Indian Statistical Institute

Semestral Examination

May 8, 2025

Total points: 35 *Brownian Motion and Diffusions, M2* Time: 3 hours

Note: This is a **closed notes/closed book** examination. Notations, if not explicitly explained, are to be interpreted as defined in class.

Unless otherwise stated, $B = (B_t)_{t\geq 0}$ denotes a standard Brownian motion defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

1. Let $(\mathcal{F}_t^B)_{t\geq 0}$ denote the natural filtration corresponding to B. $\boxed{5+3+2+3}$
   (a) Let $a < 0 < b$. Show that $\tau' := \inf\{t \geq 0 : B_t \notin (a,b)\}$ is a stopping time with respect to the filtration $(\mathcal{F}_t^B)_{t\geq 0}$ but $\tau := \inf\{t \geq 0 : B_t \notin [a,b]\}$ is not.
   (b) Prove that τ is a stopping time with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$, where $\mathcal{G}_t = \cap_{s>0}\mathcal{F}_{t+s}^B$.
   (c) Show that B_τ is measurable with respect to stopped σ-algebra $\mathcal{G}_\tau$.
   (d) Is it true that $\mathcal{G}_\tau = \cap_{s>0}\mathcal{F}_{\tau+s}^B$?

2. Let $\mathbb{P}^x$ denote the law of a Brownian motion B started at x. For $c \in \mathbb{R}$, let $\tau_c := \inf\{t \geq 0 \mid B_t = c\}$. $\boxed{2+3+3}$
   (a) Let $\alpha, \beta \in \mathbb{R}$. Show that there is a unique continuous function $h : [0,1] \to \mathbb{R}$ such that $h''(x) = 0$ for all $x \in (0,1)$, and $h(0) = \alpha$, $h(1) = \beta$. Find an explicit formula for h.
   (b) Using part (a), find an explicit formula for $\mathbb{P}^x(\tau_0 < \tau_1)$.
   (c) Using the explicit formula derived in part (b), find $\mathbb{P}^0(\tau_{-a} < \tau_b)$, where $a, b > 0$.

3. Let $X : [0,1] \times \Omega \to \mathbb{R}$ be a progressively measurable process (you may assume that the underlying filtration satisfies the usual conditions) with $\mathbb{E}\int_0^1 X_s^2\,ds < \infty$. Suppose that $\lim_{t\to 0} X_t = X_0$ in $L^2(\mathbb{P})$. Prove that $\boxed{6}$

$$\frac{1}{B_t}\int_0^t X_s\,dB_s \xrightarrow{\mathbb{P}} X_0 \quad \text{as } t \to 0.$$

4. (a) Using Itô's formula prove that $\left(\frac{1}{\sqrt{1+t}}e^{\frac{B_t^2}{2(1+t)}}\right)_{t\geq 0}$ is a martingale. $\boxed{2+(1+3+2)}$
   (b) Consider the SDE

$$dX_t = (4 - 3X_t)\,dt + 2\,dB_t, \qquad (\star)$$

   where $X_0 \sim N(0,1)$ and is independent of $(B_t)_{t\geq 0}$. Show that $(\star)$ has a unique strong solution $(X_t)_{t\geq 0}$. Derive an explicit expression for X_t, which does not involve any stochastic integrals. Show that $(X_t)_{t\geq 0}$ is a Gaussian process and compute its covariance kernel.