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Indian Statistical Institute
M.Tech. in Computer Science (2025-26)
Mid Semester Examination 2025

Paper: **Linear Algebra** Date: **12/09/2025**

Time : **2 Hrs.** Full Marks : **30**

Note: Symbols used have their usual meaning. *The figures in the margin indicate full marks.*

Answer any FIVE from the following questions. $5 \times 6 = 30$

1. (a) Consider a linear system whose augmented matrix is of the following form

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 6 & 7 & \mathbf{x} & \mathbf{y} \end{array}\right).$$

Find the consistency condition for the system. Also, find the values of $\mathbf{x}$ and $\mathbf{y}$ for which it has infinitely many solutions.

(b) Let O be the $d \times d$ matrix whose entries are all 0, $\mathbb{I}$ be the $d \times d$ identity matrix, and X be a $d \times d$ matrix with the property that $X^2 = 0$. If $A = \begin{bmatrix} O & \mathbb{I} \\ \mathbb{I} & B \end{bmatrix}$ determine the block form of $A^{-1} + A^2 + A^3$.

[3 + 3]

2. (a) Let $\mathbb{R}$ and $\mathbb{Q}$ be the fields of real and rational numbers respectively. Determine whether $\mathbf{S} = \{a + b\sqrt{2} + c\sqrt{3} | a, b, c \in \mathbb{Q}\}$ forms a vector space over the fields (i) $\mathbb{Q}$, (ii) $\mathbb{R}$. Find the dimension and a basis for each that is a vector space.

(b) Let T be a linear transformation on a finite dimensional vector space V. Show that $\dim(\text{Im}(T^2) = \dim(\text{Im}(T))$ if and only if $V = \text{Im}(T) \oplus \text{Ker}(T)$.

[3 + 3]

3. (a) Is it possible for a matrix to have the vector $(1, 1, 1)$ in its row space and $(2, 1, 1)^T$ in its null space? Explain.

(b) Use the Gram-Schmidt process to find an orthonormal basis for the space spanned by $x_1 = (2, 1, 1, 1)^T, x_2 = (1, 0, 0, 1)^T, x_3 = (3, 2, 0, 2)^T, x_4 = (0, 1, -1, 0)^T$.

[2 + 4]

4. Let $P_n[x]$ be the set of all polynomials of degree less or equal to n with real coefficient. Define a mapping $T : P_4[x] \to P_4[x]$ by

$$Tp(x) = (sx + t)\frac{d}{dx}p(x), \quad \text{for } p(x) \in P_4[x], \quad \text{where, } s, t \text{ are fixed complex number.}$$

Check T is linear or not. If, yes, then find the corresponding matrix representing T. Is T invertible? Justify your answer. [6]

P.T.O.

5. (a) Let T be a linear transformation on an inner product space V of dimension n and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an orthogonal basis of V. If

$$\langle T\alpha_i | T\alpha_i \rangle = \langle \alpha_i | \alpha_i \rangle, \quad i = 1, 2, \ldots, n,$$

is T necessarily an orthogonal transformation (i.e., $\langle u|v\rangle = \langle Tu|Tv\rangle$)?

(b) Let A be a diagonalizable matrix and let X be the diagonalizing matrix. Show that the column vectors of X that correspond to nonzero eigenvalues of A form a basis for $R(A)$.

[3 + 3]

6. (a) Let A be a $n \times n$ matrix with Schur decomposition $UTU^\dagger$. Show that if the diagonal entries of T are all distinct, then there is an upper triangular matrix R such that $X = UR$ diagonalizes A.

(b) Find an unitary diagonalizing matrix for $\begin{pmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

[3 + 3]