INDIAN STATISTICAL INSTITUTE

1st Semester Semestral Examination (2025–2026)

M. MATH. II

Advanced Functional Analysis

Date : 19.11.2025 Maximum Marks : 100 Time : $3\frac{1}{2}$ hrs.

This paper carries 110 marks. Maximum you can score is 100.

Precisely justify all your steps. Carefully state all the results you are using.

Notation : For a a Banach space X, we denote by X^* its dual, by $B(X)$ the *closed unit ball* $\{x \in X : \|x\| \leq 1\}$ and by $S(X)$ the *unit sphere* $\{x \in X : \|x\| = 1\}$. We identify $x \in X$ with its canonical image in X^{**}.

1. (a) Let X, Y be Banach spaces. Define a compact operator $T : X \to Y$. [5]

   (b) Let X, Y be Banach spaces with X reflexive. Let $T : X \to Y$ be a compact operator. Show that $T(B(X))$ is norm compact. [10]

2. (a) Let X be a Banach space. Define a denting point of a closed bounded convex set $K \subseteq X$. [5]

   (b) Show that a denting point is always an extreme point. [10]

   (c) Show that the converse holds if K is compact. [10]

   (d) Let X be a Hilbert space. Show that any $x \in S(X)$ is a denting point of $B(X)$. [10]

3. For a Banach space X, show that the following are equivalent : [20]

   (a) X has the RNP.

   (b) For every nonempty closed, bounded and convex $K \subseteq X$, $K = \overline{co}(dent(K))$, where $dent(K)$ denotes the set of denting points of K.

   (c) For every nonempty closed, bounded and convex $K \subseteq X$, $dent(K) \neq \emptyset$.

4. Let (Ω, Σ, μ) be a probability space and X^* be a dual Banach space. Show that $f : \Omega \to X^*$ is μ-measurable if and only if f is almost separably valued and for each $x \in X$, $x \circ f$ is measurable. [15]

5. (a) If each separable subspace Y of X has Y^* separable, show that X^* has the RNP. [15]

   (b) Show that the space c_0 is not isomorphic to the dual of any Banach space. [10]