Indian Statistical Institute, Kolkata

M. Math I year

Mid-Semester Examination-Autumn 2025

Algebra I

Full Marks: 30 ***Time Allotted: 2 h***

Instructions: Answer all questions. Your final score will be calculated as

$$\text{Final marks} = \min\{\text{Total marks obtained}, 30\}.$$

(1) Let R be a commutative ring with unity. Prove that if every proper ideal of R is a prime ideal, then R is a field. (6)

(2) Let R be a commutative ring with unity. Suppose every element of R is either a unit or nilpotent. Show that R is a local ring. (6)

(3) Let R be a commutative ring with unity, and let $P \subset R$ be a prime ideal. Suppose

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x]$$

is a primitive polynomial such that $a_i \in P$ for all $i = 1, \ldots, n-1$ and $a_0, a_n \notin P$. Show that $f(x)$ is irreducible in $R[x]$. (7)

(4) Let R be a PID. Show that R/P is a PID if P is a prime ideal of R. Give an example to show that if P is not prime, then R/P need not be a PID. (6 + 2)

(5) Let

$$A = \mathbb{C}[X, Y]/(X^2 + XY + Y^2),$$
$$B = \mathbb{C}[X, Y]/(X^2Y - 1),$$
$$C = \mathbb{C}[X, Y]/(X^2 + Y^5).$$

Examine whether $A \cong B$, $B \cong C$, and $A \cong C$ as rings. (3 + 3 + 3)