SURVIVAL FUNCTION ESTIMATION UNDER RANDOM CENSORING

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PREFACE

For the last three decades or so survival analysis formulated as the science of predicting, estimating or optimizing the probability of survival, the mean life or more generally, the life distribution of components or systems. During the last forty years, there has been a remarkable development in the area of statistical analysis of life data for both parametric and nonparametric models, specially in the case of one component system or in the classical competing risk in general. In this work we have mainly focussed our attention to nonparametric estimation of survival functions of components under random censoring in (a) two components parallel system and (b) $k-1$ out of k components systems ($k > 2$) with special emphasis on two out of three system. The generality of this approach is indicated by briefly outlining its extended application in slightly more complex systems.

For all the systems considered, independence of distribution is assumed and proportional hazard assumption is first made regarding the component life time and censoring time distribution. The special case of indentical component life time distribution has been addressed separately making use of the properties of the system. Finally the more general situation of all possible types (continuous) of life time distribution in a two component parallel system is tackled allowing for their dependence by developing an EM type algorithm for obtaining maximum likelihood estimators of the survival functions of component lives and other relevant quantities of interest.

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CHAPTER I

INTRODUCTION AND SUMMARY

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1.0 GENERAL INTRODUCTION

Statistical methods for life data analysis are used to measure, compare and predict characteristics of the distribution of the time to some particular event of interest, often called failure after a length of time, called life time. Failure can occur at most once for an individual. Examples of failure time include the lifetimes of machine components in industrial reliability, the duration of strikes or periods of unemployment in economic studies, the time taken by subjects to complete specified tasks in psychological experiments, the lengths of tracks on photographic plates in particle physics and the survival time of patients in clinical trials. In life testing problem, reliability or medical follow up studies and other fields, the observations on the lifetime may not be possible for some sample units because of the occurrence of some other event (say loss). For instance in reliability studies this might happen because of a measure taken to avoid destructive life testing or because of limited availability of testing facilities. In medical follow up surveys, the patient often withdraws from the survey or the experimenter may not be able to make final contact before the death of a patient. In all such cases, we say that the observation is censored. In spite of this incompleteness of data owing to censoring, it is important to estimate the survival function of the lifetime of a unit from the data set observed or collected. Censoring may be random or fixed and in the present work we consider only the case of random censoring which is identified as one where the limits of observations set are values of another random variable, usually assumed to be distributed independently of the life time distribution of the unit. Again the unit may represent a system consisting of one or more components arranged in a certain way and the system life is dependent on component lives based on this arrangement.

In our present work we have mainly focused our attention to nonparametric estimation of survival functions of components under random censoring in a (a) two components parallel system and (b) $k-1$ out of k components system ($k>2$) with special emphasis on two out of three system.

For all the systems considered independence of distributions is assumed and proportional hazard assumption is first made regarding the component lifetime and censoring time distributions. The special case of identical component life time distributions has been addressed separately as the special properties of the system in this case naturally call for a different and improved estimator. Finally, the more general situation of all possible types of (continuous) life time distributions in a two components parallel system is tackled allowing for their dependence by developing an EM type algorithm for obtaining maximum likelihood estimators of the survival functions of component lives and other relevant quantities of interest. The generality of the approach is indicated by briefly outlining its extended application in slightly more complex systems.

1.1 HISTORICAL PERSPECTIVE

During the last forty years since the first publication of a paper on life testing problems by Epstein and Sobel (1953) and particularly since the pioneering work by Kaplan and Meier (1958), there has been a remarkable development in the area of statistical analysis of life data for both parametric and nonparametric models, specially in the case of one component system or in the classical competing risk theory set up in general. Formally for a one component system we associate with an item a pair of random variables (Z,d), where Z is the observed life time of the item, censored or otherwise and d indicates its failure mode i.e. whether the observation is censored or not. Let X_1 represent the lifetime (uncensored) of the system and X_2, the censoring time. Then $Z=\text{Min}(X_1,X_2)$ and $d=(I_1,I_2)$ with $I_1=1$, $I_2=0$ when $Z=X_1$ and $I_1=0$, $I_2=1$, otherwise. In most of the cases X_1 and X_2 are assumed to be independent, so that the survival function of Z is given by $\bar{F}(Z) = \bar{F}_1(Z)\,\bar{F}_2(Z)$, $0< Z <\infty$, where $\bar{F}_i(x)$ is the survival function of X_i, $i = 1,2$. The main interest lies in inferring about $\bar{F}_1(x)$ given the data, both in the parametric and in nonparametric set-ups. The competing risk theory problem (Altschuler (1970), Tsiastis (1975)) with K causes of failure, $K\geq 2$ is an obvious extension of the same problem and the methods applied are very similar. Important references for the parametric methods developed in this connection can be found in the following standard books on life data analysis, viz, (1) David and Moeschberger (1978), Lawless (1982,1983), Kalbfleisch and Prentice (1980), Mann, Schafer and Singpurwalla (1974), Bain (1978), Nelson (1982), Cox and Oakes (1984).

The most important breakthrough in nonparametric estimation of survival functions is the paper by Kaplan and Meier (1958), where the authors first introduce the Product Limit Estimator. Kaplan Meier Product Limit Estimator of $F_1(x)$ can be described as follows :

Let n independent observations of (Z, d) be denoted as (Z_j, d_j) with $d_j = (I_{1j}, I_{2j})$, j=1 2 ...n. Then the estimator is given in one of its versions by .

$$\overset{KM}{\bar{F}_1}(x) = \prod_{j:\ Z_{(j)} \le x} \left(\frac{n-j}{n-j+1} \right)^{I_{1j}}, \text{ if } x < Z_{(n)}$$

$$= 0 \qquad , \text{ if } x > Z_{(n)} \text{ ,} \qquad \ldots\ldots\ldots\ldots\ldots (1.1.1)$$

where $Z_{(1)} \le Z_{(2)} \ldots \le Z_{(n)}$ are ordered observations in the set $\{Z_1, Z_2, \ldots Z_n\}$. [The arbitrariness in the definition of $\bar{F}_1(x)$ for $x > Z_{(n)}$ where $Z_{(n)}$ corresponds to a censored observation is being ignored in the discussions] .

The estimator $\overset{KM}{\bar{F}_1}(x)$ has been proved to be nonparametric maximum likelihood estimator, consistent and asymptotically normal. When regarded as a stochastic process in $x \in R^+$, it converges to a Gaussian process and it is known to be a self consistent estimator. Important references on the development of the above mentioned properties of Kaplan Meier Product limit Estimator are Efron (1967), Breslow and Crowley (1974), Meier (1975), Reid (1981), Major and Rejto (1988), Peterson (1977), to mention some.

Another important breakthrough in survival data model of a one component system under random censoring is through the application of counting process via martingale arguments. Important references in this are :

Aalen (1976, 1978), Anderson *et al* (1982,1985), Borgan (1984), Burke *et al* (1981), Gill (1980) and So on. The role of covariates in survival data analysis is again another area which has received considerable attention of late and the development of methods based on partial likelihood and nonparametric regression deserves special mention. A few quick references in this area are Cox (1959, 72 ,75), Nelson (1982), Cox and Oakes (1984).

The other important development over the last few years in the area is the application of nonparametric models with random censorship under proportional hazard assumption. Let us describe it briefly :

In many situations it can be assumed that $\bar{F}_1(x)$ and $\bar{F}_2(x)$ are connected by the relation, $\bar{F}_2(x) = (\bar{F}_1(x))^{\beta_1}$, $\beta_1>0$. This assumption is known as proportional hazard assumption. $\beta_1 = 0$ represents no censoring and large values of β_1 corresponds to heavy censoring in this model. This model was first introduced by Koizol and Green (1976) and subsequently used by Hollander and Proschan (1979), Csorgo and Horvath (1981), Abdushukurov (1984), Ebrahimi (1985), Cheng and Lin (1984,1987), Ghorai and Rejto (1987), Csorgo (1988), Csorgo and Mielniczuk (1988), Ghorai (1989) and Veraverbeke (1989) and others. The assumption of proportional hazard model is not unrealistic. An example where such a model is very appropriate was given by Efron (1981). Under

the proportional hazard model, let $\alpha = (\beta_1+1)^{-1}$.
Then $\bar{F}_1(x)=(\bar{F}(x))^{\alpha}$, where $\bar{F}(x)=\bar{F}_1(x)\bar{F}_2(x)$ is the survival function of Z. Abdushukurov (1984) and Cheng and Lin (1984, 1987) independently studied large sample properties of the nonparametric maximum likelihood estimator of $\bar{F}_1(x)$, which is given by

$$\bar{\bar{F}}^{a}_1(x) = (\hat{\bar{F}}_1(x))^{\hat{\alpha}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.1.2)$$

where $\hat{\alpha} = \bar{I}_1$ and $\hat{\bar{F}}(x) = n^{-1}\sum_{j=1}^{n} I(Z_j > x)$. We call $\bar{\bar{F}}^{a}_1(x)$ the ACL estimator of $\bar{F}_1(x)$. Cheng and Lin (1984, 1987) have proved the weak convergence of $\bar{\bar{F}}^{a}_1(x)$ to an appropriate Gaussian process.

Ebrahimi (1985) considered the estimation problem of $\bar{F}_1(x)$ under proportional hazard assumption from a different view point of formulation. The estimator defined by Ebrahimi (85) is as follows:

$$\begin{aligned}\bar{\bar{F}}^{e}_1(x) &= \bar{I}_1 \mathrm{Exp}(\bar{I}_1 \log\hat{S}_1(x) - \bar{I}_1\log\bar{I}_1) \\ &\quad + \bar{I}_2 \mathrm{Exp}(\bar{I}_1 \log\hat{S}_2(x) - \bar{I}_1\log\bar{I}_2)\end{aligned}$$

where
$$\hat{S}_1(x) = n^{-1}\sum_{j=1}^{n} I(Z_j > x,\ I_{1j}=1)$$

$$\hat{S}_2(x) = n^{-1}\sum_{j=1}^{n} I(Z_j > x,\ I_{2j}=1) \qquad \ldots\ldots\ldots\ldots(1.1.3)$$

We call $\bar{\bar{F}}^{e}_1(x)$, Ebrahimi's estimator of $\bar{F}_1(x)$. Ebrahimi derived large sample properties of $\bar{\bar{F}}^{e}_1(x)$ and showed that it is consistent.

It may be noted that Doss *et al* (1989) followed a different approach to the problem of component survival function estimation for a general coherent structure under continuous monitoring assumption.

In our present work, the estimators developed provide really the extensions to the slightly more general coherent systems as pointed out in the last part of section 1.0, of the estimation procedures given by Abdushukurov (1984), Cheng and Lin (1984) and Ebrahimi (1985). In the more general case without proportional hazard assumption, we exploit with modifications for the procedure developed by Dewanji *et al* (1986) for the tumor sacrifice problem, to estimate the hazard functions for the components in a two component parallel system. We also sketch the estimation procedure in a k-1 out of k components system, k≥2.

1.2 SUMMARY AND RESULTS

1.2.1 Summary of Chapter 2

In this chapter we consider the estimation of survival function of component/components of the following systems : (i) One component system and (ii) Two components parallel system.

In all these systems component lives are assumed to follow independent exponential distributions and for all of them random censoring mechanism is used. In section 2.1 (a), the available nonparametric estimators of the survival function are compared vis-a-vis the maximum likelihood estimators, when X_1 and X_2 follow independent exponential distributions. The main results are as follows :

Hurt (1982) computed the relative efficiency of KM (Kaplan Meier) estimator with respect to Maximum Likelihood estimator in this particular context. In this situation when both X_1 and X_2 follow exponential distributions, proportional hazard assumption obviously holds. Hence the estimators $\overset{a}{\bar{\bar{F}}}_1(x)$ and $\overset{e}{\bar{\bar{F}}}_1(x)$ are supposed to be more appropriate here and are expected to behave better than $\overset{KM}{\bar{F}}_1(x)$. For completing the exercise of Hurt (1982) in studying the performance of nonparametric estimators in the context of exponential distributions for X_1 and X_2, we compute the relative efficiencies of $\overset{e}{\bar{\bar{F}}}_1(x)$ and $\overset{a}{\bar{\bar{F}}}_1(x)$ with respect to the maximum likelihood estimators. It is observed that among the three nonparametric estimators, the performance of $\overset{a}{\bar{\bar{F}}}_1(x)$ happens to be the best as expected. It is to be noted that the efficiency of the estimator goes down drastically with increase in the value of $\frac{x}{\theta}$. The comparative performance of $\overset{a}{\bar{\bar{F}}}_1(x)$, vis-a-vis the maximum likelihood estimator is found to be particularly good for small values of α, i.e. low level of censoring. The same exercise is also carried out for two parameter exponential distribution under proportional hazard assumption.

It is pointed out that the methods developed for a one component system can be easily applied to the competing risk situation comprising K components series system.

In section 2.2 we assume a parallel system with two components i.e.life of the system without censoring, $X=\text{Max}(X_1,X_2)$, where X_1 and X_2 represent the lives of components 1 and 2 respectively. Let X_3 represent the censoring time. X_1, X_2 and X_3 are assumed to be independent in general. The resulting life of the system is given by $Z = \text{Min}\ (\text{Max}(X_1,X_2),X_3)$. We also introduce an indicator variable $d = (I_1,\ I_2,\ I_3,\ I_4,\ I_5\ )$, which follows a multinomial distribution with 5 mutually exclusive classes which are determined by the practically feasible and recognizable interrelationship between the variables X_1, X_2 and X_3. The classes for d are identified as :

$$A_1 = \{Z : X_1 < Z, \quad Z = X_2, \quad X_3 > Z\}$$

$$A_2 = \{Z : X_2 < Z, \quad Z = X_1, \quad X_3 > Z\}$$

$$A_3 = \{Z : X_1 < Z, \quad Z = X_3, \quad X_2 > Z\}$$

$$A_4 = \{Z : X_2 < Z, \quad Z = X_3, \quad X_1 > Z\}$$

$$A_5 = \{Z : X_1 > Z, \quad Z = X_3, \quad X_2 > Z\}$$

............(1.2.1.1)

Here $I_i = 1$ and $I_j = 0$, $\forall\ j \neq i$ if and only if $Z \in A_i$, i=1,2 ....5. Three cases are investigated in section 2.2 and they are all parametric models. In case (a), it is assumed that $X_i \sim \text{Exp}(\theta_i)$, i=1,2,3 with no relation among the parameters θ_1, θ_2 and θ_3. Case (b) is same as case (a) with the additional assumption $\theta_1 = \theta_2 = \theta$ and case (c) is a modification of case (b) as follows : when both the components 1 and 2 are functioning, X_1 and X_2 follow an identical distribution which is $\text{Exp}(\theta)$. But when one of the components fails, the distribution of the residual life time of the surviving component is again assumed to be exponential but with an expected life θ', where $\frac{\theta}{2} < \theta' < \theta$. In all these cases, some sets of adhoc estimators for the parameters are proposed and their asymptotic properties are investigated. The suggested estimators are shown to be CAN (consistent and asymptotically normal) estimators. In each case the expression for the asymptotic variance covariance matrix of each set of estimators is derived. The asymptotic variances of the estimated mean life length and the asymptotic variances of the estimated system survival functions in appropriate situations are also computed. Also explicit expressions of the likelihood equations based on (Z,d) are derived in all cases. In each of the cases outlined above we indicate an iterative method for obtaining the maximum likelihood estimates. Since it has not been possible to obtain algebraic expressions for the variances of MLE's, by generating samples from given populations via simulated experiments, variances are numerically computed for sample size 100, for some reasonable combinations of the true values assumed by the parameters of the original distributions. Another sample size, viz. 50 was also tried. But for a large number of generated samples in the latter case, the iterative method suggested did not converge or led to some wild estimates, owing to zero frequencies in one or more classes specified for d. In this context, the numerical computation of variances of the estimates, based on simulated experiments is unreliable, and the results are not worth reporting. On the contrary the adhoc estimators proposed could be computed without any difficulty in all the simulated samples of both sizes 50 and 100, and the estimates were reasonably close to the population parameters chosen. The numerical results partly justify the use of the proposed adhoc estimators and give an idea about the comparative performances of the sets of estimators proposed.

The major part of this chapter is published, viz. Mukhopadhyay and Dhar (1990), Dhar (1991), Mukhopadhyay and Dhar (1992).

1.2.2 Summary of Chapter 3

In this chapter, we consider a parallel system with two independent components 1 and 2 as explained in the preceding section. Exponentiality of distributions is not assumed here, although continuity of the distributions is assumed. To state it clearly, we observe on failure of an individual system, along with the failure time of the system, the identity of the component failed last. If the lifetime is censored, then also we observe which, if any, of the components failed before censoring. Two sets of estimators for survival functions $(\bar{F}_1(.), \bar{F}_2(.))$ are proposed under the proportional hazard assumption which says

$$\bar{F}_3(.) = (\bar{F}_1(.))^{\beta_1} = (\bar{F}_2(.))^{\beta_1\phi_1}, \ \beta_1>0, \ \phi_1>0 \ .$$

This is obviously an extension of the proportional hazard model dealt with by Abdushukurov (1984), Cheng and Lin (1984,1987), and Ebrahimi (1985) to the case of a parallel system with two components under random censoring . Two sets of adhoc estimators of the component survival functions are proposed, viz, (a) $(\overset{e}{\bar{F}}_1(.), \overset{e}{\bar{F}}_2(.))$ and (b) $(\overset{a}{\bar{F}}_1(.), \overset{a}{\bar{F}}_2(.))$, where the superscript "e" ("a ") is used to denote Ebrahimi type (ACL type) estimators. The actual expressions for the Ebrahimi type estimators are given as

$$\overset{e}{\bar{F}}_1(x) = (\bar{I}_1 + \bar{I}_3 + \alpha\bar{I}_5)\mathrm{Exp}(\hat{\phi}_1(\hat{\beta}_1\hat{\phi}_1+1)^{-1}\log\hat{C}_1(x))$$

$$+ (\bar{I}_2 + \bar{I}_4 + (1-\alpha)\bar{I}_5)\mathrm{Exp}((\hat{\beta}_1+1)^{-1}\log\hat{C}_2(x))$$

$$\overset{e}{\bar{F}}_2(x) = (\bar{I}_1 + \bar{I}_3 + \alpha\bar{I}_5)\mathrm{Exp}(\ (\hat{\beta}_1\hat{\phi}_1+1)^{-1}\log\hat{C}_1(x))$$

$$+ (\bar{I}_2 + \bar{I}_4 + (1-\alpha)\bar{I}_5)\mathrm{Exp}((\hat{\phi}_1(\hat{\beta}_1+1))^{-1}\log\hat{C}_2(x))$$

where $\hat{C}_1(x) = \hat{S}_1(x) + \hat{S}_3(x) + \hat{S}_5(x)(1+(\hat{\beta}_1\hat{\phi}_1)^{-1})$

$$\hat{C}_2(x) = \hat{S}_2(x) + \hat{S}_4(x) + \hat{S}_5(x)(1+(\hat{\beta}_1{}^{-1})$$

and α is a proper weight function lying between 0 and 1, $(\hat{\beta}_1,\hat{\phi}_1)$ is an estimator of (β_1,ϕ_1) obtained from the partial likelihood based on the observations on d and $\hat{S}_i(x) = \frac{1}{n} \sum_{j=1}^{n} I(Z_j>x, I_{ij}=1)$, $i = 1,2 \ldots 5$ . ...(1.2.2.1)

The set of estimators (b) is really based on the application of Abdushukurov (1984) and Cheng and Lin (1984,1987) procedures to the present case of a two components parallel system under random censoring. Both the set of estimators are obtained in two stages. The first stage is taken to be same for both. In the first stage, parameters β_1 and ϕ_1 are estimated from the partial likelihood of the observations on d. These estimators are substituted for the true parameters in all subsequent computations. Then the exact mathematical relationships that exist between the survival functions $\bar{F}_1(x)$ or $\bar{F}_2(x)$ and the subsurvival functions are utilized in obtaining the proposed estimators in (a). In deriving the estimators in (b) in the present context, the method consists in solving an equation that holds relating $\bar{F}_1(x)$ or $\bar{F}_2(x)$ to $S(x)$= system survival function after plugging in the estimators of β_1 and ϕ_1 for the respective true parameters in the equation which incidentally in this case is

$$\psi(\bar{F}_2(x),\ S(x)) = (\bar{F}_2(x))^{\beta_1\phi_1+1} + (\bar{F}_2(x))^{\beta_1\phi_1+\phi_1} - (\bar{F}_2(x))^{\beta_1\phi_1+\phi_1+1} - S(x) \qquad \text{........(1.2.2.2)}$$

In section 3.1, it is shown under the set up described the sets of estimators (a) and (b) are both consistent. The asymptotic variances of the proposed estimators are also derived in section 3.1.

In section 3.2, we calculate numerically the asymptotic variances of the estimated survival probability of the system life evaluated under two sets of estimators (a) and (b). For the purpose of comparison the estimation problem is taken up for those points only where the true system survival probability is (i)0.90 and (ii)0.95. The same formula for comparison is used in all the cases of estimation problems considered in chapters 4 and 5 also. Asymptotic variances are numerically calculated and compared with that of the Kaplan and Meier Product Limit Estimator for an appropriate and useful range of values of parameters from Exponential and Weibull distributions of component lives. In case of $\overset{e}{\bar{\bar{F}}}(x)$, numerical values are reported for $\alpha=0.5$ only, since it has been observed numerically that $\alpha=0.5$ gives the smallest variance of $\overset{e}{\bar{\bar{F}}}(x)$ for all sets of parameters of life distributions included.

On examining appropriate tables we observe that both $\overset{e}{\bar{\bar{F}}}(x)$ and $\overset{a}{\bar{\bar{F}}}(x)$ appear to be superior to Kaplan-Meier estimator for all degrees of censoring, whereas among the two estimators, viz,

$\overset{e}{\bar{\bar{F}}}(x)$ and $\overset{a}{\bar{\bar{F}}}(x)$, $\overset{a}{\bar{\bar{F}}}(x)$ behaves better consistently in almost all situations. But for large values of β_1, there is not much difference in the performances in general.

In section 3.3 we assume that the components 1 and 2 follow identical life distributions i.e $\bar{F}_1(.) = \bar{F}_2(.)$ or $\phi_1 = 1$ and $\bar{F}_3(.) = (\bar{F}_1(.))^{\beta_1}$, $\beta_1 > 0$. Three sets of adhoc estimators are proposed making use of the properties which hold specially as a result of the identicality of the component life distributions, for the survival function $\bar{F}_1(.)$, viz, (a) $\bar{F}_1^{eI}(.)$, (b) $\bar{F}_1^{eII}(.)$ and (c) $\bar{F}_1^{aI}(.)$, Of these (a) and (b) are based on Ebrahimi type arguments and (c) is based on Abdushukurov or Cheng and Lin type arguments. It is to be noted that these estimators do not constitute special cases of the estimators in section 3.1. Here too essentially it is the two stage method which is used. In the first stage a closed form expression can be explicitly obtained for the maximum likelihood estimator of β_1 based on the partial likelihood defined for the set of observations on d. In section 3.1, because of the nonavailability of such closed form expressions for maximum likelihood estimators of β_1 and ϕ_1 some adhoc modification was used in the relevant equation by replacing $E(\frac{n_5}{n})$ to its observed value, $\frac{n_5}{n}$, where $n_i = \sum_{j=1}^{n} I_{ij}$, $i = 1,2 .. 5$, $\sum_{i=1}^{5} n_i = n$. The very fact that component life distributions are identical led to more than one comparable estimators of $\bar{F}_1(.)$, obtained by following the procedures described in section 3.1. Some appropriate weighted average of them should be legitimately proposed as an estimator of $\bar{F}_1(.)$ in this case and this is what is done in the second stage. These estimators are shown to be consistent. Asymptotic variances of the proposed estimators are also derived in section 3.3. In section 3.4 we calculate numerically the asymptotic variances of the estimated survival probabilities of the system as in the case of nonidentical component life distributions evaluated under the three sets of estimators (a) – (c) and compare them with (d)$\overset{KM}{\bar{F}}(.)$ i.e, Kaplan-Meier estimator. An appropriate range of β_1 and of the parameters in Exponential and Weibull distributions assumed for component lives are used for numerical comparison.

On examination, the relevant tables of the chapter reveal the fact that the estimators (a) $\overset{eI}{\bar{F}}(.)$,(b) $\overset{eII}{\bar{F}}(.)$ and (c)$\overset{aI}{\bar{F}}(.)$ all appear to be superior to $\overset{KM}{\bar{F}}(.)$. The ACL type estimator, viz, $\overset{aI}{\bar{F}}(.)$, appears to behave best in comparison with the other estimators in all cases. Again the difference between $\overset{eII}{\bar{F}}(x)$ and $\overset{aI}{\bar{F}}(x)$ does not appear to be of much consequence.

1.2.3 Summary of Chapter 4

In chapter 4, the procedures developed in chapter 3 are extended and applied with necessary modifications to a more general situation. The problem considered is the nonparametric estimation of survival functions of components in a k components coherent structure, in which the system functions if and only if at least k-1 of its components function properly (k≥2). As in chapter 3, proportional hazard assumption is retained. Let X_i represent the random variable associated with life length of component i, with absolutely continuous distribution function $F_i(.)$ and survival function $\bar{F}_i(.)$, i= 1,2 ... k, X_i's are assumed to be independent. Let X_c be the censoring random variable which represents failure due to other causes, not covered by the components 1,2 ... k, with absolutely continuous distribution function $F_c(.)$ and survival function $\bar{F}_c(.)$. X_c is assumed to be distributed independently of $X_1, X_2 \ldots X_k$. What we observe in reality is the realized value of a random variable Z which represents the observed life of the system, censored or otherwise and a value of the indicator variable $d=(I_{12}, I_{13} \ldots I_{1k} \ldots I_{kc}, I_{co})$ -- I_{K-1K}, I_{1C} which follows a multinomial distribution with (k^2+1) classes determined by practically feasible and recognizable interrelationship between the variables $X_1, X_2 \ldots X_k$ and X_c i.e, along with the life of the system, censored or otherwise what is known is the identity of the components in order which failed etc. if any.

In section 4.1 we describe a general procedure for the estimation of life lengths of components 1,2 ... k under the proportional hazard assumption :

$$\bar{F}_c(.) = (\bar{F}_1(.))^{\beta_1} = (\bar{F}_2(.))^{\beta_1 \phi_1} = \ldots (\bar{F}_k(.))^{\beta_1 \phi_{k-1}} ,$$

where $\beta_1 > 0$ and $\beta_2 = \beta_1 \phi_1 > 0 \ldots \ldots \beta_k = \beta_1 \phi_{k-1} > 0$.

In section 4.2 we consider the estimation of survival functions in the special case when $X_1, X_2 \ldots X_k$ are identically distributed in addition i.e, $\phi_1 = \phi_2 \ldots = \phi_{k-1} = 1$. In this special case the common distribution function is denoted by $F_1(.)$. It is possible to find here explicitly the maximum likelihood estimator of β_1 from the partial likelihood based on the observations on d. By following procedures similar to those proposed by Ebrahimi (1985), Abdushukurov (1984) and Cheng and Lin (1984,1987), three sets of adhoc estimators of the survival function $\bar{F}_1(.)$, viz, (a) $\bar{F}_1^{e_{kI}}(.)$, (b) $\bar{F}_1^{e_{kII}}(.)$ and (c) $\bar{F}_1^{a_k}(.)$ are proposed. Of these estimators, (a) and (b) are obtained by using Ebrahimi type arguments and slightly different mathematical relations which hold, and the estimator (c) is derived by following Abdushukurov and Cheng and Lin type arguments. It is observed that all these estimators are consistent. Asymptotic variances of these proposed estimators are also derived. It is to be noted that in case of identical component life distributions, the method applied is a simple generalization and extension of the method developed in section 3.3 of chapter 3 where the special case k=2 is dealt with.

In section 4.3, to demonstrate the estimation procedure in a nonidentical set-up, we consider as a special case a two out of three system, i.e, we assume k=3. It has been pointed out that the case of general k does not lead to a generally acceptable solution which is expected to behave well for all k≥2. The estimation procedure for k=3 is essentially different from that proposed for k=2. Hence the specific development of appropriate procedures in this case becomes necessary and is related to show how the difficulty of the problem increases with k. In this case $d = (I_{12}, I_{13}, I_{1c}, I_{21}, I_{23}, I_{2c}, I_{31}, I_{32}, I_{3c}, I_{co})$ is the indicator variable, which follows a multinomial distribution with 10 mutually exclusive and exhaustive classes determined by the random variables X_1, X_2, X_3 and X_c. Two sets of estimators of survival functions associated with components 1,2 and 3 are proposed, viz, (a) $(\overset{e}{\bar{F}}{}^3_1(.), \overset{e}{\bar{F}}{}^3_2(.), \overset{e}{\bar{F}}{}^3_3(.))$ and (b) $(\overset{a}{\bar{F}}{}^3_1(.), \overset{a}{\bar{F}}{}^3_2(.), \overset{a}{\bar{F}}{}^3_3(.))$. Here the set of estimators (a) is based on Ebrahimi type arguments and the set of estimators (b) is based on Abdushukurov

(1984) and Cheng and Lin(1984,1987) type arguments. It is observed that under the above set up estimators (a) and (b) are consistent. The asymptotic variances of the proposed sets of estimators are also derived. We calculate numerically the asymptotic variances of the estimated system survival function by using the two sets of adhoc estimators proposed, viz, (a) and (b). These asymptotic variances are numerically calculated and compared with those of the corresponding Kaplan and Meier Product limit estimators. The numerical computations are carried out for useful ranges of values of parameters assuming Exponential and Weibull distributions for component lives. In general the set of estimators (b) is observed to behave better in high censoring cases and the set of estimators (a) is observed to behave better in low censoring cases.

In section 4.4 we take up the numerical investigation of the case described in section 4.2. The asymptotic variances of the estimated system survival function, viz, (a) $\bar{F}^{e_{KI}}(.)$ (b) $\bar{F}^{e_{KII}}(.)$ and (c) $\bar{F}^{a_K}(.)$ are computed in the special case k=3, under the assumption of identical distribution of component lives. The numerical values are compared with the asymptotic variances of the corresponding Kaplan Meier Product limit estimator for appropriate ranges of values of parameters, assuming Exponential and Weibull distributions for a component life. In general the ACL type estimators (c) is observed to behave better than all other rival estimators proposed.

In section 4.5, it is pointed out that the essence of the general two stage procedure of estimation dealt with in chapter 3 and earlier sections of chapter 4 can be easily extended to other coherent systems, although the details of procedures are bound to vary. To indicate the potentiality of the essential technique developed, a brief outline is provided for following the two stage approaches in deriving suitable estimators of relevant survival functions in the case of a so called series parallel system.

1.2.4 Summary of Chapter 5

In section 5.1 of chapter 5 we consider a one out of two components system under random censoring as in chapter 3. But proportional hazard assumption is done away with. Again the problem is considered to be more general, in the sense that the dependence of the life distributions of components is permitted. In this set up, a procedure is developed to estimate by maximum likelihood method of survival functions of components or other quantities of interest via EM type algorithm (Dempster *et al* (1977))as done in section 2.2 and chapter 3. The problem is first formulated as one of finding the maximum likelihood estimators of hazard rates from the given incomplete data set. The hazard rates to be estimated are :

$$\lambda_j(t) = \lim_{\Delta t \to 0+} \frac{1}{\Delta t} P(T_1 \in [t, t+\Delta t), J_1 = j \mid T_1 \geq t), \quad j = 1,2.$$

$$\lambda_{12}(t|u) = \lim_{\Delta t \to 0+} \frac{1}{\Delta t} P(T_2 \in [t, t+\Delta t), J_2 = 2 \mid T_2 \geq t, T_1 = u, J_1 = 1)$$

$$\lambda_{21}(t|u) = \lim_{\Delta t \to 0+} \frac{1}{\Delta t} P(T_2 \in [t, t+\Delta t), J_2 = 1 \mid T_2 \geq t, T_1 = u, J_1 = 2) \quad \ldots\ldots\ldots\ldots(1.2.4.1)$$

where T_1 and T_2 denote first failure and second failure times and

$J_1(J_2)$ denotes the identity of the corresponding component at the first (second) failure time point. The five classes into which the observations fall are the same as explained in section 2.2 and chapter 3. Following the partial likelihood approach introduced by cox(1975), the appropriate likelihood for the given incomplete data set is written down in terms of hazard rates in (1.2.4.1). Then methods are developed via EM algorithmic approach for finding nonparametric maximum likelihood estimates of the quantities in (1.2.4.1). Identifiability problem is tackled by imposing simple conditions leading to unique estimates of the quantities of interest. A simple method of estimation of the variance is also developed. Also indicated is an application of the method through simulated experimental data, assuming exponential life distributions of components. In section 5.2 we outline an extension of the procedure developed in section 5.1 to the more general case of $K-1$ out of K components system, $K\geq 2$. The purpose is to indicate how the same approach can be applied to more general coherent systems. But Mathematically the treatment becomes more and more difficult as the complexity of the model increases. The essential idea of the overall approach in this chapter originates from a somewhat similar problem investigated by Dewanji *et al* (1986). But the problems are different and the details have to be worked out afresh.

CHAPTER II

PARAMETRIC MODELS

CHAPTER 2

PARAMETRIC MODELS

2.0 INTRODUCTION

In this chapter we consider the estimation of survival functions for the component/components of the following systems : (1) One component system, and (2) Parallel system with two components. In both the systems component lives are assumed to follow independent exponential distributions and for all of them random censoring mechanism is assumed.

(1) One component system under random censoring which has been dealt with by various authors extensively and intensively from both the parametric and the nonparametric view points as explained in the following lines can be broadly described as :

Suppose X_1 is a random variable representing time to failure or life time of the component. Together with this random variable, is considered another random variable X_2 which represents censoring time under random censoring. What we observe in reality is the random variable (Z,d), where $Z = \text{Min}(X_1,X_2)$ and $d = (I_1,I_2)$ is the indicator, where $I_1 = 1$ if $Z = X_1$ and $I_2 = 1$ if $Z = X_2$. In general X_1 and X_2 are assumed to be independent. Suppose the random variable X_1 is distributed with survival function $\bar{F}_1(.)$ and the random variable X_2 is distributed with survival function $\bar{F}_2(.)$, both assumed to be absolutely continuous. Then if $\bar{F}(.)$ represents survival function of the random variable Z, then $\bar{F}(.) = \bar{F}_1(.)\bar{F}_2(.)$.

The usual statistical problem given the data (Z_j, d_j), $j = 1,2,\ldots..n$, is to estimate the survival function $\bar{F}_1(.)$ or in the case the distribution is known, to estimate its parameters. The most popular nonparametric estimator of the survival function is the well known Kaplan and Meier Product limit estimator which we will denote by KM estimator. For the optimal properties of the Kaplan Meier Product limit estimator, one is referred to Kaplan and Meier (1958),Efron (1967), Breslow and Crowley (1974), Aallen (1976), Gill (1980) and Reid (1981). Assuming exponential distributions for both X_1 and X_2, Hurt (1982) computed relative efficiency of the KM estimator, with respeet to the Maximum likelihood Estimator, (denoted by MLE) and investigated the performance of KM estimator in this context. In situations $\bar{F}_2(.) = (\bar{F}_1(.))^{\beta_1}$ (i.e. proportional hazard model) where β_1, the censoring parameter is >0,Abdushukurov(1984),Cheng and Lin(1984), Ebrahimi (1985), proposed other estimators of the survival function $\bar{F}_1(.)$. The estimators proposed by (i) Ebrahimi and (ii) Abdushukurov and Cheng and Lin are naturally expected to behave better than the KM estimator under proportional hazard assumption, which is satisfied when both X_1 and X_2 follow exponential distribution. The main purpose in this section is to compare the different nonparametric estimators, viz, (i) KM estimator, (ii) Ebrahimi's estimator and (iii) Abdushukurov and Cheng and Lin (denoted by ACL) estimator,vis-a-vis the MLE in the context of exponential distributions of X_1 and X_2, and to update and complete Hurt's (1982) table giving comparative efficiencies.

For the sake of completeness, we consider the expressions for the variance of Ebrahimi's (1985) estimator and ACL estimator, in general and in particular under the assumption of exponential distributions of X_1 and X_2 and compute their relative efficiencies w.r.t MLE in this case. The results giving a comparative picture of these estimators are contained in section 2.1(a). In section 2.1(b) we assume X_1 and X_2 to follow two parameter exponential distributions under proportional hazard model i.e. $\bar{F}_2(.)=(\bar{F}_1(.))^{\beta_1}$, $\beta_1>0$. In this case a similar comparison is carried out between the different estimators described and their efficiencies vis-a-vis the MLE are computed. It is pointed out that the methods employed for a one component system can be easily applied to the competing risk situation comprising a K components series system.

In section 2.2, we assume a parallel system with two components i.e. life of the system without censoring is $X = \mathrm{Max}(X_1,X_2)$, where X_1 and X_2 represent the lives of components 1 and 2 respectively. Let X_3 represent the censoring time. X_1, X_2 and X_3 are assumed to be independent in general, the resulting life of the system is given by $Z = \mathrm{Min}(\mathrm{Max}(X_1,X_2),X_3)$. We also introduce an indicator variable d, which follows a multinomial distribution with 5 mutually exclusive classes which are determined by the practically feasible and recognizable interrelationship between the variables X_1,X_2 and X_3. Three cases are investigated in section 2.2. In case (a), it is assumed that $X_i \sim \mathrm{Exp}(\theta_i)$, i=1,2,3, with no relationship among the parameters θ_1, θ_2 and θ_3, Case (b) is same as case (a) with the additional assumption $\theta_1 = \theta_2 = \theta$, say. Case (c) is a modification of case

(b) as follows: when both the components 1 and 2 are functioning, the failure times associated with them, viz, X_1 and X_2 follow the same Exp(θ) distribution. But when one of the components fails, the distribution of the failure time of the surviving component is again assumed to be exponential with parameter θ', where $\theta/2 < \theta' < \theta$.

In all these cases in section 2.2, some sets of adhoc estimators for the parameters are proposed and their asymptotic properties are investigated. The suggested estimators are shown to be CAN (Consistent Asymptotically Normal) estimators. In each case the expression for the asymptotic variance covariance matrix of each set of estimators is obtained. The asymptotic variances of the estimated mean life length and the asymptotic variances of the estimated system survival function in appropriate situations are also computed. Also explicit expressions of the likelihood equations based on the observations on (Z, d) are derived in all these cases. In each of the cases outlined above we indicate an iterative method for obtaining the maximum likelihood estimates. Since it has not been possible to obtain algebraic expressions for the variances of MLE's, by generating samples from given populations, via simulated experiments, variances are numerically computed for sample size 100 for some reasonable combinations of the true values assumed by the parameters of the original distributions. Another sample size, viz, 50 was also tried. But for a large number of generated samples, in the later case, the iterative method suggested either did not converge or led to some wild estimates, owing to zero frequencies in one or more classes specified for d. In this context the numerical compution of variances of the estimates based on simulated experiments are erratic and unreliable, not worth reporting. On the contrary adhoc estimators proposed can be computed without any difficulty from the generated samples and the estimates were reasonably close to the population parameters chosen. Thus the numerical results obtained w.r.t MLE for sample size 50, being unreliable are not reported. The numerical results partly justify the use of the proposed adhoc estimators and an idea about the performance of these adhoc estimators vis-a-vis MLE's can be had from the numerical results provided.

Possible area of application of the model in section 2.2 includes the biomedical problem of the life of a human being with two kidneys which function in parallel. The observation constitutes the life of a person and random censoring consists in the termination of life by any cause other than the failure of both the kidneys. In Industrial application of the model the system under consideration consists of two components 1 and 2 arranged in parallel and the module formed by these two components is serially connected with component 3.

The major contents of this chapter are published in Mukhopadhyay and Dhar (1990), Dhar (1991) and Mukhopadhyay and Dhar (1992).

2.1 ONE COMPONENT SYSTEM

(a) One Parameter Exponential Distribution for component life time and censoring time.

Let X_1 denote the time to failure or life time of the component with distribution function $F_1(.)$. X_2 is the censoring time, with distribution function $F_2(.)$. X_1 and X_2 are assumed to be independent. Moreover, it is assumed that $\bar{F}_1(x) = \text{Exp}(-(x/\theta))$, $x \geq 0$, $\theta > 0$, $\bar{F}_2(x) = (\bar{F}_1(x))^{\beta_1} = \text{Exp}(-(\beta_1 x)/\theta)$, $\beta_1 \geq 0$, the censoring parameter, $\beta_1 = 0$ represents no censoring. What we observe in reality is the random variable $Z = \text{Min}(X_1, X_2)$ which denotes the actual observed life of the system and an indicator variable $d = (I_1, I_2)$, $I_1, I_2 = 0,1$, $I_1+I_2=1$. Clearly d follows a bionomial distribution with two mutually exclusive classes determined by the variable X_1 and X_2. The classes for d are identified as.

$$A_1 : \quad Z = X_1,\ X_2 > Z$$

$$A_2 : \quad Z = X_2,\ X_1 > Z. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.1.1)$$

where $I_1=1$ if and only if $Z \in A_1$ and $I_2=1$ if and only if $Z \in A_2$. We write A_i for the set of Z which satisfies the conditions of A_i in (2.1.1), i=1.2 .

Thus $A_i = (Z \in R^+ \mid I_i = 1)$, i=1,2 $\ldots\ldots\ldots\ldots\ldots$ (2.1.2)

The data consists of n independent observations (Z_j, d_j), j=1,2,....n, on the random variable (Z,d). We will also write Z_{ij} as the value of Z_j if $Z_j \in A_i$, i=1,2, when needed. Let F(Z) represent the distribution function of Z. Then $\bar{F}(.) = \bar{F}_1(.).\bar{F}_2(.)$ obviously.

we observe that

$$P(I_1=1) = E(I_1) = P(Z \in A_1) = \int_0^{\infty} \bar{F}_2(y) dF_1(y)$$

$$= \int_0^{\infty} \text{Exp}((-\beta_1 y)/\theta)\ (1/\theta)\text{Exp}(-y/\theta)dy$$

$$= (\beta_1+1)^{-1} = \alpha \quad \text{(say)} \quad \ldots\ldots\ldots\ldots (2.1.3)$$

$$\text{Hence, } \bar{F}(x) = \text{Exp}(-x/\theta\alpha) \quad \ldots\ldots\ldots\ldots (2.1.4)$$

So, the likelihood based on the data (Z_j, d_j), $j=1,2,\ldots\ldots,n$ is given by $L(.) = \prod_{j=1}^{n} (dF_1(Z).\bar{F}_2(Z))^{I_{1j}} \prod_{j=1}^{n} (dF_2(Z)\bar{F}_1(Z))^{I_{2j}}$ ....(2.1.5)

Z_j's and d_j's are independently distributed in this case. MLE's for θ and α are known to be from Hurt (1982) as

$$\hat{\theta} = (\bar{Z})(\bar{I}_1)^{-1}, \; \hat{\alpha} = \bar{I}_1 \quad \ldots\ldots(2.1.6)$$

where $$\bar{Z} = \sum_{j=1}^{n} Z_j/n \text{ and } \bar{I}_1 = \sum_{j=1}^{n} I_{1j}/n \quad \ldots\ldots(2.1.7)$$

Hurt (1982) has proved asymptotic normality of $\hat{\theta}$, has obtained the expression for its bias and also has obtained the asymptotic variance of MLE of $\hat{\bar{F}}_1(x)$ of $\bar{F}_1(x)$. The results are stated below :

$$\text{Asym Var}(\hat{\bar{F}}_1(x,\theta)) = n^{-1}(\bar{F}_1(x))^2\alpha^{-1}(x/\theta)^2 + o(n^{-1}), \quad \ldots\ldots(2.1.8)$$

where the symbol o (little o'h) denotes as follows: comparing the magnitude of two functions u(x) and v(x) as the argument x tends to α, the notation $u(x) = o(v(x))$ stands for $\underset{x \to \alpha}{\text{Lt}} \frac{u(x)}{v(x)} = 0$

Kaplan Meier Product limit or KM estimator of $\bar{F}_1(x)$, is given by

$$\overset{KM}{\bar{F}_1}(x) = \prod_{Z_j \le x} \left(\frac{n-R_j}{n-R_j+1}\right)^{I_{1j}}, \text{ if } x < Z_{(n)}$$

$$= 0 \qquad , \text{if } x > Z_{(n)} \quad \ldots\ldots(2.1.9)$$

Where $Z_{(n)} = \text{Max}(Z_1,Z_2, \ldots,Z_n)$ is the largest order statistic of $Z_1,Z_2,\ldots\ldots,Z_n$ and R_j is the rank of $(Z_j, 1-I_{1j})$, $j = 1,2,\ldots,n$ in the lexicographic ordering of the sequence

$$(Z_1,1-I_{11}),(Z_2,1-I_{12}),\ldots\ldots(Z_n, 1-I_{1n}).$$

In particular the estimator is known to be a non-parametric maximum likelihood estimator and asymptotically normal. Efron

(1967), Breslow and crowley (1974), Reid (1981) and many others investigated the properties of Kaplan and Meier Product-limit estimator. Asymptotic variance of $\overset{KM}{\bar{F}}(.)$ was first derived by Breslow and crowley. From Breslow and crowley (1974), we have

$$\text{Asym var } (\overset{KM}{\bar{F}_1}(x)) = n^{-1}(\bar{F}_1(x))^2\alpha\,[\text{Exp}(x/\theta\alpha)-1]$$

where $\bar{F}_1(x) = \text{Exp}(-x/\theta),\ x \geq 0,\ \theta>0$ ....(2.1.10)

Hurt (1982) computed the asymptotic efficiency of $\overset{KM}{\bar{F}_1}(x)$ compared to $\hat{\bar{F}}_1(x)$. It has already been stated that the condition for Abdushukurov (1984) & Cheng and Lin (1984) (ACL) estimator and Ebrahimi's (1985) estimator is satisfied here. So their estimators can also be used. The estimator by Ebrahimi is given as follows:

$$\overset{e}{\bar{F}_1}(x) = \bar{I}_1\ \text{Exp}(\bar{I}_1\log \hat{S}_1(x) - \bar{I}_1\log\bar{I}_1)$$

$$+\ \bar{I}_2\text{Exp}(\bar{I}_1\log \hat{S}_2(x)-\bar{I}_1\log\bar{I}_2) \text{ .... } (2.1.11)$$

where $S_1(x) = P(\ Z>x,\ I_1 = 1\ ) =$ Subsurvival function of class

$$A_1 = \int_x^\infty dF_1(y)\bar{F}_2(y) = \alpha(\bar{F}_1(x))^{\alpha^{-1}}$$

$$S_2(x) = P(\ Z>x,\ I_2=1\ ) = \text{Subsurvival function of class } A_2$$

$$= \int_x^\infty dF_2(y)\ \bar{F}_1(Y) = (1-\alpha)(\bar{F}_1(x))^{\alpha^{-1}} \text{...............} (2.1.12)$$

where $\hat{S}_1(x), \hat{S}_2(x)$ are the empirical subsurvival functions, i.e., the sample analogues of $S_1(x)$ and $S_2(x)$ respectively, where

$$\hat{S}_1(x) = n^{-1}\sum_{j=1}^{n} I(Z_j\ >x, I_{1j}=1) \quad \text{and}$$

$$\hat{S}_2(x) = n^{-1}\sum_{j=1}^{n} I(Z_j\ >x, I_{2j}=1) \text{ .... } (2.1.13)$$

Since by the strong law of large numbers, $(\hat{S}_1(x), \hat{S}_2(x), \bar{I}_1, \bar{I}_2) \xrightarrow{P}$

$(S_1(x), S_2(x), \Pi_1, \Pi_2)$. Consequently since $\overset{e}{\bar{F}}_1(x)$ is a continuous function of $(\hat{S}_1(x), \hat{S}_2(x), \bar{I}_1, \bar{I}_2)$, we can argue that $\overset{e}{\bar{F}}_1(x)$ is a consistent estimator of $\bar{F}_1(x)$ obtained from (2.1.11) by substituting $(\hat{S}_1(x), \hat{S}_2(x), \bar{I}_1, \bar{I}_2)$ for $(S_1(x), S_2(x), \Pi_1, \Pi_2)$. The rationale for proposing the estimator (2.1.11) as described by Ebrahimi (1985) is provided by the two Mathematical relationships that exist between $\bar{F}_1(x)$ on the one hand and $S_1(x), S_2(x), \Pi_1$ and Π_2 on the other, viz, $\bar{F}_1(x) = \mathrm{Exp}(\Pi_1 \log S_1(x) - \Pi_1 \log \Pi_1)$

$= \mathrm{Exp}(\Pi_1 \log S_2(x) - \Pi_1 \log \Pi_2)$. In both the expressions, $(\Pi_1, \Pi_2, S_1(x), S_2(x))$ is replaced by $(\bar{I}_1, \bar{I}_2, \hat{S}_1(x), \hat{S}_2(x))$. If we call the corresponding estimators $\overset{e1}{\bar{F}}_1(x)$ and $\overset{e2}{\bar{F}}_1(x)$, Ebrahimi's estimator $\overset{e}{\bar{F}}_1(x)$ is the weighted average of these two, the weights suggested being the natural ones, viz, $\bar{I}_1$ and $\bar{I}_2$ respectively.

By Taylor's series expansion about $(\alpha, S_1(x), S_2(x))$ and using the fact that Z and $d = (I_1, I_2)$ are independent, we have on simplification the expression for the asymptotic variance as

$$\text{Asym var}(\overset{e}{\bar{F}}_1(x)) = n^{-1}(\bar{F}_1(x))^2(\alpha(1-\alpha)(x/\theta\alpha)^2 + \alpha^3(\mathrm{Exp}(x/\theta\alpha)-\alpha) + \alpha^2(1-\alpha)(\mathrm{Exp}(x/\theta\alpha) - (1-\alpha)) - 2\alpha^3(1-\alpha)) + o(n^{-1}) \quad \text{..... (2.1.14)}$$

The estimator proposed by Abdushukurov (1984) and Cheng and Lin (1984), 1987), called the ACL estimator in the present context is as follows : if

$$\overset{a}{\bar{F}}_1(x) = 0 \quad \text{if } x < Z_{(1)}$$
$$= 1-(l/n)^{\bar{I}_1} \quad \text{if } Z_{(l)} \le x \le Z_{(l+1)}, l=1,2.....n-1$$
$$= 1 \quad \text{if } x > Z_{(n)} \quad \text{.............(2.1.15)}$$

where $Z_{(1)} < Z_{(2)} \ldots\ldots < Z_{(n)}$ are the ordered observations in the set $\{Z_1, Z_2, \ldots\ldots, Z_n\}$. Asym var$(\overset{a}{\bar{F}}_1(x)$ from Cheng and Lin (1984) reduces to :

$$\text{Asym var}(\overset{a}{\bar{F}}_1(x)) = \bar{n}^1(\bar{F}_1(x))^2(\alpha^2(\text{Exp}(x/\theta\alpha)-1)) + \alpha(1-\alpha)\ (x/\theta\alpha)^2) + o(n^{-1}) \quad \ldots\ldots\ldots\ldots(2.1.16)$$

It is to be noted that under the proportional hazard assumption, $\overset{a}{\bar{F}}_1(x)$ is the nonparametric MLE of $\bar{F}_1(x)$. For other properties of the estimator, one is referred to Cheng and Lin (1984,1987).

Asymptotic Efficiencies of these estimators mentioned, compared with the MLE for the specific problem considered are given as follows :

$$e_1(\text{KM Compared with MLE}) = \frac{\text{Asym var}(\hat{\bar{F}}_{1\,\text{KM}}(x))}{\text{Asym var}(\bar{F}_1(x))}$$

$$= \frac{\alpha^{-1}(\bar{F}_1(x))^2(x/\theta)^2}{\alpha(\bar{F}_1(x))^2(\text{Exp}(x/\theta\alpha)-1)}$$

$$= \alpha^{-2}(x/\theta)^2\ (\text{Exp}(x/\theta\alpha)-1)^{-1} \quad \ldots.(2.1.17)$$

$$e_2(\text{Ebrahimi's estimator compared with MLE}) = \frac{\text{Asym var}(\hat{\bar{F}}_1(x))}{\text{Asym var}(\bar{F}_1^e\ (x))}$$

$$=\alpha^{-1}(\bar{F}_1(x))^2(x/\theta)^2((\bar{F}_1(x)^2(\alpha(1-\alpha)(x/\theta\alpha)^2) + \alpha^3(\text{Exp}(x/\theta\alpha)-\alpha)$$

$$+ \alpha^2(1-\alpha)\ (\text{Exp}(x/\theta\alpha)-(1-\alpha\ )) - 2\alpha^3(1-\alpha\ ))^{-1} \ldots\ldots\ldots\ldots\ldots(2..1.18)$$

$$e_3(\text{ACL estimator compared with MLE}) = \frac{\text{Asym var } ((\hat{\bar{F}}_1(x))}{\text{Asym var}(\overset{a}{\bar{F}}_1(x))}$$

$$= \frac{\alpha^{-1}(\bar{F}_1(x))^2(x/\theta)^2}{\alpha(\bar{F}_1(x))^2(\alpha(\text{Exp}(x/\theta\alpha)-1)+(1-\alpha)(x/\theta\alpha)^2)}$$

$$= \bar{\alpha}^{2}(x/\theta)^{2}(\alpha(\mathrm{Exp}(x/\theta\alpha)-1) + (1-\alpha)(x/\theta\alpha)^{2})^{-1} \quad \ldots\ldots(2.1.19)$$

e_1, e_2, e_3 are all functions of x, given θ and β_1. On writing

$y = \mathrm{Exp}(x/\theta\alpha)$, e_1, e_2, e_3 reduce to the forms

$$e_1 = (\alpha \log y)^{2}\alpha^{-2}(y - 1)^{-1}$$

$$e_2 = (\alpha \log y)^{2}((\alpha \log y)^{2}(1-\alpha)+\alpha^{4}(y-\alpha)+\alpha^{3}(1-\alpha)(y-1+\alpha)-2\alpha^{4}(1-\alpha))^{-1}$$

$$e_3 = (\log y)^{2}(\alpha(y-1)+(1-\alpha)(\log y)^{2})^{-1} \quad \ldots\ldots\ldots\ldots\ldots(2.1.20)$$

Table 2.1 gives these relative efficiencies e_1, e_2 and e_3, computed for different (x/θ) and $\alpha = P(X_1 \leq X_2)$ = level of censoring.

(Contd...)

TABLE 21

α	0.80			0.66			0.50			0.33			0.20		
x/θ	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
.50	.445	.507	.519	.504	.604	.655	.582	.736	.902	.647	.848	.967	.559	.838	.973
1.0	.627	.678	.697	.646	.734	.803	.626	.770	.954	.472	.726	.876	.170	.505	.747
1.5	.637	.687	.706	.597	.689	.748	.471	.641	.763	.228	.437	.585	.031	.138	.152
2.0	.559	.622	.628	.471	.572	.610	.298	.460	.520	.089	.211	.246	.044	.023	.023
2.5	.449	.613	.514	.339	.434	.452	.167	.290	.312	.031	.084	.087	.001	.002	.002
3.0	.339	.390	.396	.227	.306	.312	.089	.164	.171	.009	.027	.007	.001	.002	.003

On examining Table 2.1 it is found that both the ACL estimator and Ebrahimi's estimator behave better than the Kaplan Meier estimator in terms of asymptotic relative efficiency. Among the three, the performance of the ACL estimator happens to be the best as expected. It is to be noted that efficiencies of these estimators go down drastically with increase in the value of x/θ for all levels of censoring. The performance of the ACL estimator vis-a-vis the maximum likelihood estimator is found to be better for smaller values of α, i.e., low level of censoring.

The purpose of the above exercise is to make a comparative study of the different known non-parameteric estimates of $\bar{F}_1(x)$ in the simplest case of exponential distributions for X_1 and X_2. Of course in case of known distributional forms for X_1 and X_2, one is expected to use always the parametric MLE. But when the distributional form is in doubt, one uses the other more robust,(but less efficient in specific cases) nonparametric estimators. From the above exercise, we have an idea about the extent of this loss of efficiency and how it depends on the level of censoring and other variables of interest, in the simplest case when the exponential distribution of X_1 and X_2 happens to be right. Hurt (1982) computed e_1 only and the purpose here has been mainly to update Hurt's Table and extend it to e_2 and e_3 also, efficiencies of the non-parametric estimators which are more relevant in the present context.

2.1 (b) **Two Parameter Exponential Distribution for the component Life Time and Censoring Time under Proportional Hazard Assumption.**

Here we assume $\bar{F}_1(x) = \text{Exp}(-(\frac{x-\mu}{\theta}))$ $\theta \geq 0$ and $x \geq \mu$,

$\bar{F}_2(x) = (\bar{F}_1(x))^{\beta_1}$ $\beta_1 > 0$ as before, β_1 = censoring parameter, X_1 and X_2 are independent as before. Let us write in this case $Z' = \text{Min}(X_1, X_2)$. [we are writing Z' not Z as in section 2.1(a) for the sake of notational convenience, which will be clear as the devlopment follows], $d = (I_1, I_2)$ is the indicator variable,

where $I_1 = 1$ if $Z' = X_1$ and $I_2 = 1$ if $Z' = X_2$ ....(2.1.21)

So, here (Z'_i, I_{1i}, I_{2i}), $i=1,2,\ldots,n$ constitute the set of observations. Obviously, (Z'_i, I_{1i}, I_{2i}) are iid r.v's with common survival function of Z' as :

$\bar{F}(x) = P(Z' > x) = \bar{F}_1(x)\,\bar{F}_2(x)$, since X_1 and X_2 are independent.

Here too as in the earlier case, $P(I_1=1) = E(I_1) = \int_{\mu}^{\infty} dF_1(y)\bar{F}_2(y)$

$$= (\beta_1+1)^{-1} = \alpha \quad \text{(say)} \qquad \ldots\ldots(2.1.22)$$

So the MLE of (μ, θ, α) from the set of observations (Z'_i, I_{1i}, I_{2i}), $i = 1,2,\ldots,n$ is given by $\hat{\mu} = Z'_{(1)}$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} Z'_i - Z'_{(1)}}{n\bar{I}_1}$$

$$\hat{\alpha} = \bar{I}_1 \qquad \ldots\ldots(2.1.23)$$

where $Z'_{(1)} \le Z'_{(2)} \ldots\ldots \le Z'_{(n)}$ are the ordered observations in the set $\{Z'_1, Z'_2, \ldots, Z'_n\}$.

Let $Z_i = (n-i+1)(Z'_{(i)} - Z'_{(i-1)})$, $i=1,2,\ldots,n$ reading $Z'_{(0)} = \mu$, It is known that Z_i 's are iid r.v's having common survival function

$\text{Exp}(-\frac{Z}{\theta\alpha})$ $Z \geq 0$ and Z_i's are independent of $d_i = (I_{1i}, I_{2i})$, $i = 1,2,\ldots\ldots\ldots,n$. Now $\hat{\theta}$ in (2.1.23) can be re-written as

$$\hat{\theta} = \sum_{i=2}^{n} Z_i / n\bar{I}_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1.24)$$

$$E(Z) = \int_0^{\alpha} P(Z>z)dz = \int_0^{\alpha} \text{Exp}(\frac{-z}{\theta\alpha})dz = \theta\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.25)$$

writing Z as a random variable following the same distribution as that of Z_1,

$$\frac{E(Z)}{E(I_1)} = \theta$$

So, $\sqrt{n}\ (\hat{\theta} - \theta)$

$$= \sqrt{n}\ \left(\sum_{i=2}^{n} \frac{Z_i}{n\bar{I}_1} - \frac{E(Z)}{E(I_1)} \right)$$

$$= \sqrt{n}\ \left(\frac{n\bar{Z} - Z_1}{n\bar{I}_1} - \frac{E(Z)}{E(I_1)} \right)$$

$$= \sqrt{n}\ \left(\frac{\bar{Z}}{\bar{I}_1} - \frac{E(Z)}{E(I_1)} \right) - \frac{Z_1}{\sqrt{n}\ \bar{I}_1}$$

Hurt(1982) derived that $\sqrt{n}\ \left(\frac{\bar{Z}}{\bar{I}_1} - \frac{E(Z)}{E(I_1)} \right) \xrightarrow{D}$ $N(0, \frac{var(Z - \theta I_1)}{(P(X_1 \leq X_2))^2})$ Since $\frac{Z_1}{\sqrt{n}\ \bar{I}_1} \xrightarrow{P} 0$,

we have $\sqrt{n}\ (\hat{\theta} - \theta) \xrightarrow{D} N\ (0, \frac{var(Z - \theta I_1)}{(P(X_1 \leq X_2))^2})$ Since Z_i and (I_{1i}, I_{2i}) are independent, $i = 1,2,\ldots\ldots\ldots\ldots,n$ we have on Simplification

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N\ (0, \frac{\theta^2}{\alpha}) \quad \ldots\ldots\ldots\ldots(2.1.26)$$

Moreover, it is easy to show following Hurt (1982) that

Asym var$(\hat{\theta})$ = $\frac{\theta^2}{n\alpha} + o(n^{-1})$ From (2.1.23), we have $\hat{\mu} = \mu + \frac{Z_1}{n}$

or $\sqrt{n}(\hat{\mu} - \mu) = \frac{1}{\sqrt{n}} Z_1$. and $E(\sqrt{n}(\hat{\mu} - \mu)$

and $var(\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{P} 0$ as $n \longrightarrow \infty$

Since, $Z_1, Z_2, \ldots\ldots, Z_n$ are independent, $Cov(\hat{\theta}, \hat{\mu}) = 0$

Now MLE of $\bar{F}_1(x)$ in this case is given by $\hat{\bar{F}}_1(x) = Exp(-\frac{x - \hat{\mu}}{\hat{\theta}})$.

Using Taylor's series expansion about (θ, μ); retaining only the 1st order term as in section 2.1(a) and following Hurt's (1982) method we have

$$\text{Asym var}(\hat{\bar{F}}_1(x)) = n^{-1}(\bar{F}_1(x))^2\alpha^{-1}(\frac{x - \mu}{\theta})^2 + o(n^{-1}) \quad \ldots\ldots.(2.1.27)$$

where $\bar{F}_1(x) = Exp(-\frac{(x - \mu}{\theta})$, $x \geq \mu$. Here (2.1.27) gives the asymptotic variance of the MLE. The (i) Kaplan Meier estimator (ii)Ebrahimi's estimator and (iii) Abdushukurov and Cheng-Lin's (ACL) estimator in this case remain exactly same as in section 2.1(a). Hence, the asymptotic relative efficiencies e_1, e_2, e_3 are given as follows :

e_1(KM compared with MLE) $= \alpha^{-2}(\frac{x - \mu}{\theta})^2(Exp(\frac{x - \mu}{\theta\alpha}) - 1)^{-1}$

e_2(Ebrahimi's estimator compared with MLE)

$$= (\frac{x - \mu}{\theta})^2((\frac{x - \mu}{\theta})^2(1 - \alpha) + \alpha^4(Exp(\frac{x - \mu}{\theta\alpha}) - \alpha)$$

$$+ \alpha^3(1 - \alpha)\ ((Exp(\frac{x - \mu}{\theta\alpha}) - (1 - \alpha)) - 2\alpha^4(1 - \alpha))^{-1}$$

e_3 (ACL estimator compared with MLE)

$$= \alpha^{-2}(\frac{x - \mu}{\theta})^2(\alpha\ (Exp(\frac{x - \mu}{\theta\alpha}) - 1)$$

$$+ \ (1 - \alpha)(\frac{x - \mu}{\theta\alpha})^2\ )^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2.1.28)$$

Relative efficiencies e_1, e_2 and e_3 may be computed for different values of $(\frac{x-\mu}{\theta})$ and $\alpha = P(X_1 \leq X_2)$. The same table (2.1) gives the efficiencies e_1, e_2 and e_3, in this case also if we read $\frac{x-\mu}{\theta}$ for $\frac{x}{\theta}$. It may be noted that for appropriate ranges in the level of censoring, the remarks made there in connection with the problem in 2.1(a) hold in the present situation too with $\frac{x}{\theta}$ replaced by $\frac{x-\mu}{\theta}$.

Remark :

It may be pointed out that the methods of this section can be easily applied in estimating a component survival function in case of K components series system. The problem may be posed as follows :

There are K components, X_i denoting the life of the i-th component, X_i's are all independent. What is observed in reality is : $Z = \text{Min}(X_1, X_2, \ldots\ldots, X_k)$ and an indicator variable, denoted by $d = (I_1, I_2, \ldots\ldots, I_k)$ which follows a multinomial distribution with K classes, Let A_i represent the values of Z for which $Z = X_i$, $X_j > Z$, $\forall\ j \neq i$, $i=1,2,\ldots.,k$. The relations between the components of the indicator variable and the classes A_i's are as follows : $Z \in A_i \Longleftrightarrow I_1 = 1,\ I_j = 0,\ \forall_j \neq i,\ i = 1,2,\ldots\ldots,k$.

under the proportional hazard assumption $\bar{F}_i(x) = (\bar{F}_1(.))^{\beta_i}$, $\beta_i > 0$, $i = 2,3,\ldots.,k$ if our principal aim is to estimate $\bar{F}_1(x)$, the (i) Kaplan-Meier estimator (ii)Ebrahimi's estimator and (iii)Abdushukurov and Cheng and Lin 's (ACL) estimator can all be writen down exactly in the same manner as in the case of one component system under random censoring, Let us define $Z=\text{Min}(X_1, Y)$, where $Y=\text{Min}(X_2, X_3, \ldots\ldots, X_k)$, The survival function of Y is given by, $\bar{F}_y(.) = (\bar{F}_1(.))^\beta$, with $\beta = \beta_2 + \beta_3 + \ldots\ldots + \beta_k$ Also the indicator can be redefined as $d^* = (J_1, J_2)$ with $J_1 = I_1 = 1 \Leftrightarrow Z = X_1,\ Y > Z$ and $J_2 = I_2 + I_3 + \ldots\ldots + I_k = 1 \Leftrightarrow Z = Y,\ X_1 > Z$. The additional assumption of exponentiality of distributions make the MLE's and comparison of the asymptotic variances exactly same as described in this section with β_1, the censoring parameter replaced by $\sum_{i=2}^{k} \beta_i$ and $d = (I_1, I_2)$ replaced by $d^* = (J_1, J_2)$.

2.2 A PARALLEL SYSTEM WITH TWO COMPONENTS

The problem is described explicitly as : System life $Z = \mathrm{Min}(\mathrm{Max}(X_1, X_2), X_3)$, X_1, X_2 are the lives of components 1 and 2 of a parallel system and X_3 = Censoring time.

2.2.1 Case (1) : X_i's are independent and $X_i \sim \mathrm{Exp}(\theta_i)$ for $i=1,2,3$

The data consist of the observations on (Z,d), where $d = (I_1, I_2, I_3, I_4, I_5)$ is the indicator variable, for $i=1,2,3,4,5$. $I_i=1$ and $I_j = 0 \ \forall \ J \neq i$, if and only if the observation Z belongs to class A_i, the classes A_i's are as given below.

$$\begin{aligned}
A_1 &= \{ Z \in R^+ \mid X_1 < Z,\ Z = X_2\ X_3 > Z \} \\
A_2 &= \{ Z \in R^+ \mid X_2 < Z,\ Z = X_1\ X_3 > Z \} \\
A_3 &= \{ Z \in R^+ \mid X_1 < Z,\ Z = X_3\ X_2 > Z \} \\
A_4 &= \{ Z \in R^+ \mid X_2 < Z,\ Z = X_3\ X_1 > Z \} \\
A_5 &= \{ Z \in R^+ \mid X_1 > Z,\ Z = X_3\ X_2 > Z \} \quad \ldots\ldots\ldots(2.2.1)
\end{aligned}$$

For the problem stated and classes enumerated in (2.2.1), we have thus defined: $A_i = (Z \in R^+ \mid I_i = 1)$, $i=1,2,..,5$

Let $\Pi_i = P(Z \in A_i) = E(I_i)$, $i=1,2,3,4,5$ ....(2.2.2)

It is easy to show that

$$\begin{aligned}
\Pi_1 &= (B_1 - C_1)\theta_2^{-1}, \quad \Pi_2 = (D_1 - C_1)\theta_1^{-1} \\
\Pi_3 &= (B_1 - C_1)\theta_3^{-1}, \quad \Pi_4 = (D_1 - C_1)\theta_3^{-1} \\
\Pi_5 &= C_1\theta_3^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.3)
\end{aligned}$$

where

$$\begin{aligned}
D_1^{-1} &= \theta_1^{-1} + \theta_3^{-1} \\
B_1^{-1} &= \theta_2^{-1} + \theta_3^{-1} \\
C_1^{-1} &= \theta_1^{-1} + \theta_2^{-1} + \theta_3^{-1} \quad \ldots\ldots\ldots\ldots(2.2.4)
\end{aligned}$$

The conditional densities of Z given $Z \in A_i$ for $i=1,2,3,4,5$ are given by :

$$dF\ (Z|Z \in A_1) = (\Pi_1\theta_2)^{-1}(\mathrm{Exp}(\frac{-Z}{B_1}) - \mathrm{Exp}(\frac{-Z}{C_1}))$$

$$dF\ (Z|Z \in A_2) = (\Pi_2\theta_1)^{-1}(\mathrm{Exp}(\frac{-Z}{D_1}) - \mathrm{Exp}(\frac{-Z}{C_1}))$$

$$dF\ (Z|Z \in A_3) = (\Pi_3\theta_3)^{-1}(\mathrm{Exp}(\frac{-Z}{B_1}) - \mathrm{Exp}(\frac{-Z}{C_1}))$$

$$dF\ (Z|Z \in A_4) = (\Pi_4\theta_3)^{-1}(\mathrm{Exp}(\frac{-Z}{D_1}) - \mathrm{Exp}(\frac{-Z}{C_1}))$$

$$dF\ (Z|Z \in A_5) = (\Pi_5\theta_3)^{-1}(\ \mathrm{Exp}(\frac{-Z}{C_1})) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.5)$$

Let us write $E(Z|Z \in A_i) = \phi_i$ for $i = 1,2,3,4,5$, Then

$$\phi_1 = (\Pi_1\theta_2)^{-1}\ (B_1^2 - C_1^2),\ \phi_2 = (\Pi_2\theta_1)^{-1}(D_1^2 - C_1^2)$$

$$\phi_3 = (\Pi_3\theta_3)^{-1}\ (B_1^2 - C_1^2),\ \phi_4 = (\Pi_4\theta_3)^{-1}(D_1^2 - C_1^2)$$

$$\phi_5 = (\Pi_5\theta_3)^{-1}\ C_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.2.6)$$

One can easily check that

$$E(Z) = \sum_{i=1}^{5} (E(Z|Z \in A_i)\ P(Z \in A_i)$$

$$= \sum_{i=1}^{5} \Pi_i\phi_i = D_1 + B_1 - C_1 = L_1\ \text{(Say)} \quad \ldots\ldots\ldots(2.2.7)$$

$$\mathrm{Var}(Z) = 2(D_1^2 + B_1^2 - C_1^2) - L_1^2 = M_1\ \text{(Say)} \quad \ldots\ldots\ldots(2.2.8)$$

In a simple random sample of size n which constitutes the set of observations, let n_i observations belong to class A_i for $i = 1,2,3,4,5$, such that $\sum_{i=1}^{5} n_i = n$.

The likelihood can be written as : $L_1(.) = \dfrac{n!}{\prod_{i=1}^{5} n_i!} \left(\prod_{i=1}^{5} \Pi_i^{n_i} \right)$

$$\prod_{j=1}^{n} \left((\Pi_1 \theta_2)^{-1} \mathrm{Exp}(-Z_j(\theta_2^{-1} + \theta_3^{-1}))(1 - \mathrm{Exp}(\frac{-Z_j}{\theta_1})) \right)^{I_{1j}} \times$$

$$\prod_{j=1}^{n} \left((\Pi_2 \theta_1)^{-1} \mathrm{Exp}(-Z_j(\theta_1^{-1} + \theta_3^{-1}))(1 - \mathrm{Exp}(\frac{-Z_j}{\theta_2})) \right)^{I_{2j}} \times$$

$$\prod_{j=1}^{n} \left((\Pi_3 \theta_3)^{-1} \mathrm{Exp}(-Z_j(\theta_2^{-1} + \theta_3^{-1}))(1 - \mathrm{Exp}(\frac{-Z_j}{\theta_1})) \right)^{I_{3j}} \times$$

$$\prod_{j=1}^{n} \left((\Pi_4 \theta_3)^{-1} \mathrm{Exp}(-Z_j(\theta_1^{-1} + \theta_3^{-1}))(1 - \mathrm{Exp}(\frac{-Z_j}{\theta_2})) \right)^{I_{4j}} \times$$

$$\prod_{j=1}^{n} \left((\Pi_5 \theta_3)^{-1} \mathrm{Exp}(-Z_j(\theta_1^{-1} + \theta_2^{-1} + \theta_3^{-1})) \right)^{I_{5j}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.9)$$

where $d_j = (I_{1j}, I_{2j}, I_{3j}, .., I_{5j})$, $j=1,2,.....,n$ are the n observations on d. This means $\sum_{j=1}^{n} I_{ij} = n_i$ for $i = 1,2,..5$ and $I_{ij}=1$ if and only if $Z_j \in A_i$. Here MLE's for θ_1 and θ_2 cannot be expressed in a closed form and the expressions for asymptotic variances of MLE's of θ_1 and θ_2 cannot also be obtained in a closed and neat form, even though the set up is classical. It appears from the expressions of the likelihood that if one or more n_i's equal to zero (which is expected to happen, when n is not large),the maximum likelihood method may not yeild reliable estimates, making MLE's unacceptable in small samples. However, MLE of θ_3 can be obtained very easily and its asymptotic variance is very straight forward to compute.

$\dfrac{\partial \log L_1(.)}{\partial \theta_3} = 0$, yeilds obviously, the MLE of θ_3 as

$$\tilde{\theta}_3 = \bar{Z}\,(\bar{I}_3 + \bar{I}_4 + \bar{I}_5)^{-1} \quad \text{.................(2.2.10)}$$

where $\bar{Z} = \sum_{j=1}^{n} Z_j/n$ and $\bar{I}_i = \sum_{j=1}^{n} I_{ij}/n$, for $i = 1,2,3,4,5$

2.2.1.2 Adhoc Estimators Proposed for θ_1 and θ_2

We propose two sets of estimators for θ_1 and θ_2 as follows. The two sets are $(\hat{\theta}_1, \hat{\theta}_2)$ and $(\hat{\theta}_1^*, \hat{\theta}_2^*)$:

$$\hat{\theta}_1 = \bar{Z}\,\bar{I}_4(\bar{I}_2(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1}$$

$$\hat{\theta}_1^* = \bar{Z}\,(\bar{I}_4+\bar{I}_5)((\bar{I}_1+\bar{I}_2+\bar{I}_3)(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1}$$

$$\hat{\theta}_2 = \bar{Z}\,\bar{I}_3(\bar{I}_1(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1}$$

$$\hat{\theta}_2^* = \bar{Z}(\bar{I}_3+\bar{I}_5)((\bar{I}_1+\bar{I}_2+\bar{I}_4)(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1} \quad \text{...........(2.2.11)}$$

It can be shown that,

$E(\hat{\theta}_1) = \theta_1 + O(n^{-1})$, $E(\hat{\theta}_1^*) = \theta_1 + O(n^{-1})$

and $E(\hat{\theta}_2) = \theta_2 + O(n^{-1})$, $E(\hat{\theta}_2^*) = \theta_2 + O(n^{-1})$ ....(2.2.12)

The notation $u(x) = O(v(x),)$ as $x \longrightarrow L$, denotes that $|u(x)/v(x)|$ remains bounded as $x \longrightarrow$ to a limit L (not necessarily finite).

Two sets of estimators proposed are thus :

$$(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3), \ (\hat{\theta}_1^*, \hat{\theta}_2^*, \tilde{\theta}_3) \quad \text{.......................(2.2.13)}$$

Theorem 2.2.1 :

$$(\bar{Z}, \bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5) \xrightarrow{D} N(\mu_1, \frac{W_1}{n}), \text{ where ,}$$

$$\mu_1' = (E(Z), E(I_1), E(I_2), E(I_3), E(I_4), E(I_5)) \text{ and}$$

$$W_1 = \begin{bmatrix} Var(Z), Cov(Z,I_1) \ldots\ldots Cov(Z,I_5) \\ V(I_1) \ldots\ldots Cov(I_1,I_5) \\ \vdots \qquad\qquad \vdots \\ Var(I_5) \end{bmatrix}$$

where $Cov(I_i, I_j) = -\Pi_i \Pi_j$, $i \neq j$, $i,j = 1,2,3,4,5$

$Cov(Z, I_i) = \Pi_i \phi_i - \Pi_i L_1$ for $i = 1,2,3,4,5$ and

$$Var(Z) = 2(D_1^2 + B_1^2 - C_1^2) - L_1^2 = M_1 \qquad \ldots\ldots\ldots\ldots\ldots (2.2.14)$$

[The elements of μ_1 are to be found in (2.2.3) and (2.2.7)]

Proof : Obvious by the Central limit theorem for identical random variables.

2.2.1.3 Asymptotic Variances of the Proposed Estimators, viz.,

$$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3), (\hat{\theta}_1^*, \hat{\theta}_2^*, \hat{\theta}_3^*)$$

By Taylor's series expansion, about the true values ($E(Z), \Pi_1, \Pi_2, \ldots, \Pi_5$), one can write $\hat{\theta}_1$ in the following form :

$$\sqrt{n}\ (\hat{\theta}_1 - \theta_1) \approx \sqrt{n}\, l_{10}(\bar{Z} - E(Z)) + \sqrt{n} \sum_{i=1}^{5} l_{1i}\ (\bar{I}_i - E(I_i)) + R_n \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2.15)$$

where R_n converges in probability to Zero as $n \longrightarrow \alpha$, Here in particular the expressions for $(l_{10}, l_{11}, \ldots, l_{15})$ are given by.

$l_{10} = \Pi_4 (\Pi_2 . \Pi_A)^{-1}, \qquad l_{11} = 0$

$l_{12} = -\ E(Z)\ \Pi_4\ \Pi_2^{-2}\ \Pi_A^{-1},$

$l_{13} = -\ E(Z)\ \Pi_4\ \Pi_2^{-1}\ \Pi_A^{-2},$

$l_{14} = \ \ E(Z)\ \Pi_B\ \Pi_2^{-1}\ \Pi_A^{-2},$

$$l_{15} = -E(Z)\,\Pi_4\,\Pi_2^{-1}\,\Pi_A^{-2}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.16)$$

where $\Pi_A = \Pi_3 + \Pi_4 + \Pi_5$, $\Pi_B = \Pi_3 + \Pi_5$ $\ldots\ldots\ldots\ldots(2.2.17)$

By theorem 2.2.1, we can show that $\sqrt{n}(\hat{\theta}_1 - \theta_1) \xrightarrow{D} N(0, \delta_1^2)$

where $\delta_1^2 = l' W_1\, l$, $l' = (l_{10}, l_{11}, \ldots\ldots l_{15})$ $\ldots\ldots(2.2.18)$

Thus $\hat{\theta}_1$ is a CAN Estimator (consistently and asymptotically Normal) of θ_1. Proceeding exactly in similar lines, one can prove that.

$$\sqrt{n}(\hat{\theta}_2 - \theta_2) \xrightarrow{D} N(0, \delta_2^2)$$

$$\sqrt{n}(\hat{\theta}_1^* - \theta_1) \xrightarrow{D} N(0, \delta_1^{*2})$$

$$\sqrt{n}(\hat{\theta}_2^* - \theta_2) \xrightarrow{D} N(0, \delta_2^{*2})$$

where δ_2^2, δ_1^{*2}, δ_2^{*2} may be expressed in the following forms by useing Taylor's series expansions as in 2.2.15.

$\delta_2^2 = m'.W_1.\, m,$ $\qquad m' = (m_{10}, m_{11}, \ldots\ldots, m_{15})$

$\delta_1^{*2} = l^{*'}.W_1.\, l^*$ $\qquad l^{*'} = (l_{10}^*, l_{11}^*, \ldots\ldots, l_{15}^*)$

$\delta_2^{*2} = m^{*'}.W_1.\, m^*$ $\qquad m^{*'} = (m_{10}^*, m_{11}^*, \ldots\ldots, m_{15}^*)$

expressions for $m' = (m_{10}, m_{11}, \ldots m_{15})$ are given by

$m_{10} = \Pi_3(\Pi_1\,\Pi_A)^{-1}$, $\qquad m_{11} = -E(Z)\Pi_3\,\Pi_A^{-1}\Pi_1^{-2}$

$m_{12} = 0,$ $\qquad m_{13} = E(Z)\Pi_1^{-1}\,\Pi_C\,\Pi_A^{-2},$

$m_{14} = -E(Z)\Pi_3\,\Pi_1^{-1}\,\Pi_A^{-2}$, $\quad m_{15} = m_{14}$ where $\Pi_C = \Pi_4 + \Pi_5$

Expressions for $l^{*'} = (l_{10}^*, l_{11}^*, \ldots\, l_{15}^*)$ are given by

$l_{10}^* = \Pi_C\,\Pi_A^{-1}\,\Pi_D^{-1}$, $\quad l_{11}^* = -E(Z)\,\Pi_C\,\Pi_A^{-1}\,\Pi_D^{-2} = l_{12}^*$

$$l^*_{13} = - E(Z)\, \Pi_C \,[\Pi_D^{-2}\, \Pi_A^{-1} + \Pi_A^{-2}\, \Pi_D^{-1}],$$

$$l^*_{14} = E(Z)\, \Pi_D^{-1}\, \Pi_3\, \Pi_A^{-2} = l^*_{15}, \quad \text{where } \Pi_D = \Pi_1 + \Pi_2 + \Pi_3$$

Expressions for $m^{*\prime} = (m^*_{10}, m^*_{11}, \ldots m^*_{15})$ are given by

$$m^*_{10} = \Pi_B\, \Pi_E^{-1}\, \Pi_A^{-1}, \qquad m^*_{11} = m^*_{12} = - E(Z)\, \Pi_B\, \Pi_E^{-2}\, \Pi_A^{-2}$$

$$m^*_{13} = - E(Z)\, \Pi_4\, \Pi_E^{-1}\, \Pi_A^{-2} = m^*_{15}$$

$$m^*_{14} = - E(Z)\, \Pi_B\, [\Pi_E^{-2}\, \Pi_A^{-1} + \Pi_A^{-2}\, \Pi_E^{-1}]$$

$$\text{where } \Pi_E = \Pi_1 + \Pi_2 + \Pi_4 \qquad \ldots\ldots\ldots\ldots\ldots (2.2.19)$$

By Taylor's series expansion, $\tilde{\theta}_3$ can be written in the following form

$$\sqrt{n}(\tilde{\theta}_3 - \theta_3) \approx \sqrt{n}\, p_{10}(\bar{Z} - E(Z))$$

$$+ \sqrt{n}\sum_{i=1}^{5} p_{1i}(\bar{I}_i - E(I_i)) + R_n \qquad \ldots\ldots\ldots (2.2.20)$$

where R_n converges in probability to Zero as $n \longrightarrow \alpha$,

where $p_{10} = \Pi_A^{-1}, \quad p_{11} = p_{12} = 0,$

$$p_{13} = p_{14} = p_{15} = - E(Z)\, \Pi_A^{-2} \qquad \ldots\ldots\ldots\ldots\ldots (2.2.21)$$

and it can be shown that

$$\sqrt{n}(\tilde{\theta}_3 - \theta_3) \xrightarrow{D} N(0, \delta_3^2), \quad \delta_3^2 = p' W_1\, p,$$

where $p' = (p_{10}, p_{11}, \ldots p_{15})$ $\ldots\ldots\ldots\ldots\ldots\ldots (2.2.22)$

$\tilde{\theta}_3$ is also a CAN estimator for θ_3. Asymptotic covariances of the estimators can also be calculated and are given as follows :

$$\text{Asym Cov}(\hat{\theta}_1, \hat{\theta}_2) = n^{-1}\, l' W_1\, m$$

$$\text{Asym Cov}(\hat{\theta}^*_1, \hat{\theta}^*_2) = n^{-1}\, l^{*\prime} W_1\, m^*$$

$$\text{Asym Cov}(\hat{\theta}_1, \hat{\theta}_3) = n^{-1}\, l' W_1 p$$

$$\text{Asym Cov}(\hat{\theta}_2, \hat{\theta}_3) = n^{-1}\, m' W_1 p$$

$$\text{Asym Cov}(\hat{\theta}_1^*, \hat{\theta}_3) = n^{-1}\, l^{*'} W_1 p$$

$$\text{Asym Cov}(\hat{\theta}_2^*, \hat{\theta}_3) = n^{-1}\, m^{*'} W_1 p \qquad \text{.........(2.2.23)}$$

2.2.1.4 **Mean Life of the System $E(Z) = L_1$**

Assuming $(\hat{\hat{\theta}}_1, \hat{\hat{\theta}}_2, \tilde{\theta}_3)$ to be a set of consistent estimators for $(\theta_1, \theta_2, \theta_3)$ and $\hat{\hat{L}}_1$ is the estimated mean life, we have

$$\sqrt{n}(\hat{\hat{L}}_1 - L_1) \approx \sqrt{n}Q_1(\hat{\hat{\theta}}_1 - \theta_1) + \sqrt{n}Q_2(\hat{\hat{\theta}}_2 - \theta_2)$$

$$+ \sqrt{n}Q_3(\tilde{\theta}_3 - \theta_3) + R_n \qquad \text{...............(2.2.24)}$$

where R_n converges in probability to Zero as $n \longrightarrow \infty$, and

$$Q_1 = (D_1^2 - C_1^2)\,\theta_1^{-2},$$

$$Q_2 = (B_1^2 - C_1^2)\,\theta_2^{-2},$$

$$Q_3 = (D_1^2 + B_1^2 - C_1^2)\,\theta_3^{-2} \qquad \text{..........(2.2.25)}$$

Then one can show that $\sqrt{n}(\hat{\hat{L}}_1 - L_1) \xrightarrow{D} N(0, e' \Sigma e)$ where Σ is the variance covariance matrix of $(\hat{\hat{\theta}}_1, \hat{\hat{\theta}}_2, \tilde{\theta}_3)$ and

$$e' = (Q_1, Q_2, Q_3).$$

Let $\hat{L}_1$ represent the estimator of L_1 obtained by substituting $(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3)$ for $(\theta_1, \theta_2, \theta_3)$. Also let $\hat{L}_1^*$ represent the estimator of L_1 obtained by substituting $(\hat{\theta}_1^*, \hat{\theta}_2^*, \tilde{\theta}_3)$ for $(\theta_1, \theta_2, \theta_3)$. Let $\Sigma_{11}(\Sigma_{12})$ be the variance covariance matrix of

$(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3)$ $((\hat{\theta}^*_1, \hat{\theta}^*_2, \tilde{\theta}_3))$. Then Asym var$(\hat{L}_1) = n^{-1} e' \Sigma_{11} e$ and Asym var$(\hat{L}^*_1) = n^{-1} e' \Sigma_{12} e$.

2.2.1.5 Variance of the Estimated Stytem Survival Function at Time t.

System survival function, which is also called realiability at time t is given by :

$$S(t) = Exp(\frac{-t}{D_1}) + Exp(\frac{-t}{B_1}) - Exp(\frac{-t}{C_1}) \quad \text{.............. (2.2.26)}$$

For a consistent set of estimators $(\hat{\hat{\theta}}_1, \hat{\hat{\theta}}_2, \tilde{\theta}_3)$ of $(\theta_1, \theta_2, \theta_3)$, the asymptotic variance of the estimated system survival probability at time t, Say $\hat{\hat{S}}(t)$ is given as follows

$$\sqrt{n}(\hat{\hat{S}}(t) - S(t)) = \sqrt{n}\, s_1(\hat{\hat{\theta}}_1 - \theta_1) + \sqrt{n}\, s_2(\hat{\hat{\theta}}_2 - \theta_2)$$

$$+ \sqrt{n}\, s_3(\hat{\hat{\theta}}_3 - \theta_3) + R_n \quad \text{.............(2.2.27)}$$

where R_n converges in probability to zero as $n \longrightarrow \alpha$.

Proceeding as in the case of estimated variance of system life length, one can show Asym var$(\hat{S}(t)) = n^{-1} s' . \Sigma_{11} . s$ when the set of estimators used is $(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3)$ and Asym $Var(\hat{S}^*(t)) = n^{-1}.\ s' \Sigma_{12} . s$, when the set of estimators used is $(\hat{\theta}^*_1, \hat{\theta}^*_2, \tilde{\theta}_3)$.

where $\qquad s' = (s_1, s_2, s_3)$

$$s_1 = t\, \theta_1^{-2}\ (Exp(\frac{-t}{D_1}) - Exp(\frac{-t}{C_1}))$$

$$s_2 = t\, \theta_2^{-2}\ (Exp(\frac{-t}{B_1}) - Exp(\frac{-t}{C_1}))$$

$$s_3 = t\,\theta_3^{-2}\left(\mathrm{Exp}\left(\frac{-t}{D_1}\right) + \mathrm{Exp}\left(\frac{-t}{B_1}\right) - \mathrm{Exp}\left(\frac{-t}{C_1}\right)\right) \qquad (2.2.28)$$

2.2.1.6 **Maximum Likelihood Estimators.**

The maximum likelihood estimator for $(\theta_1, \theta_2, \theta_3)$ can be obtained as follows : From (2.2.9), on simplification the equations

$$\frac{\delta \log L_1}{\delta\theta_1} = 0 \quad \text{and} \quad \frac{\delta \log L_1}{\delta\theta_2} = 0, \text{ yield}$$

$$\tilde{\theta}_1 = \left[\sum_{i \in A_1^*} \sum_{j=1}^{n_i} Z_{ij} - \sum_{i \in A_2^*} \sum_{j=1}^{n_i} \left(\frac{Z_{ij}\mathrm{Exp}\left(\frac{-Z_{ij}}{\theta_1}\right)}{1 - \mathrm{Exp}\left(\frac{-Z_{ij}}{\theta_1}\right)}\right)\right] * n_2^{-1}$$

$$\tilde{\theta}_2 = \left[\sum_{i \in B_1^*} \sum_{j=1}^{n_i} Z_{ij} - \sum_{i \in B_2^*} \sum_{j=1}^{n_i} \left(\frac{Z_{ij}\mathrm{Exp}\left(\frac{-Z_{ij}}{\theta_2}\right)}{1 - \mathrm{Exp}\left(\frac{-Z_{ij}}{\theta_2}\right)}\right)\right] * n_1^{-1} \qquad (2.2.29)$$

where $A_1^* = A_2 \cup A_4 \cup A_5$, $\quad A_2^* = A_1 \cup A_3$,

$B_1^* = A_1 \cup A_3 \cup A_5$, $\quad B_2^* = A_2 \cup A_4$,

A_i's are defined in(2.2.1). It is clear that θ_1, θ_2 and θ_3 are asymptotically uncorrelated assuming that the regularity conditions hold and it is not easy from the likelihood equations to check if a set of regularity conditions hold even though the set up is classical. The estimation of the approximate variances can be carried out by the formula

$$\mathrm{Var}(\tilde{\theta}_1) = E\left(\frac{-\delta^2 \log L_1}{\delta\theta_1^2}\right)^{-1} \approx \left(\frac{-\delta^2 \log L_1}{\delta\theta_1^2} \,\Big|\, \theta_1 = \tilde{\theta}_1\right)^{-1} \quad \ldots\ldots(2.2.30)$$

$$Var(\tilde{\theta}_2) = E\left(\frac{-\delta^2 \log L_1}{\delta\theta_2^2}\right)^{-1} \approx \left(\frac{-\delta^2 \log L_1}{\delta\theta_2^2}\Big|\,\theta_2 = \tilde{\theta}_2\right)^{-1} \quad \text{......(2.2.31)}$$

Let $\tilde{L}_1(\hat{R}(t))$ represent the MLE of L_1 $(R(t))$ which is obtained by substituting $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ for $(\theta_1, \theta_2, \theta_3)$ in $L_1(R(t))$. The asym variances can be calculated numerically in the same manner as in sections 2.2.1.4 and 2.2.1.5.

2.2.1.7 **Numerical Computation**

In the following lines an attempt is made to compare the two sets of proposed adhoc estimators and also to compare them with MLE, viz $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$. The asymptotic variance covariance matrices of two proposed sets of adhoc estimators are calculated on the basis of the asymptotic formula given earlier. Also the asymptotic variance of the estimated mean life length is obtained. The asymptotic variance of estimated reliability or survival probability is computed at the points where true survival probability happens to be (i)0.90 and (ii)0.95 .

For the numerical computation of MLE's and their variances, the following procedure is adopted. Given a set of $(\theta_1, \theta_2, \theta_3)$ and a fixed sample size n (which is taken to be 100) ,a simple random sample for (Z,d) is generated. For this generated sample, MLE's of the parameters θ_1, θ_2 θ_3 are calculated. Also, the approximate estimated variance is calculated in each case. The sampling experiment is repeated 20 times. Then in each case, the mean of the computed variances for these 20 samples is taken as an estimate of the asymptotic variance. Also for each parameter there are 20 maximum likilihood estimates given by 20 samples. A variance of these 20 values is computed and presented as an estimate of the true variance.

As it is only the relative magnitude of θ_1, θ_2 and θ_3 that are important, for all computations and simulated experiments, θ_3 is chosen to be 1 .

The values of θ_1 and θ_2 are chosen to be 0.50, 1.00 and 2.00. The numerical values are computed and are presented in Tables which follow .

Table 2.2.1.1

n times variance covariance matrix of $(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3)$ for different combinations of θ_1 and θ_2 and $\theta_3 = 1$

$\theta_1 \backslash \theta_2$	0.5	1.00	2.00
	1.85,-0.49,0.07	2.68,-0.75,0.10	4.67,-1.22,-0.08
0.50	2.28,0.00	6.29,0.00	25.90, 0.00
	2.14	1.71	1.40
	6.29,-0.86,0.00	7.94,-1.43,-0.06	12.35,-2.52,-0.05
1	4.07,0.00	10.50, 0.00	39.79, 0.00
	1.71	1.50	1.30
	28.79,-1.41,-0.01	34.78,-2.64,-0.03	50.56,-5.12,-0.16
2	7.53, 0.00	18.70, 0.00	67.20,-0.00
	1.40	1.30	1.20

Table 2.2.1.2

n times variance covariance matrix of $(\hat{\theta}_1^*, \hat{\theta}_2^*, \tilde{\theta}_3)$ for different combinations of (θ_1, θ_2) and $\theta_3 = 1$

θ_1 \ θ_2	0.50	1.00	2.00
0.50	0.59,-0.08,0.00 0.59,0.00 2.14	0.70,-0.10,0.00 2.29,0.00 1.71	0.77,-0.11, 0.00 12.40, 0.00 1.40
1.00	2.29,-0.10,0.00 0.70,0.00 1.71	2.50,-0.16,-0.00 2.50, 0.00 1.50	2.70,-0.21, 0.00 12.78, 0.00 1.30
2.00	12.40,-0.11, 0.00 0.78, 0.00 1.40	12.79,-0.21, 0.00 2.70, 0.00 1.30	13.23,-0.30, 0.00 13.20, 0.00 1.20

Table 2.2.1.3

n times variance of the estimated mean life – using the two sets of estimates, viz. (a) $(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3)$ and (b) $(\hat{\theta}_1^*, \hat{\theta}_2^*, \tilde{\theta}_3)$, $\theta_3 = 1$

θ_1 \ θ_2	0.50		1.00		2.00	
	a	b	a	b	a	b
0.50	0.33	0.10	0.43	0.17	0.58	0.32
1	0.47	0.23	0.53	0.27	0.65	0.39
2	0.62	0.41	0.66	0.43	0.73	0.51

Table 2.2.1.4

n times variance of the estimated survival probability at the points where true survival probability is (i) 0.90 and (ii) 0.95. The sets of estimates used are a) $(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3)$ and (b) $(\hat{\theta}^*_1, \hat{\theta}^*_2, \tilde{\theta}_3)$, $\theta_3=1$

θ_1 \ θ_2	0.50 I	0.50 II	1.00 I	1.00 II	2.00 I	2.00 II
0.50	a 0.1062	a 0.0168	a 0.0742	a 0.0139	a 0.0322	a 0.0110
	b 0.0852	b 0.0132	b 0.0642	b 0.0121	b 0.0312	b 0.0010
1	a 0.0842	a 0.0142	a 0.0523	a 0.0123	a 0.0275	a 0.0110
	b 0.0623	b 0.0121	b 0.0412	b 0.0116	b 0.0263	b 0.0107
2	a 0.0642	a 0.0123	a 0.0321	a 0.0110	a 0.0221	a 0.0103
	b 0.0432	b 0.0109	b 0.0121	b 0.0107	b 0.0198	b 0.0102

Table 2.2.15

Variances of MLE's θ_1 and θ_2 numerically computed (i) with the formula (2.2.30) and (2.2.31) and (ii) from the observed estimates, by generating samples of size 100, $(\theta_3=1)$.

θ_1 \ θ_2		0.50 I	0.50 II	1.00 I	1.00 II	2.00 I	2.00 II
0.50	$var(\tilde{\theta}_1)$	0.0036	0.0044	0.0035	0.0045	0.0018	0.0046
	$var(\tilde{\theta}_2)$	0.0031	0.0039	0.0228	0.0223	0.1822	0.1600
1.00	$var(\tilde{\theta}_1)$	0.0242	0.0248	0.0203	0.0266	0.1100	0.0228
	$var(\tilde{\theta}_2)$	0.0044	0.0044	0.0139	0.0217	0.1061	0.1232
2.00	$var(\tilde{\theta}_1)$	0.2439	0.1609	0.1675	0.1748	0.1474	0.1451
	$var(\tilde{\theta}_2)$	0.0053	0.0048	0.0291	0.0298	0.1723	0.1408

Remark 1 A quick glance at the tables 2.2.1.1 through 2.2.1.5 reveals the fact that the proposed estimator $(\hat{\theta}^*_1, \hat{\theta}^*_2, \tilde{\theta}_3)$ is better than the estimator $(\hat{\theta}_1, \hat{\theta}_2, \tilde{\theta}_3)$. Also the estimator $\hat{\theta}^*_1$, $\hat{\theta}^*_2$ may become undefined only if atleast three of n_i's are simultaneously zero, whereas $\hat{\theta}_1$, and $\hat{\theta}_2$ may become undefined even if one of n_i's is Zero in some cases. Hence by all considerations the set of estimators $(\hat{\theta}^*_1, \hat{\theta}^*_2, \tilde{\theta}_3)$ seems to be more acceptable.

The expressions appear to be quite cumbursome and no algebraic proof has been attempted.

Remark 2.

An attempt was made to compute MLE's and their asymptotic variances for sample size 50. It was found that quite a few of the generated samples had one or more n_i's equal to zero , leading to nonconvergence of the iterative techniques used for finding MLE's based on the equation (2.2.29).

However, the proposed estimator $(\hat{\theta}^*_1, \hat{\theta}^*_2, \tilde{\theta}_3)$ could be computed without difficulty and the estimates were reasonably close to the poputation parameters chosen.

2.2.2 : Case II.

X_i's are independent and Identically Distributed,

$X_i \sim Exp(\theta)$, $i=1,2$, $X_3 \sim Exp(\theta_3)$

2.2.2.1 Preliminaries and the Proposed Estimators

The problem is the same as that considered in section 2.2.1, with the additional condition $\theta_1 = \theta_2 = \theta$. Writing $\theta_1 = \theta_2 = \theta$ in all the expressions from (2.2.1) to (2.2.9), one gets appropriate expressions of the relevant quantities in this case.

Here also MLE for θ cannot be expressed in a closed form and the expression for asymptotic variance of MLE can not be written down in a neat form. However, as noted in the previous section MLE of θ_3 can be obtained very easily and is given by the same expression $\tilde{\theta}_3$ which can be recalled as

$$\tilde{\theta}_3 = \bar{Z}(\bar{I}_3+\bar{I}_4+\bar{I}_5)^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2.32)$$

We propose three different estimators for θ as follows.

$$\hat{\theta} = \bar{Z}(\bar{I}_3+\bar{I}_4)((\bar{I}_1+\bar{I}_2)(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1}$$

$$\hat{\theta}^* = 2\ \bar{Z}\ \bar{I}_5((\bar{I}_1+\bar{I}_2+\bar{I}_3+\bar{I}_4)(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1}$$

$$\hat{\theta}^{**} = \frac{1}{2}\Big[\ \bar{Z}(\bar{I}_3+\bar{I}_5)((\bar{I}_1+\bar{I}_2+\bar{I}_4)(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1}$$
$$+\ \bar{Z}(\bar{I}_4+\bar{I}_5)((\bar{I}_1+\bar{I}_2+\bar{I}_3)(\bar{I}_3+\bar{I}_4+\bar{I}_5))^{-1}\Big] \quad ..(2.2.33)$$

It is easy to check that the estimators in (2.2.33) are actually obtained from the estimators (2.2.11), making use of the fact $\theta_1=\theta_2=\theta$ judiciously. It can be shown easily that all these estimators are asymptotically unbiased and consistent for θ. Thus the three sets of proposed estimators are : $(\hat{\theta}\ ,\tilde{\theta}_3)$, $(\hat{\theta}^*,\tilde{\theta}_3)$ and $(\hat{\theta}^{**},\tilde{\theta}_3)$. ....(2.2.34)

2.2.2.2 Asymptotic Variances of the Proposed Estimators.

The central limit Theorem 2.2.1 holds with all the expressions in μ_1 and Σ_1 modified appropriately using $\theta_1 = \theta_2 = \theta$.

By Taylor's series, expansion, $\hat{\theta}$ can be written as :

$$\sqrt{n}(\hat{\theta} - \theta) \approx \sqrt{n}\ l_{20}(\bar{Z} - E(Z)) + \sqrt{n}\sum_{i=1}^{5} l_{2i}(\bar{I}_i - E(I_i)) + R_n \ldots (2.2.35)$$

where R_n converges in probability to zero as $n \longrightarrow \alpha$. Here the expressions for $(l_{20}, l_{21}, \ldots\ldots l_{25})$ in particular are the following.

$$l_{20}=(\Pi_3+\Pi_4)(\Pi_A(\Pi_1+\Pi_2))^{-1}$$

$$l_{21}=l_{22}=-E(Z)(\Pi_3+\Pi_4)\ \Pi_A^{-1}(\Pi_1+\Pi_2)^{-2}$$

$$l_{23}=l_{24}=E(Z)(\Pi_1+\Pi_2)^{-1}\ \Pi_5\ \Pi_A^{-2}$$

$$l_{25}=-E(Z)(\Pi_3+\Pi_4)(\Pi_1+\Pi_2)^{-1}\ \Pi_A^{-2} \qquad \ldots\ldots\ldots\ldots(2.2.36)$$

Π_A is as given in (2.2.17). As in the previous section, one can write, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{D} N(0,\delta_{21}^2)$, where $\delta_{21}^2=l_2' W_1 l_2$ ..(2.2.37)

Likewise one can show that

$$\sqrt{n}(\hat{\theta}^*-\theta) \xrightarrow{D} N(0,\delta_{21}^{*2})$$

$$\delta_{21}^{*2}=l_2^{*\prime} W_1 l_2^*,\quad l_2^{*\prime}=(l_{20}^*,\ldots l_{25}^*)$$

$$\sqrt{n}(\hat{\theta}^{**}-\theta) \xrightarrow{D} N(0,\delta_{21}^{**2}),$$

$$\delta_{21}^{**2}=l_2^{**\prime} W_1 l_2^{**},\quad l_2^{**\prime}=(l_{20}^{**},\ldots l_{25}^{**})$$

$$\sqrt{n}(\tilde{\theta}_3-\theta_3) \xrightarrow{D} N(0,\delta_{23}^2)$$

$\delta_3^2 = p' W_1 p$, where $p'=(p_{10},\ldots\ldots,p_{15})$ is as given on (2.2.21) with $\theta_1=\theta_2=\theta$ ....(2.2.38)

l_2^* and l_2^{**} are obtained appropriately by Taylor's series expansion of $\sqrt{n}(\hat{\theta}^*-\theta)$ and $\sqrt{n}(\hat{\theta}^{**}-\theta)$ about the true values, respectively. Asymptotic covariances of the estimators can also be calculated and are given as follows.

(a) Asym $\mathrm{cov}(\hat{\theta}, \tilde{\theta}_3) = n^{-1} l_2' W_1 p$

(b) Asym cov$(\hat{\theta}^{*}, \tilde{\theta}_3) = n^{-1} 1_2^{*\prime} W_1 \, p$

(c) Asym cov$(\hat{\theta}^{**}, \tilde{\theta}_3) = n^{-1} 1_2^{**\prime} W_1 \, p$ ....(2.2.39)

With these variances and covariances the asymptotic variances and covariances of the estimated mean life and the estimated survival function can be written down exactly in the same manner as in sections 2.2.1.4 and 2.2.1.5.

2.2.2.3 Numerical Computation.

The maximum likelihood estimate of θ can be obtained by solving $\frac{\delta \log L}{\delta\theta} = 0$, which leads to

$$\tilde{\theta} = \frac{n\bar{Z} + \sum_{j=1}^{n_5} Z_{5j} - \sum_{i=1}^{4} \sum_{j=1}^{n_i} \left(\frac{Z_{ij} \; \mathrm{Exp}(\frac{-Z_{ij}}{\theta})}{1 - \mathrm{Exp}(\frac{-Z_{ij}}{\theta})} \right)}{n_1 + n_2} \quad ..(2.2.40)$$

As in section 2.2.1, the approximate estimation of the asymptotic variance covariance matrix of $(\tilde{\theta}, \tilde{\theta}_3)$ can be carried out as :

$$\begin{bmatrix} -\frac{\delta^2 \log L}{\delta\theta^2} \Big| \; \theta = \tilde{\theta}, \; \theta_3 = \tilde{\theta}_3, & 0 \\ & \frac{1}{\mathrm{var}(\tilde{\theta}_3)} \end{bmatrix}^{-1} \quad ...(2.2.41)$$

For numerical computation of MLE's and their variances the procedure adopted, given a set (θ, θ_3) is exactly same as in the general case dealt with in section 2.2.1. In this case too sample size 50 leads to unreliable and erratic MLE's quite a few sets of values of the parameter θ and θ_3 chosen. So the results reported correspond to the sample size n = 100 only. For the purpose of comparison θ_3 is always taken to be 1. The values of θ chosen are 0.25, 0.50, 1.00 and 2.00. The numerical results are presented in Tables 2.2.2.1 through 2.2.2.4.

Table 2.2.2.1

n times variance covariance matrix of (a) $(\hat{\theta}, \tilde{\theta}_3)$ (b) $(\hat{\theta}^*, \tilde{\theta}_3)$, (c) $(\hat{\theta}^{**}, \tilde{\theta}_3)$ for different values of θ, $\theta_3 = 1$

θ	0.25		0.50		1.00		2.00	
	0.22,	0.00	0.87,	0.00	4.50,	0.00	3.20,	0.00
a		3.46		2.14		1.50		1.20
	0.42,	0.00	1.02,	0.00	3.00,	0.00	11.20,	0.00
b		3.46		2.14		1.50		1.20
	0.07,	0.00	0.25,	0.00	2.16,	0.00	6.45,	0.00
c		3.46		2.14		1.50		1.20

Table 2.2.2.2

n times variance of the estimated mean life using the sets of estimators (a) $(\hat{\theta}, \tilde{\theta}_3)$, (b) $(\hat{\theta}^*, \tilde{\theta}_3)$ (c) $(\hat{\theta}^{**}, \tilde{\theta}_3)$, $\theta_3 = 1$

θ	0.25	0.50	1.00	2.00
a	0.1905	0.3528	0.5740	0.7847
b	0.3419	0.4034	0.4583	0.5956
c	0.0667	0.1526	0.3168	0.5507

Table 2.2.2.3

n times variance of the estimated survival probability at the point where true probability is (i)0.90 and (ii)0.95. The sets of estimators used are (a) $(\hat{\theta}, \tilde{\theta}_3)$ (b) $(\hat{\theta}^*, \tilde{\theta}_3)$ (c) $(\hat{\theta}^{**}, \tilde{\theta}_3)$.

	0.25		0.50		1.00		2.00	
	I	II	I	II	I	II	I	II
a	0.0299	0.0128	0.0169	0.0098	0.0123	0.0086	0.0104	0.0038
b	0.0257	0.0103	0.0168	0.0058	0.0120	0.0038	0.0103	0.0022
c	0.0158	0.0101	0.0132	0.0056	0.0116	0.0032	0.0103	0.0016

Table 2.2.2.4

n times variance of MLE of θ computed by generating samples of size n=100. Two sets of variances are exhibited viz. (i)Computed by the formula in (2.2.41) and (ii)Computed from the observed estimates of θ.

θ	0.25	0.50	1.00	2.00
I	0.00417	0.00262	0.01140	0.07713
II	0.00413	0.00208	0.01123	0.07682

Remark 1

As in the previous section 2.2.19 the estimator $(\hat{\theta}^{**}, \tilde{\theta}_3)$ is observed to be superior to other proposed adhoc estimators. In all respects. Small sample difficulty for MLE holds good in the present context too, whereas in such cases $\hat{\theta}^{**}$ seems to be well defined and behave satisfactorily.

2.2.3 Case (iii) :

A Modification of Case (ii), Where on Failure of One Component, Failure rate of the surviving Component Increases.

2.2.3.1 Preliminaries of the Proposed Estimators.

The problem is a modification of case II considered in section 2.2.2. Initially X_1, X_2 are iid Exp(θ), but when one of the components fails, the other follows Exp(θ') and $X_3 \sim$ Exp(θ_3). This can be explained as possibly because of the additional stress on the surviving component. For technical feasibility $\frac{\theta}{2} < \theta' < \theta$. This is the problem considered in the present section. For the present problem, the observational set up and the classes A_i's remains same as in sections 2.2.1 and 2.2.2.

$$\Pi_1 = \Pi_2 = D_3(B_3 - C_3)\ (\theta\theta')^{-1}$$

$$\Pi_3 = \Pi_4 = D_3(B_3 - C_3)\ (\theta\theta_3)^{-1}$$

$$\Pi_5 = C_3 . \theta_3^{-1} \qquad \text{..........(2.2.42)}$$

where $\quad D_3^{-1} = 2\theta^{-1} - \theta'^{-1}$

$$B_3^{-1} = \theta'^{-1} + \theta_3^{-1}$$

$$C_3^{-1} = 2\theta^{-1} + \theta_3^{-1} \qquad \text{...............(2.2.43)}$$

Conditional densities of Z given $Z \in A_i$ for i = 1,2 ...5 are given by

$$dF\ (Z|Z \in A_i) = D_3(\Pi_i \theta\ \theta')^{-1}(\mathrm{Exp}(\tfrac{-Z}{B_3}) - \mathrm{Exp}(\tfrac{-Z}{C_3}))\quad i = 1,2$$

$$dF\ (Z|Z \in A_i) = D_3(\Pi_i \theta\ \theta_3)^{-1}(\mathrm{Exp}(\tfrac{-Z}{B_3}) - \mathrm{Exp}(\tfrac{-Z}{C_3}))\quad i = 3,4$$

$$dF\ (Z|Z \in A_5) = (\Pi_5\ \theta_3)^{-1}\ \ \mathrm{Exp}(\tfrac{-Z}{C_3})) \qquad \text{.............(2.2.44)}$$

As in section 2.2.1, Let $E(Z|Z \in A_i) = \phi_i$, $i = 1,2,3,4,5.$, then

$$\phi_i = (\Pi_i \theta\theta')^{-1}(D_3(B_3^2 - C_3^3)) \quad i = 1,2$$

$$\phi_i = (\Pi_i \theta\theta_3)^{-1}(D_3(B_3^2 - C_3^3)) \quad i = 3,4$$

$$\phi_5 = (\Pi_5\theta_3)^{-1}C_3^2 \qquad \ldots\ldots\ldots\ldots(2.2.45)$$

$$E(Z) = \sum_{i=1}^{5} \Pi_i\phi_i = C_3 + 2B_3\, C_3\, \theta^{-1} = L_3 \text{ (Say)} \qquad \ldots\ldots(2.2.46)$$

$$Var(Z) = 2C_3^2 + 4B_3C_3(B_3 + C_3)\theta^{-1} - L_3^2 = M_3 \text{ (Say)} \quad \ldots(2.2.47)$$

For the present problem, the likelihood function is given by :

$$L_3() = \frac{n!}{\prod_{i=1}^{5} n_i!} \prod_{i=1}^{5} \Pi_i^{n_i} \times$$

$$\prod_{i=1}^{2} \prod_{j=1}^{n} [\, D_3(\Pi_i \theta\, \theta')^{-1} \operatorname{Exp}(-Z_j \,(\tfrac{1}{\theta'} + \tfrac{1}{\theta_3}))$$

$$(1 - \operatorname{Exp}(-Z_j(\tfrac{2}{\theta} - \tfrac{1}{\theta'})))]^{I_{ij}} \times$$

$$\prod_{i=3}^{4} \prod_{j=1}^{n} [\, D_3(\Pi_i \theta\, \theta_3)^{-1} \operatorname{Exp}(-Z_j \,(\tfrac{1}{\theta'} + \tfrac{1}{\theta_3}))$$

$$(1 - \operatorname{Exp}(-Z_j(\tfrac{2}{\theta} - \tfrac{1}{\theta'})))]^{I_{ij}} \times$$

$$\prod_{j=1}^{n} [\, (\Pi_5\, \theta_3)^{-1} \operatorname{Exp}(-Z_j \,(\tfrac{2}{\theta} + \tfrac{1}{\theta_3}))]^{I_{5j}} \quad \ldots(2.2.48)$$

MLE's for θ and θ' cannot be expressed in a closed form and it is clear from the likelihood equations that MLE's for θ and θ' are asymptotically correlated. The problem in the present context can be dealt with as in the previous sections 2.2.1 and 2.2.2. We propose adhoc estimators for θ and θ' as follows.

$$\hat{\theta}' = \bar{Z}\,(\bar{I}_3 + \bar{I}_4)((\bar{I}_1 + \bar{I}_2)(\bar{I}_3 + \bar{I}_4 + \bar{I}_5))^{-1}$$

$$\hat{\theta} = 2\,\bar{Z}\,\bar{I}_5((\bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4)(\bar{I}_3 + \bar{I}_4 + \bar{I}_5))^{-1}$$

$$\hat{\theta}^* = \frac{1}{2}\,\bar{Z}\,\bar{I}_5((\bar{I}_3 + \bar{I}_5)^{-1}(\bar{I}_3 + \bar{I}_4 + \bar{I}_5)^{-1}$$
$$+ (\bar{I}_4 + \bar{I}_5)^{-1}\,(\bar{I}_3 + \bar{I}_4 + \bar{I}_5)^{-1})$$

However, MLE of θ_3 can be found out easily as given in section 2.2.1, and this can be recalled as

$$\tilde{\theta}_3 = \bar{Z}(\bar{I}_3 + \bar{I}_4 + \bar{I}_5)^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.49)$$

It can be shown that $E(\hat{\theta}') = \theta' + O(n^{-1})$,

$$E(\hat{\theta}) = \theta + O\,(n^{-1}),$$

$$E(\hat{\theta}^*) = \theta + O\,(n^{-1}), \quad \ldots(2.2.50)$$

As in the earlier section, it can be shown easily that.

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N\,(0,\ \delta^2_{31})$$

$$\sqrt{n}(\hat{\theta}' - \theta') \xrightarrow{D} N\,(0,\ \delta'^2_{31})$$

$$\sqrt{n}(\hat{\theta}^* - \theta) \xrightarrow{D} N\,(0,\ \delta^{*2}_{31})$$

$$\sqrt{n}(\tilde{\theta}_3 - \theta_3) \xrightarrow{D} N\,(0,\ \delta^2_{33}) \quad \ldots\ldots\ldots\ldots\ldots(2.2.51)$$

where $\delta^2_{31} = s_2'\,W_3\,s_2$

$$\delta^{*2}_{31} = s_2^{*\prime}\,W_3\,s_2^*$$

Let Σ_{31} = variance covariance matrix of $(\hat{\theta}, \hat{\theta}', \tilde{\theta}_3)$, and

Σ_{32} = variance covariance matrix of $(\hat{\theta}^*, \hat{\theta}', \tilde{\theta}_3)$. Then

Asym var$(\hat{\hat{L}}_3) = n^{-1}t' \Sigma_{31}.t$, when the set of estimators used is $(\hat{\theta}, \hat{\theta}', \tilde{\theta}_3)$ and the same Asym var $= n^{-1}t' \Sigma_{32}$ t when the set of estimators used is $(\hat{\theta}^*, \hat{\theta}', \tilde{\theta}_3)$, and $t' = (t_0, t_1, t_2)$.

2.2.33 System Survival Probability at time t

System Survival Probability at time t = S(t) = P(Z > t)

$$=2D_3\, \theta^{-1}\text{Exp}(-\frac{t}{B_3}) - 2D_3\, \theta^{-1}\text{Exp}(\frac{-t}{C_3}) + \text{Exp}(\frac{-t}{C_3}) \qquad \text{.........(2.2.54)}$$

Let $\hat{\hat{S}}(t)$ be the estimate of S(t) obtained by replacing $(\theta, \theta', \theta_3)$ by a set of consistent estimators $(\hat{\hat{\theta}}, \hat{\hat{\theta}}', \tilde{\theta}_3)$. By Taylor's series expansion, we have

$$\sqrt{n}(\hat{\hat{S}}(t) - S(t)) \approx \sqrt{n}\, t_0^* (\hat{\hat{\theta}} - \theta)$$

$$+ \sqrt{n} t_1^* (\hat{\hat{\theta}}' - \theta') + \sqrt{n} t_2^* (\tilde{\theta}_3 - \theta_3) + R_n \qquad \text{....(2.2.55)}$$

where R_n converges in probability to zero as $n \longrightarrow \infty$.

where t_0^*, t_1^*, t_2^* are given by the following expressions :

$$t_0^* = 4D_3^2\, \theta^{-3}\text{Exp}(\frac{-t}{B_3}) - 2D_3\theta^{-2}\text{Exp}(\frac{-t}{B_3}) - 4D_3^2\, \theta^{-3}\text{Exp}(\frac{-t}{C_3})$$

$$+ 2D_3\, \theta^{-2}\text{Exp}(\frac{-t}{C_3}) - 4D_3 t\theta^{-3}\text{Exp}(\frac{-t}{C_3}) + 2t\theta^{-2}\text{Exp}(\frac{-t}{C_3})$$

$$t_1^* = 2D_3^2\, \theta^{-1}\theta'^{-2}\, \text{Exp}(\frac{-t}{C_3}) - 2D_3^2\, \theta^{-1}\theta'^{-2}\, \text{Exp}(\frac{-t}{B_3})$$

$$t_2^* = -\, 2D_3 t\, \theta^{-1}\theta_3^{-2}\, \text{Exp}(\frac{-t}{C_3}) + t\theta_3^{-2}\, \text{Exp}(\frac{-t}{C_3}) \qquad \text{........(2.2.56)}$$

writing $t^{*\prime} = (t_0^*, t_1^*, t_2^*)$, one can write

Asym var$(\hat{\hat{S}}(t)) = n^{-1} t^{*\prime} \Sigma_{31} t^*$, when the set of estimators used is $(\tilde{\theta}, \hat{\theta}^{\prime}, \tilde{\theta}_3)$ and the same Asym var $= n^{-1} t^{*\prime} \Sigma_{32} t^*$, when the set of estimators used is $(\hat{\theta}^*, \hat{\theta}^{\prime}, \tilde{\theta}_3)$.

For S(t)=0.90 and 0.95,the values of t are obtained for a fixed set of values of true parameters. At these points the variances of the estimated survival probabilities are computed. For different sets of values of the true parameters, these are presented in Tables 2.2.3.1 Through 2.2.3.3.

2.4.3.4 **Numerical Computation**

As in earlier sections, 2.2.1 and 2.2.2, the approximate information matrix of the MLE's of $(\tilde{\theta}\ \tilde{\theta}^{\prime}, \tilde{\theta}_3)$ of $(\theta, \theta^{\prime}, \theta_3)$ is given by

$$\begin{bmatrix} \dfrac{-\delta^2 \log L}{\delta\theta^2}, & \dfrac{-\delta^2 \log L}{\delta\theta\,\delta\theta^{\prime}}, & 0 \\ & \dfrac{-\delta^2 \log L}{\delta\theta^{\prime 2}}, & 0 \\ & & \dfrac{1}{\operatorname{var}(\tilde{\theta}_3)} \end{bmatrix} \quad \ldots\text{(2.2.57)}$$

all second order partial diffierential coefficients in (2.2.57) are evaluated at $\theta = \tilde{\theta}$, $\theta^{\prime} = \tilde{\theta}^{\prime}$, $\theta_3 = \tilde{\theta}_3$. The variance covariance matrix is obtained by taking the inverse of the information matrix. Variances and covariances of MLE's of θ, $\theta^{\prime}$, θ_3 have been calculated from the generated samples obtained by simulated experiments as done in the earlier sections 2.2.1 and 2.2.2. The numerical results pretaining to MLE's and other sets of proposed estimators are presented in Table 2.2.3.4.

Table 2.2.3.1

n times variance covariance matrix of (a) $(\hat{\theta}, \hat{\theta}', \tilde{\theta}_3)$ (b) $(\hat{\theta}^*, \hat{\theta}', \tilde{\theta}_3)$ for different combinations of (θ, θ'), $\theta_3 = 1$

θ, θ'	θ=0.50, θ'= 0.375	θ=1.00, θ'=0.75	θ=2.00, θ'=1.50
a	0.97,-0.45,0.00	2.89,-1.22,0.00	11.00,-3.75,0.00
	0.55,0.00	2.54,0.00	15.94, 0.00
	2.14	1.71	1.40
b	0.95,-0.45,0.70	1.59,-0.94, 0.49	5.17,-0.94, 0.49
	0.55,0.00	2.54, 0.00	15.94, 0.00
	2.39	1.62	1.25

Table 2.2.3.2

n times variance of the estimated mean life using the sets of estimated (a) $(\hat{\theta}, \hat{\theta}', \tilde{\theta}_3)$ (b) $(\hat{\theta}^*, \hat{\theta}', \tilde{\theta}_3)$ for different combinations of (θ, θ^*), $\theta_3 = 1$

θ, θ'	θ=0.50, θ'= 0.375	θ=1.00, θ'=0.75	θ=2.00, θ'=1.50
a	0.1559	0.3038	0.4832
b	0.2089	0.2443	0.4714

Table 2.2.3.3

n times variance of the estimated survival probability at the point where true survival probability is (i) 0.90 and (ii) 0.95. The sets of estimators used are (a) $(\hat{\theta}, \hat{\theta}', \tilde{\theta}_3)$ and (b) $(\hat{\theta}^*, \hat{\theta}', \tilde{\theta}_3)$, $\theta_3 = 1$

θ, θ'	θ=0.50,θ'=0.375		θ=1.00,θ'=0.75		θ=2.00,θ'=1.50	
	I	II	I	II	I	II
a	0.0133	0.0029	0.0014	0.0012	0.0101	0.0003
b	0.0117	0.0013	0.0012	0.0008	0.0101	0.0003

Table 2.2.3.4

n times variances and covariances of MLE's $\tilde{\theta}$, $\tilde{\theta}'$ numerically calculated by generating samples of size n=100. Two sets of variances and covariances are exhibited, viz., (i) Computed by the method in expression (2.2.57) and (ii) computed from the observed estimates $\tilde{\theta}$, $\tilde{\theta}'$ from different samples.

θ, θ'	θ=0.50,θ'=0.375		θ=1.00,θ'=0.75		θ=2.00,θ'=1.50	
	I	II	I	II	I	II
Var($\tilde{\theta}$)	0.0067	0.0070	0.0300	0.0345	0.0806	0.0998
Var($\tilde{\theta}'$)	0.0036	0.0040	0.0230	0.0312	0.0668	0.0000
Cov($\tilde{\theta}$,$\tilde{\theta}'$)	-0.0022	-0.0025	-0.0119	-0.0072	-0.4567	-0.2768

Remark

It appears from Table 2.2.3.1 through 2.2.3.4 that the set of estimates $(\hat{\theta}^*, \hat{\theta}', \tilde{\theta}_3)$ is by all consideration more acceptable than the set $(\hat{\theta}, \hat{\theta}', \tilde{\theta}_3)$. On a scrutinity of the tables it is found that the set of estimates $(\hat{\theta}^*, \hat{\theta}', \tilde{\theta}_3)$ behaves satisfactorily in comparison with MLE's . Moreover, small sample difficulty with MLE pointed out in section 2.2.1 and 2.2.2 holds good here also.

CHAPTER III

PARALLEL SYSTEM WITH TWO COMPONENTS NONPARAMETRIC MODELS UNDER PROPORTIONAL HAZARD ASSUMPTION

CHAPTER 3

PARALLEL SYSTEM WITH TWO COMPONENTS NON-PARAMETRIC MODELS UNDER PROPORTIONAL HAZARD ASSUMPTION.

3.0 INTRODUCTION

In this chapter, we consider a Parallel system with two independent components, 1 and 2 the same as in section 2.2. The life times of the components are denoted by X_1 and X_2 following independent absolutely Continuous distributions, with distribution functions F_1 (.) and F_2 (.) respectively. Exponentiality of distributions is not assumed here. X_3 is the censoring random variable which represents failure time due to other causes not covered by the components 1 and 2, distributed independently of X_1 and X_2 with absolutely continuous distribution function F_3 (.). Under random censoring what we actually observe in reality is the random variable $T = Min(Max(X_1, X_2), X_3)$ and a realization of the indicator variable $d = (I_1, I_2, I_3, I_4, I_5)$, which follows a multinomial distribution with 5 exhaustive and mutually exclusive classes, determined by practically feasible and recognizable interrelationships between the variables X_1, X_2 and X_3. To be more clear, we observe on failure of an individual system, along with the failure time of the system, the identity of the component failed last. If the life time is censored, then also we observe which, if any, of the components failed before censoring.

Two sets of estimates for survival functions, $(\bar{F}_1(.), \bar{F}_2(.))$ are proposed under the assumption that

$\bar{F}_3(.)=(\bar{F}_1(.))^{\beta_1} = (\bar{F}_2(.))^{\beta_1\phi_1}$, $\beta_1 > 0$, $\phi_1 > 0$ and β_1, $\beta_2 = \beta_1\phi_1$ represent in a way levels of censoring the random variables X_1 and X_2 are subjected to by the random variable X_3. This is obviously an extension of the proportional hazard model dealt with by Abdushukurov(1984), Cheng and Lin (1984,1987) to the case of a parallel system with two components under random censoring. Two sets of adhoc estimators of the component survival functions are proposed, viz,

(a) $(\overset{e}{\bar{F}}_1(.), \overset{e}{\bar{F}}_2(.))$ and (b) $(\overset{a}{\bar{F}}_1(.), \overset{a}{\bar{F}}_2(.))$. Both the sets of estimators are obtained in two stages and the first stage is taken to be same for both. The parameters β_1 and ϕ_1 are estimated from the partial likelihood of the observations on d. These estimators are substituted for the true parameters. Then the exact mathematical relationship that exist between the survival function $\bar{F}_1(x)$ or $\bar{F}_2(x)$ and the subsurvival functions are utilized in obtaining the proposed estimators (a). This procedure may be looked upon as an extension of Ebrahimi's (1985) method which was employed by him to the comparable case of one component under random censoring as explained in details in section 2.1. The set of estimators (b) in really based on the application of Abdushukurov's(1984)/ Cheng-Lin's(1984) procedures to the present case of a two component parallel system under random censoring. This method for the one component system under random censoring has been described in detail in section 2.1. In the present context, the method consists in solving a polynomial equation that exists relating

$$\bar{F}_1(x) \text{ or } \bar{F}_2(x) \text{ to } \bar{F}\ (x) = S(x)\ ,$$

the system survival function, after plugging in the estimators of

β_1 amd ϕ_1 for the respective true parameters in the equation.

As in section 2.1, we use $\bar{F}^{KM}(.)$ to denote Kaplan Meier (1958) Product limit estimator of $\bar{F}(.)$, the system survival function.

In section 3.1, it is shown that under the set up described, the estimators (a) and (b) are consistent. The asymptotic variances of the proposed estimators are also derived in section 3.1. In section 3.2, we calculate numerically the asymptotic variances of the estimated survival probability of the system life evaluated under two sets of estimators (a) and (b). Asymptotic variances are numerically calculated and compared with that of the Kaplan Meier Product limit estimator for an appropriate and useful range of values of parameters from Exponential and Weibull distributions of component lives.

In section 3.3 we assume that the components 1 and 2 follow identical life distributions i.e., $\bar{F}_1(.) = \bar{F}_2(.)$ or $\phi_1 = 1$ and $\bar{F}_3(.) = (\bar{F}_1(.))^{\beta_1}$, $\beta_1 > 0$, represents in a way the common degree of censoring of the the random variable X_1 or X_2 by the random variable X_3. Three sets of adhoc estimators are proposed for the survival function $\bar{F}_1(.)$. viz, (a) $\bar{F}_1^{eI}(.)$ (b) $\bar{F}_1^{eII}(.)$ and (c) $\bar{F}_1^{aI}(.)$. Of these (a) and (b) are based on Ebrahimi type arguments and (c) is based on Abdushukurov or Cheng and Lin type arguments. All these estimators are slightly different from those proposed in the case nonidentical life distributions of the components. First of all a closed form expression can be explicitly obtained for the maximum likelihood estimator of β_1 based on the partial likelihood defined for the set of observations on d, whereas in the earlier case because of the nonavailability of such closed form expressions for maximum likelihood estimators of β_1 and ϕ_1, some adhoc modifications are used for writing down their estimators. Secondly, the very fact $\bar{F}_1(.) = \bar{F}_2(.)$ leads to more than one comparable estimators of $\bar{F}_1(.)$ obtained by following the procedures described for the case of non-identical life distributions. Some appropriate weighted average of them should be legitimately proposed as an estimator of $\bar{F}_1(.)$ in this case. These estimators are shown to be consistent. Asymptotic variances of the proposed estimators are also derived in section 3.3. In section 3.4 we calculate numerically the asymptotic variance of the estimated life of the system evaluated under the three sets of estimators (a) – (c) and also the asymptotic variance of (d) $\overset{KM}{\bar{F}}(.)$. This numerical comparison is made for an appropriate range of β_1 and of the parameters from Exponential and Weibull distribution of component life .

3.1 DEVELOPMENT IN THE GENERAL CASE.

3.1.1 Preliminaries of the Proposed Estimators.

For the problem stated in section 3.0, the data consist of observations on (Z,d), where $Z=\mathrm{Min}(\mathrm{Max}(X_1,X_2),X_3)$, denotes the observed life length of the system and $d=(I_1,I_2,I_3,I_4,I_5)$ is the indicator variable. The classes for d are identified as follows :

$$A_1=\{\ Z\in R^+ |\ X_1<Z,\ Z=X_2\ ,\ X_3>Z\ \}$$

$$A_2=\{\ Z\in R^+ |\ X_2<Z,\ Z=X_1\ ,\ X_3>Z\ \}$$

$$A_3=\{ Z\in R^+ | \; X_1<Z, \; Z=X_3 \;, \; X_2>Z \}$$

$$A_4=\{ Z\in R^+ | \; X_2<Z, \; Z=X_3 \;, \; X_1>Z \}$$

$$A_5=\{ Z\in R^+ | \; X_1>Z, \; Z=X_3 \;, \; X_2>Z \}$$

.............(3.1.1)

Let us write $I_i=1$, $I_j=0$ for $j\neq i$, for an observation belonging to class A_i, $i=1,2..5$, i.e $I_i=1$, $I_j=0 \; \forall \; J\neq i \iff Z \in A_i$, $i=1,2...5$. For the problem stated and classes enumerated in (3.1.1), we have,

$$\Pi_i=P(Z\in A_i)=E(I_i), \; i=1,2..5$$

............(3.1.2)

The conditional densities of Z given $Z\in A_i$ are given by the following expressions.

$$dF(Z|Z\in A_1)=\Pi_1^{-1} \; F_1(Z) \; \bar{F}_3(Z)dF_2(Z)$$

$$dF(Z|Z\in A_2)=\Pi_2^{-1} \; F_2(Z) \; \bar{F}_3(Z)dF_1(Z)$$

$$dF(Z|Z\in A_3)=\Pi_3^{-1} \; F_1(Z) \; \bar{F}_2(Z)dF_3(Z)$$

$$dF(Z|Z\in A_4)=\Pi_4^{-1} \; F_2(Z) \; \bar{F}_1(Z)dF_3(Z)$$

$$dF(Z|Z\in A_5)=\Pi_5^{-1} \; \bar{F}_1(Z) \; \bar{F}_2(Z)dF_3(Z)$$

..........(3.1.3)

Subsurvival function of five different classes are given by the following expressions.

$$S_1(x) \; = \; P(Z \; > \; x, \; I_1=1) \; = \int_x^{\infty} F_1(Z) \; \bar{F}_3(Z) \; dF_2(Z)$$

$$S_2(x) \; = \; P(Z \; > \; x, \; I_2=1) \; = \int_x^{\infty} F_2(Z) \; \bar{F}_3(Z) \; dF_1(Z)$$

$$S_3(x) = P(Z > x,\ I_3=1) = \int_x^{\infty} F_1(Z)\ \bar{F}_2(Z)\ dF_3(Z)$$

$$S_4(x) = P(Z > x,\ I_4=1) = \int_x^{\infty} F_2(Z)\ \bar{F}_1(Z)\ dF_3(Z)$$

$$S_5(x) = P(Z > x,\ I_5=1) = \int_x^{\infty} \bar{F}_1(Z)\ \bar{F}_2(Z)\ dF_3(Z) \qquad \text{.......(3.1.4)}$$

It can be verified easily that the survival function of the random variable Z is given by :

$$S(x) = P(Z > x) = \sum_{i=1}^{5} S_i(x) = \bar{F}_3(x)\ (1-F_1(x)\ F_2(x)) \qquad \text{.......(3.1.5)}$$

under the assumption, $\bar{F}_3(.)=(\bar{F}_1(.))^{\beta_1} = (\bar{F}_2(.))^{\beta_1\phi_1}$ as introduced in the introduction, on simplification (3.1.4) reduces to the following form :

$$S_1(x)=(\beta_1\phi_1+1)^{-1}\ (\bar{F}_2(x))^{\beta_1\phi_1+1} - BB,$$

where $\quad BB=(\beta_1\phi_1+\phi_1+1)^{-1}\ (\bar{F}_2(x))^{\beta_1\phi_1+\phi_1+1}$

$$S_1(0) = \Pi_1 = E(I_1) = \phi_1(\beta_1\phi_1+1)^{-1}(\beta_1\phi_1+\phi_1+1)^{-1}$$

$$S_2(x) = (\beta_1+1)^{-1}\ (\bar{F}_2(x))^{\beta_1\phi_1+\phi_1} - \phi_1\ BB$$

$$S_2(0) = \Pi_2 = E(I_2) = (\beta_1+1)^{-1}\ (\beta_1\phi_1+\phi_1+1)^{-1}$$

$$S_3(x) = \beta_1\phi_1(\beta_1\phi_1+1)^{-1}\ (\bar{F}_2(x))^{\beta_1\phi_1+1} - \beta_1\phi_1 \cdot BB$$

$$S_3(0) = \Pi_3 = E(I_3) = \beta_1\phi_1^2(\beta_1\phi_1+1)^{-1}\ (\beta_1\phi_1+\phi_1+1)^{-1}$$

$$S_4(x) = \beta_1(\beta_1+1)^{-1}\ (\bar{F}_2(x))^{\beta_1\phi_1+\phi_1} - \beta_1\phi_1 \cdot BB$$

$$S_4(0) = \Pi_4 = E(I_4) = \beta_1(\beta_1+1)^{-1}(\beta_1\phi_1+\phi_1+1)^{-1}$$

$S_5(x) = \beta_1\phi_1.BB$

$S_5(0) = \Pi_5 = E(I_5) = \beta_1\phi_1(\beta_1\phi_1+\phi_1+1)^{-1}$ ....(3.1.6)

It can be verified easily that $\sum_{i=1}^{5}\Pi_i = 1$, as it should be. On simplification the following relations are found to hold from (3.1.6), viz,

$\bar{F}_2(x) = Exp((\beta_1\phi_1+1)^{-1}\log C_1(x)) = Exp((\phi_1(\beta_1+1))^{-1}\log C_2(x))$

$\bar{F}_1(x) = Exp(\phi_1(\beta_1\phi_1+1)^{-1}\log C_1(x)) = Exp((\beta_1+1)^{-1}\log C_2(x))$

Where $C_1(x) = S_1(x) + S_3(x) + S_5(x)(1+(\beta_1\phi_1)^{-1})$

$C_2(x) = S_2(x) + S_4(x) + S_5(x)(1+(\beta_1^{-1})$ ....(3.1.7)

It is to be noted that the identities established in (3.1.7) for $\bar{F}_1(x)$ or $\bar{F}_2(x)$ involve two expressions, one comprising the classes A_1, A_3, A_5 and the other comprising the classes A_2, A_4, A_5.

3.1.2 **Estimation of β_1 and ϕ_1**

In a simple random sample of size n, let (Z_j, d_j) with the random vector $d = (I_{1j}, I_{2j}, \ldots I_{5j})$ represent observation corresponding to the j-th sampling unit, for j=1,2,..n. Define the variable, $\bar{Z} = \sum_{j=1}^{n} Z_j/n$ and the indicator variable I_{ij}, where

$$I_{ij} = 1 \text{ if } Z_j \in A_i$$
$$= 0 \text{ if } Z_j \notin A_i,$$

and $n_i = \sum_{j=1}^{n} I_{ij}$, $\bar{I}_i = \sum_{j=1}^{n} I_{ij}/n = n_i/n$ ...(3.1.8)

Considering only the observations on d, viz.,

$d_j=(I_{1j}, I_{2j}, I_{3j}, I_{4j}, I_{5j})$, $J=1,2,...n$, the likelihood is given by :

$$L_1(.) = \prod_{i=1}^{5} \Pi_i^{n_i} \quad ....\ (3.1.9)$$

Let us try to estimate β_1 and ϕ_1 by maximizing the likelihood in (3.1.9) : The equation

$$\frac{\delta \log L_1(.)}{\delta\beta_1} = 0 \quad \text{and} \quad \frac{\delta \log L_1(.)}{\delta\phi_1} = 0 \quad \text{lead to}$$

on replacing $\Pi_5 = \beta_1\phi_1(\beta_1\phi_1+\phi_1+1]^{-1}$ by its sample analogue $\frac{n_5}{n}$, the equations (3.1.9) on simplification becomes .

$$(n_4 - n_1 - n_3) - (n_2 + n_4)\beta_1 \ (\beta_1+1]^{-1}+n_5\ \beta_1^{-1}= 0 \ .....(3.1.10)$$

$$\hat{\phi}_1 = ((\bar{I}_1+2\bar{I}_3)\hat{\beta}_1 - \bar{I}_5)(\bar{I}_5\hat{\beta}_1 - \bar{I}_3\ \hat{\beta}_1^2\ ]^{-1} \quad ....(3.1.11)$$

The replacement is conveniently done in order to obtain closed form estimators of β_1 and ϕ_1. The likelihood equations by themselves do not yield simple estimations in closed forms as desired .

(3.1.10) is a quadratic equation in β_1, leading to a unique positive solution of β_1, viz,

$$\hat{\beta}_1 = ((\bar{I}_5 + \bar{I}_4) - (\bar{I}_1 + \bar{I}_3) + \bar{A}_1)\ (2(\bar{I}_1 + \bar{I}_2 + \bar{I}_3)]^{-1} \quad \text{where}$$

$$\bar{A}_1 =(((\bar{I}_4 + \bar{I}_5)-(\bar{I}_1 + \bar{I}_3))^2+ 4\ \bar{I}_5\ (\bar{I}_1 + \bar{I}_2 + \bar{I}_3))^{0.5}.......(3.1.12)$$

We use $\hat{\beta}_1$ and $\hat{\phi}_1$ as given (3.1.12) and (3.1.11) as the estimators of β_1 and ϕ_1 respectively.

It can be checked easily that $E(\hat{\beta}_1) = \beta_1+ O(n^{-1})$ and $E(\hat{\phi}_1) = \phi_1 + O(n^{-1})$. Hence $\hat{\beta}_1$ and $\hat{\phi}_1$ are asymptotically unbiased for β_1 and ϕ_1 respectively.

3.1.3 **Asymptotic Properties β_1 and ϕ_1**

Theorem 3.1

As $n \longrightarrow \infty$, $\sqrt{n}(\hat{\beta}_1 - \beta_1)$ and $\sqrt{n}(\hat{\phi}_1 - \phi_1)$ are asymptotically normally distributed with mean zero and variances σ_1^2 and σ_2^2 respectively, where σ_1^2 and σ_2^2 are given by.,

$\sigma_1^2 = S_{11}' \, P \, S_{11}$ and $\sigma_2^2 = S_{22}' \, P \, S_{22}$ and P is the singular variance covariance matrix of $(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5)$. $S_{11}' = (a_1, a_2, a_3, a_4, a_5)$ and $S_{22}' = (b_1, b_2, b_3, b_4, b_5)$ respectively

where $a_1 = (R_3 D_1 - 2B_1) R_3^{-2}$,

$a_2 = (R_3 D_2 - 2B_1) R_3^{-2}$, $a_3 = (R_3 D_1 - 2B_1) R_3^{-2}$, $a_4 = D_3 R_3^{-1}$, $a_5 = D_4 R_3^{-1}$

and
$$b_1 = (\Pi_5 (B_1 R_3)^2 + \Pi_3 B_1^2 (R_3 B_3 D_1 - 2B_1 B_3 - B_1 R_3)$$
$$+ \Pi_5^2 R_3^2 (R_3 D_1 - 2B_1) + 2\Pi_3 \Pi_5 R_3 B_1 (2B_1 - R_3 D_1))\ XX$$

where $XX = (B_1 (\Pi_5 R_3 - \Pi_3 B_1))^{-2}$

$$b_2 = (\Pi_3 B_1^2 B_3 (R_3 D_2 - 2B_1) + \Pi_5^2 R_3^2 (R_3 D_2 - 2B_1)$$
$$+ \Pi_3 \Pi_5 R_3 B_1 (4B_1 - 2R_3 D_2))\ XX$$

$$b_3 = (\Pi_5 (B_1 R_3)^2 + \Pi_3 B_1^2 (R_3 B_3 D_1 - 2B_1 R_3 - 2B_1 B_3)$$
$$- 3\Pi_5^2 R_3^2 B_1 + \Pi_3 \Pi_5 B_1 R_3 (4B_1 - 2R_3 D_1) + B_1^3 R_3 B_3)\ XX$$

$$b_4 = (\Pi_3 B_1^2 R_3 B_3 D_3 + \Pi_5^2 R_3^3 D_3 - 2\Pi_3 \Pi_5 R_3^2 B_1 D_3)\ XX$$

$$b_5 = (\Pi_3 B_1^2 R_3 (R_3 + B_3 D_4) - 2\Pi_3 \Pi_5 R_3^2 B_1 D_4$$
$$+ \Pi_5^2 R_3^3 D_4 - B_1^2 R_3^2 B_3)\ XX$$

In the expressions above R_3, B_i's and D_i's are as given in section 3.1.5 later.

Proof

Since $\sqrt{n}(\bar{I}_1-E(I_1),\ldots \bar{I}_5 - E(I_5)) \xrightarrow{D}$ singular $N(0,P)$ as $n \longrightarrow \infty$, where $P = (\sigma_{ij})$ is the variance covariance matrix, with $\sigma_{ii}=\Pi_i(1-\Pi_i)$ and $\sigma_{ij}= -\Pi_i\Pi_j$ for $i\neq j$, $i,j=1,2,..5$, by Taylor's series expansion or delta method (Rao(1973)), we can conclude that the distributions of $\sqrt{n}(\beta_1 - \beta_1)$, and $\sqrt{n}(\phi_1 - \phi)$ are both asymptotically normal with mean zero. The asymptotic variances are $S_{11}' P S_{11}$ and $S_{22}' P S_{22}$ respectively.

3.1.4 Estimators Proposed for survival Function

Estimators of the survival functions of the components 1 and 2 are obtained following the guidelines of the procedures developed by of Abdushukurov (1984) Cheng and Lin (1984) and Ebrahimi (1985) in the context of a one component system under random censoring with proportional hazard assumption are given by :

(I) **EBRAHIMI TYPE ESTIMATORS**

First let us consider Ebrahimi type estimators indicated by the superscript e in the notation for the estimations. We can always write $\bar{F}_1(x)$ and $\bar{F}_2(x)$ in the following forms

$$n.\bar{F}_2(x) = (n_1 + n_3 + \alpha n_5)\bar{F}_2(x) + (n_2 + n_4 + (1-\alpha)n_5)\ \bar{F}_2(x)$$

$$n.\bar{F}_1(x) = (n_1 + n_3 + \alpha n_5)\bar{F}_1(x) + (n_2 + n_4 + (1-\alpha)n_5)\ \bar{F}_1(x) \quad \ldots\ldots\ldots\ldots(3.2.13)$$

where α is a weight function lying between 0 and 1 . Taking expectaions on both sides, and then using the identities in (3.1.7), one can obtain the relations :

$$\bar{F}_2(x)=(\Pi_1 + \Pi_3 + \alpha\Pi_5)\ \mathrm{Exp}((\beta_1\phi_1+1)^{-1}\log C_1(x))$$

$$+ (\Pi_2 + \Pi_4 + (1-\alpha)\Pi_5)\ \mathrm{Exp}((\phi_1(\beta_1+1))^{-1}\log C_2(x))\ldots\ldots(3.1.14)$$

$$\bar{F}_1(x)=(\Pi_1 + \Pi_3 + \alpha\Pi_5)\ \mathrm{Exp}(\phi_1(\beta_1\phi_1+1)^{-1}\log C_1(x))$$

$$+ (\Pi_2 + \Pi_4 + (1-\alpha)\Pi_5)\ \mathrm{Exp}((\beta_1+1)^{-1}\log C_2(x))\ldots\ldots\ldots(3.1.15)$$

They lead to estimators of the following forms analogues to Ebrahimi's (1985) estimator proposed in case of the one component system.

$$\bar{\bar{F}}_2(x)=(\bar{I}_1 + \bar{I}_3 + \alpha\bar{I}_5)\ \mathrm{Exp}((\hat{\beta}_1\hat{\phi}_1+1)^{-1}\log\hat{C}_1(x))$$

$$+ (\bar{I}_2 + \bar{I}_4 + (1-\alpha)\bar{I}_5)\ \mathrm{Exp}((\hat{\phi}_1(\hat{\beta}_1+1)^{-1}\log\hat{C}_2(x))\ldots\ldots(3.1.16)$$

$$\bar{\bar{F}}_1(x)=(\bar{I}_1 + \bar{I}_3 + \alpha\ \bar{I}_5)\ \mathrm{Exp}(\hat{\phi}_1(\hat{\beta}_1\phi_1+1)^{-1}\log\hat{C}_1(x))$$

$$+ (\bar{I}_2 + \bar{I}_4 + (1-\alpha)\bar{I}_5)\ \mathrm{Exp}((\hat{\beta}_1+1)^{-1}\log\hat{C}_2(x))\ldots\ldots\ldots(3.1.17)$$

where $\hat{C}_1(x) = \hat{S}_1(x) + \hat{S}_3(x) + \hat{S}_5(x)(1 + (\hat{\beta}_1\hat{\phi}_1)^{-1})$,

$\hat{C}_2(x) = \hat{S}_2(x) + \hat{S}_4(x) + \hat{S}_5(x)(1 + \hat{\beta}_1^{-1})$,

and α is a proper weight function lying between 0 and 1.

As observed earlier identity (3.1.7) is established by using the (i)sets A_1, A_3, A_5 together and (ii)sets A_2, A_4, A_5 together. Incidentally, it may be noted that the classes A_1, A_3 and A_5 represent the set of observations for which $X_2 \geq Z$ and similarly classes A_2,A_4 and A_5 represent the set of observations for which $X_1 \geq Z$.

Theorem 3.2 :

$\overset{e}{\bar{F}}_1(x)(\overset{e}{\bar{F}}_2(x))$ is a strongly consistent estimator of $\bar{F}_1(x)(\bar{F}_2(x))$ for all finite $x \geq 0$.

Proof. $\hat{\beta}_1$ and $\hat{\phi}_1$ are strongly consistent estimators of β_1 and ϕ_1 respectively, $\hat{S}_i(x) = \bar{n}^{1}\sum_{j=1}^{n} I(Z_j > X, I_{ij} = 1)$ is a strongly consistent estimator of $S_i(x)$. $i = 1,2..5$ and moreover $\overset{e}{\bar{F}}_1(x)$, $\overset{e}{\bar{F}}_2(x)$ are continuously differentiable functions of $\hat{\beta}_1$, $\hat{\phi}_1$, and $\hat{S}_i(x)$, $i=1,2..5$. Consequently $\overset{e}{\bar{F}}_1(x)(\overset{e}{\bar{F}}_2(x))$ is a strongly consistent estimator of $\bar{F}_1(x)$ $(\bar{F}_2(x))$.

(ii) **Abdushukurov and Cheng-Lin Type Estimators**

From (3.1.5), we have, $S(x) = \bar{F}_3(x)(1-F_1(x)F_2(x))$, under the assumption $\bar{F}_3(.) = (\bar{F}_1(.))^{\beta_1} = (\bar{F}_2(.))^{\beta_1\phi_1}$, $\beta_1 > 0$, $\phi_1 > 0$, the following relation would be found to hold, viz.

$$\psi(\bar{F}_2(x))=(\bar{F}_2(x))^{\beta_1\phi_1+1} + (\bar{F}_2(x))^{\beta_1\phi_1+\phi_1}$$
$$- (\bar{F}_2(x))^{\beta_1\phi_1+\phi_1+1} = S(x) \quad \text{............(3.1.18)}$$

Theorem 3.3 :

For any $S(x) \in (0,1)$, the equation in $\bar{F}_2(x)$ in (3.1.18) has a unique solution lying between 0 and 1 for any preassingned $\beta_1 > 0$, $\phi_1 > 0$.

Proof.

For notational convenience let us write $S(x) = p$ and $\bar{F}_2(x). = \gamma$. The above equation in (3.1.18) may then be expressed as

$$\psi(\gamma)-p = 0 \quad \text{..........................(3.1.19)}$$

For given $\beta_1>0$, $\phi_1>0$, $\psi'(\gamma) = \delta\psi(\gamma)/\delta\gamma$

$$= (\beta_1\phi_1+1)\gamma^{\beta_1\phi_1}(1-\gamma^{\phi_1}) + \phi_1\gamma^{\beta_1\phi_1+\phi_1-1}(1-\gamma) + \beta_1\phi_1\gamma^{\beta_1\phi_1+\phi_1-1} > 0 \text{ for all } \gamma\in(0,1).$$

Hence $\psi(\gamma)$ is monotonically increasing in $\gamma \in (0,1)$. Moreover $\psi(0) = 0$ and $\psi(1)=1$.

Equation (3.1.18) has a unique solution in $\gamma = \bar{F}_2(x)$ $\in(0,1)$ for given $p = S(x) \in (0,1)$.

The final unique solution of (3.1.18) is of course dependent on β_1,ϕ_1 and $S(x)$ and can be written as: $\bar{F}_2(x) = \psi_2(\beta_1,\phi_1,S(x))$, say, and this will lead to :

$\bar{F}_2(x)^{\phi_1}=(\psi_2(\beta_1,\phi_1,S(x))^{\phi_1} = \bar{F}_1(x) = \psi_1(\beta_1,\phi_1,S(x))$, say. They lead us to the following estimators of the form :

$$\bar{F}^{a}_1(x) = \psi_1(\hat{\beta}_1, \hat{\phi}_1, \hat{S}(x)), \text{ and } \bar{F}^{a}_2(x) = \psi_2(\hat{\beta}_1, \hat{\phi}_1,\hat{S}(x)) \quad ..(3.1.20)$$

where estimators of β_1 and ϕ_1 are as given by (3.1.12) and (3.1.11) respectively. The methods of estimation described above appear to be logical extensions to the present problem of the procedure developed by Abdushukurov(1984), Cheng and Lin (1984) in the context of one component system under random censoring with proportional hazard assumption. The estimators $\bar{F}^{a}_1(x)$ and $\bar{F}^{a}_2(x)$ will in short be referred to as ACL estimaters and that is indicated by the superscript "a" in the notation used for the estimators.

<u>Theorem 3.4</u>

Estimators $\bar{F}^{a}_1(x)$ and $\bar{F}^{a}_2(x)$ are all strongly consistent.

Proof : Similar to Theorem 3.2.

Corresponding to the two sets of estimators of $\bar{F}_1(x)$ and $\bar{F}_2(x)$, two natural estimates of $\bar{F}(x)$ are :

$$\text{(i) } \overset{e}{\bar{F}}(x) = \overset{e}{\bar{F}}_1(x) + \overset{e}{\bar{F}}_2(x) - \overset{e}{\bar{F}}_1(x)\overset{e}{\bar{F}}_2(x)$$

$$\text{(ii) } \overset{a}{\bar{\bar{F}}}(x) = \overset{a}{\bar{\bar{F}}}_1(x) + \overset{a}{\bar{\bar{F}}}_2(x) - \overset{a}{\bar{\bar{F}}}_1(x)\overset{a}{\bar{\bar{F}}}_2(x) \;, \quad \text{........(3.1.21)}$$

where $\bar{F}(x) = P(X > x)$ = Survival function of the life of the system $X = \text{Max}(X_1, X_2)$.

3.1.5 Asymptotic Variances of the Proposed Estimators,

Before deriving the asymptotic expansions, some symbols are introduced here to simplify the complicated expressions (some of them have already been used, but are included here for the sake of completeness).

$$R_1 = \sqrt{((\Pi_5 + \Pi_4 - \Pi_1 - \Pi_3)^2 + 4\Pi_5(\Pi_1 + \Pi_2 + \Pi_3))}$$

$$R_2 = \Pi_5 + \Pi_4 - \Pi_1 - \Pi_3$$

$$R_3 = 2(\Pi_1 + \Pi_2 + \Pi_3))$$

$$R_4 = \Pi_1 + \Pi_3 + \Pi_5 - \Pi_4$$

$$R_5 = \Pi_1 + \Pi_3 + \alpha\Pi_5$$

$$R_6 = \Pi_2 + \Pi_4 + (1-\alpha)\Pi_5 \quad 0 < \alpha < 1$$

$$B_1 = R_1 + R_2,\; B_2 = \Pi_1 + \Pi_3,\quad B_3 = \Pi_1 + 2\Pi_3$$

$$C_1(x) = S_1(x) + S_3(x) + S_5(x)(1 + (\beta_1\phi_1)^{-1})$$

$$C_2(x) = S_2(x) + S_4(x) + S_5(x)(1 + \beta_1^{-1})$$

$$D_1 = R_4R_1^{-1} - 1\,, \qquad D_2 = 2R_1^{-1}\Pi_5\,,$$

$$D_3 = R_2\, R_1^{-1} + 1, \qquad D_4 = R_1^{-1}(1 + \Pi_2) + 1$$

$$E_1 = (C_1(x)(\beta_1\phi_1 + 1))^{-1}S_5(x), \qquad E_2 = \phi_1 E_1,$$

$$E_3 = (\phi_1 C_2(x)\;(\beta_1 + 1))^{-1}S_5(x)\;, \qquad E_4 = \phi_1 E_3$$

$$l_{11} = (\Pi_5(2B_1B_2 - R_3B_2D_1 - R_3B_1) + \Pi_3B_1^2)\ (B_1B_2)^{-2}$$

$$l_{12} = (\Pi_5B_2(2B_1 - R_3D_2))(B_1B_2)^{-2}$$

$$l_{13} = (\Pi_5(2B_1B_2 - R_3B_2D_1 - R_3B_1)$$
$$+\ \Pi_3B_1^2 - B_1^2B_2)\ (B_1B_2)^{-2}$$

$$l_{14} = (-\ \Pi_5R_3B_2D_3)(B_1B_2)^{-2}$$

$$l_{15} = (B_1B_2R_3 - \Pi_5R_3B_2D_4)(B_1B_2)^{-2}$$

$$l^*_{11} = (\Pi_5(2B_1B_2 - R_3B_1 - R_3B_2D_1) + \Pi_3B_1^2)\ BBB\ ,$$

where $BBB = (B_1B_3 - \Pi_5R_3)^{-2}$

$$l^*_{12} = \Pi_5B_2(2B_1 - R_3D_2)\ BBB$$

$$l^*_{13} = (\Pi_5(2B_1B_2 - R_3B_1 - R_3B_2D_1) - \Pi_3{B_1}^2)\ BBB$$

$$l^*_{14} = -\ \Pi_5(R_3B_2D_3)\ BBB$$

$$l^*_{15} = (B_1B_2R_3 - \Pi_5R_3B_2D_4)\ BBB$$

$$l^{**}_{11} = (\Pi_5R_3B_1(2R_3B_2D_1 - 4B_1B_2 - R_3B_1)$$
$$+\quad B_1B_2(B_1^2R_3 + 2B_1^2\ B_3 - R_3B_3D_1) - R_3B_1^3B_3)\ LL$$

where $LL = B_1^{-4}B_2^{-2}$

$$l^{**}_{12} = (\Pi_5R_3B_1(2R_3B_2D_2-4B_1B_2) + B_1^2B_2B_3(2B_1-R_3D_2))\ LL$$

$$l^{**}_{13} = (\Pi_5R_3B_1(R_3B_1 + 2R_3B_2D_1 - 4B_1B_2) + B_1^2(2B_1B_2B_3$$
$$+\ 2B_1R_3B_2 - R_3B_2B_3D_1 - R_3B_1B_3))\ LL$$

$$l^{**}_{14} = (2\Pi_5R_3^2\ B_1B_2D_3 - R_3B_1^2B_2B_3D_3)\ LL$$

$$l^{**}_{15} = (2\Pi_5 R_3^2 B_1 B_2 D_4 - R_3 B_1^2 B_2 (R_3 + B_3 . D_4))\ LL$$

$$l^{***}_{11} = (\Pi_5^2 R_3^2 (2B_1 - R_3 D_1) + \Pi_3 \Pi_5 (B_1^2 R_3 D_1 + 2R_3^2 B_1 D_1$$

$$- 2B_1^3 - 4B_1^2 R_3) + \Pi_5 (B_1^2 R_3 B_3 D_1 - 2B_1^2 B_3 D_1$$

$$+ 2B_1^3 B_3 - B_1^3 R_3 - B_1^2 R_3^2) + \Pi_3 B_1^2 (B_1^2 + B_1 R_3$$

$$+ 2B_1 B_3 - R_3 B_3 D_1))\ XXX\ ,$$

where $XXX = (B_1 + R_3)^{-2} (B_1 B_3 - \Pi_5 R_3)^{-2}$

$$l^{***}_{12} = (\Pi_5^2 R_3^2 (2B_1 - R_3 D_2 - B_1 D_2) + \Pi_3 \Pi_5\ (2R_3^2 B_1 D_2 +$$

$$+ B_1^2 R_3 D_2 - 2B_1^3 - 4B_1^2 R_3) + B_1^2 B_3 (\Pi_3 + \Pi_5)\ (2B_1 - R_3 D_2))\ XXX$$

$$l^{***}_{13} = (\Pi_5^2 R_3^2 (2B_1 - R_3 D_1) + \Pi_3 \Pi_5 B_1 (R_3 B_1 D_1 + 2R_3^2 D_1 - 2B_1^2 - 4R_3 B_1)$$

$$+ \Pi_3 B_1^2 (2B_1^2 + B_3 D_1 + 2B_1 R_3 + 2B_1 B_3 - R_3 B_3 D_1)$$

$$+ \Pi_5 B_1^2 (2B_1 B_3 - R_3 B_3 D_1 - B_1 R_3 - R_3^2) - B_1^3 B_3 (R_3 + B_1))\ XXX$$

$$l^{***}_{14} = (-\Pi_5^2 R_3^3 D_3 + \Pi_3 \Pi_5 (2R_3^2 B_1 D_3 - B_1^2 R_3 D_3)$$

$$- \Pi_3 B_1^2 R_3 B_3 D_3 + \Pi_5 B_1^2 R_3 B_3 D_3)\ XXX$$

$$l^{***}_{15} = (-\Pi_5^2 R_3^3 D_4 + \Pi_3 \Pi_5 (2R_3^2 B_1 D_4 - B_1^2 R_3 D_4) +$$

$$+ \Pi_5 (2B_1^2 R_3 D_4 - B_1^2 R_3 B_3 D_4) - \Pi_3 B_1^2 R_3 (B_1 + B_3 D_4 + B_1^2 R_3^2)$$

$$+ B_1^2 R_3 B_3 (R_3 + B_1))\ XXX$$

$$m_{11} = (2B_1 - R_3 D_1)(B_1 + R_3)^{-2}$$

$$m_{12} = (2B_1 - R_3 D_2)(B_1 + R_3)^{-2}$$

$$m_{13} = (2B_1 - R_3 D_1)(B_1 + R_3)^{-2}$$

$$m_{14} = -R_3 D_3 (B_1 + R_3)^{-2}$$

$$m_{15} = -R_3D_4(B_1 + R_3)^{-2}$$

$$m^*_{11} = (2B_1 - R_3D_1).B_1^{-2}$$

$$m^*_{12} = (2B_1 - R_3D_2).B_1^{-2}$$

$$m^*_{13} = (2B_1 - R_3D_1).B_1^{-2}$$

$$m^*_{14} = -R_3D_3B_1^{-2},$$

$$m^*_{15} = -R_3D_4B_1^{-2}$$

$$l_{21} = (C_1(x)\ (\beta_1\phi_1 + 1))^{-1} = l_{23}$$

$$l_{22} = 0 = l_{24}$$

$$l_{25} = (\beta_1\phi_1C_1(x))^{-1}$$

$$l^*_{21} = l^*_{23} = \phi_1 l_{21},\ l^*_{22} = 0,\ l^*_{24} = 0,\ l^*_{25} = \phi_1.l_{25}$$

$$l^{**}_{21} = l^{**}_{23} = 0,\ l^{**}_{22} = l^{**}_{24} = (C_2(x)\ \phi_1(\beta_1+1))^{-1},$$

$$l^{**}_{25} = (\beta_1\phi_1C_2(x))^{-1} \qquad (3.1.22)$$

(a) **Asymptotic Variances of Ebrahimi Type Estimators**

Theorem 3.5

For any finite $x \in R^+_+$ $\sqrt{n}(\bar{I}_1-E(I_1),\ldots\ldots \bar{I}_5-E(I_5),\hat{S}_1(x)-S_1(x),\ldots\ldots\ldots\ \hat{S}_5(x)-S_5(x))$ is asymptotically normally (singular) distributed with mean vector null and variance covariance matrix $D=\begin{pmatrix} P & Q \\ Q' & R\end{pmatrix}$, where $P=(d_{ij})$ for $1 \le i, j \le 5$, $Q=(q_{ij})$, $1 \le i, j \le 5$, $R=(r_{ij})$, $1 \le i, j \le 5$, $d_{ij}=\Pi_i(1-\Pi_i)$ if $i=j$ and $d_{ij}= -\Pi_i\Pi_j$ if $i\neq j$, $q_{ij} = S_i(x)(1-\Pi_i)$ if $i=j$ and $q_{ij}=-\Pi_jS_i(x)$ if $i\neq j$, $r_{ij}= S_i(x)(1-S_i(x))$ if $i=j$ and $-S_i(x)S_j(x)$ if $i \neq j$. (3.1.23)

Proof. By Central limit theorem.

Theorem 3.6 :

$\sqrt{n}(\bar{F}^{e}_{1}(x) - \bar{F}_{1}(x))$, $(\sqrt{n}(\bar{F}^{e}_{2}(x) - \bar{F}_{2}(x))$ for finite $x \in R^{+}$ is asymptotically Normally distributed with mean zero and variance $b' D b$ $(a' D a)$, where a and b are explicitly stated in the proof that follows.

Proof. Let us recall the estimator $\bar{F}^{e}_{2}(x)$ as given in (3.1.16). By Taylor's series expansion about the true values of the parameters, viz , $(\Pi_1, \Pi_2, \ldots \Pi_5, S_1(x), \ldots S_5(x))$ and retaining only the 1st order term, we have :

$$\sqrt{n}(\bar{F}^{e}_{2}(x) - \bar{F}_{2}(x)) = \sqrt{n} \sum_{i=1}^{5} a_{1i}(\bar{I}_i - E(I_i))$$

$$+ \sqrt{n} \sum_{i=1}^{5} a_{2i}(\hat{S}_i(x) - S_i(x)) + R_n \qquad \ldots\ldots(3.1.24)$$

where R_n converges in probability to zero as $n \longrightarrow \alpha$. The expressions for a_{1i}'s, a_{2i}'s for i=1,2,....5 are given below :

$$a_{11} = \bar{F}_2(x)(l_{11}R_5 \log C_1(x) + R_5E_1 l^{*}_{11}$$

$$+ R_6 l^{***}_{11} \log C_2(x) + R_6E_3 m^{*}_{11} + 1)$$

$$a_{12} = \bar{F}_2(x)(l_{12}R_5 \log C_1(x) + R_5E_1 l^{*}_{12}$$

$$+ R_6 l^{***}_{12} \log C_2(x) + R_6E_3 m^{*}_{12} + 1)$$

$$a_{13} = \bar{F}_2(x)(l_{13}R_5 \log C_1(x) + R_5E_1 l^{*}_{13}$$

$$+ R_6 l^{***}_{13} \log C_2(x) + R_6E_3 m^{*}_{13} + 1)$$

$$a_{14} = \bar{F}_2(x)(l_{14}R_5 \log C_1(x) + R_5E_1 l^{*}_{14}$$

$$+ R_6 l^{***}_{14} \log C_2(x) + R_6E_3 m^{*}_{14} + 1)$$

$$a_{15} = \bar{F}_2(x)(l_{15}R_5\log C_1(x) + R_5E_1l^*_{15}$$

$$+ R_6l^{***}_{15}\log C_2(x) + R_6E_3m^*_{15} + 1)$$

$$a_{21} = \bar{F}_2(x)R_5l_{21} = a_{23}$$

$$a_{22} = \bar{F}_2(x)R_6l^*_{22} = a_{24}$$

$$a_{25} = \bar{F}_2(x)(R_5l_{25} + R_6l^*_{25})$$

writing $a' = (a_{11}, a_{12}, \ldots a_{15}, a_{21} \ldots a_{25})$, one can write

$$\text{Asym var}(\bar{\bar{F}}^e_2(x) = n^{-1}(a'D\,a) \text{ upto order } n^{-1}. \qquad \ldots\ldots\ldots(3.1.25)$$

Similarly one can consider the estimator $\bar{\bar{F}}^e_1(x)$ given in (3.1.17). By Taylor's series expansion, about $(\Pi_1, \Pi_2, \ldots \Pi_5, S_1(x), \ldots S_5(x))$, one can write $\bar{\bar{F}}^e_1(x)$ in the following form :

$$\sqrt{n}(\bar{\bar{F}}^e_1(x) - \bar{F}_1(x)) = \sqrt{n}\sum_{i=1}^{5} b_{1i}(\bar{I}_i - E(I_i))$$

$$+ \sqrt{n}\sum_{i=1}^{5} b_{2i}(\hat{S}_i(x) - S_i(x)) + R_n, \qquad \ldots\ldots(3.1.26)$$

where R_n converges in probability to zero as $n \longrightarrow \infty$. The expressions for b_{1i}'s, b_{2i}'s for i=1,2,....5 are given below :

$$b_{11} = \bar{F}_1(x)(R_5l^{**}_{11}\log C_1(x) + R_5E_2l^*_{11}$$

$$+ R_6m_{11}\log C_2(x) + R_6E_4m^*_{11} + 1)$$

$$b_{12} = \bar{F}_1(x)(R_5l^{**}_{12}\log C_1(x) + R_5E_2l^*_{12}$$

$$+ R_6m_{12}\log C_2(x) + R_6E_4m^*_{12} + 1)$$

$$b_{13} = \bar{F}_1(x)(R_5l^{**}_{13}\log C_2(x) + R_5E_2l^*_{13}$$

$$+ \; R_6 m_{13} \log C_2(x) + R_6 E_4 m_{13}^{*} + 1)$$

$$b_{14} = \bar{F}_1(x)(R_5 l_{14}^{**} \log C_1(x) + R_5 E_2 l_{14}^{*}$$

$$+ \; R_6 m_{14} \log C_2(x) + R_6 E_4 m_{14}^{*} + 1)$$

$$b_{15} = \bar{F}_1(x)(R_5 l_{15}^{**} \log C_1(x) + R_5 E_2 l_{15}^{*}$$

$$+ \; R_6 m_{15} \log C_2(x) + R_6 E_4 m_{15}^{*} + 1)$$

$$b_{21} = \bar{F}_1(x) R_5 l_{21}^{*} = b_{23}$$

$$b_{22} = \bar{F}_1(x) R_6 l_{22}^{***} = b_{24}$$

$$b_{25} = \bar{F}_1(x)(R_5 l_{25}^{*} + R_6 l_{25}^{***})$$

writing $b' = (b_{11}, b_{12}, \ldots\ldots b_{15}, b_{21}, \ldots . b_{25})$, one can write

$$\text{Asym Var}(\overset{e}{\bar{\bar{F}}}_1(x)) = n^{-1}(b' \, D \, b), \text{ upto order } n^{-1}. \quad \ldots\ldots\ldots(3.1.27)$$

In theorem 3.6 Asym Variance covariance matrix of (3.1.27) $(\overset{e}{\bar{\bar{F}}}_2(x), \overset{e}{\bar{\bar{F}}}_1(x)$ can be written as :

$$\begin{pmatrix} a' \, Da, & a' \, Db \\ a' \, Db, & b' \, Db \end{pmatrix}$$

From this one can write down the expression for the Asym Variance of $\overset{e}{\bar{F}}(x)$.

(b) Asymptotic Variances of ACL Type Estimators.

Writing $\hat{\gamma} = \overset{a}{\bar{\bar{F}}}_2(x) = \psi_2(\hat{\beta}_1, \hat{\phi}_1, \hat{S}(x))$ as given in (3.1.20), let us recall that $\hat{\gamma}$ is the unique solution of

$$(\hat{\gamma})^{\hat{\beta}_1 \hat{\phi}_1 + 1} + (\hat{\gamma})^{\hat{\beta}_1 \hat{\phi}_1 + \hat{\phi}_1} - (\hat{\gamma})^{\hat{\beta}_1 \hat{\phi}_1 + \hat{\phi}_1 + 1} - \hat{S}(x) = 0 \quad \ldots\ldots\ldots(3.1.28)$$

lying in (0,1).

By Taylor's series expansion about $(\beta_1, \phi_1, S(x))$ and retaining only the 1st order terms, we have.

$$0 = C_{11}\cdot\sqrt{n}(\hat{\gamma}-\gamma) + C_{22}\cdot\sqrt{n}(\hat{\beta}_1-\beta_1) + C_{33}\cdot\sqrt{n}(\hat{\phi}_1-\phi_1)$$
$$+ \quad C_{44}\cdot\sqrt{n}(\hat{S}(x) - S(x)) + R_n \qquad \ldots\ldots\ldots\ldots(3.1.29)$$

where R_n converges in probability to zero as $n \longrightarrow \infty$. The expressions for $C_{11}, C_{22}, C_{33}, C_{44}$, are :

$$C_{11} = \gamma^{\beta_1\phi_1}(\beta_1\phi_1+1+(\beta_1\phi_1+\phi_1)\gamma^{\phi_1-1} - (\beta_1\phi_1+\phi_1+1)\gamma^{\phi_1})$$

$$C_{22} = \phi_1 \log\gamma\, \gamma^{\beta_1\phi_1}(\gamma+\gamma^{\phi_1} - \gamma^{\phi_1+1})$$

$$C_{33} = \log\gamma\, \gamma^{\beta_1\phi_1}(\beta_1\gamma+(\beta_1+1)\gamma^{\phi_1} - (\beta_1+1)\gamma^{\phi_1+1})$$
$$C_{44} = -1 \qquad \ldots\ldots\ldots\ldots(3.1.30)$$

Multiplying (3.1.29) succesively by $(\hat{\gamma}-\gamma)$, $(\hat{\beta}_1-\beta_1)$, $(\hat{\phi}_1-\phi_1)$ and $(\hat{S}(x) - S(x))$ and taking expectations, we have the following systems of equations.

$$C_{11}\mathrm{Var}(\hat{\gamma}) + C_{22}\mathrm{Cov}(\hat{\gamma},\hat{\beta}_1) + C_{33}\mathrm{Cov}(\hat{\gamma},\hat{\phi}_1) + C_{44}\mathrm{Cov}(\hat{\gamma},\hat{S}(x)) = 0$$

$$C_{11}\mathrm{Cov}(\hat{\gamma},\hat{\beta}_1) + C_{22}\mathrm{Var}(\hat{\beta}_1) + C_{33}\mathrm{Cov}(\hat{\beta}_1,\hat{\phi}_1) + C_{44}\mathrm{Cov}(\hat{\beta}_1,\hat{S}(x)) = 0$$

$$C_{11}\mathrm{Cov}(\hat{\gamma},\hat{\phi}_1) + C_{22}\mathrm{Cov}(\hat{\beta}_1,\hat{\phi}_1) + C_{33}\mathrm{Var}(\hat{\phi}_1) + C_{44}\mathrm{Cov}(\hat{\phi}_1,\hat{S}(x)) = 0$$

$$C_{11}\mathrm{Cov}(\hat{S}(x),\hat{\gamma})+C_{22}\mathrm{Cov}(\hat{S}(x),\hat{\beta}_1)+C_{33}\mathrm{Cov}(\hat{S}(x),\hat{\phi}_1)+C_{44}\mathrm{Var}(\hat{S}(x)) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.1.31)$$

In order to compute the asymptotic variance of $\hat{\gamma}=\bar{\bar{F}}^{a}_2(x)$, we solve for $\mathrm{var}(\hat{\gamma})$ from (3.1.31) given the other quantities like, $\mathrm{Cov}(\hat{S}(x),\hat{\beta}_1)$, $\mathrm{Cov}(\hat{S}(x),\hat{\phi}_1)$, $\mathrm{var}(\hat{\beta}_1)$ $\mathrm{Var}(\hat{\phi}_1)$ and $\mathrm{Cov}(\hat{\beta}_1,\hat{\phi}_1)$, which are known to be as follows from Theorem 3.1 and by simple computation.

$$\text{Asym var}(\hat{\beta}_1) = n^{-1} S_{11}' . P . S_{11}$$

$$\text{Asym var}(\hat{\phi}_1) = n^{-1} S_{22}' . P . S_{22}$$

$$\text{Asym cov}(\hat{\beta}_1, \hat{\phi}_1) = n^{-1} S_{11}' . P . S_{22}$$

$$\text{Asym var}(\hat{S}(x)) = n^{-1} S(x)\ (1 - S(x))$$

$$\text{Asym cov}(\hat{S}(x), \hat{\beta}_1) = n^{-1} \left(\sum_{i=1}^{5} a_i S_i(x) - S(x) \sum_{i=1}^{5} a_i \Pi_i \right)$$

$$\text{Asym cov}(\hat{S}(x), \hat{\phi}_1) = n^{-1} \left(\sum_{i=1}^{5} b_i S_i(x) - S(x) \sum_{i=1}^{5} b_i \Pi_i \right) \quad \ldots\ldots\ldots\ldots\ldots(3.1.32)$$

One can find the asymptotic variance of $\overset{a}{\bar{\bar{F}}}_1(x) = (\overset{a}{\bar{\bar{F}}}_2(x))^{\hat{\phi}_1}$ in a manner as demonstrated earlier in connection with obtaining the Asym var of $\overset{e}{\bar{\bar{F}}}_1(x)$, after first finding the Asym var$(\overset{e}{\bar{\bar{F}}}_2(x))$. Then finding the Asym.variance of $\overset{a}{\bar{\bar{F}}}(x)$ = ACL type estimator of the system survival function is a simple exercise which can be worked out in the same manner as in the case of the Ebrahimi type estimator $\overset{e}{\bar{\bar{F}}}(x)$.

(C) **Asymptotic Variance of Kaplan. Meier Estimator, viz., $\bar{F}^{KM}(x)$.**

Following the results of Breslow and crowley (1974), one can easily write the asymptotic variance of $\overset{KM}{\bar{\bar{F}}}(x)$ by disregrading the components and treating the system as a unit. Thus,

$$\text{Asym Var}(\bar{F}^{KM}(x)) = n^{-1} \left(\int_0^x (\bar{F}(u))^{-2} (\bar{\bar{F}}_3(u))^{-1} dF(u) \right) (\bar{\bar{F}}(x))^2 \ldots (3.1.33)$$

3.2 NUMERICAL COMPUTATION IN THE GENERAL CASE

First we find the point x, when true survival probability of the system is (i) 0.90 and (ii) 0.95 for different Weibull distributions (Exponential distribution as a special case of Weibull distribution) of the component lives, with specified scale and shape parameters, α_1 and δ_1 and for different combinations of parameters connected with censorship and proportional hazard assumption viz., β_1 and ϕ_1. Asymptotic variances of (a)$\bar{\bar{F}}^{e}(x)$ (b)$\bar{\bar{F}}^{a}(x)$ and (c)$\bar{\bar{F}}^{KM}(x)$ are computed for this x by using the asymptotic formula devloped in section 3.1.5. Numerical values are reported for α = 0.5 in case of $\bar{\bar{F}}^{e}$ (x), since at α=0.5, the asymptotic variances of $\bar{\bar{F}}^{e}(x)$ is found to be smallest in all the examples considered, even though it has not been posible to prove the result algebraically in general. These results are reported in Tables 3.2.1 through 3.2.6 which follow. For the sake of comparison numerical values are reported as n times the asymptotic variance,since it has not been possible to prove algebrically the bestness or otherwise of a particular estimator.The numerical values of asymptotic variances of (a) $\bar{\bar{F}}^{e}$(.) (b)$\bar{\bar{F}}^{a}$ (.) and (c)$\bar{\bar{F}}^{KM}$(.) respectively are given for a reasonable range of parameters of the weibull distribution and covering wide range of censorship as determined by parameters. The numerical comparison leads us to some general findings which are highlighted at the end of Tables.

Table 3.2.1

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90, (ii)0.95 for $\bar{F}_2(x)=\mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1, \delta_1=1$. The sets of estimates used are (a) $\bar{\bar{F}}^{e}(.)$ (b) $\bar{\bar{F}}^{a}(.)$ and (c) $\bar{F}^{KM}(.)$

θ_1	ϕ_1	0.50 I	0.50 II	1.00 I	1.00 II	2.00 I	2.00 II
0.50	a	0.0126	0.0024	0.0120	0.0020	0.0109	0.0017
	b	0.0098	0.0012	0.0031	0.0004	0.0015	0.0003
	c	0.0441	0.0077	0.0310	0.0060	0.0198	0.0048
1.00	a	0.0033	0.0009	0.0025	0.0007	0.0017	0.0005
	b	0.0013	0.0002	0.0011	0.0001	0.0001	0.0001
	c	0.0198	0.0053	0.0117	0.0035	0.0065	0.0018
2.00	a	0.0005	0.0001	0.0004	0.00009	0.0002	0.00003
	b	0.0001	0.00003	0.00005	0.00008	0.00002	0.00001
	c	0.0065	0.0015	0.0035	0.0009	0.0018	0.0003

Table (Contd.)

Table 3.2.2

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90, (ii)0.95 for $\bar{F}_2(x)=\mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1, \delta_1=0.5$. The sets of estimates used are (a) $\bar{\bar{F}}^{e}(.)$ (b) $\bar{\bar{F}}^{a}(.)$ and (c) $\bar{F}^{KM}(.)$

β_1 | ϕ_1		0.50		1.00		2.00	
		I	II	I	II	I	II
0.50	a	0.0127	0.0025	0.0129	0.0023	0.0109	0.0027
	b	0.0051	0.0005	0.0032	0.0004	0.0016	0.0003
	c	0.0442	0.0078	0.0312	0.0062	0.0198	0.0049
1.00	a	0.0033	0.0009	0.0026	0.0007	0.0016	0.0005
	b	0.0016	0.0002	0.0011	0.0001	0.0001	0.0003
	c	0.0198	0.0052	0.0118	0.0036	0.0066	0.0020
2.00	a	0.0005	0.0001	0.0004	0.00009	0.0002	0.00003
	b	0.0001	0.00003	0.00008	0.00008	0.00002	0.00001
	c	0.0066	0.0016	0.0036	0.0009	0.0020	0.0003

Table (Contd.)

Table 3.2.3

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90, (ii)0.95 for $\bar{F}_2(x)=\mathrm{Exp}\left(\frac{-x^{\alpha_1}}{\delta_1}\right)$, $\alpha_1=1, \delta_1=2$. The sets of estimates used are (a)$\bar{F}^{e}(.)$ (b)$\bar{F}^{a}(.)$ and (c)$\bar{F}^{KM}(.)$

β_1	ϕ_1	0.50 I	0.50 II	1.00 I	1.00 II	2.00 I	2.00 II
0.50	a	0.0128	0.0026	0.0126	0.0026	0.0109	0.0024
	b	0.0056	0.0005	0.0032	0.0004	0.0016	0.0003
	c	0.0442	0.0079	0.0312	0.0062	0.0198	0.0049
1.00	a	0.0037	0.0009	0.0026	0.0007	0.0017	0.0005
	b	0.0020	0.0002	0.0012	0.0002	0.0001	0.0003
	c	0.0198	0.0053	0.0118	0.0036	0.0068	0.0022
2.00	a	0.0005	0.0001	0.0004	0.00009	0.0002	0.00003
	b	0.0001	0.00003	0.00008	0.00008	0.00002	0.00001
	c	0.0066	0.0016	0.0036	0.0009	0.0021	0.0003

Table (Contd.)

Table 3.2.4

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90, (ii)0.95 for $\bar{F}_2(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1 = 1.5$, $\delta_1=0.50$. The sets of estimates used are (a)$\overset{e}{\bar{F}}(.)$ (b)$\overset{a}{\bar{F}}(.)$ and (c)$\overset{KM}{\bar{F}}(.)$

β_1 : ϕ_1		0.50		1.00		2.00	
		I	II	I	II	I	II
0.50	a	0.0128	0.0027	0.0128	0.0025	0.0108	0.0023
	b	0.0057	0.0005	0.0036	0.0004	0.0016	0.0003
	c	0.0443	0.0079	0.0316	0.0063	0.0198	0.0048
1.00	a	0.0036	0.0009	0.0026	0.0007	0.0016	0.0005
	b	0.0020	0.0002	0.0012	0.0001	0.0001	0.0003
	c	0.0196	0.0056	0.0118	0.0036	0.0066	0.0020
2.00	a	0.0005	0.0001	0.0004	0.00009	0.0002	0.00003
	b	0.0001	0.0002	0.00008	0.00008	0.00002	0.00001
	c	0.0068	0.0016	0.0032	0.0009	0.0020	0.0002

Table (Contd.)

Table 3.2.5

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90, (ii)0.95 for $\bar{F}_2(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1 = 1.5$, $\delta_1=1.00$. The sets of estimates used are (a)$\bar{\bar{F}}^{e}(.)$ (b)$\bar{\bar{F}}^{a}(.)$ and (c)$\bar{\bar{F}}^{KM}(.)$

β_1	ϕ_1	0.50 I	0.50 II	1.00 I	1.00 II	2.00 I	2.00 II
0.50	a	0.0127	0.0027	0.0127	0.0025	0.0109	0.0022
	b	0.0096	0.0005	0.0036	0.0004	0.0016	0.0003
	c	0.0441	0.0079	0.0316	0.0062	0.0198	0.0048
1.00	a	0.0036	0.0009	0.0026	0.0007	0.0015	0.0005
	b	0.0018	0.0002	0.0012	0.0001	0.0001	0.0003
	c	0.0196	0.0056	0.0118	0.0035	0.0065	0.0020
2.00	a	0.0005	0.0001	0.0004	0.00009	0.0002	0.00003
	b	0.0001	0.0002	0.00008	0.00008	0.00002	0.00001
	c	0.0068	0.0016	0.0032	0.0009	0.0020	0.0002

Table (Contd.)

Table 3.2.6

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90, (ii)0.95 for $\bar{F}_2(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1 = 1.5$, $\delta_1=2.00$. The sets of estimates used are (a)$\overset{e}{\bar{\bar{F}}}(.)$ (b)$\overset{a}{\bar{\bar{F}}}(.)$ and (c)$\overset{KM}{\bar{\bar{F}}}(.)$

β_1 | ϕ_1		0.50 I	0.50 II	1.00 I	1.00 II	2.00 I	2.00 II
0.50	a	0.0129	0.0026	0.0123	0.0023	0.0108	0.0026
	b	0.0086	0.0005	0.0036	0.0004	0.0016	0.0003
	c	0.0443	0.0078	0.0312	0.0063	0.0198	0.0049
1.00	a	0.0036	0.0009	0.0022	0.0007	0.0016	0.0005
	b	0.0019	0.0002	0.0011	0.0001	0.0001	0.0003
	c	0.0197	0.0057	0.0118	0.0032	0.0066	0.0022
2.00	a	0.0005	0.0001	0.0004	0.00009	0.0002	0.00003
	b	0.0001	0.0002	0.00008	0.00008	0.00002	0.00001
	c	0.0067	0.0016	0.0032	0.0009	0.0018	0.0002

On examining Tables 3.2.1 through 3.2.6, we observe that both $\overset{e}{\bar{\bar{F}}}(.)$ (Ebrahimi type estimator) and $\overset{a}{\bar{\bar{F}}}(.)$ (ACL type estimator) are superior to Kaplan Meier estimator for all degrees of censoring, whereas among the two estimators, viz, $\overset{e}{\bar{\bar{F}}}(.)$ and $\overset{a}{\bar{\bar{F}}}(.)$,

$\overset{a}{\bar{F}}(.)$ behaves consistently better in almost all situations, But for large values of β_1, there is not much difference in the performances in general. Thus, in general of all the estimators included in the comparision, the ACL type estimator, $\overset{a}{\bar{F}}(.)$ is observed to behave best. Of course, it should be borne in mind that this comparison seems to be valid when the main purpose is to estimate system survival function with true system reliabilities quite high, say 0.90 or 0.95. In most of the cases the values of x with high system reliabilities are of major interest and as such the comparison is confined to this region only. But if we extend the comparison to the estimation of system survival function where true survival probability is much smaller say 0.50, it is not unexpected to arrive at a set of conclusions at variance with what we stated here.

3.3 Development in the Case of Identical Distributions of Components

In addition to the set up considered in section 3.1, we assume X_1 and X_2 are identically distributed random variables, with common distribution function $F_1(.)$ i.e., $\phi_1=1$. The present set up can of course be treated as a special case of the general problem treated in section 3.1. But the fact that in the first stage of estimation the multinomial distribution with five exhaustive and mutually exclusive classes discussed in section 3.1 will be utilized to estimate a single parameter β_1 here makes the problem much simpler and it is possible to obtain a maximum likelihood estimate of β_1 in a closed form. This is used in the second stage of estimation for estimating the survival functions. Moreover, the second stage of estimation of section 3.1 will lead to usually more than one comparable estimators in this case. So, there exists the possibility of using a socalled weighted version of the estimators proposed in section 3.1. Hence the present section calls for a specialized treatment which is elaborated in the following lines.

3.3.1 Preliminaries of Estimation Procedure.

From (3.1.4), under the assumption, $\bar{F}_1(.) = \bar{F}_2(.)$, we have,

$$S_1(x) = S_2(x) = \int_x^{\infty} F_1(Z)\bar{F}_3(Z).dF_1(Z)$$

$$S_3(x) = S_4(x) = \int_x^{\infty} F_1(Z)\bar{F}_1(Z).dF_3(Z)$$

$$S_5(x) = \int_x^{\infty} (\bar{F}_1(Z))^2\, dF_3(Z) \qquad \ldots\ldots\ldots\ldots\ldots(3.3.1)$$

under the assumption, $\bar{F}_3(.) = (\bar{F}_1(.))^{\beta_1}$, $\beta_1 > 0$ represents common degree of censoring the random variable X_1 or X_2 is subjected to by the random variable X_3, (3.3.1) reduces to :

$$S_1(x) = S_2(x) = (\beta_1+1)^{-1}(\bar{F}_1(x))^{\beta_1+1} - (\beta_1+2)^{-1}(\bar{F}_1(x))^{\beta_1+2}$$

$$S_3(x) = S_4(x) = \beta_1 S_1(x)$$

$$S_5(x) = \beta_1(\beta_1+2)^{-2}(\bar{F}_1(x))^{\beta_1+2} \qquad \ldots\ldots\ldots\ldots\ldots(3.3.2)$$

It follows that $\Pi_i = S_i(0) = E(I_i) = ((\beta_1+1)(\beta_1+2))^{-1}, i=1,2$

$$\Pi_i = S_i(0) = E(I_i) = \beta_1((\beta_1+1)(\beta_1+2))^{-1},\quad i=3,4$$

$$\Pi_5 = \beta_1(\beta_1+2)^{-1} \qquad \ldots\ldots\ldots\ldots\ldots(3.3.3)$$

From (3.3.3), on simplication, we have the following relations :

$$\bar{F}_1(x) = \mathrm{Exp}((\beta_1+1)^{-1}\log N_1(x))$$
$$= \mathrm{Exp}((\beta_1+1)^{-1}\log N_2(x))$$
$$= \mathrm{Exp}((\beta_1+1)^{-1}\log N_3(x))$$

where

$$N_1(x) = S_1(x) + S_3(x) + S_5(x)\ (1+\beta_1^{-1})$$
$$N_2(x) = S_2(x) + S_4(x) + S_5(x)\ (1+\beta_1^{-1})$$
$$N_3(x) = S(x)/2 + S_5(x)\ (1/2 + \beta_1^{-1}) \qquad \text{............(3.3.4)}$$

3.3.2 : **Estimation of β_1**

As in section 3.1.2, consider only the part of the likelihood given by the observations on d.

$$L_2(.) = \prod_{i=1}^{5} \Pi_i^{n_i} \qquad \text{.............(3.3.5)}$$

Let us try to estimate β_1 by maximizing the likelihood $L_2(.)$. The equation $\dfrac{\delta \log L_2(.)}{\delta\beta_1} = 0$, leads to the following quadratic equation in β_1, viz ,

$$\beta_1^2\ (2(n_1+n_2)+n_3+n_4) + \beta_1(3(n_1+n_2) - 2n_5) - 2(n_3+n_4+n_5) = 0 \qquad \text{..............(3.3.6)}$$

leading to a unique positive solution of β_1 say $\hat{\beta}_1$, where $\hat{\beta}_1$ is

$$\hat{\beta}_1 = (2\bar{I}_5-3(\bar{I}_1+\bar{I}_2) + \bar{M}_4)\ (2.(2(\bar{I}_1+\bar{I}_2) + \bar{I}_3+\bar{I}_4)^{-1} \qquad \text{.......(3.3.7)}$$

where

$$\bar{M}_1 = 2\bar{I}_5 - 3(\bar{I}_1 + \bar{I}_2)$$
$$\bar{M}_2 = 8(\bar{I}_3 + \bar{I}_4 + \bar{I}_5)$$
$$\bar{M}_3 = 2(\bar{I}_1 + \bar{I}_2) + \bar{I}_3 + \bar{I}_4$$
$$\bar{M}_4 = \surd(\bar{M}_1^2 + \bar{M}_2 \cdot \bar{M}_3) \qquad \text{.........(3.3.8)}$$

It can be proved easily that $E(\hat{\beta}_1) = \beta_1 + O(n^{-1})$ i.e., estimator $\hat{\beta}_1$ is asymtotically unbiased for β_1. Proceeding exactly in similar lines as in the section 3.1.2, one can show that $\sqrt{n}(\hat{\beta}_1 - \beta_1)$ is asymptotically normal with mean zero and some variance σ^2_{β}, the exact expression of which is not stated here for the sake of brevity.

3.3.3 Estimators Proposed for survival Functions

(a) Ebrahimi Type Estimators

Following estimators are proposed following the Ebrahimi type of arguments as in section 3.1.4,

$$\bar{F}_1^{eI}(x) = (\bar{I}_1+\bar{I}_3+\bar{I}_5/2)\mathrm{Exp}\left((\hat{\beta}_1+1)^{-1}\log\hat{N}_1(x)\right)$$

$$+ (\bar{I}_2+\bar{I}_4+\bar{I}_5/2)\ \mathrm{Exp}((\hat{\beta}_1+1)^{-1}\log\hat{N}_2(x))$$

$$\bar{F}_1^{eII}(x) = \mathrm{Exp}(((\hat{\beta}_1+1)^{-1}\log\hat{N}_3(x))$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3.9)$$

Where $\hat{N}_1(x)$, $\hat{N}_2(x)$ and $\hat{N}_3(x)$ are the estimators of the population quantities $N_1(x)$, $N_2(x)$ and $N_3(x)$ respectively. It can be shown easily as in the sections 3.1 that $\bar{F}_1^{eI}(x)$ are $\bar{F}_1^{eII}(x)$ are all strongly consistent estimators of $\bar{F}_1(x)$.

(b) ACL Type Estimators

Under the assumption $\bar{F}_3(.) = (\bar{F}_1(.)^{\beta_1}$, the following identity is found to hold good

$$2(\bar{F}_1(x))^{\beta_1+1} - (\bar{F}_1(x))^{\beta_1+2} - S(x) = 0 \quad \ldots\ldots\ldots\ldots(3.3.10)$$

Proceeding exactly in a manner similar to that in section 3.1.4

for ACL estimator, it can be shown that for a given $S(x) \in (0,1)$ and $\beta_1 > 0$, (3.3.10) gives a unique positive solution $\gamma_a = \psi_1^a(S(x), \beta_1)$ lying between 0 and 1. Its natural estimator $\overset{al}{\bar{F}}_1(x) = \psi_1^a(\hat{S}(x), \hat{\beta}_1)$ is a consistent estimator of $\bar{F}_1(x)$. Natural estimators of the system survival function $\bar{F}(x) = 2\bar{F}_1(x) - (\bar{F}_1(x))^2$ are derived by plugging in these different estimators of $\bar{F}_1(x)$ in the expression for $\bar{F}_1(x)$, viz,

$$\text{(a) } \overset{eI}{\bar{F}}_1(.) \quad \text{(b) } \overset{eII}{\bar{F}}_1(.) \text{ and (c) } \overset{aI}{\bar{F}}_1(.)$$

3.3.4 Asymptotic Variances of the Proposed Estimators

(a) $\overset{eI}{\bar{F}}_1(.)$ (b) $\overset{eII}{\bar{F}}_1(.)$ and (c) $\overset{aI}{\bar{F}}_1(.)$.

Before deriving the asymptotic expansions of the proposed estimators, we introduce some symbols to simplify the complicated expressions and they are given below. Some of these symbols have already been introduced but they are mentioned here for the sake of completeness.

$M_1 = 2\Pi_5 - 3(\Pi_1 + \Pi_2)$, $\quad M_2 = 8(\Pi_3 + \Pi_4 + \Pi_5)$, $\quad M_3 = 2(\Pi_1 + \Pi_2) + \Pi_3 + \Pi_4$

$M_4 = \sqrt{(M_1^2 + M_2 . M_3)}$ $\quad M_5 = M_1 + M_4$ $\quad M_6 = 8 + \Pi_1 + \Pi_2 - 6\Pi_5$

$M_7 = 8 - 4\Pi_5$, $\quad M_8 = 4 - 2(\Pi_1 + \Pi_2)$ $\quad M_9 = M_5 + 2M_3$

$K_1 = 4M_5 + 6M_3 - 2M_3(M_4)^{-1}M_6$ $\quad K_2 = 2M_5 - 2M_3\ M_4^{-1}M_7$ $\quad K_3 = -2M_3(2 + M_4^{-1}M_8)$

$W_1 = \Pi_1 + \Pi_3 + \Pi_5/2$, $\quad W_2 = \Pi_2 + \Pi_4 + \Pi_5/2$

$$N_1(x) = S_1(x) + S_3(x) + S_5(x)(1 + \beta_1^{-1})$$

$$N_2(x)=S_2(x)+S_4(x)+S_5(x)(1+\beta_1^{-1})$$

$$N_3(x)=S(x)/2 \ +S_5(x)(\tfrac{1}{2} + \beta_1^{-1})$$

$$H_1(x)=((\beta_1+1).N_1(x))^{-1}.S_5(x),$$

$$H_2(x)=((\beta_1+1).N_2(x))^{-1}.S_5(x),$$

$$H_3(x)=((\beta_1+1).\ N_3(x))^{-1}.S_5(x),$$

$$P_1=W_1\log N_1(x) \ + \ W_2\log N_2(x)$$

$$P_2=W_1H_1(x) \ + \ W_2H_2(x)$$

$$L_{11}=K_1(M_9^{-2}.P_1 \ + \ M_5^{-2}P_2) \ + \ 1 \ = \ L_{12}$$

$$L_{13}=K_2(M_9^{-2}.P_1 \ + \ M_5^{-2}P_2) \ + \ 1 \ = \ L_{14}$$

$$L_{15}=K_3(M_9^{-2}.P_1 \ + \ M_5^{-2}P_2) \ + \ 1$$

$$L_{21}=W_1((\beta_1+1)N_1(x))^{-1}=L_{23}$$

$$L_{22}=W_2((\beta_1+1)N_2(x))^{-1}=L_{24}$$

$$L_{25}=W_1((\beta_1.N_1(x))^{-1}+ \ W_2(\beta_1.N_2(x))^{-1}$$

$$L_{11}^{*}=K_1(M_9^{-2}\mathrm{Log}N_3(x) \ + \ M_5^{-2}H_3(x)) \ = \ L_{13}^{*}$$

$$L_{12}^{*}=K_2(M_9^{-2}\mathrm{Log}N_3(x) \ + \ M_5^{-2}H_3(x)) \ = \ L_{14}^{*}$$

$$L_{15}^{*}=K_3(M_9^{-2}\mathrm{Log}N_3(x) \ + \ M_5^{-2}H_3(x))$$

$$L_{21}^{*} \ = \ L_{22}^{*} \ = \ L_{23}^{*} \ = \ L_{24}^{*} \ = \ 0.5((\beta_1+1).N_3(x))^{-1}$$

$$L_{25}^{*} \ = \ (\beta_1.N_3(x))^{-1} \qquad \text{..........(3.3.10)}$$

(a) **Ebrahimi Type Estimators, viz., $\bar{\bar{F}}_1^{eI}(.), \bar{\bar{F}}_1^{eII}(.)$.**

By Taylor's series expansion about $(\Pi_1, \Pi_2 \ldots \Pi_5, S_1(x), \ldots S_5(x))$, one can write $\bar{\bar{F}}_1^{eI}(x)$ in the following form.

$$\sqrt{n}(\bar{\bar{F}}_1^{eI}(x) - \bar{F}_1(x)) = \sqrt{n}\ \bar{F}_1(x)\sum_{i=1}^{5} L_{1i}(\bar{I}_i - E(I_i))$$

$$+ \sqrt{n}\ \bar{F}_1(x)\sum_{i=1}^{5} L_{2i}(\hat{S}_i(x) - S_i(x)) + R_n \quad \ldots\ldots\ldots\ldots(3.3.11)$$

where R_n converges in probability to zero, where the expression's for L_{1i}'s and L_{2i}'s for i=1,2...5 are given in (3.3.10). Writing $L' = (L_{11}, L_{12}, \ldots L_{15}, L_{21}, \ldots L_{25})$ one can write,

$$\text{Asym Var}(\bar{\bar{F}}_1^{eI}(x)) = n^{-1}(L'.D^*.L)(\bar{F}_1(x))^2 \text{ upto order } n^{-1} \ldots\ldots\ldots(3.3.12)$$

where D^* is same as D in Theorem 3.5, with values of Π_i and $S_i(x)$ for i=1,2...5 substituted appropriately from (3.3.2) and (3.3.3) respectively. Similarly $\bar{\bar{F}}_1^{eII}(x)$ can be approximated by

$$\sqrt{n}(\bar{\bar{F}}_1^{eII}(x) - \bar{F}_1(x)) \simeq \sqrt{n}\sum_{i=1}^{5} L^*_{1i}(\bar{I}_i - E(I_i)).\bar{F}_1(x) +$$

$$\sqrt{n}\sum_{i=1}^{5} L^*_{2i}(\hat{S}_i(x) - S_i(x))\ \bar{\bar{F}}_1(x) + R_n \quad \ldots(3.3.13)$$

where R_n converges in probability to zero as $n \longrightarrow \infty$ writing $L^{*\prime} = (L^*_{11}, L^*_{12}, \ldots L^*_{15}, L^*_{21}, \ldots L^*_{25})$, one can write,

$$\text{Asym Var}(\bar{\bar{F}}_1^{eII}(x)) = n^{-1}(L^{*\prime}.D^*.L^*)(\bar{F}_1(x))^2 \text{ upto order } n^{-1} \ldots\ldots(3.3.14)$$

Asymptotic variances of the estimators of system survival function obtained by using these estimators of $\bar{F}_1(x)$ can be easily evaluated.

(b) **Asymptotic variance of ACL Type estimator, viz, $\bar{F}_1^{aI}(.)$**

From (3.3.10) on substituting β_1 from (3.3.7), the following relation is found for estimating $\bar{F}_1(x)$ on ACL lines.

$$2(\bar{F}_1^{aI}(x))^{\hat{\beta}_1+1} - (\bar{F}_1^{aI})^{\hat{\beta}_1+2} - \hat{S}(x) = 0 \quad \text{..............(3.3.15)}$$

The unique solution of (3.3.15) lying between 0 and 1 is, say, $\hat{\gamma}_a = \bar{F}_1^{aI}(x)$. From (3.3.15), as usual by Taylor's series expansion about $(S(x), \beta_1)$, upto the 1st order term, one can write.

$$0 = c_1^a(\hat{\gamma}^a - \gamma) + c_2^a(\hat{\beta}_1 - \beta_1) + c_3^a(\hat{S}(x) - S(x)) + R_n, \quad \text{.........(3.3.16)}$$

Where R_n converges in probability to zero as $n \longrightarrow \infty$ and the expressions for c_1^a, c_2^a, c_3^a are given by

$$c_1^a = (\bar{F}_1(x))^{\beta_1}(2(\beta_1+1) - (\beta_1+2)\bar{F}_1(x))$$

$$c_2^a = \log \bar{F}_1(x)(2(\bar{F}_1(x))^{\beta_1+1} - (\bar{F}_1(x))^{\beta_1+2})$$

$$c_3^a = -1 \quad \text{............(3.3.17)}$$

As in section 3.1.5(b), we have the following system of equations.

$$c_1^a \operatorname{Var}(\hat{\gamma}^a) + c_2^a \operatorname{Cov}(\hat{\gamma}^a, \hat{\beta}_1) + c_3^a \operatorname{Cov}(\hat{\gamma}^a, \hat{S}(x)) = 0$$

$$c_1^a \operatorname{Cov}(\hat{\gamma}^a, \hat{\beta}_1) + c_2^a \operatorname{Var}(\hat{\beta}_1) + c_3^a \operatorname{Cov}(\hat{\beta}_1, \hat{S}(x)) = 0$$

$$c_1^a \operatorname{Cov}(\hat{\gamma}^a, \hat{S}(x)) + c_2^a \operatorname{Cov}(\hat{S}(x), \hat{\beta}_1) + c_3^a \operatorname{Var}(\hat{S}(x)) = 0 \quad \text{...(3.3.18)}$$

As before one can easily write down the expression of Asym Var$(\bar{F}_1^{aI}(x))$ and Asym Var$(\bar{F}^{aI}(x))$ following the relations in (3.3.18).

c) The Kaplan Meier Estimator $\bar{F}^{KM}(x)$ of $\bar{F}(x)$ is as well as its Asym Variance remains same as in section 3.1.5(c)

3.4 Numerical computation in The Case of Identical Components

As in section 3.3, we find the point x, where true survival probability of the system is (i)0.90 and (ii)0.95 for different combinations of Known scale and shape parameters, α_1 and δ_1 of the Weibull distribution used for component lives, and for different degrees of censorship connected with this model. The asymptotic variances of (a)$\bar{F}^{e}(.)$ (b)$\bar{F}^{eII}(.)$ (c)$\bar{F}^{aI}(.)$ and (d)$\bar{F}^{KM}(.)$ are computed by using the asymptotic expansion developed in section 3.3. Numerical results are presented in Tables, 3.4.1 through 3.4.6.

Table 3.4.1

n times variance of the estimated survival probability at the point where true probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1, \delta_1=1$. The sets of estimates used are

(a)$\bar{F}^{eI}(.)$, (b)$\bar{F}^{eII}(.)$, (c)$\bar{F}^{aI}(.)$ and (d)$\bar{F}^{KM}(.)$

β_1	a I	a II	b I	b II	c I	c II	d I	d II
0.5	0.0310	0.0053	0.0253	0.0043	0.0248	0.0032	0.0236	0.0046
1.0	0.0054	0.0013	0.0028	0.0006	0.0022	0.0003	0.0089	0.0026
2.00	0.0004	0.0009	0.0002	0.0008	0.0002	0.0004	0.0026	0.0007

Table (Contd.)

Table 3.4.2

n times variance of the estimated survival probability at the point where true probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1, \delta_1=0.5$ The sets of estimates used are (a)$\bar{F}^{eI}(.)$, (b)$\bar{F}^{eII}(.)$, (c)$\bar{F}^{aI}(.)$ and (d)$\bar{F}^{KM}(.)$

β_1	a I	a II	b I	b II	c I	c II	d I	d II
0.5	0.0312	0.0052	0.0254	0.0042	0.0250	0.0033	0.0237	0.0046
1.0	0.0056	0.0013	0.0028	0.0007	0.0023	0.0003	0.0040	0.0026
2.00	0.0004	0.0008	0.0002	0.0008	0.0002	0.0004	0.0027	0.0007

Table 3.4.3

n times variance of the estimated survival probability at the point where true probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1, \delta_1=2$. The sets of estimates used are (a)$\bar{F}^{eI}(.)$, (b)$\bar{F}^{eII}(.)$, (c)$\bar{F}^{aI}(.)$ and (d)$\bar{F}^{KM}(.)$

β_1	a I	a II	b I	b II	c I	c II	d I	d II
0.5	0.0314	0.0056	0.0256	0.0042	0.0238	0.0033	0.0238	0.0046
1.0	0.0054	0.0012	0.0026	0.0007	0.0023	0.0003	0.0042	0.0027
2.00	0.0004	0.0009	0.0002	0.0008	0.0002	0.0004	0.0028	0.0007

Table (Contd.)

Table 3.4.4

n times variance of the estimated survival probability at the point where true probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1.5, \delta_1=0.5$ The sets of estimates used are

(a)$\bar{\bar{F}}^{eI}(.)$, (b)$\bar{\bar{F}}^{eII}(.)$, (c)$\bar{\bar{F}}^{aI}(.)$ and (d)$\bar{\bar{F}}^{KM}(.)$

β_1	a I	a II	b I	b II	c I	c II	d I	d II
0.5	0.0316	0.0054	0.0255	0.0041	0.0252	0.0032	0.0236	0.0047
1.0	0.0056	0.0014	0.0028	0.0007	0.0022	0.0003	0.0040	0.0027
2.00	0.0004	0.0008	0.0002	0.0008	0.0002	0.0004	0.0026	0.0007

Table 3.4.5

n times variance of the estimated survival probability at the point where true probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1.5, \delta_1=1$ The sets of estimates used are

(a)$\bar{\bar{F}}^{eI}(.)$, (b)$\bar{\bar{F}}^{eII}(.)$, (c)$\bar{\bar{F}}^{aI}(.)$ and (d)$\bar{\bar{F}}^{KM}(.)$

β_1	a I	a II	b I	b II	c I	c II	d I	d II
0.5	0.0314	0.0051	0.0256	0.0040	0.0251	0.0033	0.0238	0.0047
1.0	0.0053	0.0012	0.0026	0.0007	0.0022	0.0003	0.0042	0.0026
2.00	0.0004	0.0008	0.0002	0.0008	0.0002	0.0004	0.0027	0.0007

Table (Contd.)

Table 3.4.6

n times variance of the estimated survival probability at the point where true probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}\left(\frac{-x^{\alpha_1}}{\delta_1}\right)$, $\alpha_1 = 1.5$, $\delta_1 = 2$. The sets of estimates used are (a)$\bar{F}^{eI}(.)$, (b)$\bar{F}^{eII}(.)$, (c)$\bar{F}^{aI}(.)$ and (d)$\bar{F}^{KM}(.)$

β_1	a I	a II	b I	b II	c I	c II	d I	d II
0.5	0.0312	0.0055	0.0255	0.0042	0.0251	0.0033	0.0238	0.0046
1.0	0.0055	0.0016	0.0029	0.0007	0.0022	0.0003	0.0042	0.0026
2.00	0.0004	0.0008	0.0002	0.0008	0.0002	0.0004	0.0027	0.0007

On examining, Tables 3.4.1 through 3.4.6 reveal the fact that estimators (a)$\bar{F}^{eI}(.)$, (b)$\bar{F}^{eII}(.)$ and (c)$\bar{F}^{aI}(.)$ are all superior (d)$\bar{F}^{KM}(.)$. The A.C.L type estimator, viz, $\bar{F}^{aI}(.)$, appears to behave best in comparison to other estimators in all cases. Again, the difference between the estimators $\bar{F}^{eII}(x)$ and $\bar{F}^{aI}(x)$ does not appear to be much. The limitation of the comparison made, pointed out at the end of section 3.2 holds good here too.

3.5 CONCLUDING REMARKS

It may be noted that Doss *et al.* (1989) considered a different approach to the problem of reliability estimation for a general coherent structure. but they assume a completely different type of data set. They assume continuous observation of an item and the time each component fails can be ascertained until the total system failure. The data set considered in the present chapter precludes the application of their technique.

(ii) Chang (1987), Yang (1987) and Samuelson (1989) considered the non-parametric survival function estimation problem with doubly censored data. Formally, we can assume that a doubly censored data set arises from a complete data set (X_{1i}, X_{2i}, X_{3i}) i= 1,2...n, in the following way. Here X_2 is the true survival time of interest, and (X_1, X_3) is a window of observation, X_2 is left censosred at X_1 if it falls below the window and right censored at X_3 if it is above the window. The data set consists of the observations on W and d as demonstrated by Chang and Yang (1987), in the following manner.

$W = \text{Max}(X_1, \text{Min}(X_2, X_3))$ and d = censoring indicator,

$$d=1 \quad \text{if} \quad X_1 < X_2 < X_3, \quad W = X_2$$

$$d=2 \quad \text{if} \quad X_2 > X_3, \quad W = X_3$$

$$d=3 \quad \text{if} \quad X_2 < X_1, \quad W = X_1$$

Our problem dealt with in the present chapter can be colinked with this double consoring mechanism. The additional assumption of $P(X_1 < X_3) = 1$ as made by the authors cited implies $P(I_4 = 1) = P(I_5 = 1) = 0$ in our formulation of the problem and the five classes described by us will naturally reduce to three classes of their formulation with d=1,2,3 corresponding exactly to the classes $I_1=1, I_2=1$ and $I_3=1$ respectively. As such Chang and Yang procedure can be applied to our model for survival function estimation of component 2 if we make additional assumption $P(X_1 < X_3)=1$. But proportional hazard assumption cannot hold owing to the restriction $P(X_1 < X_3) = 1$.

CHAPTER IV

NONPARAMETRIC ESTIMATION OF SURVIVAL FUNCTION IN A GENERAL k-1 OUT OF k SYSTEM UNDER RANDOM CENSORING WITH PROPORTIONAL HAZARD ASSUMPTION

CHAPTER 4

NONPARAMETRIC ESTIMATION OF SURVIVAL FUNCTION IN A GENERAL k-1 OUT OF k SYSTEM UNDER RANDOM CENSORING WITH PROPORTIONAL HAZARD ASSUMPTION

4.0 INTRODUCTION

In chapter 3 we have discussed nonparametric estimation procedures for survival functions in a two component parallel system under random censoring with proportional hazard assumption on the distribution functions. In the present chapter, we extend the application of the procedures with necessary modifications to a more general situation. We consider the nonparametric estimation of survival functions of components in a k components coherent structure in which the system functions if and only if at least k-1 of its components function properly. proportional hazard assumption is retained here also. To state it explicitly :

Let X_i represent the random variable associated with the life length of component i with absolutely continuous distribution function $F_i(.)$ and survival function $\bar{F}_i(.)$, i=1,2...k. X_i's are assumed to be independent. Let X_c be the censoring random variable, which represents failure due to other causes not covered by the components 1,2 ,3 .... k , with absolutely continuous distribution function $F_c(.)$ and survival function $\bar{F}_c(.)$, X_c is assumed to be distributed independently of $X_1, X_2 \ldots X_k$. What we observe in reality is the realized value of the random variable Z which represents the observed life of the system, censored or otherwise and a value of the indicator variable $d = (I_{12}, I_{13}, \ldots I_{k-1,k}, I_{1c}, \ldots, I_{kc}, I_{co})$ which follows a multinomial distribution with $(k^2 + 1)$ classes determined by practically feasible and recognizable interrelationship between the variables $X_1, X_2 \ldots X_k$ and X_c.

In section 4.1 we describe a general procedure for the estimation of life lengths of components 1,2,...k, under the assumption $\bar{F}_c(.) = (\bar{F}_1(.))^{\beta_1} = (\bar{F}_2(.))^{\beta_1 \phi_1} \ldots = (\bar{F}_k(.))^{\beta_1 \phi_{k-1}}$, where $\beta_1 > 0$ and $\beta_2 = \beta_1 \phi_1 > 0, \ldots, \beta_k = \beta_1 \phi_{k-1} > 0$ represent in a way the amounts of censorship the random variables $X_1, X_2 \ldots X_k$ respectively are subjected to by the random variable X_c. The two stage procedure of estimation described in chapter 3 may be applied quite satisfactorily in this case. But in the case of general $k \geq 2$, to find reasonably good estimates of the parameters $\beta_1, \phi_1 \ldots \phi_{k-1}$ in simple closed forms seems to be an extremely difficult task. Moreover, a particular choice of estimators is not expected to behave equally well for all $k \geq 2$. Hence, no attempts are made to obtain explicit expressions for the estimators of survival functions in the general case for all $k \geq 2$. A few special cases are only considered as examples in the sections that follow. The essence of the methods developed there are as well applicable to the more general problems we may have to encounter and one can, it is presumed, develop appropriate expressions for the particular problems for which solutions are sought in practice by cleverly making using of the special properties and relations satisfied by the particular problems encountered.

In section 4.2 we consider the estimation of survival function in the special case, $X_1, X_2 \ldots X_k$ are identically distributed in addition i.e., $\phi_1=\phi_2 \cdots = \phi_{k-1}=1$ in the earlier model. In this special case the common distribution function is denoted by $F_1(.)$. It is possible to find here explicitly the maximum likelihood estimator of β_1 from the partial likelihood based on d. By following procedures similar to those proposed by Abdushukurov (1984) and Cheng and Lin (1984) and Ebrahimi (1985), three sets of adhoc estimators of the survival function $\bar{F}_1(.)$, viz,(a) $\{\bar{F}_1^{ekI}(.)\}$ (b) $\{\bar{F}_1^{ekII}(.)\}$ and (c) $\{\bar{F}_1^{ak}(\cdot)\}$ are proposed. It is observed that these estimators are consistent. Asymptotic variances of the proposed estimators are also derived.

It is to be noted that in case of identical component life distributions, the method applied is a simple generalization and extension of the method developed in section 3.3 , the particular case being considered there is k=2. The one out of two components system can definitely be considered as a special case of the more general problem considered in section 4.2 and the estimator $\bar{F}^{eI}(.)$ (respectively $\bar{F}^{eII}(.)$ and $\bar{F}^{aI}(.)$) of Section 3.3 exactly coincides with $\bar{F}^{ekI}(.)$ (respectively $\bar{F}^{ekII}(.)$ and $\bar{F}^{ak}(.)$) of section 4.2, for k=2. In this respect the theoretical developement in section 3.3 appears to be somewhat superfluous. But it is included there for the sake of completeness and self sufficiency of chapter 3, which considers the important problem of a one out of two components system under proportional hazard assumption. Viewing section 4.2 as a natural extension of section 3.3, Some repetitions cannot be avoided, but utmost efforts are putforth to keep them to the minimum.

In section 4.3 to demonstrate the estimation procedure in a nonidentical set up, we consider as a special case a two out of three system, i.e., we assume k=3. It has already been pointed out, the general k does not lead to a generally acceptable solution which is expected to behave well for all $k \geq 2$. The estimators proposed in case k = 3 are essentially different from those proposed for k = 2. In this case of k = 3,

$$d = (I_{12}, I_{13}, I_{1c}, I_{21}, I_{23}, I_{2c}, I_{31}, I_{32}, I_{3c}, I_{co})$$

is the indicator variable which follows a multinomial distribution with 10 mutually exclusive and exhaustive classes determined by the random variables X_1, X_2, X_3 and X_c. Two sets of estimators of survival function associated with components 1,2 and 3 are proposed , viz,

$$\text{(a)} \quad (\overset{e_3}{\bar{F}_1}(.),\ \overset{e_3}{\bar{F}_2}(.),\ \overset{e_3}{\bar{F}_3}(.)) \qquad\qquad \text{(b)} \quad (\overset{a_3}{\bar{F}_1}(.),\ \overset{a_3}{\bar{F}_2}(.),\ \overset{a_3}{\bar{F}_3}(.)).$$

Here the set of estimators (a) is obtained by using Ebrahimi type arguments and the set of estimators (b) is based on Abdushukurov (1984), Cheng and Lin (1984) type arguments. It is observed that under the above set up, estimators (a) and (b) are consistent. The asymptotic variances of the proposed sets of estimators are derived. We calculate numerically the asymptotic variances of the estimated system survival probability by using the two sets of adhoc estimators proposed, viz, (a) and (b). Asymptotic variances are numerically calculated and compared with that of the Kaplan and Meier estimator, viz, $\overset{KM}{\bar{F}}(.)$ of the system.

The numerical computations are done for useful ranges of values of parameters for Exponential and Weibull distributions. In section 4.4 we take up the numerical investigation of the case described in section 4.2. The asymptotic variances of the estimated system survival probabilities, viz, (a) $\bar{F}^{e_{kI}}(.)$ (b) $\bar{F}^{e_{kII}}(.)$ and (c) $\bar{F}^{a_k}(.)$ are computed in the special case $k = 3$ under the assumption of identical distribution of component lives. These numerical values are compared with that of $\bar{F}^{KM}(.)$ for appropriate ranges of values of parameters from Exponential and Weibull distribution.

In section 4.5., it is pointed out that the essence of the general two stage procedure of estimation dealt with in chapter 3 and in earlier sections of the present chapter can be easily extended to other coherent systems, although the details of procedures must vary, to demonstrate the potentiality of the essential technique developed.

In section 4.5,a rough outline is provided following two stage approach by means of which suitable estimators can be developed in the case of so-called series parallel system.

4.1 GENERAL PROPORTIONAL HAZARD MODEL FOR k-1 OUT OF k SYSTEM, $k \geq 2$

For identification of observation size classes along with the expressions of associated probability statements we need some necessary formulations as related in the following lines :

In a k-1 out of k components system, the system fails if two of its components fail. Let us designate them as the first and second failures. Thus the ordered pair $(i,j), i\neq j,\ i,j=1,2..k.$ identifies the first failed component of the system as i and the second failed component as j. Along with the ordered pair (i,j) there is always associated an ordered pair of time points say (u,Z), where u denotes the point of time when the first failure occurs and Z, the point of time when the second failure occurs. In our set up, u is unobserved and Z is the life length of the system, $P(u < Z) = 1$, assuming the system is uncensored. Of course the system may also be censored and the desired life of the system may be its censored life and censoring may take place before the first failure or in between the first and second failures. The ordered pair (i,c) will denote that the first failure occurs to component i at time u which is unobserved and the system life is censored, the censoring time being Z, observed, $i=1,2,...k$. The ordered pair (c,o) will be used to denote that the system life is censored at time Z, observed prior to any component failures. Thus the system failure must belong to one of the following k^2+1 classes.

$$(i,j),\ i \neq j,\ i,j = 1,2.....k.$$
$$(i,c),\qquad i = 1,2.....k.$$
$$(c,o) \qquad ....(4.1.1)$$

Under random censoring, what we observe in reality is the realized value of the random variable Z which represents the observed life of the system and a value of the indicator variable $d = (I_{12}, I_{13}....I_{1k}....I_{k-1,k}...I_{1c},...I_{kc}, I_{co})$ which follows a multinomial distribution with (k^2+1) mutually exclusive and exhaustive classes determined by practically feasible and recognizable interrelationships between the random variables $X_1, X_2........X_k$ and X_c (we write X_c for the random variable associated with censoring, which actually represents the failure of the system for all other causes not covered by the failures of the components).

Let $S = \{1,2.....k\}$

$$A_{ij} = (Z \in R^{+} \mid X_i < Z, \quad Z = X_j, \ \mathrm{Min}(X_c, X_u, u \in S - \{i,j\}) > Z)$$

$$A_{ic} = (Z \in R^{+} \mid X_i < Z, \quad Z = X_c, \ \mathrm{Min}(X_u, u \in S - \{i\}) > Z)$$

$$A_{co} = (Z \in R^{+} \mid Z = X_c, \ \mathrm{Min}(X_u, u \in S) > Z) \qquad \text{.............(4.1.2)}$$

Let us introduce the indicator variable as follows :

$$\begin{aligned} I_{ij} &= 1 \quad \text{if } Z \in A_{ij} \\ &= 0 \quad \text{if } Z \notin A_{ij}, \quad i \neq j \quad i, j \in S \end{aligned}$$

$$\begin{aligned} I_{ic} &= 1 \quad \text{if } Z \in A_{ic} \\ &= 0 \quad \text{if } Z \notin A_{ic}, \quad i \in S \end{aligned}$$

$$\begin{aligned} I_{co} &= 1 \quad \text{if } Z \in A_{co} \\ &= 0 \quad \text{if } Z \notin A_{co}, \end{aligned} \qquad \text{.............(4.1.3)}$$

Thus $A_{ij} = (Z \in R^{+} \mid I_{ij} = 1), \ i \neq j \quad i,j \in S$

$$A_{ic} = (Z \in R^{+} \mid I_{ic} = 1), \quad i \in S$$

$$A_{co} = (Z \in R^{+} \mid I_{co} = 1), \qquad \text{.............(4.1.4)}$$

Let us define .

$$\Pi_{ij} = P(Z \in A_{ij}) = P(Z \in R^{+} \mid I_{ij} = 1) \quad i \neq j, \quad i,j \in S$$

$$\Pi_{ic} = P(Z \in A_{ic}) = P(Z \in R^{+} \mid I_{ic} = 1), \quad i \in S$$

$$\Pi_{co} = P(Z \in A_{co}) = P(Z \in R^{+} \mid I_{co} = 1) \qquad \text{.............(4.1.5)}$$

and conditional densities of Z given the appropriate events $Z \in A_{ij}$, $Z \in A_{ic}$ and $Z \in A_{co}$ are given by the following expressions.

$$dF(Z \mid Z \in A_{ij}) = \bar{\Pi}^{1}_{ij}[F_i(Z).dF_j(Z). \prod_{u \in S - \{i,j\}} (\bar{F}_u(Z)).\bar{F}_c(z)] \quad i,j \in S \quad i \neq j$$

$$dF(Z \mid Z \in A_{ic}) = \bar{\Pi}^{1}_{ic}[F_i(Z) \prod_{u \in S - \{i\}} (\bar{F}_u(Z)).dF_c(Z)] \quad i \in S$$

$$dF(Z \mid Z \in A_{co}) = \bar{\Pi}^{1}_{co}[\prod_{u \in S} (\bar{F}_u(Z)).dF_c(Z)] \qquad \ldots\ldots\ldots\ldots\ldots(4.1.6)$$

(4.1.6) makes use of the fact that the random variables $X_1, X_2 \ldots X_k$ and X_c are independent. Subsurvival function of the uncensored classes and censored classes are of the form.

$$S_{ij}(x) = P(Z>x, I_{ij}=1) = \int_x^{\infty} [F_i(Z) \prod_{u \in S - \{i,j\}} (\bar{F}_u(Z)).\bar{F}_c(z)dF_j(Z)]$$

$$\Pi_{ij} = S_{ij}(0) = E(I_{ij}) = \int_0^{\infty} [F_i(Z) \prod_{u \in S - \{i,j\}} (\bar{F}_u(Z))\bar{F}_c(Z)dF_j(Z)],$$

$$S_{ic}(x) = P(Z>x, I_{ic}=1) = \int_x^{\infty} [F_i(Z) \prod_{u \in S - \{i\}} (\bar{F}_u(Z))dF_c(Z)]$$

$$\Pi_{ic} = S_{ic}(0) = E(I_{ic}) = \int_0^{\infty} [F_i(Z) \prod_{u \in S - \{i\}} (\bar{F}_u(Z))dF_c(Z)],$$

$$S_{co}(x) = P(Z>x, I_{co}=1) = \int_x^{\infty} [\prod_{u \in S} (\bar{F}_u(Z))dF_c(Z)]$$

$$\Pi_{co} = S_{co}(0) = E(I_{co}) = \int_0^{\infty} [\prod_{u \in S} (\bar{F}_u(Z))dF_c(Z)], \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.1.7)$$

It can be verified easily that, $\sum_{\substack{i,j \in S \\ i \neq j}} \sum \Pi_{ij} + \sum_{i \in S} \Pi_{ic} + \Pi_{co} = 1,$

as it should be.

Donote $S(x) = P(Z>x)$ = Survival function of the random variable Z, it follows that,

$$S(x) = \sum_{\substack{i,j \in S \\ i \neq j}} \sum S_{ij}(x) + \sum_{i \in S} S_{ic}(x) + S_{co}(x) \qquad \ldots\ldots\ldots\ldots\ldots(4.1.8)$$

Define $S_i(x) = \sum_{j \in S-\{i\}} S_{ij}(x), \quad i \in S$

$$\Pi_i = \sum_{j \in S-\{i\}} \Pi_{ij} \quad i \in S$$

$$A_i = \bigcup_{j \in S-\{i\}} A_{ij}, \quad i \in S \qquad \ldots\ldots\ldots\ldots\ldots(4.1.9)$$

under the assumption $\bar{F}_c(.)=(\bar{F}_1(.))^{\beta_1}=(\bar{F}_2(.))^{\beta_1\phi_1}= \ldots(\bar{F}_k(.))^{\beta_1\phi_{k-1}}$ it is possible to establish, k identifiable relations between $\bar{F}_1(.),\bar{F}_2(.) \ldots \bar{F}_k(.)$, using the sets

$(A_1 \cup A_{1c} \cup A_{co})$, $(A_2 \cup A_{2c} \cup A_{co})$ .... $(A_k \cup A_{kc} \cup A_{co})$.

Now for a particular choice of k, the associated parameters under the proportional hazard model, viz., $\beta_1, \phi_1, \phi_2 \ldots \phi_{k-1}$ can be estimated by using the partial likelihood based on the observations on d distributed over the (k^2+1) classes. Survival function of components, viz, $\bar{F}_1(.),\bar{F}_2(.) \ldots \bar{F}_k(.)$ can be estimated by making use of the relations indicated above, following Ebrahimi (1985) or ACL (Abdushukurov (1984) and Cheng and Lin (1984)) type arguments. In section 4.3, we will consider estimation procedure for a two out of three system (k=3) in details, for the purpose of illustration. This is taken up because for this problem each value of k has to be treated separately and no general estimation procedure can be proposed which will behave quite well for all $k \geq 2$. It will be quite clear in section 4.3, the explicit expressions for the closed form estimators β_1, ϕ_1, and ϕ_2 proposed are not the natural extensions of the estimates developed before for the case k=2. They are proposed on an adhoc basis, by critically, examining the mathematical relationship that can be established in this case i.e. k=3.

4.2 ESTIMATION IN THE IDENTICAL CASE

4.2.1 Preliminaries of the Proposed Estimators

For the problem in section 4.1 we make the additional assumption that X_i's are identically distributed for $i=1,2.....k$ for a k component system. Let $F_1(.)$ represent the common distribution function. Let us recall from section 4.0 and 4.1 that $\bar{F}_c(.)=(\bar{F}_1(.))^{\beta_1}$, i.e., $\phi_1 = \phi_2 = ...\phi_{k-1} = 1$ and $\beta_1>0$. The main problem is to estimate the survival function $\bar{F}_1(.)$. As the problem in the present section is a special case of the problem described in section 4.1, the same notation for classes, subsurvival functions and components of the indicator variable are applicable here. In this particular problem, (4.1.7) reduces to

$$S_{ij}(x)= P(Z>x, I_{ij}=1) = \int_x^{\infty} F_1(Z)\, \bar{F}_c(Z)(\bar{F}_1(Z))^{k-2} dF_1(Z)$$

$$\Pi_{ij}= S_{ij}(0)= E(I_{ij})= \int_0^{\infty} F_1(Z)\bar{F}_c(Z)(\bar{F}_1(Z))^{k-2} dF_1(Z)$$
$$\forall\ i, j \in S,\ i \neq j$$

$$S_{ic}(x) = P(Z>x, I_{ic}= 1) = \int_x^{\infty} F_1(Z)(\bar{F}_1(Z))^{k-1} dF_c(Z)\ ,$$

$$\Pi_{ic}= S_{ic}(0) = E(I_{ic}) = \int_0^{\infty} F_1(Z)\ (\bar{F}_1(Z))^{k-1} dF_c(Z)\ , \quad \forall\ i \in S$$

$$S_{co}(x) = P(Z>x, I_{co}= 1) = \int_x^{\infty} (\bar{F}_1(Z))^k dF_c(Z)$$

$$\Pi_{co}= S_{co}(o) = E(I_{co}) = \int_0^{\infty} (\bar{F}_1(Z))^k dF_c(Z)$$

........................(4.2.1)

under the assumption $\bar{F}_c(.)=(\bar{F}_1(.))^{\beta_1}$, (4.2.1) can be simplified as

$$S_{ij}(x)=(\beta_1+k-1)^{-1}.(\bar{F}_1(x))^{\beta_1+k-1} - (\beta_1+k)^{-1}.\ (\bar{F}_1(x))^{\beta_1+k} \quad i\neq j \quad i,j \in S$$

$$\Pi_{ij}= S_{ij}(0)= E(I_{ij})= ((\beta_1+k).(\beta_1+k-1))^{-1}$$

$$S_{ic}(x)=\beta_1[\,(\beta_1+k-1)^{-1}(\bar{F}_1(x))^{\beta_1+k-1} - (\beta_1+k)^{-1}(\bar{F}_1(x))^{\beta_1+k}\,],\ i \in S$$

$$\Pi_{ic}=S_{ic}(0)=E(I_{ic})=\beta_1((\beta_1+k)(\beta_1+k-1))^{-1}$$

$$S_{co}(x)=\beta_1(\beta_1+k)^{-1}(\bar{F}_1(x))^{\beta_1+k}$$

$$\Pi_{co}=S_{co}(0)=E(I_{co})=\beta_1(\beta_1+k)^{-1} \qquad (4.2.2)$$

Again from (4.1.8), we have,

$$S(x)=k(\bar{F}_1(x))^{\beta_1+k-1} - (k-1).(\bar{F}_1(x))^{\beta_1+k} \qquad (4.2.3)$$

From (4.2.2), the following identities are found to hold by utilizing the classes $(A_1\cup A_{1c}\cup A_{co})$, $(A_2\cup A_{2c}\cup A_{co})$,............ $(A_k\cup A_{kc}\cup A_{co})$ and $(\bigcup_{i\in S} A_i \ \bigcup_{i\in S} A_{ic} \cup A_{co})$, and they are, of the following form.

$$\bar{F}_1(x)=\mathrm{Exp}((\beta_1+k-1)^{-1}\log C_1(x))$$

$$=\mathrm{Exp}((\beta_1+ k-1)^{-1}\log C_2(x)) \ \ldots\ldots\ =\mathrm{Exp}((\beta_1+k-1)^{-1}\log C_k(x))$$

$$= \mathrm{Exp}((\beta_1 +k-1)^{-1}\log (\frac{S(x)}{k} + S_{co}(x)((1 - \frac{1}{k})+(k-1)\beta_1^{-1}))$$

where $C_i(x)=S_i(x)+S_{ic}(x)+S_{co}(x)(1+(k-1)\beta_1^{-1})$ $i=1,2...k.$ ...(4.2.4)

4.2.2 Estimation of β_1

In a simple random sample of size n, let n_{ij} observations belong to class A_{ij} $i \neq j$, $i,j \in S$, n_{ic} observations belong to class A_{ic}, $i \in S$ and n_{co} observations belong to class A_{co}, such that,

$$n = \sum_{i \neq j} \sum_{\in S} n_{ij} + \sum_{i \in S} n_{ic} + n_{co}$$

and define $\bar{I}_{ij} = \sum_{l=1}^{n} I_{ijl}/n, \quad i \neq j, \quad i, j \in S$

$$\bar{I}_{ic} = \sum_{l=1}^{n} I_{icl}/n, \quad i \in S, \ \bar{I}_{co} = \sum_{l=1}^{n} I_{col}/n,$$

where $d_l = (I_{12l}, I_{13l} \ldots, I_{k-1,kl}, I_{1cl}, \ldots I_{kcl}, I_{col}), l=1,2 \ldots n$

are the n observations on d and $\bar{I}_i = \sum_{j \in S-\{i\}} \bar{I}_{ij}$ ....(4.2.5)

The partial likelihood based on the observations on d is as follows

$$L_1(.) \propto [(\beta_1+k)(\beta_1+k-1)]^{-\sum_{i \neq j} \sum_{\in S} n_{ij}} * [(\beta_1+k)(\beta_1+k-1)]^{-\sum_{i \in S} n_{ic}} *$$

$$(\beta_1)^{\sum_{i \in S} n_{ic}} (\beta_1)^{n_{co}} (\beta_1+k)^{-n_{co}}$$

...............(4.2.6)

Let us try to find the estimator of β_1 which maximizes (4.2.6). The likelihood equation $\frac{\delta \log L_1(.)}{\delta \beta_1} = 0$, yeilds the followig quadratic equation in β_1, viz, $n_A \beta_1^2 + n_B \beta_1 - n_C = 0$ ......(4.2.7)

where $n_A = 2 \sum_{i \neq j} \sum_{\in S} n_{ij} + \sum_{i \in S} n_{ic}$

$n_B = (2k-1) \sum_{i \neq j} \sum_{\in S} n_{ij} - kn_{co}$

$n_C = k(k-1)(n_{co} + \sum_{i \in S} n_{ic})$ ....(4.2.8) and its unique positive solution $\hat{\beta}_1$ is given by.

$$\hat{\beta}_1 = \frac{-\bar{I}_B + \sqrt{(\bar{I}_B^2 + 4\bar{I}_A\ \bar{I}_C)}}{2\ \bar{I}_A} \qquad \text{........(4.2.9)}$$

where $\bar{I}_A = n_A/n$, $\bar{I}_B = n_B/n$, $\bar{I}_C = n_C/n$ It can be shown easily that

$$E(\hat{\beta}_1) = \beta_1 + O(n^{-1}) \qquad \text{..................(4.2.10)}$$

Thus $\hat{\beta}_1$ is asymptotically unbiased for β_1. It is to be noted that (4.2.9) reduces to (3.3.7) in the special case k=2. Before finding the asymptotic distribution of $\hat{\beta}_1$ and exploring the properties of the estimators of the survival functions to be described later, let us first observe the following :

$\bar{I}_{1c} - E(I_{1c}) \ldots \bar{I}_{kc} - E(I_{kc})$

Since $\sqrt{n}(\bar{I}_{12} - E(I_{12}), \ldots \bar{I}_{1k} - E(I_{1k}), \bar{I}_{k,k-1} - E(I_{k,k-1}), \bar{I}_{co} - E(I_{co}))$

$$\xrightarrow{D} N_{k^2+1}(0, \Sigma_I) \qquad \text{..................(4.2.11)}$$

where Σ_I (singular) represents the dispersion matrix for d. By the application of delta method, $\sqrt{n}(\hat{\beta}_1 - \beta_1) \xrightarrow{D} N(0, \sigma^2_{\beta_1})$...(4.2.12)

where $\sigma^2_{\beta_1} = L' \Sigma_I L$, where the components of L are obtained by Taylor's series expansion of $\hat{\beta}_1$ about the vector,

$$(\Pi_{12}, \Pi_{13} \ldots\ldots \Pi_{k-1,k}, \Pi_{1c}, \ldots\ldots \Pi_{kc}, \Pi_{co}).$$

4.2.3 **Estimators Proposed.**

a) **Ebrahimi Type Estimators.**

The estimators proposed for $\bar{F}_1(.)$ following the idea of Ebrahimi (1985) are :

i) $\overset{ekI}{\bar{F}_1}(.) = \sum_{i=1}^{k} W_i \text{Exp}((\hat{\beta}_1 + k - 1)^{-1} \log \hat{C}_i(x))$, where $W_i = \bar{I}_i + \bar{I}_{ic} + \alpha_i \bar{I}_{co}$,

$i=1,2....k$. $\hat{C}_i(x)=\hat{S}_i(x)+\hat{S}_{ic}(x)+\hat{S}_{co}(x)(1+(k-1)\hat{\beta}_1^{-1})$ $i=1,2....k$

and α_i's, $i=1,2....k$ are the appropriately chosen weights,i.e.,

$$\alpha_i \geq 0,\ i=1,2....k \text{ and } \sum_{i=1}^{k}\alpha_i=1 \qquad ....(4.2.13)$$

$$\text{and (ii)} \bar{F}_1^{ekII}(.) = \text{Exp}((\hat{\beta}_1+k-1)^{-1}\log(\frac{\hat{S}(x)}{k} + \hat{S}_{co}(x)((1-\frac{1}{k})+(k-1)\hat{\beta}_1^{-1}) \qquad ....(4.2.14)$$

obviously the expressions (4.2.13) and , (4.2.14) reduce to expressions (3.3.9) in the special case k=2.

$\hat{S}_{ij}(x)$, $\hat{S}_{ic}(x)$, $\hat{S}_{co}(x)$ are the empirical subsurvival functions given by.

$$\hat{S}_{ij}(x) = n^{-1}\sum_{l=1}^{n} I\ (Z_l>x,\ I_{ijl}=1),\ i\neq j,\ \ i,j \in S.$$

$$\hat{S}_{ic}(x) = n^{-1}\sum_{l=1}^{n} I\ (Z_l>x,\ I_{icl}=1),\ i \in S.$$

$$\hat{S}_{co}(x) = n^{-1}\sum_{l=1}^{n} I\ (Z_l>x,\ I_{col}=1),$$

we also write $\hat{S}_i(x)=\sum_{j\in S-\{i\}}\hat{S}_{ij}(x)$ and $I_i=\sum_{j\in S-\{i\}} I_{ij}$ ....(4.2.15)

(b) **ACL type Estimators.**

Under the assumption $\bar{F}_c(.) = (\bar{F}_1(.))^{\beta_1}$, $\beta_1 > 0$, the following relation would be found to hold from the expression (4.2.3)

$$\psi(\bar{F}_1(x),S(x)) = k(\bar{F}_1(x))^{\beta_1+k-1} - (k-1)(\bar{F}_1(x))^{\beta_1+k} - S(x) = 0 \qquad ....(4.2.16)$$

Theorem 4.1

For any given $S(x) \in (0,1)$, the equation in (4.2.16) has a unique positive solution lying between 0 and 1 for any preassgned value of $k(>2)$ and β_1.

Proof.

For notational convenience, let us write $S(x) = p$ and $\bar{F}_1(x)=\gamma$ and the expression $\psi(\bar{F}_1(x), S(x))$ in (4.2.16) can be written as $\psi(\gamma,p)=\psi_1(\gamma)-p$, where $\psi_1(\gamma)=k\gamma^{\beta_1+k-1} - (k-1)\gamma^{\beta_1+k}$ ....(4.2.17)

For given $\beta_1>0$, $k>0$, $\psi'(\gamma,\beta_1) = \dfrac{\delta\psi_1(\gamma)}{\delta\gamma}$ reduces to.

$$(\gamma)^{\beta_1+k-2}(k\beta_1(1-\gamma) + (k^2-k)(1-\gamma) + \beta_1\gamma) > 0 \text{ for } 0<\gamma<1 \quad ....(4.2.18)$$

and $\beta>0$, $k>0$ and $p \in (0,1)$, Hence, $\psi_1(\gamma)$ is monotonically increasing in $\gamma \in (0,1)$. Moreover, $\psi_1(0)=0$, $\psi_1(1)=1$.

The equation (4.2.16) i.e., $\psi_1(\gamma)=p$ has a unique solution in $\gamma \in (0,1)$ for given $p \in (0,1)$.

The final unique solution in (4.2.16) for given k is of course dependent on β_1 and $S(x)$ and can be written as

$$\bar{F}_1(x) = \psi_k(\beta_1, S(x)).$$

This leads us to the following estimator of the form $\bar{F}_1^{ak}(.) = \psi_k(\hat{\beta}_1, \hat{S}(x))$, where $\hat{\beta}_1$ is as given in (4.2.9) and $\hat{S}(x) = n^{-1}\sum_{l=1}^{n} I(Z_l > x)$, Obviously equation (4 .2.16) reduces to equation ((3.3.15) in the special case $k = 2$. Clearly, the estimator $\psi_k(\hat{\beta}_1, \hat{S}(x))$ reduces to the estimator $\psi(\hat{\beta}_1, \hat{S}(x))$ of chapter 3 when $k=2$.

Corresponding to the three sets of estimators proposed, three natural estimators of $\bar{F}(x)$ (survival function associated with the life of the system without censoring) are given by .

$$\text{(a)}\overset{e_{kI}}{\bar{F}}(.),\ \text{(b)}\overset{e_{kII}}{\bar{F}}(.)\ \text{and (c)}\overset{a_k}{\bar{F}}(.).$$

where $\bar{F}(x) = k(\bar{F}_1(x))^{k-1} - (k-1)(\bar{F}_1(x))^k$

$$\text{............}(4.2.19)$$

and $\overset{e_{kI}}{\bar{F}}(.)$ (respectively $\overset{e_{kII}}{\bar{F}}(.)$ and $\overset{a_k}{\bar{F}}(.)$) is an estimator of $\bar{F}(.)$ obtained by substituting $\overset{e_{kI}}{\bar{F}_1}(.)$ (respectively $\overset{e_{kII}}{\bar{F}_1}(.)$ and $\overset{a_k}{\bar{F}_1}(.)$) for $\bar{F}_1(.)$ in the right hand side expression of (4.2.19).

It may be noted that the estimator $\overset{a_2}{\bar{F}_1}(.)$ (i.e. $\overset{a_k}{\bar{F}_1}(.)$ in case K=2) is exactly same as the estimator $\overset{aI}{\bar{F}_1}(.)$ considered in section (3.3) of chapter 3 when k = 2. Thus with the assumption of identical distribution of component lives , $\overset{a_k}{\bar{F}_1}(.)$ is a natural extension of $\overset{a_2}{\bar{F}_1}(.)$ for the case of general k>2. Similarly $\overset{e_{kI}}{\bar{F}}(.)$ coincides with $\overset{eI}{\bar{F}}(.)$ and $\overset{e_{kII}}{\bar{F}}$ coincides with $\overset{eII}{\bar{F}}(.)$ of section (3.3) of chapter 3. Asymptotic unbiasdness and normality of the estimators $\overset{e_{kI}}{\bar{F}}(.)$, $\overset{e_{kII}}{\bar{F}}(.)$ and $\overset{a_k}{\bar{F}}(.)$ can be established using delta method in the usual manner.

4.2.4 Asymptotic Variances of the Proposed Estimators,

viz., $\bar{F}_1^{e_{kI}}(.)$, $F_1^{e_{kII}}(.)$, $F_1^{a_k}(.)$

Before deriving the asymptotic expansions of the proposed estimators (retaining only the first order terms in Taylor's series expansion), some symbols are introduced as follows.

$$\Pi_A = \sum_{\substack{i \neq j \\ i,j \in S}} \Pi_{ij}, \qquad \Pi_B = \sum_{i \in S} \Pi_{ic}, \qquad \Pi_c = \Pi_B + \Pi_{co}$$

$$P_1 = (2K-1)\Pi_A - k\,\Pi_{co}, \qquad P_2 = 2\Pi_A + \Pi_B$$

$$P_3 = (P_1^2 + 4k(k-1)\, P_2 \Pi_c)^{.5} \qquad P_4 = -\,P_1 + P_3$$

$$P_5 = -\,P_4 + 2P_2(k-1)$$

$$Q_1 = P_3^{-1}((2k-1)P_1 + 4k(k-1)\Pi_c) - (2k-1)$$

$$Q_2 = P_3^{-1}(2k(k-1)\ (P_2 + \Pi_c))$$

$$Q_3 = P_3^{-1}(2k(k-1)P_2 - kP_1) + k$$

$$Q_4 = P_5^{-2}(2P_4 - 2P_2Q_1)$$

$$Q_5 = P_5^{-2}(4P_4 - 2P_2Q_2)$$

$$Q_6 = P_5^{-2}(-2P_2Q_3)$$

$$R_1 = P_4^{-2}(k-1)(4P_4 - 2P_2Q_1)$$

$$R_2 = P_4^{-2}(k-1)(2P_4 - 2P_2Q_2)$$

$$R_3 = P_4^{-2}(k-1)(-2P_2Q_3)$$

$$W_i = \Pi_i + \Pi_{ic} + \alpha_i \Pi_{co}, \quad i=1,2 \ldots k.$$

$$\Pi_i = \sum_{j \in S-\{i\}} \Pi_{ij}, \quad 0<\alpha_i<1, \quad \text{such that} \sum_{i=1}^{k} \alpha_i = 1 .$$

$$C_i(x)=S_i(x)+S_{ic}(x)+S_{co}(x)(1+(k-1)\beta_1^{-1}), \; i=1,2\ldots.k$$

$$S_i(x)= \sum_{j\in S-\{i\}} S_{ij}(x) .$$

$$C_L(x) = \frac{S(x)}{k} + S_{co}(x)((1-\frac{1}{k}) + (k-1)\beta_1^{-1})$$

$$H_i(x) = ((\beta_1 + k-1)\; C_i(x))^{-1}, \quad i=1,2 \ldots k$$

$$H_L(x) = ((\beta_1 + k-1)\; C_L(x))^{-1},$$

Then,

$$M_1 = M_2 \ldots. = M_k = \Big(\sum_{i=1}^{k} W_i (\log C_i(x)\; Q_4 + H_i(x)\; S_{co}(x)\; R_1) + 1\Big)\bar{F}_1(x$$

$$M_{1c} = M_{2c} \ldots = M_{kc} = \Big(\sum_{i=1}^{k} W_i (\log C_i(x)\; Q_5 + H_i(x)\; S_{co}(x) R_2) + 1\Big)\bar{F}_1($$

$$M_{co} = \Big(\sum_{i=1}^{k} W_i (\log C_i(x)\; Q_6 + H_i(x)\; S_{co}(x)\; R_3) + 1\Big)\bar{F}_1(x)$$

$$N_1 = N_2 = \ldots\ldots = N_k = \Big(\sum_{i=1}^{k} W_i H_i(x)\Big)\bar{F}_1(x)$$

$$N_{1c} = N_{2c} = \ldots\ldots = N_{kc} = \Big(\sum_{i=1}^{k} W_i H_i(x)\Big)\bar{F}_1(x)$$

$$N_{co} = \Big(\Big(\sum_{i=1}^{k} W_i H_i(x)\Big)\;(1+(K-1)\beta_1^{-1})\Big)\bar{F}_1(x)$$

$$M_1^* = M_2^* \ldots. = M_k^* = \Big(Q_4 * \log C_L(x) + H_L(x)\; S_{co}(x)\; R_1\Big)\bar{F}_1(x)$$

$$M_{1c}^* = M_{2c}^* \ldots. = M_{kc}^* = \Big(Q_5 * \log C_L(x) + H_L(x)\; S_{co}(x)\; R_2\Big)\bar{F}_1(x)$$

$$M^*_{co} = \left(Q_6 * \log C_L(x) + H_L(x)\ S_{co}(x)\ R_3\right) \bar{F}_1(x)$$

$$N^*_1 = N^*_2 = \ldots\ldots = N^*_k = \left(\frac{H_L(x)}{k}\right) \bar{F}_1(x)$$

$$N^*_{1c} = N^*_{2c} = \ldots\ldots = N^*_{kc} = \left(\frac{H_L(x)}{k}\right) \bar{F}_1(x)$$

$$N_{co} = \left(H_L(x)(1+(K-1)\beta_1^{-1})\right) \bar{F}_1(x)$$

$$h^*_{11} = (\gamma)^{\beta_1+k-2} (K(\beta_1+k-1) - (k-1)(\beta_1+k)\gamma)$$

$$h^*_{12} = \log\gamma\ (\gamma)^{\beta_1+k-1} (k-(k-1)\gamma)$$

$$h^*_{13} = -1 \qquad (4.2.20)$$

Theorem 4.2 For any finite

$x \in R^+$, $\sqrt{n}$ $(\bar{I}_1 - E(I_1),\ \bar{I}_2 - E(I_2),\ \ldots.\ \bar{I}_k - E(I_k),\ \bar{I}_{1c} - E(I_{1c}),$ $\bar{I}_{kc} - E(I_{kc}),\ \bar{I}_{co} - E(I_{co}),\ \hat{S}_1(x) - S_1(x)\ \ldots\ldots\ \hat{S}_k(x) - S_k(x),$ $\hat{S}_{1c}(x) - S_{1c}(x),\ \ldots\ldots\ \hat{S}_{kc}(x) - S_{kc}(x),\ \hat{S}_{co}(x) - S_{co}(x))$ is asymptotically normally distributed with mean vector zero and variance covariance matrix $\Sigma_e = \begin{pmatrix} \Sigma_{II} & Q \\ Q' & \Sigma_{III} \end{pmatrix}$ where Σ_{II} is the variance covariance matrix of $\sqrt{n}\ d^* = \sqrt{n}\ (\bar{I}_1, \ldots \bar{I}_k,\ \bar{I}_{1c},\ \bar{I}_{kc},\ \bar{I}_{co})$ as obtained from (4.2.11) and Σ_{III} is the variance covariance matrix of $\sqrt{n}\ S' = \sqrt{n}\ (\hat{S}_1(x)\ \ldots.\ \hat{S}_k(x),\ \hat{S}_{1c}(x),\ \ldots..\ \hat{S}_{kc}(x),\ \hat{S}_{co}(x))$ and $Q = \mathrm{cov}(\sqrt{n}\ d^*, \sqrt{n}\ S)$

Proof. By central limit theorem.

Theorem 4.3.

Under the conditions of Theorem 4.2 each of,

$\sqrt{n}(\overset{e}{\bar{F}}{}_1^{kI}(x) - \bar{F}_1(x))$, and $\sqrt{n}\,(\overset{e}{\bar{F}}{}_1^{kII}(x) - \bar{F}_1(x))$ for finite $x \in R^+$ is asymptotically normally distributed with mean zero. The asymptotic variances are $L_I' \Sigma_e L_I$ and $L_{II}' \Sigma_e L_{II}$, respectively, where

$$L_I' = (M_1, M_2, \ldots\ldots M_k, M_{1c}, \ldots M_{kc}, M_{co}, \ldots N_1, N_2 \ldots N_K \ldots, N_{1c} \ldots\ldots N_{kc}, N_{co})$$

$$L_{II}' = (M_1^*, M_2^* \ldots\ldots M_k^*, M_{1c}^* \ldots M_{kc}^*, M_{co}^*, \ldots N_1^* \ldots N_k^*, \ldots N_{kc}^*, N_{co}^*)$$

Σ_e is as given in Theorem 4.2. The vectors L_I' and L_{II}' are to be obtained from (4.2.20).

Proof. By delta method.

(ii) **Asymptotic variances of ACL Type Estimators.**

From (4.2.16) we have

$$K(\hat{\gamma})^{\hat{\beta}_1+k-1} - (K-1)(\hat{\gamma})^{\hat{\beta}_1+k} - \hat{S}(x) = 0 \quad \ldots\ldots\ldots\ldots(4.2.21)$$

where we write $\hat{\gamma} = \overset{a}{\bar{F}}{}^k(x)$.

By Taylor's series expansion about $(\beta_1, S(x))$, one can write

$$0 = \sqrt{n}\, h_{11}^*(\hat{\gamma}-\gamma) + \sqrt{n}\, h_{12}^*(\hat{\beta}_1 - \beta_1) + \sqrt{n}\, h_{13}^*(\hat{S}(x) - S(x)) + R_n \quad ..(4.2.22)$$

where R_n converges in probability to zero as $n \longrightarrow \infty$ and the expressions for $h_{11}^*, h_{12}^*, h_{13}^*$ are given in (4.2.20). Multiplying (4.2.22) by $(\hat{\gamma}-\gamma)$, $(\hat{\beta}_1-\beta_1)$, $(\hat{S}(x)-S(x))$ and taking expectations, the following system of equations would be found to hold.

$$h_{11}^{*}\mathrm{Var}(\hat{\gamma}) + h_{12}^{*}\mathrm{Cov}(\hat{\gamma}, \hat{\beta}_1) + h_{13}^{*}\mathrm{Cov}(\hat{\gamma}, \hat{S}(x)) = 0$$

$$h_{11}^{*}\mathrm{Cov}(\hat{\gamma}, \hat{\beta}_1) + h_{12}^{*}\mathrm{Var}(\hat{\beta}_1) + h_{13}^{*}\mathrm{Cov}(\hat{\beta}_1, \hat{S}(x)) = 0$$

$$h_{11}^{*}\mathrm{Cov}(\hat{\gamma}, \hat{S}(x)) + h_{12}^{*}\mathrm{Cov}(\hat{\beta}_1, \hat{S}(x)) + h_{13}^{*}\mathrm{Var}(\hat{S}(x)) = 0 \quad \ldots (4.2.23)$$

In order to find the asymptotic variance of $\bar{F}_1^{a_k}(.)$, one would first compute Asym Var$(\hat{\beta}_1)$, Asym Var$(\hat{S}(x))$, and Asym Cov$(\hat{S}(x)), \hat{\beta}_1)$ and they are given by .

$$\text{Asym Var}(\hat{S}(x)) = n^{-1}S(x)\ (1-S(x))$$

$$\text{Asym Var}(\hat{\beta}_1) = n^{-1}\ L'\ \Sigma_I\ L \quad \text{as given in (4.2.12).}$$

$$\text{Asym Cov}(\hat{S}(x), \hat{\beta}_1) = n^{-1}\Big(\sum_{\substack{i\neq j\\ i,j\in S}} l_{ij}S_{ij}(x) + \sum_{i\in S} l_{ic}S_{ic}(x) + l_{co}S_{co}(x)\Big)$$

$$- \ S(x)\Big(\sum_{\substack{i,j\in S\\ i\neq j}} l_{ij}\Pi_{ij} + \sum_{i\in S} l_{ic}\Pi_{ic} + l_{co}\Pi_{co}\Big)\Big)$$

where we write $L' = (l_{11}, l_{12} \ldots\ldots l_{ic} \text{-------} l_{kc}, l_{co})$ is from (4.2.12)

Asymptotic variances of estimated system life $\bar{F}(x)$ evaluated under these sets of estimators can be derived in the usual manner.

4.3 ESTIMATION IN THE GENERAL CASE UNDER PROPORTIONAL HAZARD ASSUMPTION-TWO OUT OF THREE SYSTEM

For the general problem stated in section 4.0, with k=3, the data consist of observations on (Z,d), where Z denotes the realized value of the life of the system and

$$d = (I_{12}, I_{13}, I_{1c}, I_{21}, I_{23}, I_{2c}, I_{31}, I_{32}, I_{3c}, I_{co})$$

is the indicator variable. The classes for d, subsurvival functions and conditional densities are as given in section 4.1 in the special case k=3.

Before proceeding further, we introduce some notation below, which will be used in describing the required expressions that follow.

$K_1 = \beta_1\phi_1\phi_2 + \phi_1 + \phi_2,$ $\quad K_2 = \beta_1\phi_2 + 1 + \phi_2$

$K_3 = \beta_1\phi_1 + \phi_1 + 1$ $\quad K_4 = \beta_1\phi_1\phi_2 + \phi_1 + \phi_2 + \phi_1\phi_2$

$K_5 = \beta_1\phi_2 + \phi_2\phi_1^{-1} + 1,$ $\quad K_6 = \beta_1\phi_2 + \phi_2 + \phi_2\phi_1^{-1}$

$K_7 = \beta_1\phi_2 + \phi_2 + \phi_2\phi_1^{-1} + 1$

....................(4.3.1)

Under the assumption $\bar{F}_c(.) = (\bar{F}_1(.))^{\beta_1} = (\bar{F}_2(.))^{\beta_1\phi_1} = (\bar{F}_3(.))^{\beta_1\phi_2}$, from (4.1.7) we have ,

$$S_{12}(x) = \phi_2 K_1^{-1}(\bar{F}_3(x))^{K_5} - \phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{12} = S_{12}(0) = E(I_{12}) = \phi_1\phi_2^2(K_1K_4)^{-1}$$

$$S_{13}(x) = \phi_1 K_1^{-1}(\bar{F}_3(x))^{K_5} - \phi_1 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{13} = S_{13}(0) = E(I_{13}) = \phi_1^2\phi_2(K_1K_4)^{-1}$$

$$S_{21}(x) = \phi_2 K_2^{-1}(\bar{F}_3(x))^{K_2} - \phi_1\phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{21} = S_{21}(0) = E(I_{21}) = \phi_2^2(K_2K_4)^{-1}$$

$$S_{23}(x) = K_2^{-1}(\bar{F}_3(x))^{K_2} - \phi_1 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{23} = S_{23}(0) = E(I_{23}) = \phi_2(K_2K_4)^{-1}$$

$$S_{31}(x) = \phi_1 K_3^{-1}(\bar{F}_3(x))^{K_6} - \phi_1\phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{31} = S_{31}(0) = E(I_{31}) = \phi_1^2(K_3K_4)^{-1}$$

$$S_{32}(x) = K_3^{-1}(\bar{F}_3(x))^{K_6} - \phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{32} = S_{32}(0) = E(I_{32}) = \phi_1(K_3K_4)^{-1}$$

$$S_{1c}(x) = \beta_1\phi_1\phi_2 K_1^{-1}(\bar{F}_3(x))^{K_5} - \beta_1\phi_1\phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{1c} = S_{1c}(0) = E(I_{1c}) = \beta_1\phi_1^2\phi_2^2(K_1K_4)^{-1}$$

$$S_{2c}(x) = \beta_1\phi_2 K_2^{-1}(\bar{F}_3(x))^{K_2} - \beta_1\phi_1\phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{2c} = S_{2c}(0) = E(I_{2c}) = \beta_1\phi_2^2(K_2K_4)^{-1}$$

$$S_{3c}(x) = \beta_1\phi_1 K_3^{-1}(\bar{F}_3(x))^{K_6} - \beta_1\phi_1\phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{3c} = S_{3c}(0) = E(I_{3c}) = \beta_1\phi_1^2(K_3K_4)^{-1}$$

$$S_{co}(x) = \beta_1\phi_1\phi_2 K_4^{-1}(\bar{F}_3(x))^{K_7}$$

$$\Pi_{co} = S_{co}(0) = E(I_{co}) = \beta_1\phi_1\phi_2 K_4^{-1} \qquad \ldots\ldots\ldots\ldots\ldots(4.3.2)$$

where K_1, K_2, ... K_7 are as given in (4.3.1)

Following identities are found to hold good by utilizing the classes, as described in section 4.1.

$$F_{.}(x) = \mathrm{Exp}(\phi_2 K_5^{-1}\log C_1(x)) = \mathrm{Exp}(\phi_2 K_2^{-1}\log C_2(x)) = \mathrm{Exp}(\phi_2 K_6^{-1}\log C_3(x))$$

$$\bar{F}_2(x) = \mathrm{Exp}(\phi_2(\phi_1 K_5)^{-1}\log C_1(x)) = \mathrm{Exp}(\phi_2(\phi_1 K_2)^{-1}\log C_2(x))$$
$$= \mathrm{Exp}(\phi_2(\phi_1 K_6)^{-1}\log C_3(x))$$

$$\bar{F}_3(x) = \mathrm{Exp}(K_5^{-1}\log C_1(x)) = \mathrm{Exp}(K_2^{-1}\log C_2(x)) = \mathrm{Exp}(K_6^{-1}\log C_3(x)) \quad \ldots\ldots.(4.3.3)$$

where let us recall from section (4.1) :

$$C_1(x) = S_{12}(x) + S_{13}(x) + S_{1c}(x) + S_{co}(x)(1+(\beta_1\phi_1)^{-1}+(\beta_1\phi_2)^{-1})$$

$$C_2(x) = S_{21}(x) + S_{23}(x) + S_{2c}(x) + S_{co}(x)(1+\beta_1^{-1}+(\beta_1\phi_2)^{-1})$$

$$C_3(x) = S_{31}(x) + S_{32}(x) + S_{3c}(x) + S_{co}(x)(1+\beta_1^{-1}+(\beta_1\phi_1)^{-1})$$
$$\ldots\ldots\ldots\ldots(4.3.4)$$

4.3.1 Estimation of β_1, ϕ_1 and ϕ_2

As before in a simple random sample of size n, let n_{ij} observations belong to class A_{ij} $i \neq j$, $i,j \in S$, n_{ic} observations belong to class A_{ic}, $i \in S$ and n_{co} observations belong to class A_{co} such that,

$$\sum_{\substack{i \neq j \\ i,j \in S}} n_{ij} + \sum_{i \in S} n_{ic} + n_{co} = n \quad (\text{Here } k=3)$$

Here (Z_l, d_l) with

$$d_l = (I_{12l}, I_{13l}, I_{21l}, I_{23l}, I_{31l}, I_{32l}, I_{1cl}, I_{2cl}, I_{3cl}, I_{col}),$$

is the observation corresponding to the l-th unit of the sample, l=1,2 ..... n.

Now $\quad \bar{Z} = \sum_{l=1}^{n} Z_l/n, \qquad n_{ij} = \sum_{l=1}^{n} I_{ijl}, \qquad n_{ic} = \sum_{l=1}^{n} I_{icl},$

$$n_{co} = \sum_{l=1}^{n} I_{col} \qquad \bar{I}_{ij} = \sum_{l=1}^{n} I_{ijl}/n, \qquad \bar{I}_{ic} = \sum_{l=1}^{n} I_{icl}/n,$$

$$\bar{I}_{co} = \sum_{l=1}^{n} I_{col}/n \qquad (4.3.5)$$

The partial likelihood based on the observations on d is :

$$L_2(.) \propto \prod_{i \neq j, i, j \in S} (\Pi_{ij})^{n_{ij}} \prod_{i \in S} (\Pi_{ic})^{n_{ic}} (\Pi_{co})^{n_{co}} \quad ....(4.3.6)$$

The MLE of $(\beta_1, \phi_1, \phi_2)$ from the likelihood in (4.3.6) cannot be expressed in a closed form. We propose the following adhoc estimators of β_1, ϕ_1 and ϕ_2 based on some important relations satisfied by them.

We observe that

$$\Pi_{12} + \Pi_{13} + \Pi_{1c} = \phi_1\phi_2(\beta_1\phi_1\phi_2 + \phi_1 + \phi_2 + \phi_1\phi_2)^{-1}$$

$$\Pi_{21} + \Pi_{23} + \Pi_{2c} = \phi_2(\beta_1\phi_1\phi_2 + \phi_1 + \phi_2 + \phi_1\phi_2)^{-1}$$

$$\Pi_{31} + \Pi_{32} + \Pi_{3c} = \phi_1(\beta_1\phi_1\phi_2 + \phi_1 + \phi_2 + \phi_1\phi_2)^{-1}$$

$$\Pi_{co} = \beta_1\phi_1\phi_2(\beta_1\phi_1\phi_2 + \phi_1 + \phi_2 + \phi_1\phi_2)^{-1} \quad .....(4.3.7)$$

Hence the following estimators are proposed for β_1, ϕ_1 and ϕ_2 .

$$\hat{\beta}_1 = \bar{I}_{co}(\bar{I}_{12} + \bar{I}_{13} + \bar{I}_{1c})^{-1}$$

$$\hat{\phi}_1 = (\bar{I}_{12} + \bar{I}_{13} + \bar{I}_{1c})(\bar{I}_{21} + \bar{I}_{23} + \bar{I}_{2c})^{-1}$$

$$\hat{\phi}_2 = (\bar{I}_{12} + \bar{I}_{13} + \bar{I}_{1c})(\bar{I}_{31} + \bar{I}_{32} + \bar{I}_{3c})^{-1}$$

$$.................(4.3.8)$$

It can be shown easily that

$$E(\hat{\beta}_1) = \beta_1 + O(n^{-1}), \qquad E(\hat{\phi}_1) = \phi_1 + O(n^{-1})$$

and $$E(\hat{\phi}_2) = \phi_2 + O(n^{-1}), \quad \text{i.e.} \quad ....(4.3.9)$$

Estimators $\hat{\beta}_1$, $\hat{\phi}_1$ and $\hat{\phi}_2$ are asymptotically unbiased for β_1, ϕ_1 and ϕ_2 respectively.

Theorem 4.4. Each of $\sqrt{n}\,(\hat{\beta}_1 - \beta_1)$, $\sqrt{n}(\hat{\phi}_1 - \phi_1)$, $\sqrt{n}(\hat{\phi}_2 - \phi_2)$ is asymptotically normally distributed with mean zero. The asymptotic variances are $\sigma_{11} = u_1' \Sigma_g u_1$, $\sigma_{22} = u_2' \Sigma_g u_2$ and $\sigma_{33} = u_3' \Sigma_g u_3$ respectively, where Σ_g is the variance covariance matrix of the vector $d' = (I_{12}, I_{13}, \ldots\ldots I_{co})$ and u_1, u_2, u_3 are vectors as described in the proof.

Proof. Since $(\bar{I}_{12}, \bar{I}_{13}, \ldots\ldots \bar{I}_{co}) \xrightarrow{D} N \quad (\mu, \frac{\Sigma_g}{n})$,

where $\mu' = (\Pi_{12}, \Pi_{13} \ldots\ldots \Pi_{co})$.

Let $S^* = \{12, 13, \ldots .31, 32, c1, c2, c3, co\}$

By Taylor's series expansion, one can write :

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) \approx \sqrt{n} \sum_{k \in S^*} u_{1k}(\bar{I}_k - E(I_k)) + R_n,$$

where R_n converges in probability to zero as $n \longrightarrow \infty$

$$\sqrt{n}(\hat{\phi}_1 - \phi_1) \approx \sqrt{n} \sum_{k \in S^*} u_{2k}(\bar{I}_k - E(I_k)) + R_n,$$

where R_n converges in probability to zero as $n \longrightarrow \infty$

$$\sqrt{n}\ (\hat{\phi}_2 - \phi_2) \approx \sqrt{n} \sum_{k \in S^*} u_{3k}(\bar{I}_k - E(I_k)) + R_n,$$

where R_n converges in probability to zero as $n \longrightarrow \infty$ ...(4.3.10)

Here $u_i' = (u_{i12}, u_{i13}, \ldots\ldots\ldots u_{ico})$, $\quad i = 1,2,3$

The vectors u_i's are found by the corresponding Taylor's series expansions

The theorem follows by delta method.

4.3.2 Estimators Proposed for Survival Functions

Estimators of survival functions of components 1,2 and 3, following the guidelines of (a)Ebrahimi(1985) and (b)Abdushukurov (1984) and Cheng and Lin (1984) are presented in the following lines :

(a) Ebrahimi type Estimator.

Following estimators are proposed for the survival functions of component life lenghts 1,2 and 3 based on Ebrahimi type arguments.

$$\bar{\bar{F}}^{e}_{1}(x) = \bar{a}_{11}\mathrm{Exp}(\hat{\phi}_2\hat{K}_5^{-1}\log\hat{C}_1(x)) + \bar{a}_{12}\mathrm{Exp}(\hat{\phi}_2\hat{K}_2^{-1}\log\hat{C}_2(x))$$
$$+ \bar{a}_{13}\mathrm{Exp}(\hat{\phi}_2\hat{K}_6^{-1}\log\hat{C}_3(x))$$

$$\bar{\bar{F}}^{e}_{2}(x) = \bar{a}_{11}\mathrm{Exp}(\hat{\phi}_2(\hat{\phi}_1\hat{K}_5)^{-1}\log\hat{C}_1(x)) + \bar{a}_{12}\mathrm{Exp}(\hat{\phi}_2(\hat{\phi}_1\hat{K}_2)^{-1}\log\hat{C}_2(x))$$
$$+ \bar{a}_{13}\mathrm{Exp}(\hat{\phi}_2(\hat{\phi}_1\hat{K}_6)^{-1}\log\hat{C}_3(x))$$

$$\bar{\bar{F}}^{e}_{3}(x) = \bar{a}_{11}\mathrm{Exp}(\hat{K}_5^{-1}\log\hat{C}_1(x)) + \bar{a}_{12}\mathrm{Exp}(\hat{K}_2^{-1}\log\hat{C}_2(x))$$
$$+ \bar{a}_{13}\mathrm{Exp}(\hat{K}_6^{-1}\log\hat{C}_3(x)) \qquad \ldots\ldots\ldots\ldots\ldots(4.3.11)$$

where the expressions for $\bar{a}_{11}$, $\bar{a}_{12}$ and $\bar{a}_{13}$ are given by .

$$\bar{a}_{11}=\bar{I}_{12} + \bar{I}_{13} + \bar{I}_{1c} + \alpha_{1A}\bar{I}_{Co}$$

$$\bar{a}_{12}=\bar{I}_{21} + \bar{I}_{23} + \bar{I}_{2c} + \alpha_{2A}\bar{I}_{Co}$$

$$\bar{a}_{13}=\bar{I}_{31} + \bar{I}_{32} + \bar{I}_{3c} + (1 - \alpha_{1A} - \alpha_{2A})\bar{I}_{Co} \qquad \ldots\ldots(4.3.12)$$

α_{1A}, α_{2A} are weights, $\alpha_{iA} \geq 0$, $i=1,2$ and $0 < \alpha_{1A}+ \alpha_{2A} \leq 1$.

Here $\hat{K}_i$ is an estimator of K_i, $i=1,2 \ldots. 7$ and $\hat{C}_i(x)$ is an estimator of $C_i(x)$, $i=1,2,3$ obtained by substituting $(\hat{\beta}_1,\hat{\phi}_1,\hat{\phi}_2)$ for (β_1,ϕ_1,ϕ_2) and writing $\hat{S}_i(x)$ for $S_i(x)$, $i \in S^*$ in the corr sponding expressions.

Let us recall that

$$\hat{S}_i(x) = n^{-1}\sum_{l=1}^{n} I(Z_l > x \text{ , } I_{il}=1), \forall \quad i \in S^* .$$

Since $\hat{\beta}_1,\hat{\phi}_1$ and $\hat{\phi}_2$ are all consistent estimators of β_1, ϕ_1 and ϕ_2 respectively and $\hat{S}_i(x)$ is a consistent estimator of $S_i(x)$, $\forall\ i \in S^*$ and $\bar{F}^e_i(x)$ is a continuous differentiable function of $\hat{\beta}_1,\hat{\phi}_1,\hat{\phi}_2$ and $\hat{S}_i(x)$, $i\in S^*$ for finite $x \in R^+$, $\bar{F}^e_i(x)$ is a consistent estimator of $\bar{F}_1(x)$, which is obtained from $\bar{F}^e_i(x)$ by substituting the population parameters for their consistent estimators, $i=1,2,3$.

(b) ACL Type estimators

In the present case i.e. K=3,

$$\bar{F}(x) = \bar{F}_c(x)(\bar{F}_1(x)\bar{F}_2(x)+\bar{F}_1(x)\bar{F}_3(x)+\bar{F}_2(x)\bar{F}_3(x)-2\bar{F}_1(x)\bar{F}_2(x)\bar{F}_3(x)) \quad \ldots\ldots\ldots(4.3.13)$$

From (4.3.13) the following relation is found to hold.

$$\gamma_1^{k_2} + \gamma_1^{k_5} + \gamma_1^{k_6} - 2\gamma_1^{k_7} - S(x) = 0 \quad \ldots\ldots\ldots(4.3.14)$$

where $\gamma_1= \bar{F}_3(x)$

As in the similar cases considered in this chapter and the chapter prece ding, we can prove that for given $S(x) \in (0,1)$, the equaation (4.3.14) in $\gamma_1= \bar{F}_3(x)$ yeilds a unique positive solution lying between 0 and 1 and for any preassigned value of

β_1, ϕ_1 and ϕ_2, solution of the equation may be expressed in the form, $\gamma_1 = \psi_1(\beta_1, \phi_1, \phi_2, S(x))$. Following ACL we write estimates as

$$\overset{a}{\bar{\bar{F}}}_3(x) = \psi_1(\hat{\beta}_1, \hat{\phi}_1, \hat{\phi}_2, \hat{S}(x)), \qquad \overset{a}{\bar{\bar{F}}}_1(x) = (\overset{a}{\bar{\bar{F}}}_3(x))^{\hat{\phi}_2},$$

$$\overset{a}{\bar{\bar{F}}}_2(x) = (\overset{a}{\bar{\bar{F}}}_3(x))^{\hat{\phi}_2 \hat{\phi}_1^{-1}}$$ where $\hat{\beta}_1$, $\hat{\phi}_1$ and $\hat{\phi}_2$ are given by (4.3.8).

Consistency of ACL type estimators follow easily as in the case of Ebrahimi type estimators considered earlier. Corresponding to the two sets of estimators proposed, viz,

$$(a)(\overset{e}{\bar{\bar{F}}}_1(.),\ \overset{e}{\bar{\bar{F}}}_2(.),\ \overset{e}{\bar{\bar{F}}}_3(.)) \quad \text{and} \quad (b)(\overset{a}{\bar{\bar{F}}}_1(.),\ \overset{a}{\bar{\bar{F}}}_2(.),\ \overset{a}{\bar{\bar{F}}}_3(.)),$$

two natural estimates of system survival function are (a) $\overset{e}{\bar{\bar{F}}}(.)$ and (b) $\overset{a}{\bar{\bar{F}}}(.)$, where $\overset{e}{\bar{\bar{F}}}(.)(\overset{a}{\bar{\bar{F}}}(.))$ is obtained from (4.3.13), substituting $\overset{e}{\bar{\bar{F}}}_i(.)$ $(\overset{a}{\bar{\bar{F}}}_i(.))$ for $\bar{F}_i(.)$, $i=1,2,3$.

4.3.3 Asymptotic Variances of Ebrahimi Type and ACL Type Estimators.

Before deriving the asymptotic expansions of the proposed estimators, some symbols are introduced below which will be used in the required expressions that follow.

$$A_1 = \Pi_{12} + \Pi_{13} + \Pi_{1c} + \alpha_{1A}\Pi_{co}$$

$$A_2 = \Pi_{21} + \Pi_{23} + \Pi_{2c} + \alpha_{2A}\Pi_{co}$$

$$A_3 = \Pi_{31} + \Pi_{32} + \Pi_{3c} + (1 - \alpha_{1A} - \alpha_{2A})\Pi_{co}$$

$$B_1 = \Pi_{12} + \Pi_{13} + \Pi_{1c}$$

$$B_2 = \Pi_{21} + \Pi_{23} + \Pi_{2c}$$

$$B_3 = \Pi_{31} + \Pi_{32} + \Pi_{3c}$$

$$B_4 = 1 + (\beta_1\phi_1)^{-1} + (\beta_1\phi_2)^{-1}$$

$$B_5 = 1 + \beta_1^{-1} + (\beta_1\phi_2)^{-1}$$

$$B_6 = 1 + \beta_1^{-1} + (\beta_1\phi_1)^{-1}$$

$$E_1 = \Pi_{co} + B_2 + B_3, \qquad E_2 = \Pi_{co} + B_1 + B_3,$$

$$E_3 = \Pi_{co} + B_1 + B_2, \qquad E_4 = S_{co}(x)(B_2+B_3)\Pi_{co}^{-2}$$

$$E_5 = S_{co}(x)(B_1+B_3)\Pi_{co}^{-2}, \qquad E_6 = S_{co}(x)(B_1+B_2)\Pi_{co}^{-2}$$

$$C_1(x) = S_{12}(x) + S_{13}(x) + S_{1c}(x) + S_{co}(x)B_4$$

$$C_2(x) = S_{21}(x) + S_{23}(x) + S_{2c}(x) + S_{co}(x)B_5$$

$$C_3(x) = S_{31}(x) + S_{32}(x) + S_{3c}(x) + S_{co}(x)B_6$$

$$\log C_1(x) = XXX, \quad \log C_2(x) = YYY, \quad \log C_3(x) = ZZZ \qquad \text{...(4.3.15)}$$

Now we have,

$$\overset{g}{L}_{12} = \Big(A_1 XXX\, E_1^{-1} + A_2(YYY\, E_2^{-2}(\Pi_{co} + B_3) + \phi_2(K_2 C_2(x)\Pi_{co})^{-1} S_{co}(x))$$
$$+ A_3\big(ZZZ\, E_3^{-2}(\Pi_{co} + B_2) + \phi_2(K_6 C_3(x)\Pi_{co})^{-1} S_{co}(x)\big) + 1\Big)\bar{F}_1(x)$$
$$= \overset{g}{L}_{13} = \overset{g}{L}_{1c}$$

$$\overset{g}{L}_{21} = \Big(A_1(\phi_2(K_5 C_1(x)\Pi_{co})^{-1} S_{co}(x) - XXX\, B_1 E_1^{-2})$$
$$+ A_3(\phi_2(K_6 C_3(x)\Pi_{co})^{-1} S_{co}(x) - ZZZ\, B_1 E_3^{-2}) + 1\Big)\bar{F}_1(x)$$
$$= \overset{g}{L}_{23} = \overset{g}{L}_{2c}$$

$$\overset{g}{L}_{31} = \Big(A_1(\phi_2(K_5C_1(x)\Pi_{co})^{-1}S_{co}(x) - XXX\ B_1E_1^{-2})$$

$$+ A_2(\phi_2(K_2C_2(x)\Pi_{co})^{-1}S_{co}(x) - YYY\ B_1E_2^{-2}) + 1\Big)\bar{F}_1(x)$$

$$= \overset{g}{L}_{32} = \overset{g}{L}_{3c}$$

$$\overset{g}{L}_{co} = \Big(- A_1(XXX\ B_1E_1^{-2} + \phi_2(K_5C_1(x))^{-1}E_4)$$

$$- A_2(YYY\ B_2E_2^{-2} + \phi_2(K_2C_2(x))^{-1}E_5)$$

$$- A_3(ZZZ\ B_3E_3^{-2} + \phi_2(K_6C_3(x))^{-1}E_6) + 1\Big)\bar{F}_1(x)$$

$$\overset{g}{M}_{12} = A_1\phi_2(K_5C_1(x))^{-1}_{\bar{F}_1(x)} = \overset{g}{M}_{13} = \overset{g}{M}_{1c}$$

$$\overset{g}{M}_{21} = A_2\phi_2(K_2C_2(x))^{-1}_{\bar{F}_1(x)} = \overset{g}{M}_{23} = \overset{g}{M}_{2c}$$

$$\overset{g}{M}_{31} = A_3\phi_2(K_6C_3(x))^{-1}_{\bar{F}_1(x)} = \overset{g}{M}_{32} = \overset{g}{M}_{3c}$$

$$\overset{g}{M}_{co} = \Big(A_1\phi_2(K_5C_1(x))^{-1}B_4 + A_2\phi_2(K_2C_2(x))^{-1}B_5 + A_3\phi_2(K_6C_3(x))^{-1}B_6\Big)\bar{F}_1(x)$$

$$\overset{g_*}{L}_{12} = \Big(A_2(\phi_2(C_2(x)\phi_1K_2\Pi_{co})^{-1}S_{co}(x) - YYY\ B_2E_2^{-2})$$

$$+ A_3(\phi_2(\phi_1C_3(x)K_6\Pi_{co})^{-1}S_{co}(x)$$

$$- ZZZ\ B_2E_3^{-2}) + 1\Big)\bar{F}_2(x) = \overset{g_*}{L}_{13} = \overset{g_*}{L}_{1c}$$

$$\overset{g_*}{L}_{21} = \Big(A_1(XXX\ E_1^{-2}(B_3 + \Pi_{co}) + \phi_2(\phi_1C_1(x)K_5\Pi_{co})^{-1}S_{co}(x))$$

$$+ A_2YYY\ E_2^{-1} + A_3(ZZZ\ E_3^{-2}(B_1 + \Pi_{co})$$

$$+ \phi_2(\phi_1C_3(x)K_6\Pi_{co})^{-1}S_{co}(x)) + 1\Big)\bar{F}_2(x)$$

$$= \overset{g_*}{L}_{23} = \overset{g_*}{L}_{2c}$$

$$\overset{g}{L}{}^{*}_{31} = \left(A_1(\phi_2(\phi_1 C_1(x)K_5\Pi_{co})^{-1} S_{co}(x) - XXX\ B_2E_1^{-2}) + A_2(\phi_2(\phi_1 C_2(x)K_2\Pi_{co})^{-1} S_{co}(x) - YYY\ B_2E_2^{-2}) + 1\right)_{\wedge}^{\bar{F}_2(x)} = \overset{g}{L}{}^{*}_{32} = \overset{g}{L}{}^{*}_{3c}$$

$$\overset{g}{L}{}^{*}_{co} = \left(- A_1(\phi_2(\phi_1 C_1(x)K_5)^{-1} E_4 + XXX\ B_1E_1^{-2}) - A_2(\phi_2(\phi_1 C_2(x)K_2)^{-1} E_5 + YYY\ B_2E_2^{-2}) - A_3(\phi_2(\phi_1 C_3(x)K_6)^{-1} E_6 + ZZZ\ B_3E_3^{-2}) + 1\right) \bar{F}_2(x)$$

$$\overset{g}{M}{}^{*}_{12} = A_1\phi_2(\phi_1 K_5)^{-1}_{\bar{F}_2(x)} = \overset{g}{M}{}^{*}_{13} = \overset{g}{M}{}^{*}_{1c}$$

$$\overset{g}{M}{}^{*}_{21} = A_2\phi_2(\phi_1 K_2)^{-1}_{\bar{F}_2(x)} = \overset{g}{M}{}^{*}_{23} = \overset{g}{M}{}^{*}_{2c}$$

$$\overset{g}{M}{}^{*}_{31} = A_3\phi_2(\phi_1 K_6)^{-1}_{\bar{F}_2(x)} = \overset{g}{M}{}^{*}_{32} = \overset{g}{M}{}^{*}_{3c}$$

$$\overset{g}{M}{}^{*}_{co} = \left(A_1(\phi_1 K_5 C_1(x))^{-1}\phi_2 B_4 + A_2\phi_2(\phi_1 K_2 C_2(x))^{-1} B_5 + A_3\phi_2(\phi_1 K_6 C_3(x))^{-1} B_6\right) \bar{F}_2(x)$$

$$\overset{g}{L}{}^{**}_{12} = \left(A_2((C_2(x)\ \ K_2\Pi_{co})^{-1} S_{co}(x) - YYY\ B_3E_2^{-2}) + A_3((C_3(x)K_6\Pi_{co})^{-1} S_{co}(x) - ZZZ\ B_3E_3^{-2}) + 1\right) \bar{F}_3(x) = \overset{g}{L}{}^{**}_{13} = \overset{g}{L}{}^{**}_{1c}$$

$$\overset{g}{L}{}^{**}_{21} = \left(A_1((C_1(x)K_5\Pi_{co})^{-1} S_{co}(x) - XXX\ B_3E_1^{-2}) + A_3((C_3(x)K_6\Pi_{co})^{-1} S_{co}(x) - ZZZ\ B_3E_3^{-2}) + 1\right) \bar{F}_3(x) = \overset{g}{L}{}^{**}_{23} = \overset{g}{L}{}^{**}_{2c}$$

$$
\begin{aligned}
{}^{g}L^{**}_{31} &= \Big(A_1((C_1(x)K_5\Pi_{co})^{-1}\ S_{co}(x) + XXX\quad E_1^{-2}(\Pi_{co} + B_2)) \\
&\quad + A_2((C_2(x)K_2\Pi_{co})^{-1}S_{co}(x) \\
&\quad + YYY\ E_2^{-2}(\Pi_{co} + B_1)) + A_3 ZZZ\ E_3^{-1} + 1\Big)\bar{F}_3(x) \\
&= {}^{g}L^{**}_{32} = {}^{g}L^{**}_{3c}
\end{aligned}
$$

$$
\begin{aligned}
{}^{g}L^{**}_{co} &= \Big(-A_1((C_1(x)K_5)^{-1}\ E_4 + XXX\ B_3E_1^{-2}) \\
&\quad - A_2((C_2(x)K_2)^{-1}E_5 + YYY\ B_3E_2^{-2}) \\
&\quad - A_3(ZZZ\ B_3E_3^{-2} + (C_3(x)\ K_6)^{-1}E_6) + 1\Big)\bar{F}_3(x)
\end{aligned}
$$

$${}^{g}M^{**}_{12} = A_1(C_1(x)K_5)^{-1}\bar{F}_3(x) = {}^{g}M^{**}_{13} = {}^{g}M^{**}_{1c}$$

$${}^{g}M^{**}_{21} = A_2(C_2(x)K_2)^{-1}\bar{F}_3(x) = {}^{g}M^{**}_{23} = {}^{g}M^{**}_{2c}$$

$${}^{g}M^{**}_{31} = A_3(C_3(x)K_6)^{-1}\bar{F}_3(x) = {}^{g}M^{**}_{32} = {}^{g}M^{**}_{3c}$$

$${}^{g}M^{**}_{co} = \Big(A_1(C_1(x)K_5)^{-1}B_4 + A_2(C_2(x)K_2)^{-1}B_5 + A_3(C_3(x)K_6)^{-1}B_5\Big)\bar{F}_3(x)$$

$${}^{g}l_{11} = K_5(\bar{F}_3(x))^{K_5-1} + K_2(\bar{F}_3(x))^{K_2-1} + K_6(\bar{F}_3(x))^{K_6-1} - 2K_7(\bar{F}_3(x))^{K_7-1}$$

$${}^{g}l_{12} = \phi_2 \log\bar{F}_3(x)((\bar{F}_3(x))^{K_5} + (\bar{F}_3(x))^{K_6} + (\bar{F}_3(x))^{K_2} - 2(\bar{F}_3(x))^{K_7})$$

$${}_{3} = -\phi_2\phi_1^{-2}\log\bar{F}_3\ (x)((\bar{F}_3(x))^{K_5} + (\bar{F}_3(x))^{K_6} - 2(\bar{F}_3(x))^{K_7})$$

All the symbols used in the expression (4.3.16) are borrowed from (4.3.15).

$$\overset{g}{l}_{14} = \log(\bar{F}_3(x))((\beta_1+\phi_1^{-1})(\bar{F}_3(x))^{k_5} + (\beta_1+1)(\bar{F}_3(x))^{k_2} + (\beta_1+1+\phi_1^{-1})(\bar{F}_3(x))^{k_6} - 2(\beta_1+1+\phi_1^{-1})(\bar{F}_3(x))^{k_7})$$

$$\overset{g}{l}_{15} = -1 \; . \qquad \text{..............(4.3.16)}$$

(a) **Ebrahimi Type Estimators.**

<u>**Theorem 4.5**</u> : Under the conditions of Theorem 4.2 each of $\sqrt{n}(\overset{e}{\bar{F}}_1(x) - \bar{F}_1(x))$, $\sqrt{n}(\overset{e}{\bar{F}}_2(x) - \bar{F}_2(x))$ and $\sqrt{n}(\overset{e}{\bar{F}}_3(x) - \bar{F}_3(x))$ for finite $x \in R^+$. is asymptotically normally distributed with mean zero. The asymptotic variances are

$$\overset{g}{L}{}_1' \; \overset{o}{\Sigma}_g \; \overset{g}{L}_1 , \qquad \overset{g}{L}{}_2' \; \overset{o}{\Sigma}_g \; \overset{g}{L}_2 , \qquad \overset{g}{L}{}_3' \; \overset{o}{\Sigma}_g \; \overset{g}{L}_3$$

respectively, where $\overset{o}{\Sigma}_g$ is the variance covariance matrix of

$$\sqrt{n}(\bar{I}_{12}, \bar{I}_{13}, \; .... \; \bar{I}_{co}, \hat{S}_{12}(x), \hat{S}_{13}(x) \; .... \; \hat{S}_{co}(x)) \qquad \text{and}$$

$$\overset{g}{L}{}_1' = (\overset{g}{L}_{12}, \; .... \; \overset{g}{L}_{co}, \overset{g}{M}_{12} \; .... \; \overset{g}{M}_{co})$$

$$\overset{g}{L}{}_2' = (\overset{*g}{L}_{12}, \; .... \; \overset{*g}{L}_{co}, \overset{*g}{M}_{12} \; .... \; \overset{*g}{M}_{co})$$

$$\overset{g}{L}{}_3' = (\overset{g**}{L}_{12}, \; .... \; \overset{**g}{L}_{co}, \; \overset{**g}{M}_{12} \; .... \; \overset{**g}{M}_{co}) \qquad \text{...........(4.3.17)}$$

(b) ACL Type Estimator.

From (4.3.14), on substituting $\hat{\gamma}_1 = \overset{a}{\bar{\bar{F}}}_3(x)$, the following equations would be found to hold for the estimator $\overset{a}{\bar{\bar{F}}}_3(x)$.

$$(\hat{\gamma}_1)^{\hat{k}_5} + (\hat{\gamma}_1)^{\hat{k}_2} + (\hat{\gamma}_1)^{\hat{k}_6} - 2(\hat{\gamma}_1)^{\hat{k}_7} - \hat{S}(x) = 0$$

By Taylor's series expansion about the true parameters $(\gamma_1, \beta_1, \phi_1, \phi_2, S(x))$ and retaining only the 1st order terms, one can show the following equation hold among the asymptotic variances and covariances.

$$\overset{g}{l}_{11}\mathrm{Var}(\hat{\gamma}_1) + \overset{g}{l}_{12}\mathrm{Cov}(\hat{\beta}_1,\hat{\gamma}_1) + \overset{g}{l}_{13}\mathrm{Cov}(\hat{\gamma}_1,\hat{\phi}_1) + \overset{g}{l}_{14}\mathrm{Cov}(\hat{\gamma}_1,\hat{\phi}_2) + \overset{g}{l}_{15}\mathrm{Cov}(\hat{\gamma}_1,\hat{S}(x)) = 0$$

$$\overset{g}{l}_{11}\mathrm{Cov}(\hat{\beta}_1,\hat{\gamma}_1) + \overset{g}{l}_{12}\mathrm{Var}(\hat{\beta}_1) + \overset{g}{l}_{13}\mathrm{Cov}(\hat{\beta}_1,\hat{\phi}_1) + \overset{g}{l}_{14}\mathrm{Cov}(\hat{\beta}_1,\hat{\phi}_2) + \overset{g}{l}_{15}\mathrm{Cov}(\hat{\beta}_1,\hat{S}(x)) = 0$$

$$\overset{g}{l}_{11}\mathrm{Cov}(\hat{\phi}_1,\hat{\gamma}_1) + \overset{g}{l}_{12}\mathrm{Cov}(\hat{\beta}_1,\hat{\phi}_1) + \overset{g}{l}_{13}\mathrm{Var}(\hat{\phi}_1) + \overset{g}{l}_{14}\mathrm{Cov}(\hat{\phi}_1,\hat{\phi}_2) + \overset{g}{l}_{15}\mathrm{Cov}(\hat{\phi}_1,\hat{S}(x)) = 0$$

$$\overset{g}{l}_{11}\mathrm{Cov}(\hat{\phi}_2,\hat{\gamma}_1) + \overset{g}{l}_{12}\mathrm{Cov}(\hat{\beta}_1,\hat{\phi}_2) + \overset{g}{l}_{13}\mathrm{Cov}(\hat{\phi}_1,\hat{\phi}_2) + \overset{g}{l}_{14}\mathrm{Var}(\hat{\phi}_2) + \overset{g}{l}_{15}\mathrm{Cov}(\hat{\phi}_2,\hat{S}(x)) = 0$$

$$\overset{g}{l}_{11}\mathrm{Cov}(\hat{S}(x),\hat{\gamma}_1) + \overset{g}{l}_{12}\mathrm{Cov}(\hat{\beta}_1,\hat{S}(x)) + \overset{g}{l}_{13}\mathrm{Cov}(\hat{\phi}_1,\hat{S}(x)) + \overset{g}{l}_{14}\mathrm{Cov}(\hat{\phi}_2,\hat{S}(x)) + \overset{g}{l}_{15}\mathrm{Var}(\hat{S}(x)) = 0 \qquad \text{.........(4.3.18)}$$

From (4.3.18), one can write down the expression of the asymptotic variance of $\overset{a}{\bar{F}}_3(x)$ by using the relations .

Asymptotic variance covariance matrix of $(\hat{\beta}_1, \hat{\phi}_1, \hat{\phi}_2)$ is

$$\begin{pmatrix} \upsilon_1' \\ \upsilon_2' \\ \upsilon_3' \end{pmatrix} \Sigma_g (\upsilon_1, \upsilon_2, \upsilon_3)$$ as obtained from Theorem 4.4. and

$$\text{Asym Cov}(\hat{S}(x), \hat{\beta}_1) = n^{-1}\Big(\sum_{k \in S^*} \upsilon_{1k} S_k(x) - S(x) \sum_{k \in S^*} \Pi_k \upsilon_{1k} \Big)$$

$$\text{Asym Cov}(\hat{S}(x), \hat{\phi}_1) = n^{-1}\Big(\sum_{k \in S^*} \upsilon_{2k} S_k(x) - S(x) \sum_{k \in S^*} \Pi_k \upsilon_{2k} \Big)$$

$$\text{Asym Cov}(\hat{S}(x), \hat{\phi}_2) = n^{-1}\Big(\sum_{k \in S^*} \upsilon_{3k} S_k(x) - S(x) \sum_{k \in S^*} \Pi_k \upsilon_{3k} \Big)$$

with $S^* = \{ 12, 21, \ldots\ 31, 32, c1, c2, c3, co \}$ ....(4.3.19)

One can write down the expressions of Asym Var$(\overset{a}{\bar{F}}_2(x))$ and Asym Var$(\overset{a}{\bar{F}}_1(x))$ by using the relation, viz, $\overset{a}{\bar{F}}_1(x) = (\overset{a}{\bar{F}}_3(x))^{\hat{\phi}_2}$, $\overset{a}{\bar{F}}_2(x) = (\overset{a}{\bar{F}}_3(x))^{\hat{\phi}_2 \hat{\phi}_1^{-1}}$. Asymptotic variances of the system life evaluated under this set of estimators can be easily derived.

It follows from the results of Breslow and crowley (1974), that the asymptotic variance of $\overset{KM}{\bar{F}}(x)$ i.e. system life estimated by Kaplan and Meier's (1958) method is given by.

$$\text{Asym Var}(\overset{KM}{\bar{F}}(x)) = n^{-1} (\bar{F}(x))^2 \int_0^x (\bar{F}(s))^{-2} (\bar{F}_c(s))^{-1} dF(s)$$ ....(4.3.20)

4.3.4 Numerical Computation

First we find the point x, where true survival probability of the system is (i) 0.90 and (ii)0.95 for different weibull distributions (Exponential distribution is a special case of Weibull distribution) with specified sets of known scale and shape parameters α_1 and δ_1 and different combinations of parameters connected with the censorship under proportional hazard assumption, viz, β_1, ϕ_1 and ϕ_2 . Asymptotic variances of (a) $\overset{e}{\bar{\bar{F}}}(.)$ (b) $\overset{a}{\bar{\bar{F}}}(.)$ and (c) $\overset{KM}{\bar{\bar{F}}}(.)$ are computed by using the asymptotic formula developed in section 4.3.4. It is found in finding the asymptotic variance of $\overset{e}{\bar{\bar{F}}}(.)$, the weights α_{1A}, α_{2A} equated to 0.3333 give the smallest variance. Therefore computations are reported at $\alpha_{1A} = \alpha_{2A} =$ 0.3333 for finding the variance of $\overset{e}{\bar{\bar{F}}}(.)$. These are presented in Tables 4.3.5.1 through 4.3.5.6 which follow. For the sake of comparability the numerical values are reported as n times the asymptotic variances. It has not been possible to compare the estimators algebraically. The numerical comparison provided leads us to some general findings, which are highlighted at the end of the Tables.

Table (Contd.)

Table 4.3.5.1

n times variance of the estimated survival probability at the points where true survival probability is (i) 0.90 and (ii) 0.95 for $\bar{F}_3(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1 = 1$, $\delta_1 = 1$, $\phi_2 = 0.50$. The estimates used are (a) $\overset{e}{\bar{F}}(.)$, (b) $\overset{a}{\bar{F}}(.)$ and (c) $\overset{KM}{\bar{F}}(.)$.

$\beta_1 \backslash \phi_1$		0.50		1.00		2.00	
		I	II	I	II	I	II
0.50	a	0.0096	0.0048	0.0079	0.0029	0.0024	0.0014
	b	0.0629	0.0202	0.0341	0.0123	0.0201	0.0086
	c	0.0569	0.0208	0.0436	0.0176	0.0340	0.0141
1.00	a	0.0054	0.0024	0.0045	0.0019	0.0022	0.0040
	b	0.0270	0.0102	0.0115	0.0034	0.0060	0.0017
	c	0.0322	0.0147	0.0212	0.0076	0.0128	0.0050
2.00	a	0.0025	0.0012	0.0021	0.0013	0.0020	0.0006
	b	0.0067	0.0016	0.0023	0.0013	0.0009	0.0001
	c	0.0196	0.0028	0.0078	0.0025	0.0044	0.0019

Table (Contd.)

Table 4.3.5.2

n times variance of the estimated survival probability at the points where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_3(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1 = 1$, $\delta_1 = 1$, $\phi_2 = 1.00$. The estimates used are (a)$\overset{e}{\bar{\bar{F}}}(.)$. (b) $\overset{a}{\bar{\bar{F}}}(.)$ and (c)$\overset{KM}{\bar{F}}(.)$.

$\beta_1 \backslash \phi_1$		0.50		1.00		2.00	
		I	II	I	II	I	II
0.50	a	0.0092	0.0016	0.0052	0.0043	0.0072	0.0040
	b	0.0989	0.0278	0.0245	0.0070	0.0112	0.0032
	c	0.0432	0.0770	0.0266	0.0068	0.0168	0.0064
1.00	a	0.0067	0.0012	0.0032	0.0010	0.0028	0.0008
	b	0.0032	0.0006	0.0025	0.0005	0.0024	0.0003
	c	0.0231	0.0080	0.0045	0.0033	0.0035	0.0010
2.00	a	0.0025	0.0008	0.0009	0.0006	0.0007	0.0003
	b	0.0023	0.0004	0.0007	0.0002	0.0005	0.0001
	c	0.0126	0.0128	0.0062	0.0008	0.0034	0.0006

Table (Contd.)

Table 4.3.5.3

n times variance of the estimated survival probability at the points where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_3(x) = \text{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1 = 1$, $\delta_1 = 1$, $\phi_2 = 2.00$. The estimates used are (a)$\bar{F}^{e}(.)$, (b) $\bar{F}^{a}(.)$ and (c)$\bar{F}^{KM}(.)$.

$\beta_1 \backslash \phi_1$		0.50		1.00		2.00	
		I	II	I	II	I	II
0.50	a	0.0098	0.0029	0.0065	0.0021	0.0087	0.0033
	b	0.0248	0.0120	0.0146	0.0018	0.0056	0.0066
	c	0.0220	0.0087	0.0127	0.0027	0.0116	0.0018
1.00	a	0.0033	0.0013	0.0018	0.0004	0.0070	0.0003
	b	0.0032	0.0010	0.0016	0.0003	0.0048	0.0002
	c	0.0154	0.0053	0.0073	0.0018	0.0092	0.0009
2.00	a	0.0018	0.0010	0.0016	0.0003	0.0003	0.0002
	b	0.0016	0.0009	0.0012	0.0002	0.0002	0.0001
	c	0.0140	0.0023	0.0062	0.0010	0.0010	0.0004

Table (Contd.)

Table 4.3.5.4

n times variance of the estimated survival probability at the points where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_3(x) = \text{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, α_1=1.50, δ_1=0.50, ϕ_2=0.50 . The estimates used are (a)$\bar{F}^{e}(.)$. (b) $\bar{F}^{a}(.)$ and (c)$\bar{F}^{KM}(.)$.

$\beta_1 \backslash \phi_1$		0.50		1.00		2.00	
		I	II	I	II	I	II
0.50	a	0.0094	0.0046	0.0078	0.0028	0.0025	0.0014
	b	0.0632	0.0203	0.0342	0.0124	0.0202	0.0087
	c	0.0568	0.0206	0.0438	0.0176	0.0340	0.0141
1.00	a	0.0052	0.0026	0.0046	0.0019	0.0023	0.0010
	b	0.0272	0.0102	0.0116	0.0035	0.0060	0.0017
	c	0.0323	0.0147	0.0213	0.0075	0.0128	0.0052
2.00	a	0.0026	0.0013	0.0022	0.0012	0.0002	0.0006
	b	0.0068	0.0016	0.0024	0.0011	0.0009	0.0001
	c	0.0197	0.0049	0.0078	0.0026	0.0043	0.0018

Table (Contd.)

Table 4.3.5.5

n times variance of the estimated survival probability at the points where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_3(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1.50$, $\delta_1=1.00$, $\phi_2=1.00$. The estimates used are (a)$\bar{\bar{F}}^{e}(.)$. (b) $\bar{\bar{F}}^{a}(.)$ and (c)$\bar{\bar{F}}^{KM}(.)$.

$\beta_1 \backslash \phi_1$		0.50 I	0.50 II	1.00 I	1.00 II	2.00 I	2.00 II
	a	0.0091	0.0017	0.0053	0.0010	0.0073	0.0035
0.50	b	0.0832	0.0280	0.0246	0.0070	0.0114	0.0032
	c	0.0420	0.0771	0.0267	0.0097	0.0167	0.0064
	a	0.0066	0.0013	0.0033	0.0009	0.0023	0.0018
1.00	b	0.0031	0.0006	0.0025	0.0005	0.0020	0.0003
	c	0.0230	0.0082	0.0046	0.0034	0.0035	0.0028
	a	0.0026	0.0008	0.0009	0.0006	0.0006	0.0003
2.00	b	0.0022	0.0004	0.0006	0.0002	0.0005	0.0001
	c	0.0195	0.0129	0.0062	0.0008	0.0033	0.0008

Table (Contd.)

Table 4.3.5.6

n times variance of the estimated survival probability at the points where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_3(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$, $\alpha_1=1.50$, $\delta_1=2.00$, $\phi_2=2.00$. The estimates used are (a)$\overset{e}{\bar{F}}(.)$. (b) $\overset{a}{\bar{F}}(.)$ and (c)$\overset{KM}{\bar{F}}(.)$.

$\beta_1 \backslash \phi_1$		0.50		1.00		2.00	
		I	II	I	II	I	II
0.50	a	0.0099	0.0028	0.0064	0.0021	0.0087	0.0031
	b	0.0294	0.0122	0.0145	0.0021	0.0052	0.0010
	c	0.0221	0.0088	0.0126	0.0027	0.0116	0.0016
1.00	a	0.0033	0.0013	0.0017	0.0003	0.0012	0.0005
	b	0.0028	0.0012	0.0014	0.0002	0.0010	0.0003
	c	0.0156	0.0034	0.0072	0.0018	0.0030	0.0008
2.00	a	0.0018	0.0010	0.0013	0.0002	0.0004	0.0002
	b	0.0016	0.0008	0.0011	0.0001	0.0002	0.0001
	c	0.0140	0.0022	0.0058	0.0012	0.0012	0.0004

From Tables 4.3.5.1 through Table 4.3.5.6, we can infer that both Ebrahimi type estimator, viz, $\overset{e}{\bar{F}}(.)$ and ACL type estimator $\overset{a}{\bar{F}}(.)$ in general behave better than the Kaplan and Meier Estimator. When the degrees of censoring as determined by the parameters β_1,ϕ_1 and ϕ_2 are in general not very high, Ebrahimi type estimator in general behaves better than the ACL estimator.

On the contrary when the degrees of censoring are high, ACL type estimator behaves slightly better than Ebrahimi type estimator. On the whole the performance of Ebrahimi type estimators appears to be satisfactory.

4.4 NUMERICAL COMPUTATION IN IDENTICAL COMPONENT LIFE DISTRIBUTIONS FOR THE SPECIAL CASE K=3

The theoretical developments are given in section 4.2 for the case of general $k \geq 2$. The case k=3 is taken up in the present section as an illustration and for the sake of comparability with the computations carried out in section 4.3. This computation is taken up to investigate if the assumption of identically of distributions imposes any changes in the comparison of different estimators already carried out in section 4.3

First we find the point at which the true survival probability of the system is (i)0.90 and (ii)0.95 for different combinations of scale and shape parameters (known) of Weibull distribution and for different values of the parameter under proportional hazard assumption with censorship, viz, β_1. The estimators used are (a)$\bar{F}^{e_{3I}}(.)$ (b)$\bar{F}^{e_{3II}}(.)$ and (c)$\bar{F}^{a_3}(.)$ i.e. by writing k=3 in the estimators $\bar{F}^{e_{kI}}(.)$, $\bar{F}^{e_{kI}}(.)$, $\bar{F}^{a_k}(.)$ of section 4.2. For the Ebrahimi type estimator, $\bar{F}^{e_{3I}}$, the numerical values are reported for $\alpha_{1A} = \alpha_{2A} = 0.3333$, since these values are observed to lead to the minimum asymptotic variance of $\bar{F}^{e_{3I}}(.)$. For the sake of comparison, n times the variance of the estimated survival probabilities are computed. The numerical results are presented in Tables 4.4.1 through 4.4.3 which follow.

Table 4.4.1

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$ $\alpha_1 = 1,\ \delta_1 = 1$. The estimators used are (a) $\bar{F}^{e_{3I}}(.)$ (b) $\bar{F}^{e_{3II}}(.)$ and (c) $\bar{F}^{a_3}(.)$ and (d) $\bar{F}^{KM}(.)$

β_1 \	a		b		c		d	
	I	II	I	II	I	II	I	II
0.50	0.0452	0.0072	0.0382	0.0067	0.0203	0.0062	0.0266	0.0098
1.00	0.0099	0.0026	0.0098	0.0024	0.0061	0.0016	0.0127	0.0066
2.00	0.0007	0.0003	0.0007	0.0003	0.0006	0.0002	0.0044	0.0013

Table 4.4.2

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}(\frac{-x^{\alpha_1}}{\delta_1})$ $\alpha_1=1.5$, $\delta_1= 2$. The estimators used are (a) $\bar{F}^{e_{3I}}(.)$ (b) $\bar{F}^{e_{3II}}(.)$ and (c) $\bar{F}^{a_3}(.)$ and (d) $\bar{F}^{KM}(.)$

β_1 \	a		b		c		d	
	I	II	I	II	I	II	I	II
0.50	0.0454	0.0076	0.0383	0.0068	0.0202	0.0063	0.0267	0.0098
1.00	0.0099	0.0028	0.0098	0.0025	0.0062	0.0016	0.0128	0.0066
2.00	0.0007	0.0003	0.0007	0.0003	0.0006	0.0002	0.0045	0.0014

Table 4.4.3

n times variance of the estimated survival probability at the points, where true survival probability is (i)0.90 and (ii)0.95 for $\bar{F}_1(x) = \mathrm{Exp}\left(\frac{-x^{\alpha_1}}{\delta_1}\right)$ $\alpha_1 = 2$, $\delta_1 = 2$. The estimators used are (a) $\bar{F}^{e_{3I}}(.)$ (b) $\bar{F}^{e_{3II}}(.)$ and (c) $\bar{F}^{a_3}(.)$ and (d) $\bar{F}^{KM}(.)$

β_1	a		b		c		d	
	I	II	I	II	I	II	I	II
0.50	0.0453	0.0073	0.0386	0.0069	0.0206	0.0062	0.0266	0.0098
1.00	0.0099	0.0027	0.0092	0.0024	0.0063	0.0014	0.0126	0.0066
2.00	0.0007	0.0003	0.0007	0.0003	0.0006	0.0002	0.0044	0.0014

On examination, Tables 4.4.1 through Table 4.4.3 reveal the fact that both Ebrahimi type estimators, viz, $\bar{F}^{e_{3I}}(.)$, (b)$\bar{F}^{e_{3II}}(.)$ and ACL type estimator, viz, $\bar{F}^{a_3}(.)$ behave better than the Kaplan-Meier estimator in general. Moreover the ACL type estimator is observed to perform best among these estimators proposed in all cases. However for β_1=2, the difference between the estimatons (a), (b) and (c) appears to be very little.

4.5 . EXTENSION TO OTHER COHERENT SYSTEM STRUCTURES

The two stage method of estimation developed and discussed so far (in chapter 3 and the prece-ding sections of the present chapter) can, it is presumed, be succesfully applied to general k' out of $(0 < k' < K)$ system or to other coherent systems of practical importance, under random censoring and with proportional hazard assumption. The properties of a particular model have to be made use of properly in suggesting an appropriate estimator. For each model the indicator variable has to be properly defined given the data set, the classes for the indicator variable as well as the conditional density function are to be appropriately determined and the Mathematical relationships that exist between the subsurvival functions and parameters have to be cleverly made use of in proposing reasonable estimators for the required survival functions of interest. All these steps are to be worked out in details for the different models which might be under consideration and a general estimation procedure, even if it is attempted for a reasonably wide class of models, is not expected to behave well for all of them in the class. However, to demonstrate the inherent potentiality of the fundamental essence of the two stage procedure a model essentially simple in nature, but of a slightly different kind than the ones considered so far is taken up for investigation in the present section.

The problem considered for example is the estimatoin of survival functions of component lives of a system structure, which in realiability terminology is called a series parallel system as follows :

There are two main components I and II arranged in a parallel set up and each of the main components is comprised of two subcomponents, which are assumed to be serially connected. Let X_1 , X_2 , X_3 , X_4 be the random variables associated with the lives of subcomponents 1,2,3,4 respectively. The subcomponents 1 and 2 together constitute the main component I and the subcomponents 3 and 4 constitute the main component II. X_i's are assumed to be independent, X_i following an absolutely continuous distribution with distribution function $F_i(.)$, $i=1,2,3,4$. The system so formed is censored by another random variable X_c, distributed independently of X_1, X_2, X_3, X_4. X_c follows an absolutely continuous distribution with distribution function $F_c(.)$. We observe in reality the value of the random variable $Z = \mathrm{Min}\{\mathrm{Max}(\mathrm{Min}(X_1,X_2),\ \mathrm{Min}(X_3,X_4)), X_c\}$ and a value of the indicator variable, d which follows a multinomial distribution with 13 mutually exclusive classes determined by the interrelationship among the random variables X_1, X_2, X_3, X_4 and X_c. The classes for d are identified as :

$$A_{13} = \{ Z \in R^+ \mid X_1 = \mathrm{Min}(X_1,X_2) < Z,\ \ X_3 = \mathrm{Min}(X_3,X_4) = Z,\ X_c > Z \}$$

$$A_{14} = \{ Z \in R^+ \mid X_1 = \mathrm{Min}(X_1,X_2) < Z,\ \ X_4 = \mathrm{Min}(X_3,X_4) = Z,\ X_c > Z \}$$

$$A_{23} = \{ Z \in R^+ \mid X_2 = \mathrm{Min}(X_1,X_2) < Z,\ \ X_3 = \mathrm{Min}(X_3,X_4) = Z,\ X_c > Z \}$$

$$A_{24} = \{ Z \in R^+ \mid X_2 = \mathrm{Min}(X_1,X_2) < Z,\ \ X_4 = \mathrm{Min}(X_3,X_4) = Z,\ X_c > Z \}$$

$$A_{31} = \{ Z \in R^+ \mid X_3 = \mathrm{Min}(X_3,X_4) < Z,\ \ X_1 = \mathrm{Min}(X_1,X_2) = Z,\ X_c > Z \}$$

$$A_{32} = \{ Z \in R^+ \mid X_3 = \mathrm{Min}(X_3,X_4) < Z,\ \ X_2 = \mathrm{Min}(X_1,X_2) = Z,\ X_c > Z \}$$

$$A_{41} = \{ Z \in R^+ \mid X_4 = \mathrm{Min}(X_3,X_4) < Z,\ \ X_1 = \mathrm{Min}(X_1,X_2) = Z,\ X_c > Z \}$$

$$A_{42} = \{ Z \in R^+ \mid X_4 = \mathrm{Min}(X_3,X_4) < Z,\ \ X_2 = \mathrm{Min}(X_1,X_2) = Z,\ X_c > Z \}$$

$$A_{1c} = \{ Z \in R^+ \mid X_1 = \mathrm{Min}(X_1,X_2) < Z,\ Z = X_c,\ \mathrm{Min}(X_3,X_4) > Z \}$$

$A_{2c} = \{ Z \in R^+ | \ X_2 = \mathrm{Min}(X_1, X_2) < Z \ , \ Z = X_c \ , \ \mathrm{Min}(X_3, X_4) > Z \}$

$A_{3c} = \{ Z \in R^+ | \ X_3 = \mathrm{Min}(X_3, X_4) < Z \ , \ Z = X_c \ , \ \mathrm{Min}(X_1, X_2) > Z \}$

$A_{4c} = \{ Z \in R^+ | \ X_4 = \mathrm{Min}(X_3, X_4) < Z \ , \ Z = X_c \ , \ \mathrm{Min}(X_1, X_2) > Z \}$

$A_{co} = \{ Z \in R^+ | \ \mathrm{Min}(X_1, X_2) > Z \ , \ Z = X_c \ , \ \mathrm{Min}(X_3, X_4) > Z \}$.

....................(4.5.1)

Here d is a 13 tuple , written as

$$(I_{13}, I_{14}, I_{23}, I_{24}, I_{31}, I_{32}, I_{41}, I_{42}, I_{1c}, I_{2c}, I_{3c}, I_{4c}, I_{co})$$

Let $S^* = \{ 13,14,23,24,31,32,41,42,1c,2c,3c,4c,co \}$.

...........(4.5.2)

For the subcomponent labels 1,2,3,4 let us write $S_1 = \{1,2\}$ and $S_2 = \{ 3,4 \}$.

The components of the indicator variable, d are given as follows .

$I_{ij} = 1$ if and only if $Z \in A_{ij}, \ \forall (i,j) \in S^*$, when S^* is as defined in (4.5.2)

...................(4.5.3)

The primary interest here lies in estimating the survival functions of the components 1,2,3,4 on observed realization on (Z,d) under the proportional hazard assumption, $\bar{F}_c(.) = (\bar{F}_1(.) \ \bar{F}_2(.))^{\beta_1} = (\bar{F}_3(.)\bar{F}_4(.))^{\beta_1 \phi_1}$, and $\bar{F}_2(.) = (\bar{F}_1(.))^{\delta_1}$, $\bar{F}_4(.) = (\bar{F}_3(.))^{\delta_2}$, where β_1, ϕ_1, δ_1 and δ_2 are the parameters associated with censorship and or proportional hazard assumption.

One can obtain the subsurvival function pertaining to 13 different classes as follows :

$$S_{13}(x) = P(Z > x, \ I_{13} = 1) = \int_x^{\infty} \left(\int_0^z dF_1(u) \bar{F}_2(u) \right) \bar{F}_4(z) \bar{F}_c(z) dF_3(z)$$

$$S_{14}(x)=P(Z>x,\ I_{14}=1) = \int_x^{\infty}\left(\int_0^z dF_1(u)\bar{F}_2(u)\right)\bar{F}_3(z)\bar{F}_c(z)dF_4(z)$$

$$S_{23}(x)=P(Z>x,\ I_{23}=1) = \int_x^{\infty}\left(\int_0^z dF_2(u)\bar{F}_1(u)\right)\bar{F}_4(z)\bar{F}_c(z)dF_3(z)$$

$$S_{24}(x)=P(Z>x,\ I_{24}=1) = \int_x^{\infty}\left(\int_0^z dF_2(u)\bar{F}_1(u)\right)\bar{F}_3(z)\bar{F}_c(z)dF_4(z)$$

$$S_{31}(x)=P(Z>x,\ I_{31}=1) = \int_x^{\infty}\left(\int_0^z dF_3(u)\bar{F}_4(u)\right)\bar{F}_2(z)\bar{F}_c(z)dF_1(z)$$

$$S_{32}(x)=P(Z>x,\ I_{32}=1) = \int_x^{\infty}\left(\int_0^z dF_3(u)\bar{F}_4(u)\right)\bar{F}_1(z)\bar{F}_c(z)dF_2(z)$$

$$S_{41}(x)=P(Z>x,\ I_{41}=1) = \int_x^{\infty}\left(\int_0^z dF_4(u)\bar{F}_3(u)\right)\bar{F}_2(z)\bar{F}_c(z)dF_1(z)$$

$$S_{42}(x)=P(Z>x,\ I_{42}=1) = \int_x^{\infty}\left(\int_0^z dF_4(u)\bar{F}_3(u)\right)\bar{F}_1(z)\bar{F}_c(z)dF_2(z)$$

$$S_{1c}(x)=P(Z>x,\ I_{1c}=1) = \int_x^{\infty}\left(\int_0^z dF_1(u)\bar{F}_2(u)\right)\bar{F}_3(z)\bar{F}_4(z)dF_c(z)$$

$$S_{2c}(x)=P(Z>x,\ I_{2c}=1) = \int_x^{\infty}\left(\int_0^z dF_2(u)\bar{F}_1(u)\right)\bar{F}_3(z)\bar{F}_4(z)dF_c(z)$$

$$S_{3c}(x)=P(Z>x,\ I_{3c}=1) = \int_x^{\infty}\left(\int_0^z dF_3(u)\bar{F}_4(u)\right)\bar{F}_1(z)\bar{F}_2(z)dF_c(z)$$

$$S_{4c}(x)=P(Z>x,\ I_{4c}=1) = \int_x^{\infty}\left(\int_0^z dF_4(u)\bar{F}_3(u)\right)\bar{F}_1(z)\bar{F}_2(z)dF_c(z)$$

$$S_{co}(x)=P(Z>x,\ I_{co}=1) = \int_x^{\infty} \bar{F}_1(z)\bar{F}_2(z))\bar{F}_3(z)\bar{F}_4(z)dF_c(z)$$

....................(4.5.4)

(Contd.)

It can be verified easily that the survival function of the random variable Z is given by.

$$S(x) = P(Z > x)$$

$$= \bar{F}_c(x)(\bar{F}_1(x)\bar{F}_2(x) + \bar{F}_3(x)\bar{F}_4(x) - \bar{F}_1(x)\bar{F}_2(x)\bar{F}_3(x)\bar{F}_4(x))$$

$= \sum_{k \in S^*} S_k(x)$, where S^* is as defined in (4.5.2). ......(4.5.5)

Before proceeding further, we introduce some symbols to expressions that simplify and they are.

$$K_1^* = (\beta_1\phi_1 + 1)(1 + \delta_2)$$

$$K_2^* = (\beta_1\phi_1 + \phi_1 + 1)(1 + \delta_2)$$

$$K_3^* = \phi_1(1 + \delta_2)(\beta_1 + 1)$$

$$K_4^* = (1 + \delta_1)(1 + \delta_2)$$

$$K_5^* = \beta_1\phi_1 + 1, \quad K_6^* = \beta_1 + 1$$

$$K_7^* = \beta_1\phi_1 + \phi_1 + 1 \qquad \text{.........................(4.5.6)}$$

Under the assumption that

$$\bar{F}_c(.) = (\bar{F}_1(.)\ \bar{F}_2(.))^{\beta_1} = (\bar{F}_3(.)\ \bar{F}_4(.))^{\beta_1\phi_1},$$

$$\bar{F}_2(.) = (\bar{F}_1(.))^{\delta_1},\ \bar{F}_4(.) = (\bar{F}_3(.))^{\delta_2},$$

(4.5.4) reduces to the following expressions :

$$S_{13}(x) = (K_4^*.K_5^*)^{-1}(\bar{F}_3(x))^{K_1^*} - (K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{13} = S_{13}(0) = E(I_{13}) = \phi_1(K_4^*.K_5^*.K_7^*)^{-1}$$

$$S_{14}(x) = \delta_2(K_4^*.K_5^*)^{-1}(\bar{F}_3(x))^{K_1^*} - \delta_2(K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{14} = S_{14}(0) = E(I_{14}) = \delta_2\phi_1(K_4^*.K_5^*.K_7^*)^{-1}$$

$$S_{23}(x) = \delta_1(K_4^*.K_5^*)^{-1}(\bar{F}_3(x))^{K_1^*} - \delta_1(K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{23} = S_{23}(0) = E(I_{23}) = \delta_1\phi_1(K_4^*.K_5^*.K_7^*)^{-1}$$

$$S_{24}(x) = \delta_1\delta_2(K_4^*.K_5^*)^{-1}(\bar{F}_3(x))^{K_1^*} - \delta_1\delta_2(K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{24} = S_{24}(0) = E(I_{24}) = \delta_1\delta_2\phi_1(K_4^*.K_5^*.K_7^*)^{-1}$$

$$S_{31}(x) = (K_4^*.K_6^*)^{-1}(\bar{F}_3(x))^{K_3^*} - \phi_1(K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{31} = S_{31}(0) = E(I_{31}) = (K_4^*.K_6^*.K_7^*)^{-1}$$

$$S_{32}(x) = \delta_1(K_4^*.K_6^*)^{-1}(\bar{F}_3(x))^{K_3^*} - \delta_1\phi_1(K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{32} = S_{32}(0) = E(I_{32}) = \delta_1(K_4^*.K_6^*.K_7^*)^{-1}$$

$$S_{41}(x) = \delta_2(K_4^*.K_6^*)^{-1}(\bar{F}_3(x))^{K_3^*} - \delta_2\phi_1(K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{41} = S_{41}(0) = E(I_{41}) = \delta_2(K_4^*.K_6^*.K_7^*)^{-1}$$

$$S_{42}(x) = \delta_1\delta_2(K_4^*.K_6^*)^{-1}(\bar{F}_3(x))^{K_3^*} - \delta_1\delta_2\phi_1(K_4^*.K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*}$$

$$\Pi_{42} = S_{42}(0) = E(I_{42}) = \delta_1\delta_2(K_4^*.K_6^*.K_7^*)^{-1}$$

$$S_{1c}(x) = \beta_1\phi_1(\delta_1 + 1)^{-1}((K_5^*)^{-1}(\bar{F}_3(x))^{K_1^*} - (K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*})$$

$$\Pi_{1c} = S_{1c}(0) = E(I_{1c}) = \beta_1\phi_1^2(\delta_1+1)^{-1}(K_5^*.K_7^*)^{-1}$$

$$S_{2c}(x) = \delta_1\beta_1\phi_1(\delta_1+1)^{-1}((K_5^*)^{-1}(\bar{F}_3(x))^{K_1^*} - (K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*})$$

$$\Pi_{2c} = S_{2c}(0) = E(I_{2c}) = \delta_1\beta_1\phi_1^2(\delta_1+1)^{-1}(K_5^*.K_7^*)^{-1}$$

$$S_{3c}(x) = \beta_1(\delta_2+1)^{-1}((K_6^*)^{-1}(\bar{F}_3(x))^{K_3^*} - \phi_1(K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*})$$

$$\Pi_{3c} = S_{3c}(0) = E(I_{3c}) = \beta_1(\delta_2+1)^{-1}(K_6^*.K_7^*)^{-1}$$

$$S_{4c}(x) = \delta_2\beta_1(\delta_2+1)^{-1}((K_6^*)^{-1}(\bar{F}_3(x))^{K_3^*} - \phi_1(K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*})$$

$$\Pi_{4c} = S_{4c}(0) = E(I_{4c}) = \delta_2\beta_1(\delta_2+1)^{-1}(K_6^*.K_7^*)^{-1}$$

$$S_{co}(x) = \beta_1\phi_1(K_7^*)^{-1}(\bar{F}_3(x))^{K_2^*})$$

$$\Pi_{co} = S_{co}(0) = E(I_{co}) = \beta_1\phi_1 K_7^{*^{-1}} \qquad \ldots\ldots\ldots\ldots\ldots (4.5.7)$$

It can be verified easily that $\sum_{k \in S^*} \Pi_k = 1$, as it should be. The following relations are found to hold among the subsurvival functions and parameters.

$$\bar{F}_1(x) = \mathrm{Exp}(\phi_1(1+\delta_2)(1+\delta_1)^{-1}K_1^{*^{-1}}\log C_1^*(x))$$

$$= \mathrm{Exp}(\phi_1(1+\delta_2)(1+\delta_1)^{-1}K_3^{*^{-1}}\log C_2^*(x))$$

$$\bar{F}_2(x) = \operatorname{Exp}(\phi_1\delta_1(1+\delta_2)(1+\delta_1)^{-1}K_1^{*^{-1}}\log C_1^*(x))$$

$$= \operatorname{Exp}(\phi_1(1+\delta_2)(1+\delta_1)\delta_1^{-1}K_3^{*^{-1}}\log C_2^*(x))$$

$$\bar{F}_3(x) = \operatorname{Exp}(K_1^{*^{-1}}\log C_1^*(x)) = \operatorname{Exp}(K_3^{*^{-1}}\log C_2^*(x))$$

$$\bar{F}_4(x) = \operatorname{Exp}(\delta_2 K_1^{*^{-1}}\log C_1^*(x)) = \operatorname{Exp}(\delta_2 K_3^{*^{-1}}\log C_2^*(x))$$

where

$$C_1^*(x) = \sum_{i\in S_1}\sum_{j\in S_2} S_{ij}(x) + \sum_{i\in S_1} S_{ic}(x) + S_{co}(x)(1+(\beta_1\phi_1)^{-1})$$

$$C_2^*(x) = \sum_{i\in S_1}\sum_{j\in S_2} S_{ji}(x) + \sum_{j\in S_2} S_{jc}(x) + S_{co}(x)(1+(\beta_1)^{-1})$$

$$\dots\dots\dots\dots(4.5.8)$$

where $S_1 = \{1,2\}$ and $S_2 = \{3,4\}$ as defined earlier.

In a simple random sample of size n, let n_k observations belong to class A_k, $k \in S^*$, where S^* is as defined in (4.5.2) and $\sum_{k\in S^*} n_k = n$

The partial likelihood based on the observations on d, the indicator variable is

$$L_{sp}(.) \propto \prod_{k\in S^*} \Pi_k^{n_k} \qquad \dots\dots\dots\dots(4.5.9)$$

Let us try to estimate $\beta_1, \phi_1, \delta_1, \delta_2$ by maximizing the likelihood $L_{sp}(.)$. Before proceeding further, let us introduce the following symbols :

$$n_a = \sum_{i\in S_1}\sum_{j\in S_2} n_{ij} + 2\sum_{i\in S_1} n_{ic} + n_{co}$$

$$n_b = \sum_{i\in S_1}\sum_{j\in S_2} n_{ij} + \sum_{i\in S_1}\sum_{j\in S_2} n_{ji} + \sum_{i\in S_1} n_{ic}$$

$$n_c = \sum_{i \in S_1} \sum_{j \in S_2} n_{ij} + \sum_{i \in S_1} \sum_{j \in S_2} n_{ji} + \sum_{j \in S_2} n_{jc}$$

$$n_d = \sum_{i \in S_1} \sum_{j \in S_2} n_{ij} + \sum_{i \in S_1} n_{ic}$$

$$n_e = \sum_{i \in S_1} \sum_{j \in S_2} n_{ji} + \sum_{j \in S_2} n_{jc}$$

$$n_f = \sum_{i \in S_1} n_{ic} + \sum_{j \in S_2} n_{jc} + n_{co}$$

$$n_m = n_f - n_a$$

..................(4.5.10)

The equations $\dfrac{\delta \log L_{sp}(.)}{\delta \beta_1} = 0$,

$$\frac{\delta \log L_{sp}(.)}{\delta \phi_1} = 0 , \quad \frac{\delta \log L_{sp}(.)}{\delta \delta_1} = 0 ,$$

and $\dfrac{\delta \log L_{sp}(.)}{\delta \delta_2} = 0$ lead to the following estimators on replacing $\Pi_{co} = \beta_1\phi_1(\beta_1\phi_1 + \phi_1 + 1)^{-1}$ by its sample analogue n_{co}/n as was done in case of a one out of two system in chapter 3.

Let us write

$$\hat{I}_k = n^{-1} \sum_{l=1}^{n} I_{kl}, \quad k \in S^* \qquad ....(4.5.11)$$

Then , $\hat{\delta}_1 = \dfrac{\bar{I}_{23} + \bar{I}_{24} + \bar{I}_{32} + \bar{I}_{42} + \bar{I}_{2c}}{\bar{I}_{13} + \bar{I}_{14} + \bar{I}_{31} + \bar{I}_{41} + \bar{I}_{1c}}$

$$\hat{\delta}_2 = \frac{\bar{I}_{14} + \bar{I}_{24} + \bar{I}_{41} + \bar{I}_{42} + \bar{I}_{4c}}{\bar{I}_{13} + \bar{I}_{23} + \bar{I}_{31} + \bar{I}_{32} + \bar{I}_{3c}}$$

$$\hat{\beta}_1 = \frac{-(\bar{I}_m + \bar{I}_{co}) + \sqrt{((\bar{I}_m + \bar{I}_{co})^2 + 4\bar{I}_{co}(\bar{I}_m - \bar{I}_e))}}{2(\bar{I}_m - \bar{I}_e)}$$

$$\hat{\phi}_1 = \frac{(\bar{I}_a - \bar{I}_{co}) - (\bar{I}_{co})(\hat{\beta}_1)^{-1}}{\hat{\beta}_1(\bar{I}_d + \bar{I}_{co} - \bar{I}_a) + \bar{I}_{co}}$$

...........(4.5.12)

Asymptotic properties of these estimators (4.5.11) through (4.5.12) can be easily established as in the other models in preceding sections .

Making use of these estimators for $\beta_1, \phi_1, \delta_1$, and δ_2 and the relations given in (4.5.8) , appropriate Ebrahimi type estimators for component survival functions can be easily developed as in the preceding sections. From (4.5.5), we have $S(x) = \bar{F}_c(x)\,[\,\bar{F}_1(x)\bar{F}_2(x) + \bar{F}_3(x)\bar{F}_4(x) - \bar{F}_1(x)\bar{F}_2(x)\bar{F}_3(x)\bar{F}_4(x)]$ under the assumption that,

$$\bar{F}_c(.) = (\bar{F}_1(.)\bar{F}_2(.))^{\beta_1} = (\bar{F}_3(.)\bar{F}_4(.))^{\beta_1\phi_1} ,$$

$$\bar{F}_2(.) = (\bar{F}_1(.))^{\delta_1} , \quad \bar{F}_4(.) = (\bar{F}_3(.))^{\delta_2} ,$$

the following equation is found to hold, viz,

$$\psi(\bar{F}_3(x), S(x)) = (\bar{F}_3(x))^{\phi_1(1+\beta_1)(1+\delta_2)}$$

$$+ (\bar{F}_3(x))^{(\beta_1\phi_1+1)(1+\delta_2)} - (\bar{F}_3(x))^{(\beta_1\phi_1+\phi_1+1)(1+\delta_2)} - S(x) = 0$$

.............(4.5.13)

using the estimators $(\hat{\beta}_1, \hat{\phi}_1, \hat{\delta}_1, \hat{\delta}_2)$ for $(\beta_1, \phi_1, \delta_1, \delta_2)$ and making use of the equation (4.5.13) appropriate, ACL type estimator for $\bar{F}_3(x)$ $\forall$ finite $x \in R^+$ can be developed as in chapter 3 and in the preceding sections of the present chapter.

Asymptotic properties of the estimators can again be worked out in a similar manner as in all the cases dealt with in chapter 3 and in the preceding sections of the present chapter.

CHAPTER V

GENERAL NONPARAMETRIC MODELS WITHOUT PROPORTIONAL HAZARD ASSUMPTION - ESTIMATION OF SURVIVAL FUNCTION BY MAXIMUM LIKELIHOOD METHOD EM ALGORITHM APPROACH

CHAPTER 5

GENERAL NONPARAMETERIC MODELS WITHOUT PROPORTIONAL HAZARD ASSUMPTION - ESTIMATION OF SURVIVAL FUNCTION BY MAXIMUM LIKELIHOOD METHOD - EM ALGORITHM APPROACH.

5.0 INTRODUCTION

In chapters 3 and 4 we have discussed nonparametric estimation procedure of survival functions in a two components parallel system and its simple extensions under random censoring, with proportional hazard assumption on the distribution functions. Also, independence is assumed regarding the life distributions. In the present chapter we first consider a one out of two components system under random censoring. But proportional hazard assumption is done away with. Again the problem is considered to be more general, where the assumption of independence of the life distributions of components is permitted to be violated. In this set up, a procedure is developed to estimate by maximum likelihood method of Survival functions of components or other quantities of interest via EM algorithm approach of Dempster *et al* (1977). The problem is first formulated as one of finding maximum likelihood estimators of hazard rates from an incomplete data set, where the first failure time point when it occurs is unobserved . It may be recalled in this connection what was pointed out in details in chapter 2. While in large samples, the maximum likelihood estimators are expected to perform very well, in small samples, they are often observed to behave badly. So it should be borne in mind that the methods of the present chapter are meant to be applied in reasonably large samples only and section 5.1 deals with the problem in details, establishing the identifiability conditions, developing a method of estimation of the variance and indicating an application of the method through simulated experimental data, assuming exponential life distribution. In section 5.2 we outline an extension of the procedure developed in section 5.1 to the more general case of $k-1$ out of k components system, $k\geq 2$. The purpose is to indicate a rough outline of how the same approach can be applied to more general coherent systems. But Mathematically the treatment becomes more and more difficult as the complexity of the model increases.

5.1 TWO COMPONENTS PARALLEL SYSTEM.

Consider a parallel system with two components under random censoring. We observe the lives, censored or otherwise of n such systems along with the identity of the component failed last if the system is uncensored. If the life time is censored then along with the censoring time point it is also noted which, if any, of the components failed before censoring. So the data set coincides with that in chapter 3. As in chapter 3, let X_1 and X_2 represent the life times of the components 1 and 2 respectively and let X_3 denote the censoring time point. The observed life Z of the system belongs to one of the five following class types as described in chapter 3. In the present chapter we will denote the indicator variable by d which takes the value i, if and only if the corresponding observation belongs to class A_i, $i = 1,2...5$.

Classes type	Observation type	d=Indicator variable.
A_1	$X_1 \le X_2 = Z < X_3$	1
A_2	$X_2 \le X_1 = Z < X_3$	2
A_3	$X_1 \le X_3 = Z < X_2$	3
A_4	$X_2 \le X_3 = Z < X_1$	4
A_5	$X_3 = Z < X_1 < X_2$	5

.........(5.1.1)

Here $Z = \mathrm{Min}(\mathrm{Max}(X_1, X_2),\ X_3)$ denotes the observed life of the system, censored or otherwise . So for the i th system, the observation consists of $(Z_i,\ d_i)$, the copy of $(Z,\ d)$ for the i th system, i = 1,2...n. Based on these observations our primary interest lies in estimating the distribution functions or survival functions of X_1 and X_2.

It is to be noted that in the present model, X_1 and X_2 are not necessarily independent. The estimation procedure is developed based on the partial likelihood of the observations, ignoring the part of the likelihood contribution by censoring time distribution. For general properties of this type of a partial likelihood approach, a reference may be made to Cox (1972) . Thus, the actual distribution of X_3 is not explicitly included in formulating the problem and in devising the appropriate estimators of the survival functions of X_1 and X_2 . There is some similarity between the problem considered here and the time of tumor problem of Dewanji *et al* (1986) in which observation on some life time indicates termination of another life at an earlier but unobserved time. Hence there is a similarity in approaches, although the methods differ substantially in details.

In section 5.1.1, the partial likelihood is written down and the identifiability problem is addressed. In section 5.1.2. We devlop an EM type algorithm to find the maximum likelihood estimators of the life time distributions of component survival functions . In section 5.1.3 we discuss variance estimation procedure and in section 5.1.4 we illustrate the procedure developed with a simulated data set.

5.1.1 Likelihood and Identifiability

A parallel system with two components fail when both the components fail, one after another. Let us call them the first and the second failures. Until the first failure takes place, the two life times X_1 and X_2 of the components constitute a competing risk framewark allowing for dependence between them. After the first failure, the living component carries on possibly with a different life distribution depending on the first failure time (unobserved). In order to describe all these, let us denote the first failure times and the second failure times and the identity of the corresponding components by T_1, T_2 and J_1, J_2 respectively and define the cause specific hazard rates as :-

$$\lambda_j(t) = \lim_{\Delta t \longrightarrow 0+} \frac{1}{\Delta t} P(T_1 \in [t, t+\Delta t), J_1 = j \mid T_1 \geq t) \quad j = 1, 2$$

$$\lambda_{12}(t|u) = \lim_{\Delta t \longrightarrow 0+} \frac{1}{\Delta t} P(T_2 \in [t, t+\Delta t), J_2=2 \mid T_2 \geq t, T_1=u, J_1=1)$$

$$\lambda_{21}(t|u) = \lim_{\Delta t \longrightarrow 0+} \frac{1}{\Delta t} P(T_2 \in [t, t+\Delta t), J_2=1 \mid T_2 \geq t, T_1=u, J_1=2),$$

$$\dots\dots\dots\dots\dots\dots (5.1.2)$$

where $\lambda_j(t)$ is the cause specific hazard rate of T_1 for the j th component, and $\lambda_{12}(t|u)$, $\lambda_{21}(t|u)$ are in conditional hazard rates for T_2 given T_1 with $j_2 = 2$ and $j_2 = 1$ respectively for $t \geq u > 0$, where u and t denote the first and second failure times respectively.

Under independent censoring mechanism, the relevant likelihood contribution from five possible class types of an observation at any time t is proportional to

$$\int_0^t \lambda_1(u)S(u)\lambda_{12}(t|u)\ S_{12}(t|u)du \quad \text{for class type } A_1$$

$$\int_0^t \lambda_2(u)S(u)\lambda_{21}(t|u)\ S_{21}(t|u)du \quad \text{for class type } A_2$$

$$\int_0^t \lambda_1(u)S(u)S_{12}(t+|u) \quad \text{for class type } A_3$$

$$\int_0^t \lambda_2(u)S(u)\ S_{21}(t+|u) \quad \text{for class type } A_4$$

$$\text{and} \quad S(t+), \qquad \text{for class type } A_5, \qquad \text{.................(5.1.3)}$$

$$\text{Where } S(t) = P(T_1 \geq t) = \mathrm{Exp}\left(-\int_0^t (\lambda_1(u) + \lambda_2(u))du\right)$$

$$S_{12}(t|u) = P(T_2 > t \mid T_1 = u,\ j_1 = 1) = \mathrm{Exp}\left(-\int_u^t \lambda_{12}(v|u)dv\right)$$

$$S_{21}(t|u) = P(T_2 > t \mid T_1 = u,\ j_1 = 2) = \mathrm{Exp}\left(-\int_u^t \lambda_{21}(v|u)dv\right) \qquad \text{..................(5.1.4)}$$

Consider n observation times Z_i, $i = 1,2....n$ and order them. Let $t_1 < t_2 .. < t_m$ be the distinct ordered observation times. It is clear from (5.1.3) that in order to maximize the likelihood function, it is enough to restrict the distribution of $(T_1, J_1=j_1, T_2, J_2=j_2)$ for (j_1, j_2) a permutation of (1,2), to have discrete mass on the observation times $t_1, t_2, \ldots, t_m$ and arbitrary time t_{m+1} much larger than t_m. In otherwords, the " nonparametric maximum likelihood estimate " of the quantities in (5.1.2) will be nonzero at the time points t_1, t_2, t_m, t_{m+1}. For notational convenience let us label these time points by 1,2...m, m+1 respectively. For this restricted class of distributions, the survival function in (5.1.4) can be written as.

$$S(t) = \prod_{l=1}^{t-1} (1-\lambda_1(l)-\lambda_2(l))$$

$$S_{12}(t|u) = \prod_{l=u}^{t-1} (1-\lambda_{12}(l|u))$$

$$S_{21}(t|u) = \prod_{l=u}^{t-1} (1-\lambda_{21}(l|u)) \qquad \ldots\ldots\ldots\ldots \quad (5.1.5)$$

for $u < t = 1,2 \ldots\ldots m+1$ and define $S_{12}(t|t) = S_{21}(t|t) = 1$ for all t.

The likelihood contribution in (5.1.3) is then proportional to ,

$$\sum_{u=1}^{t} \lambda_1(u)S(u)\lambda_{12}(t|u)S_{12}(t|u) \text{ for class type } A_1$$

$$\sum_{u=1}^{t} \lambda_2(u)S(u)\lambda_{21}(t|u)S_{21}(t|u) \text{ for class type } A_2$$

$$\sum_{u=1}^{t} \lambda_1(u)S(u)S_{12}(t+1|u) \text{ for class type } A_3$$

$$\sum_{u=1}^{t} \lambda_2(u)S(u)S_{21}(t+1|u) \text{ for class type } A_4$$

and $S(t+1)$ for class type A_5

where $t = 1,2 \ldots\ldots m$. $\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.1.6)$

Denote the two first two expressions in (5.1.6) by $\Pi_1(t)$ and $\Pi_2(t)$ respectively and write $\alpha_t = \sum_{u \le t} \Pi_1(u)$, $\beta_t = \sum_{u \le t} \Pi_2(u)$. Also denote the third and fourth expression by γ_t and δ_t respectively, so that the fifth expression can be written as $(1-\alpha_t-\beta_t-\gamma_t-\delta_t)$. The relevant partial likelihood can now be written as

$$L_2 \propto \prod_{t=1}^{m} [\{ \alpha_t - \alpha_{t-1} \}^{n_1(t)} \{ \beta_t - \beta_{t-1} \}^{n_2(t)}$$

$$\gamma_t^{n_3(t)} \delta_t^{n_4(t)} (1-\alpha_t-\beta_t-\gamma_t-\delta_t)^{n_5(t)}] \quad ....(5.1.7)$$

where $n_j(l) = n_j(t_l)$ is the number of j th class type observation i.e. observations belonging to class A_j at the time t_l for $j = 1,2 \ldots 5$ and $l = 1,2 \ldots m$. The parameter space is the convex region given by.

$$0 \le \alpha_1 \le \alpha_2 \quad ...... \le \alpha_m \le 1$$
$$0 \le \beta_1 \le \beta_2 \quad .... \le \beta_m \le 1$$
$$0 \le \gamma_t \; , \quad \delta_t \le 1 \; , \quad \text{for } t = 1,2 \ldots m.$$
$$0 \le \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \le \; ..... \; \alpha_m + \beta_m + \gamma_m + \delta_m \le 1$$

The negative of the second derivatives of the log-likelihood function with respect to parameters $(\alpha_t, \beta_t, \gamma_t, \delta_t)$, t=1,2 ....m is obtained from (5.1.7) :

$$H = \begin{bmatrix} UAU' + E & E & E & E \\ E & UBU' + E & E & E \\ E & E & C+E & E \\ E & E & E & D+E \end{bmatrix}$$

where ,

$$U_{m\times m} = \begin{bmatrix} 1 & -1 & 0 \cdots 0 & 0 \\ 0 & 1 & -1 \cdots 0 & 0 \\ \vdots & \vdots & \vdots \quad\; \vdots & \\ 0 & 0 & 0 \cdots 1 & -1 \\ 0 & 0 & 0 \cdots 0 & 1 \end{bmatrix}$$

$$A_{m\times m} = \operatorname{diag}\left\{ \frac{n_1(t)}{(\alpha_t - \alpha_{t-1})^2}, t = 1,2 \ldots\ldots m \right\}$$

$$B_{m\times m} = \operatorname{diag}\left\{ \frac{n_2(t)}{(\beta_t - \beta_{t-1})^2}, t = 1,2 \ldots\ldots m \right\}$$

$$C_{m\times m} = \operatorname{diag}\left\{ \frac{n_3(t)}{\gamma_t^2}, t = 1,2 \ldots\ldots m \right\}$$

$$D_{m\times m} = \operatorname{diag}\left\{ \frac{n_4(t)}{\delta_t^2}, t = 1,2 \ldots\ldots m \right\}$$

$$E_{m\times m} = \operatorname{diag}\left\{ \frac{n_5(t)}{(1-\alpha_t-\beta_t-\gamma_t-\delta_t)^2}, t = 1,2 \ldots\ldots m \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.1.8)$$

Using arguments similar to these in Dewanji *et al* (1986), H can be proved to be strictly positive definite at least in a strictly reduced parameter space which will suffice for our purpose, if $n_3(t) > 0$ and $n_4(t) + n_5(t) > 0$ or $n_4(t) > 0$ and $n_3(t) + n_5(t) > 0$.

This holds if at least any two of $n_3(t)$, $n_4(t)$ and $n_5(t)$ are positive for all $t = 1,2 \ldots\ldots m$. This extends the identifiability result of Dewanji *et al* (1986). Following their proof, it is now easy to see that the likelihood function (5.1.7) has a unique maximum under the above sufficient condition, which may occur in the boundary of the parameter space, i.e., the estimates $(\tilde{\alpha}_t, \tilde{\beta}_t, \tilde{\gamma}_t, \tilde{\delta}_t,\ t = 1,2 \ldots m)$ obtained by maximizing the likelihood function are unique. In other words it may be noted that in general there is a class C of maximum likelihood estimators in the original parameter space Λ of

$$\underset{\sim}{\lambda} = \{ \lambda_1(u),\ \lambda_2(u),\ \lambda_{12}(t|u),\ \lambda_{21}(t|u),\ \text{for } t \geq u = 1,2 \ldots m \}$$

for which $\alpha_t = \tilde{\alpha}_t$, $\beta_t = \tilde{\beta}_t$, $\gamma_t = \tilde{\gamma}_t$, $\delta_t = \tilde{\delta}_t$ for $t=1,2 \ldots m$. Thus the only identifiable quantities are $\{ \alpha_t, \beta_t, \gamma_t, \delta_t,\ t = 1,2 \ldots m \}$ and the estimates of $\underset{\sim}{\lambda}$ may not be unique.

By noting that,

$$\gamma_t = \lambda_1(t)S(t) + \gamma_{t-1} - (\alpha_t - \alpha_{t-1}) \quad \text{and} \quad \text{that}$$

$$S(t) = 1 - \alpha_{t-1} - \beta_{t-1} - \gamma_{t-1} - \delta_{t-1},$$

We see that $\lambda_1(t)$ is uniquely estimable. Similarly by considering δ_t, one can verify the unique estimability of $\lambda_2(t)$ as well. Thus all the elements of C will have same values of $\lambda_1(t)$ and $\lambda_2(t)$ for $t = 1,2 \ldots m$. According to the result of Meilijson (1981), identifiability for the component life time distributions for a parallel system fails because, as noted in that paper, his incidence matrix M (Meilijson's notation) will have two identical columns. The data on censored observations makes up for that by introducing new rows in M which makes it full rank, a requirement for identifiability. This explains the importance of the censored observations for identifiability in the present model.

For the estimability of conditional hazards, one needs further modelling. As in Dewanji *et al* (1986), one may consider the Markov model given by

$$\lambda_{12}(t|u) = \lambda_{12}(t)$$

$$\lambda_{21}(t|u) = \lambda_{21}(t) \qquad \text{.............(5.1.9)}$$

or the Semi-Markov-model given by .

$$\lambda_{12}(t|u) = \lambda_{12}(t-u)$$

$$\lambda_{21}(t|u) = \lambda_{21}(t-u) \qquad \text{...........(5.1.10)}$$

for $u \le t = 1,2 \ldots m$.

The unique estimability of the conditional hazards for these two models is easy to verify .

5.1.2 Application of EM Algorithm

In order to use the EM algorithm (Dempster *et al* (1977), we consider the following complete data representation.

In addition to the observations in (5.1.1), suppose the failure times T_1 (X_1 or X_2) is also observable when it did occur, i.e., for the first four observation types, we also observe the corresponding first failure time. Then the likelihood contribution from this complete version can be written in terms of the discrete version of the distribution as :

$\lambda_1(u)S(u)\lambda_{12}(t|u)S_{12}(t|u)$ for class type A_1

$\lambda_2(u)S(u)\lambda_{21}(t|u)S_{21}(t|u)$ for class type A_2

$\lambda_1(u)S(u)S_{12}(t+1|u)$ for class type A_3

$\lambda_2(u)S(u)S_{21}(t+1|u)$ for class type A_4

and $S(t+1)$ for class type A_5 ....(5.1.11)

for the 5 different class types, where u refers to the first failure point of time $u \le t$, $t=1,2 \ldots m$.

Let $n_1^c(t|u)$ = No of observations with $T_1 = u$, $J_1 = 1$, $T_2 = t$, $J_2 = 2$

Let $n_2^c(t|u)$ = No of observations with $T_1 = u$, $J_1 = 2$, $T_2 = t$, $J_2 = 1$

Let $n_3^c(t|u)$ = No of observations with $T_1 = u$, $J_1 = 1$, $T_2 > t$

Let $n_4^c(t|u)$ = No of observations with $T_1 = u$, $J_1 = 2$, $T_2 > t$

for $u \le t$, $t = 1,2 \ldots m$.

Also write $n_j(t) = \sum_{u \le t} n_j^c(t|u)$, $J=1,2,3,4$ and

$n_j^c(u)$= No of observations with ($T_1=u$, $J_1=j$) for u = 1,2 ..m and

j = 1,2. The superscript "c" in $n_j^c(u)$ stands for the complete unobserved frequency.

Note that
$$n_1^c(u) = \sum_{t \ge u} (n_1^c(t|u) + n_3^c(t|u))$$
$$n_2^c(u) = \sum_{t \ge u} (n_2^c(t|u) + n_4^c(t|u)) \qquad \ldots\ldots\ldots\ldots\ldots(5.1.12)$$

Then using (5.1.7) complete data likelihood can be written as

$$L_2 = \prod_{u=1}^{m} \Big\{ [\{\lambda_1(u)S(u)\}^{n_1^c(u)} \{\lambda_2(u)S(u)\}^{n_2^c(u)} \{S(u+1)\}^{n_5^c(u)}] \; *$$
$$\prod_{t \ge u} [\{\lambda_{12}(t|u)S_{12}(t|u)\}^{n_1^c(t|u)} \{\lambda_{21}(t|u)S_{21}(t|u)\}^{n_2^c(t|u)}$$
$$\{S_{12}(t+1|u)\}^{n_3^c(t|u)} \{S_{21}(t+1|u)\}^{n_4^c(t|u)}] \Big\}$$

$$= \prod_{u=1}^{m} \Big\{ [\{\lambda_1(u)\}^{n_1^c(u)} \lambda_2(u)\}^{n_2^c(u)} \{1-\lambda_1(u)-\lambda_2(u)\}^{r^c(u) - n_1^c(u) - n_2^c(u)}]$$
$$\prod_{t \ge u} [\{\lambda_{12}(t|u)\}^{n_1^c(t|u)} \{1-\lambda_{12}(t|u)\}^{r_1^c(t|u) - n_1^c(t|u)}$$
$$\{\lambda_{21}(t|u)\}^{n_2^c(t|u)} \{1-\lambda_{21}(t|u)\}^{r_2^c(t|u) - n_2^c(t|u)} \Big\} \qquad \ldots\ldots\ldots\ldots\ldots(5.1.13)$$

where $\quad r^{c}(u) = \sum_{l \geq u} (n_1^c(l) + n_2^c(l) + n_5(l))$

$$r_1^c(t|u) = \sum_{l \geq t} (n_1^c(l|u) + n_3^c(l|u))$$

$$r_2^c(t|u) = \sum_{l \geq t} (n_2^c(l|u) + n_4^c(l|u)) \qquad \ldots\ldots\ldots\ldots(5.1.14)$$

note that the conditional distribution of $\{ n_j^c(t|u),\ u=1,2 \ldots . t\}$ given $n_j(t)$ is multinomial with parameters $\{n_j(t),\ \Pi_j(t|u)),\ u = 1,2 \ldots . t\}$, for $j = 1,2.3,4$ where

$$\Pi_1(t|u) = \frac{\lambda_1(u)S(u)\lambda_{12}(t|u)S_{12}(t|u)}{\sum_{l \leq t} \lambda_1(l)S(l)\lambda_{12}(t|l)S_{12}(t|l)}$$

$$\Pi_2(t|u) = \frac{\lambda_2(u)S(u)\lambda_{21}(t|u)S_{21}(t|u)}{\sum_{l \leq t} \lambda_2(l)S(l)\lambda_{21}(t|l)S_{21}(t|l)}$$

$$\Pi_3(t|u) = \frac{\lambda_1(u)S(u)S_{12}(t+1|u)}{\sum_{l \leq t} \lambda_1(l)S(l)S_{12}(t+1|l)}$$

$$\Pi_4(t|u) = \frac{\lambda_2(u)S(u)S_{21}(t+1|u)}{\sum_{l \leq t} \lambda_2(l)S(l)S_{21}(t+1|l)}$$

$$u \leq t = 1,2 \ldots m. \qquad \ldots\ldots\ldots\ldots(5.1.15)$$

Thus, given an initial estimate $\underset{\sim}{\lambda}^{o}$ of $\underset{\sim}{\lambda}$, we have conditional expectation of $n_j^c(t|u)$ given the observed data as .

$$E(n_j^c(t|u) \mid n_j(t)) = n_j(t)\overset{o}{\Pi}_j(t|u) = n_j^{c(o)}(t|u) \text{ say } j=1,2,3,4, \qquad \ldots\ldots\ldots\ldots(5.1.16)$$

for $t \geq u = 1,2 \ldots m$, where $\overset{o}{\Pi}_j(t|u)$ is $\Pi_j(t|u)$ evaluated at $\underset{\sim}{\lambda} = \underset{\sim}{\lambda}^{o}$.

This is the Estep of the EM algorithm, where we are required to compute $E(\text{Log } L_2 \mid \text{observed data}, \underset{\sim}{\lambda} = \underset{\sim}{\lambda}^{o})$. In M step we need to maximize this quantity with respect to $\underset{\sim}{\lambda}$, which from (5.1.13), leads to improved estimate as

$$\hat{\lambda}_j(t|u) = \frac{n_j^{c(o)}}{r^{c(o)}(u)} , \qquad j = 1,2$$

$$\hat{\lambda}_{12}(t|u) = \frac{n_1^{c(o)}(t|u)}{r_1^{c(o)}(t|u)}$$

$$\hat{\lambda}_{21}(t|u) = \frac{n_2^{c(o)}(t|u)}{r_2^{c(o)}(t|u)} \qquad \text{for } u \le t \le m. , \qquad \text{.................(5.1.17)}$$

where the additional superscript "(o)" along with c means the conditional expectation of the corresponding quantities given the observed date and $\underset{\sim}{\lambda} = \underset{\sim}{\lambda}^{o}$ obtained by using (5.1.16), (5.1.12) and (5.1.14).

These improved estimates are then used again in the E step, and the same steps are repeated until convergence.

Using Theorem 1 of Dempster *et al* (1977) and the fact that the observed likelihood has a unique maximum, one can argue that the EM algorithm described above converges to the unique maximum (Wu, 1983)

, that is, the algorithm converges to a fixed point in the convex region in (5.1.17).

under suitable Markov or semi-Markov assumption regarding conditional hazards.

The component life time distribution may be expressed in terms of cause specific hozard rates, $\lambda_j(\ )$ $j = 1, 2$ and conditional hazard rates $\lambda_{j_1,j_2}(t|u), u \le t$ $(j_1,j_2)=(1,2)$ or $(2,1)$.

It may be of interest to provide estimates of the cumulative hazards $\Lambda_j(t) = \sum_{u=1}^{t} \lambda_j(u)$, $j=1,2$ ....(5.1.18) and from them , the estimates of the component life survival functions upto first failure time follow easily and other quantities of interest can similarly be computed.

5.1.3 Variance Estimation

The variance estimates for estimable quantities can be obtained from the observed information matrix and for that purpose, we assume the Markov model (5.1.9) in which full parameter vector $\underset{\sim}{\lambda}_M = (\lambda_1(u), \lambda_2(u), \lambda_{12}(u), \lambda_{21}(u), u=1,2 \ldots m)$ is uniquely estimable. The observed information in the case of complete data using the results of Louis (1982) can be obtained by the method of Dewanji *et al* (1986). Since the data have grouped multinomial structure , this requires calculation of within sum of sqares for each combination of j and t, based on the estimated complete scores $\hat{S}_j(u,t)$ which corresponds to the score contribution of j class type observation at time t with $T_1 = u$ and is evaluated at the maximum likelihood estimate $\underset{\sim}{\lambda}_M$, $j=1,2,3,4$, $u \le t$, $t=1,2,\ldots m$. The forms of these scores are very similar to those in Dewanji *et al* (1986) and easy to obtain. One can verify that the observed information matrix is.

$$I = I^{c}(\underset{\sim}{\tilde{\lambda}}_M) - \sum_{j=1}^{4} \sum_{t=1}^{m} \sum_{u \le t} \hat{n}_j^c(t|u) \, (\hat{S}_j(u,t) - \hat{S}_j(t))(\hat{S}_j(u,t) - \hat{S}_j(t))',$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(5.1.19)$$

where $\hat{n}_j^c(t|u)$ is the imputed frequency of j class type observations at time t with $T_1 = u$ (this is $n_j(t)\ \Pi_j(t|u)$)

evaluated at $\underset{\sim}{\lambda}_M = \underset{\sim}{\lambda}_M^{\sim}$ as obtained in the last step of the EM algorithm.

$$\hat{S}_j(t) = \frac{1}{n_j(t)} \sum_{u \le t} \hat{n}_j^c(t|u)\hat{S}_j(u,t) \text{ for } j=1,2,3,4 \quad .....(5.1.20)$$

and $I^c(\underset{\sim}{\tilde{\lambda}}_m)$ is the observed information matrix from the complete data likelihood (5.1.13) using imputed frequencies. The expressions are easy to obtain and details are avoided for the sake of brevity. Variance estimates for the quantities of interest can now be obtained by delta method.

5.1.4 Illustration with simulated data

In order to illustrate the estimation procedure, described in the preceding sections, we consider a simulated data set for which we take a simple exponential model with

$$\lambda_1(t) = 0.014, \quad \lambda_2(t) = 0.016 \ , \ \forall\ t$$

$$\text{and} \quad \lambda_{12}(t|u) = \lambda_{21}(t|u) = 0.03\ \forall\ t,\ u.$$

Each individual system is subjected to a random censoring plan with prabability of censoring q_j at time 10j ,for j= 1,2 ... 10. For our simulation we take q_j's as 0.08, 0.08, 0.08, 0.08, 0.10, 0,10, 0.10,0.12,0.12 and 0.14 respectively , so that 100 is the terminal censoring time. The observations on the lifetime of the systems are grouped in the intervals (10(j−1), 10j), for j=1,2 ..... 10. Table 5.1.4.1 presents the simulated data for 500 individual systems.

Table 5.4.1

Frequency distribution of the simulated data

t	Observation class type 1	2	3	4	5
0 - 10	16	8	1	4	24
10 - 20	21	20	4	4	21
30 - 40	20	24	2	13	15
40 - 50	20	20	10	7	7
50 - 60	16	14	2	8	6
60 - 70	10	9	6	5	10
70 - 80	5	8	5	9	13
80 - 90	2	4	2	8	5
90 - 100	6	3	3	5	2

It is to be noted that the hazard rates $\lambda_1(.)$ and $\lambda_2(.)$ are estimable at the censoring times along with other time points when system failure takes place (section 5.1.2) with groupings used, we allow $\lambda_1(.)$ and $\lambda_2(.)$ to be non zero only at the group boundary points. This is so under the simiplifying assumption that any failure occuring within a group interval occures at the end of the interval. The resulting estimates from this simulated data set is to be viewed as estimates for the corresponding interval hazards, as argued in Dewanji *et al* (1986). In this case, for example, the true interval hazard rate of component 1 to the first failure for ith interval is constant and given by

$$\frac{0.014}{0.014 + 0.016} (1 - e^{-(0.014 + 0.016)*10})$$

(Kalbfleisch and Prentice (1980)). The above data set is analyzed by using the Markov model (5.1.9). Note that the estimates $\lambda_1(.)$ and $\lambda_2(.)$ are also the same assuming the general model (5.1.2). This has been verified in this example. The estimates of cumulative hazards (5.1.8) are presented in Table 5.1.4.2 along with corresponding standard errors in parentheses. The true values of the corresponding cumulative hazards are also presented for comparison.

Table 5.1.4.2

Estimates of the cumulative hazards for the data in Table 5.1.4.1

t	$\hat{A}_1(t)$	$\hat{A}_2(t)$	$A_1(t)$	$A_2(t)$
0 - 10	0.0648(0.0331)	0.1472(0.0611)	0.1210	0.1382
10 - 20	0.2312(0.0726)	0.1863(0.0725)	0.2419	0.2765
20 - 30	0.3529(0.0649)	0.4056(0.1076)	0.3629	0.4147
30 - 40	0.3529(0.0702)	0.6556(0.1335)	0.4838	0.5529
40 - 50	0.7871(0.1721)	0.6558(0.1909)	0.6048	0.6912
50 - 60	0.7871(0.3145)	0.6816(0.2460)	0.7257	0.8294
60 - 70	0.8029(0.2343)	0.61816(0.2415)	0.8467	0.9676
70 - 80	0.8029(0.2147)	0.8039(0.2400)	0.9676	1.1058
80 - 90	0.8029(0.3833)	1.1515(0.2460)	1.0886	1.2411
90 - 100	1.5378(0.6266)	1.1515(0.9269)	1.2095	1.3828

Remarks :

1\. It may appear from the illustrative example because of grouping of data, that the only interval hazards derived from the original model are estimable. This is not so in principle. The method can be applied to ungrouped data as well to obtain pointwise hazard estimates. But this will mean a large number of parameters and hence require a large number of censored observations for identifiability. This will make the large sample results inappropriate for use.

In addition to the Markov or Semi-Markov model given by (5.1.9) and (5.1.10) respectively, one can think of other models based on intutive appeal. For example, when the two components are believed to have identical life distributions, one can assume,

$$\lambda_1(t) = \lambda_2(t) = \lambda(t),$$

$$\lambda_{12}(t|u) = \lambda_{21}(t|u) = \lambda^*(t|u)$$

which also states that the conditional hazards for T_2 given T_1 is different from the hazard of T_1 because of possibly the extra load on the single surviving component. One can think of further modelling $\lambda^*(t|u)$ in Markov or Semi-Markov fashion, like

$$\lambda^*(t|u) = \lambda_M(t) \text{ or } \lambda_{SM}(t-u)$$

respectively. If X_1 and X_2 are believed to be independent, then one can assume

$$\lambda_{21}(t|u) = \lambda_1(t), \quad \lambda_{12}(t|u) = \lambda_2(t)$$

which is a necessary condition for independence. The method described in the section 5.1 is applicable to these different models as well.

2.

It may be recalled that the problem of estimation of component life survival functions in the general framework of a coherent structure has received attention of various authors including Doss *et al* (1989) and voelkel *et al* (1984). In all these papers, continuous monitoring of the system is assumed. To the best of our knowledge, our data set is different and the available methods do not seem to be applicable to the problems considered here.

5.2 k-1 OUT OF k SYSTEM ($k \geq 2$)

5.2.1. Preliminaries of Estimation Procedure.

The method of section 5.1 can possibly be extended to more general coherent systems of interest. Further assumption like independence of life distribution of components may become helpful in some cases. But in any case difficulty in treatment increases with the complexity of structure. But it is noted with satisfaction that the method developed in section. 5.1 can be extended without much difficulty to a more general class of coherent systems, viz, k-1 out of k components system $\forall\ k \geq 2$.

In a k-1 out of k components system the system functions if and only if at least k-1 of its components function, i.e. the system fails if two of its components fail, one after another. Let us call them the first and second failures. Let us recall what was introduced in chapter 4 in the same connection. X_i represents the life of the ith component, $i = 1,2 \ldots\ldots k$ and X_c denotes the time of censoring. What we observe in reality is a realized value, Z which represents the observed life of the system, censored or otherwise and a value of the indicator variable

$$d = (I_{12}, I_{13}, \ldots I_{1k}, \ldots I_{k1}, \ldots I_{k-1\,k}, I_{1c}, \ldots I_{kc}, I_{co})$$

which represents a multinomial distribution with (k^2+1) classes determined by the relationships between the variables X_1, X_2 .... X_k and X_c. We observe Z = the life, censored or otherwise on n such systems along with the identity of the two components whose failure has led to the system failure, when the system is uncensored and if the life time is censored, we note which, of any of the components failed before censoring. This is what comprises the data set. The classes for d are identified as

$$A_{ij} = \{Z \in R^+ \mid X_i < X_j = Z,\ \mathrm{Min}(X_u,\ u \in S - \{i,j\}, X_c) > Z\}\quad i \neq j,\ i,j \in S.$$

$$A_{1c} = \{Z \in R^+ \mid X_i < X_c = Z,\ \mathrm{Min}(X_u, u \in S - \{i\}) > Z\},\ i \in S.$$

$$A_{co} = \{Z \in R^+ \mid X_c = Z, \mathrm{Min}(X_u, u \in S) > Z\} \qquad (5.2.1)$$

Here $S = \{1,2, \ldots k\}$ as in section 4.2 of chapter 4. Based on the observations (Z_l, d_l), where $d_l = (I_{12l}, I_{13l}, \ldots I_{1kl}, I_{21l}, \ldots I_{k-1\,k,l} \ldots I_{kcl}, I_{col})$, $l = 1,2 \ldots n$, the primary interest lies in estimating the distribution functions or survival functions of $X_1, X_2, \ldots X_k$.

5.2.2. Formulation of the Problem, and Solution Procedure

Once we attempt to formulate the problem in terms of the first and second failures, the framework is a very straight forward generalization of the one out of two system in section 5.1 and exactly the same approach and same procedure applies here. Here too, until the first failure takes place, k lifetimes of the components constitute a competing risk framework allowing for dependence between them. After the first failure, the living components carry on possibly with a different life time distribution (because of the additional stress possibly) depending on the first failure time (unobserved). Let us denote the cause specific hazard rates as in section 5.1 by

$$\lambda_j(t) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} P(T_1 \in [t, t+\Delta t), J_1 = j \mid T_1 \geq t), \ j \in \mathcal{S}$$

$$\lambda_{ij}(t|u) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} P(T_2 \in [t, t+\Delta t), J_2 = j \mid T_2 \geq t, T_1 = u, J_1 = i)$$

$$\forall \ i,j \in \mathcal{S} \text{ and } i \neq j, \qquad \text{.............(5.2.2)}$$

where

(1) T_i = ith failure time, i=1,2, and (2) J_i = component which fails at ith failure and (J_1, J_2) indicates one of the ${}^kP_2$ permutations of the symbols in $\mathcal{S} = \{1,2 \ldots k\}$.

$\lambda_j(t)$ for $j \in \mathcal{S}$ is the cause specific hazard rate of T_1 for the jth component and $\lambda_{ij}(t|u)$ is the cause specific conditional hazard rate of T_2 given T_1 with $J_2 = j$ and $J_1 = i \ \forall \ i,j \in \mathcal{S}$ and $i \neq j$, for $t \geq u > 0$. Under independent censoring mechanism as in section 5.1, the partial likelihood contribution from the (k^2+1) possible observational class types at any time t is proportional to

(i) $$L_{ij}(.) = \int_0^t \lambda_i(u) S(u) \lambda_{ij}(t|u) S_{i(.)}(t|u) \qquad \text{.....(5.2.3)}$$

for an observation belonging to class type A_{ij}, $i \neq j$, $i,j \in \mathcal{S}$

(ii) $$L_{ic}(.) = \int_0^t \lambda_i(u) S(u) S_{i(.)}(t^+|u) \qquad \text{.............(5.2.4)}$$

for an observation belonging to class type A_{ic}.

(iii) $$L_{co}(.) = S(t^+) \qquad \text{.............(5.2.5)}$$

for an observation belonging to class type A_{co}.

where ,

$$S(t) = P(T_1 \geq t) = \operatorname{Exp}\left(-\int_0^t \sum_{j \in \mathcal{S}} \lambda_j(u) du\right)$$

$$S_{i(.)}(t|u) = P(T_2 \geq t | T_1 = u, j_1 = i)$$

$$= \mathrm{Exp}\left(-\int_u^t \sum_{j \in \mathcal{S}-\{i\}} \lambda_{ij}(v|u)dv\right) \quad \ldots\ldots(5.2.6)$$

consider n observation times Z_1, $1 = 1,2 \ldots n$ as in section 5.1 and order them. Let $t_1 < t_2 \ldots\ldots < t_m$ be the distinct ordered observation times. It is clear from (5.2.5) that in order to maximize the likelihood function, it is enough to restrict the distribution of $(T_1, J_1 = j_1, T_2, J_2 = j_2)$ for $J_1 \neq J_2$, $J_1, J_2 \in \mathcal{S}$ to have discrete masses on the observation times $t_1, t_2 \ldots\ldots t_m$ and an arbitrary time t_{m+1} larger than t_m. In other words, the "nonparametric maximum likelihood estimate" of the quantities in (5.2.2) will be non zero at the time points $t_1, t_2 \ldots\ldots t_{m+1}$. For notational convenience let us label the time points by $1,2 \ldots m+1$ respectively. For this restricted class of distributions the survival function in (5.2.6) can be written as

$$S(t) = \prod_{1=1}^{t-1} \left(1 - \sum_{j \in \mathcal{S}} \lambda_j(1)\right)$$

$$S_{i(.)}(t|u) = \prod_{1=u}^{t-1} \left(1 - \sum_{j \in \mathcal{S}-\{i\}} \lambda_{ij}(1|u)\right) \quad \forall\ i \in \mathcal{S}$$

for $u < t = 1,2, \ldots\ldots m+1$ and $S_{i(.)}(t|t) = 1$, $\forall\ i \in \mathcal{S}$ $\quad\ldots\ldots\ldots\ldots(5.2.7)$

Proceeding as in section 5.1 and writing,

$$\Pi_{ij}(t) = \sum_{u=1}^{t} \lambda_i(u) S(u) \lambda_{ij}(t|u) S_{i(.)}(t|u) \quad i \neq j, \quad i,j \in \mathcal{S}$$

$$\gamma_{ict} = \sum_{u=1}^{t} \lambda_i(u) S(u) S_{i(.)}(t+1|u), \ i \in \mathcal{S} \quad \ldots\ldots(5.2.8)$$

and writing $\alpha_{ijt} = \sum_{u \leq t} \Pi_{ij}(u)$, one can show that

$S(t+1) = 1 - \sum_{\substack{i,j \in \mathcal{S} \\ i \neq j}} \alpha_{ijt} - \sum_{i \in s} \gamma_{ict}$. The likelihood is

$$L_k \propto \prod_{t=1}^{m} \Big[\prod_{\substack{i,j \in \mathcal{S} \\ i \neq j}} (\alpha_{ijt} - \alpha_{ij,t-1})^{n_{ij}(t)}$$

$$\prod_{i \in \mathcal{S}} \gamma_{ict}^{\,n_{ic}(t)} \Big(1 - \sum_{i,j \in \mathcal{S}} \alpha_{ijt} - \sum_{i \in \mathcal{S}} \gamma_{ict}\Big)^{n_{co}(t)} \Big] \quad \ldots .(5.2.9)$$

where $n_{ij}(t_1)$ = No of observation belonging to class A_{ij} among whom the value of Z, i.e. observed life, censored or otherwise is

t_1, $(i,j) \in \mathcal{S}^* = \{(i,j) \in \mathcal{S}\ i \neq j\} \cup \{(i\ c),\ i \in \mathcal{S}\} \cup \{co\}$

Then proceeding exactly as in section 5.1, identifiability can be established under exactly similar conditions i.e. $\lambda_i(t)$'s are uniquely estimable always and $\lambda_{ij}(t|u)$'s are uniquely estimable under Markovian , Semi Markovian or other reasonable assumptions cited in section 5.1. Also the application of EM algorithm centred on the unobserved first failure time (if there is a first failure) follows exactly in similar lines, for which further elaborations are considered redundant. Hence the detailed expressions are not included in the present section for the general k-1 out of k components system to avoid repetition.

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