INDIAN STATISTICAL INSTITUTE
Semester Examination: 2025-26 (First Semester)

M. MATH. II YEAR
Commutative Algebra I

Date : 28.11.25 Maximum Marks : 60 Duration : 3 Hours

Throughout the paper, k will denote a field and R a commutative ring with unity.

ANSWER ANY FOUR QUESTIONS.
Each question carries 16 marks

1. Examine with justification whether the following statements are true. Answer ANY FOUR.

   (i) There exists an $\mathbb{N}$-graded structure of the ring $R = k[[X]]$ with $R_0 = k$.

   (ii) A finitely generated flat $\mathbb{Z}[X]$-module is projective.

   (iii) $k[X,Y]/(Y^2 - X(X+1))$ is a normal domain.

   (iv) Any maximal ideal of $\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1)$ is generated by at most 3-elements.

   (v) $k[t]$ is a free $k[t^2,t^3]$-module. [4 × 4=16]

2. (i) Let I, P_1, P_2 be ideals of R and P_3 a prime ideal of R such that $I \subseteq P_1 \cup P_2 \cup P_3$. Prove that $I \subseteq P_j$ for some j, $1 \leq j \leq 3$.

   (ii) Give an example of a ring R to show that the hypothesis P_3 is prime is necessary.

   (iii) Prove that if R contains an infinite field then the hypothesis "P_3 is prime" is not required. [8+4+4=16]

3. (i) Let $R = k \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$ be a graded ring. Prove that if R is Noetherian, then R is a finitely generated algebra over k.

   (ii) Find the integral closure of the ring $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{5}]$. [8+8=16]

4. (i) State Noether's Normalisation Lemma.

   (ii) Deduce that if a field L is a finitely generated k-algebra, then L is a finite dimensional vector space over k.

   (iii) Let $A = \mathbb{C}[X,Y,Z]/(X^2+YZ+1, Z^2-XY)$ and let x, y, z respectively denote the images of X, Y, Z in A. Find $\lambda_1, \lambda_2 \in \mathbb{C}$ such that A is integral over $\mathbb{C}[x+\lambda_1 y+\lambda_2 z]$. [4+4+8=16]

5. Let $k \subseteq R \subseteq A = k[X_1, \ldots, X_n]$ be such that A is integral over R.

(i) Prove that there exists a finitely generated k-subalgebra $R_0 \subseteq R$ such that A is integral over R_0.

(ii) Deduce that R is a finitely generated k-algebra.

(iii) Deduce that any k-subalgebra B of $k[X_1]$ is a Noetherian ring.

[6+6+4=16]

6. (i) Suppose that f and g are two polynomials in $k[X, Y]$ such that $\gcd(f, g) = 1$. Show that $\mathcal{Z}(f, g)$ is either $\emptyset$ or a finite set in $\mathbb{A}^2_k$.

(ii) Let $\{f_\alpha \mid \alpha \in \Delta\}$ be a collection of polynomials in $\mathbb{Q}[X_1, \cdots, X_n]$. Show that if there exist complex numbers $z_1, \cdots, z_n$ such that $f_\alpha(z_1, \cdots, z_n) = 0 \ \forall \ \alpha \in \Delta$, then there exist algebraic numbers $a_1, \cdots, a_n$ such that $f_\alpha(a_1, \cdots, a_n) = 0 \ \forall \ \alpha \in \Delta$.

[8+8=16]