

INDIAN STATISTICAL INSTITUTE

Semestral Examination: (2025–2026)

M. Stat 2nd Year

Statistical Computing I

Date: ~~27/11/2024~~ ²⁰ Marks: 100. Duration: 3 hours.

Attempt all questions

1. (a) Provide details of an example, with suitable mathematical justifications, where the usual EM algorithm strategy fails. [15]
- (c) Given univariate, real-valued observations y_1, \dots, y_n and real-valued differentiable regression functions $\mu_1(\theta), \dots, \mu_n(\theta)$, where θ is a real-valued parameter, consider the least absolute deviation regression. Show that the iteratively re-weighted least squares method for estimating θ coincides with an MM algorithm. [10]

2. Let V be the collection of functions f with $f'' \in L^2[0, 1]$. Consider the subspace

$$W_2^0 = \{f(x) \in V : f, f' \text{ absolutely continuous and } f(0) = f'(0) = 0\}.$$

Define an inner product on W_2^0 as

$$\langle f, g \rangle = \int_0^1 f''(t)g''(t)dt.$$

- (i) Show that W_2^0 is a reproducing kernel Hilbert space. [15]
 - (ii) Hence, obtain the reproducing kernel of W_2^0 . [10]
3. Consider the generalized accept-reject method: At iteration n ($n \geq 1$)
 1. Generate $X_n \sim g_n$ and $U_n \sim Uniform(0, 1)$, independently.
 2. If $U_n \leq \theta_n f(X_n)/g_n(X_n)$, accept $X_n \sim f$.
 3. Otherwise, move to iteration $n + 1$.

Let Z denote the random variable that is output by this algorithm.

(a) Show that Z has the distribution function

$$P(Z \leq z) = \sum_{n=1}^{\infty} p_n \int_{-\infty}^z f(x) dx$$

where $p_1 = \theta_1$ and $p_n = \theta_n \prod_{m=1}^{n-1} (1 - \theta_m)$ for $n = 2, 3, \dots$ [5]

(b) Show that

$$\sum_{n=1}^{\infty} p_n = 1 \text{ if and only if } \sum_{n=1}^{\infty} \log(1 - \theta_n) \text{ diverges.}$$

[10]

(c) Give one example of a sequence $\{\theta_n\}$ that satisfies $\sum_{n=1}^{\infty} p_n = 1$ and one example of a sequence $\{\theta_n\}$ that does not satisfy it.

[10]

4. Consider a general independence Metropolis sampler with target density $\pi(x)$ and proposal density $p(x)$. Suppose that both these densities are continuous and strictly positive on the support \mathcal{X} of $\pi(x)$. Assume that there exists a constant $M > 0$ such that $\pi(x)/p(x) < M$ for all $x \in \mathcal{X}$. Then show that

$$\|P^n(x, \cdot) - \pi(\cdot)\| \leq \left(1 - \frac{1}{M}\right)^n.$$

where $P^n(x, \cdot)$ is the n -th step Markov transition kernel started at x , and $\|\cdot\|$ is the total variation norm. [25]