Back Paper Examination, M. Math (Academic Year: 2025-2026)
Topology-1

Maximum Marks: 45 **Time: 3 hours**

Provide answers with complete justifications. It is not an open-book exam.

1. (a) Is $[0,1)$ homeomorphic to $[0,1]$ with subspace topology of $\mathbb{R}$? (2)
   (b) Let $p \in \mathbb{S}^1$, then what is $\pi_1(\mathbb{S}^1, p) \cong$? (2)
   (c) Give a Covering space of $\mathbb{S}^1 \times \mathbb{S}^1$. (2)
   (d) State the Urysohn lemma. (2)
   (e) What is the relation between *compactness, sequential compactness* and *limit point compactness* for Metrizable topological spaces? (4)
2. Give an example of a topological space which is not compact but limit point compact. Justify your answer. (10)
3. Show that if X is compact Hausdorff under both τ and τ', then either τ and τ' are equal or they are not comparable. (6)
4. Show that a closed subspace of a normal space is normal. (6)
5. Let $\mathbb{R}_{cf}$ denote $\mathbb{R}$ with the co-finite topology. Prove or disprove: the product topology on $\mathbb{R}_{cf} \times \mathbb{R}_{cf}$ is the same as the co-finite topology on $\mathbb{R}^2$. (6)
6. Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$. (6)
7. Let A be a path connected subspace of a topological space X. Prove or disprove: The interior of A in X is also path-connected. (5)
8. Let $C_1 \supset C_2 \supset \cdots$ be a nested sequence of nonempty closed sets in the complete metric space X . If diam $C_n \to 0$, then $\bigcap_n C_n \neq \phi$. (6)
9. Show that a subspace of a first-countable space is first-countable. (6)
10. Let (X, d) be a metric space. Show that (X, d) is complete if and only if $(X, \bar{d})$ is complete, where $\bar{d}(x, y) = \min\{d(x, y), 1\}$. (8)
11. Show that every locally compact Hausdorff space is regular. (6)

12. Consider the covering map $p : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 - \{(0,0)\}$ defined as $p(x,t) := te^{2\pi ix}$. Let $f : [0,1] \to \mathbb{R}^2 - \{(0,0)\}$ be a path defined as $f(s) = (2-s, 0)$. Find a lifting of the path f. (5)

13. Show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point [*Hint:* Use fixed point theorem for B^2]. (6)

14. Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β, respectively, with $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ a lifting of $\alpha * \beta$. (6)

15. Let $A \subset X$; suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. If $a_0 \in A$, show that $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective. (6)