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I express my gratitude to *Razi University*, Kermanshah, Iran, *Ministry of Culture and Higher Education*, Tehran, Iran, and *Indian Council for Cultural Relationship*, New Delhi, India for arranging scholarships which enabled me to study at the *Indian Statistical Institute, Delhi.* My thanks also go to Dr. *Ebrahim Hajizadeh*, Head, Science and Education Section, Embassy of Iran, New Delhi and other *staff members* in this section for their complete official support and excellent executive affairs. I am very thankful to Professor *Aloke Dey*, Head, Stat-Math Unit, ISID, for providing me excellent working conditions throughout this study. My thanks also go to other *faculty members* of the Stat-Math Unit of ISID for their invaluable teaching and advice. I would like to thank all my *friends* in ISID hostel and *staff* of ISID for their fruitful cooperation which enabled me do research work for the Ph.D. degree. I am very grateful to Dr. *M. R. Shamsaddini* and Dr. *M. Behbodnia* and their *families* for their company, help and kindness to me and my family during our stay in India.

(Baha-Eldin Khaledi)

B.Khaledi

Stochastic Comparisons and Dependence among Order Statistics, Spacings and Concomitants of Order Statistics

Baha-Eldin Khaledi

Indian Statistical Institute, Delhi centre

Thesis submitted to the Indian Statistical Institute

in partial fulfilment of the requirements for

the award of the degree of **Doctor of Philosophy**

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To my wife, Yeganeh

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Contents

Chapter 1

Introduction and Summary

1.1 Introduction and definitions

The simplest and the most common way of comparing two random variables is through their means and variances. It may happen that in some cases the median of X is larger than the median of Y, while the mean of X is smaller than the mean of Y. However, this confusion will not arise if the random variables are stochastically ordered. Similarly, the same may happen if one would like to compare the variability of X with that of Y based only on numerical measures of variability. Besides, these characteristics of distributions might not exist in some cases. In most cases one can express various forms of knowledge about the underlying distributions in terms of their survival functions, hazard rate functions, mean residual functions, quantile functions and other suitable functions of probability distributions. These methods are much more informative than those based on comparing only few numerical characteristics of distributions. Comparisons of random variables based on such functions usually establish partial orders among them. We call them as stochastic orders.

Stochastic models are usually sufficiently complex in various fields of statistics, particularly in reliability theory. Obtaining bounds and approximations for their characteristics is of practical importance. That is, the approximation of a stochastic model either by a simpler model or by a model with simple constituent components might lead to convenient bounds and approximations for some particular and desired characteristics of the model. The study of changes in the properties of a model, as the constituent components vary, is also of great interest. Accordingly, since the stochastic components of models involve random variables, the topic of stochastic orders and dependence among random variables plays an important role in these areas.

Now we introduce the notation and give some definitions of various types of stochastic orders and dependence among random variables. Throughout this thesis *increasing* means *nondecreasing* and *decreasing* means *nonincreasing*. We assume that expectations are well defined and multiple integrals can be evaluated irrespective of order. Let X and Y be univariate random variables with distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, density functions f and g; and hazard rates r_F $(= f/\overline{F})$ and r_G $(= g/\overline{G})$, respectively.

Stochastic orderings

Definition 1.1.1 *X is said to be stochastically smaller than Y (denoted by $X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all x.*

This is equivalent to saying that $Eg(X) \leq Eg(Y)$ for any increasing function g.

Definition 1.1.2 *X is said to be smaller than Y in hazard rate ordering (denoted by $X \leq_{hr} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in x.*

It is worth noting that $X \leq_{hr} Y$ is equivalent to the inequalities

$$P[X - t > x | X > t] \leq P[Y - t > x | Y > t], \quad \text{for all } x \geq 0 \text{ and } t.$$

In other words, the conditional distributions, given that the random variables are at least of a certain size, are all stochastically ordered (in the standard sense) in the same direction. Thus, if X and Y represent the survival times of different models of an appliance that satisfy this ordering, one model is better (in the sense of stochastic ordering) when the appliances are new, the same appliance is better when both are one month old, and in fact is better no matter how much time has elapsed. It is clearly useful to know when this strong type of stochastic ordering holds since quantities judgements are then easy to make. In case the hazard rates exist, it is easy to see that $X \leq_{hr} Y$, if and only if, $r_G(x) \leq r_F(x)$ for every x. The hazard rate ordering is also known as uniform stochastic ordering in the literature.

Definition 1.1.3 *X is said to be smaller than Y in likelihood ratio ordering (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x.*

Definition 1.1.4 *X is said to be smaller than Y in mean residual life (MRL) ordering (denoted by $X \leq_{mrl} Y$) if $\int_t^{+\infty} \overline{G}(x)dx / \int_t^{+\infty} \overline{F}(x)\,dx$ is increasing in t.*

Note that $X \leq_{mrl} Y \Leftrightarrow \mu_F(x) \leq \mu_G(x)$ for every x, where $\mu_F(x) = E[X - x | X > x]$ denotes the mean residual life function of X. When the supports of X and Y have a common left end-point, we have the following chain of implications among the above stochastic orders :

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y.$$

Also

$$X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y.$$

For more details on stochastic orderings, see Chapter 1 of Shaked and Shanthikumar (1994).

The above notions of stochastic orderings between X and Y are based only on their marginal distributions and they ignore the dependence information contained in their joint distribution. These may not be appropriate when there is a dependence between them. When confronted with the problem of comparing dependent variables X and Y, Shanthikumar and Yao (1991) introduced the following criteria. Let

$G_{st} = \{g : I\!R^2 \to I\!R : g(x,y) - g(y,x) \text{ increasing in } x \ \forall y\}$,

$G_{hr} = \{g : I\!R^2 \to I\!R : g(x,y) - g(y,x) \text{ increasing in } x \ \forall y \leq x\}$,

$G_{lr} = \{g : I\!R^2 \to I\!R : g(x,y) \geq g(y,x) \ \ \forall y \leq x\}$.

Definition 1.1.5 *X is said to be smaller than Y according to*

(*i*) *joint stochastic ordering (denoted by $X \overset{st:j}{\preceq} Y$) if*

$$E[g(X,Y)] \leq E[g(Y,X)], \tag{1.1.1}$$

for all $g \in G_{st}$;

(*ii*) *joint hazard rate ordering (denoted by $X \overset{hr:j}{\preceq} Y$) if* (1.1.1) *holds for all $g \in G_{hr}$;*

(*iii*) *joint likelihood ratio ordering (denoted by $X \overset{lr:j}{\preceq} Y$) if* (1.1.1) *holds for all $g \in G_{lr}$.*

We have the following chain of implications : $X \overset{lr:j}{\preceq} Y \Rightarrow X \overset{hr:j}{\preceq} Y \Rightarrow X \overset{st:j}{\preceq} Y$.

As pointed out by Shanthikumar and Yao (1991), unless the random variables are independent, neither joint likelihood ratio ordering nor joint hazard rate ordering imply the corresponding usual ordering between their marginal distributions. However all of these joint orderings imply $X \leq_{st} Y$. They have also extended these concepts to the multivariate case. Below we give the extension of the joint likelihood ratio ordering to the multivariate case.

Let $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ be two vectors. We say that $\mathbf{x}$ *is better arranged than* $\mathbf{y}$ ($\mathbf{x} \overset{a}{\succeq} \mathbf{y}$) if $\mathbf{x}$ can be obtained from $\mathbf{y}$ through successive pairwise interchanges of its components, with each interchange resulting in an increasing order of the two interchanged components. A function $g : \mathbb{R}^n \to \mathbb{R}$ that preserves the ordering $\overset{a}{\succeq}$ is called an *arrangement increasing* function denoted by $g \in \mathcal{AI}$ if $\mathbf{x} \overset{a}{\succeq} \mathbf{y} \Rightarrow g(x) \geq g(y)$. See Marshall and Olkin (1979, p. 169) for more discussion of such functions.

Definition 1.1.6 *Let f denote the joint density of* $\mathbf{X}$. *Then*

$$X_1 \overset{lr:j}{\preceq} X_2 \overset{lr:j}{\preceq} \cdots \overset{lr:j}{\preceq} X_n \Leftrightarrow f \in \mathcal{AI}.$$

One of the basic criteria for comparing variability in probability distributions is that of dispersive ordering. Let F^{-1} and G^{-1} be the right continuous inverses (quantile functions) of F and G, respectively. We say that X is less *dispersed* than Y (denoted by $X \leq_{disp} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$, for all $0 \leq \alpha \leq \beta \leq 1$. From this one can easily obtain that

$$X \leq_{disp} Y \Longleftrightarrow g(x) \leq f\left(F^{-1}G(x)\right) \;\; \forall\, x \qquad (1.1.2)$$

when the random variables X and Y admit densities. A consequence of $X \leq_{disp} Y$ is that $|X_1 - X_2| \leq_{st} |Y_1 - Y_2|$ and which in turn implies $var(X) \leq var(Y)$ as well as $E[|X_1 - X_2|] \leq E[|Y_1 - Y_2|]$, where X_1, X_2 (Y_1, Y_2) are two independent copies of X (Y). For details, see Saunders and Moran (1978), Lewis and Thompson (1981), Deshpande and Kochar (1983), Bagai and Kochar

(1986), Bartoszewicz (1986, 1987); and Section 2.B of Shaked and Shanthikumar (1994).

A related concept is that of star-ordering. X is said to be *star-ordered* with respect to Y (denoted by $X \overset{*}{\leq} Y$) if $G^{-1}F(x)/x$ is increasing in x. It is easy to see that $X \leq_{disp} Y \Leftrightarrow e^X \overset{*}{\leq} e^Y$. It is well known that a distribution F is $IFRA$ (increasing failure rate average) if and only if it is star-ordered with respect to exponential distribution. Also $X \overset{*}{\leq} Y$ implies that the Lorenz curve of G is uniformly smaller than that of F. For more details on this topic see Section 3.C. of Shaked and Shanthikumar (1994).

We end the subsection on the definitions of various kinds of stochastic orderings with the definition of multivariate stochastic ordering.

Definition 1.1.7 *The random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is smaller than the random vector* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *in the multivariate stochastic order (denoted by* $\mathbf{X} \overset{st}{\preceq} \mathbf{Y}$*) if* $h(\mathbf{X}) \leq_{st} h(\mathbf{Y})$ *for all increasing functions* h.

It is easy to see that multivariate stochastic ordering implies component-wise usual stochastic ordering. Section 4.B. of Shaked and Shanthikumar (1994) gives a comprehensive discussion of this ordering.

Notions of dependence

There are several notions of positive and negative dependence between random variables and these have been discussed in detail in Lehmann (1966), Esary and Proschan (1972), Barlow and Proschan (1981), Shaked (1977), Block and Ting (1981), Lee (1985), and Shaked and Spizzichino (1998). For a brief introduction, see Boland et al. (1996). The following concepts of dependence will be used in this thesis.

Definition 1.1.8 *(Karlin, 1968)* *We say that a function $h(x,y)$ is Sign-Regular of order 2 (SR_2) if $\varepsilon_1 h(x,y) \geq 0$ and*

$$\varepsilon_2 \begin{vmatrix} h(x_1,y_1) & h(x_1,y_2) \\ h(x_2,y_1) & h(x_2,y_2) \end{vmatrix} \geq 0, \tag{1.1.3}$$

whenever $x_1 < x_2$, $y_1 < y_2$, and $\varepsilon_i \in \{-1,1\}$ for $i = 1,2$.

If the above relations hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$ then h is said to be *Totally Positive of order 2* (TP_2); and if they hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$ then h is said to be *Reverse Regular of order* 2 (RR_2).

Let $X_1, \ldots, X_n$ be random variables with joint distribution function F and density f. For $s > 0$, let $\gamma^{(s)}(t)$ be defined as follows :

$$\gamma^{(s)}(t) = \begin{cases} (-t)^{s-1}/\Gamma(s) & \text{if } t \leq 0 \\ 0 & \text{if } t > 0. \end{cases}$$

Define the n fold integral $\psi_{k_1,\ldots,k_n}$ by

$$\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \gamma^{(k_i)}(x_i - t_i) dF(t_1,\ldots,t_n)$$

and define $\psi_{0,\ldots,0} = f$. Also define $\psi_{0,\ldots,0,k_{i+1},\ldots,k_n}$ to be the $(n-i)$ fold integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=i+1}^{n} \gamma^{(k_j)}(x_j - t_j) g_i(x_1,\ldots,x_i) dF(t_{i+1},\ldots,t_n | x_1,\ldots,x_i)$$

where g_i is the joint density of $(X_1,\ldots,X_i)$ and $F(t_{i+1},\ldots,t_n|x_1,\ldots,x_i)$ is the conditional distribution function of $(X_{i+1},\ldots,X_n)$ given $X_1 = x_1,\ldots,X_i = x_i$, for $k_{i+1} > 0,\ldots,k_n > 0$. Similarly we can define $\psi_{k_1,\ldots,k_n}$ with any subset of $\{k_1,\ldots,k_n\}$ consisting of zeros.

Lee (1985) introduced the following concept of positive dependence for the multivariate case which is an extension of the one studied by Shaked (1977) for the bivariate case.

Definition 1.1.9 *The random vector* $(X_1, \ldots, X_n)$ *is said to be dependent by total positivity with degree* $(k_1, \ldots, k_n)$*, denoted by* $DTP(k_1, \ldots, k_n)$*, if* $\psi_{k_1,\ldots,k_n}(x_1, \ldots, x_n)$ *is* TP_2 *in pairs of* $\{x_1, \ldots, x_n\}$*.*

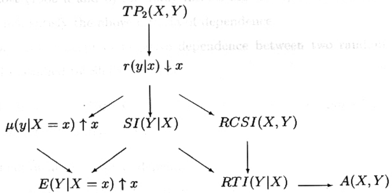

Figure 1.1.1. Implications among notions of positive dependence

As pointed out by Shaked (1977), two random variables X and Y are likelihood ratio (or TP_2) dependent if and only if X and Y are $DTP(0,0)$ dependent. They are $DTP(0,1)$ dependent if the conditional hazard rate of Y given $X = x$, $r(y|x)$, is decreasing in x. The random variables X and Y are $DTP(1,1)$ dependent if the joint survival function $\overline{F}(x,y) = P[X > x, Y > y]$ of (X,Y) is TP_2. In this case the random variables X and Y are also said to be *right corner set increasing (RCSI)*. The random variables X and Y are $DTP(0,2)$ dependent if the conditional mean residual life function of Y given $X = x$, $\mu(y|X = x)$, is increasing in x. We say that Y is stochastically increasing in X (denoted by $SI(Y|X)$ if $P[Y > y|X = x]$ is increasing in x for all y. Y is right tail increasing in X (denoted by $RTI(Y|X)$) if $P[Y > y|X > x]$ is increasing in x for all y. Two random variables X and Y are said to be associated (denoted by $A(X,Y)$) if $Cov(u(X,Y),v(X,Y)) \geq 0$ for all increasing binary functions u and v. Figure 1.1.1. shows the chain of implications that hold among the above notions of positive dependence. There are many other notions of positive dependence, but we will not be discussing them here. See Karlin and Rinott (1980 a and b) for many interesting examples of bivariate distributions which satisfy the above criteria of dependence.

The corresponding concept of negative dependence between two random variables was also studied by Shaked (1977).

Definition 1.1.10 *We say that (X,Y) is dependent by reverse regular of degree k_1 and k_2, denoted by $DRR(k_1,k_2)$, if $\psi_{k_1,k_2}(x,y)$ is RR_2.*

The concepts of bivariate positive dependence can be easily extended to the multivariate case. A function $\psi : \mathbb{R}^n \rightarrow [0,\infty)$ is said to be *multivariate total positivity of order 2* (denoted by MTP_2) if

$$\psi(\mathbf{x})\psi(\mathbf{y}) \leq \psi(\mathbf{x}\wedge\mathbf{y})\psi(\mathbf{x}\vee\mathbf{y})$$

for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, where
$\mathbf{x}\wedge\mathbf{y} = (min(x_1,y_1),\ldots,min(x_n,y_n))$ and $\mathbf{x}\vee\mathbf{y} = (max(x_1,y_1),\ldots,max(x_n,y_n))$.

Definition 1.1.11 *Random variables $X_1,\ldots,X_n$ are said to be MTP_2 dependent if their joint density function is MTP_2.*

It is shown in Block and Ting (1981) that if the support of a random vector $\mathbf{X} = (X_1,\ldots,X_n)$ is a lattice (that is, if $\mathbf{x}$ and $\mathbf{y}$ are in the support of $\mathbf{X}$ then so are $\mathbf{x}\wedge\mathbf{y}$ and $\mathbf{x}\vee\mathbf{y}$) then X is MTP_2, if and only if, its density function f is TP_2 in each pair of its variables when the other $(n-2)$ variables are held fixed. See Karlin and Rinott (1980 a) for more details on properties of MTP_2 functions.

Definition 1.1.12 *Random variables* $X_1, \ldots, X_n$ *are* conditionally increasing in sequence *if* $P[X_i > x | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}]$ *is increasing in* $x_1, \ldots, x_{i-1}$ *for* $i = 2, \ldots, n$ *and each fixed* x.

Definition 1.1.13 *A set of random variables* $\mathbf{X} = (X_1, \ldots, X_n)$ *are associated if* $cov(u(\mathbf{X}), v(\mathbf{X})) \geq 0$ *for all increasing binary functions* u *and* v.

Karlin and Rinott (1980 a) proved that if a set of random variables are MTP_2 dependent then they are conditionally increasing in sequence and which in turn implies that they are associated (cf. Barlow and Proschan, 1981, p. 146), a result which partly extends the implications in Figure 1.1.1. to the multivariate case.

Notions of Majorization and related orderings

One of the basic tools in establishing various inequalities in statistics and probability is the notion of majorization. Let $\{x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Definition 1.1.14 *The vector* $\mathbf{x}$ *is said to majorize the vector* $\mathbf{y}$ *(written* $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$*) if* $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ *for* $j = 1, \ldots, n-1$ *and* $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Functions that preserve the majorization ordering are called Schur-convex functions. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ weakly (written $\mathbf{x} \overset{w}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$. Marshall and Olkin (1979) provides extensive and comprehensive details on the theory of majorization and its applications in statistics.

Recently Bon and Paltanea (1999) have considered a pre-order on $\mathbb{R}^{+^n}$, which they call as a *p-larger order*.

Definition 1.1.15 *A vector* $\mathbf{x}$ *in* $\mathbb{R}^{+n}$ *is said to be p-larger than another vector* $\mathbf{y}$ *also in* $\mathbb{R}^{+n}$ *(written* $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$*) if* $\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, j = 1, \ldots, n$.

Let $\log(\mathbf{x})$ denote the vector of logarithms of the coordinates of $\mathbf{x}$. It is easy to verify that

$$\mathbf{x} \overset{p}{\succeq} \mathbf{y} \Leftrightarrow \log(\mathbf{x}) \overset{w}{\succeq} \log(\mathbf{y}). \tag{1.1.4}$$

It is known that $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \implies (g(x_1), \ldots, g(x_n)) \overset{w}{\succeq} (g(y_1), \ldots, g(y_n))$ for all concave functions g (cf. Marshal and Olkin, 1979, p. 115). From this and (1.1.4), it follows that when $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{+n}$

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}.$$

The converse is, however, not true. For example, the vectors $(0.2, 1, 5) \overset{p}{\succeq} (1, 2, 3)$ but majorization does not hold between these two vectors.

Notions of Aging

Let X be a random variable with distribution function F and let X_t denote a random variable with the same distribution as that of $X - t | X > t$. We will use the following notions of aging in this thesis.

(*a*) X is said to have an increasing failure rate (denoted by IFR) distribution if $X_t \leq_{st} X_{t'}$, for $t > t'$. This is equivalent to saying that $\overline{F}(x+t)/\overline{F}(t)$ decreasing in t for $x > 0$. It is easy to see that in case the random variable X admits density, F is IFR if and only if, the hazard rate $r_F(t) = f(t)/\overline{F}(t)$ is increasing in t.

(*b*) X is said to have a decreasing failure rate (denoted by DFR) distribution if $X_t \geq_{st} X_{t'}$, for $t > t'$. This is equivalent to $\overline{F}(x+t)/\overline{F}(t)$ increasing in t for $x > 0$.

(c) X is said to have a decreasing mean residual life (denoted by $DMRL$) distribution if $\mu_F(t) = E[X_t]$ is a decreasing function of t.

(d) X is said to have an increasing mean residual life (denoted by $IMRL$) distribution if $\mu_F(t)$ is an increasing function of t.

It is known that a random variable with log-concave (log-convex) density is IFR (DFR) and which in turn implies it is $DMRL$ ($IMRL$). The reader is referred to Barlow and Proschan (1981, Ch. 3) for these observations and more details on the various notions of aging.

Order statistics, spacings and concomitants of order statistics are of great interest in many areas in statistics and they have received a lot of attention from many researchers. Let $X_1, \ldots, X_n$ be n random variables. The ith order statistic, the ith smallest of X_i's, is denoted by $X_{i:n}$. A k-out-of-n system of n components functions if at least k of n components function. The time of a k-out-of-n system of n components with life times $X_1, \ldots, X_n$ corresponds to the $(n-k+1)$th order statistic. Thus, the study of lifetimes of k-out-of-n systems is equivalent to the study of the stochastic properties of order statistics. Spacings, the differences between successive order statistics, and their functions are also important in statistics, in general, and in particular in the context of life testing and reliability models. Lot of work has been done in the literature on different aspects of order statistics and spacings. For a glimpse of this, see the books by David (1981), and Arnold, Balakrishnan and Nagaraja (1992); and two volumes of papers on this topic by Balakrishnan and Rao (1998 a and b). But most of this work has been confined to the case when the observations are i.i.d. In many practical situations, like in reliability theory, the observations are not necessarily i.i.d. Because of the complicated nature of the problem, not much work has been done for the non i.i.d. case.

Some references for this case are Sen (1970), David (1981, p.22), Shaked and Tong (1984), Bapat and Beg (1989), Boland et al. (1996), and Nappo and Spizzichino (1998).

Some interesting partial ordering results on order statistics and spacings from independent but non-identically random variables have been obtained by Pledger and Proschan (1971), Proschan and Sethuraman (1976), Bapat and Kochar (1994), Boland, El-Neweihi, and Proschan (1994 a), Kochar and Kirmani (1995), Kochar and Korwar (1996), Kochar and Rojo (1996), Dykstra, Kochar, and Rojo (1997), Kochar (1998), Kochar and Ma (1999) and Bon and Paltanea (1999). Boland, Shaked and Shanthikumar (1998/1995) give a good survey of the area of stochastic comparisons of order statistics.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample of size n from a continuous bivariate distribution. If we arrange the X's in the ascending order as $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ then the Y's associated with these order statistics are denoted by $Y_{[1]}, Y_{[2]}, \ldots, Y_{[n]}$ and are called concomitants of order statistics. They are also known as induced order statistics in the literature. The concomitants are of interest in selection and prediction problems based on the ranks of the X's. They are also of interest in a variety of estimation problems. Their general distribution theory has been studied in Yang (1977). Under the assumption that X and Y are linearly related, apart from an independent error term, the small sample theory of concomitants of order statistics has been discussed by David (1973) and Kim and David (1990). See David and Nagaraja (1998) for an excellent review of this topic. While we are not aware of any previous results on stochastic orderings among concomitants of order statistics, some results on dependence among them are known for certain special types of models (cf. Kim and David, 1990).

In this thesis, stochastic comparisons and dependence among order statistics, spacings and concomitants of order statistics are studied.

1.2 Summary of results on stochastic orders for order statistics

Exponential distribution plays a very important role in statistics, and in reliability theory and life testing, in particular. Because of its non-aging property, it has many nice properties and it often gives very convenient bounds on survival probabilities and other characteristics of interest for systems with non-exponential components.

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate $\lambda_i^*, i = 1 \ldots, n$. Pledger and Proschan (1971) showed that if $\boldsymbol{\lambda} \stackrel{m}{\succeq} \boldsymbol{\lambda}^*$ then

$$X_{i:n} \geq_{st} Y_{i:n}, \quad i = 1, \ldots, n.$$

In Chapter 2, it is proved that for the largest order statistic the same result continues to hold under the weaker p-larger ordering. It is proved that if $\boldsymbol{\lambda} \stackrel{p}{\succeq} \boldsymbol{\lambda}^*$ then

$$X_{n:n} \geq_{st} Y_{n:n}. \tag{1.2.1}$$

Boland, El-Neweihi, and Proschan (1994 a) proved that $X_{2:2} \geq_{hr} Y_{2:2}$. They showed with the help of counterexample that for $n > 2$, (1.2.1) cannot be strengthened from stochastic ordering to hazard rate ordering.

Dykstra, Kochar and Rojo (1997) studied the problem of stochastically comparing the largest order statistic of a set of n independent and non-identically distributed exponential random variables with that corresponding to a set of n independent and identically distributed exponential random variables. Let $Z_1, \ldots, Z_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^{n}\lambda_i/n$, the arithmetic mean of the λ_i's. They proved that $X_{n:n}$ is greater than $Z_{n:n}$ according to dispersive as well as hazard rate orderings. Kochar and Rojo (1996) considered the same problem for spacings and proved that $X_{n:n} - X_{1:n} \geq_{st} Z_{n:n} - Z_{1:n}$. In Chapter 2, it is shown that similar results hold if one replaces $\bar{\lambda}$ with $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of the λ_i's. These results lead to an improved lower bound for the variance of $X_{n:n}$, a better upper bound on the hazard rate function of $X_{n:n}$, and an improved upper bound on the distribution function of the sample range of X_i's in terms of $\tilde{\lambda}$. It is also shown that these bounds are better than those obtained by Dykstra, Kochar and Rojo (1997) and Kochar and Rojo (1996) which are in terms of the arithmetic mean of the λ_i's.

Kochar and Ma (1999) considered the problem of stochastically comparing convolutions of exponential random variables with possibly different hazard rates. They proved that, $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^*$ implies $\sum_{i=1}^{n} X_i \geq_{disp} \sum_{i=1}^{n} Z_i$. They also showed that this result can be immediately extended to convolutions of Erlang random variables. In Chapter 2, it is shown that these results hold under the weaker ordering, $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ and which in turn give better bounds on variances and other measures of variability of convolutions of independent exponential as well as Erlang random variables . Some related work on convolutions of non-identically distributed exponential random variables is by Tong (1988), Boland, El-Neweihi, and Proschan (1994 b), and Bon and Paltanea (1999). The results reported below are based on Khaledi and Kochar (2000 a).

Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution with distribution function F. David and Groeneveld (1982) proved that if F is a DFR distribution, then $var(X_{i:n}) \leq var(X_{j:n})$ for $i \leq j$. Kochar (1996) strengthened this result to prove that under the same condition, $X_{i:n} \leq_{disp} X_{j:n}$ for $i \leq j$. In Chapter 2, these results are extended to compare the variabilities of order statistics based on samples of possibly different sizes. Both, the one-sample as well as the two-sample problems are discussed. It is proved that if F is DFR, then $X_{i:n} \leq_{disp} X_{j:m}$ for $i \leq j$ and $n - i \geq m - j$. Let $Y_{j:m}$ denote the jth order statistic of a random sample of size m taken from a probability distribution with continuous distribution function G. We also prove that if $X \leq_{disp} Y$ and if either F or G is DFR, then $X_{i:n} \leq_{disp} Y_{j:m}$ for $i \leq j$ and $n - i \geq m - j$. This result also holds if instead, one assumes that $X \leq_{hr} Y$ and either F or G is DFR .

1.3 Summary of results on dependence and stochastic orders for spacings

Let $X_1, \ldots, X_n$ be n random variables. For $i = 1, \ldots, n$, we shall denote by $D_{i:n} = X_{i:n} - X_{i-1:n}$ and $D^*_{i:n} = (n - i + 1)D_{i:n}$ the ith spacing and the ith normalized spacing, respectively. It is well known that if $X_1, \ldots, X_n$ is a random sample from an exponential distribution, then $D_{1:n}, \ldots, D_{n:n}$ are independent. If we have a random sample from a DFR distribution then spacings are conditionally increasing in sequence (cf. Barlow and Proschan, 1981, p. 151). It is remarked in Karlin and Rinott (1980 a) that in case the random sample is from a distribution with log-convex density, then the spacings are MTP_2 dependent. In Chapter 3, we extend this result to the case when the random variables $X_1, \ldots, X_n$ are dependent. It is proved that if the joint pdf of X_i's is permutation symmetric, TP_2 in pairs, and log-convex in each argument, then their spacings are MTP_2 dependent. A consequence of this result is that in this case $var(X_{1:n}) \leq var(X_{2:n}) \leq \ldots \leq var(X_{n:n})$.

We also study the dependence properties of spacings of independent but non-identically distributed exponential random variables. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i \in \{1, \ldots, n\}$. It is shown with the help of a counterexample that in this case, in general, the spacings may not be MTP_2 dependent. In fact, for $n = 3$, even $RTI(D_{3:3}|D_{2:3})$ does not hold for some values of the parameters. We show that, however, $cov(D_{2:3}, D_{3:3}) \geq 0$.

Let $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$ and $\lambda_n = \lambda^*$. Such a model is known as a single outlier exponential model with parameters λ and λ^* and it has been studied extensively in the literature. See Gross, Hunt and Odeh (1986), Balakrishnan (1994), and Arnold (1994) for more details and various applications of this model. It is proved in Chapter 3 that for this model, the spacings are MTP_2 dependent. It is also proved that in the case of multiple-outlier model, when $\lambda_1 = \ldots = \lambda_k = \lambda$ and $\lambda_{k+1} = \ldots = \lambda_n = \lambda^*$, $1 < k < n-1$, any pair of consecutive spacings, $D_{i:n}$ and $D_{i+1:n}$ are TP_2 dependent for $i = 1, \ldots, n-1$.

These results on dependence among spacings lead to some interesting results on variances and on covariances of order statistics. We prove that, when $X_1, \ldots, X_n$ follow the single outlier exponential model with parameters λ and λ^* and $Y_1, \ldots, Y_n$ are i.i.d. exponential with common hazard rate $(\lambda^* + (n-1)\lambda)/n$, then $cov(X_{i:n}, X_{j:n}) \geq cov(Y_{i:n}, Y_{j:n})$ and $var(X_{i:n}) \geq var(Y_{i:n})$. These results give us convenient lower bounds on the variance of total time on test statistic and on covariances between order statistics. Bartoszewicz (1985) obtained similar results when $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are random samples on X and Y and $X \geq_{disp} Y$.

Kim and David (1990) proved that when $X_1, \ldots, X_n$ is a random sample from a DFR distribution, then $cov(X_{i:n}, X_{j:n})$ is increasing in i as well as j. It is shown that this result continues to hold if instead, random variables X_i, $i = 1, \ldots, n$ follow the single outlier exponential model.

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i \in \{1, \ldots, n\}$. Kochar and Korwar (1996) conjectured that for $i = 1, \ldots, n-1$, $D^*_{i+1:n} \geq_{hr} D^*_{i:n}$. In Chapter 3, we prove their conjecture for a special case when X_i's follow a single outlier exponential model. Pledger and Proschan (1971) proved that for $i \in \{1, \ldots, n\}$, $D_{i:n}$ is stochastically larger when the parameters are unequal than when they are all equal. This prompted them to examine the question whether the survival function of $D_{i:n}$ is Schur-convex in $(\lambda_1, \ldots, \lambda_n)$. They came up with a counterexample to show that this is not true in general. Kochar and Korwar (1996) proved that in case of second spacing, whereas the survival function of $D_{2:n}$ is Schur-convex in $(\lambda_1, \ldots, \lambda_n)$, its hazard rate is not Schur-concave. We examine this question in Chapter 3, for the single outlier model and prove that for $i \in \{1, \ldots, n\}$, the hazard rate of $D_{i:n}$ is Schur-concave in λ's. The results in Chapter 3 are mainly based on Khaledi and Kochar (2000 c).

1.4 Summary of results on dependence and stochastic orders for concomitants of order statistics

In Chapter 4, the problems of stochastic comparisons and dependence among concomitants of order statistics are discussed. Kim and David (1990) considered the model $Y = g(X) + Z$, where random variable Z is independent of X. They proved that if g is increasing, then $(Y_{[1]}, \dots, Y_{[n]})$ are associated. Under the additional condition that Z has a log-concave density, they proved that the joint density of $(Y_{[1]}, \dots, Y_{[n]})$ is MTP_2. In this thesis we substantially improve upon their results.

The results obtained in Chapter 4, are general in the sense that they apply to any distribution with monotone dependence between variables X and Y. By assuming different kinds of dependence between X and Y, successively stronger dependence and stochastic ordering results among the concomitant variables $Y_{[1]}, \dots, Y_{[n]}$ are established. It is seen that monotone (positive or negative) dependence between X and Y implies positive dependence among $Y_{[i]}$'s. We prove that if (X, Y) is $DRR(0, m)$ or $DTP(0, m)$, then $(Y_{[1]}, \dots, Y_{[n]})$ is $DTP(m, \dots, m)$ for all nonnegative integers m. Particularly, the following results are proved in this chapter.

(a) If Y is stochastically increasing in X, then the concomitant variables $Y_{[i]}$'s are stochastically increasing according to usual stochastic ordering and they are associated.

(b) If X and Y are TP_2 dependent, then $Y_{[i]}$'s are increasing according to likelihood ratio ordering and the joint density of $Y_{[i]}$'s is TP_2 in pairs.

(c) If the hazard rate of the conditional distribution of Y given $X = x$ is decreasing in x, then for $1 \le i < j \le n$, $Y_{[i]}$ is smaller than $Y_{[j]}$ according to hazard rate ordering and $Y_{[i]}$'s are $DTP(1, \dots, 1)$. In particular $Y_{[i]}$ and $Y_{[j]}$ are $RCSI$ for $i \ne j \in \{1, \dots, n\}$.

(d) If the conditional mean residual life of Y given $X = x$ is increasing in x, then for any $1 \le i < j \le n$, $Y_{[i]}$'s are increasing according to mean residual life ordering and they are $DTP(2, \ldots, 2)$.

(e) If $E[Y|X = x]$ is increasing in x, then for any $1 \leq i < j \leq n$, $E[Y_{[i]}] \leq E[Y_{[j]}]$ and $Cov(Y_{[i]}, Y_{[j]}) \geq 0$.

It is also proved that under the above conditions in (a), (b), and (c) the concomitants of order statistics are increasing according to joint stochastic ordering, joint likelihood ratio ordering, and joint hazard rate ordering, respectively. Analogous results on stochastic orders among the concomitants of order statistics are obtained when the dependence between X and Y is monotone negative.

It is also proved in Chapter 4 that if $r(y|x)$ is decreasing in y for each fixed x, then $Y_{[i]}$ has DFR distribution for $1 \leq i \leq n$. If $r(y|x)$ is also decreasing in x then $Y_{[i]} \leq_{disp} Y_{[j]}$ for $i < j$. The inequality is reversed if instead, one assumes that $r(y|x)$ is increasing in x.

Again for the model $Y = g(X)+Z$, where Z is independent of X, assuming that the function g is increasing (decreasing), it is proved in Chapter 4, that

(i) $Y_{[i]}$'s are associated and they are increasing (decreasing) according to usual stochastic ordering;

(ii) if Z has a log-concave density, then the joint density of $Y_{[i]}$'s is TP_2 in pairs and they are increasing (decreasing) according to likelihood ratio ordering;

(iii) if Z is IFR, then $Y_{[i]}$'s are $DTP(1, \ldots, 1)$ and they are increasing (decreasing) according to hazard rate ordering and

(iv) if Z is $DMRL$, then $Y_{[i]}$'s are $DTP(2, \ldots, 2)$ and they are increasing (decreasing) according to mean residual life ordering.

In Case (ii) above, if instead of assuming that Z is log-concave, we assume that Z is log-convex, it is shown that the joint density of $Y_{[i]}$'s is still TP_2 in pairs, but they are decreasing (increasing) according to likelihood ratio ordering. Similarly, in case (iii), if we assume that Z is DFR, then $Y_{[i]}$'s are $DTP(1,\ldots,1)$ and they are decreasing (increasing) according to hazard rate ordering. Finally, by assuming that Z is $IMRL$ in (iv), it is shown that $Y_{[i]}$'s are $DTP(2,\ldots,2)$ and they are decreasing (increasing) according to mean residual life ordering. The results reported in Chapter 4 are mainly based on Khaledi and Kochar (2000 b).

Chapter 2

Stochastic Comparisons of Order Statistics

2.1 Introduction

Order statistics are of great interest in statistics, in general and in reliability theory, in particular. A k-out-of-n system with n components is said to be functional if at least k of n components function. The time of a k-out-of-n system of n components with lifetimes $X_1, \ldots, X_n$ corresponds to the $(n-k+1)$th order statistic. In particular, the lifetime of a parallel system is the same as the largest order statistic. Series and parallel systems are the simplest examples of coherent systems and they have been studied in detail in the literature in case the components are independent and identically distributed. But in real life, oftenly, the systems are made up of components with non-identically distributed lifetimes. Since the distribution theory becomes quite complicated then, relatively fewer results are available for the general case.

The exponential distribution plays a very important role in statistics. Because of its non-aging properties it often gives very convenient bounds on survival probabilities and other characteristics of interest for systems with non-exponential components.

In this chapter we give some new results on stochastic comparisons of order statistics and their functions, when the underlying random variables are independent but non-identically distributed as exponentials. In Section 2.2, some new results on dispersive as well as hazard rate orderings for the largest order statistic are given. In Section 2.3, we stochastically compare the sample range from independent exponential random variables having possibly different hazard rates with that corresponding to a random sample from an exponential distribution. In Section 2.4, we prove that if the hazard rates of independent exponential random variables are more dispersed according to p-larger ordering, then their convolution is more dispersed in the sense of dispersive ordering. Section 2.5 is devoted to the study of dispersive ordering among order statistics in one-sample and two-sample problems.

2.2 Stochastic comparisons of parallel systems

Some interesting partial ordering results on order statistics from independent but non-identically exponential random variables have been obtained by Pledger and Proschan (1971), Proschan and Sethuraman (1976), Bapat and Kochar (1994), Boland , El-Neweihi and Proschan (1994 a), Dykstra, Kochar and Rojo (1997). Boland, Shaked and Shanthikumar (1998/1995) and Kochar (1998) give good surveys of the area of stochastic comparisons of order statistics.

Pledger and Proschan (1971) considered the problem of stochastically comparing the order statistics from independent random variables with proportional hazard rates. In particular, they proved the following Schur-type result.

THEOREM **2.2.1** *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$ and let $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate λ_i^*, $i = 1, \ldots, n$. Then*

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{i:n} \geq_{st} Y_{i:n}, \quad i = 1, \ldots, n.$$

Proschan and Sethuraman (1976) strengthened Theorem 2.2.1 from componentwise stochastic ordering to multivariate stochastic ordering. Boland, El-Neweihi and Proschan (1994 a) proved that for $n = 2$ the above result can be extended from stochastic ordering to hazard rate ordering. They also showed with the help of a counterexample that for $n > 2$, Theorem 2.2.1 cannot be strengthened from stochastic ordering to hazard rate ordering.

Dykstra, Kochar and Rojo (1997) studied the problem of stochastically comparing the largest order statistic of a set of n independent and non-identically distributed exponential random variables with that corresponding to a set of n independent and identically distributed exponential random variables. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i / n$, the arithmetic mean of the λ_i's. They proved that $X_{n:n}$ is greater than $Y_{n:n}$ according to dispersive as well as hazard rate orderings. In Theorem 2.2.3 below we prove that similar results hold if instead, we assume that for $i = 1, \ldots, n$, the random variable Y_i has exponential distribution with hazard rate $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of the λ_i's. To prove dispersive ordering between $X_{n:n}$ and $Y_{n:n}$ in Theorem 2.2.3 we shall need the following lemma.

LEMMA 2.2.1 *For $z > 0$, the functions $g(z) = (1-e^{-z})/z$ and $h(z) = (z^2e^{-z})/(1-e^{-z})^2$ are both decreasing.*

PROOF : The numerator of the derivative of $g(z)$ is $k(z) = (1+z)e^{-z} - 1$, which is a decreasing function of z. This implies that $k(z) < 0$ for $z > 0$, since $k(0) = 0$.

It is easy to see after some simplifications that

$$\frac{d}{dz}(\log(h(z))) = \frac{2 - 2e^{-z} - z - ze^{-z}}{z(1-e^{-z})}. \tag{2.2.1}$$

Using the fact that $k(z)$ is negative, one can verify that the numerator of (2.2.1) is decreasing, from which the required result follows.

■

Next theorem due to Bagai and Kochar (1986) and Bartoszewicz (1987) establishes a connection between dispersive ordering and hazard rate ordering.

THEOREM **2.2.2** *Let X and Y be random variables with distribution function F and G, respectively. Then,*

(a) $X \leq_{hr} Y$ and F or G being DFR implies $X \leq_{disp} Y$;

(b) $X \leq_{disp} Y$ and F or G being IFR implies $X \leq_{hr} Y$.

Now we prove Theorem 2.2.3.

THEOREM **2.2.3** *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. Then*

(a) $X_{n:n} \geq_{disp} Y_{n:n}$;

(b) $X_{n:n} \geq_{hr} Y_{n:n}$.

PROOF : (a) The distribution function of $X_{n:n}$ is

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n}\left(1 - e^{-\lambda_i x}\right),$$

with density function as

$$f_{X_{n:n}}(x) = \sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}\prod_{i=1}^{n}\left(1 - e^{-\lambda_i x}\right). \tag{2.2.2}$$

Replacing λ_i with $\tilde{\lambda}$ in (2.2.2), we see that the distribution function and the density function of $Y_{n:n}$ are

$$F_{Y_{n:n}}(x) = \left(1 - e^{-\tilde{\lambda} x}\right)^n \quad \text{and} \quad f_{Y_{n:n}}(x) = n\tilde{\lambda}e^{-\tilde{\lambda} x}\left(1 - e^{-\tilde{\lambda} x}\right)^{n-1},$$

respectively. It is easy to verify that $F^{-1}_{Y_{n:n}}(x) = -\frac{1}{\tilde{\lambda}}\log\left(1 - x^{1/n}\right)$. Using these observations, it follows that

$$f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}}F_{X_{n:n}}(x)\right) = n\tilde{\lambda}\left(1 - \prod_{i=1}^{n}(1 - e^{-\lambda_i x})^{1/n}\right)\left(\prod_{i=1}^{n}(1 - e^{-\lambda_i x})^{1/n}\right)^{n-1}. \tag{2.2.3}$$

To prove that $X_{n:n} \geq_{disp} Y_{n:n}$, it follows from relation (1.1.2) that it is sufficient to show that

$$f_{X_{n:n}}(x) \leq f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}}F_{X_{n:n}}(x)\right) \quad \forall x > 0. \tag{2.2.4}$$

Using expressions (2.2.2) and (2.2.3) in (2.2.4), one can see after some simplifications that (2.2.4) is equivalent to

$$\sum_{i=1}^{n}\frac{\lambda_i}{1 - e^{-\lambda_i x}} - n\prod_{i=1}^{n}\left(\frac{\lambda_i}{1 - e^{-\lambda_i x}}\right)^{1/n} \leq \sum_{i=1}^{n}\lambda_i - n\prod_{i=1}^{n}(\lambda_i)^{1/n}. \tag{2.2.5}$$

To prove that (2.2.5) holds for all $\lambda_i > 0, i = 1, \ldots, n$, it is sufficient to show that the L.H.S. of (2.2.5) (denoted by $h(x)$) is increasing in x since for $x > 0$,

$$h(x) \leq lim_{x\to+\infty}h(x) = \sum_{i=1}^{n}\lambda_i - n\prod_{i=1}^{n}(\lambda_i)^{1/n},$$

the right hand side of (2.2.5).

The derivative of $h(x)$ is

$$\begin{aligned} h'(x) &= \left(\sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}}\right)\left(\prod_{i=1}^{n}\frac{\lambda_i}{1-e^{-\lambda_i x}}\right)^{1/n} - \sum_{i=1}^{n}\frac{\lambda_i^2 e^{-\lambda_i x}}{(1-e^{-\lambda_i x})^2} \\ &\geq \left(\sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}}\right)\left(\frac{n}{\sum_{i=1}^{n}\frac{1-e^{-\lambda_i x}}{\lambda_i}}\right) - \sum_{i=1}^{n}\frac{\lambda_i^2 e^{-\lambda_i x}}{(1-e^{-\lambda_i x})^2}, \end{aligned}$$

since the geometric mean of a set of numbers is always greater than or equal to its harmonic mean. Now $h'(x) \geq 0$ if and only if,

$$n\sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i x}}{1-e^{-\lambda_i x}} \geq \left(\sum_{i=1}^{n}\frac{\lambda_i^2 e^{-\lambda_i x}}{(1-e^{-\lambda_i x})^2}\right)\left(\sum_{i=1}^{n}\frac{1-e^{-\lambda_i x}}{\lambda_i}\right). \tag{2.2.6}$$

Multiplying both sides of (2.2.6) by $x(>0)$ and replacing the $\lambda_i x$ with z_i for $i = 1, \ldots, n$, it is enough to prove that

$$n\sum_{i=1}^{n}\frac{z_i e^{-z_i}}{1-e^{-z_i}} \geq \left(\sum_{i=1}^{n}\frac{z_i^2 e^{-z_i}}{(1-e^{-z_i})^2}\right)\left(\sum_{i=1}^{n}\frac{1-e^{-z_i}}{z_i}\right). \tag{2.2.7}$$

The inequality in (2.2.7) follows immediately from Čebyšev's inequality (Theorem 1, p. 36 of Mitrinović, 1970), Lemma 2.2.1 and by writing

$$\frac{z_i e^{-z_i}}{1-e^{-z_i}} = \left(\frac{z_i^2 e^{-z_i}}{(1-e^{-z_i})^2}\right)\left(\frac{1-e^{-z_i}}{z_i}\right).$$

This proves that $h(x)$ is increasing in x and hence the result.

(b) It follows from Theorem 5.8 of Barlow and Proschan (1981) that $Y_{n:n}$ is *IFR*. Using this and part (a), the required result follows from Theorem 2.2.2.

■

From the above results, we get the following convenient bounds on the hazard rate and the variance of $X_{n:n}$.

COROLLARY 2.2.1 *Under the conditions of Theorem 2.2.3,* *(a) the hazard rate $r_{X_{n:n}}$ of $X_{n:n}$ satisfies*

$$r_{X_{n:n}}(x;\boldsymbol{\lambda}) \leq \frac{n\tilde{\lambda}\left(1-exp(-\tilde{\lambda}x)\right)^{n-1} exp(-\tilde{\lambda}x)}{1-\left(1-exp(-\tilde{\lambda}x)\right)^{n}} ,$$

(b)

$$var(X_{n:n};\boldsymbol{\lambda}) \geq \frac{1}{\tilde{\lambda}^2}\sum_{i=1}^{n}\frac{1}{(n-i+1)^2} .$$

Dykstra, Kochar and Rojo (1997) proved a result similar to Theorem 2.2.3 by assuming that the random variables Y_i's are exponential with common hazard rate $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i/n$ and obtained bounds on the hazard rate and the variance of $X_{n:n}$ in terms of $\overline{\lambda}$. The new bounds given in Corollary 2.2.1 are better because $r_{Y_{n:n}}$ and $var(Y_{n:n})$ are increasing and decreasing function of $\tilde{\lambda}$, respectively, and the fact that the geometric mean of λ_i's is smaller than their arithmetic mean.

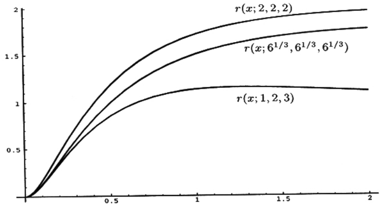

Figure 2.2.1. Graphs of hazard rates of $X_{3:3}$

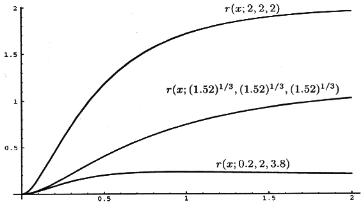

Figure 2.2.2. Graphs of hazard rates of $X_{3:3}$

In Figures 2.2.1. and 2.2.2. above, we plot the hazard rates of parallel systems of three exponential components along with the upper bounds as given by Dykstra, Kochar and Rojo (1997) and the one's given by Corollary 2.2.1 (a). The vector of parameters in Figure 2.2.1. is $\boldsymbol{\lambda_1} = (1,2,3)$ and that in Figure 2.2.2 is $\boldsymbol{\lambda_2} = (0.2,2,3.8)$. Note that $\boldsymbol{\lambda_2} \overset{m}{\succeq} \boldsymbol{\lambda_1}$. It appears from these figures that the improvements in the bounds are relatively more if λ_i's are more dispersed in the sense of majorization. This is true because the geometric mean is Schur-concave and the hazard rate of a parallel system of i.i.d. exponential components with a common parameter $\tilde{\lambda}$ is increasing in $\tilde{\lambda}$.

In Theorem 2.2.6 below we prove that for the largest order statistic the conclusion of Theorem 2.2.1 holds under the weaker p-larger ordering. The proof of this theorem hinges on the following results.

THEOREM **2.2.4** *(Marshall and Olkin, 1979, p. 57) Let $I \subset I\!R$ be an open interval and let $\phi : I^n \rightarrow I\!R$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are ϕ is symmetric on I^n*

and for all $i \neq j$,

$$(z_i - z_j)[\phi_{(i)}(z_i) - \phi_{(j)}(z_j)] \geq 0 \quad \textit{for all } z \in I^n,$$

where $\phi_{(i)}(z)$ denotes the partial derivative of ϕ with respect to its ith argument.

THEOREM **2.2.5** *(Marshall and Olkin, 1979, p. 59) A real-valued function ϕ on the set $A \subset I\!R^n$ satisfies*

$$\mathbf{x} \overset{w}{\succeq} \mathbf{y} \textit{ on } A \Longrightarrow \phi(\mathbf{x}) \geq \phi(\mathbf{y})$$

if and only if ϕ is decreasing and Schur-convex on A.

LEMMA 2.2.2 *The function $\psi : I\!R^{+n} \rightarrow I\!R$ satisfies*

$$\mathbf{x} \overset{p}{\succeq} \mathbf{y} \Longrightarrow \psi(\mathbf{x}) \geq \psi(\mathbf{y}) \tag{2.2.8}$$

if and only if,

(i) $\psi(e^{a_1}, \ldots, e^{a_n})$ is Schur-convex in $(a_1, \cdots, a_n)$

(ii) $\psi(e^{a_1}, \ldots, e^{a_n})$ is decreasing in a_i, for $i = 1, \ldots, n$,

where $a_i = \log x_i$, for $i = 1, \ldots, n$.

PROOF : Using relation (1.1.4), we see that (2.2.8) is equivalent to

$$\mathbf{a} \overset{w}{\succeq} \mathbf{b} \Longrightarrow \psi(e^{a_1}, \ldots, e^{a_n}) \geq \psi(e^{b_1}, \ldots, e^{b_n}), \tag{2.2.9}$$

where $a_i = \log x_i$ and $b_i = \log y_i$, for $i = 1, \ldots, n$.

Taking $\phi(a_1, \cdots, a_n) = \psi(e^{a_1}, \ldots, e^{a_n})$ in Theorem 2.2.5, we get the required result.

■

Next we prove Theorem 2.2.6.

Theorem 2.2.6 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate λ_i^*, $i = 1, \ldots, n$. Then*

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{n:n} \geq_{st} Y_{n:n}.$$

Proof : The survival function of $X_{n:n}$ can be written as

$$\overline{F}_{X_{n:n}}(x) = 1 - \prod_{i=1}^{n}(1 - e^{-e^{a_i x}}), \tag{2.2.10}$$

where $a_i = \log \lambda_i$, $i = 1, \ldots, n$.

Using Lemma 2.2.2, we find that it is enough to show that the function $\overline{F}_{X_{n:n}}$ given by (2.2.10) is Schur-convex and decreasing in a_i's. To prove its Schur-convexity, it follows from Theorem 2.2.4 that, we have to show that for $i \neq j$, $(a_i - a_j)(\frac{\partial \overline{F}_{X_{n:n}}}{\partial a_i} - \frac{\partial \overline{F}_{X_{n:n}}}{\partial a_j}) \geq 0$. That is,

$$x(a_i - a_j)\left(\prod_{i=1}^{n}(1 - e^{-e^{a_i x}})\right)\left(\frac{e^{a_j}e^{-e^{a_j x}}}{1 - e^{-e^{a_j x}}} - \frac{e^{a_i}e^{-e^{a_i x}}}{1 - e^{-e^{a_i x}}}\right) \geq 0, \text{for } i \neq j \tag{2.2.11}$$

since

$$\frac{\partial \overline{F}_{X_{n:n}}}{\partial a_i} = -\prod_{i=1}^{n}(1 - e^{-e^{a_i x}})\left(\frac{xe^{a_i}e^{-e^{a_i x}}}{1 - e^{-e^{a_i x}}}\right).$$

It is easy to see that the function $be^{-bx}/(1 - e^{-bx})$ is decreasing in b, for each fixed $x > 0$. Replacing b with e^{a_i}, it follows that the function $e^{a_i}e^{-e^{a_i x}}/(1 - e^{-e^{a_i x}})$ is also decreasing in a_i for $i = 1, \ldots, n$. This proves that (2.2.11) holds. The partial derivative of $\overline{F}_{X_{n:n}}$ with respect to a_i is negative and which in turn implies that the survival function of $X_{n:n}$ is decreasing in a_i for $i = 1, \ldots, n$. This completes the proof.

■

Boland, El-Neweihi and Proschan (1994 a) pointed out that for $n > 2$, Theorem 2.2.1 cannot be strengthened from stochastic ordering to hazard rate ordering. Since majorization implies p-larger ordering, it follows that, in general, Theorem 2.2.6 cannot be strengthened to hazard rate ordering.

As shown in the next example, a result similar to Theorem 2.2.6 may not hold for other order statistics.

EXAMPLE 2.2.1 : Let X_1, X_2, X_3 be independent exponential random variables with $\boldsymbol{\lambda} = (0.1, 1, 7.9)$ and Y_1, Y_2, Y_3 be independent exponential random variables with $\boldsymbol{\lambda}^* = (1, 2, 5)$. It is easy to see that $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$. The $X_{1:3}$ and $Y_{1:3}$ have exponential distributions with respective hazard rates 3 and 8/3 and which implies that $Y_{1:3} \geq_{st} X_{1:3}$.

2.3 Stochastic ordering for sample range

Sample range is one of the criteria for comparing variabilities among distributions and hence it is important to study its stochastic properties. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\overline{\lambda}$, the arithmetic mean of the λ_i's. Finally, let $R_X = X_{n:n} - X_{1:n}$ and $R_Y = Y_{n:n} - Y_{1:n}$ denote the sample ranges of X_i's and Y_i's, respectively. Kochar and Rojo (1996) proved that $R_X \geq_{st} R_Y$. In the next theorem we prove that a similar result holds if instead, one assumes that for $i = 1, \ldots, n$, the random variable Y_i has an exponential distribution with hazard rate $\tilde{\lambda}$, the geometric mean of λ_i's.

THEOREM **2.3.1** *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda}$.*

Then,

$$R_X \geq_{st} R_Y.$$

PROOF : The distribution function of R_X (see David, 1981, p. 26) is

$$F_{R_X}(x) = \frac{1}{\sum_{i=1}^n \lambda_i} \sum_{i=1}^n \frac{\lambda_i}{1-e^{-\lambda_i x}} \prod_{i=1}^n (1-e^{-\lambda_i x}). \tag{2.3.1}$$

and that of R_Y is

$$G_{R_Y}(x) = \left(1-e^{-\tilde{\lambda}x)}\right)^{n-1}. \tag{2.3.2}$$

Using (2.3.1) and (2.3.2), we have to show that

$$\sum_{i=1}^n \frac{\lambda_i}{1-e^{-\lambda_i x}} \prod_{i=1}^n (1-e^{-\lambda_i x}) \leq \sum_{i=1}^n \lambda_i \left(1-e^{-\tilde{\lambda}x)}\right)^{n-1}. \tag{2.3.3}$$

Multiplying both sides of (2.3.3) by $x(>0)$, it is sufficient to prove that

$$\sum_{i=1}^n \frac{\lambda_i x}{1-e^{-\lambda_i x}} \prod_{i=1}^n (1-e^{-\lambda_i x}) \leq \left(\sum_{i=1}^n \lambda_i x\right) \left(1-e^{-\tilde{\lambda}x)}\right)^{n-1}. \tag{2.3.4}$$

Dykstra, Kochar and Rojo (1997) proved that

$$\sum_{i=1}^n \frac{y_i}{1-e^{-y_i}} \leq \left(\sum_{i=1}^n y_i\right) \prod_{i=1}^n (1-e^{-y_i})^{-\frac{1}{n}},$$

where $y_i > 0$ for $i = 1, \ldots, n$. Making use of this inequality on the L.H.S. of (2.3.4), we get

$$\sum_{i=1}^n \frac{\lambda_i x}{1-e^{-\lambda_i x}} \prod_{i=1}^n (1-e^{-\lambda_i x}) \leq \left(\sum_{i=1}^n \lambda_i x\right) \prod_{i=1}^n \left(1-e^{-\lambda_i x}\right)^{\frac{n-1}{n}} \tag{2.3.5}$$

A consequence of Theorem 2.2.3 (*b*) is that $X_{n:n} \geq_{st} Y_{n:n}$, which is equivalent to $\prod_{i=1}^n (1-e^{-\lambda_i x})^{1/n} \leq 1-e^{-\tilde{\lambda}x}$. Using this result, we find that the expression on the R.H.S. of (2.3.5) is less than or equal to that on the R.H.S. of (2.3.4) and from which the required result follows.

■

As a consequence of this result we get the following upper bound on the distribution function of R_X in terms $\tilde{\lambda}$.

Corollary 2.3.1 *Under the conditions of Theorem 2.3.1, for $x > 0$,*

$$P[X_{n:n} - X_{1:n} \leq x] \leq \left[1 - e^{-\tilde{\lambda}x}\right]^{n-1}. \tag{2.3.6}$$

This bound is better than the one obtained in Kochar and Rojo (1996) in terms of $\overline{\lambda}$, since the expression on the R.H.S. of (2.3.6) is increasing in $\tilde{\lambda}$ and $\tilde{\lambda} \leq \overline{\lambda}$.

2.4 Dispersive ordering among convolutions of independent exponential random variables

In this section we will concentrate on convolutions of independent random variables differing in their scale parameters and obtain some new dispersive ordering results for them. Boland, El-Neweihi and Proschan (1994 b) proved that a convolution of independent exponential random variables with unequal hazard rates is stochastically larger with respect to likelihood ratio ordering when the parameters of the exponential distributions are more dispersed in the sense of majorization. Kochar and Ma (1999) established a dispersive ordering result for the convolution of independent exponential random variables under the same conditions. These results can be immediately extended to the convolutions of Erlang random variables. Some related work on this problem is by Tong (1988) and Bon and Paltanea (1999), among others.

We pursue this problem further in this section and obtain some dispersive ordering results for convolutions of heterogeneous exponential random variables under p-larger ordering which is weaker than majorization. These results lead to better bounds on various quantities of interest associated with these statistics.

To prove the desired results in this section we need the following theorems.

THEOREM 2.4.1 *(Saunders and Moran, 1978) Let X_a be a random variable with distribution function F_a for each $a \in I\!R$ such that*

(i) F_a is supported on some interval $(x_-^{(a)}, x_+^{(a)}) \subseteq (0, \infty)$ and has density f_a which does not vanish on any sub-interval of $(x_-^{(a)}, x_+^{(a)})$,

(ii) derivative of F_a with respect to a exists and denoted by F_a'.

Then,

$$X_a \geq_{disp} X_{a^*} \quad \text{for} \quad a,\ a^* \in I\!R \quad \text{and } a > a^*, \tag{2.4.1}$$

if and only if,

$$F_a'(x)/f_a(x) \quad \text{is decreasing in} \quad x. \tag{2.4.2}$$

THEOREM 2.4.2 *(Lewis and Thompson, 1981) Let Z be a random variable independent of random variables X and Y. If $X \geq_{disp} Y$ and Z has a log-concave density, then*

$$X + Z \geq_{disp} Y + Z.$$

This result leads to the following lemma.

LEMMA 2.4.1 *Let $X_1, X_2; Y_1, Y_2$ be independent random variables with log-concave densities. Then $X_i \leq_{disp} Y_i$ for $i = 1, 2$ implies*

$$X_1 + X_2 \leq_{disp} Y_1 + Y_2.$$

PROOF : Since X_2 is independent of X_1 and Y_1 and it has a log-concave density, it follows from Theorem 2.4.2 that $X_1 \leq_{disp} Y_1$ implies

$$X_1 + X_2 \leq_{disp} Y_1 + X_2. \tag{2.4.3}$$

Using the same argument it follows that $X_2 \leq_{disp} Y_2$ implies

$$Y_1 + X_2 \leq_{disp} Y_1 + Y_2. \tag{2.4.4}$$

Combining (2.4.3) and (2.4.4), we get the required result.

■

THEOREM **2.4.3** *(Lewis and Thompson, 1981) Let for $n \geq 1$, X_n, Y_n, X and Y be random variables such that $X_n \to X$ and $Y_n \to Y$, weakly. Then $X_n \geq_{disp} Y_n$, $n \geq 1$ implies $X \geq_{disp} Y$.*

We first consider convolutions of two independent exponential random variables.

THEOREM **2.4.4** *Let $X_{\lambda_1}, X_{\lambda_2}$ be independent exponential random variables with respective hazard rate λ_1, λ_2. Then $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ implies $S(\lambda_1, \lambda_2) \geq_{disp} S(\lambda_1^*, \lambda_2^*)$, where $S(\lambda_1, \lambda_2) \overset{dist}{=} X_{\lambda_1} + X_{\lambda_2}$*

PROOF : Without loss of generality we assume that $\lambda_1 \geq \lambda_2$ and $\lambda_1^* \geq \lambda_2^*$. To prove the result we need to consider the following four cases.

Case (a) $\lambda_1 > \lambda_2$ and $\lambda_1^* > \lambda_2^*$.

Let us first prove the result when $\lambda_1 \neq \lambda_1^*$ and $\lambda_2 \neq \lambda_2^*$. Then we will discuss the other possibilities later. It is easy to see that the density function and the distribution function of $S(\lambda_1, \lambda_2)$ are, respectively,

$$h(\lambda_1, \lambda_2; x) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 x} - e^{-\lambda_1 x}\right) \tag{2.4.5}$$

and

$$H(\lambda_1, \lambda_2; x) = 1 + \frac{\lambda_2 e^{-\lambda_1 x} - \lambda_1 e^{-\lambda_2 x}}{\lambda_1 - \lambda_2}. \tag{2.4.6}$$

To prove the required result, in the light of Lemma 2.2.2, it is sufficient to show that for $0 < x \leq y < 1$,

(i) $H^{-1}(e^{a_1}, e^{a_2}; x) - H^{-1}(e^{a_1}, e^{a_2}; y)$ is Schur-convex in (a_1, a_2),

(ii) $H^{-1}(e^{a_1}, e^{a_2}; x) - H^{-1}(e^{a_1}, e^{a_2}; y)$ is decreasing in a_1 and a_2, where $a_i = \log \lambda_i$, $i = 1, 2$.

From the definition of dispersive ordering it easy to see that (i) is equivalent to

$$(a_1, a_2) \overset{m}{\succeq} (a_1^*, a_2^*) \Longrightarrow S(e^{a_1}, e^{a_2}) \geq_{disp} S(e^{a_1^*}, e^{a_2^*}), \tag{2.4.7}$$

where $a_i^* = \log \lambda_i^*$, $i = 1, 2$. Hence to prove (i), we show that (2.4.7) holds. $\lambda_1 > \lambda_2$ and $\lambda_1^* > \lambda_2^*$ respectively imply that $a_1 > a_2$ and $a_1^* > a_2^*$. Let $a_1 + a_2 = c$, . Using (2.4.5), (2.4.6) and these assumptions, the density function and the distribution function of $S(\lambda_1, \lambda_2)$, respectively can be written as

$$h_{a_1}(x) = \frac{e^c}{e^c - e^{c-a_1}} \left(e^{-e^{c-a_1}x} - e^{-e^{a_1}x} \right), \quad a_1 \in (c/2, c) \tag{2.4.8}$$

and

$$H_{a_1}(x) = 1 + \frac{e^{-e^{a_1}x} e^{c-2a_1} - e^{-e^{c-a_1}x}}{1 - e^{c-2a_1}}, \quad a_1 \in (c/2, c). \tag{2.4.9}$$

The derivative of $H_{a_1}(x)$ with respect to a_1 is

$$H'_{a_1}(x) = \frac{1}{(1 - e^{c-2a_1})^2} \Big\{ \Big(-2e^{c-2a_1} e^{-e^{a_1}x} - x e^{a_1} e^{-e^{a_1}x} e^{c-2a_1} - x e^{c-a_1} e^{-e^{c-a_1}x} \Big)$$
$$\times \left(1 - e^{c-2a_1} \right) - 2e^{c-2a_1} \left(e^{-e^{a_1}x} e^{c-2a_1} - e^{-e^{c-a_1}x} \right) \Big\}. \tag{2.4.10}$$

Using (2.4.8) and (2.4.10), we get after some simplifications,

$$\frac{H'_{a_1}(x)}{h_{a_1}(x)} = \frac{e^{a_1} - e^{c-a_1}}{e^c(1 - e^{c-2a_1})^2} \left\{ 2e^{c-2a_1} + \frac{x(e^{-e^{a_1}x} + e^{-e^{c-a_1}x})(e^{c-2a_1} - 1)(e^{c-a_1})}{e^{-e^{c-a_1}x} - e^{-e^{a_1}x}} \right\}. \tag{2.4.11}$$

To prove that expression in (2.4.11) is decreasing in x, we have to show that the function

$$g(x) = \frac{x\{e^{-e^{a_1}x} + e^{-e^{c-a_1}x}\}}{e^{-e^{c-a_1}x} - e^{-e^{a_1}x}}$$

is increasing in x, since $a_1 \in (c/2, c)$ implies $e^{c-2a_1} < 1$. After some simplifications we find that the numerator of g', the derivative of g, is

$$k(x) = e^{-e^{a_1}x} e^{-e^{c-a_1}x} \left\{ e^{x(e^{a_1} - e^{c-a_1})} - e^{-x(e^{a_1} - e^{c-a_1})} - 2x(e^{a_1} - e^{c-a_1}) \right\}. \tag{2.4.12}$$

Let us define $z(y) = e^y - e^{-y} - 2y$. It is easy to see that for $y > 0$, $z'(y) > 0$ and $z(0) = 0$. Using this observation and replacing y with $x(e^{a_1} - e^{c-a_1})$ in the function z, we have shown that the function $k(x)$ given in (2.4.12) is positive for $x > 0$ and which in turn implies that $H'_{a_1}(x)/h_{a_1}(x)$ is decreasing in x. Using this result, (2.4.7) follows from Theorem 2.4.1.

It is worth noting that (ii) is equivalent to saying that $S(e^{a_1}, e^{a_2})$ is decreasing in a_1 and a_2 according to dispersive ordering. Now let $a_1 > a'_1$. It is easy to see that $X_{e^{a'_1}} \geq_{disp} X_{e^{a_1}}$. The random variables X's are independent and $X_{e^{a_2}}$ has a log-concave density. Combining these facts, it follows from Theorem 2.4.2 that $S(e^{a'_1}, e^{a_2}) \geq_{disp} S(e^{a_1}, e^{a_2})$. Similarly one can prove that $S(e^{a_1}, e^{a_2})$ is decreasing in a_2, from which the required result follows.

Now if $\lambda_1 = \lambda_1^*$, then $(\lambda_1, \lambda_2) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*)$ implies that $\lambda_2 < \lambda_2^*$ and which in turn implies that $X_{\lambda_2} \geq_{disp} X_{\lambda_2^*}$. Now the required result in this case follows from Theorem 2.4.2, since the random variables X's have log-concave densities and they are independent.

The last possibility is $\lambda_1 = \lambda_2^*$. In this case $\lambda_2 < \lambda_1 = \lambda_2^* < \lambda_1^*$. Again the required result follows from Theorem 2.4.2. This completes the proof of case (a).

Case (b) $\quad \lambda_1 = \lambda_2$ and $\lambda_1^* > \lambda_2^*$.

Noting that $\lambda_1 = \lambda_2^* < \lambda_1^*$ or $\lambda_1 < \lambda_2^* < \lambda_1^*$, the result follows from Theorem 2.4.2.

Case (c) $\quad \lambda_1 > \lambda_2$ and $\lambda_1^* = \lambda_2^*$. Again the required result for the case when $\lambda_1 = \lambda_1^*$ immediately follows from Theorem 2.4.2. Now let $\lambda_1 \neq \lambda_1^*$. In this case $(\lambda_1, \lambda_2) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*)$ implies that $\lambda_1^* \geq \tilde{\lambda}$, where $\tilde{\lambda} = (\lambda_1\lambda_1)^{1/2}$, the geometric mean of λ_1, λ_2. First we prove the result for the case when $\lambda_1^* = \tilde{\lambda}$. It is easy to see that, for $n \geq 1$, $(\lambda_1, \lambda_2) \overset{p}{\succeq} (\tilde{\lambda}, \tilde{\lambda} + 1/n)$. Using this observation, it follows from case (a) that, for $n \geq 1$,

$$X_{\lambda_1} + X_{\lambda_2} \geq_{disp} X_{\tilde{\lambda}} + X_{\tilde{\lambda}+1/n}.$$

Using the fact that $X_{\tilde{\lambda}+1/n} \to X_{\tilde{\lambda}}$, weakly, it follows that $X_{\tilde{\lambda}} + X_{\tilde{\lambda}+1/n} \to Y$ weakly, where Y is a gamma random variable with shape parameter 2 and scale parameter $\tilde{\lambda}$. Combining these observations, the required result in this case follows from Theorem 2.4.3. The result for the case when $\lambda_1^* > \tilde{\lambda}$ follows from the above case and the fact that gamma random variables with equal shape parameters are decreasing according to dispersive ordering with respect to their scale parameters. This completes the proof of this case.

Case (d) $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$.

In this case $S(\lambda_1, \lambda_2)$ and $S(\lambda_1^*, \lambda_2^*)$ have gamma distribuions with equal shape parameters and respective scale parameters λ_1 and λ_1^*. Now the result again follows from the fact that gamma random variables with equal shape parameters are dispersive ordered with respect to their scale parameters.

■

In the next theorem we extend Theorem 2.4.4 from $n = 2$ to $n > 2$.

THEOREM **2.4.5** *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has exponential distribution with hazard rates λ_i, for $i = 1, \ldots, n$. Then, $\boldsymbol{\lambda} \stackrel{p}{\succeq} \boldsymbol{\lambda}^*$ implies $S(\lambda_1, \ldots, \lambda_n) \geq_{disp} S(\lambda_1^*, \ldots, \lambda_n^*)$.*

PROOF : As in the proof of Theorem 2.4.4, we show that

(i)

$$\mathbf{a} \stackrel{m}{\succeq} \mathbf{a}^* \Longrightarrow S(e^{a_1}, \ldots, e^{a_n}) \geq_{disp} S(e^{a_1^*}, \ldots, e^{a_n^*}).$$

where $a_i = \log \lambda_i$ and $a_i^* = \log \lambda_i^*$, $i = 1, , \ldots, n$,

(ii) $S(e^{a_1}, \ldots, e^{a_n})$ is decreasing in a_i, for $i = 1, \ldots, n$ according to dispersive ordering.

To prove (i), it is sufficient to consider the case when $(a_1, a_2) \overset{m}{\succeq} (a_1^*, a_2^*)$, and $a_i = a_i^*$, $i = 3, \ldots, n$. Then it follows from Theorem 2.4.4 that $S(e^{a_1}, e^{a_2}) \geq_{disp} S(e^{a_1^*}, e^{a_2^*})$. The random variable $S(e^{a_3}, \ldots, e^{a_n})$ has a log-concave density, since the class of distributions with log-concave densities is closed under convolutions (cf. Shaked and Shanthikumar, 1994, p. 439). Adding $S(e^{a_3}, \ldots, e^{a_n})$ to both sides of the above inequality, we find that the required result follows from Theorem 2.4.2.

The proof of (ii) here is similar to that of (ii) in Theorem 2.4.4. Using (i) and (ii), again the main result follows from Lemma 2.2.2

■

The following result also proved by Bon and Paltanea (1999) immediately follows from the above result.

COROLLARY 2.4.1 *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has exponential distribution with hazard rates λ_i for $i = 1, \ldots, n$. Then, $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ implies $S(\lambda_1, \ldots, \lambda_n) \geq_{hr} S(\lambda_1^*, \ldots, \lambda_n^*)$.*

PROOF : Since $S(\lambda_1, \ldots, \lambda_n)$ has a log-concave density, it is IFR. From Theorem 2.4.5 $S(\lambda_1, \ldots, \lambda_n) \geq_{disp} S(\lambda_1^*, \ldots, \lambda_n^*)$. The required result then follows from Theorem 2.2.2.

■

Kochar and Ma (1999) also proved that $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^*$ implies $S(\lambda_1, \ldots, \lambda_n) \geq_{disp} S(\lambda_1^*, \ldots, \lambda_n^*)$ when random variable X_{λ_i} follows the Erlang-m distribution with mean m/λ_i, for $i = 1, \ldots, n$. In the next theorem we show that this result continues to hold if, we replace $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^*$ with the weaker partial ordering $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$.

THEOREM **2.4.6** *Let $Y_{\lambda_1}, \ldots, Y_{\lambda_n}$ be independent random variables such that Y_{λ_i} has Erlang distribution with shape parameter m and mean m/λ_i, for $i =$*

$1, \ldots, n$. *Then,* $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ *implies*

$$\sum_{i=1}^{n} Y_{\lambda_i} \geq_{disp} \sum_{i=1}^{n} Y_{\lambda_i^*}.$$

PROOF : It is known that $Y_{\lambda_i} \overset{dist}{=} \sum_{j=1}^{m} X_{\lambda_i, j}$, $i = 1, \ldots, n$, where for every j, $X_{\lambda_1, j}, \ldots, X_{\lambda_n, j}$ are independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$. By Theorem 2.4.5, under the given conditions, for $j = 1, \ldots, m$,

$$\sum_{i=1}^{n} X_{\lambda_{i,j}} \geq_{disp} \sum_{j=1}^{n} X_{\lambda_{i,j}^*}.$$

The required result follows by repeatedly using Lemma 2.4.1.

■

As a consequence of this result we get the following convenient lower bound for the variance of $\sum_{i=1}^{n} Y_{\lambda_i}$,

$$var\left(\sum_{i=1}^{n} Y_{\lambda_i}\right) \geq \frac{mn}{\tilde{\lambda}^2},$$

since $\boldsymbol{\lambda} \overset{p}{\succeq} (\tilde{\lambda}, \ldots, \tilde{\lambda})$, where $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$.

It is clear that this bound is better than the one obtained by Kochar and Ma (1999) which is in terms of the arithmetic mean of the λ_i's, since it is known that the geometric mean is smaller than or equal to the arithmetic mean.

Remark : Pareto distribution of the first kind with distribution function $F(x) = 1 - x^{-\lambda}$, $x \geq 1$ is of practical important in the economic. Using relation between dispersive ordering and star ordering, one can easily see that

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Rightarrow \prod_{i=1}^{n} X_{\lambda_i} \overset{*}{\succeq} \prod_{i=1}^{n} X_{\lambda_i^*}$$

where X_{λ_i} has Pareto distribution of the first kind with parameter λ_i, for $i = 1, \ldots, n$. This result is also an extension of Corollary 2.2 of Kochar and Ma (1999) from majorization to weaker p-larger ordering.

2.5 Dispersive ordering among order statistics from DFR distributions

In this section we discuss the problem of dispersive ordering among order statistics when the observations are independent and identically distributed from a DFR distribution. We consider both, the one-sample as well as the two-sample problems when random samples are of possibly different sizes. Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution with distribution function F. David and Groeneveld (1982) proved that if F is a DFR distribution, then

$$var(X_{1:n}) \leq var(X_{2:n}) \leq \cdots \leq var(X_{n:n})$$

Boland, Shaked and Shanthikumar (1998/1995) proved that if F is exponential distribution, then

$$X_{1:n} \leq_{disp} X_{2:n} \leq_{disp} \cdots \leq_{disp} X_{n:n}. \tag{2.5.1}$$

Kochar (1996) strengthened these results to prove that (2.5.1) continues to hold for a DFR distribution.

In Lemma 2.5.1 below we extend the result of Boland, Shaked and Shanthikumar (1998/1995) to the case when the order statistics are based on samples of possibly different sizes. To prove it, we need the following theorem.

THEOREM **2.5.1** *(Lewis and Thompson, 1981) The random variable X satisfies $X \leq_{disp} X + Y$, for any random variable Y independent of X if and only if, X has a log-concave density.*

Now we are ready to prove Lemma 2.5.1

LEMMA 2.5.1 *Let $X_{i:n}$ be the ith order statistic of a random sample of size n from an exponential distribution. Then*

$$X_{i:n} \leq_{disp} X_{j:m} \quad \text{for } i \leq j \text{ and } n-i \geq m-j. \tag{2.5.2}$$

PROOF : Suppose we have two independent random samples, $X_1, \ldots, X_n$ and $X'_1, \ldots, X'_m$ of sizes n and m from an exponential distribution with failure rate λ. The *ith* order statistic $X_{i:n}$ can be written as a convolutions of sample spacings as

$$\begin{aligned} X_{i:n} &= (X_{i:n} - X_{i-1:n}) + \cdots + (X_{2:n} - X_{1:n}) + X_{1:n} \\ &\stackrel{dist}{=} \sum_{k=1}^{i} E_{n-i+k} \end{aligned}$$

where for $k = 1, \ldots, i$, E_{n-i+k} is an exponential random variable with failure rate $(n - i + k)\lambda$. It is a well known fact that E_{n-i+k}'s are independent. Similarly we can express $X'_{j:m}$ as

$$X'_{j:m} \stackrel{dist}{=} \sum_{k=1}^{j} E'_{m-j+k}$$

where again for $k = 1, \ldots, j$, E'_{m-j+k} is an exponential random variable with failure rate $(m - j + k)\lambda$ and E'_{m-j+k}'s are independent. It is easy to verify that $E_{n-i+k} \leq_{disp} E'_{m-j+k}$ for $n - i \geq m - j$ and $1 \leq k \leq j$.

Since an exponential random variable has a log-concave density it follows from Lemma 2.4.1 that

$$\sum_{k=1}^{i} E_{n-i+k} \leq_{disp} \sum_{k=1}^{i} E'_{m-j+k}. \tag{2.5.3}$$

Again since $\sum_{k=i+1}^{j} E'_{m-j+k}$, being the sum of independent exponential random variables is independent of $\sum_{k=1}^{i} E'_{n-i+k}$ and which has a log-concave density, it follows from Theorem 2.5.1 that the R.H.S of (2.5.3) is less dispersed than

$\sum_{k=1}^{j} E'_{m-j+k}$ for $i \leq j$. That is,

$$X_{i:n} \stackrel{dist}{=} \sum_{k=1}^{i} E_{n-i+k} \leq_{disp} \sum_{k=1}^{j} E'_{m-j+k} \stackrel{dist}{=} X'_{j:m}.$$

Since $X_{j:m}$ and $X'_{j:m}$ are stochastically equivalent, (2.5.2) follows from this.

■

The proof of the next lemma can be found in Bartoszewicz (1987).

LEMMA 2.5.2 *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\phi(0) = 0$ and $\phi(x) - x$ is increasing. Then for every convex and strictly increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ the function $\psi\phi\psi^{-1}(x) - x$ is increasing.*

In the next theorem we extend Lemma 2.5.1 to the case when F is a DFR distribution. This result is also an extension of the result given in Kochar (1996) to the case when the order statistics are based on samples of possibly different sizes.

THEOREM **2.5.2** *Let $X_{i:n}$ be the ith order statistic of a random sample of size n from a DFR distribution F. Then*

$$X_{i:n} \leq_{disp} X_{j:m} \quad \textit{for } i \leq j \textit{ and } n - i \geq m - j.$$

PROOF : The distribution function of $X_{j:m}$ is $F_{j:m}(x) = B_{j:m}F(x)$, where $B_{j:m}$ is the distribution function of the beta distribution with parameters $(j, m - j + 1)$.

Let G denote the distribution function of a unit mean exponential random variable. Then $H_{j:m}(x) = B_{j:m}G(x)$ is the distribution function of the jth order statistic in a random sample of size m from a unit mean exponential distribution. We can express $F_{j:m}$ as

$$\begin{aligned} F_{j:m}(x) &= B_{j:m}GG^{-1}F(x) \\ &= H_{j:m}G^{-1}F(x). \end{aligned}$$

To prove the required result, we have to show that for $i \leq j$ and $n - i \geq m - j$,

$$F_{j:m}^{-1} F_{i:n}(x) - x \quad \text{is increasing in } x$$
$$\Leftrightarrow \quad F^{-1} G H_{j:m}^{-1} H_{i:n} G^{-1} F(x) - x \quad \text{is increasing in } x.$$

By Lemma 2.5.1, $H_{j:m}^{-1} H_{i:n}(x) - x$ is increasing in x for $i \leq j$ and $n - i \geq m - j$. Also the function $\psi(x) = F^{-1}G(x)$ is strictly increasing and it is convex if F is DFR. The required result now follows from Lemma 2.5.2.

■

Remark : A consequence of Theorem 2.5.2 is that if we have random samples from a DFR distribution, then

$$X_{i:n+1} \leq_{disp} X_{i:n} \leq_{disp} X_{i+1:n+1}, \qquad \text{for } i = 1, \ldots, n.$$

Now we consider the two-sample case. Let $X_1, \ldots, X_n$ and $Y_1 \ldots, Y_n$ be two random samples from continuous distribution functions F and G, respectively. Bartoszewicz (1986) proved that,

$$X \leq_{disp} Y \Rightarrow X_{i:n} \leq_{disp} Y_{i:n}, \qquad \text{for } i = 1, \ldots, n. \tag{2.5.4}$$

In the next theorem we extend this result to establish dispersive ordering between order statistics when the random samples are possibly of different sizes and at least one of the two distributions is DFR.

THEOREM **2.5.3** *Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution F and let $Y_1 \ldots, Y_m$ be a random sample of size m from another continuous distribution G. If either F or G is DFR, then*

$$X \leq_{disp} Y \Rightarrow X_{i:n} \leq_{disp} Y_{j:m} \quad \text{for } i \leq j \text{ and } n - i \geq m - j.$$

PROOF : Let F be a DFR distribution. The proof for the case when G is DFR is similar. By Theorem 2.5.2, $X_{i:n} \leq_{disp} X_{j:m}$ for $i \leq j$ and $n-i \geq m-j$. Using (2.5.4), it follows that $X_{j:m} \leq_{disp} Y_{j:m}$. Combining these observations, we get the required result.

∎

Since from the property $X \leq_{hr} Y$ together with the condition that either F or G is DFR implies that $X \leq_{disp} Y$ (cf. Theorem 2.2.2), we get the following result from the above theorem.

COROLLARY 2.5.1 *Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution F and $Y_1 \ldots, Y_m$ be a random sample of size m from another continuous distribution G. If either F or G is DFR, then*

$$X \leq_{hr} Y \Rightarrow X_{i:n} \leq_{disp} Y_{j:m} \quad \textit{for } i \leq j \textit{ and } n-i \geq m-j.$$

The material of Section 2.5 is mainly based on Khaledi and Kochar (2000 a).

Chapter 3

Dependence and Stochastic Orders among Spacings

3.1 Introduction

Let $X_1, \ldots, X_n$ be n random variables. The random variables $D_{i:n} = X_{i:n} - X_{i-1:n}$ and $D^*_{i:n} = (n-i+1)D_{i:n}$ are respectively called spacings and normalized spacings, for $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$. They are of great interest in various areas of statistics, in particular, in characterizations of distributions, goodness-of-fit tests, life testing and reliability models. In the reliability context they correspond to times elapsed between successive failures of components in a system. It is well known that the normalized spacings of a random sample of size n from an exponential distribution are i.i.d. random variables having the same exponential distribution. This characterization may not generally hold for other distributions and much of the reliability theory deals with this aspect of spacings. In this chapter we investigate the dependence and stochastic ordering properties of spacings when the original random variables are not necessarily i.i.d. In Section 3.2, we obtain some new results on dependence among spacings when observations follow an exchangeable distribution. Section 3.3 deals with this problem when the parent observations are independent and exponentially distributed with unequal parameters. In particular, in this section we obtain some interesting results on dependence among spacings in single-outlier as well as multiple-outlier exponential models. Section 3.4 is devoted to the study of hazard rate ordering among spacings when observations follow the single outlier exponential model. Section 3.5 is concerned with some inequalities among variances and covariances of order statistics for this model. In Section 3.6, we discuss some open problems on dependence among spacings.

3.2 Dependence among spacings of exchangeable random variables

It is well known that if $X_1, \ldots, X_n$ is a random sample from an exponential distribution, then $D_{1:n}, \ldots, D_{n:n}$ are independent. If we have a random sample from a DFR distribution then spacings are conditionally increasing in sequence (cf. Barlow and Proschan, 1981, p. 151). As pointed out in Karlin and Rinott (1980 a, p. 483) the spacings of a random sample from a distribution with log-convex density are MTP_2 dependent. In the next theorem we extend this result to the case when X_i's are exchangeable and their joint density is TP_2 in pairs.

THEOREM **3.2.1** *Let $X_1, \ldots, X_n$ be exchangeable random variables with absolutely continuous joint pdf $f_{\mathbf{X}}(x_1, \ldots, x_n)$ which is positive on $\prod_{i=1}^{n} \Omega_i^n$, $\Omega_i \subset \mathbb{R}^1$, $i = 1, \ldots, n$ and satisfies the following conditions :*

(a) $f_{\mathbf{X}}$ *is TP_2 in pairs,*

(b) $f_{\mathbf{X}}$ *is log-convex in each argument when the remaining arguments are held fixed and*

(c) *the first partial derivative of* $f_{\mathbf{X}}(\mathbf{x})$ *with respect to* x_i *exists for* $i = 1, \ldots, n$.

Then $D_{1:n}, \ldots, D_{n:n}$ *are* MTP_2 *dependent.*

PROOF : The joint pdf of $D_{1:n}, \ldots, D_{n:n}$ is

$$f_{\mathbf{D}}(d_1, \ldots, d_n) = n! f_{\mathbf{X}}(d_1, \sum_{j=1}^{2} d_j, \ldots, \sum_{j=1}^{i} d_j, \ldots, \sum_{j=1}^{n} d_i).$$

By Theorem 1.5 page 158 of Karlin (1968), $f_{\mathbf{D}}(d_1, \ldots, d_n)$ will be TP_2 in pairs of $d_1, \ldots, d_n$ if and only if for any $i \neq j,\ 1 \leq i,\ j \leq n$, $(\partial/\partial d_i) \log f_{\mathbf{D}}(d_1, \ldots, d_n)$ is increasing in d_j. Let $i < j$. By the chain rule of differentiation

$$(\partial/\partial d_i) \log f_{\mathbf{D}}(d_1, \ldots, d_n) = \sum_{k=i}^{n} (\partial/\partial x_k) \log f_{\mathbf{X}}(d_1, \sum_{j=1}^{2} d_j, \ldots, \sum_{j=1}^{i} d_j, \ldots, \sum_{j=1}^{n} d_i),$$

where $x_k = \sum_{l=1}^{k} d_l$ for $k \in \{1, \ldots, n\}$. The term $(\partial/\partial x_k) \log f_{\mathbf{X}}(\mathbf{x})$ is increasing in x_k for $k \in \{1, \ldots, n\}$, since $f_{\mathbf{X}}$ is log-convex in x_k for each k. It is increasing in x_m, $m \neq k$, $m \in \{1, \ldots, n\}$ since $f_{\mathbf{X}}$ is TP_2 in pairs. Now x_m and x_k are both increasing functions of d_j. This implies that $(\partial/\partial d_i) \log f_{\mathbf{D}}(d_1, \ldots, d_n)$ is an increasing function of d_j. Hence $f_{\mathbf{D}}(d_1, \ldots, d_n)$ is TP_2 in pairs. Clearly the support of spacings is a lattice under the given conditions. Combining these facts, we get the required result.

■

Remark : In Theorem 3.2.1 if instead of conditions (a) and (b) we assume that $f_{\mathbf{X}}$ is RR_2 in pairs and $f_{\mathbf{X}}$ is log-concave in each argument, then one can prove that the joint pdf of spacings is RR_2 in pairs.

LEMMA 3.2.1 *For a bivariate random vector* (X, Y),

$$cov(Y - X, X) \geq 0 \Longrightarrow var(X) \leq var(Y). \tag{3.2.1}$$

PROOF : The inequality $cov(Y - X, X) \geq 0$ implies $cov(X, Y) \geq var(X)$ which in turn implies that

$$\{var(X)/var(Y)\} \leq \rho^2(X, Y) \leq 1,$$

where $\rho(X, Y)$ is the correlation coefficient between X and Y. The required result follows from this.

■

This lemma and Theorem 3.2.1 lead to the following corollary.

COROLLARY 3.2.1 *Under the assumptions of Theorem 3.2.1,*

$$var(X_{1:n}) \leq var(X_{2:n}) \leq \ldots \leq var(X_{n:n}).$$

PROOF : Since under the given conditions $D_{i:n}$'s are MTP_2 dependent, they are associated. This implies that for $j = 2, \ldots, n$,

$$cov(X_{j:n} - X_{j-1:n}, X_{j-1:n}) \equiv cov(D_{j:n}, \sum_{i=1}^{j-1} D_{i:n}) \geq 0, \tag{3.2.2}$$

since $\sum_{i=1}^{j-1} D_{i:n}$ and $D_{j:n}$ are increasing functions of $(D_{1:n}, \ldots, D_{n:n})$. The required result follows from Lemma 3.2.1.

■

EXAMPLE 3.2.1 (INVERTED DIRICHLET DISTRIBUTION) : Let X_i, $i = 0, \ldots, n$ be independent gamma random variables each with unit scale parameter and such that X_0 has shape parameter β and X_i has shape parameter α, for $i \in \{1, \ldots, n\}$. Then the joint pdf of $Y_i = X_i/X_0$, $i = 1, \ldots, n$ is

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \frac{\Gamma(n\alpha + \beta)}{(\Gamma(\alpha))^n \Gamma(\beta)} \cdot \frac{(\prod_{i=1}^n y_i)^{\alpha - 1}}{(1 + \sum_{i=1}^n y_i)^{n\alpha + \beta}} \text{ for } y_i \geq 0.$$

It is easy to see that $f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n)$ is exchangeable, TP_2 in pairs and log-convex in each argument when $0<\alpha<1$ and $n\alpha+\beta\geq 1$. Thus the conditions of Theorem 3.2.1 are satisfied and as a result the spacings of $Y_1,\ldots,Y_n$ are MTP_2 dependent. By Corollary 3.2.1 the variances of the successive order statistics increase as i goes from 1 to n.

3.3 Dependence among spacings of heterogeneous exponential random variables

Let $X_1,\ldots,X_n$ be n independent random variables with X_i having exponential distribution with hazard rate λ_i, for $i=1,\ldots,n$. Gross, Hunt and Odeh (1986) briefly discussed the dependence properties of spacings for the single-outlier exponential model with parameters λ and λ^* in which $\lambda_1=\ldots\lambda_{n-1}=\lambda$ and $\lambda_n=\lambda^*$, $\lambda\neq\lambda^*$. They incorrectly pointed out that in this case the spacings $D_{i:n}$ and $D_{j:n}$ are independent for $j-i\geq 2$. While it is true that $D_{1:n}$ is independent of $(D_{2:n},\ldots,D_{n:n})$, the other D_i's are not independent. In fact, for $n=4$,

$$cov(D_{2:4},D_{4:4})=\frac{2\lambda^*(\lambda^*-\lambda)^2}{(\lambda^*+\lambda)(2\lambda^*+\lambda)^2(3\lambda^*+\lambda)^2},$$

which is positive unless $\lambda^*=\lambda$. In this section we extensively investigate this problem for the independent exponential random variables when λ_i's are possibly different. In particular we obtain some interesting results on dependence among spacings when random variables $X_1,\ldots,X_n$ follow the single-outlier as well as the multiple-outlier exponential models. In the latter case, for $1<k<n-1$, $X_1,\ldots,X_k$ are i.i.d. exponential random variables with common hazard rate λ and $X_{k+1},\ldots,X_n$ are i.i.d. exponentials with hazard rate λ^*. For more details and various applications of these models the reader

is referred to Balakrishnan (1994) and Arnold (1994).

The joint density function of spacings when λ_i's are possibly different is given by (cf. Kochar and Korwar, 1996),

$$f_{D_{1:n},\ldots,D_{n:n}}(x_1,\ldots,x_n) = \sum_{(\mathbf{r})} \frac{\prod_{i=1}^{n}\lambda_i}{\prod_{i=1}^{n}\sum_{j=i}^{n}\lambda(r_j)} \prod_{i=1}^{n}(\sum_{j=i}^{n}\lambda(r_j))exp\{-x_i\sum_{j=i}^{n}\lambda(r_j)\}, \tag{3.3.1}$$

for $x_i \geq 0$, $i = 1,\ldots,n$, where $(\mathbf{r}) = (r_1,\ldots,r_n)$ is a permutation of $(1,\ldots,n)$ and $\lambda(i) = \lambda_i$. It is a mixture of products of exponential random variables. From (3.3.1) it is easy to find that the joint pdf of $(D_{i:n}, D_{j:n})$ for $1 \leq i < j \leq n$, is

$$\begin{aligned} f_{D_{i:n},D_{j:n}}(x,y) &= \sum_{(\mathbf{r})} \frac{\prod_{i=1}^{n}\lambda_i}{\prod_{i=1}^{n}\sum_{j=i}^{n}\lambda(r_j)} \\ &\times (\sum_{m=i}^{n}\lambda(r_m))exp\{-x\sum_{m=i}^{n}\lambda(r_m)\}(\sum_{m=j}^{n}\lambda(r_m))exp\{-y\sum_{m=j}^{n}\lambda(r_m)\}, \end{aligned} \tag{3.3.2}$$

for $x,\ y \geq 0$.

The next example shows that the spacings may not be MTP_2 dependent if λ_i's are all different.

EXAMPLE 3.3.1 : Let X_1, X_2, X_3 be independent exponential random variables with respective hazard rates 5,4 and 1. Using (3.3.2), we find after some simplifications that

$$\begin{aligned} h(y) &= P(D_{3:3} > 2|D_{2:3} > y) \\ &= \frac{20\left((\frac{1}{9e^{9y}} + \frac{1}{6e^{6y}})(\frac{1}{5})e^{-10} + (\frac{1}{9e^{9y}} + \frac{1}{5e^{5y}})(\frac{1}{4})e^{-8} + (\frac{1}{6e^{6y}} + \frac{1}{5e^{5y}})e^{-2}\right)}{e^{-20y}(e^{11y} + 4e^{14y} + 5e^{15y})} \end{aligned}$$

It is clear from the plot of $h(y)$ below that the function $h(y)$ is not monotonically increasing, proving thereby that $D_{3:3}$ is even not RTI in $D_{2:3}$. Hence $D_{2:3}$ and $D_{3:3}$ are not TP_2 dependent.

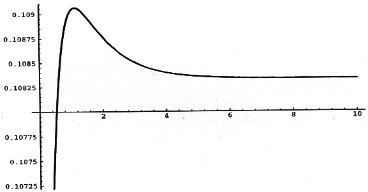

Figure 3.2.1. Graph of $h(y)$.

The covariance between $D_{i:n}$ and $D_{j:n}$ for $i < j$ is

$$cov(D_{i:n}, D_{j:n}) = \sum_{(\mathbf{r})} \frac{\prod_{i=1}^{n} \lambda_i}{\prod_{i=1}^{n} \sum_{j=i}^{n} \lambda(r_j)} \left\{ \sum_{m=i}^{n} \lambda(r_m) \right\}^{-1} \left\{ \sum_{m=j}^{n} \lambda(r_m) \right\}^{-1}$$

$$- \left[\sum_{(\mathbf{r})} \frac{\prod_{i=1}^{n} \lambda_i}{\prod_{i=1}^{n} \sum_{j=i}^{n} \lambda(r_j)} \left\{ \sum_{m=i}^{n} \lambda(r_m) \right\}^{-1} \right] \left[\sum_{(\mathbf{r})} \frac{\prod_{i=1}^{n} \lambda_i}{\prod_{i=1}^{n} \sum_{j=i}^{n} \lambda(r_j)} \left\{ \sum_{m=j}^{n} \lambda(r_m) \right\}^{-1} \right]$$

We conjecture that, in general, the covariance between $D_{i:n}$ and $D_{j:n}$ for $i < j$ is nonnegative. We prove this conjecture for $n = 3$ in Corollary 3.3.1. In fact we prove in the next theorem that the covariance between $D_{2:3}$ and $D_{3:3}$ is Schur-convex in λ_i's.

THEOREM 3.3.1 *Let X_1, X_2, X_3 be independent exponential random variables having hazard rates $\lambda_1, \lambda_2, \lambda_3$ respectively. Then $cov(D_{2:3}, D_{3:3})$ is Schur-convex in λ_i's.*

PROOF : The covariance between $D_{2:3}$ and $D_{3:3}$ is

$$\phi(\lambda_1, \lambda_2, \lambda_3) = cov(D_{2:3}, D_{3:3})$$

$$
\begin{aligned}
= \quad & (\lambda_1\lambda_2\lambda_3)(\lambda_1+\lambda_2+\lambda_3)^{-1}\left\{(\lambda_1^{-2}+\lambda_2^{-2})(\lambda_1+\lambda_2)^{-2}\right. \\
& \left.+ (\lambda_1^{-2}+\lambda_3^{-2})(\lambda_1+\lambda_3)^{-2} + (\lambda_2^{-2}+\lambda_3^{-2})(\lambda_2+\lambda_3)^{-2}\right\} \\
& - \left\{(\lambda_1+\lambda_2+\lambda_3)^{-1}\left(\lambda_3(\lambda_1+\lambda_2)^{-1}\right.\right. \\
& \left.\left.+\lambda_2(\lambda_1+\lambda_3)^{-1} + \lambda_1(\lambda_2+\lambda_3)^{-1}\right)\right\} \qquad (3.3.3) \\
& \times \left\{(\lambda_1\lambda_2\lambda_3)(\lambda_1+\lambda_2+\lambda_3)^{-1}\{(\lambda_2^{-2}+\lambda_3^{-2})(\lambda_2+\lambda_3)^{-1}\right. \\
& \left.+ \left(\lambda_1^{-2}+\lambda_3^{-2})(\lambda_1+\lambda_3)^{-1} + (\lambda_1^{-2}+\lambda_2^{-2})(\lambda_1+\lambda_2)^{-1}\right\}\right\}.
\end{aligned}
$$

After some simplifications, we find that

$$(\lambda_1-\lambda_2)\left\{\phi_{(1)}(\lambda_1,\lambda_2,\lambda_3) - \phi_{(2)}(\lambda_1,\lambda_2,\lambda_3)\right\}$$

is equal to

$$\frac{8(\lambda_1-\lambda_2)^2\lambda_3^2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)},$$

which is nonnegative for all $\lambda_1, \lambda_2, \lambda_3 > 0$. Since the function ϕ is symmetric in $(\lambda_1, \lambda_2, \lambda_3)$, the required result follows from Theorem 2.2.4.

COROLLARY 3.3.1 *Under the assumptions of Theorem 3.3.1,* $cov(D_{2:3}, D_{3:3}) \geq 0$ *and* $var(X_{1:3}) \leq var(X_{2:3}) \leq var(X_{3:3})$.

PROOF : Let $\overline{\lambda}$ be the average of λ_i's. Since $(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}) \overset{m}{\preceq} (\lambda_1, \lambda_2, \lambda_3)$, it follows from Theorem 3.3.1 that

$$\phi(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}) \leq \phi(\lambda_1, \lambda_2, \lambda_3), \qquad (3.3.4)$$

where the function ϕ is given by (3.3.3). The L.H.S of (3.3.4) is zero, since spacings of a random sample from an exponential distribution are independent. This proves that $cov(D_{2:3}, D_{3:3}) \geq 0$. Since $D_{1:3}$ is independent of $D_{2:3}$ and $D_{3:3}$, it follows that $cov(X_{3:3} - X_{2:3}, X_{2:3}) \geq 0$. The required result follows from Lemma 3.2.1.

■

To prove the other results of this section we shall be repeatedly using the following known result.

THEOREM 3.3.2 *(Shaked and Spizzichino, 1998) Let the joint distribution function of* $\mathbf{X} = (X_1, \ldots, X_n)$ *be*

$$F(x_1, \ldots, x_n) = \int_{-\infty}^{+\infty} \prod_{i=1}^{n} F_i(x_i|\theta_i) dG(\theta),$$

where $F_i(.|\theta)$ *is an absolutely continuous distribution function with respect to Lebesgue measure on* R *for each* θ *in the support of* Θ *with density function* $f_i(.|\theta)$ *for* $i = 1, \ldots, n$. *Suppose that the support of* $(X_1, \ldots, X_n)$ *is a lattice. If* $f_i(x|\theta)$ *is* TP_2 (RR_2) *in* (x, θ) *for all* $i \in \{1, \ldots, n\}$, *then* $(X_1, \ldots, X_n)$ *is* MTP_2.

Now we examine the problem of dependence among spacings when random variables $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ and λ^*. Theorem 3.3.3 which follows, replaces the incorrect result of Gross, Hunt and Odeh (1986) for the single-outlier exponential model.

THEOREM 3.3.3 *Let* $X_1, \ldots, X_n$ *be independent exponential random variables such that* X_i *has hazard rate* λ *for* $i \in \{1, \ldots, n-1\}$ *and* X_n *has hazard rate* λ^*. *Then* $(D_{1:n}, \ldots, D_{n:n})$ *is* MTP_2 *dependent.*

PROOF : Using (3.3.2), we find that the joint pdf of $(D_{1:n}, \ldots, D_{n:n})$ in this case is

$$f_{D_{1:n},\ldots,D_{n:n}}(x_1, \ldots, x_n) = \sum_{\theta=1}^{n} \frac{(n-1)!\lambda^*(\lambda)^{n-1}}{\prod_{i=1}^{\theta}((n-i)\lambda + \lambda^*) \prod_{i=\theta+1}^{n}(n-i+1)\lambda}$$

$$\times \prod_{i=1}^{\theta}((n-i)\lambda + \lambda^*) e^{-((n-i)\lambda+\lambda^*)x_i} \prod_{i=\theta+1}^{n} (n-i+1)\lambda e^{-(n-i+1)\lambda x_i}, \qquad (3.3.5)$$

which can be expressed as

$$f_{D_{1:n},\ldots,D_{n:n}}(x_1,\ldots,x_n) = \int_{-\infty}^{+\infty} \prod_{i=1}^{n} f_{D_{i:n}}(x_i|\theta)dH(\theta),$$

where $H(\theta)$ is distribution function of a discrete random variable Θ with following probability mass function,

$$h(\theta) = \frac{(n-1)!\lambda^*(\lambda)^{n-1}}{\prod_{i=1}^{\theta}((n-i)\lambda+\lambda^*)\prod_{i=\theta+1}^{n}(n-i+1)\lambda}, \qquad \textit{for} \;\; \theta = 1,\ldots,n; \quad (3.3.6)$$

and

$$f_{D_{i:n}}(x|\theta) = \begin{cases} ((n-i)\lambda+\lambda^*)e^{-((n-i)\lambda+\lambda^*)x}, & i \leq \theta \\ (n-i+1)\lambda e^{-(n-i+1)\lambda x}, & i \geq \theta+1. \end{cases} \quad (3.3.7)$$

We show that the conditional densities as given by (3.3.7) are all TP_2 if $\lambda^* < \lambda$ and are all RR_2 if $\lambda^* > \lambda$. Suppose $\theta_1 < \theta_2$ and $\theta_1, \theta_2 \in \{1,\ldots,n\}$. Then the ratio,

$$\frac{f_{D_{i:n}}(x|\theta_2)}{f_{D_{i:n}}(x|\theta_1)} = \begin{cases} 1, & i \leq \theta_1 \\ \frac{((n-i)\lambda+\lambda^*)e^{-((n-i)\lambda+\lambda^*)x}}{(n-i+1)\lambda e^{-(n-i+1)\lambda x}}, & \theta_1 < i \leq \theta_2 \\ 1, & \theta_2 < i \end{cases}$$

is a constant function of x, for $i \leq \theta_1$ or $i > \theta_2$ and is an increasing function of x for $\theta_1 < i \leq \theta_2$, if $\lambda^* < \lambda$. That is $f_{D_{i:n}}(x|\theta)$ is TP_2 in (x,θ), if $\lambda^* < \lambda$, for $i = 1,\ldots,n$. Similarly it is RR_2 in (x,θ), if $\lambda^* > \lambda$. The required result follows from Theorem 3.3.2.

■

The following corollary is an immediate result of Theorem 3.3.3 and Corollary 3.2.1

COROLLARY 3.3.2 *Under the assumptions of Theorem 3.3.3,*

$$var(X_{1:n}) \leq var(X_{2:n}) \leq \ldots \leq var(X_{n:n}).$$

In the next theorem we consider the multiple-outliers model. We prove that in this case $D_{i:n}$ and $D_{i+1:n}$ are TP_2 dependent for $i = 1, \ldots, n-1$. It is not known whether the spacings are MTP_2 dependent in this case.

THEOREM **3.3.4** *Let $X_1, \ldots, X_n$ be independent exponential random variables such that X_i has hazard rate λ for $i \in \{1, \ldots, k\}$ and hazard rate λ^* for $i \in \{k+1, \ldots, n\}$, $k \in \{2, \ldots, n-2\}$. Then $D_{i:n}$ and $D_{i+1:n}$ are TP_2 dependent.*

PROOF : Without loss of generality we assume that $k \leq n-k$.

Case (i) Let $k < i \leq n-k$.

From (3.3.2) the joint pdf of $(D_{i:n}, D_{i+1:n})$ for this set of λ_i's can be expressed as

$$f_{D_{i:n},D_{i+1:n}}(x,y) = \int_{-\infty}^{+\infty} f_{D_{i:n}}(x|\theta) f_{D_{i+1:n}}(y|\theta) dH(\theta),$$

where $H(\theta)$ here is distribution function of a discrete random variable Θ taking values $0, 1, 2, \ldots, 2k$ with following probability mass function. For $\theta = 0, 2, 4, \ldots, 2k$,

$$h(\theta) = \lambda^k (\lambda^*)^{n-k} k!(n-k)! \sum_{(r_\theta)} \frac{1}{\prod_{i=1}^n \sum_{j=i}^n \lambda(r_j)},$$

where summation is being taken over all permutations of

$$(\mathbf{r}_\theta) = (\overbrace{\lambda, \ldots, \lambda}^{k-\theta/2}, \overbrace{\lambda^*, \ldots, \lambda^*}^{i-1-k+\theta/2}, \overbrace{\lambda^*}^{1}, \overbrace{\lambda, \ldots, \lambda}^{\theta/2}, \overbrace{\lambda^*, \ldots, \lambda^*}^{n-i-\theta/2}) \qquad (3.3.8)$$

for which the ith component of $(\mathbf{r}_\theta)$ is λ^* and its last $(n-i)$ components consist of $(\theta/2)$ λ's and $(n-i-\theta/2)$ λ^*'s.

For $\theta = 1, 3, 5, \ldots, 2k-1$,

$$h(\theta) = \lambda^k (\lambda^*)^{n-k} k!(n-k)! \sum_{(\mathbf{r}'_\theta)} \frac{1}{\prod_{i=1}^n \sum_{j=i}^n \lambda(r'_j)},$$

where summation is being taken over all permutations of

$$(\mathbf{r}'_\theta) = (\overbrace{\lambda,\ldots,\lambda}^{k-(\theta+1)/2},\ \overbrace{\lambda^*,\ldots,\lambda^*}^{i-1-k+(\theta+1)/2},\ \overbrace{\lambda}^{1},\ \overbrace{\lambda,\ldots,\lambda}^{(\theta+1)/2-1},\ \overbrace{\lambda^*,\ldots,\lambda^*}^{n-i-(\theta+1)/2+1}\ ) \qquad (3.3.9)$$

for which the ith component of $(\mathbf{r}'_\theta)$ is λ and the last $(n-i)$ components of $(\mathbf{r}'_\theta)$ consist of $((\theta+1)/2-1)$ λ's and $(n-i-(\theta+1)/2+1)$ λ^*'s.

For $\theta \in \{0,\ldots,2k\}$,

$$f_{D_{i:n}}(x|\theta) = \{(n-i-[(\theta+1)/2]+1)\lambda^* + [(\theta+1)/2]\lambda\} \times e^{-\{(n-i-[(\theta+1)/2]+1)\lambda^*+[(\theta+1)/2]\lambda\}x}, \qquad (3.3.10)$$

and

$$f_{D_{i+1:n}}(x|\theta) = \{(n-i-[\theta/2])\lambda^* + [\theta/2]\lambda\}e^{-\{(n-i-[\theta/2])\lambda^*+[\theta/2]\lambda\}x}, \qquad (3.3.11)$$

where $[x]$ denote the greatest integer less than or equal to x.

To prove the required result we show that $f_{D_{i:n}}(x|\theta)$ and $f_{D_{i+1:n}}(x|\theta)$ are all TP_2 if $\lambda < \lambda^*$ and are all RR_2 if $\lambda > \lambda^*$.

$$\frac{f_{D_{i:n}}(x|\theta+1)}{f_{D_{i:n}}(x|\theta)} = \frac{\{(n-i-[(\theta+2)/2]+1)\lambda^* + [(\theta+2)/2]\lambda\}e^{-\{(n-i-[(\theta+2)/2]+1)\lambda^*+[(\theta+2)/2]\lambda\}x}}{\{(n-i-[(\theta+1)/2]+1)\lambda^* + [(\theta+1)/2]\lambda\}e^{-\{(n-i-[(\theta+1)/2]+1)\lambda^*+[(\theta+1)/2]\lambda\}x}}$$

$$= \begin{cases} 1 & \text{if } \theta = 1,3,5,\ldots,2k-1 \\ \frac{\{(n-i-\theta/2)\lambda^*+(\theta/2+1)\lambda\}}{\{(n-i-\theta/2+1)\lambda^*+(\theta/2)\lambda\}}e^{-(\lambda-\lambda^*)x} & \text{if } \theta = 0,2,4,\ldots,2k-2. \end{cases} \qquad (3.3.12)$$

From (3.3.12) we conclude that if $\lambda < \lambda^*$ $(\lambda > \lambda^*)$ then $f_{D_{i:n}}(x|\theta)$ is TP_2 (RR_2) for $i = 1\ldots,n$. Similarly for $f_{D_{i+1:n}}(x|\theta)$ we have,

$$\frac{f_{D_{i+1:n}}(x|\theta+1)}{f_{D_{i+1:n}}(x|\theta)} =$$

$$\frac{\{(n-i-[(\theta+1)/2])\lambda^* + [(\theta+1)/2]\lambda\}e^{-\{(n-i-[(\theta+1)/2])\lambda^*+[(\theta+1)/2]\lambda\}x}}{\{(n-i-[\theta/2])\lambda^* + [\theta/2]\lambda\}e^{-\{(n-i-[\theta/2])\lambda^*+[\theta/2]\lambda\}x}}$$
$$= \begin{cases} 1 & \text{if } \theta = 0,2,4,\ldots,2k-2 \\ \frac{\{(n-i-(\theta+1)/2)\lambda^*+((\theta+1)/2)\lambda\}}{\{(n-i-(\theta-1)/2)\lambda^*+((\theta-1)/2)\lambda\}} e^{-(\lambda-\lambda^*)x} & \text{if } \theta = 1,3,5,\ldots,2k-1. \end{cases} \tag{3.3.13}$$

Again from (3.3.13), it follows that $f_{D_{i+1:n}}(x|\theta)$ is TP_2 (RR_2) if $\lambda < \lambda^*$ $(\lambda > \lambda^*)$. Using these observations, the required result follows from Theorem 3.3.2.

Case (ii) $\quad i > n-k$.

In this case for $\theta \in \{0, 2, \ldots, 2(n-i)\}$, $(\mathbf{r}_\theta)$ is given by (3.3.8) and for $\theta \in \{1, 3, \ldots, 2(n-i)+1\}$, $(\mathbf{r}'_\theta)$ is given by (3.3.9). Hence for $\theta \in \{0, 1, 2, \ldots, 2(n-i+1)-1\}$, $f_{D_{i:n}}(x|\theta)$ and $f_{D_{i+1:n}}(x|\theta)$ are the same as given by (3.3.10) and (3.3.11) respectively. The required result follows from the same kind of arguments as in case (i).

Case (iii) $\quad i \le k$.

The proof is similar to the previous case. The vectors $(\mathbf{r}_\theta)$ and $(\mathbf{r}'_\theta)$ corresponding to (3.3.8) and (3.3.9) are as follows. For $\theta = 0, 2, \ldots, 2i-2$

$$(\mathbf{r}_\theta) = (\overbrace{\lambda,\ldots,\lambda}^{i-1-\theta/2}, \overbrace{\lambda^*,\ldots,\lambda^*}^{\theta/2}, \overbrace{\lambda}^{1}, \overbrace{\lambda,\ldots,\lambda}^{k-i+\theta/2}, \overbrace{\lambda^*,\ldots,\lambda^*}^{n-k-\theta/2}),$$

for which the ith component of $(\mathbf{r}_\theta)$ is λ and the last $(n-i)$ components of $(\mathbf{r}_\theta)$ consist of $(k-i+\theta/2)$ λ's and $(n-k-\theta/2)$ λ^*'s.

For $\theta = 1, 3, \ldots, 2i-1$

$$(\mathbf{r}'_\theta) = (\overbrace{\lambda,\ldots,\lambda}^{i-((\theta+1)/2)}, \overbrace{\lambda^*,\ldots,\lambda^*}^{((\theta+1)/2)-1}, \overbrace{\lambda^*}^{1}, \overbrace{\lambda,\ldots,\lambda}^{k-i+((\theta+1)/2))}, \overbrace{\lambda^*,\ldots,\lambda^*}^{n-k-((\theta+1)/2))}),$$

for which the ith component of $(\mathbf{r}'_\theta)$ is λ^* and the last $(n-i)$ components of $(\mathbf{r}'_\theta)$ consist of $(k-i+(\theta+1)/2)$ λ's and $(n-k-(\theta+1)/2)$ λ^*'s.

Therefore, For $\theta \in \{0, \ldots, 2i-1\}$,

$$f_{D_{i:n}}(x|\theta) = \{(k-i+1+[\theta/2])\lambda + (n-k-[\theta/2])\lambda^*\} \times$$

$$e^{-\{(k-i+1+[\theta/2])\lambda+(n-k-[\theta/2])\lambda^*\}x},$$

and

$$f_{D_{i+1:n}}(x|\theta) = \{(k-i-[(\theta+1)/2])\lambda + (n-k-[(\theta+1)/2])\lambda^*\} \times e^{-\{(k-i-[(\theta+1)/2])\lambda+(n-k-[(\theta+1)/2])\lambda^*\}x}.$$

The required result follows from the same kind of arguments as in case (i).

■

3.4 Hazard rate ordering among spacings

Many authors have studied the stochastic properties of spacings from restricted families of distributions. Barlow and Proschan (1966) proved that if $X_1, \ldots, X_n$ is a random sample from a DFR distribution, then the successive normalized spacings are stochastically increasing. Kochar and Kirmani (1995) strengthened this result from stochastic ordering to hazard rate ordering, that is, for $i = 1, \ldots, n-1$

$$D^*_{i+1:n} \geq_{hr} D^*_{i:n}. \tag{3.4.1}$$

The corresponding problem when the random variables are not identically distributed, has also been studied by many researchers, including Pledger and Proschan (1971), Shaked and Tong (1984), Kochar and Korwar (1996), Kochar and Rojo (1996), Nappo and Spizzichino (1998), among others. For a review of this topic see Kochar (1998).

Kochar and Korwar (1996) conjectured that a result similar to (3.4.1) holds in the case when $X_1, \ldots, X_n$ are independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. In the next theorem we prove this conjecture when random variables X_i's follow a single outlier model with parameters λ and λ^*. Using (3.3.5), the joint density function of $(D_{1:n}, \ldots, D_{n:n})$ can be expressed as

$$f_{D_{1:n},\ldots,D_{n:n}}(x_1, \ldots, x_n) = \sum_{\theta=1}^{n} h(\theta) \prod_{i=1}^{\theta} \alpha_i^* e^{-\alpha_i^* x_i} \prod_{i=\theta+1}^{n} \alpha_i e^{-\alpha_i x_i},$$

where $\alpha_i = (n-i+1)\lambda$, $\alpha_i^* = (n-i)\lambda + \lambda^*$, $i = 1, \ldots, n$ and $h(\theta)$ is given by (3.3.6). Using α_i and α_i^*, the function h can be written as

$$h(\theta) = \frac{(n-1)!\lambda^{n-1}\lambda^*}{\prod_{i=1}^{\theta} \alpha_i^* \prod_{i=\theta+1}^{n} \alpha_i}, \quad \theta = 1, \ldots, n. \tag{3.4.2}$$

The marginal density function of $D_{i:n}$ can be expressed as

$$f_{D_{i:n}}(x) = H_i \alpha_i e^{-\alpha_i x} + \overline{H}_i \alpha_i^* e^{-\alpha_i^* x}, \tag{3.4.3}$$

where

$$H_i = \sum_{\theta=1}^{i-1} h(\theta), \; i = 1, \ldots, n. \tag{3.4.4}$$

Thus, the density function of $D_{i:n}$ is a mixture of two exponential random variables with parameters α_i and α_i^*.

THEOREM **3.4.1** *Let $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ and λ^*. Then*

$$D_{i+1:n}^* \geq_{hr} D_{i:n}^*, \quad i = 1, \ldots, n-1.$$

PROOF : We prove the result when $\lambda^* > \lambda$. The proof for the case $\lambda^* < \lambda$ follows using the same kind of arguments. From (3.4.3) we find that the survival function of $D_{i:n}^*$ is $\overline{F}_{D_{i:n}^*}(x) = H_i e^{-\lambda x} + \overline{H}_i e^{-\eta_i x}$, where $\eta_i = \frac{(n-i)\lambda + \lambda^*}{n-i+1}$. To prove the theorem we have to show that for any $i \in \{1, \ldots, n-1\}$,

$$g(x) = \frac{\overline{F}_{D_{i+1:n}^*}(x)}{\overline{F}_{D_{i:n}^*}(x)}$$

is increasing in x. The numerator of $g'(x)$ the derivative of $g(x)$ is

$$\begin{aligned}
A(x) &= [H_i e^{-\lambda x} + \overline{H}_i e^{-\eta_i x}][-\lambda H_{i+1} e^{-\lambda x} - \eta_{i+1}\overline{H}_{i+1} e^{-\eta_{i+1} x}] \\
&\quad + [H_{i+1} e^{-\lambda x} + \overline{H}_{i+1} e^{-\eta_{i+1} x}][\lambda H_i e^{-\lambda x} + \eta_i \overline{H}_i e^{-\eta_i x}] \\
&= (\lambda^* - \lambda)\left\{ \frac{\overline{H}_i H_{i+1}}{n-i+1} e^{-(\eta_i+\lambda)x} \right. \\
&\quad \left. - \frac{\overline{H}_{i+1} H_i}{n-i} e^{-(\eta_{i+1}+\lambda)x} - \frac{\overline{H}_i \overline{H}_{i+1}}{(n-i+1)(n-i)} e^{(\eta_i+\eta_{i+1})x} \right\} \\
&\geq (\lambda^* - \lambda)\left\{ \left(\frac{\overline{H}_i H_{i+1}}{n-i+1} - \frac{\overline{H}_{i+1} H_i}{n-i} \right) e^{-(\eta_{i+1}+\lambda)x} \right. \qquad (3.4.5) \\
&\quad \left. - \frac{\overline{H}_i \overline{H}_{i+1}}{(n-i+1)(n-i)} e^{(\eta_i+\eta_{i+1})x} \right\} \\
&= \frac{(\lambda^* - \lambda)}{(n-i)(n-i+1)} \left\{ \left\{ (n-i)\overline{H}_i - (n-i+1)\overline{H}_{i+1} + \overline{H}_i \overline{H}_{i+1} \right\} \right. \\
&\quad \times \left. e^{-(\eta_{i+1}+\lambda)x} - \overline{H}_i \overline{H}_{i+1} e^{-(\eta_i+\eta_{i+1})x} \right\}. \qquad (3.4.6)
\end{aligned}$$

The inequality in (3.4.5) follows, since $\lambda^* > \lambda$ implies $\eta_{i+1} > \eta_i$.

Again $\lambda^* > \lambda$ implies $\lambda < \eta_i$ and which in turn implies $e^{-(\eta_{i+1}+\lambda)x} \geq e^{-(\eta_i+\eta_{i+1})x}$ for every $x \geq 0$. Also for $\lambda^* > \lambda$,

$$\begin{aligned}
\left\{ (n-i)\overline{H}_i - (n-i+1)\overline{H}_{i+1} \right\} &= (n-i)h(i) - \overline{H}_{i+1} \\
&\geq 0, \qquad (3.4.7)
\end{aligned}$$

since for $\lambda^* > \lambda$, $h(j)$ is a decreasing function of j. Using these results in (3.4.6) we find that $A(x)$ and hence $g'(x)$ is nonnegative for $x \geq 0$. This proves the required result.

■

Let $X_1, \ldots, X_n$ be independent exponential random variables with unequal hazard rates. Pledger and Proschan (1971) proved that for $i \in \{1, \ldots, n\}$, $D_{i:n}$ is stochastically larger when the hazard rates are unequal than when they are all equal. This prompted them to examine the question whether the survival function of $D_{i:n}$ is Schur-convex in $(\lambda_1, \ldots, \lambda_n)$. They came up with a counter example to show that this is not true in general. Kochar and Korwar (1996) proved that in the special case of second spacing, whereas the survival function of $D_{2:n}$ is Schur-convex in $(\lambda_1, \ldots, \lambda_n)$, its hazard rate is not Schur-concave. They proved, however, that the hazard rate of $D_{2:2}$ is Schur-concave. We now examine this question when $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters λ and λ^*. In the rest of this section, we assume that $\lambda^* < \lambda$. We will treat it as a part of the model. It is easy to see that in this case, $(\lambda_1^*, \lambda_1, \ldots, \lambda_1) \overset{m}{\succeq} (\lambda_2^*, \lambda_2, \ldots, \lambda_2)$ if and only if $\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1$ and $\lambda_1^* + (n-1)\lambda_1 = \lambda_2^* + (n-1)\lambda_2$. We prove later in this section that for the single outlier model, for $i \in \{1, \ldots, n\}$, the hazard rate of $D_{i:n}$ is Schur-concave in λ's. To prove it we need the following lemmas.

LEMMA 3.4.1 *Let $X_1, \ldots, X_n$ follow the single outlier exponential model with parameters λ and λ^*. Then*

$$\lambda^* < \lambda \Longrightarrow H_i \le \frac{i-1}{n}, \text{ for } i = 1, \ldots, n, \tag{3.4.8}$$

where H_i is given by (3.4.4). *The inequality in* (3.4.8) *is reversed for $\lambda^* > \lambda$.*

PROOF : $\lambda^* < \lambda$ implies that the function $h(j)$ in (3.4.2) is increasing in j, $j = 1, \ldots, n$. Note that

$$(h(1), h(2), \ldots, h(n)) \overset{m}{\succeq} (1/n, \ldots, 1/n).$$

The required result follows from the definition of majorization.

The proof for the case $\lambda^* > \lambda$ follows from the same kind of arguments.

■

LEMMA 3.4.2 *Let $X_1, \ldots, X_n$ follow the single outlier exponential model with parameters λ_1 and λ_1^*. Let $Y_1, \ldots, Y_n$ be another set of random variables following the single outlier exponential model with parameters λ_2 and λ_2^*. If* *(i)* $\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1$, *then* $\Theta_1 \geq_{lr} \Theta_2$,

(ii) $\lambda_1 < \lambda_2 < \lambda_2^* < \lambda_1^*$, *then* $\Theta_1 \leq_{lr} \Theta_2$,

where Θ_1 *and* Θ_2 *correspond to random variable* Θ *with probability mass function* $h(j)$ *in* (3.4.2) *for* X_i*'s and* Y_i*'s, respectively.*

PROOF : *(i)* We prove that

$$\frac{h_2(\theta+1)}{h_1(\theta+1)} \leq \frac{h_2(\theta)}{h_1(\theta)},$$

where h_1 and h_2 are probability mass functions of Θ_1 and Θ_2, respectively. This inequality holds if and only if

$$\frac{(n-\theta-1)\lambda_1 + \lambda_1^*}{(n-\theta-1)\lambda_2 + \lambda_2^*} \leq \frac{\lambda_1}{\lambda_2}. \tag{3.4.9}$$

Since $\lambda_1^* < \lambda_2^*$ and $\lambda_2 < \lambda_1$, it is easy to see that (3.4.9) is true.

(ii) In this case the inequality in (3.4.9) is reversed which in turn implies that $\Theta_1 \leq_{lr} \Theta_2$. This proves the result.

■

THEOREM 3.4.2 *Let* $X_1, \ldots, X_n$ *follow the single outlier exponential model with parameters* λ_1 *and* λ_1^* *with* $\lambda_1^* < \lambda_1$. *Then for* $i \in \{1, \ldots, n\}$, *the hazard rate of* $D_{i:n}$ *is Schur-concave in* $\{\lambda_1, \ldots, \lambda_1, \lambda_1^*\}$.

PROOF : Let $Y_1, \ldots, Y_n$ be another set of random variables following the single outlier exponential model with parameters λ_2 and λ_2^* $(\lambda_2^* < \lambda_2)$ such that $(\lambda_1^*, \lambda_1, \ldots, \lambda_1) \stackrel{m}{\succeq} (\lambda_2^*, \lambda_2, \ldots, \lambda_2)$. As discussed above this holds if and only if $\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1$ and $\lambda_1^* + (n-1)\lambda_1 = \lambda_2^* + (n-1)\lambda_2$. Without loss of generality, let us assume that $\lambda_1^* + (n-1)\lambda_1 = 1$. We have to prove that under the given conditions for $i = 1, \ldots, n$,

$$D_{i:n}^{(1)} \geq_{hr} D_{i:n}^{(2)},$$

where $D^{(1)}_{i:n}$ ($D^{(2)}_{i:n}$) denotes the ith spacing of X_i's (Y_i's). From (3.4.3) the survival functions of $D^{(1)}_{i:n}$ and $D^{(2)}_{i:n}$ are

$$\overline{F}_{D^{(1)}_{i:n}}(x) = P_i e^{-\alpha_{i1}x} + \overline{P}_i e^{-\alpha^*_{i1}x},$$

$$\overline{F}_{D^{(2)}_{i:n}}(x) = Q_i e^{-\alpha_{i2}x} + \overline{Q}_i e^{-\alpha^*_{i2}x},$$

where P_i and Q_i correspond to H_i in (3.4.3) for $D^{(1)}_{i:n}$ and $D^{(2)}_{i:n}$, respectively and $\alpha_{i1} = (n-i+1)\lambda_1$, $\alpha^*_{i1} = (n-i)\lambda_1 + \lambda^*_1$, $\alpha_{i2} = (n-i+1)\lambda_2$ and $\alpha^*_{i2} = (n-i)\lambda_2 + \lambda^*_2$.

We have to show that

$$\phi(x) = \frac{\overline{F}_{D^{(1)}_{i:n}}(x)}{\overline{F}_{D^{(2)}_{i:n}}(x)}$$

is increasing in x. After some simplifications the numerator of $\phi'(x)$ the derivative of $\phi(x)$ is

$$\begin{aligned} g(x) &= -(\alpha_{i1}-\alpha_{i2})P_iQ_ie^{-(\alpha_{i1}+\alpha_{i2})x} + (\alpha^*_{i2}-\alpha^*_{i1})\overline{P}_i\overline{Q}_ie^{-(\alpha^*_{i1}+\alpha^*_{i2})x} \\ &- (\alpha^*_{i1}-\alpha_{i2})Q_i\overline{P}_ie^{-(\alpha_{i2}+\alpha^*_{i1})x} + (\alpha^*_{i2}-\alpha_{i1})\overline{Q}_iP_ie^{-\alpha_{i1}+\alpha^*_{i2})x}, \end{aligned} \quad (3.4.10)$$

Using the assumption $\lambda^*_1 < \lambda^*_2 < \lambda_2 < \lambda_1$ and the fact the $\lambda^*_i + (n-1)\lambda_i = 1$, $i = 1, 2$, it follows, $\alpha_{i1}+\alpha^*_{i2} < \alpha_{i1}+\alpha_{i2}$, $\alpha_{i1}+\alpha^*_{i2} > \alpha^*_{i1}+\alpha^*_{i2}$, $\alpha_{i1}+\alpha^*_{i2} > \alpha^*_{i1}+\alpha_{i2}$ and all $(\alpha_{i1}-\alpha_{i2})$, $(\alpha^*_{i2}-\alpha^*_{i1})$, $(\alpha_{i2}-\alpha^*_{i1})$, are nonnegative. Using these observations in (3.4.10), we see

$$\begin{aligned} g(x) &\geq e^{-(\alpha_{i1}+\alpha^*_{i2})x}\{-(\alpha_{i1}-\alpha_{i2})P_iQ_i + (\alpha^*_{i2}-\alpha^*_{i1})\overline{P}_i\overline{Q}_i \\ &\quad -(\alpha_{i1}-\alpha^*_{i2})\overline{Q}_iP_i + (\alpha_{i2}-\alpha^*_{i1})Q_i\overline{P}_i\} \\ &= \frac{e^{-(\alpha_{i1}+\alpha^*_{i2})x}}{n-1}\{Q_i - P_i - (nQ_i-(i-1))\lambda^*_2 + (nP_i-(i-1))\lambda^*_1\} \\ &\geq \frac{e^{-(\alpha_{i1}+\alpha^*_{i2})x}}{n-1}\{Q_i - P_i - n(Q_i-P_i)\lambda^*_2\} \qquad (3.4.11) \\ &= \frac{e^{-(\alpha_{i1}+\alpha^*_{i2})x}}{n-1}(Q_i-P_i)(1-n\lambda^*_2) \\ &\geq 0. \qquad (3.4.12) \end{aligned}$$

The inequality in (3.4.11) follows, since by Lemma 3.4.1 $P_i \leq \frac{i-1}{n}$ and $\lambda_1^* < \lambda_2^*$. From Lemma 3.4.2 it follows that $Q_i \geq P_i$, since it is known the likelihood ratio ordering implies usual stochastic ordering. This observation along with the fact that $\lambda_2^* \leq 1/n$ implies the inequality in (3.4.12).

■

Remark : The result of Theorem 3.4.2 holds if instead of $\lambda_1^* < \lambda_1$ and $\lambda_2^* < \lambda_2$ we assume that $\lambda_1^* > \lambda_1$ and $\lambda_2^* > \lambda_2$.

It is known that spacings of independent exponential random variables have DFR distributions (cf. Kochar and Korwar, 1996). Combining this observation with Theorem 2.2.2, we have proved the following corollary.

COROLLARY 3.4.1 *Under the assumptions of Theorem 3.4.2,*

$$D_{i:n}^{(1)} \geq_{disp} D_{i:n}^{(2)}.$$

A consequence of Corollary 3.4.1 is that $var(D_{i:n}^{(1)}) \geq var(D_{i:n}^{(2)})$, $i = 1, \ldots, n$.

3.5 Moment inequalities for order statistics in a single-outlier exponential model

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be random samples on continuous random variables X and Y, respectively. Bartoszewicz (1985) showed that if $X \geq_{disp} Y$ then for $i, j \in \{1, \ldots, n\}$ and $i < j$,

$$var(X_{i:n}) \geq var(Y_{i:n}), \tag{3.5.1}$$

$$cov(X_{i:n}, X_{j:n}) \geq cov(Y_{i:n}, Y_{j:n}), \tag{3.5.2}$$

The results on dependence among spacings in Section 3.3 lead us to extend these results to Theorem 3.5.2 below. To prove it we use the following result.

Theorem 3.5.1 *(Kochar and Korwar, 1996) Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with hazard rate $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i/n$. Then, for $i = 1, \ldots, n$,*

$$D_{i:n}^{(1)} \geq_{disp} D_{i:n}^{(2)},$$

where $D_{i:n}^{(1)}$ ($D_{i:n}^{(2)}$) denotes the ith spacing of X_i's (Y_i's).

Theorem 3.5.2 *Let $X_1, \ldots, X_n$ follow the single outlier exponential model with parameters λ and λ^* and $Y_1, \ldots, Y_n$ be i.i.d exponential random variables with common hazard rate $\overline{\lambda} = (\lambda^* + (n-1)\lambda)/n$. Then, the inequalities* (3.5.1) *and* (3.5.2) *continue to hold.*

Proof : It follows from Theorem 3.3.3 that, $D_{i:n}^{(1)}$'s are MTP_2 dependent and which in turn implies that they are associated implying thereby that $cov(D_{k:n}^{(1)}, D_{l:n}^{(1)})$ is nonnegative, for $k < l$. Now

$$\begin{aligned} var(X_{i:n}) &= var\{D_{1:n}^{(1)} + D_{2:n}^{(1)} + \ldots + D_{i:n}^{(1)}\} \\ &= \sum_{k=1}^{i} var(D_{k:n}^{(1)}) + 2\sum\sum_{k<l} cov(D_{k:n}^{(1)}, D_{l:n}^{(1)}) \\ &\geq \sum_{k=1}^{i} var(D_{k:n}^{(1)}) && (3.5.3) \\ &\geq \sum_{k=1}^{i} var(D_{k:n}^{(2)}) && (3.5.4) \\ &= var(Y_{i:n}). && (3.5.5) \end{aligned}$$

The inequality (3.5.4) follows from Theorem 3.5.1. The equality (3.5.5) follows since the spacings of a random sample from an exponential distribution are independent. This proves inequality (3.5.1).

To prove inequality (3.5.2), without loss of generality assume that $i < j$.

$$\begin{aligned} cov(X_{i:n}, X_{j:n}) &= cov(\sum_{k=1}^{i} D_{k:n}^{(1)}, \sum_{k=1}^{i} D_{k:n}^{(1)} + \sum_{k=i+1}^{j} D_{k:n}^{(1)}) \\ &= var(X_{i:n}) + 2\sum_{k=1}^{i}\sum_{l=i+1}^{j} cov(D_{k:n}^{(1)}, D_{l:n}^{(1)}) \\ &\geq var(Y_{i:n}) + 2\sum_{k=1}^{i}\sum_{l=i+1}^{n} cov(D_{k:n}^{(2)}, D_{l:n}^{(2)}) \qquad (3.5.6) \\ &= cov(Y_{i:n}, Y_{j:n}). \end{aligned}$$

The inequality (3.5.6) follows from (3.5.1), the fact that $cov(D_{k:n}^{(1)}, D_{l:n}^{(1)})$ is non-negative and since the spacings of a random sample from an exponential distribution are independent.

∎

These results give us convenient lower bounds on the variance of the total time on test statistic and on covariances between order statistic.

COROLLARY 3.5.1 *Under the assumptions of Theorem 3.5.2,*

(a) for $i < j$, $cov(X_{i:n}, X_{j:n}) \geq \frac{1}{\lambda^2}\sum_{k=1}^{i}\frac{1}{(n-k+1)^2}$,

(b) $var(\tau(X_{i:n})) \geq \frac{i}{\lambda}$, *where* $\tau(t) = \sum_{k=1}^{i}(n-k+1)D_{k:n}^{(1)} + (n-i)(t - X_{i:n})$ *is the total time on test statistic.*

PROOF : (a)

$$\begin{aligned} cov(Y_{i:n}, Y_{j:n}) &= cov(\sum_{k=1}^{i} D_{k:n}^{(2)}, \sum_{k=1}^{j} D_{k:n}^{(2)}) \\ &= var(\sum_{k=1}^{i} D_{k:n}^{(2)}) \\ &= \sum_{k=1}^{i} var(D_{k:n}^{(2)}) \\ &= \frac{1}{\lambda^2}\sum_{k=1}^{i}\frac{1}{(n-k+1)^2}. \end{aligned}$$

Using part (b) of Theorem 3.5.2, the result follows.

(b)

$$\begin{aligned} var(\tau(X_{i:n})) &\geq \sum_{k=1}^{i}(n-k+1)^2 var(D_{k:n}^{(1)}) \\ &\geq \sum_{k=1}^{i}(n-k+1)^2 var(D_{k:n}^{(2)}) \\ &= \frac{i}{\overline{\lambda}^2} \end{aligned} \tag{3.5.7}$$

The first and second inequalities in (3.5.7) follow from the association property of $D_{k:n}^{(1)}$'s and Theorem 3.5.1. Now the required result follows, since

$$var(D_{k:n}^{(2)}) = \frac{1}{\overline{\lambda}^2(n-k+1)^2}.$$

■

Kim and David (1990) proved that $cov(X_{i:n}, X_{j:n})$ is increasing in i as well as j when $X_1, \ldots, X_n$ is a random sample drawn from a DFR distribution. In the next theorem we prove that the same result holds if instead, the observations follow a single-outlier exponential model.

THEOREM **3.5.3** *Let $X_1, \ldots, X_n$ follow the single outlier exponential model with parameters λ and λ^*. Then*

(a) $cov(X_{i:n}, X_{j:n})$ is increasing in i as well as j.

(b) $var(X_{i:n}) \leq cov(X_{i:n}, X_{i+1:n}) \leq var(X_{i+1:n})$

PROOF : (a)

$$\begin{aligned} cov(X_{i:n}, X_{j+1:n}) - cov(X_{i:n}, X_{j:n}) &= cov(X_{i:n}, X_{j+1:n} - X_{j:n}) \\ &= cov(\sum_{k=1}^{i} D_{k:n}, D_{j+1:n}) \\ &\geq 0 \end{aligned}$$

The last inequality follows, since spacings from a single outlier model are associated.

(b) The proof directly follows from (a).

■

3.6 Concluding remarks

In this chapter we have obtained some new results on dependence among spacings of heterogeneous independent exponential random variables. Whereas in the case of a single-outlier exponential model, the spacings are shown to be MTP_2 dependent, it is not known whether the same result holds for the multiple outliers model. In the latter case, we are only able to establish TP_2 dependence between consecutive spacings. Another unsettled question is to examine whether in the case of independent exponential random variables, in general, the spacings are positively correlated. We have given a proof of this conjecture for $n = 3$.

The results of this chapter are mainly based on Khaledi and Kochar (2000 c).

Chapter 4

Stochastic Comparisons and Dependence among Concomitants of Order Statistics

4.1 Introduction

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample of size n from a continuous bivariate distribution. If we arrange the X's in ascending order as $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ then the Y's associated with these order statistics are denoted by $Y_{[1]}, Y_{[2]}, \ldots, Y_{[n]}$ and are called concomitants of order statistics. They are also known as induced order statistics in the literature. The concomitants are of interest in selection and prediction problems based on the ranks of the X's. For example, when k $(< n)$ individuals having the highest X-scores are selected, we may wish to know the behavior of the corresponding Y-scores. They are also of interest in a variety of estimation problems.

Let $f(y|x)$ denote the conditional pdf of Y given $X = x$ and let

$f_{r_1,\ldots,r_k}(x_1,\ldots,x_k)$ denote the joint pdf of $X_{(r_1)},\ldots,X_{(r_k)}$ with $1 \le r_1 \le \ldots \le r_k \le n$. Then, as discussed in Yang (1977), the joint pdf of the k-concomitants $(Y_{[r_1]},\ldots,Y_{[r_k]})$ $(1 \le k \le n)$ is

$$f_{Y_{[r_1]},\ldots,Y_{[r_k]}}(y_1,\ldots,y_k) = \int_{-\infty}^{+\infty}\int_{-\infty}^{x_k}\cdots\int_{-\infty}^{x_2}\left\{\prod_{i=1}^{k} f(y_i|x_i)\right\} f_{r_1,\ldots,r_k}(x_1,\ldots,x_k)\prod_{i=1}^{k} dx_i \,. \tag{4.1.1}$$

From this we obtain the marginal pdf of the rth concomitant $Y_{[r]}$ as

$$f_{Y_{[r]}}(y) = \int_{-\infty}^{+\infty} f(y|x)f_r(x)dx \,, \tag{4.1.2}$$

where f_r is the density of $X_{(r)}$, $r = 1,\ldots,n$.

Under the assumption that X and Y are linearly related, apart from an independent error term, the small sample theory of the concomitants of order statistics has been discussed in David (1973) and Kim and David (1990). For a comprehensive review of this topic see David and Nagaraja (1998).

In this chapter we consider the problems of stochastic comparisons and dependence among concomitants of order statistics. While we are not aware of any previous results on stochastic orderings among concomitants of order statistics, some results on dependence among them are known for certain special types of models. (cf. Kim and David, 1990). Intuitively, it is clear that when X and Y are positively (negatively) dependent, the $Y_{[i]}$'s should be increasing (decreasing) in some stochastic sense. By assuming different kinds of dependence between X and Y, we obtain various types of stochastic ordering and dependence results among the $Y_{[i]}$'s. The results obtained are general in the sense that they apply to any bivariate distribution with monotone dependence between the variables X and Y. In Section 4.2, we discuss the dependence properties of the concomitants of order statistics. Section 4.3 is devoted to the study of stochastic comparisons among them. In Section 4.4, we apply the results obtained in these two sections to the model $Y = g(X) + Z$, where the random variable Z is independent of the random variable X and the function g is an arbitrary monotone function.

4.2 Dependence among concomitants of order statistics

Kim and David (1990) studied the problem of dependence among concomitants of order statistics for the model

$$Y = g(X) + Z,$$

where random variables X and Z are mutually independent. They proved that if g is increasing, then $(Y_{[1]}, \ldots, Y_{[n]})$ are associated. Under the additional condition that Z has a log-concave density, they proved that the joint density of $(Y_{[1]}, \ldots, Y_{[n]})$ is MTP_2. In this section, we discuss in detail the dependence properties of the concomitants of order statistics. The results obtained, are general in the sense that they apply to any distribution with monotone (positive or negative) dependence between the variables X and Y. By assuming different kinds of dependence between X and Y, successively stronger dependence results among the concomitant variables $Y_{[1]}, \ldots, Y_{[n]}$ are established. We shall see that both positive as well as negative dependence between X and Y imply positive dependence among $Y_{[i]}$'s. The following lemmas will be found useful in this development.

LEMMA 4.2.1 *(Karlin, 1968, p. 99) Let A, B and C be subsets of the real line and let $L(x, z)$ be SR_2 for $x \in A$, $z \in B$ and $M(z, y)$ be SR_2 for $z \in B$, $y \in C$. Then $K(x, y) = \int L(x, z)M(z, y)\, d\mu(z)$ is SR_2 for $x \in A$, $y \in C$ and $\varepsilon_i(K) = \varepsilon_i(L) \times \varepsilon_i(M)\ \forall\ i = 1, 2$. Here μ is a sigma-finite measure.*

Thus according to Lemma 4.2.1 the composition of two TP_2 functions or two RR_2 functions is TP_2 and the composition of an RR_2 function and a TP_2 function is RR_2.

LEMMA 4.2.2 *(Karlin, 1968, p. 123) Let λ, x, ζ traverse the linear sets Λ, X and Z, respectively; and consider functions $f(\lambda, x, \zeta)$ and $g(\lambda, \zeta)$ satisfying the conditions*

(a) $f(\lambda, x, \zeta) > 0$ and f is TP_2 in each pair of variables when the third variable is held fixed; and

(b) $g(\lambda, \zeta)$ is TP_2.

Then the function

$$h(\lambda, x) = \int_Z f(\lambda, x, \zeta) g(\lambda, \zeta) d\mu(\zeta) \tag{4.2.1}$$

defined on $\Lambda \times X$ is TP_2. Here μ represents a σ-finite measure.

For establishing positive dependence among $Y_{[i]}$'s when X and Y are negatively dependent, we prove another lemma which may be of independent interest.

LEMMA 4.2.3 *Suppose λ, x, ζ traverse the ordered sets Λ, X and Z, respectively and consider the function $f(\lambda, x, \zeta)$ satisfying the following conditions (a) $f(\lambda, x, \zeta) > 0$ and f is TP_2 in (λ, x); (b) $f(\lambda, x, \zeta)$ is RR_2 in (λ, ζ) as well as in (x, ζ) for all λ, x and ζ . Then the function $h(\lambda, x) = \int_Z f(\lambda, x, \zeta) d\mu(\zeta)$, defined on $\Lambda \times X$ is TP_2 in (λ, x). Here μ represents a σ-finite measure.*

PROOF : We have to prove that for $\lambda_1 < \lambda_2$ and $x_1 < x_2$, $h(\lambda_2, x_2)h(\lambda_1, x_1) - h(\lambda_2, x_1)h(\lambda_1, x_2) \geq 0$. That is,

$$\int_Z f(\lambda_2, x_2, \zeta) d\mu(\zeta) \int_Z f(\lambda_1, x_1, \zeta) d\mu(\zeta) - \int_Z f(\lambda_2, x_1, \zeta) d\mu(\zeta) \int_Z f(\lambda_1, x_2, \zeta) d\mu(\zeta) \geq 0. \tag{4.2.2}$$

The L.H.S. of (4.2.2) is

$$\int\int f(\lambda_2,x_2,\zeta)f(\lambda_1,x_1,u)d\mu(\zeta)d\mu(u)-\int\int f(\lambda_2,x_1,\zeta)f(\lambda_1,x_2,u)d\mu(\zeta)d\mu(u),$$

which is equal to :

$$\int_{-\infty}^{+\infty}\int_{u}^{+\infty}[f(\lambda_2,x_2,\zeta)f(\lambda_1,x_1,u)-f(\lambda_2,x_1,\zeta)f(\lambda_1,x_2,u)]d\mu(\zeta)d\mu(u)$$

$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{u}[f(\lambda_2,x_2,\zeta)f(\lambda_1,x_1,u)-f(\lambda_2,x_1,\zeta)f(\lambda_1,x_2,u)]d\mu(\zeta)d\mu(u). \tag{4.2.3}$$

By changing the order of integration in the second term of (4.2.3) and renaming the variables $\zeta\to u$, $u\to\zeta$, (4.2.3) becomes

$$\int_{-\infty}^{+\infty}\int_{u}^{+\infty}[f(\lambda_2,x_2,\zeta)f(\lambda_1,x_1,u)-f(\lambda_2,x_1,\zeta)f(\lambda_1,x_2,u)$$
$$+f(\lambda_2,x_2,u)f(\lambda_1,x_1,\zeta)-f(\lambda_2,x_1,u)f(\lambda_1,x_2,\zeta)]d\mu(\zeta)d\mu(u). \tag{4.2.4}$$

We shall show that the expression inside the square bracket in (4.2.4) is nonnegative. By assumption (a)

$$\frac{f(\lambda_2,x_2,\zeta)}{f(\lambda_2,x_1,\zeta)}-\frac{f(\lambda_1,x_2,\zeta)}{f(\lambda_1,x_1,\zeta)}\geq 0,$$

Hence

$$\frac{f(\lambda_2,x_2,\zeta)}{f(\lambda_2,x_1,\zeta)}-\frac{f(\lambda_1,x_2,u)}{f(\lambda_1,x_1,u)}+\frac{f(\lambda_2,x_2,u)}{f(\lambda_2,x_1,u)}-\frac{f(\lambda_1,x_2,\zeta)}{f(\lambda_1,x_1,\zeta)}\geq 0.$$

That is,

$$[f(\lambda_2,x_2,\zeta)f(\lambda_1,x_1,u)-f(\lambda_2,x_1,\zeta)f(\lambda_1,x_2,u)]+\frac{f(\lambda_2,x_1,\zeta)f(\lambda_1,x_1,u)}{f(\lambda_2,x_1,u)f(\lambda_1,x_1,\zeta)}$$
$$\times[f(\lambda_2,x_2,u)f(\lambda_1,x_1,\zeta)-f(\lambda_2,x_1,u)f(\lambda_1,x_2,\zeta)]\geq 0. \tag{4.2.5}$$

Note that for $\zeta>u$ the ratio in the L.H.S. of (4.2.5) is at most one since f is RR_2 in λ and ζ. Now since f is TP_2 in (λ,x) and RR_2 in (x,ζ), we get the following inequalities for $x_1<x_2$, $\lambda_1<\lambda_2$ and $u<\zeta$,

$$\frac{f(\lambda_2, x_2, u)}{f(\lambda_2, x_1, u)} \geq \frac{f(\lambda_1, x_2, u)}{f(\lambda_1, x_1, u)} \geq \frac{f(\lambda_1, x_2, \zeta)}{f(\lambda_1, x_1, \zeta)},$$

implying thereby that,

$$\frac{f(\lambda_2, x_2, u)}{f(\lambda_2, x_1, u)} - \frac{f(\lambda_1, x_2, \zeta)}{f(\lambda_1, x_1, \zeta)} \geq 0$$

and which in turn implies that the quantity inside the square brackets in the second term of (4.2.5) is nonnegative. Combining these results we find that the quantity inside the square brackets in (4.2.4) is nonnegative for $\zeta > u$, from which the result follows.

■

The next lemma will be found very useful in proving the various results in this section.

LEMMA 4.2.4 *Let*

$$z(y_1, \ldots, y_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} K(x_i, x_{i+1}) \prod_{i=1}^{n} h(x_i, y_i) \prod_{i=1}^{n} dx_i, \tag{4.2.6}$$

where

$$K(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x \geq y \end{cases}, \tag{4.2.7}$$

and where $x_{n+1} \equiv \infty$. *Then* $h(x, y)$ RR_2 *or* TP_2 *in* (x, y) *implies that the function* z *is* TP_2 *in pairs.*

PROOF : Suppose that $h(x, y)$ is RR_2. By Lemma 4.2.1 the innermost integral in (4.2.6),

$$g_1(x_2, y_1) = \int_{-\infty}^{+\infty} K(x_1, x_2) h(x_1, y_1)\, dx_1,$$

is RR_2 in (x_2, y_1) since K is TP_2 and h is RR_2. The next integral in (4.2.6) is

$$g_2(x_3, y_1, y_2) = \int_{-\infty}^{+\infty} K(x_2, x_3) g_1(x_2, y_1) h(x_2, y_2)\, dx_2. \tag{4.2.8}$$

Again by Lemma 4.2.1, the function g_2 in (4.2.8) is TP_2 in (y_1, y_2), RR_2 in (x_3, y_1) and in (x_3, y_2). We prove the desired result by induction. Define for $i = 1, \ldots, n$

$$g_i(x_{i+1}, y_1, \ldots, y_i) = \int_{-\infty}^{+\infty} g_{i-1}(x_i, y_1, \ldots, y_{i-1}) h(x_i, y_i) K(x_i, x_{i+1})\, dx_i \quad (4.2.9)$$

Assume that g_{i-1} is TP_2 in (y_j, y_k), RR_2 in (x_i, y_j) for $j, k \in \{1, \ldots, i-1\}$. Using Lemma 4.2.1, the function g_i is RR_2 in (y_i, x_{i+1}), TP_2 in (y_i, y_j) and RR_2 in (x_{i+1}, y_j) for $j \in \{1, \ldots, i-1\}$. It remains to show that g_i is TP_2 in (y_j, y_k) for $j, k \in \{1, 2, \ldots, i-1\}$. For fixed $(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{k-1}, y_{k+1}, \ldots, y_{i-1}, y_i)$ and x_{i+1} the function

$$m(x_i, y_j, y_k) = h(x_i, y_i) \times K(x_i, x_{i+1}) \times g_{i-1}(x_i, y_1, \ldots, y_{i-1})$$

is TP_2 in (y_j, y_k), RR_2 in (x_i, y_j) and (x_i, y_k). Now from Lemma 4.2.3 it follows that g_i is TP_2 in (y_j, y_k) for $j, k \in \{1, \ldots, i-1\}$. That is, $g_n(x_{n+1}, y_1, \ldots, y_n) = z(y_1, \ldots, y_n)$ is TP_2 in (y_i, y_j) for $i, j \in \{1, \ldots, n\}$. This proves the required result.

The proof when h is TP_2 follows on the same lines using Lemma 4.2.1, Lemma 4.2.2.

■

Using this lemma we prove a general result on positive dependence among the concomitants of order statistics.

THEOREM 4.2.1 *If (X, Y) is $DRR(0, m)$ or $DTP(0, m)$ then $(Y_{[1]}, \ldots, Y_{[n]})$ is $DTP(m, \ldots, m)$ for all nonnegative integers m.*

PROOF : Suppose that (X, Y) is $DRR(0, m)$. Then by Definition 1.1.9, for $m > 0$ we have

$$\psi_{m,\ldots,m}(y_1, \ldots, y_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \gamma^{(m)}(y_i - t_i) f_{Y_{[1]},\ldots,Y_{[n]}}(t_1, \ldots, t_n) \prod_{i=1}^{n} dt_i$$

$$= n! \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} K(x_i, x_{i+1}) \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left\{ \prod_{i=1}^{n} \gamma^{(m)}(y_i - t_i) f(t_i|x_i) f(x_i)\, dt_i \right\} \prod_{i=1}^{n} dx_i$$

$$= n! \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} K(x_i, x_{i+1}) \prod_{i=1}^{n} \psi_{0,m}(x_i, y_i) \prod_{i=1}^{n} dx_i,$$

where

$$\psi_{0,m}(x_i, y_i) = \int_{-\infty}^{+\infty} \gamma^{(m)}(y_i - t_i) f(t_i|x_i) f(x_i) dt_i,$$

and the function K is given by (4.2.7).

Since (X, Y) being $DRR(0, m)$ is equivalent to $\psi_{0,m}(x, y)$ being RR_2 in (x, y), it follows from Lemma 4.2.4 with $h(x_i, y_i)$ replaced by $\psi_{0,m}(x_i, y_i)$, that $\psi_{m,\ldots,m}(y_1, \ldots, y_n)$ is TP_2 in pairs.

Now let us consider the case when $m = 0$. In this case the function

$$\begin{aligned}\psi_{0,\ldots,0}(y_1, \ldots, y_n) &= f_{Y_{[1]},\ldots,Y_{[n]}}(t_1, \ldots, t_n) \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} K(x_i, x_{i+1}) \prod_{i=1}^{n} f(x_i, y_i) \prod_{i=1}^{n} dx_i\end{aligned}$$

is clearly seen to be $DTP(0, \cdots, 0)$. The proof follows from Lemma 4.2.4 with $h(x, y) = f(x, y)$.

When $(X,\ Y)$ is $DTP(0, m)$, then the function $\psi_{0,m}(x, y)$ is TP_2. The required result follows similarly using the TP_2 part of Lemma 4.2.4.

■

The following results are immediate consequences of the above theorem.

COROLLARY 4.2.1 *(i) If X and Y are TP_2 or RR_2 dependent, then the joint density of $(Y_{[1]}, \ldots, Y_{[n]})$ is TP_2 in pairs.*

(ii) If the conditional hazard rate of Y given $X = x$ is monotone in x, then the concomitants $(Y_{[1]}, \ldots, Y_{[n]})$ are $DTP(1, \ldots, 1)$. In particular $Y_{[i]}$ and $Y_{[j]}$ are $RCSI$ for $i \neq j \in \{1, \ldots, n\}$.

Remark : If the conditions of Corollary 4.2.1 (i) are satisfied and if the support of $(Y_{[1]}, \ldots, Y_{[n]})$ ia a lattice, then $Y_{[1]}, \ldots, Y_{[n]}$ are MTP_2 dependent.

We show in the next theorem that if Y is either stochastically increasing or stochastically decreasing in X, then the $Y_{[i]}$'s are associated.

THEOREM **4.2.2** *If Y is stochastically monotone in X, then $Y_{[1]}, \ldots, Y_{[n]}$ are associated.*

PROOF : We give the proof for the case when Y is stochastically decreasing in X. The proof for the other case follows on the same lines.

Consider arbitrary increasing real-valued functions M and N defined on R^n. Then by the definition of associated variables it is enough to show that

$$Cov(M(\mathbf{Y}_{[]}), N(\mathbf{Y}_{[]})) \geq 0,$$

whenever it exists. Now

$$\begin{aligned} Cov(M(\mathbf{Y}_{[]}), N(\mathbf{Y}_{[]})) &= Cov(E[M(\mathbf{Y}_{[]})|\mathbf{X}_{()}], E[N(\mathbf{Y}_{[]})|\mathbf{X}_{()}]) \\ &\quad + E(Cov[M(\mathbf{Y}_{[]})|\mathbf{X}_{()}, N(\mathbf{Y}_{[]})|\mathbf{X}_{()}]) \quad (4.2.10) \end{aligned}$$

where $\mathbf{Y}_{[]} = (Y_{[1]}, \ldots, Y_{[n]})$ and $\mathbf{X}_{()} = (X_{(1)}, \ldots, X_{(n)})$.

Note that the concomitants $(Y_{[1]}, \ldots, Y_{[n]})$ given the order statistics are conditionally independent and for $1 \leq i \leq n$ the conditional distribution of $Y_{[i]}$ given $X_{(i)} = x$ is the same as that of Y given $X = x$. Therefore,

$$\{\mathbf{Y}_{[]}|\mathbf{X}_{()} = \mathbf{x}_{()}\} \overset{st}{\succeq} \{\mathbf{Y}_{[]}|\mathbf{X}_{()} = \mathbf{x}'_{()}\} \qquad (4.2.11)$$

for $\mathbf{x}_{()} \leq \mathbf{x}'_{()}$ if Y is stochastically decreasing in X.

It follows from (4.2.11) that $E[M(\mathbf{Y}_{[]})|\mathbf{X}_{()} = \mathbf{x}]$ is decreasing in $\mathbf{x}$ since M is increasing. The first term in the R.H.S. of (4.2.10) is the covariance between two decreasing functions of order statistics (which are associated) and hence is nonnegative.

The second term in the R.H.S. of (4.2.10) is also nonnegative since covariance between two increasing functions of independent random variables is nonnegative.

■

Note that the concomitant $Y_{[j]}$'s are associated under the conditions of Corollary 4.2.1 and Theorems 4.2.2. As a consequence $Cov(Y_{[i]}, Y_{[j]}) \geq 0$ for $i, j \in \{1, \ldots, n\}$. However, as shown below this result holds under a rather weaker condition that $E[Y|X = x]$ is monotone in x.

THEOREM **4.2.3** *Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from a distribution for which $E[Y|X = x]$ is monotone in x. Then $Cov(Y_{[i]}, Y_{[j]}) \geq 0$ for all $i, j \in \{1, \ldots, n\}$.*

PROOF : From Yang (1977),

$$\begin{aligned} Cov(Y_{[i]}, Y_{[j]}) &= Cov[E(Y_1|X_1 = X_{(i)}), E(Y_1|X_1 = X_{(j)})] \\ &= Cov[\psi(X_{(i)}), \psi(X_{(j)})] \geq 0 \end{aligned}$$

since $\psi(x) = E(Y|X = x)$ is monotone and $X_{(i)}$ and $X_{(j)}$ are associated.

■

4.3 Stochastic orderings among concomitants of order statistics

In this section we consider the problem of stochastically comparing the concomitant $Y_{[i]}$'s under different kinds of dependence between X and Y. Intuitively, it is clear that when X and Y are positively (negatively) dependent, the $Y_{[i]}$'s should be increasing (decreasing) in some stochastic sense.

It is proved in the next theorem that if Y is stochastically increasing (decreasing) in X, then the concomitant variables $Y_{[i]}$'s are stochastically increasing (decreasing).

THEOREM 4.3.1

$$(a)\quad SI(Y|X) \Rightarrow Y_{[i]} \overset{st:j}{\preceq} Y_{[j]} \Rightarrow Y_{[i]} \leq_{st} Y_{[j]} \quad \textit{for } 1 \leq i < j \leq n, \tag{4.3.1}$$

$$(b)\quad SD(Y|X) \Rightarrow Y_{[i]} \overset{st:j}{\succeq} Y_{[j]} \Rightarrow Y_{[i]} \geq_{st} Y_{[j]} \quad \textit{for } 1 \leq i < j \leq n. \tag{4.3.2}$$

PROOF : (a) Let g be any element of G_{st} as defined in Section 1.1. That is, g is such that

$$g(y_2, y_1) - g(y_1, y_2) \text{ is increasing in } y_2\ \forall y_1. \tag{4.3.3}$$

It is enough to show that for every such function g,

$$E[g(Y_{[j]}, Y_{[i]})] - E[g(Y_{[i]}, Y_{[j]})] \geq 0. \tag{4.3.4}$$

The L.H.S. of (4.3.4) after changing the order of integration is

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_2}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{g(y_2, y_1) - g(y_1, y_2)\}\, f(y_1|x_1) f(y_2|x_2) dy_1 dy_2\right]$$
$$\times f_{i,j}(x_1, x_2)\, dx_1 dx_2$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{x_2}[E(g(Y_2|X_2 = x_2, Y_1|X_1 = x_1)) - E(g(Y_1|X_1 = x_1, Y_2|X_2 = x_2))]$$
$$\times f_{i,j}(x_1, x_2) dx_1 dx_2 \tag{4.3.5}$$

where (X_1, Y_1) and (X_2, Y_2) are two independent copies of (X, Y). Now $SI(Y|X)$ implies that for any $x_1 < x_2$, the expression inside the square brackets in (4.3.5) is nonnegative for all $x_1 < x_2$. The required result now follows from this. The result $Y_{[i]} \leq_{st} Y_{[j]}$ for $i < j$ follows from Theorem 4.9 of Shanthikumar and Yao

(1991) as joint stochastic ordering between two random variables implies usual stochastic ordering between their marginal distributions.

(*b*) By the definition of $SD(Y|X)$, in this case the expression inside the square brackets in (4.3.5) is nonpositive and hence the inequality in (4.3.4) will be reversed.

■

In the next theorem, we make a stronger assumption on the dependence between X and Y and establish hazard rate ordering among the concomitants of order statistics.

THEOREM **4.3.2** *Let $r(y|x)$, the hazard rate of the conditional distribution of Y given $X = x$, be decreasing in x. Then for $1 \leq i < j \leq n$,*

(a) $Y_{[i]} \overset{hr:j}{\preceq} Y_{[j]}$,

(b) $Y_{[i]} \leq_{hr} Y_{[j]}$.

The inequalities in (a) and (b) are reversed in case $r(y|x)$ is increasing in x.

(Note that (*b*) does not follow from (*a*) since, as shown in Shanthikumar and Yao (1991), joint hazard rate ordering may not imply usual hazard rate ordering.)

PROOF : (*a*) We prove the result for $r(y|x)$ decreasing in x. The proof is similar when it is increasing in x. We have to prove that under the given condition

$$E[g(Y_{[j]}, Y_{[i]})] - E[g(Y_{[i]}, Y_{[j]})] \geq 0 \tag{4.3.6}$$

for any bivariate function $g \in G_{hr}$, that is, for a function g satisfying $g(y_2, y_1) - g(y_1, y_2)$ increasing in y_2 for $y_2 \geq y_1$ and for which the expectations exist. As seen in Theorem 3.1, the L.H.S. of (4.3.6) is

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_2}[E(g(Y_2|X_2=x_2,Y_1|X_1=x_1))-E(g(Y_1|X_1=x_1,Y_2|X_2=x_2))] \times f_{i,j}(x_1,x_2)\,dx_1\,dx_2 \tag{4.3.7}$$

where (X_1, Y_1) and (X_2, Y_2) are two independent copies of (X, Y). By the assumption that $r(y|x)$ is decreasing in x, $\{Y|X = x_2\} \geq_{hr} \{Y|X = x_1\}$ for $x_1 < x_2$. Hence the expression inside the square brackets in (4.3.7) is nonnegative for $x_1 \leq x_2$. The required result follows from this.

(*b*) The survival function of $Y_{[i]}$ is

$$\begin{aligned}\overline{F}_{Y_{[i]}}(y) &= \int_{y}^{+\infty}\left[\int_{-\infty}^{+\infty} f(y|x)\, f_i(x)\, dx\right] dy \\ &= \int_{-\infty}^{+\infty}\overline{F}(y|x)\, f_i(x)\, dx\end{aligned}$$

where f_i is the pdf of $X_{(i)}$. Since the successive order statistics are increasing according to likelihood ratio ordering (cf. Shaked and Shanthikumar, 1994, p. 22), the function $f_i(x)$ is TP_2 in (x, i). Also the survival function $\overline{F}(y|x)$ is TP_2 (RR_2) in (x, y) if $r(y|x)$ is decreasing (increasing) in x. It follows from Lemma 4.2.1, that $\overline{F}_{Y_{[i]}}(y)$ is TP_2 or RR_2 in (i, y) depending upon whether $r(y|x)$ is decreasing or increasing in x. That is, $Y_{[i]} \leq_{hr} (\geq_{hr}) Y_{[j]}$ for $1 \leq i < j \leq n$ if $r(y|x)$ is decreasing (increasing) in x.

■

In case X and Y are TP_2 (likelihood ratio dependent) or RR_2 dependent, we get the following stronger result on the stochastic monotonicity of the $Y_{[i]}$'s.

THEOREM 4.3.3 *Suppose that X and Y are TP_2 dependent. Then*

(*a*) $Y_{[i]} \leq_{lr} Y_{[j]}$ *for* $i < j$,

(*b*) $Y_{[1]} \overset{lr:j}{\preceq} \cdots \overset{lr:j}{\preceq} Y_{[n]}$.

The inequalities in (a) and (b) are reversed in case X and Y are RR_2 dependent.

PROOF : (*a*) The density function of $Y_{[i]}$ is

$$f_{Y_{[i]}}(y) = \int_{-\infty}^{+\infty} f(y|x)\, f_i(x) dx.$$

As noted in the previous theorem, the function $f_i(x)$ is TP_2 in (x, i). The TP_2 (RR_2) condition on f is equivalent to $f(y|x)$ being TP_2 (RR_2) in (y, x). The required result immediately follows from Lemma 4.2.1.

(*b*) First we consider the case when the joint density of (X, Y) is TP_2. We have to prove that under this condition the joint density of $(Y_{[1]}, \dots, Y_{[n]})$ is *arrangement increasing*. That is, if $\mathbf{u}$ and $\mathbf{y}$ are vectors of order n such that $\mathbf{u} \overset{a}{\succeq} \mathbf{y}$, then

$$f_{Y_{[1]},\dots,Y_{[n]}}(\mathbf{u}) - f_{Y_{[1]},\dots,Y_{[n]}}(\mathbf{y}) \geq 0. \tag{4.3.8}$$

Clearly the L.H.S. of (4.3.8) is

$$n! \int_{-\infty}^{+\infty} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \left[\prod_{i=1}^{n} f(x_i, u_i) - \prod_{i=1}^{n} f(x_i, y_i) \right] \prod_{i=1}^{n} dx_i.$$

Now the function $\prod_{i=1}^{n} f(x_i, y_i)$ is arrangement increasing if $f_{X,Y}(x, y)$ is TP_2 in (x, y) (cf. Marshall and Olkin, 1979, F. 9. (a), p. 163). Therefore,

$$\prod_{i=1}^{n} f(x_i, u_i) \geq \prod_{i=1}^{n} f(x_i, y_i) \tag{4.3.9}$$

for all $\mathbf{x}$ such that $x_1 \leq x_2 \leq \cdots \leq x_n$. This implies that $f_{Y_{[1]},\dots,Y_{[n]}}(\mathbf{u}) \geq f_{Y_{[1]},\dots,Y_{[n]}}(\mathbf{y})$ for all $\mathbf{u} \overset{a}{\succeq} \mathbf{y}$. That is, $f_{Y_{[1]},\dots,Y_{[n]}} \in \mathcal{AI}$. This proves that $Y_{[1]} \overset{lr:j}{\preceq} \cdots \overset{lr:j}{\preceq} Y_{[n]}$.

Now consider the case when (X, Y) are RR_2 dependent. To establish the required result we have to prove that (4.3.9) holds whenever $\mathbf{y} \overset{a}{\succeq} \mathbf{u}$.

Suppose $x_1 \leq \cdots \leq x_n$, $y_1 \leq \cdots \leq y_n$. Without loss of generality assume that $\mathbf{u} = (y_2, y_1, \ldots, y_n)$. Then $\mathbf{y} \overset{a}{\succeq} \mathbf{u}$ and

$$\prod_{i=1}^{n} f(x_i, y_i) - \prod_{i=1}^{n} f(x_i, u_i) = \prod_{i=3}^{n} f(x_i, u_i)[f(x_1, y_1)f(x_2, y_2) - f(x_1, y_2)f(x_2, y_1)]. \tag{4.3.10}$$

By the RR_2 property of (X, Y), the quantity inside the square brackets in (4.3.10) is nonpositive. The rest of the proof follows on the lines of TP_2 part proving thereby that $Y_{[1]} \overset{lr:j}{\succeq} \cdots \overset{lr:j}{\succeq} Y_{[n]}$. ■

Remark : In the above theorem, (a) does not follows from (b) and vice-versa since, as discussed in Shanthikumar and Yao (1991), joint likelihood ratio ordering may not imply the usual likelihood ratio ordering among the marginal distributions of the components of the random vector.

THEOREM **4.3.4** *Suppose that the conditional mean residual life of* Y *given* $X = x$, $\mu(y|x)$, *is increasing in* x. *Then for any* $1 \leq i < j \leq n$, $Y_{[i]} \leq_{mrl} Y_{[j]}$. *The inequality is reversed in case* $\mu(y|x)$ *is decreasing in* x.

PROOF : We give proof for the case when $\mu(y|x)$, is increasing in x. The proof is similar when it is decreasing. Since $Y_{[i]} \leq_{mrl} Y_{[j]}$ iff

$$\frac{\int_t^{+\infty} \overline{F}_{[j]}(y)\, dy}{\int_t^{+\infty} \overline{F}_{[i]}(y)\, dy} \quad \text{is increasing in } t,$$

it is enough to prove that the function

$$h(i,t) = \int_t^{+\infty} \overline{F}_{[i]}(y)\, dy = \int_{-\infty}^{+\infty} \left[\int_t^{+\infty} \overline{F}(y|x)\, dy\right] f_i(x)\, dx$$

is TP_2 in (i, t) if (X, Y) is $DTP(0, 2)$.

The required result follows from Lemma 4.2.1 since the function $\int_t^{+\infty} \overline{F}(y|x)\, dy$ is TP_2 in (t, x) if $\mu(y|x)$ is increasing in x and $f_i(x)$ is TP_2 in (x, i).

■

Obviously $Y_{[i]} \leq_{mrl} Y_{[j]}$ implies that $E[Y_{[i]}] \leq E[Y_{[j]}]$ for $i < j$. However, as proved in the next theorem, this inequality holds under the weaker condition that $E[Y|X = x]$ is increasing in x.

THEOREM **4.3.5** *Suppose $E[Y|X = x]$ is increasing in x. Then*

$$E[Y_{[i]}] \leq E[Y_{[j]}] \quad \text{for } i < j. \tag{4.3.11}$$

The inequality in (4.3.11) *is reversed in case $E[Y|X = x]$ is decreasing in x.*

PROOF : $E[Y_{[i]}] = \int_{-\infty}^{+\infty} E[Y|X = x] f_{(i)}(x)\, dx = E[\psi(X_{(i)})]$, where $\psi(x) = E[Y|X = x]$. The required result now follows from this since $X_{(i)} \leq_{st} X_{(j)}$ for $i < j$ and $\psi(x)$ is assumed to be increasing in x. The inequality in (4.3.11) is reversed in case $\psi(x)$ is decreasing in x.

■

Next we show that if the conditional distribution of Y given $X = x$ is DFR for each fixed x, then the concomitant $Y_{[i]}$'s are also DFR for $1 \leq i \leq n$.

THEOREM **4.3.6** *If $r(y|x)$, the conditional hazard rate of Y given $X = x$, is decreasing in y for each fixed x, then $Y_{[i]}$ has DFR distribution for $1 \leq i \leq n$,*

PROOF : Since a mixture of DFR distributions is DFR (cf. Barlow and Proschan, 1981, p. 103), it follows from (4.1.2) and the assumption that $r(y|x)$ is decreasing in y for each fixed x that $Y_{[i]}$ has DFR distribution for $1 \leq i \leq n$.

■

Using Theorem 2.2.2 and the above theorem, we get the following result.

THEOREM **4.3.7** *Suppose $r(y|x)$ is decreasing in x and y and the left end-point of the support of the conditional distribution of Y given $X = x$ does not depend on x. Then*

$$Y_{[i]} \leq_{disp} Y_{[j]} \quad \text{for } i < j. \tag{4.3.12}$$

The inequality in (4.3.12) *is reversed in case* $r(y|x)$ *is increasing in* x *for each fixed* y.

Here is an example of a bivariate distribution which satisfies the conditions of this theorem.

EXAMPLE 4.3.1 : Let (X_i, Y_i), $i = 1, \ldots, n$ be a random sample from *bivariate Pareto distribution* (see Johnson and Kotz, 1972, p. 285), with density

$$f(x,y) = a(a+1)(\theta_1\theta_2)^{a+1}(\theta_2 x + \theta_1 y - \theta_1\theta_2)^{-(a+2)},$$

for $a > 0$, $x > \theta_1 > 0$, $y > \theta_2 > 0$. The conditional hazard rate of Y given X is

$$r(y|x) = \frac{\theta_1(a+1)}{\theta_1 y + \theta_2 x - \theta_1\theta_2},$$

which is decreasing in x as well as in y. It follows from Theorem 4.3.6 and Theorem 4.3.7 that each $Y_{[i]}$ has DFR distribution and that $Y_{[i]} \leq^{disp} Y_{[j]}$ for $i < j$.

4.4 Some results for the model Y=g(X)+Z

We again consider the following model discussed earlier by Kim and David (1990),

$$Y = g(X) + Z, \tag{4.4.1}$$

where random variable Z is independent of random variable X. They proved that if g is increasing, then $(Y_{[1]}, \ldots, Y_{[n]})$ are associated. Under the additional condition that Z has a log-concave density, they proved that the joint density of $(Y_{[1]}, \ldots, Y_{[n]})$ is MTP_2. In this section, we substantially improve upon the results of Kim and David (1990). The next lemma establishes dependence of different types between X and Y under various conditions on the distribution of Z.

LEMMA 4.4.1 *Assume the model given by* (4.4.1) *with* g *increasing (decreasing). Then*

(a) Y *is stochastically increasing (decreasing) in* X,

(b) *if* Z *has a log-concave density, then* X *and* Y *are* TP_2 (RR_2) *dependent,*

(c) *if* Z *is* IFR, *then* X *and* Y *are* $DTP(0,1)$ $(DRR(0,1))$ *dependent,*

(d) *if* Z *is* $DMRL$, *then* X *and* Y *are* $DTP(0,2)$ $(DRR(0,2))$ *dependent.*

PROOF : *(a)* Let f_Z denote the density of Z. Then

$$\begin{aligned} P[Y > y|X = x] &= \int_y^{+\infty} f_Z(w - g(x))dw \\ &= \overline{F}_Z(y - g(x)), \end{aligned}$$

is increasing (decreasing) in x since g is an increasing (decreasing) function.

(b) As the conditional density of Y given X is $f(y|x) = f_Z(y - g(x))$, it follows that the joint density of X and Y is

$$f_{X,Y}(x, y) = f_Z(y - g(x)).f_X(x).$$

Since f_Z being log-concave is equivalent to $h(y, x) = f_Z(y - x)$ being TP_2 , it follows from Theorems $A.3$ and $A.2$ of Marshall and Olkin (1979 p. 488) that $f_{X,Y}$ is TP_2 when g is increasing and RR_2 when g is decreasing.

(c) As seen in the proof of part *(a)*, $\overline{F}(y|x) = \overline{F}_Z(y - g(x))$. This is clearly TP_2 (RR_2) in (x, y) as $\overline{F}_Z(y - x)$ is TP_2 if Z is IFR and g is an increasing (decreasing) function.

(d) The proof is similar to part *(c)* and is omitted.

■

Remark : Assume the model given by (4.4.1) with g increasing (decreasing). Then

(a) if Z has a log-convex density, then X and Y are RR_2 (TP_2) dependent,

(b) if Z is DFR, then X and Y are $DRR(0,1)$ $(DTP(0,1))$ dependent,

(c) if Z is $IMRL$, then X and Y are $DRR(0,2)$ $(DTP(0,2))$ dependent.

In the next theorem we extend the results of Kim and David (1990) to a great extent.

THEOREM **4.4.1** *Let* $(X_1, Y_1), \ldots, (X_n, Y_n)$ *be a random sample from a bivariate distribution satisfying the model* $Y = g(X) + Z$, *where* g *is monotone and* Z *is independent of* X. *Then*

(a) *if the density of* Z *is log-concave, then the joint density of* $(Y_{[1]}, \ldots, Y_{[n]})$ *is* TP_2 *in pairs,*

(b) *if* Z *is* IFR, *then* $(Y_{[1]}, \ldots, Y_{[n]})$ *is* $DTP(1, \ldots, 1)$,

(c) *if* Z *is* $DMRL$, *then* $(Y_{[1]}, \ldots, Y_{[n]})$ *is* $DTP(2, \ldots, 2)$.

PROOF : The proofs follow immediately from Lemma 4.4.1, Theorem 4.2.1 and Corollary 4.2.1.

■

In Theorem 4.4.2 below we prove some new results on stochastic comparisons among concomitants for the above model.

THEOREM **4.4.2** *Let* $(X_1, Y_1), \ldots, (X_n, Y_n)$ *be a random sample from a bivariate distribution satisfying the model* $Y = g(X) + Z$, *where* g *is increasing (increasing) and* Z *is independent of* X. *Then*

(i) $Y_{[i]}$*'s are increasing (decreasing) according to usual stochastic ordering;*

(ii) *if Z has a log-concave density, then $Y_{[i]}$'s are increasing (decreasing) according to likelihood ratio ordering;*

(iii) *if Z is IFR, $Y_{[i]}$'s are increasing (decreasing) according to hazard rate ordering and*

(iv) *if Z is $DMRL$, then $Y_{[i]}$'s are increasing (decreasing) according to mean residual life ordering.*

PROOF : Using Lemma 4.4.1, the proofs follow from Theorem 4.3.1, 4.3.3, 4.3.2 and 4.3.4, respectively.

■

Remark : In (ii) of Theorem 4.4.2, if instead of assuming that Z is log-concave, we assume that Z is log-convex, then the joint density of $Y_{[i]}$'s is still TP_2 in pairs, but they are decreasing (increasing) according to likelihood ratio ordering. Similarly, in case (iii), if we assume that Z is DFR, then $Y_{[i]}$'s are $DTP(1, \ldots, 1)$ and they are decreasing (increasing) according to hazard rate ordering. Finally, by assuming that Z is $IMRL$ in (iv), it follows that $Y_{[i]}$'s are $DTP(2, \ldots, 2)$ and they are decreasing (increasing) according to mean residual life ordering.

Remark : Under the conditions of Theorem 4.4.2, the corresponding joint stochastic orderings hold among $Y_{[i]}$'s.

4.5 Concluding remarks

In this chapter we have obtained some new results on stochastic orderings and dependence among the concomitants of order statistics from bivariate distributions which have various types of monotone dependence structures. While we are not aware of any previous results on stochastic monotonicity among concomitants, the results on dependence among concomitants of order statistics were known only for certain special types of models. The results obtained in this chapter are general in the sense that they apply to any particular distribution with any monotone dependence between the variables X and Y. It is proved that if Y is stochastically increasing in X, then $Y_{[i]}^{\prime}s$ are stochastically increasing and are associated. However, under a stronger condition that the conditional hazard rate of Y given $X = x$ is decreasing in x, it is proved that the $Y_{[i]}$'s are increasing according to hazard rate ordering and they are dependent according to $DTP(1, \ldots, 1)$ criteria. In particular, in this case, $Y_{[i]}$ and $Y_{[j]}$ are $RCSI$ for $i \neq j \in \{1, \ldots, n\}$. In case X and Y are TP_2 dependent, the successive $Y_{[i]}^{\prime}s$ are increasing according to likelihood ratio ordering and their joint density is TP_2 in pairs. Analogous results on stochastic orderings among concomitants of order statistics are obtained when the variables X and Y have monotone negative dependence. Surprisingly, in this case also the $Y_{[i]}$'s are positively dependent. We also prove that when the conditional hazard rate of Y given $X = x$ is decreasing in y for each fixed x, then $Y_{[i]}$'s have DFR (decreasing failure rate) distributions. If in addition, the above conditional hazard rate is monotone in x as well for each y, then the concomitants are ordered according to dispersive ordering. These results may have potential applications in the study of small sample properties of various estimates and tests for independence based on concomitants of order statistics.

The results obtained in this chapter are mainly based on Khaledi and Kochar (2000 b).

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