

Indian Statistical Institute
Second Semester Examinations: 2025-26

Course Name: M. Math, 2nd year
Subject Name : Probability Theory
Maximum Marks: 50, Duration: Three hours
Date: 22.11.2025, 2:30 PM – 5:30 PM

- You may use any results proved in class. Any other results require proof.
- You will be awarded the full marks for a question if you justify using correct arguments that there is a mistake in the question.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

(a) [2 marks] $\mathbb{E}[X|\sigma(X^2)] = |X| \cdot \mathbb{E}[\mathbb{1}_{\{X>0\}} - \mathbb{1}_{\{X<0\}}|\sigma(X^2)]$.

(b) [4 marks] If X has a continuous probability density function f , that is $\mathbb{P}[X \in (a, b)] = \int_a^b f(x)dx$, then

$$\mathbb{E}[\mathbb{1}_{(0, \infty)} \circ X|\sigma(X^2)](\omega) = \frac{f(|X|(\omega))}{f(|X|(\omega)) + f(-|X|(\omega))}, \quad \mathbb{P} - a.s.$$

2. Let $(X_i)_{i=1}^{\infty}$ be iid RVs with characteristic function $\varphi_{X_1}(t) = \mathbb{E}[e^{itX_1}]$. Write $S_n = \sum_{k=1}^n X_k$. Show the following.

(a) [2 marks] Show that $\varphi_{S_n/n}(t) = \varphi_{X_1}(t/n)^n$.

(b) [2 mark] Use $\lim_{n \rightarrow \infty} (1 + x/n)^n = e^x$ to show that if $\varphi'_{X_1}(0) = ia$ then $S_n/n \xrightarrow{n \rightarrow \infty} a$ in distribution.

(c) [3 marks] If $\varphi'_{X_1}(0) = ia$, then $S_n/n \xrightarrow{n \rightarrow \infty} a$ in \mathbb{P} (in probability).

(d) [2 marks] If $S_n/n \xrightarrow{n \rightarrow \infty} a$ in \mathbb{P} then $\varphi(t/n)^n \xrightarrow{n \rightarrow \infty} e^{iat}$.

3. Consider the Markov chain in \mathbb{Z} with transition probability

$$P_x = c\delta_0 + \frac{1-c}{2}(\delta_{x-1} + \delta_{x+1}) \quad \text{for } x \in S.$$

(a) [5 marks] Describe the stationary measure explicitly when $c = 1/5$.

(b) [1 mark] Let $\tau_a^{(k)}$ be the k -th time of hitting $a \in \mathbb{Z}$. Argue that $\lim_{k \rightarrow \infty} \tau_a^{(k)}/k$ exists \mathbb{P}_y -a.s.

(c) [1 mark] Determine the \mathbb{P}_y -a.s limit $\lim_{k \rightarrow \infty} \tau_a^{(k)}/k$ when $c = 1/5$ in terms of a .

(d) [3 marks] Show that for all $n \in \mathbb{N}$, $\mathbb{P}_0[\tau_0 > n] \leq (1-c)^n$.

(e) [2 marks] Prove or disprove that $\frac{\tau_0^{(n)} - n\mu}{\sigma\sqrt{n}}$ converges in distribution for some $\mu, \sigma > 0$.

4. Let (\mathbb{Z}, P) be the simple random walk in the integers, so

$$P_x = \frac{1}{2}(\delta_{x+1} + \delta_{x-1}), \quad \text{for all } x \in S.$$

Let (\mathbb{Z}, Q) be a different random walk in the integers, where the stochastic matrix is

$$Q_{xy} = \begin{cases} 1/4, & y \in \{x-1, x+1\}, x \equiv 0 \pmod{2} \\ 1/2, & y \in \{x-1, x+1\}, x \equiv 1 \pmod{2} \\ 1/2, & y = x, x \equiv 0 \pmod{2} \\ 0, & y \notin \{x-1, x, x+1\}. \end{cases} \quad (1)$$

- (a) [3 mark] Show that every state is recurrent in both the Markov processes.
- (b) [3 marks] Show that no state is positively recurrent in either of the processes.
- (c) [2 marks] Let $(Z_n)_{n=0}^\infty$ be the stochastic process corresponding to the simple random walk (\mathbb{Z}, P) . Let $p \in \mathbb{N}$ and for any $n \in \mathbb{N}_0$, set

$$R_n \in \mathbb{Z}_p := \{0, \dots, p-1\} \text{ be such that } R_n \equiv Z_n \pmod{p}.$$

Compute for $j \in \mathbb{Z}_p$,

$$\mathbb{E}_0[\mathbb{1}_{\{j\}} \circ R_{n+1} | \sigma(R_n)](\omega)$$

in terms of $R_n(\omega)$.

- (d) [2 marks] Describe the stochastic matrix $M : \mathbb{Z}_p^2 \rightarrow [0, \infty)$ such that the corresponding Markov chain is R_n .
- (e) [2 marks] Is (\mathbb{Z}_p, M) irreducible? Explain.
- (f) [2 marks] Is (\mathbb{Z}_p, M) aperiodic? Explain.
- (g) [2 marks] Describe explicitly the stationary distribution for (\mathbb{Z}_p, M) ?
- (h) [2 mark] Does

$$\lim_{n \rightarrow \infty} \mathbb{P}_0[Z_n \equiv 0 \pmod{p}]$$

exist? Explain.

- (i) [1 mark] Compute

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{P}_0[Z_k \equiv 0 \pmod{p}].$$

- (j) [4 marks] Does the stochastic process $(R_n)_{n=1}^\infty$ correspond to a Markov chain, when now $(Z_n)_{n=1}^\infty$ corresponds to (\mathbb{Z}, Q) instead? Explain.