Indian Statistical Institute, Kolkata

Midterm (Supplementary): Second semester 2024 - '25

B.Stat I year

Subject	:	**Analysis II**
Time	:	2 hours
Maximum score	:	30

Work independently. Attempt all the problems. Please use a new page to answer each problem and make sure that the problem number in the margin can be read, even after stapling. If you attempt the same problem several times, please strike out all the attempts except the correct one before submitting your answer script. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answer script. Points will be deducted for missing or incomplete arguments.

(1) Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2n}{n^2+k^2}$ by interpreting this as a Riemann sum corresponding to an integral.

[4 marks]

(2) (i) If a function f is continuous on a closed interval $[a, b]$ and $\int_a^b x^n f(x)dx = 0$ for every $n \in \mathbb{N}$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(ii) Let $0 \leq a < b, 0 \leq c < d$ and f be a strictly increasing function on $[a, b]$ onto $[c, d]$ and let ϕ be its inverse. Interpreting the integral as an area, show that $\int_c^d \varphi(x)dx = bd - ac - \int_a^b f(x)dx$.
[Hint : The graph of φ is same as that of f seen from a different perspective.]
Use the above relation to obtain an expression for $\int_a^b \log x dx$ for $1 < a < b$.

[4 + 4 = 8 marks]

(3) (i) Prove that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is uniformly convergent for every real x. If $f(x)$ is the sum of the series, prove that f is continuous on $\mathbb{R}$.

(ii) Also prove that $\int_0^{\pi} f(x)dx = 2(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots)$

[3 + 2 = 5 marks]

(4) Evaluate the following improper integrals:

(i) $\displaystyle\int_0^{\infty} \frac{1}{x^2 + \sqrt{x}} dx$

(ii) $\displaystyle\int_1^{\infty} \frac{1}{x(\sqrt{(\ln x)} + (\ln x)^2)} dx$

[4 + 4 = 8 marks]

(5) For $n \geq 1$, let $f_n : [-1, 1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{1+x^2n^2}$. Show that $\{f_n\}$ converges uniformly on $[-1, 1]$, the limit function f is differentiable, but $f'(x) = \lim_n f_n'(x)$ does not hold for some $x \in [-1, 1]$.

[5 marks]