INDIAN STATISTICAL INSTITUTE, KOLKATA
FINAL EXAMINATION: FIRST SEMESTER 2025 - '26
M.STAT II YEAR

Subject	:	**Functional Analysis**
Time	:	3 hours
Maximum score	:	50

Work independently. Attempt all the problems. Please use a new page to answer each problem and make sure that the problem number in the margin can be read, even after stapling. If you attempt the same problem several times, please strike out all the attempts except the correct one before submitting your answer script. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answer script. Points will be deducted for missing or incomplete arguments.

(1) Show that the (usual) norms of the following Banach spaces cannot be induced by inner products.
   (a) $C([0,1])$
   (b) $L^1([0,1])$ with respect to the Lebesgue measure

[3+3 = 6 marks]

(2) Let $\{e_i\}_{i\in I}$ be an orthonormal basis of a Hilbert space and let x be a unit vector. For a fixed $k \in \mathbb{N}$, show that the set $\{i \in I : |\langle x, e_i\rangle| \geq 1/k\}$ has at most k^2 elements.

[5 marks]

(3) Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$.
   (i) Show that this orthonormal basis gives an example of a closed and bounded set which is not compact.
   (ii) Let

$$Q = \{x \in \mathcal{H} : x = \sum_{n=1}^{\infty} c_n e_n \text{ where } c_n \in \mathbb{C}, |c_n| \leq 1/n\}$$

   Prove that Q is compact subset of $\mathcal{H}$.
   (iii) Suppose $\{\delta_n\}_n$ is a sequence and

$$R = \{x \in \mathcal{H} : x = \sum_{n=1}^{\infty} c_n e_n \text{ where } c_n \in \mathbb{C}, |c_n| \leq \delta_n\}$$

   Prove that R is compact if and only if $\{\delta_n\}_n$ is an ℓ^2 sequence.

[3+3+4 = 10 marks]

(4) Define $T : \ell^2 \to \ell^2$ by the formula $T(a_1, a_2, a_3, ...) = (0, a_1, 0, a_2, \ldots)$ Find T^*.

[5 marks]

(5) Define $T : \ell^2 \to \ell^2$ by the formula $T(a_1, a_2, a_3, \ldots) = (a_1, a_2/2, a_3/3, \ldots)$
   (i) Prove that T is a compact self adjoint operator.
   (ii) Is the image of T closed? Justify your answer.

[6 + 2 = 8 marks]

(6) Show that the dual of the space c, of all convergent scalar sequences equipped with the sup norm, can be isometrically identified with the space ℓ_1 under the action

$$\boldsymbol{a}(\boldsymbol{x}) = a_1 \lim x_n + \sum_{n=1}^{\infty} a_{n+1} x_n \text{ where } \boldsymbol{a} = \{a_n\} \in \ell_1 \text{ and } \boldsymbol{x} = \{x_n\} \in c.$$

Identify the canonical image of c inside ℓ_∞. This shows that a dual Banach space can have two different preduals. Are c_0 and c isomorphic? Are they isometric?

[4 + 2 + 2 = 8 marks]

(7) Let $\mathcal{H}$ be a Hilbert space, $x \in \mathcal{H}$ and $C \subset \mathcal{H}$ be a closed convex set. Show that the distance between x and C is attained at a unique point of C.
(*Hint: Pick a sequence* $\{x_k\}_{k\in\mathbb{N}}$ *in* C *such that* $\|x_k - x\|$ *decreases to the distance of* C *from* x *as* k *tends to* ∞. *Show that* $\{x_k\}_{k\in\mathbb{N}}$ *is Cauchy using convexity of* C *and parallelogram law.*)

[8 marks]