INDIAN STATISTICAL INSTITUTE

SEMESTRAL EXAMINATION
B.STAT - III YEAR

DESIGN AND ANALYSIS OF ALGORITHMS

Date: 17.11.2025 Maximum marks: 100 Duration: 3 hours

The paper contains 120 marks. Answer as much as you can, the maximum you can score is 100.

1. (a) Let f and g be functions from the set of real numbers to itself, and suppose that $f(x) \to \infty$ as $x \to \infty$. If $f(x) = o(g(x))$ (i.e. $f(x)$ is little-oh of $g(x)$), then show that $e^{f(x)} = o(e^{g(x)})$, where e is the base of natural logarithms.

   (b) Suppose n is an even integer. Write the pseudo-code of an algorithm which simultaneously finds the maximum and the minimum in an array of numbers of length n and requires $3n/2 - 2$ comparisons.

   (c) Describe an algorithm to construct a permutation from an inversion vector.

   (d) Is radix sort a stable sorting algorithm? Justify your answer.

   (e) Let L be a list of elements from a finite non-empty set X. For $L = (x_1, \ldots, x_n)$, where $n \geq 0$ and x an element of X, the operation FindInsert(L, x) updates L as follows: if $x = x_i$ for some $1 \leq i \leq n$, then L is updated to $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i)$. while if x is not present in L, then L is updated to $(x, x_1, \ldots, x_n)$. Suppose Rand() is a function which samples with replacement from X following some *non-uniform* probability distribution. Consider the following code fragment.

```
L = ()
while (true) {
    x ← Rand()
    FindInsert(L, x)
}
```

   Explain with justification whether a linked list or an array would be more suitable to represent L. Describe an efficient algorithm to implement the operation FindInsert. Justify your answer.

$(5 + 5 + 5 + 5 + 10 = 30)$

2. (a) Describe with justification an efficient data structure to represent a directed graph.

   (b) Write the pseudo-code of the algorithm to perform topological sort of a directed acyclic graph.

   (c) With reference to depth first search on a directed graph, define the notion of low point LOWPT[v] for a node v. Suppose v is the root of a strongly connected component.

Show that it is not possible for LOWPT[v] to be equal to LOWPT[u] for some tree edge (v, u).

(d) What data structures will you use to implement Kruskal's algorithm to find a minimum spanning tree in a weighted undirected graph? Using the data structures obtain the time complexity of the algorithm.

$(3 + 7 + (3 + 7) + 10 = 30)$

3. (a) Consider the following pseudo-code defining two functions Generate() and next().

```
Generate(n) {
    for i ← 0 to n − 1 {X[i] ← 0}
    output X
    flg ← 1; b ← 0
    while (flg == 1) {
        b ← 1 − b
        flg ← next(X, n, b)
        output X
    }
}
next(X, n, b) {
    j ← 0
    if (b == 0) {
        while (X[j] == 0) {j ← j + 1}
        j ← j + 1
    }
    if (j == n), return 0
    else {
        X[j] ← 1 − X[j]
        return 1
    }
}
```

Describe with justification the output that will be produced by the call Generate(n) for some positive integer n.

(b) For a positive integer n and an integer w with $0 \leq w \leq n$, let $S_{n,w}$ be the set of all binary strings of length n and weight w. Describe with justification an $O(n)$ time algorithm to sample uniformly at random from the set $S_{n,w}$.

$(7 + 8 = 15)$

4. (a) Use the extended Euclidean algorithm to find the multiplicative inverse of 125 modulo 58.

(b) Write the pseudo-code of an $O(n)$ time algorithm to perform the multiplication of two $n \times n$ permutation matrices.

(c) For positive integers m, n and p assume that multiplying an $m \times n$ matrix with an $n \times p$ matrix with entries from some ring require $m \cdot n \cdot p$ ring multiplications. Use dynamic programming technique to find a way of multiplying the following sequence of matrices such that the total number of ring multiplications is minimised.

$$A_{20\times 30} \cdot B_{30\times 10} \cdot C_{10\times 1} \cdot D_{1\times 100}$$

(d) Let [M] denote the cost of multiplying two polynomials of degrees at most 3, and let [A] denote the cost of an addition or a subtraction of two polynomials of degrees at most 3. Use Karatsuba's algorithm to determine the cost of multiplying two polynomials of degrees at most 15 in terms of [M] and [A].

(e) Describe the main algebraic ideas behind the Fast Fourier Transform (FFT) algorithm.

(f) Let $a(x) = 2x^3 - 4x^2 + x$ and $b(x) = x^3 - 5x^2 + 7$. It is desired to multiply $a(x)$ and $b(x)$ over the integers using the FFT algorithm. Specify with justification a suitable ring R such that multiplying $a(x)$ and $b(x)$ over R provides the product over the integers.

$(5 + 5 + 5 + 5 + 10 + 5 = 35)$

5. (a) If π is an NP-complete problem and π' is another decision problem such that $\pi \leq_m^p \pi'$, then show that π' is NP-hard.

(b) Let u_1, u_2, u_3, u_4 and u_5 be Boolean valued variables and consider the formula

$$F = (u_1 \vee \overline{u}_2 \vee u_3 \vee \overline{u}_4 \vee u_5).$$

Construct a formula F' in the 3-SAT format such that F is satisfiable if and only if F' is satisfiable.

$(5 + 5 = 10)$