# INDIAN STATISTICAL INSTITUTE

Semestral Examination:(2024-2025)

B.Stat III YEAR

## Subject Name: Quantum Physics

Maximum Marks: 60 Duration: 3 hours Date: 03.05.2023

### Answer any five of the following eight questions

1. Consider a particle in the potential $V(x) = \infty$ for $x < 0$ and $x > L$ and $V(x) = 0$ for $0 \leq x < L$.

a. Show that for such system, the momentum operator ($-i\hbar\frac{\partial}{\partial x}$) is Hermitian when the wave functions have same value at end points.

b. Argue why eigenfunction of momentum operator cannot be a possible state of the particle.

c. The wave function of the particle at $t = 0$ is the following:

$$\psi(x) = \frac{2}{\sqrt{L}}\cos(\frac{2\pi x}{L})\sin(\frac{5\pi x}{L})$$

(i) Find the possible results of energy measurement and their corresponding probabilities.

(ii) Find the wave function at $t = T$.

[3 + 2 + (5 + 2)]

2. Two operators $a$ and $a^\dagger$ can be defined in terms of position and momentum operators in such a way that the Hamiltonian of linear harmonic oscillator can be expressed as

$$H = [a^\dagger a + \frac{1}{2}I]\omega\hbar$$

i) Show that $[a, H] = \hbar\omega a$

ii) Find the eigenvalues of $H$ by finding the eigenvalues of $a^\dagger a$.

iii) Show that for nth energy eigenstate:

$$\Delta X \Delta P = (n + \frac{1}{2})\hbar$$

[4 + 4 + 4]

3. Consider a two qubit pure state $|\psi\rangle_{AB}$ and four unitary operators $\{U_i, i = 1, 2, 3, 4\}$ acting on the first particle $A$ such that $\{U_i \otimes I|\psi\rangle_{AB}, i = 1, 2, 3, 4\}$ form an orthogonal set.

(a) Show that the state $|\psi\rangle_{AB}$ is necessarily entangled.

(b) Show the states $\{U_i \otimes I|\phi^+\rangle_{AB}, i = 1, 2, 3, 4\}$ where $U_1 = I, U_2 = \sigma_x, U_3 = \sigma_y$ and $U_4 = \sigma_z$ form an orthogonal set, where $|\phi^+\rangle_{AB}$ is a the following Bell state:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

c) Discuss how the result can be used to send two bits of information by transferring a single qubit if the qubit is entangled with another qubit with receiver represented by the above mentioned Bell state.

[2 + 4 + 6]

4. Consider a function $f : \{0, 1\}^n \to \{0, 1\}$

Let there be a unitary gate $U_f$ which acts in the following way:

$$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

where $x \in \{0, 1\}^n$ and $y \in \{0, 1\}$.

a) Show that the following relation holds.

$$U_f|x\rangle\frac{1}{\sqrt{2}}[|0\rangle - |1\rangle] = (-1)^{f(x)}|x\rangle\frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$$

b) Consider a function $f : \{0, 1\}^n \to \{0, 1\}$, where the function $f$ is either constant or balanced ($f(x) = 0$ for half of the possible input values). Discuss how the number of queries increases with $n$ to learn the nature of the function by using deterministic classical computation.

c) Describe the quantum algorithm by which the function can be shown to be either constant or balanced without calculating the function at various points.

[4 + 2 + 6]

5. a) Consider the following two qubits density matrix:

$$\rho_{AB} = p|\phi^+\rangle\langle\phi^+| + (1 - p)|00\rangle\langle 00|, \quad 0 < p < 1$$

Show the range of $p$ for which the state is entangled.
b) Consider the following two qubits density matrix:

$$\rho_{AB} = p|\phi^+\rangle\langle\phi^+| + (1-p)|\phi^+\rangle\langle\phi^+|, \quad 0 \leq p \leq 1$$

where

$$|\phi^+\rangle = \frac{1}{2}[|00\rangle + |11\rangle]$$

Find the range of $p$ for which the state violates Bell's inequality.

[6 + 6]

6. Consider a source that produces pair of particles that reach Alice and Bob stationed in labs that are far apart. Alice and Bob both can perform one of the three possible measurements namely $M_1$, $M_2$ and $M_3$ chosen randomly at each turn. The possible measurement outcomes are +1 and −1. The statistics collected satisfy the following conditions.
a) For same measurements chosen on both sides, the outcomes are same.
b) The probability of having same outcomes on both sides for different choice of measurements is $\frac{1}{4}$.
i) Show that this statistics is incompatible with local realistic theory.
ii) Provide an example of quantum state that can produce the above statistics.

[6 + 6]

7. Consider the following three qubits state shared between Alice, Bob and Charlie stationed at distant laboratories:

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}[|000\rangle - |111\rangle],$$

where $|0\rangle$ and $|1\rangle$ form an orthogonal basis in two dimensional Hilbert space.
a) Let Alice is given another qubit whose state is not known to her. She has to prepare the state in Bob's lab. Show how this task can be made successful if all three parties cooperate by local operation and classical communication.
b) Show that $|\psi\rangle_{ABC}$ is an eigenstate of the following operators: $\sigma_x\sigma_y\sigma_y$, $\sigma_y\sigma_x\sigma_y$, $\sigma_y\sigma_y\sigma_x$, and $\sigma_x\sigma_x\sigma_x$. Find the corresponding eigenvalues.
c) Argue why no local realistic theory can reproduce measurement statistics of those spin measuremets performed locally.

[1 + 2 + 6]

8. Consider the following general state shared between Alice, Bob and Eve

$$|\psi\rangle_{ABE} = a|00\rangle_{AB}|\phi_{00}\rangle_E + b|01\rangle_{AB}|\phi_{01}\rangle_E + c|10\rangle_{AB}|\phi_{10}\rangle_E + d|11\rangle_{AB}|\phi_{11}\rangle_E$$

where Alice and Bob each has one qubit and $|\phi_{ij}\rangle_E$'s refer to states of Eve's system which may be associated with higher dimensional Hilbert space. Let the statistics obtained by Alice and Bob satisfy the following conditions:

i) The results of local spin measurements by Alice and Bob along any directions, are completely random.

ii) Alice and Bob's results are fully correlated when they both measure spin either along $Z$-axis or $X$-axis.

a) Show that Alice and Bob actually share a two-qubit maximally entangled state which is completely uncorrelated with Eve's system.

b) Argue how a protocol can be designed to guarantee secure key between Alice and Bob by exploiting the above results.

[7 + 5]