$. Consequently $\alpha = 0$. This contradicts the hypothesis that α is non-zero. Therefore, $ker(\alpha(1)) = ker(\phi(1))$.

Now to prove the other implication, assume that the CP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is not quasi-pure, then we have $h_0 \in \mathcal{H}$ such that $0 \neq Vh_0$ is not cyclic for π. Let $\widetilde{\mathcal{K}} = \overline{\text{span}}\{\pi(X)Vh_0 : X \in \mathcal{A}\}$ and let P be the projection of $\mathcal{K}$ onto this reducing subspace $\widetilde{\mathcal{K}}$. Set

$$\alpha(X) = V^*\pi(X)(I - P)V, \ X \in \mathcal{A}.$$

Then $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defines a CP map dominated by ϕ. Clearly, $\alpha(1)h_0 = 0$ as $Vh_0 \in \widetilde{\mathcal{K}}$. But $\phi(1)h_0 = V^*Vh_0 \neq 0$ as $\langle h_0, V^*Vh_0\rangle = \|Vh_0\|^2 \neq 0$. Hence $\ker(\alpha(1)) \neq \ker(\phi(1))$.

■

As an immediate consequence of this theorem we have the following corollary.

Corollary 4.1.4. Suppose $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a quasi-pure CP map and $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a CP map dominated by ϕ. Then α is quasi-pure.

Proof. Clear from Theorem 4.1.3. ■

For CP maps on matrix algebras quasi-purity can be described in terms of Choi-Kraus coefficients.

Theorem 4.1.5. Let $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ be a CP map with a minimal Choi-Kraus decomposition:

$$\phi(X) = \sum_{j=1}^{k} L_j^* X L_j, \quad X \in M_{d_1}(\mathbb{C}) \tag{4.1.2}$$

where $k \in \mathbb{N}$ and $L_1, L_2, \ldots, L_k$ are $d_1 \times d_2$ matrices. Then ϕ is *quasi-pure iff the collection* $\{L_1h_0, L_2h_0, \ldots, L_kh_0\}$ *is linearly independent, whenever* $L_ih_0 \neq 0$ *for some* i *in* $\{1, 2, \ldots, k\}$ *and* $h_0 \in \mathbb{C}^{d_2}$.

Proof. Here ϕ has the minimal Stinespring representation, $(\mathcal{K}, \pi, V)$ where $\mathcal{K} = \mathbb{C}^{d_1} \otimes \mathbb{C}^k$, $\pi(X) = X \otimes I_{\mathbb{C}^k}$ and $V : \mathbb{C}^{d_2} \to \mathcal{K}$, is defined by $Vh = \sum_{j=1}^{k} L_jh \otimes e_j$, where $e_1, e_2, \ldots, e_k$ is an orthonormal basis of $\mathbb{C}^k$. Assume that there exists a nonzero $h_0 \in \mathbb{C}^{d_2}$ such that the collection $\{L_1h_0, L_2h_0, \ldots, L_kh_0\}$ is linearly dependent, and $L_ih_0 \neq 0$ for some i in $\{1, 2, \ldots, k\}$. Then Vh_0 is non-zero and there exist complex numbers a_j's (not all zero) such that

$$\sum_{j=1}^{k} a_j L_j h_0 = 0. \tag{4.1.3}$$

Let $\widetilde{\mathcal{K}} = \overline{\text{span}}\{\pi(X)Vh_0 : X \in M_{d_1}(\mathbb{C})\} = \overline{\text{span}}\{\sum_{j=1}^{k} X(L_jh_0) \otimes e_j : X \in M_{d_1}(\mathbb{C})\}$, a subspace of the minimal dilation space, $\mathbb{C}^{d_1} \otimes \mathbb{C}^k$. Now the condition (4.1.3) implies that every vector in $\widetilde{\mathcal{K}}$ is orthogonal to vectors of the form $\sum_{j=1}^{k} \bar{a}_j v \otimes e_j$ for any $v \in \mathbb{C}^{d_1}$. In particular, $\widetilde{\mathcal{K}}$ is not whole of the dilation space and ϕ is not quasi-pure.

Now to prove the other implication, suppose h_0 is an element of $\mathbb{C}^{d_2}$ such that the collection $\{L_1h_0, L_2h_0, \ldots, L_kh_0\}$ is a linearly independent subset of $\mathbb{C}^{d_1}$. Extend it to a basis $\{f_1, f_2, \ldots, f_{d_1}\}$ of $\mathbb{C}^{d_1}$, where $f_i = L_ih_0, i = 1, 2, \ldots, k$. Fix a basis $\{e_1, e_2, \ldots, e_k\}$ for $\mathbb{C}^k$. Define linear maps $X_{ir} \in M_{d_1}(\mathbb{C})$ by

$$X_{ir}(f_j) = \delta_{r,j} f_i : i, j, r \in \{1, 2, \ldots, d_1\},$$

where δ is the Kronecker delta function. Now

$$\begin{aligned}
\mathbb{C}^{d_1} \otimes \mathbb{C}^k &\supseteq \overline{\text{span}}\{\pi(X)Vh_0 : X \in M_{d_1}(\mathbb{C})\} \\
&= \overline{\text{span}}\{\sum_{j=1}^{k} X(L_jh_0) \otimes e_j : X \in M_{d_1}(\mathbb{C})\} \\
&= \{\sum_{j=1}^{k} Xf_j \otimes e_j : X \in M_{d_1}(\mathbb{C})\} \\
&\supseteq \{\sum_{j=1}^{k} X_{ir}f_j \otimes e_j : 1 \leq i, r \leq d_1\} \\
&= \{f_i \otimes e_j : 1 \leq i \leq d_1, 1 \leq j \leq k\} \\
&= \mathbb{C}^{d_1} \otimes \mathbb{C}^k.
\end{aligned}$$

Therefore, Vh_0 is a cyclic vector for the representation π, as required. ■

Recall that the Choi rank of a CP map ϕ on matrix algebras is the minimum number of L_j's needed in expressing it in the form $\phi(X) = \sum_j L_j^* X L_j$. The Choi rank of a CP map $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ is at most $d_1 d_2$.

Corollary 4.1.6. Let $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ be a quasi-pure CP map. Then the Choi rank of ϕ is at most d_1.

Proof. This follows from Theorem 4.1.5 due to linear independence of $\{L_1 h_0, L_2 h_0, \dots, L_k h_0\}$ in $\mathbb{C}^{d_1}$. ■

Corollary 4.1.7. Let $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ be a quasi-pure CP map with Choi-Kraus decomposition as in (4.1.2). Then $\ker(L_i) = \ker(L_j)$ for $1 \leq i, j \leq k$.

Proof. From Theorem 4.1.5, if $L_i h_0 \neq 0$ for some i, then $L_j h_0 \neq 0$ for all j. ■

A minimal Choi-Kraus decomposition of the CP map of Example 4.1.2 is given by,

$$\phi(X) = \sum_{j=1}^{k} L_j^* X L_j, \quad X \in M_{d_1}(\mathbb{C}),$$

where $L_j = |\sqrt{p_j} u_j\rangle\langle v|$, is the map $w \mapsto \sqrt{p_j} u_j \langle v, w\rangle$. This shows that the Choi rank of a quasi-pure map from $M_{d_1}(\mathbb{C})$ to $M_{d_2}(\mathbb{C})$ can be any number in $\{1, 2, \dots, d_1\}$.

A CP map $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ is defined to be *entanglement breaking* (EB) if it has a (not-necessarily minimal) Choi-Kraus decomposition of the form $\phi(X) = \sum_{j=1}^{k} L_j^* X L_j$, where L_j's are rank one operators. Entanglement breaking maps have a special role in quantum information theory and there is plenty of literature on the same. See the influential paper of Horodecki, Shor and Ruskai [HSR03], and its references as well as citations for further information. It is clear from this definition that the quasi-pure CP map considered in Example 4.1.2 is an EB map. Conversely, from Theorem 4.1.3 or from Theorem 4.1.5 it follows that every quasi-pure EB map $\phi : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ is necessarily of the form 4.1.1 for some positive matrix $\rho \in M_{d_1}(\mathbb{C})$ and unit vector $v \in \mathbb{C}^{d_2}$.

4.2 Minimal completion theorem

Consider a 2×2 block operator matrix of operators on Hilbert spaces,

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assume that A is strictly positive, that is, it is positive and invertible. Then N is positive iff $C = B^*$ and $D \geq CA^{-1}B$ (See Lemma 1.4.1 of [BB23]). Now consider a partially given block operator matrix:

$$M = \begin{bmatrix} A & * \\ C & * \end{bmatrix},$$

where $*$ indicates that the particular entry hasn't been specified. Now it is clear that if A is strictly positive, then M can always be completed to a positive block operator matrix and in fact there is a minimal positive completion given by

$$\begin{bmatrix} A & C^* \\ C & CA^{-1}C^* \end{bmatrix}.$$

Little more care is needed when A is positive but not strictly positive.

Lemma 4.2.1. Let $\mathcal{K}$ be a proper closed subspace of a Hilbert space $\mathcal{H}$. Suppose $A \in \mathcal{B}(\mathcal{K}), C \in \mathcal{B}(\mathcal{K}, \mathcal{K}^{\perp})$ are given.Then there exists a positive operator, $M \in \mathcal{B}(\mathcal{H})$ such that with respect to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$, it has the form:

$$M = \begin{bmatrix} A & * \\ C & * \end{bmatrix},$$

where $*$ denotes unspecified entries, if and only if there exists $q > 0$ such that

$$C^*C \leq qA.$$

Moreover, in such a case, there exists unique minimal D such that $\begin{bmatrix} A & C^* \\ C & D \end{bmatrix}$ is positive.

Proof. Suppose

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

is a positive operator. Then by a standard result (See Lemma 1.5.1 of [BB23]), A, D are positive, and there exists a contraction $K \in \mathcal{B}(\mathcal{K}, \mathcal{K}^{\perp})$ such that $C = D^{\frac{1}{2}} K A^{\frac{1}{2}}$. Therefore,

$$C^*C = A^{\frac{1}{2}} K^* D K A^{\frac{1}{2}} \leq qA$$

where $q = \|K^*DK\|$. Conversely, suppose $C^*C \leq qA$ for some $q > 0$. Let $C = V|C|$ be the polar decomposition of C. Then

$$\begin{bmatrix} A & C^* \\ C & qVV^* \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{q}}|C| & 0 \\ 0 & \sqrt{q}I \end{bmatrix} \cdot \begin{bmatrix} I & V^* \\ V & VV^* \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{q}}|C| & 0 \\ 0 & \sqrt{q}I \end{bmatrix} + \begin{bmatrix} A - \frac{1}{q}C^*C & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

As V is a partial isometry $qVV^* \leq qI$. By functional calculus for $0 < s < t < \infty$, $(A + s), (A + t)$ are invertible and $(A + s)^{-1} \geq (A + t)^{-1}$. Also,

$$\begin{bmatrix} A + t & C^* \\ C & qI \end{bmatrix} \geq 0$$

implies that

$$C(A + t)^{-1}C^* \leq qI.$$

This follows from the standard Schur complement argument for block operator matrices:

$$\begin{bmatrix} I & 0 \\ -C(A+t)^{-1} & I \end{bmatrix} \begin{bmatrix} A + t & C^* \\ C & qI \end{bmatrix} \begin{bmatrix} I & -(A+t)^{-1}C^* \\ 0 & I \end{bmatrix} = \begin{bmatrix} A + t & 0 \\ 0 & qI - C(A+t)^{-1}C^* \end{bmatrix}.$$

Therefore, as $t \downarrow 0$, the family

$$C(A + t)^{-1}C^*$$

is monotonically increasing and bounded above by qI. Consequently,

$$C(A + t)^{-1}C^* \uparrow D$$

in the strong operator topology for some positive bounded operator D.

Observe that, since

$$C(A + t)^{-1}C^* \leq D$$

for every $t > 0$, we have

$$\begin{bmatrix} A + t & C^* \\ C & D \end{bmatrix} \geq 0.$$

Taking the strong operator topology limit as $t \downarrow 0$, it follows that

$$\begin{bmatrix} A & C^* \\ C & D \end{bmatrix} \geq 0.$$

Now suppose there exists another operator D_1 such that

$$\begin{bmatrix} A & C^* \\ C & D_1 \end{bmatrix} \geq 0.$$

Then, again by the Schur complement argument,

$$C(A+t)^{-1}C^* \leq D_1$$

for every $t > 0$. Passing to the strong operator topology limit as $t \downarrow 0$, we obtain

$$D \leq D_1.$$

Thus, D is the unique minimal solution. ■

These observations lead to some necessary conditions for the existence of CP completion of linear maps. Here and elsewhere, by $\mathcal{A}_+$ we mean positive elements of $\mathcal{A}$.

Theorem 4.2.2. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Fix a projection $R \in \mathcal{B}$. Let $\beta : \mathcal{A} \to \mathcal{B}.R$ be a linear map. If β is CP completable then the following properties hold:

(i) The map $X \mapsto R\beta(X)$ is completely positive;

(ii) There exists $q \geq 0$ such that,

$$\beta(X)^*(1-R)\beta(X) \leq q\|X\|R\beta(X), \quad \forall X \in \mathcal{A}_+.$$

(iii) For $X = [X_{ij}]_{1\leq i,j\leq n}$ in $M_n(\mathcal{A})_+$, $[\beta(X_{ij})]_{1\leq i,j\leq n}$ in $M_n(\mathcal{B}R)$ is positive completable to an operator in $M_n(\mathcal{B})$.

Proof. Here $\beta : \mathcal{A} \to \mathcal{B}.R$ is given to be CP completable. Let, $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be one such completion for β, i.e $\beta(X) = \phi(X)R, \ \ \forall X \in \mathcal{A}$.

(i) Then the map $X \mapsto R\beta(X) = R\phi(X)R$ is completely positive.

(ii) For every $X \in \mathcal{A}_+$, with respect to the decomposition $\mathcal{H} = R(\mathcal{H}) \oplus (I-R)(\mathcal{H})$,

$$\phi(X) = \begin{bmatrix} R\phi(X)R & R\phi(X)(1-R) \\ (1-R)\phi(X)R & (1-R)\phi(X)(1-R) \end{bmatrix},$$

is positive. Now as in the proof of Lemma 4.2.1,

$$R\phi(X)^*(1-R)\phi(X)R \leq \|(1-R)\phi(X)(1-R)\|R\phi(X)R \leq q\|X\|R\phi(X)R,$$

where $q = \|\phi\|$. Hence we have,

$$\beta(X)^*(1-R)\beta(X) \leq q\|X\|R\beta(X), \quad \forall X \in \mathcal{A}_+.$$

(iii) For $X = [X_{ij}]_{1\leq i,j\leq n} \geq 0$ in $M_n(\mathcal{A})$, $[\phi(X_{ij})]_{1\leq i,j\leq n}$ in $M_n(\mathcal{B})$ is positive and $[\beta(X_{ij})]_{1\leq i,j\leq n} = [\phi(X_{ij})R]_{1\leq i,j\leq n} = [\phi(X_{ij})]_{1\leq i,j\leq n}(1 \otimes R)$. Therefore, $[\beta(X_{ij})]_{1\leq i,j\leq n}$ is positive completable. ■

In the following result we observe that the condition (iii) of this theorem is also sufficient in some special cases.

Theorem 4.2.3. Let $\mathcal{A} = M_d(\mathbb{C})$ for some $d \geq 1$. Let $\mathcal{B}$ be a C^*-algebra with $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Fix a projection $R \in \mathcal{B}$. Let $\beta : \mathcal{A} \to \mathcal{B}.R$ be a linear map. Then β is CP completable if and only if $[\beta(E_{ij})]_{1 \leq i,j \leq d}$ can be completed to a positive matrix of $M_d(\mathcal{B})$ where $E_{i,j}, 1 \leq i,j \leq d$ are the matrix units of $M_d(\mathbb{C})$.

Proof. If $[Y_{ij}]$ is a positive completion of $[\beta(E_{ij})]$, define a linear map $\phi : M_d \to \mathcal{B}$ by setting $\phi(E_{ij}) = Y_{ij}, 1 \leq i,j \leq d$. Then by a well-known result of Choi (See [Cho75]), ϕ defines a completely positive map. By linearity it is clear that ϕ is a completion of β. The converse has been observed in the previous Theorem. ∎

So far we do not have a general necessary and sufficient condition for CP completability. However, the following result shows that if a linear map admits a CP completion then there is a unique minimal CP completion. This is the main result of this Section.

Theorem 4.2.4. (Minimal Completion theorem): Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Fix $R \in \mathcal{B}(\mathcal{H})$. Let $\beta : \mathcal{A} \to \mathcal{B}.R$ be CP completable. Then there exists a unique CP completion α such that if ψ is any CP completion of β then α is dominated by ψ.

Proof. Let $\phi_i : \mathcal{A} \to \mathcal{B}$ be CP completions of β for $i = 1, 2$. Let $(\mathcal{K}_i, \pi_i, V_i)$ be the unique minimal Stinespring representations of ϕ_i, so that

$$\begin{aligned} \phi_i(X) &= V_i^* \pi_i(X) V_i, \quad X \in \mathcal{A}; \\ \mathcal{K}_i &= \overline{\text{span}}\{\pi_i(X) V_i h : X \in \mathcal{A}, h \in \mathcal{H}\}, \end{aligned}$$

for $i = 1, 2$. Now take $\widetilde{\mathcal{K}_i} = \overline{\text{span}}\{\pi_i(X) V_i R h : X \in \mathcal{A}, h \in \mathcal{H}\}$. Note that $\widetilde{\mathcal{K}_i}$ is a reducing subspace for the representation π_i. Let Q_i be the projection of $\mathcal{K}_i$ to $\widetilde{\mathcal{K}_i}$. Set

$$\alpha_i(X) = V_i^* \pi_i(X) Q_i V_i, \ X \in \mathcal{A}.$$

Then α_i is a CP map dominated by ϕ_i. Also $\alpha_i(X)R = \phi_i(X)R = \beta(X)$. Hence it is a completion of β. We need to show that α_i is independent of i. Define $U : \widetilde{\mathcal{K}_1} \to \widetilde{\mathcal{K}_2}$ by setting

$$U \pi_1(X) V_1 R h = \pi_2(X) V_2 R h, X \in \mathcal{A}, h \in \mathcal{H}.$$

By direct computation, U is isometric as:

$$\begin{aligned} \langle \pi_i(X) V_i R g, \pi_i(Y) V_i R h \rangle &= \langle R g, V_i^* \pi_i(X^* Y) V_i R h \rangle \\ &= \langle R g, \phi_i(X^* Y) R h \rangle \\ &= \langle R g, \beta(X^* Y) R h \rangle, \end{aligned}$$

for every $X, Y \in \mathcal{A}, g, h \in \mathcal{H}$. From the definition of $\widetilde{\mathcal{K}_1}, \widetilde{\mathcal{K}_2}$ and U, U extends to a unitary and satisfies the following identity :

$$U \pi_1(X) Q_1 = Q_2 \pi_2(X) U, \ \forall \ X \in \mathcal{A}.$$

Using the definition of U, and the fact that $\beta(X)h = \phi_1(X)R = \phi_2(X)R$, we have,

$$\beta(X)h = V_2^* Q_2 \pi_2(X) V_2 R h = V_2^* Q_2 U \pi_1(X) V_1 R h = V_1^* Q_1 \pi_1(X) V_1 R h, \forall X \in \mathcal{A}, h \in \mathcal{H}.$$

Since the collection of vectors of the form $\pi_1(X) V_1 R h$ is total in $\widetilde{K}_1$, we get $V_2^* Q_2 U = V_1^* Q_1$ or equivalently $U^* Q_2 V_2 = Q_1 V_1$. Hence $Q_2 V_2 = U Q_1 V_1$. Now, for all $X \in \mathcal{A}$,

$$\alpha_2(X) = V_2^* Q_2 \pi_2(X) Q_2 V_2 = V_2^* Q_2 \pi_2(X) U Q_1 V_1 = V_2^* Q_2 U \pi_1(X) Q_1 V_1 = V_1^* Q_1 \pi_1(X) Q_1 V_1 = \alpha_1(X).$$

∎

Instead of assuming that $\mathcal{B}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and using the Stinespring representation, we could have used the Hilbert C^*-module language and Paschke's version of Stinespring's theorem (See [Pas73]) in this proof. We have not opted for such an approach to keep the presentation more accessible.

4.3 Almost everywhere equivalence for CP maps

Generalizing the notion of equal almost everywhere with respect to a state for CP maps, we have the following definition.

Definition 4.3.1. Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{H}$ be a Hilbert space. Let ϕ, ψ be linear maps from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Suppose $R \in \mathcal{B}(\mathcal{H})$. Then ψ is said to be R-equivalent to ϕ if

$$\phi(X)R = \psi(X)R, \quad \forall X \in \mathcal{A}. \tag{4.3.1}$$

This is denoted by $\phi \underset{R}{=} \psi$.

Comparing with Definition 5.2.1, ϕ is equal almost everywhere to ψ with respect to a CP map ξ, if and only if $\phi \underset{P}{=} \psi$ where P is the support projection of ξ. Note that $\phi \underset{R}{=} 0$ if and only if $\phi(X)R = 0$ for every $X \in \mathcal{A}$.

Theorem 4.3.2. Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{H}$ be a Hilbert space and let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then for any $R \in \mathcal{B}(\mathcal{H})$, there exists a unique CP map $\alpha := \alpha_R$ with following two decomposition properties:

1. ϕ decomposes as $\phi = \alpha + \phi_1$ where α, ϕ_1 are CP maps satisfying
$$\phi \underset{R}{=} \alpha, \quad \phi_1 \underset{R}{=} 0. \tag{4.3.2}$$
2. If $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is any CP map such that $\phi \underset{R}{=} \psi$, then ψ decomposes as $\psi = \alpha + \psi_1$ with these CP maps satisfying
$$\phi \underset{R}{=} \alpha \underset{R}{=} \psi, \quad \phi_1 \underset{R}{=} \psi_1 \underset{R}{=} 0. \tag{4.3.3}$$

Proof. Define $\beta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by $\beta(X) = \phi(X)R$. Now β is CP completable with ϕ being one such completion. Hence Theorem 4.2.4 is applicable. Let $(\mathcal{K}, \pi, V)$ be a minimal Stinespring representation of ϕ. Take $\widetilde{\mathcal{K}} = \overline{\text{span}}\{\pi(X)VRh : X \in \mathcal{A}, h \in \mathcal{H}\}$. Let Q be the projection of $\mathcal{K}$ onto the reducing subspace $\widetilde{\mathcal{K}}$. Set

$$\alpha(X) = V^*\pi(X)QV, \ X \in \mathcal{A}.$$

Then α is a CP map dominated by ϕ. Now Theorem 4.2.4 implies that α is the minimal CP completion of β. Take $\phi_1 = \phi - \alpha$ and $\psi_1 = \psi - \alpha$. Now decomposition properties are easy verifications. The uniqueness follows from the uniqueness of minimal completion. ■

This leads to the main theorem of this Section. It shows some kind of rigidity of quasi-pure maps, in the sense that under some mild conditions any CP map which is R-equivalent to a quasi-pure CP map is the map itself.

Theorem 4.3.3. Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{H}$ be a Hilbert space. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely positive maps, where ϕ is quasi-pure and $\phi(I) = \psi(I)$. Suppose some $R \in \mathcal{B}(\mathcal{H})$ is given such that ϕ is $R-$equivalent to ψ, that is,

$$\phi(X)R = \psi(X)R, \quad \forall X \in \mathcal{A}. \tag{4.3.4}$$

Assume that there exists $X_0 \in \mathcal{A}$ such that $\phi(X_0)R \neq 0$. Then $\phi = \psi$.

Proof. We view ϕ, ψ as CP completions of the map β defined on $\mathcal{A}$ by $\beta(X) = \phi(X)R = \psi(X)R$. Let $(\mathcal{K}, \pi, V)$ be a minimal Stinespring representation of ϕ. The existence of X_0 as above implies that $\pi(X_0)VR \neq 0$, Hence the reducing subspace $\{\pi(X)VRh : X \in \mathcal{A}, h \in \mathcal{H}\}$ is non-trivial and the quasi-purity of ϕ implies that it is whole of $\mathcal{K}$. Therefore the minimal CP completion of the map β is same as ϕ. In particular, ϕ is dominated by ψ. Since $\phi(I) = \psi(I)$, it follows that $\phi = \psi$.

∎

In this theorem the condition of quasi-purity plays a very crucial role as the following example shows.

Example 4.3.4. Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{H}$ be a Hilbert space. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely positive map which is not quasi-pure. Then by Theorem 4.1.3 there exists a non-zero CP map α dominated by ϕ, with an $h_0 \in \mathcal{H}$ such that $\alpha(1)h_0 = 0$ and $\phi(1)h_0 \neq 0$.

Suppose we can choose $Z \in \mathcal{B}(\mathcal{H})$ such that $Zh_0 = 0$ and $Z^*\alpha(X_1)Z \neq \alpha(X_1)$ for some $X_1 \in \mathcal{A}$, then $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by

$$\psi(X) = Z^*\alpha(X)Z + (\phi - \alpha)(X), \quad X \in \mathcal{A},$$

satisfies $\phi(X)R = \psi(X)R$ where $R = |h_0\rangle\langle h_0|$, but ψ is different from ϕ. Typically, it is possible to choose such a Z, even with the additional restriction $Z^*\alpha(1)Z = \alpha(1)$ so that $\psi(1) = \phi(1)$ and all the conditions of the previous theorem are satisfied except quasi-purity of ϕ. However this may not be possible in some special cases as the following example demonstrates.

Example 4.3.5. Consider $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$\phi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a+d & b+c \\ b+c & a+d \end{bmatrix}.$$

Then ϕ is CP. It is not hard to see that ϕ is not quasi-pure.Take,

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easily seen that if ψ is a CP map satisfying $\phi(X)R = \psi(X)R$ for all X and $\psi(1) = \phi(1)$, then $\psi = \phi$. In other words, the conclusion of the previous theorem may hold for some R even though ϕ is not quasi-pure.

Corollary 4.3.6. Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{B}, \mathcal{C}$ be von Neumann algebras. Let $\xi : \mathcal{B} \to \mathcal{C}$ be a non-zero normal completely positive map. Let $\phi, \psi : \mathcal{A} \to \mathcal{B}$ be two completely positive maps. Suppose ϕ is a quasi-pure completely positive map, $\xi \circ \phi$ is a non-zero map, $\phi(I) = \psi(I)$ and $\phi \underset{\xi}{=} \psi$. Then $\phi = \psi$.

Proof. As $\mathcal{B}$ is a von Neumann algebra, $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In order to apply the previous results we may view ϕ, ψ as maps from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Let P be the support projection of ξ. Then by (4.0.1) $\mathcal{N}_\xi = \mathcal{B}(I-P)$. From $\phi \underset{\xi}{=} \psi$, we get $\phi(X) - \psi(X) \in \mathcal{B}(I-P)$ for all $X \in \mathcal{A}$. In particular, $(\phi(X) - \phi(X))P = 0$, or $\phi(X)P = \psi(X)P$ for all $X \in \mathcal{A}$. Now the result is immediate from the previous theorem. ∎

Remark 4.3.7. This corollary generalizes results of [PR23] in the following ways. The algebras are no longer just matrix algebras. The identity map has been replaced by quasi-pure maps and the states are replaced by general completely positive maps.

The following example shows that in the last result the condition of normality on the CP map ξ is not redundant.

Example 4.1. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathcal{K}(\mathcal{H})$ denote the algebra of compact operators. Consider the quotient map

$$\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

Fix a state ξ_0 on the *Calkin algebra* $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and define a state on $\mathcal{B}(\mathcal{H})$ by

$$\xi := \xi_0 \circ \pi.$$

This gives a *non-normal* state on $\mathcal{B}(\mathcal{H})$.

Now, let $Q \in \mathcal{B}(\mathcal{H})$ be a non-zero finite-rank projection, and define the pure completely positive map

$$\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), \quad \phi(X) = QXQ \quad \text{for all } X \in \mathcal{B}(\mathcal{H}).$$

Next, define another completely positive map

$$\psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), \quad \psi(X) = \frac{1}{2}\phi(X) + \frac{1}{2}\xi(X)Q.$$

Observe that $\phi(I) = \psi(I) = Q$, and for every $X \in \mathcal{B}(\mathcal{H})$,

$$\phi(X) - \psi(X) \in \mathcal{N}_\xi,$$

so that $\phi \underset{\xi}{=} \psi$.

However, evaluating at Q gives $\phi(Q) \neq \psi(Q)$, showing that ϕ and ψ are indeed distinct, even though they are equal almost everywhere with respect to ξ.

Chapter 5

Disintegration

Recent developments in the categorical formulation of probability theory have inspired a renewed study of its quantum counterparts. Within this framework, several classical constructions have been revisited through the lens of operator algebras, among which disintegration plays a fundamental role.

In classical probability, disintegration provides a systematic method of decomposing measures into conditional components possessing certain desirable properties. This notion has proved essential in diverse areas such as probability theory, optimal transport, variational analysis, and ergodic theory. In particular, within ergodic theory, the disintegration of a measure is closely tied to the ergodic decomposition of invariant measures—objects that capture the asymptotic behavior of dynamical systems [OV14]. The concept of disintegration, however, extends far beyond ergodic theory and appears throughout modern mathematics, including in probability [Par67; CP97; Kal02; Fre06], geometry [PR25], and related fields [BM17].

The idea of disintegrating a measure dates back to von Neumann's 1932 work [Neu32], and since then, several formulations and generalizations have appeared, for instance, in [DM78; CP97; Fre06]. Although we do not aim to delve deeply into the classical aspects here, the interested reader may refer to [Bog07; Fre06] for a comprehensive treatment of this topic and its developments within probability theory. The quantum analogue of disintegration, however, is considerably more subtle. It intertwines with key ideas such as conditional expectations, regular conditional probabilities, and perfect error-correcting codes—concepts central to quantum probability and quantum information theory. Among various attempts to formalize quantum disintegration [CJ19; Fri20; BFL11; Lei06], the formulation proposed by Parzygnat and Russo [PR23] provides a particularly natural and consistent framework in the operator-algebraic setting.

In this chapter, we begin by recalling the classical notion of disintegration, followed by the necessary preliminaries and a discussion of its non-commutative counterpart. We show that in the case of matrix algebras, the problem of disintegration reduces to the question of the existence of a left inverse. Building on this insight, we investigate the structure of left-invertible completely positive maps, providing a general structural characterization in Theorem 5.2.4 and Corollary 5.2.6. Finally, we extend the theory to the infinite-dimensional setting in Theorem 5.2.7, thereby formulating a quantum version of disintegration consistent with the classical theory.

5.1 Classical Disintegration

Before delving into the non-commutative disintegration framework proposed by Parzygnat and Russo, let us first revisit the classical notion of disintegration.

Definition 5.1.1 (Transition kernel)**.** Let $(X, \mathcal{O}(X))$ and $(Y, \mathcal{O}(Y))$ be two measurable spaces. A map $r : X \times \mathcal{O}(Y) \to [0, \infty]$ is called a *transition kernel* from X to Y if it satisfies:

(i) for each $x \in X$, the map $A \mapsto r(x, A)$ defines a positive measure on $(Y, \mathcal{O}(Y))$;

(ii) for each $A \in \mathcal{O}(Y)$, the map $x \mapsto r(x, A)$ is non-negative $\mathcal{O}(X)$-measurable.

In particular, if for each $x \in X$, the map $r_x(A) := r(x, A)$ defines a probability measure on $(Y, \mathcal{O}(Y))$, then r is called a *Markov kernel*.

Example 5.1.2. (i) Any measure μ on a measurable space $(Y, \mathcal{O}(Y))$ can be viewed as a transition kernel from a singleton space $\{*\}$ to $(Y, \mathcal{O}(Y))$ defined by $r(*, A) = \mu(A)$.

(ii) Every measurable map $f : (X, \mathcal{O}(X)) \to (Y, \mathcal{O}(Y))$ defines a transition kernel given by

$$r(x, A) = 1_A(f(x)).$$

Motivated by the notion of measure-preserving maps, the composition of transition kernels provides a natural generalization in the categorical formulation of probability theory. In fact, the definition of a measure-preserving map can be recovered as a special case of this composition.

Definition 5.1.3 (Composition of transition kernels)**.** Let $(X, \mathcal{O}(X))$, $(Y, \mathcal{O}(Y))$, and $(Z, \mathcal{O}(Z))$ be measurable spaces. Let $r : X \to Y$ and $s : Y \to Z$ be two transition kernels. Their composition $r \circ s : X \to Z$ is defined by

$$(r \circ s)(x, E) := \int_Y r_y(E)\, ds_x(y), \quad \text{for all } x \in X,\ E \in \mathcal{O}(Z). \tag{5.1.1}$$

The equation (5.1.1) is known as the *Chapman–Kolmogorov equation*.

The fact that this construction indeed defines a transition kernel follows from the monotone convergence theorem for measurable functions.

Definition 5.1.4 (μ-a.e. equality of transition kernels)**.** Let $(X, \mathcal{O}(X), \mu)$ and $(Y, \mathcal{O}(Y), \nu)$ be measurable spaces, and let $r, s : X \to Y$ be two transition kernels. We say that r and s are *equal μ-almost everywhere* (written $r \underset{\mu}{=} s$) if for every $A \in \mathcal{O}(Y)$ there exists a μ-null set $N_A \in \mathcal{O}(X)$ such that

$$r_x(A) = s_x(A), \quad \forall\, x \in X \setminus N_A,$$

where $\mu(N_A) = 0$.

Remark 5.1.5. In the preceding discussion, the transition kernel was regarded as a function on $X \times \mathcal{O}(Y)$. However, to align with the standard convention for measurable maps, we shall henceforth consider it as a map on $Y \times \mathcal{O}(X)$.

It is also important to note that, in the subsequent definitions, whenever we refer to the equality of two such maps, it is understood in the sense of equality of *kernels*, rather than equality of functions or measures in the conventional sense.

Below we present the definition of disintegration for classical probability in a more abstract framework, following the appendix of [PR23]; interested readers may also consult [Bog07; Cle+17] for further details.

Definition 5.1.6. [Disintegration] Let $(X, \mathcal{O}(X), \mu)$ and $(Y, \mathcal{O}(Y), \nu)$ be two measurable spaces, and let $f : X \to Y$ be a measurable, measure-preserving map; that is,

$$(f \circ \mu)(E) = \mu(f^{-1}(E)) = \nu(E), \qquad \forall\, E \in \mathcal{O}(Y),$$

where we view f as the deterministic kernel

$$f : X \times \mathcal{O}(Y) \to [0, \infty), \qquad (x, B) \mapsto \mathbf{1}_B(f(x)) = \mathbf{1}_{f^{-1}(B)}(x).$$

We say that μ *admits a disintegration over* ν *consistent with* f if there exists a *transition kernel*

$$r : Y \times \mathcal{O}(X) \to [0, \infty]$$

such that the following two conditions hold.

(i) **Disintegration formula** ($\mu = r \circ \nu$)**.** We view ν as the kernel

$$\nu : \{*\} \times \mathcal{O}(Y) \to [0, \infty], \qquad (\star, B) \mapsto \nu(B).$$

Then the composition $r \circ \nu : \{*\} \times \mathcal{O}(X) \to [0, \infty]$, given by

$$(r \circ \nu)(*, A) = \int_Y r_y(A)\, d\nu(y), \qquad A \in \mathcal{O}(X),$$

satisfies

$$\boxed{r \circ \nu = \mu, \quad \textit{i.e.,} \quad \mu(A) = \int_Y r_y(A)\, d\nu(y), \qquad \forall\, A \in \mathcal{O}(X),}$$

where we write $r_y := r(y, \cdot)$ for the measure on X at $y \in Y$.

(ii) **Fiber support condition** $\Big(f \circ r \underset{\nu}{=} \mathrm{id}_Y\Big)$**.** The composition $f \circ r : Y \times \mathcal{O}(Y) \to [0, \infty]$, given by

$$(f \circ r)(y, B) = \int_X \mathbf{1}_{f^{-1}(B)}(x)\, dr_y(x) = r_y(f^{-1}(B)) = r(y, f^{-1}(B)),$$

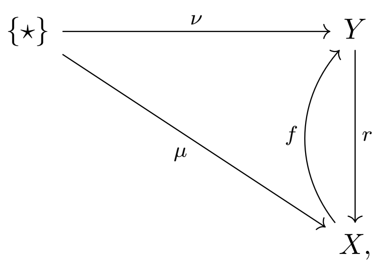
must equal the identity kernel $\mathrm{id}_Y : Y \times \mathcal{O}(Y) \to [0, \infty]$, $(y, B) \mapsto \mathbf{1}_B(y)$, ν-almost everywhere:

$$\boxed{f \circ r \underset{\nu}{=} \mathrm{id}_Y, \quad \textit{i.e.,} \quad r(y, f^{-1}(B)) = \mathbf{1}_B(y) \quad \nu\text{-a.e. in } y, \quad \forall\, B \in \mathcal{O}(Y).}$$

Equivalently, the measure r_y is concentrated on the fiber $f^{-1}(y) = \{x \in X : f(x) = y\}$ for ν-almost every $y \in Y$:

$$r_y\big(f^{-1}(y)\big) = 1 \qquad \nu\text{-a.e.}$$

In the language of Markov kernels:



where the triangle commutes ($r \circ \nu = \mu$) and $f \circ r = \mathrm{id}_Y$ holds ν-almost everywhere.

Remark 5.1.7. The measure-preserving assumption on f is not independent: it is in fact a *consequence* of the existence of a disintegration. Indeed, setting $A = f^{-1}(B)$ in condition (i) and applying condition (ii):

$$\begin{aligned} \mu(f^{-1}(B)) &= \int_Y r(y, f^{-1}(B))\, d\nu(y) && \text{(condition (i))} \\ &= \int_Y \mathbf{1}_B(y)\, d\nu(y) && \text{(condition (ii), } \nu\text{-a.e.)} \\ &= \nu(B), \end{aligned}$$

recovering $(f \circ \mu)(B) = \nu(B)$ for all $B \in \mathcal{O}(Y)$. This explains why f is assumed to be measure preserving.

The following theorem, concerning the disintegration of the underlying measure along its fibres for Polish spaces, follows immediately from [Bog07, Proposition 10.4.12].

Theorem 5.1.8. Let $(X, \mathcal{O}(X), \mu)$ be a Polish probability space and $(Y, \mathcal{O}(Y), \nu)$ a probability space. Suppose that $f : X \to Y$ is a measurable, measure-preserving map, that is, $\nu = \mu \circ f^{-1}$. Then there exists a disintegration $r : Y \to X$ of μ over ν consistent with f. Moreover, if $r' : Y \to X$ is another disintegration of μ over ν, then $r = r'$ almost everywhere with respect to ν.

5.2 Non-commutative Disintegration

To introduce non-commutative disintegration, we need the concept of equal almost everywhere (a.e.) for completely positive maps with respect to a given state. We recall this notion from the Chapter 4 (Definition 2.9, [PR23]).

Definition 5.2.1. Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras, and let $\xi : \mathcal{B} \to \mathbb{C}$ be a state. Two linear maps $\phi, \psi : \mathcal{A} \to \mathcal{B}$ are said to be *equal almost everywhere (a.e.)* with respect to ξ if $\phi(X) - \psi(X) \in \mathcal{N}_\xi$ for every $X \in \mathcal{A}$, where $\mathcal{N}_\xi$ denotes the null-ideal of the state ξ. This relation is denoted by $\phi \underset{\xi}{=} \psi$.

Next, we present the definition of non-commutative probability spaces, which form the foundational setting for the operator-algebraic formulation of quantum probability.

Definition 5.2.2 (Quantum probability space)**.** A *non-commutative* (or *quantum*) probability space is a pair $(\mathcal{A}, \omega)$, where $\mathcal{A}$ is a C^*-algebra and $\omega : \mathcal{A} \to \mathbb{C}$ is a state on $\mathcal{A}$.

With the preliminaries in place, we now formally introduce the definition of non-commutative disintegration (Definition 4.1 in [PR23]).

Definition 5.2.3 (Non-commutative Disintegration)**.** Let $(\mathcal{A}, \xi)$ and $(\mathcal{B}, \omega)$ be quantum probability spaces, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a state-preserving (i.e. $\omega \circ \phi = \xi$) UCP map. A *disintegration* of ω over ξ consistent with ϕ is a UCP map $\psi : \mathcal{B} \to \mathcal{A}$ satisfying the following:

(i) ψ is *state-preserving*, i.e. $\xi \circ \psi = \omega$ and

(ii) $\psi \circ \phi \underset{\xi}{=} Id_{\mathcal{A}}$.

Following this definition, a disintegration theorem was established in the matrix algebra case (Theorem 4.3 in [PR23]), and it was also shown that the classical case involving finite sets arises as a natural restriction of their theorem to commutative quantum spaces. Using the next result, we will extend their theory to the infinite-dimensional setting.

As an application of the Corollary 4.3.6 in Chapter 4, we can conclude that whenever the identity map becomes a quasi-pure completely positive (CP) map, the problem of finding a disintegration is equivalent to finding a left inverse of the CP map. In the following, we extend our analysis to the question of existence of a left inverse for normal CP maps on $\mathcal{B}(\mathcal{H})$. We approach the problem from a more general perspective and then recover the left inverse case as a special instance. Our technique is inspired by the approach used in Theorem 2.1 of [NS07], where the authors derived the structure of left-invertible completely positive trace-preserving maps. However, in the present context, we describe the structure of both maps involved in the invertibility relation in a more general setting.

Given a normal completely positive (CP) map $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, consider the map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{P}), \phi(X) = \tau(X) \otimes I_{\mathcal{P}}$ for some Hilbert space $\mathcal{P}$. Then we can easily recover τ from ϕ as follows. Take $\psi : \mathcal{B}(\mathcal{H}\otimes\mathcal{P}) \to \mathcal{B}(\mathcal{H})$ defined by $\psi(A\otimes B) = \mathrm{Tr}(\rho B)A, \forall A \in \mathcal{B}(\mathcal{H}),\ B \in \mathcal{B}(\mathcal{P})$, where ρ is a fixed density operator on $\mathcal{P}$. Then $\tau = \psi\circ\phi$. Thus, whenever τ arises as the compression of a left–invertible CP map, it admits such a recovery. Interestingly, if $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a normal pure CP map, the converse also holds: whenever such a map admits a decomposition into two normal CP maps, these constituent maps necessarily exhibit a structure analogous to ϕ and ψ described above. In essence, ϕ appears as a composition of a left–invertible CP map with τ, up to a suitable distortion, while ψ functions as the left inverse of the corresponding $*$-homomorphism. This structural characterization forms the main theme of the following theorem.

Theorem 5.2.4. Let $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be normal completely positive maps. Assume that τ is pure, $\tau(I_{\mathcal{H}})$ is a projection and ϕ is unital. Suppose there exist a normal completely positive map $\psi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ such that

$$\psi \circ \phi = \tau.$$

Then there exist Hilbert spaces $\mathcal{H}_0, \mathcal{P}, \mathcal{Q}$, and a unitary operator

$$U : (\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q} \longrightarrow \mathcal{K}$$

such that, for all $X \in \mathcal{B}(\mathcal{H})$,

$$U^*\phi(X)U = (\tau(X) \otimes I_{\mathcal{P}}) \oplus \eta(X),$$

where $\mathcal{H}_0 = \tau(I_{\mathcal{H}})\mathcal{H} \subseteq \mathcal{H}$, and $\eta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{Q})$ is a normal unital completely positive map. (Note: Since $\tau(I_{\mathcal{H}})$ is a projection, the range of τ is naturally contained in $\mathcal{B}(\mathcal{H}_0)$. Accordingly, viewing $\tau(X)$ as an operator on $\mathcal{H}_0$ within the tensor product involves only a mild abuse of notation.)

Furthermore, let $P : (\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q} \to \mathcal{H}_0 \otimes \mathcal{P}$ denote the projection. Then $\psi(UYU^*) = \psi(UPYPU^*)$. Moreover, there exists a density operator $\rho \in \mathcal{B}(\mathcal{P})$ such that

$$\psi\big(U(A \otimes B)U^*\big) = \mathrm{Tr}(\rho B)\,\tau(I_{\mathcal{H}})A\tau(I_{\mathcal{H}}), \quad \forall\, A \in \mathcal{B}(\mathcal{H}_0),\ B \in \mathcal{B}(\mathcal{P}).$$

Proof. Let ϕ, ψ, τ be normal CP maps with minimal Choi–Kraus representations as discussed in Corollary 1.2.16 and Remark 1.2.17,

$$\phi(X) = \sum_{i\in I} L_i X L_i^*, \qquad \psi(X) = \sum_{j\in J} M_j X M_j^*, \qquad \tau(X) = KXK^*,$$

where X denotes a bounded operator on the appropriate Hilbert space, $L_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ for each $i \in I$, $M_j \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ for each $j \in J$, and $K \in \mathcal{B}(\mathcal{H})$. (Here I and J are the index sets of the minimal Kraus decompositions; they may be finite or countable.)

Let $P_0 = \tau(1_{\mathcal{H}})$, which is the projection onto $\mathcal{H}_0 \subseteq \mathcal{H}$. As $\tau = \psi \circ \phi$ there exist scalars $\{\alpha_{i,j}\}$ such that

$$M_j L_i = \alpha_{i,j} K \quad \text{and} \quad \sum_{i\in I, j\in J} |\alpha_{i,j}|^2 = 1.$$

From this, we have $M_j L_i L_i^* M_{j'}^* = \alpha_{i,j}\overline{\alpha_{i,j'}} KK^*$. Define

$$\beta_{j,j'} := \sum_{i\in I} \alpha_{i,j}\overline{\alpha_{i,j'}} \quad \forall\, j, j' \in J.$$

Then, for every $j, j' \in J$,

$$\begin{aligned}\beta_{j,j'} P_0 &= \sum_{i\in I} \alpha_{i,j}\overline{\alpha_{i,j'}} KK^*,\\ &= \sum_{i\in I} M_j L_i L_i^* M_{j'}^*\\ &= M_j M_{j'}^*,\end{aligned}$$

since ϕ is unital. Let $\mathcal{P}'$ be a separable Hilbert space with orthonormal basis $\{e_n : n \in J\}$.

$$T : \mathcal{P}' \to \mathcal{P}' \quad \text{such that} \quad \langle e_j, T(e_{j'})\rangle = \beta_{j,j'} \quad \forall\, j, j' \in J.$$

Since $\sum_{i\in I, j\in J} |\alpha_{i,j}|^2 = 1$, it follows that $T \in \mathcal{B}(\mathcal{P}')$. Moreover, the conditions $\beta_{j,j} \geq 0$ and $\sum_{j\in J} \beta_{j,j} = 1$ imply that T is a density operator. By the spectral theorem, there exists a unitary operator V and a diagonal operator $D \in \mathcal{B}(\mathcal{P}')$ such that $VTV^* = D$, and $D(e_j) = \lambda_j e_j$, $\forall\, j \in J$, with $\lambda_j \geq 0, \sum_j \lambda_j = 1$.

Define $N_k = \sum_{j\in J} v_{kj} M_j$, where $v_{kj} = \langle e_k, V e_j\rangle$. Then we have

$$\begin{aligned}N_k N_{k'}^* &= \sum_{j\in J} v_{kj} M_j \sum_{j'\in J} \overline{v}_{k'j'} M_{j'}^*\\ &= \sum_{j,j'\in J} v_{kj}\overline{v}_{k'j'} (M_j M_{j'}^*)\\ &= \sum_{j,j'\in J} v_{kj}\beta_{j,j'}\overline{v}_{k'j'}\, P_0\\ &= D_{kk'}\, P_0\\ &= \delta_{kk'}\lambda_k\, P_0,\end{aligned}$$

where $\delta_{kk'}$ denotes the Kronecker delta. In particular, $N_k = 0$ if $\lambda_k = 0$. Since V is a unitary in $\mathcal{B}(\mathcal{P}')$, a direct computation yields, for every $X \in \mathcal{B}(\mathcal{K})$,

$$\sum_{k\in\mathcal{O}} N_k X N_k^* = \psi(X),$$

where $\mathcal{O} = \{k \in J : \lambda_k \neq 0\}$. Let $\mathcal{P}$ denote the subspace $\ker(T)^\perp$ of $\mathcal{P}'$, and $\rho = T|_{\mathcal{P}}$. Define, $f_k = V^* e_k$ for $k \in \mathcal{O}$. Note that $\{f_k : k \in \mathcal{O}\}$ is an orthonormal basis of $\mathcal{P}$. Define an operator $W : \mathcal{H}_0 \otimes \mathcal{P} \to \mathcal{K}$, by $W(x \otimes f_i) = \frac{1}{\sqrt{\lambda_i}} N_i^*(x)$, then $\langle W(x\otimes f_i), W(y \otimes f_j)\rangle = \frac{1}{\sqrt{\lambda_i\lambda_j}}\langle N_j N_i^*(x), y\rangle = \langle x\otimes f_i, y\otimes f_j\rangle$ for every $x, y \in \mathcal{H}_0$ and $i, j \in \mathcal{O}$. Thus W is an isometric embedding of $\mathcal{H}_0 \otimes \mathcal{P}$ into $\mathcal{K}$. Therefore, we obtain the unitary operator

$$U := W \oplus I_{\mathcal{Q}} : (\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q} \to \mathcal{K},$$

where $\mathcal{Q}$ is the orthogonal complement of $W(\mathcal{H}_0 \otimes \mathcal{P})$ in $\mathcal{K}$.

Consequently, we may identify $\mathcal{K}$ with $(\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q}$ and the normal CP map ψ as $\psi'(X) = \sum_{k\in\mathcal{O}} N_k' X N_k'^*$, i.e. $\psi'(X) = \psi(UXU^*)$ for $X \in \mathcal{B}((\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q})$.where each operator $N_k'^* : \mathcal{H} \to (\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q}$ is defined by $N_k' = N_k U$. Indeed for $x \in \mathcal{H}, y \in \mathcal{H}_0, l \in \mathcal{O}$ and $q \in \mathcal{Q}$ we have $\langle N_k'^*(x), (y \otimes f_l) \oplus q\rangle = \langle U^* N_k^*(x), (y \otimes f_l) \oplus q\rangle = \langle N_k^*(x), \frac{1}{\sqrt{\lambda_l}} N_l^*(y)\rangle =$

$\langle P_0x \otimes \sqrt{\lambda_k}f_k, (y \otimes f_l) \oplus q\rangle$. Thus, $N_k'^*(x) = P_0x \otimes \sqrt{\lambda_k}f_k,\ x \in \mathcal{H}$. We would like to mention here that for $y \in \mathcal{H}_0, f \in \mathcal{P}, q \in \mathcal{Q}$

$$\sum_k N_k^* N_k(x \otimes y \oplus q) = \sum_k y \otimes \lambda_k \langle f, f_k\rangle = ((I_{\mathcal{H}_0} \otimes \rho) \oplus 0)(y \otimes f \oplus q). \tag{5.2.1}$$

Therefore, we can identify $\sum_k N_k'^* N_k'$ with $I_{\mathcal{H}_0} \otimes \rho$ on $\mathcal{H}_0 \otimes \mathcal{P}$. Let $P : (\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q} \to \mathcal{H}_0 \otimes \mathcal{P}$ denote the orthogonal projection onto the subspace $\mathcal{H}_0 \otimes \mathcal{P}$. Then, by the definition of ψ', it follows that

$$\psi'(X) = \psi'(PXP), \quad \forall\, X \in \mathcal{B}\big((\mathcal{H}_0 \otimes \mathcal{P}) \oplus \mathcal{Q}\big).$$

For $A \in \mathcal{B}(\mathcal{H}_0), B \in \mathcal{B}(\mathcal{P})$,

$$\begin{aligned}\psi'(A \otimes B)(x) &= \sum_{k \in \mathcal{O}} N_k'(A \otimes B) N_k'^*(x)\\ &= \sum_{k \in \mathcal{O}} \sqrt{\lambda_k} N_k'(AP_0x \otimes Bf_k)\\ &= P_0AP_0x \sum_{k \in \mathcal{O}} \lambda_k \langle f_k, Bf_k\rangle\\ &= P_0AP_0x \operatorname{Tr}(\rho B).\end{aligned}$$

Hence, $\psi'(A \otimes B) = P_0AP_0 \operatorname{Tr}(\rho B)$ i.e. $\psi(U(A \otimes B)U^*) = \operatorname{Tr}(\rho B)\tau(I_{\mathcal{H}})A\tau(I_{\mathcal{H}})$, for all $A \in \mathcal{B}(\mathcal{H}_0), B \in \mathcal{B}(\mathcal{P})$.

We can write $\psi \circ \phi = \tau$ as $\psi' \circ \phi' = \tau$, where $\phi'(X) = U^*\phi(X)U$. Further one can realize the previous identity as $\psi' \circ P\phi'P = \tau$, since $\psi'(X) = \psi'(PXP)$. We claim that $P\phi'(X)P = \tau(X) \otimes 1_{\mathcal{P}}$, This implies that $\phi'(X) = (\tau(X) \otimes 1_{\mathcal{P}}) \oplus \eta(X)$, where $\eta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{Q})$ is a normal UCP map.

Proof of the claim: Let the normal UCP map $P_{\psi'}\phi' P_{\psi'}$ has the minimal Choi-Kraus representation as

$$P\phi'(X)P = \sum_{i \in I'} L_i' X L_i'^*,$$

where $L_i' \in \mathcal{B}(\mathcal{H}, \mathcal{H}_1 \otimes \mathcal{P}), \forall i \in I'$. $\psi' \circ P\phi'P = \tau \implies$ there exists scalars $\{\alpha_{i,j}'\}$ such that

$$N_j' L_i' = \alpha_{i,j}' K$$

and

$$\sum_{i \in I', j \in \mathcal{O}} |\alpha_{i,j}'|^2 = 1.$$

From the discussion following the Equation 5.2.1, we have

$$\begin{aligned} N_j'L_i' = \alpha_{i,j}'K &\implies L_i'^* N_j'^* = \overline{\alpha}_{i,j}' K^*\\ &\implies L_i'^* \sum_{j \in \mathcal{O}} N_j'^* N_j' = K^* \sum_{j \in \mathcal{O}} \overline{\alpha}_{i,j}' N_j'\\ &\implies (I_{\mathcal{H}_0} \otimes \rho) L_i'(x) = \sum_{j \in \mathcal{O}} \alpha_{i,j}' \sqrt{\lambda_j} (K(x) \otimes f_j), \forall\ x\ \in \mathcal{H}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(I_{\mathcal{H}_0} \otimes \rho)\sum_{i \in I'} L_i' X L_i'^* (I_{\mathcal{H}_0} \otimes \rho)(x \otimes y) &= (I_{\mathcal{H}_0} \otimes \rho)\sum_{i \in I'} L_i'\Big(\sum_{j \in \mathcal{O}} (\langle f_j, y\rangle \overline{\alpha_{i,j}'} \sqrt{\lambda_j} X K^* x)\Big)\\ &= \sum_{i \in I'} \sum_{j \in \mathcal{O}} (\langle f_j, y\rangle \overline{\alpha_{i,j}'} \sqrt{\lambda_j}) \sum_{k \in \mathcal{O}} (\alpha_{i,k}' \sqrt{\lambda_k} (KXK^*x \otimes f_k))\\ &= KXK^*x \otimes \sum_{i \in I'} \Big(\langle \Big(\sum_{j \in \mathcal{O}} \alpha_{i,j}' \sqrt{\lambda_j} f_j\Big), y\rangle \sum_{j \in \mathcal{O}} \alpha_{i,j}' \sqrt{\lambda_j} f_j\Big)\\ &= (\tau(X) \otimes \rho')(x \otimes y)\ \forall\ x \in \mathcal{H}_0\ \&\ \ y \in \mathcal{P},\end{aligned}$$

where $\rho'(y) = \sum_{i\in I'} (\langle (\sum_{j\in\mathcal{O}} \alpha'_{i,j}\sqrt{\lambda_j} f_j), y\rangle \sum_{j\in\mathcal{O}} \alpha'_{i,j}\sqrt{\lambda_j} f_j)$ and $\rho' \in \mathcal{B}(\mathcal{P})$. Hence, $P\varphi' P = \tau(X)\otimes 1_{\mathcal{P}}$.

■

Remark 5.2.5. We remark that the proof of Theorem 5.2.4 extends without essential changes when one further assumes that $\phi(1)$ is a projection. Under this additional hypothesis, the resulting structural conclusions remain unchanged.

As a direct application of Theorem 5.2.4, we obtain the following corollary.

Corollary 5.2.6. Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a left invertible normal CP map i.e. there exists a normal CP map $\psi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ such that $\psi\circ\phi(X) = X$. Then there exist Hilbert spaces $\mathcal{P}, \mathcal{Q}$, and a unitary $U : (\mathcal{H}\otimes\mathcal{P})\oplus\mathcal{Q} \to \mathcal{K}$ such that for all $X \in \mathcal{B}(\mathcal{H})$,

$$\lim_{n\to\infty} (\phi(1_{\mathcal{H}}) + \tfrac{1}{n})^{-\frac{1}{2}} \phi(X) (\phi(1_{\mathcal{H}}) + \tfrac{1}{n})^{-\frac{1}{2}} = U[(X\otimes 1_{\mathcal{P}}) \oplus \eta(X)]U^*,$$

where $\eta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{Q})$ is a normal CP map.

Proof. Since the CP map ϕ need not be unital, we first modify it to obtain a CP map ϕ' such that $\phi'(1_{\mathcal{H}})$ is a projection to which the Remark 5.2.5 can be applied. Define a map $\phi' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ by

$$\phi'(X) = \lim_{n\to\infty} \big(\phi(1_{\mathcal{H}}) + \tfrac{1}{n}\big)^{-1/2} \phi(X) \big(\phi(1_{\mathcal{H}}) + \tfrac{1}{n}\big)^{-1/2}, \qquad X \in \mathcal{B}(\mathcal{H}).$$

It is straightforward to verify that $\phi'(1_{\mathcal{H}})$ is a projection. Note that $\psi'\circ\phi'(X) = X$ for all $X \in \mathcal{B}(\mathcal{H})$, where $\psi' : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ is given by $\psi'(Y) = \psi(\big(\phi(1_{\mathcal{H}})\big)^{1/2} Y \big(\phi(1_{\mathcal{H}})\big)^{1/2})$, $Y \in \mathcal{B}(\mathcal{K})$. Consequently, following Remark 5.2.5, there exist separable Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$, and a unitary operator

$$U : (\mathcal{H}\otimes\mathcal{P})\oplus\mathcal{Q} \longrightarrow \mathcal{K}$$

such that

$$\phi'(X) = U\big[(X\otimes 1_{\mathcal{P}}) \oplus \eta(X)\big]U^*, \qquad X \in \mathcal{B}(\mathcal{H}),$$

for some normal CP map $\eta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{Q})$.

■

In the setting of the algebra of bounded operators on a Hilbert space, and employing Theorem 5.2.4, the following theorem provides a complete characterization of maps that admit a disintegration. This result fully extends the disintegration theorem of Parzygnat and Russo for $*$-homomorphisms (see Theorem 4.3 in [PR23]).

Theorem 5.2.7. Let $(\mathcal{B}(\mathcal{H}), \xi)$ and $(\mathcal{B}(\mathcal{K}), \omega)$ be quantum spaces with normal states. Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a normal unital completely positive (UCP) map, and let $(\mathcal{B}(\mathcal{H}), \xi) \xrightarrow{\phi} (\mathcal{B}(\mathcal{K}), \omega)$ be a normal UCP state-preserving map. Then a disintegration ψ of ω over ξ consistent with ϕ exists if and only if there exist separable Hilbert spaces $\mathcal{P}, \mathcal{Q}$, a density operator $\rho \in \mathcal{B}(\mathcal{P})$, a unitary operator $U : (\mathcal{H}\otimes\mathcal{P})\oplus\mathcal{Q} \longrightarrow \mathcal{K}$, and a normal UCP map $\eta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{Q})$ such that

$$\phi(X) = U\big((X\otimes I_{\mathcal{P}}) \oplus \eta(X)\big)U^*, \qquad \forall\, X \in \mathcal{B}(\mathcal{H}),$$

and

$$\omega(UYU^*) = \omega(UPYPU^*), \text{for all } Y \in \mathcal{B}((\mathcal{H}\otimes\mathcal{P})\oplus\mathcal{Q})$$

where $P : (\mathcal{H} \otimes \mathcal{P}) \oplus \mathcal{Q} \to \mathcal{H} \otimes \mathcal{P}$ denotes the orthogonal projection. Moreover, for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{P})$, the state ω satisfies

$$\omega\big(U(A \otimes B)U^*\big) = \mathrm{Tr}(\rho B)\,\xi(A).$$

If a disintegration from $(\mathcal{B}(\mathcal{K}), \omega)$ to $(\mathcal{B}(\mathcal{H}), \xi)$ exists then it is unique.

Proof. Since $Id_{\mathcal{B}(\mathcal{H})} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a quasi-pure map, Corollary 4.3.6 implies that the relation $\psi \circ \phi \underset{\xi}{=} Id_{\mathcal{B}(\mathcal{H})}$ reduces to the equality $\psi \circ \phi = Id_{\mathcal{B}(\mathcal{H})}$. Hence, the existence of a disintegration is equivalent to the existence of a left inverse for ϕ. In particular, Theorem 5.2.4 ensures the abstract structure of such a left inverse. Consequently, there exist Hilbert spaces $\mathcal{P}, \mathcal{Q}$ and a unitary $U : (\mathcal{H} \otimes \mathcal{P}) \oplus \mathcal{Q} \longrightarrow \mathcal{K}$ such that $\phi(X) = U\big((X \otimes I_{\mathcal{P}}) \oplus \eta(X)\big)U^*$, $\forall X \in \mathcal{B}(\mathcal{H})$, and $\psi(U^*(A \otimes B)\,U) = \mathrm{tr}(\rho B)\,A$, $\forall\, A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{P})$ where ρ a density operator in $\mathcal{B}(\mathcal{P})$.

Since $\psi(UYU^*) = \psi(UPYPU^*)$ and $\xi \circ \psi = \omega$, we obtain $\omega(UYU^*) = \omega(UPYPU^*)$, and in particular, $\omega(U(A \otimes B)U^*) = \mathrm{Tr}(\rho B)\,\xi(A)$.

It remains to establish the uniqueness part. Without loss of generality, assume that

$$\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}\big((\mathcal{H} \otimes \mathcal{P}) \oplus \mathcal{Q}\big)$$

is a normal UCP map of the form $\phi(X) = (X \otimes I_{\mathcal{P}}) \oplus \eta(X)$, $\forall\, X \in \mathcal{B}(\mathcal{H})$, and let $\omega : \mathcal{B}\big((\mathcal{H} \otimes \mathcal{P}) \oplus \mathcal{Q}\big) \to \mathbb{C}$ be a normal state satisfying $\omega(X) = \omega(PXP)$, $\forall\, X$, where $P : (\mathcal{H} \otimes \mathcal{P}) \oplus \mathcal{Q} \to \mathcal{H} \otimes \mathcal{P}$ denotes the orthogonal projection. For all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{P})$, the state ω satisfies $\omega(A \otimes B) = \mathrm{Tr}(\rho B)\,\xi(A)$, for some density operator $\rho \in \mathcal{B}(\mathcal{P})$ and a state ξ on $\mathcal{B}(\mathcal{H})$.

Let $\psi_i : \mathcal{B}\big((\mathcal{H} \otimes \mathcal{P}) \oplus \mathcal{Q}\big) \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$, be two normal UCP maps acting as disintegrations of ω over ξ. By the state-preserving condition, we have $\xi \circ \psi_i = \omega$, $i = 1, 2$. Since each ψ_i serves as a left inverse for ϕ, by a similar argument as in Theorem 5.2.4, and using the state-preserving condition, we obtain $\psi_i(X) = \psi_i(PXP)$, $\forall\, X \in \mathcal{B}((\mathcal{H} \otimes \mathcal{P}) \oplus \mathcal{Q})$, and for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{P}), \psi_i(A \otimes B) = \mathrm{Tr}(\rho B)\,A$, $i = 1, 2$. Hence $\psi_1 = \psi_2$, establishing uniqueness. ∎

Remark 5.2.8. Corollary 5.2.7 settles the existence and uniqueness question of disintegration for arbitrary normal UCP maps on the algebra of bounded linear operators on Hilbert spaces. In particular, the case studied by Parzygnat and Russo appears as a special instance of Corollary 5.2.7, corresponding to the situation where ϕ is a $*$-homomorphism on a matrix algebra. From the above discussion, it is also apparent that the map ϕ is ω a.e. equivalent to a $*$-homomorphism.

List of Publications

The material presented in this thesis is primarily based on the following published paper and the accompanying preprint:

1. B.V. Rajarama Bhat, and Arghya Chongdar,
   A minimal completion theorem and almost everywhere equivalence for Completely Positive maps,
   https://doi.org/10.1090/proc/16921, Proc. Amer. Math. Soc (2025).

2. B.V. Rajarama Bhat, Arghya Chongdar and Sruthymurali,
   Understanding Quantum Instruments Through the Analysis of C^*-Convexity and Their Marginals,
   preprint, arxiv:2509.11785v2 (2025).

The contents of Chapter 2 follow the exposition in Paper 1. Similarly, the material presented in Chapter 4 is based on Paper 2.

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