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**SEARCH DESIGNS AND OTHER TOPICS IN
FRACTIONAL REPLICATION**

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INTRODUCTION AND SUMMARY

0.1 Introduction

This thesis primarily deals with search designs (see Srivastava (1975)) of various types for general symmetric and asymmetric factorials. The thesis also considers some other types on fractional factorials. We have made use of Kronecker products and various other results from matrix theory.

In the first four chapters, different types of search designs have been studied. Chapter 5A considers the construction of orthogonal main-effect plans for general factorial experiments under the presence of a linear time trend. Chapter 5B extends the calculus for factorial arrangements to a fractional factorial set-up and examines critically the proportional frequency plans from the point of view of estimability and efficiency.

Section 0.2 contains a review of the literature on search designs and also on trend-free plans and proportional frequency plans. While reviewing the literature, we have restricted ourselves to the results pertaining to the topics discussed in this thesis. Section 0.3A,B present the main notations, plan of the thesis and a detailed chapterwise summary. The motivation for the different chapters has been discussed and all main results have been recalled. For ease of reference, the serial numbers of these theorems and definitions are the same as those in the main chapters.

0.2 Review of the Literature

Through the contributions of numerous researchers, the area of fractional replication is now so rich that it is difficult to make a comprehensive review of the entire literature in a limited space. We, therefore, begin with a brief overview of the general area of fractional replication and indicate some of the key references including review articles and monographs. This has been done in the next paragraph. The rest of this section presents a detailed review of

the literature on the topics pertaining to this thesis.

The idea of fractional replication in factorial experiments was introduced and popularized by Finney (1945). Among the early authors, Plackett and Burman (1946), Kempthorne (1947) and Rao (1947) studied the problem in more detail and gave several fractional plans. Rao (1947) developed the theory of fractional replication for symmetric factorials through the use of orthogonal arrays. Box and Hunter (1961a,b) introduced the notion of resolution which was subsequently generalized by Webb (1968). The development of orthogonal fractional factorial plans, including studies on their optimality properties (vide Cheng (1980)), attracted the attention of many authors over the last four decades and for a review, together with a comprehensive list of references, we refer to Dey (1985). Also, we refer to Raghavarao (1971), Rao (1973) and Hedayat and Stufken (1988) (and the references therein) for discussion on orthogonal arrays which play a crucial role in the context of orthogonal fractional factorial plans. Another very important area was opened up by Bose and Srivastava (1964) who considered fractional factorial plans based on balanced arrays - for further results in this line, together with studies on optimality, see Srivastava and Chopra (1971), Shirakura (1976) and Yamamoto, Shirakura and Kuwada (1976), among others. A detailed list of references on fractional factorial plans based on balanced arrays is available in Kuwada and Nishii (1988). As for review articles on fractional replication, reference is made to Addelman (1963, 1972) for informative reviews up to those stages. For more up-to-date excellent reviews, together with detailed lists of references and discussion on open problems, we refer to Srivastava (1978), Raktue, Hedayat and Federer (1981), Dey (1985) and Srivastava (1987). *

In the usual fractional factorial designs, the object of the design is to estimate certain factorial effects, assuming that the remaining (or a subset of

the remaining) effects are negligible. One may wonder, in general, as to what extent the assumption regarding higher order effects being zero, is valid. This problem was first raised by Srivastava who pioneered the new concept of **search designs** in a paper presented at the International Symposium on Statistical Design and Linear Models in 1973. Srivastava (1975) considered the following general linear model :

$$E(Y) = X_1 \tau_1 + X_2 \tau_2, \text{Disp}(Y) = \sigma^2 I \quad (0.2.1)$$

where $Y^{(N \times 1)}$ is a vector of observations, $X_i^{(N \times n_i)}$, ($i=1,2$), are the so called design matrices and $\tau_i^{(N \times n_i)}$, ($i = 1, 2$), are vectors of fixed unknown parameters, with $\tau_i' = (\tau_{i1}, \tau_{i2}, \dots, \tau_{in_i})$. Suppose we want to estimate the elements of τ_1 . Also suppose it is given that the elements of τ_2 are all negligible, except possibly for a set of at most k^* elements, where k^* would be considered quite small compared to n_2 . Srivastava (1975) described a search design with resolving power $\{\tau_1, \tau_2, k^*\}$ as one which enables the searching of the non-negligible elements of τ_2 and which also permits the estimation of them together with the elements of τ_1 . Following Srivastava (1975), the case when $\sigma^2 = 0$ will be called the noiseless case.

Theorem 0.2.1 (Srivastava (1975)). Consider the model (0.2.1) together with the assumptions mentioned above and let $\sigma^2 = 0$. Then a design S , corresponding to the observations Y , is a search design of resolving power $\{\tau_1, \tau_2, k^*\}$ if and only if for every submatrix $X_{20}^{(N \times 2k^*)}$ of X_2 , we have

$$\text{Rank} [X_1, X_{20}] = n_1 + 2k^* \quad (0.2.2)$$

Srivastava (1975) also developed a detailed theory covering some aspects of 2^m search designs. Search designs in the presence of noise were also considered in that paper. Further results on search designs were reported by Srivastava (1976).

The path breaking works of Srivastava (1975, 1976) led to a considerable interest and a resulting growth in the literature on search designs. It will be convenient to recall the following definitions while reviewing the literature.

Definition 0.2.1 (Srivastava (1975)). A search design S for a 2^m factorial experiment is said to be a 'main effect plus k^* (MEP. k^*)' plan, when in the search model (0.2.1) τ_1 consists of the general mean and all main-effects and τ_2 consists of the two-factor and higher order interactions.

Definition 0.2.2 (Srivastava and Ghosh (1976)). A search design S for a 2^m factorial experiment is said to be a 'balanced resolution V plus one (BARE 5.1)' plan if it is a balanced resolution V plan which, with reference to the model (0.2.1), allows the search and estimation of at best one non-negligible parameter in τ_2 , which consists of the three-factor and higher order interactions, and also which permits the estimation of the parameters of τ_1 containing the general mean, all main-effects and all two-factor interactions.

For 2^m factorials, Srivastava and Ghosh (1976) showed that if S consists of the N treatments of the set $S_1, S_{m-2}, S_{m-1}, S_m$ where S_v ($v = 0, 1, \dots, m$) is the set of $\binom{m}{v}$ treatments in which exactly v factors occur at level 1 and $(m-v)$ factors occur at level 0, then S is a BARE 5.1 plan in $N = 1 + 2\binom{m}{1} + \binom{m}{2}$ runs. Srivastava and Ghosh (1977) reported some further interesting findings on BARE 5.1 plans (including some necessary and sufficient conditions for existence) for 2^m factorials on the basis of the following general result:

Theorem 0.2.2 (Srivastava and Ghosh (1977)). Let S be a design of resolution V and with reference to model (0.2.1), let $\text{rank}(X_1) = n_1$. Then the search design problem, where τ_1 contains the general mean, all main-effects and two-factor interactions and τ_2 contains all three-factor and higher order interactions, can be completely solved in the noiseless case if and only if for every $(N \times 2k^*)$ submatrix X_{20} of X_2 , the relation

$$\text{Rank}(W) = 2k^*$$

holds where

$$W = X'_{20} X_{20} - X'_{20} X_1 (X'_1 X_1)^{-1} X'_1 X_{20}.$$

Srivastava (1977) recognized that, because of possible bias in the underlying model, the usual optimal designs may not be optimal and that the theory of search models offers possibility of 'searching' the bias and 'correcting' the same. In this paper the concept of optimal search designs and the criteria of AD-optimality and DD-optimality were introduced and studied.

With reference to the model (0.2.1) with $\sigma^2 = 0$, Srivastava and Gupta (1979) derived an interesting reduction of the necessary and sufficient condition in the Theorem 0.2.1. Suppose X_1 has rank n_1 and X_{11} be an $(n_1 \times n_1)$ matrix of X_1 of rank n_1 . Accordingly partition Y , X_1 , X_2 as

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad X_1 = \begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix}, \quad X_2 = \begin{bmatrix} X_{12} \\ X_{22} \end{bmatrix}$$

$$\text{and let } X_{22}^* = X_{22} - X_{21} X_{11}^{-1} X_{12}.$$

Theorem 0.2.3 (Srivastava and Gupta (1979)). A necessary and sufficient condition for X_2 to satisfy condition (0.2.2) for all X_{20} is that every set of $2k^*$ distinct columns of X_{22}^* is linearly independent.

In the set-up of Theorem 0.2.3, let S be the set of treatment combinations constituting a search design and S_1^* , S_2^* be subsets of S corresponding to Y_1 and Y_2 respectively. Considering 2^m factorials in particular, Srivastava and Gupta (1979) applied Theorem 0.2.3 to obtain a necessary and sufficient condition for S to be an MEP. 1 plan (vide definition 0.2.1) for given S_1^* . Ghosh (1980) obtained further interesting results on MEP. 1 plans for 2^m factorials, where $m = 2^{h-1}(h \geq 2)$, in $N = 1 + m + (m-h)(h+1)$ runs. He gave a necessary and sufficient condition for S , as defined above, to be a good MEP. 1 plan when S_1^* is a good main-effect plan.

Anderson and Thomas (1980) obtained a series of weakly resolvable (cf. Srivastava (1975)) search designs for the s^m factorial experiment for which the mean and all main-effects are estimable in the presence of any number of two-factor interactions and for which any combination of three or fewer pairs of factors that interact may be detected. They assumed s to be a prime or a prime power and considered designs in $N = s + ms(s-1)$ runs except in some cases where additional runs are required. Ghosh (1981) obtained MEP.1 search designs with reasonably small number of treatments for 2^m factorial experiments, $3 \leq m \leq 8$. There he gave a necessary and sufficient condition on the existence of MEP.1 search plans. Gupta and Carvajal (1984a) derived a necessary condition for the existence of MEP.1 search plan S for 2^m factorials, where S_1^* consists of S_0, S_1 and S_m (here S, S_1^*, S_0, S_1, S_m are as already defined in the context of Srivastava and Ghosh (1976) and Srivastava and Gupta (1979)). Gupta and Carvajal (1984b, 1987), Morris (1984) and Gupta (1988) gave several lower bounds on the number of runs in MEP.1 search designs for 2^m factorials. Shirakura and Ohnishi (1985) generalized the ideas in Srivastava and Ghosh (1976, 1977). Based on Theorem 0.2.2, they reported a simple necessary and sufficient condition for a balanced 2^m factorial design of resolution $2l + 1$ to be a BARE $(2l + 1)$. 1 plan. They also gave BARE 5.1 plans for various values of m and N on the basis of AD-optimality criterion defined in Srivastava (1977). Ohnishi and Shirakura (1985) reported some further interesting results in continuation of Ghosh (1981). Ghosh and Zhang (1987) presented two series of search designs for 3^m factorial experiments in $N = 1+3m$ and $N = (1+m+2m^2)$ runs respectively. In the first series of designs they considered a set-up where, with reference to (0.2.1), τ_1 contains the general mean and all main-effects and τ_2 contains two-factor interactions while in the second series τ_1 contains the general mean, all main-effects and two-factor interactions and τ_2 contains all three-factor interactions. Either series is capable of searching

one among the n_2 elements of τ_2 and estimating it along with the elements of τ_1 . For a comprehensive review and further references on various aspects of search designs reference is made to Gupta (1990). Search designs of the above types are discussed in this thesis in chapters 1-4.

In industrial and agricultural experimentation, when the observations are taken over a period of time, it happens in many practical situations that the observations are affected by a time trend in addition to the effects of the treatments under consideration. The problem of factorial experimentation in the presence of such time trend was considered among others by Hill (1960), Daniel and Wilcoxon (1966), Draper and Stoneman (1968), Phillips (1968) and Joiner and Campbell (1976) (see also Dey (1985) in this context) and more recently by Cheng (1985), Coster and Cheng (1988) and Cheng and Jacroux (1988). These authors discussed in detail the philosophical basis of the problem, cited numerous practical applications and suggested relevant experimental plans mainly with reference to 2^m factorials. Chapter 5A of the thesis deals with the problem of constructing trend-free main-effect plans. In addition to trend-free fractional factorials, there is a vast literature on trend-free block designs for varietal trials. This is, however, beyond the scope of this thesis and we only refer to Bradley and Yeh (1980), Yeh and Bradley (1983) and Stufken (1988) and the references therein.

Plackett (1946) introduced the criterion of proportional frequency in a multifactor setting. It was contended by Addelman (1962) that fractional factorial plans based on this criterion lead to orthogonal estimation of lower order factorial effects when higher order effects are assumed to be absent. Lewis and John (1976) and later, in more detail, Kuwada and Nishii (1979), Mukerjee (1980) had shown that the contention of Addelman regarding orthogonality in proportional frequency plans is not in general true and established, in the general case, the criterion of equal frequency (in terms of orthogonal arrays) as a necessary

and sufficient condition for orthogonal fractional factorial plans. However, there are situations where orthogonal plans based on the equal frequency criterion are non-existent (cf. Mukerjee (1980)). Hence in factorial experimentation under such circumstances, one has no other alternative than to forgo orthogonality and search for possibly non-orthogonal plans. In this context, because of their wide availability (Addelman (1963, 1972)), it may be worthwhile to critically examine the proportional frequency plans in terms of estimability and efficiency. This has been attempted in Chapter 5B of this thesis.

0.3A Major Notations Used in the Thesis

This thesis primarily deals with search designs of various types. A detailed chapterwise summary has been presented in Section 0.3B. For ease in presentation, it will be convenient to introduce the major notations. This is done below first considering the case of symmetric factorials. Then the natural extension of the notations to the asymmetric case has also been indicated.

With reference to a symmetric (s^m) factorial set-up $m \geq 2$, $s \geq 2$, the following notational system will be helpful:

- (i) Define the $(m \times 1)$ vectors $\mathbf{a}_j^{(1)}$ ($0 \leq j \leq s-1$) such that in $\mathbf{a}_j^{(1)}$, all the elements are j ; let $A_j^{(1)}$ be an $(m \times s)$ array whose columns are given by $\mathbf{a}_j^{(1)}$ for all j .
- (ii) For $0 \leq j \leq s-1$, define the $(m \times 1)$ vectors $\mathbf{a}_{j,u,i_u}^{(2)}$ ($1 \leq u \leq m$, $0 \leq i_u \leq s-1$, $i_u \neq j$) such that in $\mathbf{a}_{j,u,i_u}^{(2)}$, the u th element is i_u and the remaining elements all equal j ; let $A_j^{(2)}$ be an $(m \times m(s-1))$ array whose columns are $\mathbf{a}_{j,u,i_u}^{(2)}$ for all u and i_u .
- (iii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $\mathbf{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ ($1 \leq u_1 < u_2 \leq m$, $0 \leq \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}) \leq s-1$) such that in $\mathbf{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} respectively and the remaining elements all equal j ; let $A_j^{(3)}$ be an $(m \times [m(m-1)j(s-j-1)])$ array whose columns are $\mathbf{a}_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(3)}$ is defined only when $s \geq 3$.

(iv) For $0 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ ($1 \leq u_1 < u_2 \leq m$, $j+1 \leq i_{u_1}, i_{u_2} \leq s-1$) such that in $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} respectively and the remaining elements all equal j ; let $A_j^{(4)}$ be an $(m \times [m(m-1)(s-1)^2/2])$ array whose columns are $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(4)}$ is defined only when $s \geq 3$.

(v) Define the $(m \times 1)$ vectors $a_{j,u_1,u_2}^{(5)}$ ($1 \leq j \leq s-1$, $1 \leq u_1 < u_2 \leq m$) such that in $a_{j,u_1,u_2}^{(5)}$, the u_1 th and u_2 th elements are 0 and the remaining elements all equal j ; let $A^{(5)}$ be an $(m \times [m(m-1)(s-1)/2])$ array whose columns are given by $a_{j,u_1,u_2}^{(5)}$ for all j, u_1 and u_2 .

(vi) Define the $(m \times 1)$ vectors $a_{j,u_1,u_2,u_3}^{(6)}$ ($1 \leq j \leq s-1$, $1 \leq u_1 < u_2 < u_3 \leq m$) such that in $a_{j,u_1,u_2,u_3}^{(6)}$, the u_1 th, u_2 th and u_3 th elements are 0 and the remaining elements all equal j ; let $A^{(6)}$ be an $(m \times [m(m-1)(m-2)(s-1)/6])$ array whose columns are given by $a_{j,u_1,u_2,u_3}^{(6)}$ for all j, u_1, u_2 and u_3 .

(vii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(7)}$ ($1 \leq u_1 < u_2 < u_3 \leq m$, $i_{(u_1)} = 0$, $i_{(u_2)} = 0$, $j+1 \leq i_{(u_3)} \leq s-1$, $i_{(u_1)}, i_{(u_2)}$ and $i_{(u_3)}$ represent the non-decreasingly ordered arrangement of i_{u_1}, i_{u_2} and i_{u_3} , e.g. if $i_{u_1} = 0$, $i_{u_2} = 2$, $i_{u_3} = 0$, then $i_{(u_1)} = i_{(u_2)} = 0$, $i_{(u_3)} = 2$) such that in $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(7)}$, the u_1 th, u_2 th and u_3 th elements are i_{u_1}, i_{u_2} and i_{u_3} respectively and the remaining elements all equal j ; let $A_j^{(7)}$ be an $(m \times [m(m-1)(m-2)(s-j-1)/2])$ array whose columns are given by $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(7)}$ for all $u_1, u_2, u_3, i_{u_1}, i_{u_2}$ and i_{u_3} . Note that $A_j^{(7)}$ is defined only when $s \geq 3$.

(viii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j, u_1, u_2, u_3}^{(8)}$, i_{u_1} , i_{u_2} , i_{u_3} ($1 \leq u_1 < u_2 < u_3 \leq m$, $0 \leq i_{(u_1)} \leq j-1$, $j+1 \leq i_{(u_2)} \leq s-1$, $i_{(u_3)} \leq s-1$, $i_{(u_1)}$, $i_{(u_2)}$ and $i_{(u_3)}$ are as defined above) such that in $a_{j, u_1, u_2, u_3}^{(8)}$, the u_1 th, u_2 th and u_3 th elements are i_{u_1} , i_{u_2} and i_{u_3} respectively and the remaining elements all equal j ; let $A_j^{(8)}$ be an $(m \times [m(m-1)(m-2)j(s-j-1)^2/2])$ array whose columns are given by $a_{j, u_1, u_2, u_3}^{(8)}$ for all $u_1, u_2, u_3, i_{u_1}, i_{u_2}$ and i_{u_3} . Note that $A_j^{(8)}$ is defined only when $s \geq 3$.

(ix) For $2 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j, u_1, u_2, u_3}^{(9)}$, i_{u_1} , i_{u_2} , i_{u_3} ($1 \leq u_1 < u_2 < u_3 \leq m$, $i_{(u_1)} = 0$, $1 \leq i_{(u_2)} \leq j-1$, $j+1 \leq i_{(u_3)} \leq s-1$, $i_{(u_1)}$, $i_{(u_2)}$ and $i_{(u_3)}$ are as defined above) such that in $a_{j, u_1, u_2, u_3}^{(9)}$, the u_1 th, u_2 th and u_3 th elements are i_{u_1} , i_{u_2} and i_{u_3} respectively and the remaining elements all equal j ; let $A_j^{(9)}$ be an $(m \times [m(m-1)(m-2)(j-1)(s-j-1)])$ array whose columns are given by $a_{j, u_1, u_2, u_3}^{(9)}$ for all $u_1, u_2, u_3, i_{u_1}, i_{u_2}$, i_{u_3} . Note that $A_j^{(9)}$ is defined only when $s \geq 4$.

(x) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j, u_1, u_2, u_3}^{(10)}$, i_{u_1} , i_{u_2} , i_{u_3} ($1 \leq u_1 < u_2 < u_3 \leq m$, $j+1 \leq i_{u_1}, i_{u_2}, i_{u_3} \leq s-1$) such that in $a_{j, u_1, u_2, u_3}^{(10)}$, the u_1 th, u_2 th and u_3 th elements are i_{u_1} , i_{u_2} and i_{u_3} respectively and the remaining elements all equal j ; let $A_j^{(10)}$ be an $(m \times [m(m-1)(m-2)(s-j-1)^3/6])$ array whose columns are given by $a_{j, u_1, u_2, u_3}^{(10)}$ for all $u_1, u_2, u_3, i_{u_1}, i_{u_2}$ and i_{u_3} . Note that $A_j^{(10)}$

is defined only when $s \geq 3$.

Turning now to more general factorials, not necessarily symmetric, consider an $s_1 \times s_2 \times \dots \times s_m$, $m \geq 2$, factorial, where, without loss of generality, let $s_{v-1} \leq s_v$ ($2 \leq v \leq m$). Also, for notational convenience, define $s_0 = 0$.

(i) For $1 \leq v \leq m-1$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $\mathbf{b}_j^{(1,v)}$ ($s_{v-1} \leq j \leq s_v - 1$) such that in $\mathbf{b}_j^{(1,v)}$, the first $(v-1)$ elements equal 0 and the remaining elements all equal j ; let $B^{(1,v)}$ be an $(m \times (s_v - s_{v-1}))$ array whose columns are given by $\mathbf{b}_j^{(1,v)}$ for all j . Note that $B^{(1,v)}$ is defined only when $s_{v-1} < s_v$.

(ii) For $1 \leq v \leq m-2$, if $s_v < s_{v+1}$, then define the $(m \times 1)$ vectors $\mathbf{b}_{j,u,i_u}^{(2,v)}$ ($s_v \leq j \leq s_{v+1} - 1$, $1 \leq u \leq v$, $1 \leq i_u \leq s_u - 1$) such that in $\mathbf{b}_{j,u,i_u}^{(2,v)}$, the u th element is i_u , the last $(m-v)$ elements all equal j and the remaining elements all equal 0; let $B^{(2,v)}$ be an $(m \times [\sum_{u=1}^v (s_u - 1)(s_{v+1} - s_v)])$ array whose columns are given by $\mathbf{b}_{j,u,i_u}^{(2,v)}$ for all j, u and i_u . Note that $B^{(2,v)}$ is defined only when $s_v < s_{v+1}$ and $m \geq 3$.

(iii) For $1 \leq v \leq m-2$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $\mathbf{b}_{j,u,i_u}^{(3,v)}$ ($s_{v-1} \leq j \leq s_v - 1$, $v \leq u \leq m$, $0 \leq i_u \leq s_u - 1$, $i_u \neq j$) such that in $\mathbf{b}_{j,u,i_u}^{(3,v)}$, the first $(v-1)$ elements equal 0, the u th element is i_u and the remaining elements all equal j ; let $B^{(3,v)}$ be an $(m \times [\sum_{u=v}^m (s_u - s_{v-1})(s_u - 1)])$ array whose columns are given by $\mathbf{b}_{j,u,i_u}^{(3,v)}$ for all j, u and i_u . Note that $B^{(3,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 3$.

(iv) If $s_{m-2} < s_{m-1} - 1$, then define the $(m \times 1)$ vectors $\mathbf{b}_{jk}^{(4)}$ ($s_{m-2} \leq j < s_{m-1} - 1$, $s_{m-2} \leq k \leq s_{m-1} - 1$, $j \neq k$) such that in $\mathbf{b}_{jk}^{(4)}$, the last two elements are j and k and all other elements equal 0; let $B^{(4)}$ be an $(m \times [(s_{m-1} - s_{m-2} - 1)(s_{m-1} - s_{m-2} - 1)])$ array whose columns are given by $\mathbf{b}_{jk}^{(4)}$ for all j and k . Note that $B^{(4)}$ is defined only when $s_{m-2} < s_{m-1} - 1$.

(v) If $s_{m-2} < s_m$, then define the $(m \times 1)$ vectors $\mathbf{b}_k^{(5)}$ ($s_{m-2} \leq k \leq s_{m-1} - 1$, $k \neq s_{m-1} - 1$) such that in $\mathbf{b}_k^{(5)}$, the last two elements are $s_{m-1} - 1$ and k and the remaining elements

all equal 0; let $B^{(5)}$ be an $(m \times (s_m - s_{m-2} - 1 + \delta_{s_{m-2}s_{m-1}}))$ array, where $\delta_{s_{m-2}s_{m-1}}$ is the Kronecker delta, whose columns are $b_k^{(5)}$ for all k . Note that $B^{(5)}$ is defined only for $s_{m-2} < s_m$.

(vi) If $s_{m-1} < s_m$, then define the $(m \times 1)$ vectors $b_{u,i_u,k}^{(6)}$ ($1 \leq u \leq m-2$, $1 \leq i_u \leq s_u - 1$, $s_{m-1} \leq k \leq s_m - 1$) such that in $b_{u,i_u,k}^{(6)}$, the u th element is i_u , the $(m-1)$ th element is $s_{m-1} - 1$, the m th element is k and the remaining elements all equal 0; let $B^{(6)}$ be an $(m \times [(s_m - s_{m-1}) \sum_{u=1}^{m-2} (s_u - 1)])$ array whose columns are $b_{u,i_u,k}^{(6)}$ for all u , i_u and k . Note that $B^{(6)}$ is defined only when $s_{m-1} < s_m$.

(vii) If $s_{m-3} < s_{m-2}$, then define the $(m \times 1)$ vectors $b_{j,u,i_u}^{(7)}$ ($m-2 \leq u \leq m$, $s_{m-3} \leq j \leq s_{m-2} - 1$, $1 \leq i_u \leq s_u - 1$, $i_u \neq j$) such that in $b_{j,u,i_u}^{(7)}$, the first $(m-3)$ elements equal 0, the u th element is i_u and the remaining elements equal j ; let $B^{(7)}$ be an $(m \times [(s_{m-2} - s_{m-3}) \sum_{u=m-2}^m (s_u - 2)])$ array whose columns are given by $b_{j,u,i_u}^{(7)}$ for all j , u and i_u . Note that $B^{(7)}$ is defined only when $s_{m-3} < s_{m-2}$ and $m \geq 3$.

(viii) For $1 \leq v \leq m-3$, if $s_v < s_{v+1}$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ ($s_v \leq j \leq s_{v+1} - 1$, $1 \leq u_1 \leq v$, $1 \leq i_{u_1} \leq s_{u_1} - 1$, $v+1 \leq u_2 \leq m$, $j+1 \leq i_{u_2} \leq s_{u_2} - 1$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$, the u_1 th element is i_{u_1} , the remaining of the first v elements equal 0, the u_2 th element is i_{u_2} and the remaining elements all equal j ; let $B^{(8,v)}$ be an $(m \times [\sum_{u_1=1}^v (s_{u_1} - 1) [\sum_{j=s_v}^{s_{v+1}-1} \sum_{u_2=v+1}^m (s_{u_2} - j - 1)]])$ array whose columns are given by $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(8,v)}$ is defined only when $s_v < s_{v+1}$ and $m \geq 4$.

(ix) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$

($s_{v-1} \leq j \leq s_v - 1$, $v \leq u_1 < u_2 \leq m$, $j+1 \leq i_{u_1} \leq s_{u_1} - 1$, $j+1 \leq i_{u_2} \leq s_{u_2} - 1$) such that in

$b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$, the first $(v-1)$ elements equal 0, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} and the remaining elements all equal j ; let $B^{(9,v)}$ be an $(m \times [\sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m (s_{u_1}-j-1)(s_{u_2}-j-1)])$ array whose columns are given by $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$ with $j = 0$ and $u_1 < u_2$

for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(9,v)}$ is defined only when $s_{v-1} < s_v$ and

$m \geq 4$. For $v = 1$, we further define the $(m \times [\sum_{j=1}^{s_1-1} \sum_{u_1=1}^m \sum_{u_2=1}^m (s_{u_1}-j-1)(s_{u_2}-j-1)])$ array $\bar{B}^{(9,1)}$ which we obtain from $B^{(9,1)}$ after deleting the $(m \times 1)$ vectors

$b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,1)}$ with $j = 0$

(x) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$ ($s_{v-1} \leq j \leq s_v - 1$, $j \neq 0$, $v \leq u_1 < u_2 \leq m$, $0 \leq i_{u_1} \leq s_{u_1} - 1$, $0 \leq i_{u_2} \leq s_{u_2} - 1$, $\min(i_{u_1}, i_{u_2}) < j$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$, the first $(v-1)$ elements equal 0, the u_1 th

and u_2 th elements are i_{u_1} and i_{u_2} and the remaining elements all equal j ; let $B^{(10,v)}$ be an $(m \times [\sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m j(s_{u_1} + s_{u_2} - 2j - 2)])$ array whose columns are given by $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$ for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(10,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 4$.

(xi) If $s_{m-2} < s_{m-1}$, then define the $(m \times 1)$ vectors $b_{j,k,u,i_u}^{(11)}$ ($s_{m-2} \leq j \leq s_{m-1} - 1$, $s_{m-2} \leq k \leq s_{m-1} - 1$, $j \neq k$, $1 \leq u \leq m-2$, $1 \leq i_u \leq s_u - 1$) such that in $b_{j,k,u,i_u}^{(11)}$, the u th element is i_u , the last two elements are j and k and the remaining elements all equal 0; let $B^{(11)}$ be an $(m \times [\sum_{u=1}^{m-2} (s_u - 1)(s_{m-1} - s_{m-2})(s_m - s_{m-2} - 1)])$ array whose columns are given by $b_{j,k,u,i_u}^{(11)}$ for all j, k, u and i_u . Note that $B^{(11)}$ is defined only when $s_{m-2} < s_{m-1}$.

(xii) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2}^{(12,v)} (s_{v-1} \leq j \leq s_v-1, j \neq 0, v \leq u_1 < u_2 \leq m)$ such that in $b_{j,u_1,u_2}^{(12,v)}$, the first $(v-1)$ elements, the u_1 th and u_2 th elements all equal 0 and the remaining elements all equal j ; let $B^{(12,v)}$ be an $(m \times [((m-v+1)(m-v)/2)(s_v-s_{v-1}-\delta_{1v})])$ array whose columns are $b_{j,u_1,u_2}^{(12,v)}$ for all possible j, u_1 and u_2 . Note that $B^{(12,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 4$.

(xiii) If $s_{m-3} < s_{m-2}$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(13)} (s_{m-3} \leq j \leq s_{m-2}-1, m-2 \leq u_1 < u_2 \leq m, 0 \leq i_{u_1} \leq s_{u_1}-1, 0 \leq i_{u_2} \leq s_{u_2}-1, \min(i_{u_1}, i_{u_2})=0, \max(i_{u_1}, i_{u_2}) > j)$ such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(13)}$, the first $(m-3)$ elements equal 0, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} and the remaining element equals j ; let $B^{(13)}$ be an $(m \times [\sum_{j=s_{m-3}}^{s_{m-2}-1} \sum_{u_1=m-2}^m \sum_{u_2=m-2}^m (s_{u_1} + s_{u_2} - 2j-2)])$ array whose columns are $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(13)}$ for all possible j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(13)}$ is defined only when $s_{m-3} < s_{m-2}$ and $m \geq 3$.

0.3B Detailed Chapterwise Summary of the Thesis

As mentioned earlier, this thesis primarily deals with search designs; the first four chapters of this thesis considers search designs of various types while the last chapter, which is divided into two parts, deals with linear trend-free plans and proportional frequency plans.

The results on search designs relate to general symmetric and asymmetric factorials and we may emphasize that the numbers of levels of the different factors

have not been assumed to be primes or prime powers. Admittedly, the proofs are often lengthy and they have been presented in the appendices in order to ensure greater readability. In Chapter 1, most of the proofs have been given in detail. In the subsequent chapters, occasionally, we have presented condensed version of the proofs to save space. In Chapters 1-3, we start with symmetric factorials to motivate the ideas which have subsequently been extended to the asymmetric case. All the main results have been illustrated with examples.

In the concluding remarks in Chapters 1-3, we have compared our search designs with the existing ones whenever possible. However, very often, especially in the asymmetric case, no search plan other than ours is available in the literature. It is a standard practice (see e.g. Anderson and Thomas (1980), Ghosh and Zhang (1987)) to compare search designs also with the corresponding regular fractional plans in terms of numbers of runs. Under such comparisons, our plans turn out to be quite satisfactory as illustrated in tabular form in some of the chapters.

We have attempted to retain notational uniformity as far as practicable. In the beginning of each chapter, the notation has been explained in full. Although this leads to some repetition, it is hoped that this will ensure greater readability.

The main results in this thesis are presented below:

Chapter 1 : In Chapter 1 we consider search designs for searching one among the two-factor interactions in the general asymmetric factorial set-up through the symmetric one. A fractional factorial design will be said to be a search design for searching one among the two-factor interactions if it is capable of estimating the general mean and all main-effects and permits the detection and estimation of the possibly present at best one among the two-factor interactions.

The linear model considered in this chapter and also in Chapters 2-4 resembles closely to that of Srivastava (1975) (vide (0.2.1)). Let $Y^{(N \times 1)}$ be the observational vector with

$$E(Y) = X\tau; \text{Disp}(Y) = \sigma^2 I \quad (1.2.1)$$

where

$$X = [X_1^{(N \times n_1)}, X_2^{(N \times n_2)}]$$

is the known design matrix and $\tau = (\tau_1', \tau_2')'$, τ_1 being an $(n_1 \times 1)$ vector of unknown fixed parameters and τ_2 being an $(n_2 \times 1)$ vector of fixed parameters which is partially known in the following sense. It is known that τ_2 may be partitioned as :

$$\tau_2 = (\beta_1', \beta_2', \dots, \beta_\eta')' \quad (1.2.2)$$

and among $\beta_1, \beta_2, \dots, \beta_\eta$, at most k^* are non-null, k^* being small compared to η .

The problem is to search the non-null vectors among $\beta_1, \beta_2, \dots, \beta_\eta$ and to draw inference regarding them and τ_1 . Let the partitioning of X_2 corresponding to (1.2.2) be $X_2 = [X_{21}, X_{22}, \dots, X_{2\eta}]$. Then we have the following theorem which serves as an important tool in obtaining the results in Chapters 1-4 :

Theorem 1.2.1. Consider the model (1.2.1), together with the assumptions stated above. A necessary and sufficient condition that the above search design problem will be completely solved in the noiseless case is that for arbitrary $g_1, g_2, \dots, g_{2k^*}$ ($1 \leq g_1 < g_2 < \dots < g_{2k^*} \leq \eta$), the matrix $[X_1, X_{2g_1}, X_{2g_2}, \dots, X_{2g_{2k^*}}]$ has full column rank.

The above theorem is virtually another version of Theorem 1 in Srivastava (1975) (presented as Theorem 0.2.1 above) and may be proved along the same line. Unlike Srivastava (1975), we consider the case where the β_i 's are possibly vector-valued- the change is essential as we shall be considering general factorials where an interaction may be represented by more than one independent parameters.

Section 1.4 deals with the construction of search designs in the general symmetric factorial set-up. The following are the main results of this section.

Theorem 1.4.1. If $m = 3$, then the following level combinations $\{(i,j,i), (j,i,j), (j,j,i); 0 \leq i, j \leq s-1, j \neq i\}$ give a search design of the required type in $3s(s-1)$ runs.

Theorem 1.4.2. If $m \geq 4$, then the level combinations $\{(0, 0, \dots, 0), (i, j, \dots, j), (j, i, \dots, j), \dots, (j, j, \dots, i); 0 \leq i, j \leq s-1, j \neq i\}$ give a search design of the required type in $1 + ms(s-1)$ runs.

In Section 1.5, we consider the search design problem for $s^{m_1} \times (s^*)^{m-m_1}$ factorial set-up where $m_1 < m$. A two-factor interaction $F_{v_1} F_{v_2}$ will be said to be of type (11) if $1 \leq v_1 < v_2 \leq m_1$, of type (22) if $m_1+1 \leq v_1 < v_2 \leq m$, and of type (12) if $1 \leq v_1 \leq m_1, m_1+1 \leq v_2 \leq m$. This section develops some results on the estimation of the general mean and all main-effects and the detection and estimation of the possibly present at best one two-factor interaction when among the two-factor interactions at most one is present and that one is of (i) type (12), (ii) type (11), (iii) type (22), (iv) type (11) or type (22), (v) type (11) or type (12), (vi) type (12) or type (22) and (vii) any of the three types. The designs appropriate for cases (i)-(vii) will be called search designs of types A_1 - A_7 respectively. Obviously, case (vii) above is the most general, although, as explained in Chapter 1, the other cases may also be of practical relevance.

Definition 1.5.1. A subset of the level combinations of $F_1, F_2, \dots, F_{m_1}$ alone (ignoring $F_{m_1+1}, \dots, F_m$) will be called

- (a) a type B_{11} design if it represents a saturated main-effect plan in terms of $F_1, F_2, \dots, F_{m_1}$,
- (b) a type B_{12} (search) design if given that among the main-effects $F_1, F_2, \dots, F_{m_1}$, at most one is present, it allows the estimation of the general mean and detection and estimation of the possibly present main-effect,
- (c) a type B_{13} (search) design if given that among two-factor interactions $F_{v_1} F_{v_2}$ ($1 \leq v_1 < v_2 \leq m_1$) at most one is present, it allows the estimation of the general mean and the main-effects and detection and estimation of the possibly present two-factor interaction.

Similarly, considering subsets of the level combinations of $F_{m_1+1}, \dots, F_m$ alone (ignoring $F_1, F_2, \dots, F_{m_1}$) one may define a type B_{21} (saturated) design and (search) designs of types B_{22} and B_{23} . Then we have the following results:

Theorem 1.5.1. The level combinations $\{(0,0, \dots, 0), (i,0, \dots, 0), (0,i, \dots, 0), \dots, (0,0, \dots, i); 0 \leq i \leq s-1\}$ of the factors $F_1, F_2, \dots, F_{m_1}$ constitute a type B_{11} design in $1+m_1(s-1)$ runs.

For $m_1 \geq 4$ and $1 \leq i_1, i_2 \leq s-1$, define the $(m_1 \times m_1)$ circulant matrix

$$E_{i_1 i_2} = \begin{bmatrix} 0 & i_1 & i_2 & \dots & i_2 \\ i_2 & 0 & i_1 & \dots & i_2 \\ i_2 & i_2 & 0 & \dots & i_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i_1 & i_2 & i_2 & \dots & 0 \end{bmatrix}$$

Then we have:

Theorem 1.5.2. If $s(=2\mu+1)$ be odd and $m_1 \geq 4$, the columns of the matrix $[0, E_{21}, E_{43}, \dots, E_{2\mu-2\mu-1}]$, interpreted as the level combinations of $F_1, F_2, \dots, F_{m_1}$, give a type B_{12} search design in $m_1\mu+1(=h_{12})$ runs.

Theorem 1.5.3. If $s(=2\mu)$ be even and $m_1 \geq 4$, the columns of the matrix $[0, E_{21}, E_{43}, \dots, E_{2\mu-2\mu-3}, E_{2\mu-1\ 2\mu-2}]$, interpreted as the level combinations of $F_1, F_2, \dots, F_{m_1}$, give a type B_{12} search design in $m_1\mu+1(=h_{12})$ runs.

Theorem 1.5.4. If $m_1 = 3$, the level combinations $\{(0,0,0), (i,0,i), (i,i,0); 1 \leq i \leq s-1\}$ give a type B_{12} search design in $2s-1(=h_{12})$ runs.

Let $S_{11}, S_{12}, S_{13} (S_{21}, S_{22}, S_{23})$ be not necessarily disjoint subsets of distinct level combinations of $F_1, F_2, \dots, F_{m_1} (F_{m_1+1}, \dots, F_m)$ forming designs of types $B_{11}, B_{12}, B_{13} (B_{21}, B_{22}, B_{23})$ respectively. For each t_1, t_2 ($t_1 = 1, 2; t_2 = 1, 2, 3$), $h_{t_1 t_2}$ is the

cardinality of $S_{t_1 t_2}$; $S'_{t_1 t_2}$ is any singleton subset of $S_{t_1 t_2}$ and $\bar{S}_{t_1 t_2} = S_{t_1 t_2} - S'_{t_1 t_2}$. The (symbolic) direct product, $S_1 \otimes S_2$, of a subset S_1 of the level combinations of $F_1, F_2, \dots, F_{m_1}$ (ignoring $F_{m_1+1}, F_{m_1+2}, \dots, F_m$) and a subset S_2 of the level combinations of $F_{m_1+1}, F_{m_1+2}, \dots, F_m$ (ignoring $F_1, F_2, \dots, F_{m_1}$) will be a subset of the level combinations of $F_1, F_2, \dots, F_m$ obtained by combining each member of S_1 with each member of S_2 . Then we have the following theorems:

Theorem 1.5.5. Each of the unions (a) $(S_{11} \otimes S'_{21}) \cup (S_{12} \otimes \bar{S}_{21})$, (b) $(S'_{11} \otimes S_{21}) \cup (\bar{S}_{11} \otimes S_{22})$ gives a type A_1 search design in $h_{11} + h_{12}(h_{21} - 1)$ and $h_{21} + h_{22}(h_{11} - 1)$ runs respectively.

Theorem 1.5.6. (a) The union $(S'_{13} \otimes S_{21}) \cup (\bar{S}_{13} \otimes S'_{21})$ gives a type A_2 search design in $h_{21} + h_{13} - 1$ runs.

(b) The union $(S_{11} \otimes S'_{23}) \cup (S'_{11} \otimes \bar{S}_{23})$ gives a type A_3 search design in $h_{11} + h_{23} - 1$ runs.

Theorem 1.5.7. Each of the unions (a) $(S_{13} \otimes S'_{23}) \cup (S'_{13} \otimes \bar{S}_{23})$, (b) $(S'_{13} \otimes S_{23}) \cup (\bar{S}_{13} \otimes S'_{23})$ gives a type A_4 search design in $h_{13} + h_{23} - 1$ runs.

Theorem 1.5.8. (a) The union $(S'_{13} \otimes S_{21}) \cup (\bar{S}_{13} \otimes S_{22})$ gives a type A_5 search design in $h_{21} + h_{22}(h_{13} - 1)$ runs.

(b) The union $(S_{11} \otimes S'_{23}) \cup (S_{12} \otimes \bar{S}_{23})$ gives a type A_6 search design in $h_{11} + h_{12}(h_{23} - 1)$ runs.

Theorem 1.5.9. Each of the unions (a) $(S_{13} \otimes S'_{23}) \cup (S_{12} \otimes \bar{S}_{23})$, (b) $(S'_{13} \otimes S_{23}) \cup (\bar{S}_{13} \otimes S_{22})$ gives a type A_7 search design in $h_{13} + h_{12}(h_{23} - 1)$ and $h_{23} + h_{22}(h_{13} - 1)$ runs respectively.

In this connection, one may note that in any particular situation, among the two designs suggested in Theorems 1.5.5 and 1.5.9, the one with the smaller number of runs should be used.

Section 1.6 considers the construction of search designs for the general asymmetric factorial set-up that allow the estimation of the general mean, all main-effects and permit the detection and estimation of at most one possibly present non-zero two-factor interaction.

Let

$$N = \sum_{v=1}^{m-1} (s_v - s_{v-1}) + \sum_{v=1}^{m-2} \sum_{u=1}^v (s_u - 1)(s_{v+1} - s_v) + \sum_{v=1}^{m-2} \sum_{u=v}^m (s_v - s_{v-1}) \times \\ (s_u - 1) + (s_{m-1} - s_{m-2} - 1)(s_m - s_{m-2} - 1) + (s_m - s_{m-2} - 1 + \\ \delta_{s_{m-1} s_{m-2}}) + (s_m - s_{m-1}) \sum_{u=1}^{m-2} (s_u - 1)$$

where $\delta_{s_{m-1} s_{m-2}}$ is the Kronecker delta and with reference to Section 0.3A, define the following $(m \times N)$ array

$$Q = [B^{(1,1)}, B^{(1,2)}, \dots, B^{(1,m-1)}, B^{(2,1)}, B^{(2,2)}, \dots, B^{(2,m-2)}, \\ B^{(3,1)}, B^{(3,2)}, \dots, B^{(3,m-2)}, B^{(4)}, B^{(5)}, B^{(6)}] \quad (1.6.1)$$

where in Q any subarray of the form $B^{(u,v)}$ or $B^{(u)}$ is included provided it is defined. Then we have the following theorem with reference to the search design problem stated above :

Theorem 1.6.1. The columns of the array Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N runs.

Chapter 2. A fractional factorial design will be said to be a search design for searching one among the three-factor interactions **if it is capable of estimating the general mean, all main-effects and two-factor interactions and permits the search and estimation of the possibly present at best one among the three-factor interactions.** This chapter considers search designs of the above type.

Section 2.3 describes the construction of search designs for the given problem in the general symmetric factorial set up. Let

$$N_1 = s + ms(s-1) + m(m-1)s(s-1)(s-2)/6 + m(m-1)s(s-1)(2s-1)/12,$$

$$N_2 = 1 + ms(s-1) + m(m-1)s(s-1)(s-2)/6 + m(m-1)s(s-1)(2s-1)/12$$

and define the following arrays :

$$Q_1 = [A^{(1)}_0, A^{(2)}_0, A^{(2)}_1, \dots, A^{(2)}_{s-1}, A^{(3)}_1, \dots, A^{(3)}_{s-2}, A^{(4)}_0, A^{(4)}_1, \dots, A^{(4)}_{s-2}] \quad (2.3.1)$$

$$Q_2 = [0, A^{(2)}_0, A^{(2)}_1, \dots, A^{(2)}_{s-1}, A^{(3)}_1, A^{(3)}_2, \dots, A^{(3)}_{s-2}, A^{(4)}_0, A^{(4)}_1, \dots, A^{(4)}_{s-2}] \quad (2.3.2)$$

Then we have the following theorems :

Theorem 2.3.1. For $m \leq 6$, the columns of the matrix Q_1 , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N_1 runs.

Theorem 2.3.2. For $m > 6$, the columns of the matrix Q_2 , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the search design of the type considered in this section in N_2 runs.

Section 2.4 develops search designs of the type mentioned above in the general asymmetric factorial set-up. In order to describe the search plan presented in this section, we first define

$$\begin{aligned} N = & \sum_{v=1}^{m-2} (s_v - s_{v-1}) + \sum_{v=1}^{m-2} \sum_{u=1}^v (s_u - 1)(s_{v+1} - s_v) + \sum_{v=1}^{m-3} \sum_{u=v}^m (s_v - s_{v-1})(s_u - 1) + \\ & (s_{m-2} - s_{m-3}) \sum_{u=m-2}^m (s_u - 2) + \sum_{v=1}^{m-3} \left\{ \sum_{u_1=1}^v (s_{u_1} - 1) \right\} \left\{ \sum_{j=s_v}^{s_{v+1}-1} \sum_{u_2=v+1}^m (s_{u_2} - j - 1) \right\} + \\ & \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{\substack{u_2=v \\ u_1 < u_2}}^m (s_{u_1} - j - 1)(s_{u_2} - j - 1) + \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{\substack{u_2=v \\ u_1 < u_2}}^m j(s_{u_1} + \\ & s_{u_2} - 2j - 2) + (s_{m-1} - s_{m-2})(s_{m-2} - s_{m-3}) \sum_{u=1}^{m-2} (s_u - 1) \end{aligned}$$

and the following $(m \times N)$ array

$$Q = [B^{(1,1)}, \dots, B^{(1,m-2)}, B^{(2,1)}, \dots, B^{(2,m-2)}, B^{(3,1)}, \dots, \\ B^{(3,m-3)}, B^{(7)}, B^{(8,1)}, \dots, B^{(8,m-3)}, B^{(9,1)}, \dots, B^{(9,m-3)}, \\ B^{(10,1)}, \dots, B^{(10,m-3)}, B^{(11)}] \quad \dots \quad (2.4.1)$$

where in Q any subarray of the form $B^{(u,v)}$ or $B^{(u)}$ is included provided it is defined.

The main result in this section is the following :

Theorem 2.4.1. The columns of the matrix Q , as defined in (2.4.1), interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in the general asymmetric factorial set-up in N runs.

Chapter 3. This chapter extends the search design problem considered in Chapter 1. A fractional factorial design will be said to be a search design for searching one among the two and three-factor interactions if it is capable of estimating the general mean, all the main-effects and permits the detection and estimation of the possibly present at best one among the two and three-factor interactions. This chapter considers search designs of this type.

In Section 3.3, we present the above mentioned search designs in the general symmetric factorial set-up. Let

$$N = s + ms(s-1) + m(m-1)(s-1)/2 + m(m-1)s(s-1)(s-2)/6 + m(m-1)(s-1)(s-2)(2s-3)/12$$

and define the following $(m \times N)$ array :

$$Q = [A^{(1)}, A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}, A_1^{(3)}, A_2^{(3)}, \dots, A_{s-2}^{(3)}, A_1^{(4)}, \\ A_2^{(4)}, \dots, A_{s-2}^{(4)}, A^{(5)}] \quad (3.3.1)$$

The main result in this section is :

Theorem 3.3.1. The columns of the matrix Q , as defined in (3.3.1), interpreted

as the level combinations of $F_1, F_2, \dots, F_m$, give the required search runs.

Section 3.4 describes search designs for searching one among three-factor interactions in the general asymmetric factorial set-up.

$$\begin{aligned}
 N = & \sum_{v=1}^{m-1} (s_v - s_{v-1}) + \sum_{v=1}^{m-2} \sum_{u=1}^v (s_u - 1)(s_{v+1} - s_v) + \sum_{v=1}^{m-2} \sum_{u=v}^m (s_v - s_{v-1}) \\
 & (s_{m-1} - s_{m-2} - 1)(s_m - s_{m-2} - 1) + (s_m - s_{m-2} - 1 + \delta_{s_{m-1} s_{m-2}} \\
 & \sum_{v=1}^{m-3} \left[\sum_{u_1=1}^v (s_{u_1} - 1) \left(\sum_{j=s_v}^{s_{v+1}-1} \sum_{u_2=v+1}^m (s_{u_2} - j - 1) \right) \right] + \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m : \\
 & \quad j \neq 0 \quad u_1 < \dots \\
 & - j - 1)(s_{u_2} - j - 1) + \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m j(s_{u_1} + s_{u_2} - 2j - 2) + \dots \\
 & \quad u_1 < u_2 \\
 & s_{m-2})(s_m - s_{m-2} - 1) \sum_{u=1}^{m-2} (s_u - 1) + \sum_{v=1}^{m-3} [(m-v+1)(m-v)(s_v - s_{v-1} - \dots \\
 & \sum_{j=s_{m-3}}^{s_{m-2}-1} \sum_{u_1=m-2}^m \sum_{u_2=m-2}^m (s_{u_1} + s_{u_2} - 2j - 2), \\
 & \quad u_1 < u_2
 \end{aligned}$$

and the following $(m \times N)$ array :

$$\begin{aligned}
 Q = & [B^{(1,1)}, \dots, B^{(1,m-1)}, B^{(2,1)}, \dots, B^{(2,m-2)}, B^{(3,1)}, \dots, B^{(3,m-3)}, \\
 & B^{(4)}, B^{(5)}, B^{(8,1)}, \dots, B^{(8,m-3)}, B^{(9,1)}, \dots, B^{(9,m-3)}, \\
 & B^{(10,1)}, \dots, B^{(10,m-3)}, B^{(11)}, B^{(12,1)}, \dots, B^{(12,m-3)}, B^{(13)}
 \end{aligned}$$

where in Q any subarray of the form $B^{(u,v)}$ or $B^{(u)}$ is included

it is defined. The main result in this section is :

Theorem 3.4.4. The columns of the matrix Q , as defined in (3.4.1), inter level combinations of $F_1, F_2, \dots, F_m$, give the required search design

Chapter 4. This Chapter further extends the search design problem considered in Chapters 1 and 3. We consider search designs in the general symmetric factorial set-up for searching one among the two, three and four-factor interactions. A fractional factorial design will be said to be a search design for searching one among the two, three and four-factor interactions if it is capable of searching one among the two, three and four factor interactions and also permits its estimation together with the general mean and all main-effects. The available literature on this search design problem relates to 2^m factorials only. So in the general symmetric factorial set-up our plans are the only plans available in the literature. Let

$$N = s + ms(s-1) + m(m-1)(s-1)/2 + m(m-1)s(s-1)(s-2)/6 + \\ m(m-1)(s-1)(s-2)(2s-3)/12 + m(m-1)(m-2)(s-1)/6 + \\ m(m-1)(m-2)(s-1)(s-2)(s^2-2)/12 ,$$

and define the following $(m \times N)$ array :

$$Q = [A^{(1)}_1, A^{(2)}_0, A^{(2)}_1, \dots, A^{(2)}_{s-1}, A^{(3)}_1, A^{(3)}_2, \dots, A^{(3)}_{s-2}, A^{(4)}_1, \\ A^{(4)}_2, \dots, A^{(4)}_{s-2}, A^{(5)}_1, A^{(6)}_1, A^{(7)}_1, A^{(7)}_2, \dots, A^{(7)}_{s-2}, A^{(8)}_1, \\ A^{(8)}_2, \dots, A^{(8)}_{s-2}, A^{(9)}_1, \dots, A^{(9)}_{s-2}, A^{(10)}_1, A^{(10)}_2, \dots, A^{(10)}_{s-2}] \quad (4.3.1)$$

Then we have the following theorem :

Theorem 4.3.7. The columns of the matrix Q , given by (4.3.1), interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N runs.

Chapter 5A. This chapter deals with the construction of orthogonal main-effect plans for factorial experiments under the presence of a linear time trend. Consider an $s_1 \times s_2 \times \dots \times s_m$ factorial, a typical level combination being denoted by $(i_1, i_2, \dots, i_m)$; $0 \leq i_v \leq s_v - 1, 1 \leq v \leq m$. Let $S = \{ (i_1^{(t)}, i_2^{(t)}, \dots, i_m^{(t)}) : 0 \leq t \leq N-1 \}$

be a set of level combinations such that the level combination $(i_1^{(t)}, i_2^{(t)}, \dots, i_m^{(t)})$ is applied at time point t . Here we have assumed the N points to be equispaced as happens in most practical situations. Let y_t denote the observation corresponding to the level combination $(i_1^{(t)}, i_2^{(t)}, \dots, i_m^{(t)})$. Then assuming a linear time trend and the absence of all interactions, we take the following fixed effects linear model:

$$E(y_t) = \alpha_0 + \alpha_1 t + \gamma_{1i_1}(t) + \gamma_{2i_2}(t) + \dots + \gamma_{mi_m}(t), \quad (5A.1.1)$$

where $\alpha_0, \alpha_1, \gamma_{vi_v}(t)$ ($1 \leq v \leq m$) are unknown parameters. It is assumed that the N observations are independent and homoscedastic.

A fraction, involving N observations or runs will be called a **Trend-Free Orthogonal Main-Effect** plan (TFOME plan) if under the model (5A.1.1) it allows estimation of the main-effect contrasts on γ_{vj} orthogonally among themselves and also orthogonally to $\hat{\alpha}_0$ and $\hat{\alpha}_1$, the best linear unbiased estimators of α_0 and α_1 respectively. In the absence of a time trend (i.e., if $\alpha_1 = 0$ in (5A.1.1)) it is well-known (cf. Rao (1947, 1973)) that an orthogonal array of strength 2, with its columns interpreted as level combinations, represents an orthogonal main-effect plan and such plans are universally optimal for the estimation of the main-effects (see e.g., Cheng (1980), Mukerjee (1982)). It is, therefore, natural to examine the conditions under which such an array represents a TFOME plan under the model (5A.1.1).

For $0 \leq j \leq s_v - 1$, $1 \leq v \leq m$, let $R_{vj} = \sum t$, the summation being taken over those t for which $i_v^{(t)}$ equals j . Then we have the following theorem which is useful in proving the subsequent results:

Theorem 5A.1.1. Assume that the N level combinations included is S form an orthogonal array of strength 2 and further that

$$R_{v0} = R_{v1} = \dots = R_{vs_v-1} \quad \text{for each } v \ (1 \leq v \leq m) \quad (5A.1.2)$$

Then S represents a TFOME plan for an $s_1 \times s_2 \times \dots \times s_m$ factorial.

A comprehensive discussion on the available methods for the construction of orthogonal arrays of strength 2 is available in Raghavarao (1971). The remaining sections of this chapter investigate the appropriate modifications of these methods so as to generate TFOME plans.

Section 5A.2 investigates orthogonal arrays obtained through the use of projective geometry. Let B_1 be an OA $[s^r, m+r, s, 2]$ where $m, r \geq 2$. Suppose B_1 can be partitioned as $B_1 = [B_2', B_3']'$, where B_2 is $(r \times s^r)$ and B_3 is $(m \times s^r)$ such that for $0 \leq l_1, l_2, \dots, l_r \leq s-1$, the $(l_1 s^{r-1} + l_2 s^{r-2} + \dots + l_r)$ th column of B_2 is given by $(l_1, l_2, \dots, l_r)$. Then in this section it has been proved that the columns of B_3 , interpreted as the level combinations, represent a TFOME plan in s^r runs for an s^m factorial.

Orthogonal arrays obtained by the method of differences are studied in Section 5A.3 to obtain TFOME plans. Let V be a module with s elements $u_0, u_1, \dots, u_{s-1}$, the zero element being given by u_0 . Let A_0 be an $((m+1) \times r)$ array with entries given by the members of V such that among the differences arising out of the corresponding elements in any two distinct rows of A_0 , every element of V occurs exactly λ times (i.e., $r = \lambda s$). Let $A = [A_0, A_1, \dots, A_{s-1}]$ where A_j ($1 \leq j \leq s-1$) be an $((m+1) \times r)$ array obtained by adding u_j to each element of A_0 . Further let $\bar{A}$ be an $(m \times rs)$ array obtained by deleting the top row of A . Then the following holds :

Theorem 5A.3.1. The columns of $\bar{A}$, interpreted as level combinations, represent a TFOME plan in $rs (= \lambda s^2)$ runs for an s^m factorial.

In Section 5A.4, we have investigated Modified Addelman-Kempthorne (1961) plans. Let L_0 be an OA $[2s^r, m+r, s, 2]$, where $m, r \geq 2$. Suppose L_0 can be partitioned as :

$$L_0 = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}$$

where L_1, L_2, L_3, L_4 are of orders $(r \times s^r), (r \times s^r), (m \times s^r), (m \times s^r)$ respectively.

Then under suitable conditions on L_1, L_2, L_3 and L_4 , it has been shown that the columns of $\bar{L}_0 = [L_3, L_4]$, interpreted as level combinations, represent a TFOME plan in $2s^r$ runs for an s^m factorial.

Some further results, including those based on direct products, have also been considered in this chapter. It is thus seen that corresponding to many of the well-known systematic methods for the construction of orthogonal arrays of strength 2 (cf. Raghavarao (1971), Dey (1985)), there exist suitable modifications which yield TFOME plans.

Chapter 5B : In this chapter the calculus of factorial arrangements is extended to a fractional factorial set-up in order to investigate critically the worth of proportional frequency plans in terms of estimability and efficiency. Although such plans are not necessarily orthogonal (vide Lewis and John (1976), Mukerjee (1980)), they are available in abundance (Addelman (1963, 1972)) and, therefore, when orthogonal plans are non-existent, it may be practically useful to study the utility of proportional frequency plans.

Consider an unblocked m -factor design in which the i th level combination is replicated r_i (≥ 0) times. Let for $1 \leq v_1 < \dots < v_q \leq m, 1 \leq q \leq m, 0 \leq i_{v_\ell} \leq s_{v_\ell} - 1$ ($1 \leq \ell \leq q$), $n_{i_{v_1}^{(v_1)}, i_{v_2}^{(v_2)}, \dots, i_{v_q}^{(v_q)}}$ represent the number of times the level combination $(i_{v_1}, i_{v_2}, \dots, i_{v_q})$ of the factors $F_{v_1}, F_{v_2}, \dots, F_{v_q}$ occur among the level combinations included in the experiment. The chosen level combinations will be defined to represent a proportional frequency plan of strength d ($\leq m$) if for every $v_1, v_2, \dots, v_d$ and for every $i_{v_1}, i_{v_2}, \dots, i_{v_d}$ ($1 \leq v_1 < \dots < v_d \leq m; 0 \leq i_{v_\ell} \leq s_{v_\ell} - 1, 1 \leq \ell \leq d$)

$$n_{i_{v_1}, i_{v_2}, \dots, i_{v_d}} = n^{-(d-1)} \prod_{\ell=1}^d n_{i_{v_\ell}}^{(v_\ell)} > 0 \quad (5B.3.1)$$

where $n = \sum r_i (> 0)$.

Here we consider the usual fixed effects linear model with a constant error variance.

In Section 5B.3, we have studied the worth of proportional frequency plans from the point of view of estimability. In this section, we have the following theorem :

Theorem 5B.3.1. If the chosen level combinations form a proportional frequency plan of strength $2g$ ($\leq m$) then all contrasts belonging to effects involving g or less factors are estimable when all effects involving more than g factors are absent.

One would naturally like to investigate the possible extension of the above theorem to the case $g < t$ i.e. to examine whether all contrasts belonging to effects involving g or less factors are estimable when all effects involving more than t factors are absent given that the chosen level combinations form a proportional frequency plan of strength $(g+t)$. The answer, however, will, in general, be in the negative and in support of this an example has been presented there.

Finally, in Section 5B.4, we consider the case $g = t$ and study the condition under which a proportional frequency plan has high efficiency with respect to the factorial effects of interest.

Chapter 1

SEARCH DESIGNS FOR SEARCHING ONE AMONG THE TWO-FACTOR INTERACTION EFFECTS IN THE GENERAL SYMMETRIC AND ASYMMETRIC FACTORIAL SET-UP

1.1 Introduction

In this chapter, the method of Kronecker products is applied to construct search designs for searching one among the two-factor interactions in the general asymmetric set-up through the symmetric one. In the symmetric set-up, the designs presented here do not require that the number of levels of a factor to be a prime or a prime power. A fractional factorial design will be said to be a search design for searching one among the two-factor interactions if it is capable of estimating the general mean, all the main-effects and permits the detection and estimation of the possibly present at best one among the two-factor interactions.

The concept of search designs was pioneered by Srivastava (1975) in a paper presented at the International Symposium on Statistical Design and Linear Models in 1973. Perhaps the best description of the concept is from the summary of that paper. "We introduce a new concept, called search designs, and study the basic related problems. Briefly, consider a linear model situation, where the location parameters are divided into categories I and II, with (say) n_1 and n_2 parameters respectively. All parameters of category I are estimated. Regarding the parameters in category II, it is known that all but k^* of them are zero, where k^* is small compared to n_2 . However, it is not known which k^* parameters in category II are possibly non-zero. A 'Search Design' is a design which would enable us to estimate the n_1 parameters in category I, and search the non-negligible set of parameters in category II, and estimate them".

The basic mathematical formulation of the search design problem and some necessary/necessary and sufficient conditions for such designs were given by Srivastava (1975,1976). Some examples for the 2^m series were given in those papers.

Srivastava (1977) also introduced the concept of optimal search designs and in this context gave the criteria of AD-optimality and DD-optimality. In that paper, Srivastava also considered the bias and some interesting applications of search models.

Since 1975, a good number of research papers on search designs by various authors had appeared in the literature. But most of the work which had been done under the present search linear model set-up, so far, were on constructing the search designs for the 2^m factorials. Srivastava and Gupta (1979) gave a necessary and sufficient condition for the existence of search designs for searching at best one among the two-factor and higher order interactions. They also gave comprehensive construction procedures for such designs in the 2^m factorial setting. Ghosh (1980, 1981) obtained further interesting results on search designs of the above type for 2^m factorials, where $m = 2^h - 1$ ($h \geq 2$), in $N = 1 + m + (m-h)(h+1)$ runs. He also gave a necessary and sufficient condition on the existence of good search designs for searching at best one among the two-factor and higher order interactions. Anderson and Thomas (1980) considered methods of constructing a series of weakly resolvable search designs for the s^m factorial experiment, where s is a prime or a prime power, which permits the estimation of the general mean and all main-effects in the presence of any number of two-factor interactions and for which any combination of three or fewer pairs of factors that interact may be detected. In most cases, the proposed designs require $N = s + ms(s-1)$ runs. Gupta and Carvajal (1984a) suggested a necessary condition for the existence of search designs for the 2^m factorials that permit the estimation of the general mean, all main-effects and to search and estimate possibly present at best one non-zero effect among the two-factor and higher order interactions. Several lower bounds on search designs of the above mentioned type for 2^m factorials were given among others by Gupta and Carvajal (1984b, 1987), Morris (1984) and Gupta (1988).

Shirakura and Ohnishi (1985) reported further interesting results on search designs. Ohnishi and Shirakura (1985) gave some further results in continuation of Ghosh (1981). Recently, Ghosh and Zhang (1987) gave a new series of search designs with small number of runs that can search at best one parameter corresponding to a non-zero two-factor interaction and estimate it along with the general mean and all main-effects for 3^m factorials.

The aim of this chapter is to give search designs in the general asymmetric factorial set-up through the symmetric one. The proposed designs can search at best one non-zero two-factor interaction and estimate it along with the general mean and all main-effects.

In Sections 1.2 and 1.3 we have given the search linear model and the notations and preliminaries. Section 1.4 describes the construction of search designs in the general symmetric factorial set-up. Section 1.5 considers the construction of search designs for $s_1^{m_1} \times (s_2^*)^{m-m_1}$ factorial set-up while Section 1.6 provides the construction for general asymmetric factorials. Section 1.7 gives a brief discussion on the contents of this chapter. The proofs of the theorems and the lemmas are given in Section 1.8.

1.2 The Search Linear Model

The linear model considered in this chapter and also in Chapters 2-4 resembles closely to that of Srivastava (1975) (vide (0.2.1)). Let $\mathbf{Y}^{(N \times 1)}$ be the observational vector with

$$E(\mathbf{Y}) = \mathbf{X} \boldsymbol{\tau} ; \text{Disp}(\mathbf{Y}) = \sigma^2 \mathbf{I} \quad (1.2.1)$$

where

$$\mathbf{X} = [\mathbf{X}_1^{(N \times n_1)}, \mathbf{X}_2^{(N \times n_2)}]$$

is the known design matrix and $\boldsymbol{\tau} = (\boldsymbol{\tau}_1', \boldsymbol{\tau}_2')'$, $\boldsymbol{\tau}_1$ being an $(n_1 \times 1)$ vector of unknown

fixed parameters and τ_2 being an $(n_2 \times 1)$ vector of fixed parameters which is partially known in the following sense. It is known that τ_2 may be partitioned as :

$$\tau_2 = (\beta_1', \beta_2', \dots, \beta_\eta')' \quad (1.2.2)$$

and among $\beta_1, \beta_2, \dots, \beta_\eta$, at most k^* are non-null, k^* being small compared to η . The problem is to search the non-null vectors among $\beta_1, \beta_2, \dots, \beta_\eta$ and to draw inference regarding them and τ_1 . It is at this point where we deviate slightly from Srivastava (1975) who considered a set-up where each of $\beta_1, \beta_2, \dots, \beta_\eta$ involves a single component. This change, however, is imperative since in the general asymmetric factorial set-up, a factorial effect may be represented by more than one independent parameters.

Let the partitioning of X_2 corresponding to (1.2.2) be $X_2 = [X_{21}, X_{22}, \dots, X_{2\eta}]$ and assume that the error variance is negligible (i.e. $\sigma^2 = 0$ which corresponds to the noiseless case). Then we have the following fundamental result concerning this problem which is virtually another version of Theorem 1 of Srivastava (1975) (see also Chatterjee and Mukerjee (1986)) and may be proved along the same line.

Theorem 1.2.1. Consider the model (1.2.1), together with the assumptions stated above. A necessary and sufficient condition that the above search design problem will be completely solved in the noiseless case is that for arbitrary $g_1, g_2, \dots, g_{2k^*}$ ($1 \leq g_1 < g_2 < \dots < g_{2k^*} \leq n$), the matrix $[X_1, X_{2g_1}, X_{2g_2}, \dots, X_{2g_{2k^*}}]$ has full column rank.

1.3 The Set-up

Consider a factorial experiment involving m ($m \geq 2$) factors $F_1, F_2, \dots, F_m$ at $s_1, s_2, \dots, s_m$ levels respectively ($s_v \geq 2, 1 \leq v \leq m$), where without loss of generality, suppose

$$s_1 \leq s_2 \leq \dots \leq s_m \quad (1.3.1)$$

A typical level combination will be denoted by $(i_1, i_2, \dots, i_m)$; $0 \leq i_v \leq s_v - 1$, $1 \leq v \leq m$. Let $\phi_v = (0, 1, 2, \dots, s_v - 1)'$ be a vector whose elements denote the levels of F_v , $1 \leq v \leq m$. Then the symbolic direct product $\phi_1 \otimes \phi_2 \otimes \dots \otimes \phi_m$ may be used to order lexicographically the $\prod_{v=1}^m s_v$ level combinations. Throughout this work, unless otherwise specified, the level combinations will always be written in this standard order. For any $(i_1, i_2, \dots, i_m)$; $0 \leq i_v \leq s_v - 1$, $1 \leq v \leq m$, $\tau(i_1, i_2, \dots, i_m)$ will represent the fixed effect corresponding to the level combination $(i_1, i_2, \dots, i_m)$. Here τ will stand for a vector with elements $\tau(i_1, i_2, \dots, i_m)$ taken in standard order.

A treatment contrast

$$\sum_{i_1} \dots \sum_{i_m} \ell_{i_1 \dots i_m} \tau(i_1, i_2, \dots, i_m)$$

where $\ell_{i_1 \dots i_m}$ are real numbers which are such that

$$\sum_{i_1} \dots \sum_{i_m} \ell_{i_1 \dots i_m} = 0$$

will be said to belong to the factorial effect $F_1 F_2 \dots F_r$ ($1 \leq r \leq m$) if for every $i_1, i_2, \dots, i_r$,

$$\begin{aligned} \ell_{i_1 \dots i_r i_{r+1} \dots i_m} &= \ell_{i_1 \dots i_r} \forall i_{r+1}, \dots, i_m; \sum_{i_1} \ell_{i_1 i_1 \dots i_r} = 0 \forall i_2, \dots, i_r; \\ &\dots; \sum_{i_r} \ell_{i_1 \dots i_r} = 0 \forall i_1, i_2, \dots, i_{r-1}. \end{aligned}$$

Similarly contrasts belonging to other factorial effects can be defined.

Let for $1 \leq v \leq m$, $1_v^{(s_v \times 1)} = (1, 1, \dots, 1)'$ and P_v be an $s_v \times (s_v - 1)$ matrix with full column rank such that $P_v' 1_v = 0$. Note that the choice of P_v ($1 \leq v \leq m$) may be non-unique. It will be clear from our subsequent discussion (see e.g. the proofs of the theorems of this chapter and of the following chapters) that our results will be valid for every choice of P_v ($1 \leq v \leq m$) satisfying the stated conditions. Define for $1 \leq v \leq m$,

$$P_v(x_v) = 1_v, \text{ if } x_v = 0 \\ = P_v, \text{ if } x_v = 1.$$

Let

$$1 = \times_{w=1}^m 1_w, \quad Z_v = \times_{w=1}^m P_w(\delta_{wv}) : 1 \leq v \leq m \quad \text{and}$$

$$Z_{v_1 v_2} = \times_{w=1}^m P_w(\delta_{wv_1} + \delta_{wv_2}) : 1 \leq v_1 < v_2 \leq m.$$

where δ_{wv_1} , δ_{wv_2} are kronecker deltas and $\times$ denotes the Kronecker product. Suppose prior information is available regarding the absence of all interactions involving three or more factors. Throughout this chapter these interactions will be assumed to be absent. Then it is easy to observe that if one observation is made for each of the level combinations of $F_1, F_2, \dots, F_m$, the resulting design matrix will be of the form

$$[1, Z_1, Z_2, \dots, Z_m, Z_{12}, Z_{13}, \dots, Z_{m-1 m}] \quad (1.3.2)$$

where 1 corresponds to the general mean and the columns of Z_v and $Z_{v_1 v_2}$ corresponds to the main-effect F_v ($1 \leq v \leq m$) and interaction $F_{v_1} F_{v_2}$ ($1 \leq v_1 < v_2 \leq m$) respectively. Of course (1.3.2) incorporates $Z_{v_1 v_2}$ only when the associated interaction $F_{v_1} F_{v_2}$ is included in the model. Let

$$H_1 = [1, Z_1, Z_2, \dots, Z_m]$$

$$H_2 = [Z_{12}, Z_{13}, \dots, Z_{m-1 m}]$$

and S be a subset of N distinct level combinations of $F_1, F_2, \dots, F_m$ with the corresponding sub-matrices of H_1 and H_2 as, say, $T_1 = [\epsilon, L_1, L_2, \dots, L_m]$ and $T_2 = [G_{12}, G_{13}, \dots, G_{m-1 m}]$ respectively. From **Theorem 1.2.1** one observes that S represents a search design for the above problem with $k^* = 1$ if and only if for each

q_1, q_2, q_3, q_4 ($1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 \leq m, (q_1, q_2) \neq (q_3, q_4)$), the matrix $[T_1, G_{q_1 q_2}, G_{q_3 q_4}]$ has full column rank.

1.4 Search Designs for the General Symmetric Factorial Set-up

This section deals with the construction of search designs for general symmetric factorials that allow the estimation of the general mean, the main-effects and detect at most one two-factor interaction. In order to construct search designs for the above set-up, consider the case where $s_1 = s_2 = \dots = s_m = s$, say, where s is not necessarily a prime or prime power. We have the following theorems the proofs of which are given in the Appendix.

Theorem 1.4.1. If $m = 3$, then the following level combinations $\{(i, j, j), (j, i, j), (j, j, i); 0 \leq i, j \leq s-1, j \neq i\}$ give a search design of the required type in $3s(s-1)$ runs.

Theorem 1.4.2. If $m \geq 4$, then the level combinations $\{(0, 0, \dots, 0), (i, j, \dots, j), (j, i, \dots, j), \dots, (j, j, \dots, i); 0 \leq i, j \leq s-1, j \neq i\}$ give a search design of the required type in $1 + ms(s-1)$ runs.

The following examples will illustrate the above theorems:

Example 1.4.1. Let $m=3$ and $s=4$. Then the following level combinations, written as columns, give a search design in 36 runs that will provide the estimation of the general mean, all the main-effects and will permit the detection and estimation of possibly present one two-factor interaction.

```
123000000023111111013222222012333333
000123000111023111222013222333012333
000000123111111023222222013333333012
```

Example 1.4.2. Let $m=5$ and $s=3$. Then the following level combinations, written as columns, give a search design of the required type in 31 runs.

0120000000002111111110122222222
 0001200000011021111112201222222
 0000012000011110211112222012222
 0000000120011111102112222220122
 000000000121111111102222222201

1.5 Search Designs for $s^{m_1} \times (s^*)^{m-m_1}$ Factorial set up

In many practical situations there may be a natural partitioning of the set of factors into two groups, namely $F_1, \dots, F_{m_1}$ and $F_{m_1+1}, \dots, F_m$ ($m_1 < m$) such that the factors within the same group have the same number of levels. To that effect, let $s_1 = s_2 = \dots = s_{m_1} = s$, and $s_{m_1+1} = s_{m_1+2} = \dots = s_m = s^*$. This section develops some results on search designs for $s^{m_1} \times (s^*)^{m-m_1}$ factorials. The proofs of all the results in this section have been presented in the Appendix. A two-factor interaction $F_{v_1} F_{v_2}$ will be said to be of type (11) if $1 \leq v_1 < v_2 \leq m_1$, of type (22) if $m_1 + 1 \leq v_1 < v_2 \leq m$, and of type (12) if $1 \leq v_1 \leq m_1$, $m_1 + 1 \leq v_2 \leq m$.

In the above situation the presence or absence of interaction between two factors may depend on whether they belong to the same group or not. Taking this possibility into account, this section develops some results on the estimation of the general mean and all main-effects and the detection and estimation of the possibly present at best one two-factor interaction when among the two-factor interactions at most one is present and that one is of (i) type (12), (ii) type (11), (iii) type (22), (iv) type (11) or type (22), (v) type (11) or type (12), (vi) type (12) or type (22) and (vii) any of the three types. It may be remarked that among the seven cases mentioned above, the last one is the most general and is meaningful even when there is no

grouping of factors. Still then a study of the first six cases remains important since they may be relevant in particular practical situations. For example, the m factors may represent m fertilizers of which the first m_1 are traditional and the last $(m-m_1)$ are newly introduced ones. Now if prior knowledge is available that at most one two-factor interaction is present and further that traditionally used fertilizers do not interact among themselves then an investigation of case (vi) becomes appropriate.

In the following, designs appropriate for cases (i) - (vii) will be called search designs of types A_1 - A_7 respectively. This section describes in a sense the construction of higher order search designs from lower order ones. To that effect, the following definition is introduced.

Definition 1.5.1. A subset of the level combinations of $F_1, F_2, \dots, F_{m_1}$ alone (ignoring $F_{m_1+1}, \dots, F_m$) will be called

- (a) a type B_{11} design if it represents a saturated main-effect plan in terms of $F_1, F_2, \dots, F_{m_1}$,
- (b) a type B_{12} (search) design if given that among the main-effects $F_1, F_2, \dots, F_{m_1}$, at most one is present, it allows the estimation of the general mean and detection and estimation of the possibly present main-effect,
- (c) a type B_{13} (search) design if given that among two-factor interactions $F_{v_1} F_{v_2}$ ($1 \leq v_1 < v_2 \leq m_1$) at most one is present, it allows the estimation of the general mean and the main-effects and detection and estimation of the possibly present two-factor interaction.

Similarly, considering subsets of the level combinations of $F_{m_1+1}, \dots, F_m$ alone (ignoring $F_1, F_2, \dots, F_{m_1}$) one may define a type B_{21} (saturated) design and (search) designs of types B_{22} and B_{23} .

The (symbolic) direct product, $S_1 \otimes S_2$, of a subset S_1 of the level combinations of $F_1, \dots, F_{m_1}$ (ignoring $F_{m_1+1}, \dots, F_m$) and a subset S_2 of the level combinations of

$F_{m_1+1}, \dots, F_m$ (ignoring $F_1, \dots, F_{m_1}$) will be a subset of the level combinations of $F_1, \dots, F_m$ obtained by combining each member of S_1 with each member of S_2 . The following notations will be helpful in describing the construction of search designs of types $A_1 - A_7$. Define

$$Z_v^{(1)} = \times_{w=1}^{m_1} P_w(\delta_{wv}), \quad 1 \leq v \leq m_1,$$

$$Z_v^{(2)} = \times_{w=m_1+1}^m P_w(\delta_{wv}), \quad m_1 + 1 \leq v \leq m,$$

$$Z_{v_1 v_2}^{(1)} = \times_{w=1}^{m_1} P_w(\delta_{wv_1} + \delta_{wv_2}), \quad 1 \leq v_1 < v_2 \leq m_1$$

$$Z_{v_1 v_2}^{(2)} = \times_{w=m_1+1}^m P_w(\delta_{wv_1} + \delta_{wv_2}), \quad m_1 + 1 \leq v_1 < v_2 \leq m,$$

$$\mathbf{1}^{(1)} = \times_{w=1}^{m_1} \mathbf{1}_w, \quad \mathbf{1}^{(2)} = \times_{w=m_1+1}^m \mathbf{1}_w$$

$$H_1^{(1)} = [\mathbf{1}^{(1)}, Z_1^{(1)}, \dots, Z_{m_1}^{(1)}],$$

$$H_1^{(2)} = [\mathbf{1}^{(2)}, Z_{m_1+1}^{(2)}, \dots, Z_m^{(2)}],$$

$$H_2^{(1)} = [Z_{12}^{(1)}, Z_{13}^{(1)}, \dots, Z_{m_1-1 \ m_1}^{(1)}] \text{ and}$$

$$H_2^{(2)} = [Z_{m_1+1 \ m_1+2}^{(2)}, Z_{m_1+1 \ m_1+3}^{(2)}, \dots, Z_{m-1 \ m}^{(2)}]$$

It is easy to observe that a subset of N_1 (N_2) level combinations of $F_1, F_2, \dots, F_{m_1}$ ($F_{m_1+1}, F_{m_1+2}, \dots, F_m$) dictates N_1 (N_2) rowed submatrices of $H_1^{(1)}$ and $H_2^{(1)}$ ($H_1^{(2)}$ and $H_2^{(2)}$). Our ultimate objective is to construct search designs of types

$A_1 - A_7$ for which consideration of designs of types B_{t1}, B_{t2}, B_{t3} ($t = 1, 2$) will be helpful.

In this connection, we have the following lemma, part (a) of which is well known and parts (b) and (c) of which follow as corollaries of Theorem 1.2.1.

Lemma 1.5.1. Let S_1 be a subset of N_1 level combinations of $F_1, F_2, \dots, F_{m_1}$ with the corresponding submatrices of $H_1^{(1)}, H_2^{(1)}$ as, say, $T_1^{(1)} = [\epsilon^{(1)}, L_1^{(1)}, \dots, L_{m_1}^{(1)}]$ and $T_2^{(1)} = [G_{12}^{(1)}, G_{13}^{(1)}, \dots, G_{m_1-1, m_1}^{(1)}]$ respectively. Then S_1 represents

(a) a type B_{11} design if and only if $T_1^{(1)}$ has full column rank and S_1 has cardinality

$$1 + \sum_{v=1}^{m_1} (s_v - 1) = 1 + m_1(s-1) \text{ (so that } T_1^{(1)} \text{ becomes a square matrix).}$$

(b) a type B_{12} design if and only if for each $v_1, v_2 (1 \leq v_1 < v_2 \leq m_1)$, the matrix $[\epsilon^{(1)}, L_{v_1}^{(1)}, L_{v_2}^{(1)}]$ has full column rank, and

(c) a type B_{13} design if and only if for each $v_1, v_2, v_3, v_4 (1 \leq v_1 < v_2 \leq m_1, 1 \leq v_3 < v_4 \leq m_1, (v_1, v_2) \neq (v_3, v_4))$, the matrix $[T_1^{(1)}, G_{v_1 v_2}^{(1)}, G_{v_3 v_4}^{(1)}]$ has full column rank.

The existence of a dual of this lemma associated with the factors $F_{m_1+1}, \dots, F_m$ is obvious. The following theorems will suggest designs of types $B_{t_1 t_2}$ ($t_1 = 1, 2, t_2 = 1, 2, 3$).

Theorem 1.5.1. The level combinations $\{(0, 0, \dots, 0), (i, 0, \dots, 0), (0, i, \dots, 0), \dots, (0, 0, \dots, i); 0 \leq i \leq s-1\}$ of the factors $F_1, F_2, \dots, F_{m_1}$ constitute a type B_{11} designs in $1 + m_1(s-1)$ runs.

For $m_1 \geq 4$ and $1 \leq i_1, i_2 \leq s-1$, define the $(m_1 \times m_1)$ circulant matrix

$$E_{i_1 i_2} = \begin{bmatrix} 0 & i_1 & i_2 & \dots & i_2 \\ i_2 & 0 & i_1 & \dots & i_2 \\ i_2 & i_2 & 0 & \dots & i_2 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ i_1 & i_2 & i_2 & \dots & 0 \end{bmatrix}$$

Theorem 1.5.2. If $s(= 2\mu + 1)$ be odd and $m_1 \geq 4$, the columns of the matrix $[0, E_{21}, E_{43}, \dots, E_{2\mu-2\mu-1}]$, interpreted as the level combinations of $F_1, F_2, \dots, F_{m_1}$, give a type B_{12} search design in $m_1\mu + 1 (= h_{12})$ runs.

Theorem 1.5.3. If $s(= 2\mu)$ be even and $m_1 \geq 4$, the columns of the matrix $[0, E_{21}, E_{43}, \dots, E_{2\mu-2\mu-3}, E_{2\mu-1\ 2\mu-2}]$, interpreted as the level combinations of $F_1, F_2, \dots, F_{m_1}$, give a type B_{12} search design in $m_1\mu + 1 (= h_{12})$ runs.

Theorem 1.5.4. If $m_1 = 3$, the level combinations $\{(0,0,0), (i,0,i), (i,i,0); 1 \leq i \leq s-1\}$ give a type B_{12} search design in $2s-1 (= h_{12})$ runs.

Following Theorems 1.4.1 and 1.4.2 one can obtain search designs of type B_{13} and its dual yielding search design of type B_{23} for the factors $F_{m_1+1}, F_{m_1+2}, \dots, F_m$. It is also easy to develop duals of Theorems 1.5.1-1.5.4 giving a type B_{21} design and search designs of type B_{22} for the factors $F_{m_1+1}, \dots, F_m$.

EXample 1.5.1. Let $s = 7(= 2 \times 3 + 1)$, $m_1 = 5$. The following level combinations, written as columns, of the factors $F_1, F_2, \dots, F_5$, give a type B_{12} search design in 16 runs.

```
0021110433306555
0102113043350655
0110213304355065
0111023330455506
0211104333065550
```

Example 1.5.2. Let $s = 8(= 2 \times 4)$, $m_1 = 5$. The following level combinations, written as columns, of the factors $F_1, F_2, \dots, F_5$, give a type B_{12} search design in 21 runs.

```
002111043330655507666
010211304335065560766
011021330435506566076
011102333045550666607
021110433306555076660
```

Let $S_{11}, S_{12}, S_{13} (S_{21}, S_{22}, S_{23})$ be not necessarily disjoint subsets of distinct

level combinations of $F_1, F_2, \dots, F_{m_1}$ ($F_{m_1+1}, F_{m_1+2}, \dots, F_m$) forming designs of types B_{11}, B_{12}, B_{13} (B_{21}, B_{22}, B_{23}) respectively. For each t_1, t_2 ($t_1 = 1, 2; t_2 = 1, 2, 3$), $h_{t_1 t_2}$ is the cardinality of $S_{t_1 t_2}$; $S'_{t_1 t_2}$ is any singleton subset of $S_{t_1 t_2}$ and $\bar{S}_{t_1 t_2} = S_{t_1 t_2} - S'_{t_1 t_2}$. The following theorems present the construction of search designs of types $A_1 - A_7$ employing direct products involving designs of types $B_{t_1}, B_{t_2}, B_{t_3}$ ($t = 1, 2$).

Theorem 1.5.5. Each of the unions (a) $(S_{11} \otimes S'_{21}) \cup (S_{12} \otimes \bar{S}_{21})$, (b) $(S'_{11} \otimes S_{21}) \cup (\bar{S}_{11} \otimes S_{22})$ gives a type A_1 search design in $h_{11} + h_{12}(h_{21} - 1)$ and $h_{21} + h_{22}(h_{11} - 1)$ runs respectively.

Theorem 1.5.6. (a) The union $(S'_{13} \otimes S_{21}) \cup (\bar{S}_{13} \otimes S'_{21})$ gives a type A_2 search design in $h_{21} + h_{13} - 1$ runs.

(b) The union $(S_{11} \otimes S'_{23}) \cup (S'_{11} \otimes \bar{S}_{23})$ gives a type A_3 search design in $h_{11} + h_{23} - 1$ runs.

Theorem 1.5.7. Each of the unions (a) $(S_{13} \otimes S'_{23}) \cup (S'_{13} \otimes \bar{S}_{23})$, (b) $(S'_{13} \otimes S_{23}) \cup (\bar{S}_{13} \otimes S'_{23})$ gives a type A_4 search design in $h_{13} + h_{23} - 1$ runs.

Theorem 1.5.8. (a) The union $(S'_{13} \otimes S_{21}) \cup (\bar{S}_{13} \otimes S_{22})$ gives a type A_5 search design in $h_{21} + h_{22}(h_{13} - 1)$ runs.

(b) The union $(S_{11} \otimes S'_{23}) \cup (S_{12} \otimes \bar{S}_{23})$ gives a type A_6 search design in $h_{11} + h_{12}(h_{23} - 1)$ runs.

Theorem 1.5.9. Each of the unions (a) $(S_{13} \otimes S'_{23}) \cup (S_{12} \otimes \bar{S}_{23})$, (b) $(S'_{13} \otimes S_{23}) \cup (\bar{S}_{13} \otimes S_{22})$ gives a type A_7 search design in $h_{13} + h_{12}(h_{23} - 1)$ and $h_{23} + h_{22}(h_{13} - 1)$ runs respectively.

In this connection, one may note that in any particular situation, among the two designs suggested in Theorems 1.5.5 and 1.5.9, the one with the smaller number of runs should be used. The examples given below will be helpful in illustrating the plans discussed above.

1.6 Search Designs for the General Asymmetric Factorial Set-up

This section describes the construction of search designs for the general asymmetric factorial set-up that allow the estimation of the general mean, all main-effects and detect and estimate possibly present at best one non-zero two-factor interaction. The following notations will be helpful in describing the search designs considered here. Admittedly, the notation is somewhat involved, but it may be seen that the plan described below arises through a kind of natural generalisation of the case of symmetric factorials. Anyway, it is expected that the examples stated later in this section would be helpful in illustrating the plan under consideration.

Let $s_0 = 0$ and recall (1.3.1). We now define the following :

- (i) For $1 \leq v \leq m-1$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $\mathbf{b}_j^{(1,v)}$ ($s_{v-1} \leq j \leq s_v - 1$) such that in $\mathbf{b}_j^{(1,v)}$, the first $(v-1)$ elements equal 0 and the remaining elements all equal j ; let $B^{(1,v)}$ be an $(m \times (s_v - s_{v-1}))$ array whose columns are given by $\mathbf{b}_j^{(1,v)}$ for all j . Note that $B^{(1,v)}$ is defined only when $s_{v-1} < s_v$.
- (ii) For $1 \leq v \leq m-2$, if $s_v < s_{v+1}$, then define the $(m \times 1)$ vectors $\mathbf{b}_{j,u,i_u}^{(2,v)}$ ($s_v \leq j \leq s_{v+1}-1$, $1 \leq u \leq v$, $1 \leq i_u \leq s_u - 1$) such that in $\mathbf{b}_{j,u,i_u}^{(2,v)}$, the u th element is i_u , the last $(m-v)$ elements all equal j and the remaining elements all equal 0; let $B^{(2,v)}$ be an $(m \times [\sum_{u=1}^v (s_u - 1)(s_{v+1} - s_v)])$ array whose columns are given by $\mathbf{b}_{j,u,i_u}^{(2,v)}$ for all j, u and i_u . Note that $B^{(2,v)}$ is defined only when $s_v < s_{v+1}$ and $m \geq 3$.
- (iii) For $1 \leq v \leq m-2$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $\mathbf{b}_{j,u,i_u}^{(3,v)}$ ($s_{v-1} \leq j \leq s_v - 1$, $v \leq u \leq m$, $0 \leq i_u \leq s_u - 1$, $i_u \neq j$) such that in $\mathbf{b}_{j,u,i_u}^{(3,v)}$, the first $(v-1)$ elements equal 0, the u th element is i_u and the remaining elements all equal j ; let $B^{(3,v)}$ be an $(m \times [\sum_{u=v}^m (s_v - s_{v-1})(s_u - 1)])$ array whose columns are given by $\mathbf{b}_{j,u,i_u}^{(3,v)}$ for all j, u and i_u . Note that $B^{(3,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 3$.

(iv) If $s_{m-2} < s_{m-1} - 1$, then define the $(m \times 1)$ vectors $\mathbf{b}_{jk}^{(4)}$ ($s_{m-2} \leq j < s_{m-1} - 1$, $s_{m-2} \leq k \leq s_m - 1$, $j \neq k$) such that in $\mathbf{b}_{jk}^{(4)}$, the last two elements are j and k and all other elements equal 0; let $B^{(4)}$ be an $(m \times [(s_{m-1} - s_{m-2} - 1)(s_m - s_{m-2} - 1)])$ array whose columns are given by $\mathbf{b}_{jk}^{(4)}$ for all j and k . Note that $B^{(4)}$ is defined only when $s_{m-2} < s_{m-1} - 1$.

(v) If $s_{m-2} < s_m$, then define the $(m \times 1)$ vectors $\mathbf{b}_k^{(5)}$ ($s_{m-2} \leq k \leq s_m - 1$, $k \neq s_{m-1} - 1$) such that in $\mathbf{b}_k^{(5)}$, the last two elements are $s_{m-1} - 1$ and k and the remaining elements all equal 0; let $B^{(5)}$ be an $(m \times (s_m - s_{m-2} - 1 + \delta_{s_{m-2}s_{m-1}}))$ array, where $\delta_{s_{m-2}s_{m-1}}$ is the Kronecker delta, whose columns are $\mathbf{b}_k^{(5)}$ for all k . Note that $B^{(5)}$ is defined only for $s_{m-2} < s_m$.

(vi) If $s_{m-1} < s_m$, then define the $(m \times 1)$ vectors $\mathbf{b}_{u,i_u,k}^{(6)}$ ($1 \leq u \leq m-2$, $1 \leq i_u \leq s_u - 1$, $s_{m-1} \leq k \leq s_m - 1$) such that in $\mathbf{b}_{u,i_u,k}^{(6)}$, the u th element is i_u , the $(m-1)$ th element is $s_{m-1} - 1$, the m th element is k and the remaining elements all equal 0; let $B^{(6)}$ be an $(m \times [(s_m - s_{m-1}) \sum_{u=1}^{m-2} (s_u - 1)])$ array whose columns are $\mathbf{b}_{u,i_u,k}^{(6)}$ for all u , i_u and k . Note that $B^{(6)}$ is defined only when $s_{m-1} < s_m$.

Let

$$N = \sum_{v=1}^{m-1} (s_v - s_{v-1}) + \sum_{v=1}^{m-2} \sum_{u=1}^v (s_u - 1)(s_{v+1} - s_v) + \sum_{v=1}^{m-2} \sum_{u=v}^m (s_v - s_{v-1})(s_u - 1) + (s_{m-1} - s_{m-2} - 1)(s_m - s_{m-2} - 1) + (s_m - s_{m-2} - 1 + \delta_{s_{m-1}s_{m-2}}) + (s_m - s_{m-1}) \sum_{u=1}^{m-2} (s_u - 1)$$

and define the following $(m \times N)$ array

$$Q = [B^{(1,1)}, B^{(1,2)}, \dots, B^{(1,m-1)}, B^{(2,1)}, B^{(2,2)}, \dots, B^{(2,m-2)}, \\ B^{(3,1)}, B^{(3,2)}, \dots, B^{(3,m-2)}, B^{(4)}, B^{(5)}, B^{(6)}] \quad (1.6.1)$$

where in Q any subarray of the form $B^{(u,v)}$ or $B^{(u)}$ is included provided it is defined. Then we have the following theorem with reference to the search design problem stated in the beginning of this section.

Theorem 1.6.1. The columns of the array Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N runs.

The proof of the above theorem follows from Theorem 1.2.1 and the following Lemma which has been proved in the Appendix.

Lemma 1.6.1. The level combinations of the factors $F_1, F_2, \dots, F_m$, given by the columns of the array Q , are such that the matrices

- (a) $[T_1, G_{q_1 q_2}, G_{q_1 q_3}]$, where $q_1 < q_2 < q_3$,
- (b) $[T_1, G_{q_1 q_3}, G_{q_2 q_3}]$, where $q_1, q_2 < q_3, q_1 \neq q_2$,
- (c) $[T_1, G_{q_1 q_2}, G_{q_2 q_3}]$, where $q_1 < q_2 < q_3$, and
- (d) $[T_1, G_{q_1 q_2}, G_{q_3 q_4}]$, where $q_1 < q_2, q_3 < q_4, q_1 \neq q_3, q_4 \neq q_2 \neq q_3, q_4$,

have full column rank.

As indicated earlier, depending on $s_1, s_2, \dots, s_m$, some of the subarrays in the definition of Q (see (1.6.1)) may not arise in practice. This is illustrated by the following examples. In Example 1.6.1, all the subarrays arise while in the next two examples some of the subarrays are not meaningful and hence do not arise.

Example 1.6.1. Let $m = 5, s_1 = 3, s_2 = 4, s_3 = 5, s_4 = 7, s_5 = 9$. Then the following level combinations taken together as in (1.6.1), which are given as columns, give

the search design in 164 runs that will permit the estimation of the general mean, all the main-effects and allow the search and estimation of possibly present one nonzero two-factor interaction.

	012	0	0	00	12	12000
$B^{(1,1)} : 012 ; B^{(1,2)} : 3 ; B^{(1,3)} : 0 ; B^{(1,4)} : 00 ; B^{(2,1)} : 33 ; B^{(2,2)} : 00123$						
012	3	4	00	33	44444	
012	3	4	56	33	44444	
012	3	4	56	33	44444	
$B^{(2,3)} : 120000000120000000 ; B^{(3,1)} : 0012300000001230000 ;$						
000001234000001234						
555555556666666666						
555555556666666666						

(Contd.)

01222222222222222222	00000000000000000000	00000000000000000000
22013222222222222222 ; $B^{(3,2)} : 01233333333333333333 ; B^{(3,3)} : 00000000000000000000 ;$		
2222201342222222222222	33301243333333333333	01234444444444444444
2222222220134562222222	3333333012456333333333	44440123564444444444
22222222222222201345678	333333333333301245678	4444444444401235678

$B^{(4)} : 000 ; B^{(5)} : 000 ; B^{(6)} : 120000000120000000 ;$		
000	000	000001234000001234
555	666	666666666666666666
678	578	777777778888888888

Example 1.6.2. Let $m = 5$, $s_1 = 3$, $s_2 = 5$, $s_3 = 5$, $s_4 = 7$, $s_5 = 7$. Then as before the following level combinations, written as columns, give a search design of the required type in 139 runs.

(Contd.)

01233333333333333333333333333333	000000000000000000000000	000000000000000000000000
33301233333333333333333333333333	000000000000000000000000	000000000000000000000000
33333301243333333333333333333333 ; $B^{(3,3)}$	: 012344444444444444444444 ; $B^{(3,4)}$	: 000000000000000000000000 ;
33333333330124533333333333333333	444401235444444444444444	012345555555555555555555
33333333333333330124533333333333	444444444012354444444444	555550123455555555555555
33333333333333333333333301245	444444444444444401235	5555555555501234

Here $B^{(1,2)}$, $B^{(1,5)}$, $B^{(2,1)}$, $B^{(2,4)}$, $B^{(3,2)}$, $B^{(4)}$, $B^{(5)}$, $B^{(6)}$ are not defined.

It may be remarked that the regular resolution V plans corresponding to Examples 1.6.1, 1.6.2, 1.6.3 require 224, 211 and 284 runs respectively.

1.7 Discussion

Search designs considered in this chapter involve considerably small number of runs compared to corresponding regular resolution V plans. These regular plans are, indeed, capable of estimating many more parameters than those considered in this chapter. Anyway, if prior knowledge is available that at most one among all two-factor interactions is present then such elaborate estimation is not required and, in such situations, the designs presented here appear to be economic (vide Table 1.7.1).

For symmetric s^m factorials, Anderson and Thomas (1980) described search designs in at least $s + ms(s-1)$ runs. Although, their designs are capable of detecting even more than one two-factor interactions, unlike us they require that s must be a prime or a prime power. The plan described by Ghosh and Zhang (1987) for 3^m factorials cannot be compared with our plan in Section 1.4, since our plan is capable of searching more parameters than their plan. With reference to the search linear model given in Section 1.2, the plan due to Ghosh and Zhang is capable of searching one among the n_2 elements of τ_2 and estimating it together with the elements of τ_1 while our plan is capable of searching one among $\beta_1, \beta_2, \dots, \beta_n$ (where in the 3^m factorial set-up, each of $\beta_1, \beta_2, \dots, \beta_n$ involve more than a single element of τ_2) and

estimating it along with the elements of τ_1 .

As for 2^m factorial set-up, our plan requires 6 runs when $m = 3$ and $1 + 2m$ runs when $m \geq 4$ (vide Theorems 1.4.1 and 1.4.2, with $s = 2$) which at best equals the number of runs in the corresponding plans due to Srivastava (1975), Ghosh (1980, 1981), Shirakura and Ohnishi (1985) and Ohnishi and Shirakura (1985) for 2^m factorials. However, our plans are inferior to those of Srivastava (1975), Ghosh (1980, 1981) and Ohnishi and Shirakura (1985) since the later plans are capable of searching and estimating one among two-factor and higher order interactions. Notwithstanding the above, it appears from a review of the literature that there is no obvious way of generalizing the available results in the 2^m setting (vide Gupta (1990)) to more general symmetric and asymmetric factorials and, as indicated earlier, for such situations our plans seem to be the only ones available in the literature.

TABLE 1.7.1

SOME EXAMPLES ILLUSTRATING THE NUMBER OF RUNS REQUIRED FOR THE PLANS PRESENTED IN SECTIONS 1.4 AND 1.6 AND THE CORRESPONDING RESOLUTION V PLAN.

Serial no.	Set-up	Number of runs	
		Our plan	Regular plan
1	2^7	15	29
2	2^8	17	37
3	2^9	19	46
4	2^{12}	25	79
5	3^7	43	99
6	$3^3 \times 4^3 \times 5$	93	173
7	3^{10}	61	201
8	4^7	85	211
9	5^6	121	265

(Contd. Table 1.7.1)

Serial no.	Set-up	Number of runs	
		Our plan	Regular plan
10	4^8	97	277
11	2×5^6	130	290
12	$4^3 \times 5^3 \times 6$	153	315
13	$4^2 \times 6^2 \times 7^2$	191	351
14	5^7	141	365
15	$2 \times 6^3 \times 8 \times 9$	246	418
16	$4^3 \times 6^3 \times 8$	232	437
17	$4 \times 5 \times 6 \times 7 \times 8 \times 9$	276	479
18	5^8	161	481
19	$3 \times 6^5 \times 7$	227	496
20	$5^3 \times 6^3 \times 7$	227	499
21	$6^4 \times 8^2$	254	514
22	$2 \times 6^5 \times 9$	276	518
23	$2 \times 6^4 \times 7 \times 9$	282	548
24	$3^5 \times 5^7$	195	695
25	$3^5 \times 5^5 \times 7^3$	343	1097
26	$2^5 \times 3^5 \times 5^{10}$	280	1476
27	$3^{10} \times 4^7 \times 5^5$	310	1831
28	$3^7 \times 5^{13}$	335	2127

1.8 Appendix

This section considers the proofs of some of the theorems stated in the earlier sections. The proofs of the remaining theorems follow in a similar manner. Before going into the proofs of the theorems, we first prove the following lemma.

Lemma 1.8.1. Let $P = (p_0, p_1, \dots, p_{s-1})'$ be an $(s \times (s-1))$ matrix such that $P'1_s = 0$ and P is of full column rank. If $(p_j' - p_0')\phi = 0$, $j = 1, 2, \dots, s-1$, then $\phi = 0$.

Proof : Let $\tilde{P} = (p_1, \dots, p_{s-1})'$, and $P^* = \begin{bmatrix} 1 & p_0' \\ 1_{s-1} & \tilde{P}' \end{bmatrix}$. Then by the definition of P , the square matrix P^* is non-singular. Premultiplying P^* by the non-singular matrix

$$\begin{bmatrix} 1 & 0' \\ -1_{s-1} & I_{s-1} \end{bmatrix}$$

and noting that

$$\begin{bmatrix} 1 & 0' \\ -1_{s-1} & I_{s-1} \end{bmatrix} P^* = \begin{bmatrix} 1 & p_0' \\ 0 & \tilde{P} - 1_{s-1} p_0' \end{bmatrix},$$

it follows that the square matrix $\tilde{P} - 1_{s-1} p_0'$ is non-singular. Now, if $(p_j' - p_0')\phi = 0$, $j = 1, 2, \dots, s-1$, then $(\tilde{P} - 1_{s-1} p_0')\phi = 0$ and the non-singularity of $\tilde{P} - 1_{s-1} p_0'$ implies that $\phi = 0$.

Proof of Theorem 1.4.2. Under the notation of Section 1.3, consider the matrix

$[T_1, G_{q_1 q_2}, G_{q_3 q_4}] = [\epsilon, L_1, L_2, \dots, L_m, G_{q_1 q_2}, G_{q_3 q_4}]$, where without loss of generality, suppose

$$q_1 < q_2, q_3 < q_4, q_2 \leq q_4 \text{ and } (q_1, q_2) \neq (q_3, q_4). \quad (1.8.1)$$

By Theorem 1.2.1, it is enough to prove the linear independence of the columns of the above matrix. Since the stated plan is invariant under the renaming of factors, it is enough to verify the linear independence of the columns of the above mentioned matrix for $q_1 = 1, q_2 = 2, q_3 = 1, q_4 = 3$ and $q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 4$. We first start with $q_1 = 1, q_2 = 2, q_3 = 1, q_4 = 3$. To prove the linear independence of the columns of the above matrix, consider

$$[\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{13}]\theta = 0, \quad (1.8.2)$$

where $\theta = (\theta_0, \theta_1', \theta_2', \dots, \theta_m', \phi_1', \phi_2')'$, say. Let p_{vj}' ($1 \leq v \leq m, 0 \leq j \leq s-1$) denote the j th row of P_v . Recalling the definitions of $L_1, L_2, \dots, L_m, G_{12}$ and G_{13} , one gets from (1.8.2), the equations (1.8.3a - e) stated below :

$$\theta_0 + \sum_{w=1}^m p'_{w0} \theta_w + (p'_{10} \times p'_{20}) \phi_1 + (p'_{10} \times p'_{30}) \phi_2 = 0, \quad (1.8.3a)$$

$$\theta_0 + p'_{1i} \theta_1 + \sum_{w=2}^m p'_{wj} \theta_w + (p'_{1i} \times p'_{2j}) \phi_1 + (p'_{1i} \times p'_{3j}) \phi_2 = 0, \quad (1.8.3b)$$

$$\theta_0 + p'_{2i} \theta_2 + \sum_{\substack{w=1 \\ w \neq 2}}^m p'_{wj} \theta_w + (p'_{1j} \times p'_{2i}) \phi_1 + (p'_{1j} \times p'_{3j}) \phi_2 = 0, \quad (1.8.3c)$$

$$\theta_0 + p'_{3i} \theta_3 + \sum_{\substack{w=1 \\ w \neq 3}}^m p'_{wj} \theta_w + (p'_{1j} \times p'_{2j}) \phi_1 + (p'_{1j} \times p'_{3i}) \phi_2 = 0, \quad (1.8.3d)$$

$$\theta_0 + p'_{ti} \theta_t + \sum_{\substack{w=1 \\ w \neq t}}^m p'_{wj} \theta_w + (p'_{1j} \times p'_{2j}) \phi_1 + (p'_{1j} \times p'_{3j}) \phi_2 = 0, \quad (1.8.3e)$$

Where $0 \leq i, j \leq s-1, i \neq j, 4 \leq t \leq m$. Here it is to be noted that $P_1 = \dots = P_m$.

For any fixed $t, 4 \leq t \leq m$, if one subtracts (1.8.3a) from (1.8.3e) with $j = 0$, one gets

$$(p'_{ti} - p'_{t0}) \theta_t = 0, \quad (1.8.4)$$

where $1 \leq i \leq s-1$. Then from **Lemma 1.8.1** and (1.8.4), one obtains

$$\theta_t = 0, \quad 4 \leq t \leq m. \quad (1.8.5)$$

For any fixed $r, 1 \leq r \leq s-1$, if one subtracts (1.8.3a) from (1.8.3c) with $i = r, j = 0$, one gets

$$(p'_{2r} - p'_{20}) \theta_2 + [p'_{10} \times (p'_{2r} - p'_{20})] \phi_1 = 0. \quad (1.8.6)$$

For any fixed $r, 1 \leq r \leq s-1$, if one subtracts (1.8.3b) with $i = r, j = 0$ from (1.8.3d) with $i = 0, j = r$, one obtains on employing (1.8.5),

$$(p'_{2r} - p'_{20}) \theta_2 + [p'_{1r} \times (p'_{2r} - p'_{20})] \phi_1 = 0. \quad (1.8.7)$$

For any fixed $r, 1 \leq r \leq s-1$, if one subtracts (1.8.3c) with $i = 0, j = h$ from (1.8.3c) with $i = r, j = h$, one gets

$$(p'_{2r} - p'_{20}) \theta_2 + [p'_{1h} \times (p'_{2r} - p'_{20})] \phi_1 = 0, \quad (1.8.8)$$

where $1 \leq h \leq s-1$, $h \neq r$. Summing the s equations that (1.8.6), (1.8.7) and (1.8.8) together, one obtains

$$(p'_{2r} - p'_{20}) \theta_2 = 0, \quad 1 \leq r \leq s-1. \quad (1.8.9)$$

So that analogously to (1.8.5), (1.8.9) implies

$$\theta_2 = 0.$$

Proceeding in a similar manner, one obtains from the equations (1.8.3a-3d)

$$\theta_3 = 0.$$

Again from the equations (1.8.6), (1.8.7) and (1.8.8), one gets

$$[p'_{1h} \times (p'_{2r} - p'_{20})] \phi_1 = 0, \quad (1.8.10)$$

where $0 \leq h \leq s-1$, $1 \leq r \leq s-1$. Now (1.8.10) implies that $(P_1 \times P_2^*) \phi_1 = 0$, where P_2^* is an $((s-1) \times (s-1))$ matrix with rows $(p'_{2r} - p'_{20})$, $1 \leq r \leq s-1$. Since P_1 and P_2^* are both of full column rank, one obtains

$$\phi_1 = 0.$$

In a similar manner, one can obtain

$$\phi_2 = 0.$$

From the equation (1.8.3a), it is now easy to see that $\theta_0 = 0$. Hence $\theta = 0$.

Next suppose $q_1 = 1$, $q_2 = 2$, $q_3 = 3$ and $q_4 = 4$. To prove the linear independence of the columns of the matrix stated at the beginning of the proof of this theorem, consider

$$[\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{34}] \theta = 0, \quad (1.8.11)$$

where θ is defined as earlier. If one recalls the definition of $L_1, L_2, \dots, L_m, G_{12}, G_{34}$, (1.8.11) gives the equations (1.8.12a-f)

$$\theta_0 + \sum_{w=1}^m p'_{w0} \theta_w + (p'_{10} \times p'_{20}) \phi_1 + (p'_{30} \times p'_{40}) \phi_2 = 0, \quad (1.8.12a)$$

$$\theta_0 + p'_{1i} \theta_1 + \sum_{w=2}^m p'_{wj} \theta_w + (p'_{1i} \times p'_{2j}) \phi_1 + (p'_{3j} \times p'_{4i}) \phi_2 = 0, \quad (1.8.12b)$$

$$\theta_0 + p'_{2i} \theta_2 + \sum_{\substack{w=1 \\ w \neq 2}}^m p'_{wj} \theta_w + (p'_{1j} \times p'_{2i}) \phi_1 + (p'_{3j} \times p'_{4i}) \phi_2 = 0, \quad (1.8.12c)$$

$$\theta_0 + p'_{3i} \theta_3 + \sum_{\substack{w=1 \\ w \neq 3}}^m p'_{wj} \theta_w + (p'_{1j} \times p'_{2j}) \phi_1 + (p'_{3i} \times p'_{4j}) \phi_2 = 0, \quad (1.8.12d)$$

$$\theta_0 + p'_{4i} \theta_4 + \sum_{\substack{w=1 \\ w \neq 4}}^m p'_{wj} \theta_w + (p'_{1j} \times p'_{2j}) \phi_1 + (p'_{3j} \times p'_{4i}) \phi_2 = 0, \quad (1.8.12e)$$

$$\theta_0 + p'_{ti} \theta_t + \sum_{\substack{w=1 \\ w \neq t}}^m p'_{wj} \theta_w + (p'_{1j} \times p'_{2j}) \phi_1 + (p'_{3j} \times p'_{4j}) \phi_2 = 0, \quad (1.8.12f)$$

where $0 \leq i, j \leq s-1, i \neq j, 5 \leq t \leq m$.

Here first suppose $m = 4$. For any fixed r , subtracting (1.8.12a) from (1.8.12b) with $i = r, j = 0$, one obtains

$$(p'_{1r} - p'_{10}) \theta_1 + [(p'_{1r} - p'_{10}) \times p'_{20}] \phi_1 = 0, \quad 1 \leq r \leq s-1, \quad (1.8.13)$$

For any fixed $r, 1 \leq r \leq s-1$, if one subtracts (1.8.12c) with $i = r, j = 0$ from (1.8.12e)

with $i = 0, j = r$, one gets

$$(p'_{1r} - p'_{10}) \theta_1 + (p'_{3r} - p'_{30}) \theta_3 + [(p'_{1r} - p'_{10}) \times p'_{2r}] \phi_1 + [(p'_{3r} - p'_{30}) \times p'_{40}] \phi_2 = 0. \quad (1.8.14)$$

If one subtracts (1.8.12a) from (1.8.12d) with $i = r, j = 0$, one gets, for any fixed r ,

$1 \leq r \leq s-1$,

$$(p'_{3r} - p'_{30}) \theta_3 + [(p'_{3r} - p'_{30}) \times p'_{40}] \phi_2 = 0. \quad (1.8.15)$$

If one subtracts (1.8.15) from (1.8.14), one gets for any fixed $r, 1 \leq r \leq s-1$,

$$(p'_{1r} - p'_{10}) \theta_1 + [(p'_{1r} - p'_{10}) \times p'_{2r}] \phi_1 = 0. \quad (1.8.16)$$

For any fixed $r, 1 \leq r \leq s-1$, subtracting (1.8.12b) with $i = 0, j = h$ from (1.8.12b) with

$i = r, j = h$, one gets

$$(\mathbf{p}'_{1r} - \mathbf{p}'_{10}) \boldsymbol{\theta}_1 + [(\mathbf{p}'_{1r} - \mathbf{p}'_{10}) \times \mathbf{p}'_{2h}] \boldsymbol{\phi}_1 = 0, \quad (1.8.17)$$

where $1 \leq h \leq s-1$, $h \neq r$. The s equations that (1.8.13), (1.8.16) and (1.8.17), when added gives

$$(\mathbf{p}'_{1r} - \mathbf{p}'_{10}) \boldsymbol{\theta}_1 = 0, \quad 1 \leq r \leq s-1. \quad (1.8.18)$$

Hence analogously to (1.8.5), one obtains

$$\boldsymbol{\theta}_1 = 0. \quad (1.8.19)$$

Then from the equations (1.8.13), (1.8.16), (1.8.17) and (1.8.19), one obtains

$$[(\mathbf{p}'_{1r} - \mathbf{p}'_{10}) \times \mathbf{p}'_{2h}] \boldsymbol{\phi}_1 = 0, \quad (1.8.20)$$

where $1 \leq r \leq s-1$, $0 \leq h \leq s-1$. Now (1.8.20) implies that $(P_1^* \times P_2) \boldsymbol{\phi}_1 = 0$ where P_1^* is an $((s-1) \times (s-1))$ matrix with rows $(\mathbf{p}'_{1r} - \mathbf{p}'_{10})$, $1 \leq r \leq s-1$. Since P_1^* and P_2 are both of full column rank, one obtains

$$\boldsymbol{\phi}_1 = 0.$$

Proceeding in a similar manner it can be shown that $\boldsymbol{\theta}_2 = 0$, $\boldsymbol{\theta}_3 = 0$, $\boldsymbol{\theta}_4 = 0$, $\boldsymbol{\phi}_2 = 0$ and $\boldsymbol{\theta}_0 = 0$. Hence from (1.8.2), one obtains

$$\boldsymbol{\theta} = 0.$$

For $m \geq 5$, the proof follows along the line as the proof of the earlier case.

Proof of Theorem 1.5.2. As in Lemma 1.5.1(b), for $1 \leq q_1 < q_2 \leq m_1$, consider the matrix $[\boldsymbol{\epsilon}^{(1)}, L_{q_1}^{(1)}, L_{q_2}^{(1)}]$. To prove the linear independence of the columns of the above matrix, consider

$$[\boldsymbol{\epsilon}^{(1)}, L_{q_1}^{(1)}, L_{q_2}^{(1)}] \boldsymbol{\theta}_1^* = 0, \quad (1.8.21)$$

where $\boldsymbol{\theta}_1^* = (\theta_0, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)'$, say.

First suppose, $q_2 = q_1 + 1$, $1 \leq q_1 \leq m_1 - 1$. Then by (1.8.21) and from the definitions of L_{q_1} and L_{q_1+1} , one gets the equations (1.8.22a-e) as stated below :

$$\theta_o + p'_{q_1 0} \phi_1 + p'_{q_1+1 0} \phi_2 = 0, \quad (1.8.22a)$$

$$\theta_o + p'_{q_1 2r} \phi_1 + p'_{q_1+1 0} \phi_2 = 0, \quad (1.8.22b)$$

$$\theta_o + p'_{q_1 2r-1} \phi_1 + p'_{q_1+1 2r} \phi_2 = 0, \quad (1.8.22c)$$

$$\theta_o + p'_{q_1 0} \phi_1 + p'_{q_1+1 2r-1} \phi_2 = 0, \quad (1.8.22d)$$

$$\theta_o + p'_{q_1 2r-1} \phi_1 + p'_{q_1+1 2r-1} \phi_2 = 0, \quad (1.8.22e)$$

where $1 \leq r \leq \mu$.

If one subtracts (1.8.22a) from (1.8.22b), one gets, for any fixed r , $1 \leq r \leq \mu$,

$$(p'_{q_1 2r} - p'_{q_1 0}) \phi_1 = 0. \quad (1.8.23)$$

Again, if one subtracts (1.8.22d) from (1.8.22e), one obtains for any fixed r , $1 \leq r \leq \mu$,

$$(p'_{q_1 2r-1} - p'_{q_1 0}) \phi_1 = 0. \quad (1.8.24)$$

Hence as before, one gets from (1.8.23) and (1.8.24),

$$\phi_1 = 0. \quad (1.8.25)$$

From (1.8.22a,c,d) and (1.8.25), it is easy to observe that $\phi_2 = 0$, $\theta_o = 0$ and hence

$$\theta_1^* = 0.$$

Next suppose q_1 and q_2 be such that $q_2 \neq q_1+1$. Here (1.8.21) gives the equations

1.8.26a-g)

$$\theta_o + p'_{q_1 0} \phi_1 + p'_{q_2 0} \phi_2 = 0, \quad (1.8.26a)$$

$$\theta_o + p'_{q_1 0} \phi_1 + p'_{q_2 2r-1} \phi_2 = 0, \quad (1.8.26b)$$

$$\theta_o + p'_{q_1 2r} \phi_1 + p'_{q_2 2r-1} \phi_2 = 0, \quad (1.8.26c)$$

$$\theta_0 + p'_{q_1}{}_{2r-1} \phi_1 + p'_{q_2}{}_0 \phi_2 = 0, \quad (1.8.26d)$$

$$\theta_0 + p'_{q_1}{}_{2r-1} \phi_1 + p'_{q_2}{}_{2r} \phi_2 = 0, \quad (1.8.26e)$$

$$\theta_0 + p'_{q_1}{}_0 \phi_1 + p'_{q_2}{}_{2r} \phi_2 = 0, \quad (1.8.26f)$$

$$\theta_0 + p'_{q_2}{}_{2r-1} \phi_1 + p'_{q_2}{}_{2r-1} \phi_2 = 0, \quad (1.8.26g)$$

where $1 \leq r \leq \mu$. Note that the equation (1.8.26f) arises when $q_1 = 1$ and $q_2 = m$ and in this case the equation (1.8.26b) does not arise. First suppose $(q_1, q_2) \neq (1, m)$. If one subtracts (1.8.26a) from (1.8.26b), one gets for any fixed r , $1 \leq r \leq \mu$,

$$(p'_{q_2}{}_{2r-1} - p'_{q_2}{}_0) \phi_2 = 0, \quad (1.8.27)$$

Again subtracting (1.8.26d) from (1.8.26e), one obtains for any fixed r , $1 \leq r \leq \mu$,

$$(p'_{q_2}{}_{2r} - p'_{q_2}{}_0) \phi_2 = 0. \quad (1.8.28)$$

Then as before, (1.8.27) and (1.8.28) implies

$$\phi_2 = 0. \quad (1.8.29)$$

From (1.8.26a,c,d) and (1.8.29), one can easily obtain $\phi_1 = 0$, $\theta_0 = 0$. Hence (1.8.21) gives

$$\theta_1^* = 0.$$

The proof of the case $(q_1, q_2) = (1, m)$ follows similarly.

Proof of Theorem 1.5.5. First consider the proof of part (a) of the theorem. The submatrix of $H_1^{(2)}$ dictated by S_{21} is, say,

$$[\epsilon_1^{(2)}, L_{m_1+1}^{(2)}, L_{m_1+2}^{(2)}, \dots, L_{m_1}^{(2)}],$$

which may be partitioned as

$$\begin{bmatrix} 1 & \mathbf{l}_{m_1+1}^{(2)'} & \mathbf{l}_{m_1+2}^{(2)'} & \dots & \mathbf{l}_{m_1}^{(2)'} \\ \epsilon_1^{(2)*} & \mathbf{L}_{m_1+1}^{(2)*} & \mathbf{L}_{m_1+2}^{(2)*} & \dots & \mathbf{L}_{m_1}^{(2)*} \end{bmatrix} \quad (1.8.30)$$

Similarly for $j' = 1, 2$, the submatrices of $H_1^{(1)}$ dictated by $S_{1j'}$ may be denoted by

$$[\epsilon_{j'}^{(1)}, \mathbf{L}_{1j'}^{(1)}, \mathbf{L}_{2j'}^{(1)}, \dots, \mathbf{L}_{m_1j'}^{(1)}].$$

Then under the absence of two-factor interactions of types (11) and (22), the design matrix of the plan (a) may, analogously to (1.2.1), be expressed as $X = [X_1, X_2]$, where

$$X_1 = \begin{bmatrix} X_{11} & \epsilon_1^{(1)} \times \mathbf{l}_{m_1+1}^{(2)'} & \epsilon_1^{(1)} \times \mathbf{l}_{m_1+2}^{(2)'} & \dots & \epsilon_1^{(1)} \times \mathbf{l}_{m_1}^{(2)'} \\ X_{12} & \epsilon_2^{(1)} \times \mathbf{L}_{m_1+1}^{(2)*} & \epsilon_2^{(1)} \times \mathbf{L}_{m_1+2}^{(2)*} & \dots & \epsilon_2^{(1)} \times \mathbf{L}_{m_1}^{(2)*} \end{bmatrix}$$

with

$$X_{11} = [\epsilon_1^{(1)}, \mathbf{L}_{11}^{(1)}, \mathbf{L}_{21}^{(1)}, \dots, \mathbf{L}_{m_11}^{(1)}],$$

$$X_{12} = [\epsilon_2^{(1)} \times \epsilon_1^{(2)*}, \mathbf{L}_{12}^{(1)} \times \epsilon_1^{(2)*}, \dots, \mathbf{L}_{m_12}^{(1)} \times \epsilon_1^{(2)*}] \text{ and}$$

$$X_2 = \begin{bmatrix} \mathbf{L}_{11}^{(1)} \times \mathbf{l}_{m_1+1}^{(2)'} , \dots , \mathbf{L}_{11}^{(1)} \times \mathbf{l}_{m_1}^{(2)'} , \dots , \mathbf{L}_{m_11}^{(1)} \times \mathbf{l}_{m_1+1}^{(2)'} , \dots , \mathbf{L}_{m_11}^{(1)} \times \mathbf{l}_{m_1}^{(2)'} \\ \mathbf{L}_{12}^{(1)} \times \mathbf{L}_{m_1+1}^{(2)*} , \dots , \mathbf{L}_{12}^{(1)} \times \mathbf{L}_{m_1}^{(2)*} , \dots , \mathbf{L}_{m_11}^{(1)} \times \mathbf{L}_{m_1+1}^{(2)*} , \dots , \mathbf{L}_{m_11}^{(1)} \times \mathbf{L}_{m_1}^{(2)*} \end{bmatrix}$$

By Theorem 1.2.1, it is enough to show that for every q_1, q_2, q_3, q_4 , where $1 \leq q_1, q_2 \leq m_1, m_1 + 1 \leq q_3, q_4 \leq m$, and $(q_1, q_3) \neq (q_2, q_4)$, the matrix

$$\left[X_1 \begin{vmatrix} \mathbf{L}_{q_11}^{(1)} \times \mathbf{l}_{q_31}^{(2)'} & \mathbf{L}_{q_21}^{(1)} \times \mathbf{l}_{q_41}^{(2)'} \\ \mathbf{L}_{q_12}^{(1)} \times \mathbf{L}_{q_31}^{(2)*} & \mathbf{L}_{q_22}^{(1)} \times \mathbf{L}_{q_41}^{(2)*} \end{vmatrix} \right] \quad (1.8.31)$$

has full column rank. Observe that by elementary column transformations (1.8.31) can be reduced to the matrix

$$\begin{bmatrix} X_{11} & 0 & \dots & 0 & 0 & 0 \\ X_{12} & \epsilon_2^{(1)} \times M_{m_1+1}^{(2)*} & \dots & \epsilon_2^{(1)} \times M_{m_1}^{(2)*} & L_{q_1 2}^{(1)} \times M_{q_3 1}^{(2)*} & L_{q_2 2}^{(1)} \times M_{q_4 1}^{(2)*} \end{bmatrix}, \quad (1.8.32)$$

where $M_{t1}^{(2)*} = L_{t1}^{(2)*} - \epsilon_1^{(2)*} L_{t1}^{(2)'} (m_1+1 \leq t \leq m)$. Since S_{11} is a type B_{11} design, by Lemma 1.5.1(a), X_{11} has full column rank and hence by (1.8.32), it remains to show that the matrix

$$[\epsilon_2^{(1)} \times M_{m_1+1}^{(2)*}, \dots, \epsilon_2^{(1)} \times M_{m_1}^{(2)*}, L_{q_1 2}^{(1)} \times M_{q_3 1}^{(2)*}, L_{q_2 2}^{(1)} \times M_{q_4 1}^{(2)*}] \quad (1.8.33)$$

has all columns linearly independent.

First suppose $q_1 \neq q_2$. Since S_{12} is a type B_{12} design, by Lemma 1.5.1(b), the matrix

$$[\epsilon_2^{(1)}, L_{q_1 2}^{(1)}, L_{q_2 2}^{(1)}]$$

is of full column rank. Further, since, S_{21} is a type B_{21} design, by the dual of Lemma 1.5.1(a) for $F_{m_1+1}, \dots, F_m$, the matrix (1.8.30) and hence the matrix

$$\begin{bmatrix} 1 & 0' & \dots & 0' \\ \epsilon_1^{(2)*} & M_{m_1+1}^{(2)*} & \dots & M_{m_1}^{(2)*} \end{bmatrix}$$

which may be obtained from (1.8.30) through elementary column transformations, is of full column rank. Therefore $[M_{m_1+1}^{(2)*}, M_{m_1+2}^{(2)*}, \dots, M_{m_1}^{(2)*}]$ and consequently

$$[\epsilon_2^{(1)}, L_{q_1 2}^{(1)}, L_{q_2 2}^{(1)}, \dots] \times [M_{m_1+1}^{(2)*}, \dots, M_{m_1}^{(2)*}] \quad (1.8.34)$$

has full column rank. Since (1.8.33) may be obtained from (1.8.34) by deleting some of the columns of the latter, linear independence of the columns of (1.8.33) is immediate.

Next suppose $q_1 = q_2$ ($q_3 \neq q_4$). This case may be similarly treated noting that $[\epsilon_2^{(1)}, L_{q_1 2}^{(1)}]$ has full column rank. This proves the result for the design (a). Since the design (b) is the dual of (a), obtained by interchanging the roles of the two groups of factors, clearly (b) also represents a type A_1 search design.

Proof of Theorem 1.5.6. Consider first (a). The submatrices of $H_1^{(1)}$ and $H_1^{(2)}$ dictated by S_{13} are, say,

$$[\epsilon_3^{(1)}, L_{13}^{(1)}, L_{23}^{(1)}, \dots, L_{m_1 3}^{(1)}] \text{ and } G_{12}^{(1)}, G_{13}^{(1)}, \dots, G_{m_1-1 \ m_1}^{(1)},$$

which may be partitioned as

$$\begin{bmatrix} 1 & \mathbf{l}_{13}^{(1)'} & \mathbf{l}_{23}^{(1)'} & \dots & \mathbf{l}_{m_1 3}^{(1)'} \\ \epsilon_3^{(1)*} & L_{13}^{(1)*} & L_{23}^{(1)*} & \dots & L_{m_1 3}^{(1)*} \end{bmatrix} \quad (1.8.35a)$$

and

$$\begin{bmatrix} \mathbf{g}_{12}^{(1)'} & \mathbf{g}_{13}^{(1)'} & \dots & \mathbf{g}_{m_1-1 \ m_1}^{(1)'} \\ G_{12}^{(1)*} & G_{13}^{(1)*} & \dots & G_{m_1-1 \ m_1}^{(1)*} \end{bmatrix} \quad (1.8.35b)$$

Similarly the submatrix of $H_2^{(1)}$ dictated by S_{21} is given by (1.8.30). Then under the absence of two factor interactions of the types (12) and (22), the design matrix of the plan (a) may, analogously to (1.2.1), be expressed as $X = [X_1, X_2]$, where

$$X_1 = \begin{bmatrix} \epsilon_1^{(2)} & \mathbf{l}_{13}^{(1)'} \times \epsilon_1^{(2)} & \dots & \mathbf{l}_{m_1 3}^{(1)'} \times \epsilon_1^{(2)} & X_{11}^* \\ \epsilon_3^{(1)*} & L_{13}^{(1)*} & \dots & L_{m_1 3}^{(1)*} & X_{12}^* \end{bmatrix}$$

with

$$X_{11}^* = [L_{m_1+1 \ 1}^{(2)}, L_{m_1+2 \ 1}^{(2)}, \dots, L_{m_1}^{(2)}],$$

$$X_{12}^* = [\epsilon_3^{(1)*} \times \mathbf{l}_{m_1+1 \ 1}^{(2)'}, \epsilon_3^{(1)*} \times \mathbf{l}_{m_1+2 \ 1}^{(2)'}, \dots, \epsilon_3^{(1)*} \times \mathbf{l}_{m_1}^{(2)'}]$$

$$X_2 = \begin{bmatrix} g_{12}^{(1)'} \times \epsilon_1^{(2)} & g_{13}^{(1)'} \times \epsilon_1^{(2)} & \dots & g_{m_1-1, m_1}^{(1)'} \times \epsilon_1^{(2)} \\ G_{12}^{(1)*} & G_{13}^{(1)*} & \dots & G_{m_1-1, m_1}^{(1)*} \end{bmatrix}$$

By Theorem 1.2.1, it is enough to show that for every q_1, q_2, q_3, q_4 , where $1 \leq q_1 < q_2 \leq m_1$, $1 \leq q_3 < q_4 \leq m_1$, $(q_1, q_2) \neq (q_3, q_4)$, the matrix

$$\left[\begin{array}{c|cc} X_1 & g_{q_1 q_2}^{(1)'} \times \epsilon_1^{(2)} & g_{q_3 q_4}^{(1)'} \times \epsilon_1^{(2)} \\ \hline & G_{q_1 q_2}^{(1)*} & G_{q_3 q_4}^{(1)*} \end{array} \right] \quad (1.8.36)$$

has full column rank. Observe that by elementary column transformations (1.8.36) can be reduced to the matrix

$$\left[\begin{array}{ccccccccc} \epsilon_1^{(2)} & 0 & 0 & \dots & 0 & X_{11}^* & 0 & 0 \\ \epsilon_3^{(1)*} & M_{13}^{(1)*} & M_{23}^{(1)*} & \dots & M_{m_1 3}^{(1)*} & X_{12}^* & R_{q_1 q_2}^{(1)*} & R_{q_3 q_4}^{(1)*} \end{array} \right], \quad (1.8.37)$$

where $M_{t3}^{(1)*} = L_{t3}^{(1)*} - l_{t3}^{(1)'} \times \epsilon_3^{(1)*}$ ($1 \leq t \leq m_1$) and $R_{q_1 q_2}^{(1)*} = G_{q_1 q_2}^{(1)*} - g_{q_1 q_2}^{(1)'} \times \epsilon_3^{(1)*}$,

$R_{q_3 q_4}^{(1)*} = G_{q_3 q_4}^{(1)*} - g_{q_3 q_4}^{(1)'} \times \epsilon_3^{(1)*}$. Since S_{21} is a B_{21} design, by the dual of Lemma 1.5.1(a), $[\epsilon_1^{(2)}, X_{11}^*]$ has full column rank and hence by (1.8.37), it remains to show that the matrix

$$[M_{13}^{(1)*} \ M_{23}^{(1)*} \ \dots \ M_{m_1 3}^{(1)*} \ R_{q_1 q_2}^{(1)*} \ R_{q_3 q_4}^{(1)*}] \quad (1.8.38)$$

has all columns linearly independent. Since S_{13} is a type B_{13} design, by Lemma 1.5.1(c), the matrix

$$\left[\begin{array}{cccccc} 1 & l_{13}^{(1)'} & l_{23}^{(1)'} & \dots & l_{m_1 3}^{(1)'} & g_{q_1 q_2}^{(1)'} & g_{q_3 q_4}^{(1)'} \\ \epsilon_3^{(1)*} & L_{13}^{(1)*} & L_{23}^{(1)*} & \dots & L_{m_1 3}^{(1)*} & G_{q_1 q_2}^{(1)*} & G_{q_3 q_4}^{(1)*} \end{array} \right] \quad (1.8.39)$$

and hence the matrix

$$\begin{bmatrix} 1 & 0 & 0 & & 0 & 0 & 0 \\ \epsilon_3^{(1)*} & M_{13}^{(1)*} & M_{23}^{(1)*} & \dots & M_{m3}^{(1)*} & R_{q_1 q_2}^{(1)*} & R_{q_3 q_4}^{(1)*} \end{bmatrix} \quad (1.8.40)$$

which may be obtained from (1.8.39) through elementary column transformations, is of full column rank. Consequently (1.8.38) has full column rank. This proves the design (a). Proceeding in a similar way it can be proved that the design (b) of Theorem 1.5.6 gives a type A_3 search design.

Proof of Lemma 1.6.1. (a) To prove this Lemma, consider

$$[T_1, G_{q_1 q_2}, G_{q_1 q_3}] \theta = [\epsilon, L_1, L_2, \dots, L_m, G_{q_1 q_2}, G_{q_1 q_3}] \theta = 0, \quad (1.8.41)$$

where $\theta = (\theta_0, \theta_1', \theta_2', \dots, \theta_m', \phi_1', \phi_2')'$, say. If one recalls the definitions of $L_1, L_2, \dots, L_m, G_{q_1 q_2}, G_{q_1 q_3}$, (1.8.41) and (1.6.1) give the equations (1.8.42a-f) as stated below :

$$\begin{aligned} \theta_0 + \sum_{w=1}^{v-1} p_{w0}' \theta_w + \sum_{w=v}^m p_{wj}' \theta_w + [\beta(v, q_1) \{ \beta(v, q_2) (p_{q_1 0}' \times p_{q_2 0}') + (1 - \beta(v, q_2) (p_{q_1 0}' \times p_{q_2 j}') \} + (1 - \beta(v, q_1)) (p_{q_1 j}' \times p_{q_2 j}')] \phi_1 + [\beta(v, q_1) \{ \beta(v, q_3) (p_{q_1 0}' \times p_{q_3 0}') + (1 - \beta(v, q_3) (p_{q_1 0}' \times p_{q_3 j}') \} + (1 - \beta(v, q_1)) (p_{q_1 j}' \times p_{q_3 j}')] \phi_2 = 0, \end{aligned} \quad (1.8.42a)$$

where $1 \leq v \leq m-1$, $s_{v-1} \leq j \leq s_v - 1$, $\beta(v_1, v_2) = 0$ if $v_1 \leq v_2$; $= 1$ if $v_1 > v_2$ and $\times$ is the Kronecker product. Note that for given v , the equations (1.8.42a) are defined only when $s_{v-1} < s_v$.

$$\begin{aligned} \theta_0 + \sum_{\substack{w=1 \\ w \neq u}}^v p_{w0}' \theta_w + p_{ui}' \theta_u + \sum_{w=v+1}^m p_{wj}' \theta_w + [\beta(v+1, q_1) \{ \beta(v+1, q_2) \{ \delta_{uq_1} (p_{q_1 i}' \times p_{q_2 0}') + \delta_{uq_2} (p_{q_1 0}' \times p_{q_2 i}') \} + (1 - \delta_{uq_1} - \delta_{uq_2}) (p_{q_1 0}' \times p_{q_2 0}') \} + (1 - \beta(v+1, q_2)) \{ \delta_{uq_1} (p_{q_1 i}' \times p_{q_2 j}') + (1 - \delta_{uq_1}) (p_{q_1 0}' \times p_{q_2 j}') \} + (1 - \beta(v+1, q_1)) (p_{q_1 j}' \times p_{q_2 j}')] \phi_2 = 0, \end{aligned}$$

$$\begin{aligned}
& p'_{q_2 j} \} \Phi_1 + [\beta(v+1, q_1) \{ \beta(v+1, q_3) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_3 0}) + \delta_{uq_3} (p'_{q_1 0} \times p'_{q_3 i_{q_3}}) + \\
& (1 - \delta_{uq_1} - \delta_{uq_3}) (p'_{q_1 0} \times p'_{q_3 0}) \} + (1 - \beta(v+1, q_3)) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_3 j}) + (1 - \\
& \delta_{uq_1}) (p'_{q_1 0} \times p'_{q_3 j}) \} + (1 - \beta(v+1, q_3)) (p'_{q_1 j} \times p'_{q_3 j}) \} \Phi_2 = 0, \quad (1.8.42b)
\end{aligned}$$

where $1 \leq v \leq m-2$, $1 \leq u \leq v$, $0 \leq i_u \leq s_u-1$, $s_v \leq j \leq s_{v+1}-1$ and δ is the Kronecker delta. Note that for given v , the equations given by (1.8.42b) are defined only when $s_v < s_{v+1}$ and $m \geq 3$.

$$\begin{aligned}
& \theta_0 + \sum_{w=1}^{v-1} p'_{w0} \theta_w + \sum_{\substack{w=v \\ w \neq u}}^m p'_{wj} \theta_w + p'_{ui_u} \theta_u + [\beta(v, q_1) \{ \beta(v, q_2) (p'_{q_1 0} \times p'_{q_2 0}) + \\
& (1 - \beta(v, q_2)) \{ \delta_{uq_2} (p'_{q_1 0} \times p'_{q_2 i_{q_2}}) + (1 - \delta_{uq_2}) (p'_{q_1 0} \times p'_{q_2 j}) \} + (1 - \beta(v, \\
& q_1)) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 j}) + \delta_{uq_2} (p'_{q_1 j} \times p'_{q_2 i_{q_2}}) + (1 - \delta_{uq_1} - \delta_{uq_2}) (p'_{q_1 j} \times \\
& p'_{q_2 j}) \} \} \Phi_1 + [\beta(v, q_1) \{ \beta(v, q_3) (p'_{q_1 0} \times p'_{q_2 0}) + (1 - \beta(v, q_3)) \{ \delta_{uq_3} (p'_{q_1 0} \times \\
& p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_3}) (p'_{q_1 0} \times p'_{q_3 j}) \} + (1 - \beta(v, q_1)) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_3 j}) + \\
& \delta_{uq_3} (p'_{q_1 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_1} - \delta_{uq_3}) (p'_{q_1 j} \times p'_{q_3 j}) \} \} \Phi_2 = 0, \quad (1.8.42c)
\end{aligned}$$

where $1 \leq v \leq m-2$, $v \leq u \leq m$, $0 \leq i_u \leq s_u-1$, $s_{v-1} \leq j \leq s_v-1$, $i_u \neq j$. Note that for given v , the equations given by (1.8.42c) are defined only when $s_{v-1} < s_v$ and $m \geq 3$.

$$\begin{aligned}
& \theta_0 + \sum_{w=1}^{m-2} p'_{w0} \theta_w + p'_{m-1 j} \theta_{m-1} + p'_{mk} \theta_m + [\beta(m-1, q_1) \{ \beta(m-1, q_2) (p'_{q_1 0} \times \\
& p'_{q_2 0}) + (1 - \beta(m-1, q_2)) \{ \delta_{m-1 q_2} (p'_{q_1 0} \times p'_{q_2 j}) + \delta_{mq_2} (p'_{q_1 0} \times p'_{q_2 k}) \} + (1 -
\end{aligned}$$

$$\begin{aligned}
& \beta(m-1, q_1)) (\dot{p}'_{q_1 j} \times \dot{p}'_{q_2 k}) \} \phi_1 + [\beta(m-1, q_1) \{ \beta(m-1, q_3) (\dot{p}'_{q_1 0} \times \dot{p}'_{q_3 0}) + (1 - \\
& \beta(m-1, q_3)) \{ \delta_{m-1} q_3 (\dot{p}'_{q_1 0} \times \dot{p}'_{q_3 j}) + \delta_{mq_3} (\dot{p}'_{q_1 0} \times \dot{p}'_{q_3 k}) \} \} + (1 - \beta(m-1, \\
& q_1)) (\dot{p}'_{q_1 j} \times \dot{p}'_{q_3 k}) \} \phi_2 = 0, \quad (1.8.42d)
\end{aligned}$$

where $s_{m-2} \leq j < s_{m-1}-1$, $s_{m-2} \leq k \leq s_{m-1}-1$, $j \neq k$. Note that the equations given by (1.8.42d) are defined only when $s_{m-2} < s_{m-1} - 1$.

$$\begin{aligned}
& \theta_0 + \sum_{w=1}^{m-2} \dot{p}'_{w0} \theta_w + \dot{p}'_{m-1} s_{m-1}-1 \theta_{m-1} + \dot{p}'_{mk} \theta_m + [\beta(m-1, q_1) \{ \beta(m-1, q_2) (\dot{p}'_{q_1 0} \times \\
& \dot{p}'_{q_2 0}) + (1 - \beta(m-1, q_2)) \{ \delta_{m-1} q_2 (\dot{p}'_{q_1 0} \times \dot{p}'_{q_2 s_{m-1}-1}) + \delta_{mq_2} (\dot{p}'_{q_1 0} \times \dot{p}'_{q_2 k}) \} \} + \\
& (1 - \beta(m-1, q_1)) (\dot{p}'_{q_1 s_{m-1}-1} \times \dot{p}'_{q_2 k}) \} \phi_1 + [\beta(m-1, q_1) \{ \beta(m-1, q_3) (\dot{p}'_{q_1 0} \times \dot{p}'_{q_3 0}) + \\
& (1 - \beta(m-1, q_3)) \{ \delta_{m-1} q_3 (\dot{p}'_{q_1 0} \times \dot{p}'_{q_3 s_{m-1}-1}) + \delta_{mq_3} (\dot{p}'_{q_1 0} \times \dot{p}'_{q_3 k}) \} \} + (1 - \\
& \beta(m-1, q_1)) (\dot{p}'_{q_1 s_{m-1}-1} \times \dot{p}'_{q_3 k}) \} \phi_2 = 0, \quad (1.8.42e)
\end{aligned}$$

where $s_{m-2} \leq k \leq s_{m-1}-1$, $k \neq s_{m-1}-1$. Note that the equations given by (1.8.42e) are defined only when $s_{m-2} < s_m$ and

$$\begin{aligned}
& \theta_0 + \sum_{\substack{w=1 \\ w \neq u}}^{m-2} \dot{p}'_{w0} \theta_w + \dot{p}'_{ui} \theta_u + \dot{p}'_{m-1} s_{m-1}-1 \theta_{m-1} + \dot{p}'_{mk} \theta_m + [\beta(m-1, q_1) \{ \beta(m-1, \\
& q_2) \{ \delta_{uq_1} (\dot{p}'_{q_1 i_{q_1}} \times \dot{p}'_{q_2 0}) + \delta_{uq_2} (\dot{p}'_{q_1 0} \times \dot{p}'_{q_2 i_{q_2}}) + (1 - \delta_{uq_1} - \delta_{uq_2}) (\dot{p}'_{q_1 0} \times \\
& \dot{p}'_{q_2 0}) \} + (1 - \beta(m-1, q_2)) \{ \delta_{uq_1} \{ \delta_{m-1} q_2 (\dot{p}'_{q_1 i_{q_1}} \times \dot{p}'_{q_2 s_{m-1}-1}) + \delta_{mq_2} (\dot{p}'_{q_1 i_{q_1}} \times \\
& \dot{p}'_{q_2 k}) \} + (1 - \delta_{uq_1}) \{ \delta_{m-1} q_2 (\dot{p}'_{q_1 0} \times \dot{p}'_{q_2 s_{m-1}-1}) + \delta_{mq_2} (\dot{p}'_{q_1 0} \times \dot{p}'_{q_2 k}) \} \} \} + \\
& (1 - \beta(m-1, q_1)) (\dot{p}'_{q_1 s_{m-1}-1} \times \dot{p}'_{q_2 k}) \} \phi_1 + [\beta(m-1, q_1) \{ \beta(m-1, q_3) \{ \delta_{uq_1} (\dot{p}'_{q_1 i_{q_1}} \times
\end{aligned}$$

$$\begin{aligned}
& p'_{q_3 0}) + \delta_{u q_3} (p'_{q_1 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u q_1} - \delta_{u q_3}) (p'_{q_1 0} \times p'_{q_3 0}) + (1 - \\
& \beta(m-1, q_3)) \{ \delta_{u q_1} \{ \delta_{m-1} q_3 (p'_{q_1 i_{q_1}} \times p'_{q_3 s_{m-1}-1}) + \delta_{m q_3} (p'_{q_1 i_{q_1}} \times p'_{q_3 k}) \} + \\
& (1 - \delta_{u q_1}) \{ \delta_{m-1} q_3 (p'_{q_1 0} \times p'_{q_3 s_{m-1}-1}) + \delta_{m q_3} (p'_{q_1 0} \times p'_{q_3 k}) \} \} + (1 - \\
& \beta(m-1, q_1)) (p'_{q_1 s_{m-1}-1} \times p'_{q_3 k}) \} \phi_2 = 0, \quad (1.8.42f)
\end{aligned}$$

where $1 \leq u \leq m-2$, $1 \leq i_u \leq s_u-1$, $s_{m-1} \leq k \leq s_m-1$. Note that the equations given by (1.8.42f) are defined only when $s_{m-1} < s_m$.

For any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3$), subtracting (1.8.42a) with $v = 1$, $j = 0$ from (1.8.42c) with $u = t$, $v = 1$, $j = 0$, one gets,

$$(p'_{t i_t} - p'_{t 0}) \theta_t = 0, \quad 1 \leq i_t \leq s_t-1. \quad (1.8.43)$$

Then from (1.8.43) and from Lemma 1.8.1, one obtains

$$\theta_t = 0, \quad (1.8.44)$$

where $1 \leq t \leq m$, $t \neq q_1, q_2, q_3$. For any fixed r ($1 \leq r \leq s_{q_2}-1$), if one subtracts (1.8.42a) with $v = 1$, $j = 0$ from (1.8.42c) with $u = q_2$, $i_{q_2} = r$, $v = 1$, $j = 0$, then one obtains

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 0} \times (p'_{q_2 r} - p'_{q_2 0})] \phi_1 = 0. \quad (1.8.45)$$

If $s_{v-1} < s_v$, then for any fixed r ($s_{v-1} \leq r \leq s_v-1$, $r \neq 0$), if one subtracts (1.8.42c) with $u = q_2$, $i_{q_2} = 0$, $j = r$ from (1.8.42a) with $j = r$, one obtains

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 r} \times (p'_{q_2 r} - p'_{q_2 0})] \phi_1 = 0, \quad (1.8.46)$$

where $1 \leq v \leq q_1$. For any fixed r ($1 \leq r \leq s_{q_2}-1$), subtracting (1.8.42c) with $u = q_2$, $i_{q_2} = 0$, $j = h$ from (1.8.42c) with $u = q_2$, $i_{q_2} = r$, $j = h$ when $s_{v-1} < s_v$, one obtains

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 h} \times (p'_{q_2 r} - p'_{q_2 0})] \phi_1 = 0, \quad (1.8.47)$$

where $s_{v-1} \leq h \leq s_v-1$, $1 \leq v \leq q_1$, $h \neq 0$, r . Note that the equations (1.8.46) and

(1.8.47) are defined only when $s_{v-1} < s_v$. The s_{q_1} equations that (1.8.45), (1.8.46) and (1.8.47), when added gives, for any fixed r ($1 \leq r \leq s_{q_2} - 1$)

$$(\mathbf{p}'_{q_2 r} - \mathbf{p}'_{q_2 0}) \boldsymbol{\theta}_{q_2} = 0. \quad (1.8.48)$$

Then from (1.8.48) and from Lemma 1.8.1, one gets

$$\boldsymbol{\theta}_{q_2} = 0. \quad (1.8.49)$$

From (1.8.45), (1.8.46), (1.8.47) and (1.8.49), one gets

$$[\mathbf{p}'_{q_1 h} \times (\mathbf{p}'_{q_2 r} - \mathbf{p}'_{q_2 0})] \boldsymbol{\phi}_1 = 0, \quad (1.8.50)$$

where $0 \leq h \leq s_{q_1} - 1$, $1 \leq r \leq s_{q_2} - 1$. Now (1.8.50) implies that $(\mathbf{P}_{q_1} \times \mathbf{P}_{q_2}^*) \boldsymbol{\phi}_1 = 0$,

where $\mathbf{P}_{q_2}^*$ is an $((s_{q_2} - 1) \times (s_{q_2} - 1))$ matrix with rows $(\mathbf{p}'_{q_2 r} - \mathbf{p}'_{q_2 0})$, $1 \leq r \leq s_{q_2} - 1$.

Since $\mathbf{P}_{q_1}$ and $\mathbf{P}_{q_2}^*$ are both of full column rank, one gets

$$\boldsymbol{\phi}_1 = 0. \quad (1.8.51)$$

Analogously to the derivations of (1.8.49), (1.8.51), it follows from (1.8.42a - f) that

$$\boldsymbol{\theta}_{q_3} = 0 \quad \text{and} \quad \boldsymbol{\phi}_2 = 0.$$

Now from the equations (1.8.42a) and (1.8.42c), it is easy to verify that $\boldsymbol{\theta}_0 = 0$, $\boldsymbol{\theta}_{q_1} = 0$.

This completes the proof of part (a) of this lemma.

Next consider the proof of part (b) of this lemma. To prove this, consider

$$[\mathbf{T}_1, \mathbf{G}_{q_1 q_3}, \mathbf{G}_{q_2 q_3}] \boldsymbol{\theta} = [\boldsymbol{\epsilon}, \mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_m, \mathbf{G}_{q_1 q_3}, \mathbf{G}_{q_2 q_3}] \boldsymbol{\theta} = 0 \quad (1.8.52)$$

where $\boldsymbol{\theta}$ is defined as earlier. Recalling the definitions of $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_m, \mathbf{G}_{q_1 q_3}, \mathbf{G}_{q_2 q_3}$ and using (1.6.1), (1.8.52) gives the equations (1.8.53 a-f) as stated below:

$$\begin{aligned} \boldsymbol{\theta}_0 + \sum_{w=1}^{v-1} \mathbf{p}_{w0} \boldsymbol{\theta}_w + \sum_{w=v}^m \mathbf{p}'_{wj} \boldsymbol{\theta}_w + [\beta(v, q_1) \{\beta(v, q_3) (\mathbf{p}'_{q_1 0} \times \mathbf{p}'_{q_3 0}) + (1 - \beta(v, q_3)) (\mathbf{p}'_{q_1 0} \times \mathbf{p}'_{q_3 j})\} + (1 - \beta(v, q_1)) (\mathbf{p}'_{q_1 j} \times \mathbf{p}'_{q_3 j})] \boldsymbol{\phi}_1 + [\beta(v, q_2) \{\beta(v, q_3) (\mathbf{p}'_{q_2 0} \times \mathbf{p}'_{q_3 0}) + (1 - \beta(v, q_3)) (\mathbf{p}'_{q_2 0} \times \mathbf{p}'_{q_3 j})\} + (1 - \beta(v, q_2)) (\mathbf{p}'_{q_2 j} \times \mathbf{p}'_{q_3 j})] \boldsymbol{\phi}_2 = 0 \end{aligned}$$

$$p'_{q_3 0}) + (1 - \beta(v, q_3)) (p'_{q_2 0} \times p'_{q_3 j}) + (1 - \beta(v, q_2)) (p'_{q_2 j} \times p'_{q_3 j}) \} \Phi_2 = 0, \quad (1.8.53a)$$

where $1 \leq v \leq m-1$, $s_{v-1} \leq j \leq s_v - 1$ and $\beta(v_1, v_2)$ is defined as earlier. Note that for given v , the equations given in (1.8.53a) are defined only when $s_{v-1} < s_v$.

$$\begin{aligned} \theta_0 + \sum_{\substack{w=1 \\ w \neq u}}^v p'_{w0} \theta_w + p'_{ui_u} \theta_u + \sum_{w=v+1}^m p'_{wj} \theta_w + [\beta(v+1, q_1) \{ \beta(v+1, q_3) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_3 0}) + \delta_{uq_3} (p'_{q_1 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_1} - \delta_{uq_3}) (p'_{q_1 0} \times p'_{q_3 0}) \} + (1 - \beta(v+1, q_3)) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_3 j}) + (1 - \delta_{uq_1}) (p'_{q_1 0} \times p'_{q_3 j}) \} \} + (1 - \beta(v+1, q_1)) (p'_{q_1 j} \times p'_{q_3 j}) \} \Phi_1 + [\beta(v+1, q_2) \{ \beta(v+1, q_3) \{ \delta_{uq_2} (p'_{q_2 i_{q_2}} \times p'_{q_3 0}) + \delta_{uq_3} (p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_2} - \delta_{uq_3}) (p'_{q_2 0} \times p'_{q_3 0}) \} + (1 - \beta(v+1, q_3)) \{ \delta_{uq_2} (p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + (1 - \delta_{uq_2}) (p'_{q_2 0} \times p'_{q_3 j}) \} \} + (1 - \beta(v+1, q_2)) (p'_{q_2 j} \times p'_{q_3 j}) \} \Phi_2 = 0, \quad (1.8.53b) \end{aligned}$$

where $1 \leq v \leq m-2$, $1 \leq u \leq v$, $0 \leq i_u \leq s_u - 1$, $s_v \leq j \leq s_{v+1} - 1$. Note that for given v , the equations given by (1.8.53b) are defined only when $s_v < s_{v+1}$ and $m \geq 3$.

$$\begin{aligned} \theta_0 + \sum_{w=1}^{v-1} p'_{w0} \theta_w + \sum_{\substack{w=v \\ w \neq u}}^m p'_{wj} \theta_w + p'_{ui_u} \theta_u + [\beta(v, q_1) \{ \beta(v, q_3) (p'_{q_1 0} \times p'_{q_3 0}) + (1 - \beta(v, q_3)) \{ \delta_{uq_3} (p'_{q_1 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_3}) (p'_{q_1 0} \times p'_{q_3 j}) \} \} + (1 - \beta(v, q_1)) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_3 j}) + \delta_{uq_3} (p'_{q_1 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_1} - \delta_{uq_3}) (p'_{q_1 j} \times p'_{q_3 j}) \} \Phi_1 + [\beta(v, q_2) \{ \beta(v, q_3) (p'_{q_2 0} \times p'_{q_3 0}) + (1 - \beta(v, q_3)) \{ \delta_{uq_3} (p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_3}) (p'_{q_2 0} \times p'_{q_3 j}) \} \} + (1 - \beta(v, q_2)) \{ \delta_{uq_2} (p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + (1 - \delta_{uq_2}) (p'_{q_2 0} \times p'_{q_3 j}) \} \Phi_2 = 0, \end{aligned}$$

$$\delta_{uq_3} (p'_{q_2j} \times p'_{q_3i_{q_3}}) + (1 - \delta_{uq_2} - \delta_{uq_3}) (p'_{q_2j} \times p'_{q_3j}) \} \phi_2 = 0, \quad (1.8.53c)$$

where $1 \leq v \leq m-2$, $v \leq u \leq m$, $0 \leq i_u \leq s_u - 1$, $s_{v-1} \leq j \leq s_v - 1$, $j \neq i_u$. It is to be noted that for given v , the equations given in (1.8.53c) are defined only when $s_{v-1} < s_v$ and $m \geq 3$.

$$\begin{aligned} & \theta_0 + \sum_{w=1}^{m-2} p'_{w0} \theta_w + p'_{m-1j} \theta_{m-1} + p'_{mk} \theta_m + [\beta(m-1, q_1) \{ \beta(m-1, q_3) (p'_{q_10} \times \\ & p'_{q_30}) + (1 - \beta(m-1, q_3)) \{ \delta_{m-1q_3} (p'_{q_10} \times p'_{q_3j}) + \delta_{mq_3} (p'_{q_10} \times p'_{q_3k}) \} \} + (1 - \\ & \beta(m-1, q_1)) (p'_{q_1j} \times p'_{q_3k}) \} \phi_1 + [\beta(m-1, q_2) \{ \beta(m-1, q_3) (p'_{q_20} \times p'_{q_30}) + (1 - \\ & \beta(m-1, q_3)) \{ \delta_{m-1q_3} (p'_{q_20} \times p'_{q_3j}) + \delta_{mq_3} (p'_{q_20} \times p'_{q_3k}) \} \} + (1 - \beta(m-1, \\ & q_2)) (p'_{q_2j} \times p'_{q_3k}) \} \phi_2 = 0, \end{aligned} \quad (1.8.53d)$$

where $s_{m-2} \leq j < s_{m-1}^{-1}$, $s_{m-2} \leq k \leq s_m^{-1}$, $j \neq k$. The equations given by (1.8.53d) are defined only when $s_{m-2} < s_{m-1}^{-1}$.

$$\begin{aligned} & \theta_0 + \sum_{w=1}^{m-2} p'_{w0} \theta_w + p'_{m-1s_{m-1}^{-1}} \theta_{m-1} + p'_{mk} \theta_m + [\beta(m-1, q_1) \{ \beta(m-1, \\ & q_3) (p'_{q_10} \times p'_{q_30}) + (1 - \beta(m-1, q_3)) \{ \delta_{m-1q_3} (p'_{q_10} \times p'_{q_3s_{m-1}^{-1}}) + \\ & \delta_{mq_3} (p'_{q_10} \times p'_{q_3k}) \} \} + (1 - \beta(m-1, q_1)) (p'_{q_1s_{m-1}^{-1}} \times p'_{q_3k}) \} \phi_1 + [\beta(m-1, \\ & q_2) \{ \beta(m-1, q_3) (p'_{q_20} \times p'_{q_30}) + (1 - \beta(m-1, q_3)) \{ \delta_{m-1q_3} (p'_{q_20} \times p'_{q_3s_{m-1}^{-1}}) + \\ & \delta_{mq_3} (p'_{q_20} \times p'_{q_3k}) \} \} + (1 - \beta(m-1, q_2)) (p'_{q_2s_{m-1}^{-1}} \times p'_{q_3k}) \} \phi_2 = 0, \end{aligned} \quad (1.8.53e)$$

where $s_{m-2} \leq k \leq s_m^{-1}$, $k \neq s_{m-1}^{-1}$. The equations given by (1.8.53e) are defined only when $s_{m-2} < s_m$.

$$\begin{aligned}
& \theta_0 + \sum_{\substack{w=1 \\ w \neq u}}^{m-2} p'_{w0} \theta_w + p'_{ui_u} \theta_u + p'_{m-1} s_{m-1}^{-1} \theta_{m-1} + p'_{mk} \theta_m + [\beta(m-1, \\
& q_1) \{ \beta(m-1, q_3) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_3 0}) + \delta_{uq_3} (p'_{q_1 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_1} - \\
& \delta_{uq_3}) (p'_{q_1 0} \times p'_{q_3 0}) \} + (1 - \beta(m-1, q_3)) \{ \delta_{uq_1} \{ \delta_{m-1} q_3 (p'_{q_1 i_{q_1}} \times p'_{q_3 s_{m-1}^{-1}}) + \\
& \delta_{mq_3} (p'_{q_1 i_{q_1}} \times p'_{q_3 k}) \} + (1 - \delta_{uq_1}) \{ \delta_{m-1} q_3 (p'_{q_1 0} \times p'_{q_3 s_{m-1}^{-1}}) + \\
& \delta_{mq_3} (p'_{q_1 0} \times p'_{q_3 k}) \} \} + (1 - \beta(m-1, q_1)) (p'_{q_1 s_{m-1}^{-1}} \times p'_{q_3 k}) \} \phi_1 + [\beta(m-1, \\
& q_2) \{ \beta(m-1, q_3) \{ \delta_{uq_2} (p'_{q_2 i_{q_2}} \times p'_{q_3 0}) + \delta_{uq_3} (p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_2} - \\
& \delta_{uq_3}) (p'_{q_2 0} \times p'_{q_3 0}) \} + (1 - \beta(m-1, q_3)) \{ \delta_{uq_2} \{ \delta_{m-1} q_3 (p'_{q_2 i_{q_2}} \times p'_{q_3 s_{m-1}^{-1}}) + \\
& \delta_{mq_3} (p'_{q_2 i_{q_2}} \times p'_{q_3 k}) \} + (1 - \delta_{uq_2}) \{ \delta_{m-1} q_3 (p'_{q_2 0} \times p'_{q_3 s_{m-1}^{-1}}) + \delta_{mq_3} (p'_{q_2 0} \times \\
& p'_{q_3 k}) \} \} + (1 - \beta(m-1, q_2)) (p'_{q_2 s_{m-1}^{-1}} \times p'_{q_3 k}) \} \phi_2 = 0, \quad (1.8.53f)
\end{aligned}$$

where $1 \leq u \leq m-2$, $1 \leq i_u \leq s_u^{-1}$, $s_{m-1} \leq k \leq s_m^{-1}$. It is to be noted that the equations given by (1.8.53f) are defined only when $s_{m-1} < s_m$.

For any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3$), subtracting (1.8.53a) with $v = 1$, $j = 0$ from (1.8.53c) with $u = t$, $v = 1$, $j = 0$, one gets

$$(p'_{ti_t} - p'_{t0}) \theta_t = 0, \quad 1 \leq i_t \leq s_t^{-1}. \quad (1.8.54)$$

Hence as before, one gets from (1.8.54) and Lemma 1.8.1

$$\theta_t = 0, \quad 1 \leq t \leq m, \quad t \neq q_1, q_2, q_3.$$

First suppose $q_3 \leq m-1$. For any fixed r ($1 \leq r \leq s_{q_1}^{-1}$), if one subtracts (1.8.53a)

with $v = 1, j = 0$ from (1.8.53c) with $v = 1, u = q_1, i_{q_1} = r, j = 0$, then one obtains

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 0}] \phi_1 = 0. \quad (1.8.55)$$

If $s_{v-1} < s_v$, then for any fixed r ($s_{v-1} \leq r \leq s_v - 1, r \neq 0$), if one subtracts (1.8.53c)

with $u = q_1, i_{q_1} = 0, j = r$ from (1.8.53a) with $j = r$, one obtains

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 r}] \phi_1 = 0, \quad (1.8.56)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed r ($1 \leq r \leq s_{q_1} - 1$), subtracting

(1.8.53c) with $u = q_1, i_{q_1} = 0, j = h$ from (1.8.53c) with $u = q_1, i_{q_1} = r, j = h$, one gets

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 h}] \phi_1 = 0, \quad (1.8.57)$$

where $s_{v-1} \leq h \leq s_v - 1, 1 \leq v \leq q_1, h \neq 0, r$. Note that for any given v the equations given by (1.8.56) and (1.8.57) are defined only when $s_{v-1} < s_v$. For any fixed r ($1 \leq r \leq s_{q_1} - 1$), if one subtracts (1.8.53a) with $j = h$ for $s_{v-1} \leq h \leq s_v - 1, q_1 + 1 \leq v \leq q_3$ from (1.8.53b) with $u = q_1, i_{q_1} = r, j = h$ for $s_v \leq h \leq s_{v+1} - 1, q_1 \leq v \leq q_3 - 1$, then one obtains

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 h}] \phi_1 = 0. \quad (1.8.58)$$

It is to be noted that for given v ($q_1 \leq v \leq q_3 - 1$), the equations given by (1.8.58) are defined only when $s_v < s_{v+1}$. Summing the s_{q_3} equations that (1.8.55), (1.8.56), (1.8.57) and (1.8.58), one gets for any fixed r ($1 \leq r \leq s_{q_1} - 1$)

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} = 0. \quad (1.8.59)$$

Hence as before, (1.8.59) and the Lemma 1.8.1 gives

$$\theta_{q_1} = 0. \quad (1.8.60)$$

Hence from (1.8.55), (1.8.56), (1.8.57), (1.8.58) and (1.8.60), one gets

$$[(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 h}] \phi_1 = 0, \quad (1.8.61)$$

where $1 \leq r \leq s_{q_1} - 1$, $0 \leq h \leq s_{q_3} - 1$. Now (1.8.61) implies that $(P_{q_1}^* \times P_{q_3}) \phi_1 = 0$, where $P_{q_1}^*$ is an $((s_{q_1} - 1) \times (s_{q_1} - 1))$ matrix with rows $(p'_{q_1 r} - p'_{q_1 0})$, $1 \leq r \leq s_{q_1} - 1$. Since $P_{q_1}^*$ and P_{q_3} are both of full column rank, one obtains

$$\phi_1 = 0. \quad (1.8.62)$$

Analogously to the derivation of (1.8.60) and (1.8.62), it follows from (1.8.53a-f) that

$$\theta_{q_2} = 0 \text{ and } \phi_2 = 0.$$

Next suppose $q_3 = m$ and $q_2 \leq m-2$. For any fixed r ($1 \leq r \leq s_{q_1} - 1$), if one subtracts (1.8.53a) with $j = h$ for $s_{v-1} \leq h \leq s_v - 1$, $q_1 + 1 \leq v \leq m-1$ from (1.8.53b) with $u = q_1$, $i_{q_1} = r$, $j = h$ for $s_v \leq h \leq s_{v+1} - 1$, $q_1 \leq v \leq m-2$, one gets

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 h}] \phi_1 = 0. \quad (1.8.63)$$

Here it is to be noted that for given v , $q_1 \leq v \leq m-2$, the equations given by (1.8.63) are defined only when $s_v < s_{v+1}$. For any fixed r ($1 \leq r \leq s_{q_1} - 1$), subtracting (1.8.53e) with $k = h$ from (1.8.53f) with $u = q_1$, $i_{q_1} = r$, $k = h$, one gets

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 h}] \phi_1 = 0, \quad (1.8.64)$$

where $s_{m-1} \leq h \leq s_m - 1$. Note that the equations given by (1.8.64) are defined only when $s_{m-1} < s_m$. The s_{q_3} equations (1.8.55), (1.8.56), (1.8.57), (1.8.63) and (1.8.64), when added gives for any fixed r ($1 \leq r \leq s_{q_1} - 1$),

$$(p'_{q_1 r} - p'_{q_1 0}) \theta_{q_1} = 0. \quad (1.8.65)$$

Hence as before, from (1.8.65) and from Lemma 1.8.1, one gets

$$\theta_{q_1} = 0. \quad (1.8.66)$$

From (1.8.55), (1.8.56), (1.8.57), (1.8.63), (1.8.64) and (1.8.66), it follows that

$$[(p'_{q_1 r} - p'_{q_1 0}) \times p'_{q_3 h}] \phi_1 = 0, \quad (1.8.67)$$

where $1 \leq r \leq s_{q_1} - 1$, $0 \leq h \leq s_{q_3} - 1$. Now (1.8.67) implies that $(P_{q_1}^* \times P_{q_3}) \phi_1 = 0$. Since $P_{q_1}^*$ and P_{q_3} are both of full column rank, one obtains

$$\phi_1 = 0. \quad (1.8.68)$$

Analogously to the derivation of (1.8.66) and (1.8.68), one can obtain from (1.8.53a-f)

$$\theta_{q_2} = 0 \quad \text{and} \quad \phi_2 = 0.$$

Finally suppose $q_2 = m-1$. Here the proofs of $\theta_{q_1} = 0$ and $\phi_1 = 0$ follow the same line as earlier. The proofs of $\theta_{q_2} = 0$ and $\phi_2 = 0$ are shown below:

For any fixed r ($1 \leq r \leq s_{q_2} - 1$), if one subtracts (1.8.53a) with $v = 1$, $j = 0$ from (1.8.53c) with $v = 1$, $u = q_2$, $i_{q_2} = r$, $j = 0$, one obtains

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [(p'_{q_2 r} - p'_{q_2 0}) \times p'_{q_3 0}] \phi_2 = 0. \quad (1.8.69)$$

If $s_{v-1} < s_v$, for any fixed r ($s_{v-1} \leq r \leq s_v - 1$, $r \neq 0$), if one subtracts (1.8.53c) with $u = q_2$, $i_{q_2} = 0$, $j = r$ from (1.8.53a) with $j = r$, one gets

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [(p'_{q_2 r} - p'_{q_2 0}) \times p'_{q_3 r}] \phi_2 = 0, \quad (1.8.70)$$

where $1 \leq v \leq m-2$. Note that for given v , the equations given by (1.8.70) are defined only when $s_{v-1} < s_v$. For any fixed r ($s_{m-2} \leq r \leq s_{m-1} - 1$), subtracting (1.8.53c) with $v = 1$, $j = 0$, $u = q_3$, $i_{q_3} = r$ from (1.8.53a) with $v = q_2$, $j = r$, one obtains

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [(p'_{q_2 r} - p'_{q_2 0}) \times p'_{q_3 r}] \phi_2 = 0. \quad (1.8.71)$$

Here one may note that the equations given by (1.8.71) are defined only when $s_{m-2} < s_{m-1}$. If $s_{v-1} < s_v$, then for any fixed r ($1 \leq r \leq s_{q_2} - 1$), if one subtracts (1.8.53c) with $u = q_2$, $i_{q_2} = 0$, $j = h$ from (1.8.53c) with $u = q_2$, $i_{q_2} = r$, $j = h$, one gets

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [(p'_{q_2 r} - p'_{q_2 0}) \times p'_{q_3 h}] \phi_2 = 0, \quad (1.8.72)$$

where $s_{v-1} \leq h \leq s_v - 1$, $1 \leq v \leq m-2$, $h \neq 0$, r . Note that for given v , the equations

given by (1.8.72) are defined only when $s_{v-1} < s_v$. If one subtracts (1.8.53c) with $v = 1$, $j = 0$, $u = q_3$, $i_{q_3} = h$ from (1.8.53d) with $j = r$, $k = h$, one gets for any fixed r ($s_{m-2} \leq r < s_{m-1} - 1$),

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} + [(p'_{q_2 r} - p'_{q_2 0}) \times p'_{q_3 h}] \phi_2 = 0, \quad (1.8.73)$$

where $s_{m-2} \leq h \leq s_{m-1}$, $h \neq r$. Note that the equations given by (1.8.73) are defined only when $s_{m-2} < s_{m-1} - 1$. Subtracting (1.8.53c) with $v = 1$, $j = 0$, $u = q_3$, $i_{q_3} = h$ from (1.8.53e) with $k = h$, one gets

$$(p'_{q_2 s_{m-1}-1} - p'_{q_2 0}) \theta_{q_2} + [(p'_{q_2 s_{m-1}-1} - p'_{q_2 0}) \times p'_{q_3 h}] \phi_2 = 0, \quad (1.8.74)$$

where $s_{m-2} \leq h \leq s_{m-1}$, $h \neq s_{m-1} - 1$. Note that the equations given by (1.8.74) are defined only when $s_{m-2} < s_m$. Summing the s_{q_3} equations that (1.8.69 - 73), one gets for any fixed r ($1 \leq r \leq s_{q_2} - 1$)

$$(p'_{q_2 r} - p'_{q_2 0}) \theta_{q_2} = 0. \quad (1.8.75)$$

From (1.8.75) and Lemma 1.8.1, one gets

$$\theta_{q_2} = 0. \quad (1.8.76)$$

Hence from (1.8.69 - 73) and (1.8.76), one obtains

$$[(p'_{q_2 r} - p'_{q_2 0}) \times p'_{q_3 h}] \phi_2 = 0, \quad (1.8.77)$$

where $1 \leq r \leq s_{q_2} - 1$, $0 \leq h \leq s_{q_3} - 1$. Now (1.8.77) implies that $(P_{q_2}^* \times P_{q_3}) \phi_2 = 0$. Since $P_{q_2}^*$ and P_{q_3} are both of full column rank, one obtains

$$\phi_2 = 0$$

From the equations (1.8.53a) and (1.8.53c), it is now easy to verify that $\theta_0 = 0$, $\theta_{q_3} = 0$.

This completes the proof.

The proofs of parts (c) and (d) of the lemma will be similar.

Chapter 2

DESIGNS FOR SEARCHING ONE AMONG THE THREE-FACTOR INTERACTION EFFECTS IN THE GENERAL SYMMETRIC AND ASYMMETRIC FACTORIAL SET-UP.

2.1 Introduction

A fractional factorial design will be said to be a search design for searching one among the three-factor interactions if it is capable of estimating the general mean, all the main-effects and the two-factor interactions and permits the search and estimation of the possibly present at best one among the three-factor interactions. This chapter deals with the above mentioned search design problems.

As mentioned earlier, the basic mathematical formulation of the problem was due to Srivastava (1975). The present problem was considered among others by Srivastava and Ghosh (1976, 1977) and Shirakura and Ohnishi (1985) who confined themselves in 2^m factorial set-up. Ghosh and Zhang (1987) considered the construction problem for 3^m factorial set-up. They gave a new series of search designs that can search one parameter corresponding to a non-zero three-factor interaction and estimate it along with the general mean, all main-effects and all two-factor interactions for the 3^m factorial set-up.

The aim of this chapter is to give search designs of the above type in the general asymmetric factorial set-up through symmetric one by the use of the method of Kronecker products. In Section 2.2 we have given the notations and preliminaries. Section 2.3 describes the construction of search designs for the given problem in the general symmetric factorial set-up. Section 2.4 considers the construction in the general asymmetric factorial set-up. A brief discussion has been given in Section 2.5. The proofs of some of the theorems have been given in Section 2.6. The proofs of the remaining theorems and of the lemmas follow exactly in a similar way.

2.2 Notations and Preliminaries

Consider a factorial set-up involving $m(m \geq 3)$ factors $F_1, F_2, \dots, F_m$ at $s_1, s_2, \dots, s_m$ levels respectively ($s_v \geq 2, 1 \leq v \leq m$) where without loss of generality, suppose

$$s_1 \leq s_2 \leq \dots \leq s_m. \quad (2.2.1)$$

In addition to the notations given in Section 1.3 of Chapter 1, we further define

$$Z_{v_1 v_2 v_3} = \sum_{w=1}^m P_w (\delta_{wv_1} + \delta_{wv_2} + \delta_{wv_3}), \quad 1 \leq v_1 < v_2 < v_3 \leq m,$$

where δ 's are Kronecker deltas and $\times$ is the Kronecker product. Suppose prior information is available regarding the absence of all interactions involving three or more factors. Throughout this chapter these interactions will be assumed to be absent. It is now easy to observe that if one observation is made for each of $\prod_{v=1}^m s_v$ level combinations of $F_1, F_2, \dots, F_m$, then analogously to (1.3.2), the resulting design matrix will be of the form :

$$[1, Z_1, Z_2, \dots, Z_m, Z_{12}, Z_{13}, \dots, Z_{m-1 m}, Z_{123}, Z_{124}, \dots, Z_{m-2 m-1 m}], \quad (2.2.2)$$

where 1 corresponds to the general mean and the columns of $Z_v, Z_{v_1 v_2}, Z_{v_1 v_2 v_3}$ correspond to the main-effect F_v ($1 \leq v \leq m$), the two-factor interaction $F_{v_1} F_{v_2}$ ($1 \leq v_1 < v_2 \leq m$) and the three-factor interaction $F_{v_1} F_{v_2} F_{v_3}$ ($1 \leq v_1 < v_2 < v_3 \leq m$) respectively. Of course (2.2.2) incorporates $Z_{v_1 v_2 v_3}$ only when the associated interaction $F_{v_1} F_{v_2} F_{v_3}$ is included in the model. Let

$$H_1^* = [1, Z_1, Z_2, \dots, Z_m, Z_{12}, \dots, Z_{m-1 m}] \text{ and}$$

$$H_2^* = [Z_{123}, Z_{124}, \dots, Z_{m-2 m-1 m}]$$

Also let S be a subset of N level combinations of $F_1, F_2, \dots, F_m$ with the corres-

ponding sub-matrices of H_1^* and H_2^* as, say, $T_1^* = [\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1\ m}]$ and $T_2^* = [V_{123}, V_{124}, \dots, V_{m-2\ m-1\ m}]$ respectively. From Theorem 1.2.1 of Chapter 1, one observes that S represents a search design of the problem stated in Section 2.1, if and only if for each $q_i, i = 1, 2, \dots, 6$ ($1 \leq q_1 < q_2 < q_3 \leq m, 1 \leq q_4 < q_5 < q_6 \leq m, (q_1, q_2, q_3) \neq (q_4, q_5, q_6)$), the matrix $[T_1^*, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}]$ has full column rank.

2.3 Main Result in the Symmetric Case

This section considers the construction of search designs for the general symmetric factorial set-up that allow the estimation of the general mean, all main-effects and the two-factor interactions and are capable of detecting and estimating the possibly present at best one among the three-factor interactions. Here we consider $s_1 = s_2 = \dots = s_m = s$, say, in (2.2.1). The following notations will be helpful in describing the search designs considered here.

- (i) Define the $(m \times 1)$ vectors $a_j^{(1)}$ ($0 \leq j \leq s-1$) such that in $a_j^{(1)}$, all the elements are j ; let $A^{(1)}$ be an $(m \times s)$ array whose columns are given by $a_j^{(1)}$ for all j .
- (ii) For $0 \leq j \leq s-1$, define the $(m \times 1)$ vectors $a_{j,u,i_u}^{(2)}$ ($1 \leq u \leq m, 0 \leq i_u \leq s-1, i_u \neq j$) such that in $a_{j,u,i_u}^{(2)}$, the u th element is i_u and the remaining elements all equal j ; let $A_j^{(2)}$ be an $(m \times m(s-1))$ array whose columns are $a_{j,u,i_u}^{(2)}$ for all u and i_u .
- (iii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ ($1 \leq u_1 < u_2 \leq m, 0 \leq \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}) \leq s-1$) such that in $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} respectively and the remaining elements all equal j ; let $A_j^{(3)}$ be an $(m \times [m(m-1)j(s-j-1)])$ array whose columns are given by $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(3)}$ is defined only when $s \geq 3$.

(iv) For $0 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ ($1 \leq u_1 < u_2 \leq m$), $j+1 \leq i_{u_1}, i_{u_2} \leq s-1$) such that in $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} respectively and the remaining elements all equal j ; let $A_j^{(4)}$ be an $(m \times [m(m-1)(s-j-1)^2/2])$ array whose columns are $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(4)}$ is defined only when $s \geq 3$.

Let

$$N_1 = s + ms(s-1) + m(m-1)s(s-1)(s-2)/6 + m(m-1)s(s-1)(2s-1)/12,$$

$$N_2 = 1 + ms(s-1) + m(m-1)s(s-1)(s-2)/6 + m(m-1)s(s-1)(2s-1)/12$$

and define the following arrays :

$$Q_1 = [A^{(1)}, A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}, A_1^{(3)}, A_2^{(3)}, \dots, A_{s-2}^{(3)}, \\ A_0^{(4)}, A_1^{(4)}, \dots, A_{s-2}^{(4)}] \quad (2.3.1)$$

$$Q_2 = [0, A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}, A_1^{(3)}, A_2^{(3)}, \dots, A_{s-2}^{(3)}, \\ A_0^{(4)}, A_1^{(4)}, \dots, A_{s-2}^{(4)}] \quad (2.3.2)$$

Then we have the following theorems:

Theorem 2.3.1. For $m \leq 6$, the columns of the matrix Q_1 , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N_1 runs.

Theorem 2.3.2. For $m > 6$, the columns of the matrix Q_2 , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the search design of the type considered in this section in N_2 runs.

Remark: That Theorem 2.3.2 does not hold for $m \leq 6$ follows from (2.6.19) in Section 2.6

The example given below will be helpful in illustrating the plan described above.

22222211111111111111
 21111122222111111111
 12111121111222211111
 $A_1^{(4)} : 112111121112111222111 ;$
 11121111211121121221
 111121111211121121212
 111112111121112112122

Here one may note that the corresponding regular resolution VII plan requires a total of 379 runs.

2.4 Main Result in the Asymmetric Case

This section presents the construction of search designs for the general asymmetric factorial set-up that are capable of detecting the possibly present at best one non-zero three-factor interaction and estimating it together with the general mean, all main-effects and two-factor interactions. Generalizing the ideas for the symmetric case, we first introduce the following notations.

Let $s_0 = 0$ and recall (2.2.1). Then define :

(i) For $1 \leq v \leq m-2$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_j^{(1,v)}$ ($s_{v-1} \leq j \leq s_v - 1$) such that in $b_j^{(1,v)}$, the first $(v-1)$ elements equal 0 and the remaining elements all equal j ; let $B^{(1,v)}$ be an $(m \times (s_v - s_{v-1}))$ array whose columns are given by $b_j^{(1,v)}$ for all j . Note that $B^{(1,v)}$ is defined only when $s_{v-1} < s_v$.

(ii) For $1 \leq v \leq m-2$, if $s_v < s_{v+1}$, then define the $(m \times 1)$ vectors $b_{j,u,i_u}^{(2,v)}$ ($s_v \leq j \leq s_{v+1} - 1$, $1 \leq u \leq v$, $1 \leq i_u \leq s_u - 1$) such that in $b_{j,u,i_u}^{(2,v)}$, the u th element is i_u , the last $(m-v)$ elements all equal j and the remaining elements all equal 0; let $B^{(2,v)}$ be an $(m \times [\sum_{u=1}^v (s_u - 1) (s_{v+1} - s_v)])$ array whose columns are given by $b_{j,u,i_u}^{(2,v)}$ for all j , u and i_u . Note that $B^{(2,v)}$ is defined only when $s_v < s_{v+1}$.

(iii) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u,i_u}^{(3,v)}$ ($s_{v-1} \leq$

$j \leq s_v - 1$, $v \leq u \leq m$, $0 \leq i_u \leq s_u - 1$, $i_u \neq j$) such that in $b_{j,u,i_u}^{(3,v)}$, the first $(v-1)$ elements equal 0, the u th element is i_u and the remaining elements all equal j ; let $B^{(3,v)}$ be an $(m \times [\sum_{u=v}^m (s_v - s_{v-1})(s_u - 1)])$ array whose columns are given by $b_{j,u,i_u}^{(3,v)}$ for all j , u and i_u . Note that $B^{(3,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 4$.

(iv) If $s_{m-3} < s_{m-2}$, then define the $(m \times 1)$ vectors $b_{j,u,i_u}^{(7)}$ ($m-2 \leq u \leq m$, $s_{m-3} \leq j \leq s_{m-2} - 1$, $i \leq i_u \leq s_u - 1$, $i_u \neq j$) such that in $b_{j,u,i_u}^{(7)}$, the first $(m-3)$ elements equal 0, the u th element is i_u and the remaining elements equal j ; let $B^{(7)}$ be an $(m \times [(s_{m-2} - s_{m-3}) \sum_{u=m-2}^m (s_u - 2)])$ array whose columns are given by $b_{j,u,i_u}^{(7)}$ for all j , u and i_u . Note that $B^{(7)}$ is defined only when $s_{m-3} < s_{m-2}$.

(v) For $1 \leq v \leq m-3$, if $s_v < s_{v+1}$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ ($s_v \leq j \leq s_{v+1} - 1$, $1 \leq u_1 \leq v$, $1 \leq i_{u_1} \leq s_{u_1} - 1$, $v+1 \leq u_2 \leq m$, $j+1 \leq i_{u_2} \leq s_{u_2} - 1$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$, the u_1 th element is i_{u_1} , the remaining of the first v elements equal 0, the u_2 th element is i_{u_2} and the remaining elements all equal j ; let $B^{(8,v)}$ be an $(m \times [\sum_{u_1=1}^v (s_{u_1} - 1)] [\sum_{j=s_v}^{s_{v+1}-1} \sum_{u_2=v+1}^m (s_{u_2} - j - 1)])$ array whose columns are given by $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ for all j , u_1 , u_2 , i_{u_1} and i_{u_2} . Note that $B^{(8,v)}$ is defined only when $s_v < s_{v+1}$ and $m \geq 4$.

(vi) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$ ($s_{v-1} \leq j \leq s_v - 1$, $v \leq u_1 < u_2 \leq m$, $j+1 \leq i_{u_1} \leq s_{u_1} - 1$, $j+1 \leq i_{u_2} \leq s_{u_2} - 1$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$, the first $(v-1)$ elements equal 0, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} and the remaining elements all equal j ; let $B^{(9,v)}$ be an $(m \times [\sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m (s_{u_1} - j - 1)(s_{u_2} - j - 1)])$ array whose columns are given by $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$ for all j , u_1 , u_2 , i_{u_1} and i_{u_2} . Note that $B^{(9,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 4$.

$b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$ for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(9,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 4$.

(vii) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$ ($s_{v-1} \leq j \leq s_v - 1, j \neq 0, v \leq u_1 < u_2 \leq m, 0 \leq i_{u_1} \leq s_{u_1} - 1, 0 \leq i_{u_2} \leq s_{u_2} - 1, \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2})$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$, the first $(v-1)$ elements equal 0, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} and the remaining elements all equal j ; let $B^{(10,v)}$ be an $(m \times [\sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m j(s_{u_1} + s_{u_2} - 2j - 2)])$ array whose columns are given by $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$ for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(10,v)}$ is defined only when $s_{v-1} < s_v$ and $m \geq 4$.

(viii) If $s_{m-2} < s_{m-1}$, then define the $(m \times 1)$ vectors $b_{j,k,u,i_u}^{(11)}$ ($s_{m-2} \leq j \leq s_{m-1} - 1, s_{m-2} \leq k \leq s_{m-1} - 1, j \neq k, 1 \leq u \leq m-2, 1 \leq i_u \leq s_u - 1$) such that in $b_{j,k,u,i_u}^{(11)}$, the u th element is i_u , the last two elements are j and k and the remaining elements all equal 0; let $B^{(11)}$ be an $(m \times [(\sum_{u=1}^{m-2} (s_u - 1)) (s_{m-1} - s_{m-2}) (s_{m-1} - s_{m-2} - 1)])$ array whose columns are given by $b_{j,k,u,i_u}^{(11)}$ for all j, k, u and i_u . Note that $B^{(11)}$ is defined only when $s_{m-2} < s_{m-1}$.

Let

$$N = \sum_{v=1}^{m-2} (s_v - s_{v-1}) + \sum_{v=1}^{m-2} \sum_{u=1}^v (s_u - 1) (s_{v+1} - s_v) + \sum_{v=1}^{m-3} \sum_{u=v}^m (s_v - s_{v-1}) (s_u - 1) + (s_{m-2} - s_{m-3}) \sum_{u=m-2}^m (s_u - 2) + \sum_{v=1}^{m-3} \sum_{u_1=1}^v (s_{u_1} - 1) \{ \sum_{j=s_v}^{s_{v+1}-1} \sum_{u_2=v+1}^m (s_{u_2} - j - 1) \} + \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m (s_{u_1} - j - 1) \}$$

$$j-1)(s_{u_2} - j-1) + \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m j(s_{u_1} + s_{u_2} - 2j - 2) +$$

$$u_1 < u_2$$

$$(s_{m-1} - s_{m-2})(s_m - s_{m-2} - 1) \sum_{u=1}^{m-2} (s_u - 1)$$

and define the following $(m \times N)$ array

$$Q = [B^{(1,1)}, \dots, B^{(1,m-2)}, B^{(2,1)}, \dots, B^{(2,m-2)}, B^{(3,1)}, \dots,$$

$$B^{(3,m-3)}, B^{(7)}, B^{(8,1)}, \dots, B^{(8,m-3)}, B^{(9,1)}, \dots, B^{(9,m-3)},$$

$$B^{(10,1)}, \dots, B^{(10,m-3)}, B^{(11)}]$$
(2.4.1)

where in Q any subarray of the form $B^{(u,v)}$ or $B^{(u)}$ is included provided it is defined. Then we have the following theorem :

Theorem 2.4.1. The columns of the matrix Q , as defined in (2.4.1), interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in the general asymmetric factorial set-up in N runs.

Example 2.4.1. Let $m = 6$, $s_1 = 2$, $s_2 = 2$, $s_3 = 2$, $s_4 = 3$, $s_5 = 4$, $s_6 = 5$. Then according to **Theorem 2.4.1**, the following level combinations, written as columns, give the search design in 152 runs that permits the estimation of the general mean, all main-effects and all two-factor interactions and allow the search and estimation of possibly present one non-zero three-factor interaction.

01	0	100	10000
01	0	010	01000
$B^{(1,1)} : 01 ;$	$B^{(1,4)} : 0 ;$	$B^{(2,3)} : 001 ;$	$B^{(2,4)} : 00100 ;$
01	2	222	00012
01	2	222	33333
01	2	222	33333

$B^{(1,2)}$, $B^{(1,3)}$, $B^{(2,1)}$, $B^{(2,2)}$, $B^{(2,4)}$, $B^{(3,2)}$, $B^{(3,3)}$, $B^{(8,1)}$, $B^{(8,2)}$, $B^{(8,3)}$, $B^{(9,2)}$, $B^{(9,3)}$, $B^{(10,2)}$, $B^{(10,3)}$, $B^{(11)}$ are not defined.

Here one may note that the regular resolution VII plans corresponding to the Examples 2.4.1 and 2.4.2 require 199 and 414 runs respectively.

2.5 Discussion

The present chapter describes a method for constructing search designs for the general symmetric and asymmetric factorials that allow the estimation of the general mean, all main-effects and two-factor interactions and search and estimate the possibly present at best one among the three-factor interactions. The proposed designs are useful when $m \geq 5$, where m is the number of factors; then, as in the last chapter, our plans require considerably smaller number of runs compared to the corresponding regular resolution VII fractional factorials.

In the 2^m setting, when $m \leq 6$, our plan requires $(m^2 + 3m + 4)/2$ runs while the corresponding plans due to Srivastava and Ghosh (1976) involve $(m^2 + 3m + 2)/2$ runs, but when $m > 6$, both the plans require equal number of runs. However, compared to our plans the scope of search is broader in the plans due to Srivastava and Ghosh (1976). The plans due to Shirakura and Ohnishi (1985) for 2^m factorial set-up are slightly better than ours in respect of the number of runs.

For 3^m factorial set-up, our plan cannot be compared with those in Ghosh and Zhang (1987) for reasons as described in Section 1.7 of the previous chapter.

For the search problem considered in this chapter, with more general symmetric and asymmetric factorials no plan other than ours has been reported in the literature.

2.6 Appendix

This section considers the proofs of the Theorems 2.3.2 and 2.4.1. The proof of Theorem 2.3.1 follows from Theorem 2.4.1 as a special case and is, therefore, omitted here.

Proof of Theorem 2.3.2. Under the notations of Section 1.3 and 2.2, consider the matrix $[T_1^*, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}] = [\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}]$, where without loss of generality, suppose

$$q_1 < q_2 < q_3, q_4 < q_5 < q_6, q_3 \leq q_6 \text{ and } (q_1, q_2, q_3) \neq (q_4, q_5, q_6). \quad (2.6.1)$$

By Theorem 1.2.1, it is enough to prove the linear independence of the columns of the above matrix. The proof of this involves a number of cases.

First suppose $q_1 = q_4, q_2 = q_5$ and $q_3 < q_6$. Then consider

$$[\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, V_{q_1 q_2 q_6}] \theta^* = 0, \quad (2.6.2)$$

where $\theta^* = (\theta_0, \theta_1', \theta_2', \dots, \theta_m', \theta_{12}', \theta_{13}', \dots, \theta_{m-1 m}', \phi_1', \phi_2')$, say. Let $p'_{vj} (1 \leq v \leq m, 0 \leq j \leq s-1)$ denote the j th row of P_v . Recalling the definitions of $L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}$ and $V_{q_1 q_2 q_6}$, one gets from (2.3.2) and (2.6.2), for a typical level combination $(i_1, i_2, \dots, i_m)$ the following equation:

$$\begin{aligned} \theta_0 + \sum_{w=1}^m p'_{wi} \theta_w + \sum_{w_1=1}^m \sum_{\substack{w_2=1 \\ w_1 < w_2}}^m (p'_{w_1 i_{w_1}} \times p'_{w_2 i_{w_2}}) \theta_{w_1 w_2} + (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) \phi_1 + (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_6 i_{q_6}}) \phi_2 = 0, \\ (i_1, i_2, \dots, i_m) \in S_2(Q_2), \end{aligned} \quad (2.6.3)$$

where $S_2(Q_2)$ is the set of level combinations given by the array Q_2 given by (2.3.2). Recalling the structure of Q_2 , as given by (2.3.2), it is evident that the system of equations (2.6.3) can be split up into subsystems (2.6.3a), (2.6.3b), (2.6.3c) and (2.6.3d) which arise from the level combinations given by the columns of 0 , $[A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}]$, $[A_1^{(3)}, A_2^{(3)}, \dots, A_{s-2}^{(3)}]$ and $[A_0^{(4)}, A_1^{(4)}, \dots, A_{s-2}^{(4)}]$ respectively. For example, taking care of the definition of the column vectors $a_{j,u,i_u}^{(2)}$ which constitute the columns of $[A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}]$, the subsystem of equations (2.6.3b) corresponding to the columns of $[A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}]$ is given by

$$\begin{aligned}
 & \theta_0 + \sum_{\substack{w=1 \\ w \neq u}}^m p'_{wj} \theta_w + p'_{ui_u} \theta_u + \sum_{w_1=1}^m \sum_{\substack{w_2=1 \\ w_1 < w_2 \neq u}}^m (p'_{w_1j} \times p'_{w_2j}) \theta_{w_1 w_2} + \\
 & \sum_{w_1 < u} (p'_{w_1j} \times p'_{ui_u}) \theta_{w_1 u} + \sum_{w_2 > u} (p'_{ui_u} \times p'_{w_2j}) \theta_{u w_2} + [\delta_{uq_1} (p'_{q_1 i_{q_1}} \times \\
 & p'_{q_2 j} \times p'_{q_3 j}) + \delta_{uq_2} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + \delta_{uq_3} (p'_{q_1 j} \times p'_{q_2 j} \times \\
 & p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_1} - \delta_{uq_2} - \delta_{uq_3}) (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 j})] \phi_1 + \\
 & [\delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_6 j}) + \delta_{uq_2} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_6 j}) + \\
 & \delta_{uq_6} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{uq_1} - \delta_{uq_2} - \delta_{uq_6}) (p'_{q_1 j} \times \\
 & p'_{q_2 j} \times p'_{q_6 j})] \phi_2 = 0, \quad (2.6.3b)
 \end{aligned}$$

where $0 \leq j \leq s-1$, $1 \leq u \leq m$, $0 \leq i_u \leq s-1$, $i_u \neq j$. Similarly the other equations can be written down in detail.

For any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_6$), if one subtracts (2.6.3a) from (2.6.3b) with $j = 0$, $u = t$, one gets

$$\begin{aligned} & (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - \\ & p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} = 0, \end{aligned} \quad (2.6.4)$$

where $1 \leq i_t \leq s_t - 1$. For any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_6$), if one subtracts (2.6.3b) with $j = 0$, $u = r$ from (2.6.3d) with $j = 0$, $u_1 = r$, $u_2 = t$, one obtains

$$\begin{aligned} & (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 \leq t, w_1 \neq r} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - \\ & p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} + [p'_{ri_r} \times (p'_{ti_t} - p'_{t0})] \theta_{rt} = 0, \end{aligned} \quad (2.6.5)$$

where $1 \leq r < t$, $1 \leq i_t$, $i_r \leq s-1$. Similarly, for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_6$), if one subtracts (2.6.3b) with $j = 0$, $u = r$ from (2.6.3d) with $j = 0$, $u_1 = t$, $u_2 = r$, one obtains

$$\begin{aligned} & (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t, w_2 \neq r} [(p'_{ti_t} - \\ & p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} + [(p'_{ti_t} - p'_{t0}) \times p'_{ri_r}] \theta_{tr} = 0, \end{aligned} \quad (2.6.6)$$

where $t < r \leq m$, $1 \leq i_t$, $i_r \leq s-1$. Now summing the s equations that (2.6.4) and (2.6.5), one gets for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_6$),

$$\begin{aligned} & (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t, w_1 \neq r} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - \\ & p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} = 0, \end{aligned} \quad (2.6.7)$$

where $1 \leq r < t$, $1 \leq i_t \leq s-1$, if one subtracts (2.6.7) from (2.6.4) and (2.6.5), one obtains

$$[p'_{ri_r} \times (p'_{ti_t} - p'_{t0})] \theta_{rt} = 0, \quad (2.6.8)$$

where $1 \leq r < t \leq m$, $t \neq q_1, q_2, q_3, q_6$, $0 \leq i_r \leq s-1$ and $1 \leq i_t \leq s-1$. Similarly, from equations (2.6.4) and (2.6.6), one obtains

$$[(p'_{ti_t} - p'_{t0}) \times p'_{ri_r}] \theta_{tr} = 0, \quad (2.6.9)$$

where $1 \leq t < r \leq m$, $t \neq q_1, q_2, q_3, q_6$, $0 \leq i_r \leq s-1$ and $1 \leq i_t \leq s-1$. Now (2.6.8) and (2.6.9) implies that for $1 \leq r < t \leq m$, $t \neq q_1, q_2, q_3, q_6$, $[P_r \times P_t^*] \theta_{rt} = 0$ and for $1 \leq t < r \leq m$, $t \neq q_1, q_2, q_3, q_6$, $[P_t^* \times P_r] \theta_{tr} = 0$, where P_t^* is an $((s-1) \times (s-1))$ matrix with rows $(p'_{ti_t} - p'_{t0})$, $1 \leq i_t \leq s-1$. Since P_r and P_t^* are both of full column rank, one obtains for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_6$),

$$\theta_{rt} = 0, \quad 1 \leq r < t \leq m \quad \text{and} \quad \theta_{tr} = 0, \quad 1 \leq t < r \leq m. \quad (2.6.10)$$

Then from (2.6.4) and (2.6.10), one obtains $(p'_{ti_t} - p'_{t0}) \theta_t = 0$, $1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_6$. From Lemma 1.8.1 of Chapter 1, one obtains

$$\theta_t = 0, \quad 1 \leq t \leq m, \quad t \neq q_1, q_2, q_3, q_6. \quad (2.6.11)$$

For any fixed k_3 ($1 \leq k_3 \leq s-1$), if one subtracts (2.6.3a) from (2.6.3b) with $j = 0$, $u = q_3$, $i_{q_3} = k_3$, one obtains

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + \sum_{w_1 < q_3} [p'_{w_1 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{w_1 q_3} + \sum_{w_2 > q_3} [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{w_2 0}] \theta_{q_3 w_2} + [p'_{q_1 0} \times p'_{q_2 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \quad (2.6.12)$$

For any fixed k_3, k_6 , $1 \leq k_3, k_6 \leq s-1$, if one subtracts (2.6.3b) with $j = 0$, $u = q_6$, $i_{q_6} = k_6$ from (2.6.3d) with $j = 0$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_6$, $i_{q_6} = k_6$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + \sum_{w_1 < q_3} [p'_{w_1 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{w_1 q_3} + \sum_{w_2 > q_3 \neq q_6} [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{w_2 0}] \theta_{q_3 w_2} + [p'_{q_1 0} \times p'_{q_2 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0.$$

$$\begin{aligned}
& p'_{q_3 0} \times p'_{w_2 0}] \theta_{q_3 w_2} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_6 k_6}] \theta_{q_3 q_6} + [p'_{q_1 0} \times \\
& p'_{q_2 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0 .
\end{aligned} \tag{2.6.13}$$

Now summing the s equations that (2.6.12) and (2.6.13), one gets for any fixed k_3 ,
 $1 \leq k_3 \leq s-1$,

$$\begin{aligned}
& (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + \sum_{w_1 < q_3} [p'_{w_1 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{w_1 q_3} + \sum_{w_2 > q_3 \neq q_6} [(p'_{q_3 k_3} - \\
& p'_{q_3 0}) \times p'_{w_2 0}] \theta_{q_3 w_2} + [p'_{q_1 0} \times p'_{q_2 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0 .
\end{aligned} \tag{2.6.14}$$

If one subtracts (2.6.14) from (2.6.12) and (2.6.13), one obtains

$$[(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_6 k_6}] \theta_{q_3 q_6} = 0 , \tag{2.6.15}$$

where $1 \leq k_3 \leq s-1$, $0 \leq k_6 \leq s-1$. Now (2.6.15) implies that $[P'_{q_3} \times P'_{q_6}] \theta_{q_3 q_6} = 0$, where $P'^{*}_{q_3}$ is an $((s-1) \times (s-1))$ matrix with rows $(p'_{q_3 k_3} - p'_{q_3 0})$, $1 \leq k_3 \leq s-1$.

Since $P'^{*}_{q_3}$ and P'_{q_6} are both of full column rank, one obtains

$$\theta_{q_3 q_6} = 0 . \tag{2.6.16}$$

For any fixed k_1, k_3 , $1 \leq k_1, k_3 \leq s-1$, if one subtracts (2.6.3b) with $j = 0$, $u = q_1$, $i_{q_1} = k_1$ from (2.6.3d) with $j = 0$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = k_3$, then one gets on using (2.6.10) and (2.6.16),

$$\begin{aligned}
& (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 0} \times (p'_{q_3 k_3} - \\
& p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0 .
\end{aligned} \tag{2.6.17}$$

For any fixed k_2, k_3 , $1 \leq k_2, k_3 \leq s-1$, if one subtracts (2.6.3b) with $j = 0$, $u = q_2$,

$i_{q_2} = k_2$ from (2.6.3d) with $j = 0$, $u_1 = q_2$, $i_{q_2} = k_2$, $u_2 = q_3$, $i_{q_3} = k_3$, then one gets on using (2.6.10) and (2.6.16),

$$\begin{aligned} & (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 0} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - \\ & p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 0} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \end{aligned} \quad (2.6.18)$$

For any fixed k_3 , $1 \leq k_3 \leq s-1$, if one subtracts (2.6.3b) with $j = k_3$, $u = q_3$, $i_{q_3} = 0$ from (2.6.3b) with $j = k_3$, $u = t$, $i_t = 0$, $t \neq q_1, q_2, q_3, q_6$, one obtains on using (2.6.10), (2.6.11) and (2.6.16),

$$\begin{aligned} & (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_3} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_3} \times (p'_{q_3 k_3} - \\ & p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_3} \times p'_{q_2 k_3} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \end{aligned} \quad (2.6.19)$$

Here, one may note that the equations (2.6.20-26) stated below are obtained on using (2.6.10) and (2.6.16). For any fixed k_1, k_3 , $1 \leq k_1, k_3 \leq s-1$, $k_1 \neq k_3$, if one subtracts (2.6.3b) with $j = k_1$, $u = q_3$, $i_{q_3} = 0$ from (2.6.3b) with $j = k_1$, $u = q_3$, $i_{q_3} = k_3$, then one gets

$$\begin{aligned} & (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_1} \times (p'_{q_3 k_3} - \\ & p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \end{aligned} \quad (2.6.20)$$

For any fixed k_1, k_3 , $1 \leq k_3 < k_1 \leq s-1$, if one subtracts (2.6.3c) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = 0$ from (2.6.3b) with $j = k_3$, $u = q_1$, $i_{q_1} = k_1$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_3} \times (p'_{q_3 k_3} -$$

$$p'_{q_3 0} \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 k_3} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \quad (2.6.21)$$

For any fixed k_2, k_3 , $1 \leq k_3 < k_2 \leq s-1$, if one subtracts (2.6.3c) with $j = k_3$, $u_1 = q_2$, $i_{q_2} = k_2$, $u_2 = q_3$, $i_{q_3} = 0$ from (2.6.3b) with $j = k_3$, $u = q_2$, $i_{q_2} = k_2$, then one obtains

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_3} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_3} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \quad (2.6.22)$$

For any fixed k_1, k_2, k_3 , $1 \leq k_3 < k_2 < k_1 \leq s-1$, if one subtracts (2.6.3c) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = 0$ from (2.6.3c) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = k_3$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \quad (2.6.23)$$

For any fixed k_1, k_2, k_3 , $1 \leq k_3 < k_1 < k_2 \leq s-1$, if one subtracts (2.6.3c) with $j = k_1$, $u_1 = q_2$, $i_{q_2} = k_2$, $u_2 = q_3$, $i_{q_3} = 0$ from (2.6.3c) with $j = k_1$, $u_1 = q_2$, $i_{q_2} = k_2$, $u_2 = q_3$, $i_{q_3} = k_3$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \quad (2.6.24)$$

For any fixed k_1, k_2, k_3 , $1 \leq k_2 \leq s-2$, $k_2 < k_1$, $k_3 \leq s-1$, if one subtracts (2.6.3c) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = 0$ from (2.6.3d) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = k_3$, one gets

$$\begin{aligned}
& (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - \\
& p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0 . \quad (2.6.25)
\end{aligned}$$

For any fixed $k_1, k_2, k_3, 1 \leq k_1 \leq s-2, k_1 < k_2, k_3 \leq s-1$, if one subtracts (2.6.3c) with $j = k_1, u_1 = q_2, i_{q_2} = k_2, u_2 = q_3, i_{q_3} = 0$ from (2.6.3d) with $j = k_1, u_1 = q_2, i_{q_2} = k_2, u_2 = q_3, i_{q_3} = k_3$, one obtains

$$\begin{aligned}
& (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - \\
& p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0 . \quad (2.6.26)
\end{aligned}$$

Now combining the equations (2.6.12), (2.6.17-26) and using (2.6.10) and (2.6.16), one gets for any fixed $k_1, k_2, k_3, 1 \leq k_3 \leq s-1, 0 \leq k_1, k_2, \leq s-1$,

$$\begin{aligned}
& (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - \\
& p'_{q_3 0})] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0 . \quad (2.6.27)
\end{aligned}$$

For any fixed $k_2, k_3, 0 \leq k_2 \leq s-1, 1 \leq k_3 \leq s-1$, the s equations that (2.6.27), when added, gives

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_2 q_3} = 0 . \quad (2.6.28)$$

Again summing the s equations that (2.6.28) one gets for any fixed $k_3, 1 \leq k_3 \leq s-1, (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} = 0$. Then from Lemma 1.8.1, one obtains

$$\theta_{q_3} = 0 . \quad (2.6.29)$$

From (2.6.28) and (2.6.29), one gets for any fixed $k_2, k_3, 0 \leq k_2 \leq s-1, 1 \leq k_3 \leq s-1$,

$$[p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_2 q_3} = 0 \quad (2.6.30)$$

which implies $[P_{q_2} \times P_{q_3}^*] \theta_{q_2 q_3} = 0$. Since P_{q_2} and $P_{q_3}^*$ are of full column rank, one obtains

$$\theta_{q_2 q_3} = 0. \quad (2.6.31)$$

For any fixed k_1 and $k_3, 0 \leq k_1 \leq s-1, 1 \leq k_3 \leq s-1$, summing the s equations that (2.6.27) one gets on using (2.6.29) and (2.6.31)

$$[p'_{q_1 k_1} \times (p'_{q_3 k_3} - p'_{q_3 0})] \theta_{q_1 q_3} = 0 \quad (2.6.32)$$

which again implies $[P_{q_1} \times P_{q_3}^*] \theta_{q_1 q_3} = 0$. Since P_{q_1} and $P_{q_3}^*$ are both of full column rank, one obtains

$$\theta_{q_1 q_3} = 0. \quad (2.6.33)$$

From (2.6.27), (2.6.29), (2.6.31) and (2.6.33), one gets for any fixed $k_1, k_2, k_3, 0 \leq k_1, k_2 \leq s-1, 1 \leq k_3 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times (p'_{q_3 k_3} - p'_{q_3 0})] \phi_1 = 0. \quad (2.6.34)$$

Equations (2.6.34) implies that $[P_{q_1} \times P_{q_2} \times P_{q_3}^*] \phi_1 = 0$. Since P_{q_1}, P_{q_2} and $P_{q_3}^*$ are of full column rank, one obtains

$$\phi_1 = 0. \quad (2.6.35)$$

In a similar way, a set of equations similar to (2.6.12), (2.6.17-27) may be obtained from which one can get

$$\theta_{q_6} = 0, \theta_{q_2 q_6} = 0, \theta_{q_1 q_6} = 0, \phi_2 = 0.$$

It is then easy to prove that $\theta_{q_1 q_2} = 0$, $\theta_{q_1} = 0$, $\theta_{q_2} = 0$ and $\theta_o = 0$. This completes the proof in this case. The proofs of other cases follow exactly the same way.

Proof of Theorem 2.4.1. Under the notation of section 2, consider the matrix

$$[T_1^*, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}] = [\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, \\ V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}], \text{ where without loss of generality, suppose}$$

$$q_1 < q_2 < q_3, q_4 < q_5 < q_6, q_3 \leq q_6 \text{ and } (q_1, q_2, q_3) \neq (q_4, q_5, q_6). \quad (2.6.36)$$

By Theorem 1.2.1, it is enough to prove the linear independence of the columns of the above matrix. First suppose $q_2 = q_5, q_3 = q_6, q_1 \neq q_4$. We then consider

$$[\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, \\ V_{q_4 q_2 q_3}] \theta^* = 0 \quad (2.6.37)$$

where $\theta^* = (\theta_o, \theta'_1, \dots, \theta'_m, \theta'_{12}, \theta'_{13}, \dots, \theta'_{m-1 m}, \phi'_1, \phi'_2)'$, say. Recalling the definitions of $L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, V_{q_4 q_2 q_3}$, (2.4.1) and (2.6.37) give, for a typical level combination $(i_1, i_2, \dots, i_m)$, the following equation

$$\theta_o + \sum_{w=1}^m p'_{w i_w} \theta_w + \sum_{w_1=1}^m \sum_{\substack{w_2=1 \\ w_1 < w_2}}^m (p'_{w_1 i_{w_1}} \times p'_{w_2 i_{w_2}}) \theta_{w_1 w_2} + (p'_{q_1 i_{q_1}} \times \\ p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) \phi_1 + (p'_{q_4 i_{q_4}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) \phi_2 = 0, \\ (i_1, i_2, \dots, i_m) \in S(Q), \quad (2.6.38)$$

where $S(Q)$ is the set of level combinations given by the columns of the array Q . Recalling the structure of Q , as given by (2.4.1), it is evident that the system of equations (2.6.38) can be split up into subsystems (2.6.38a), (2.6.38b), $\dots$, (2.6.38h) which

arise from the level combinations given by the columns of $[B^{(1,1)}, \dots, B^{(1,m-2)}], \dots, [B^{(3,1)}, \dots, B^{(3,m-3)}], B^{(7)}, [B^{(8,1)}, \dots, B^{(8,m-3)}], \dots, [B^{(10,1)}, \dots, B^{(10,m-3)}], B^{(11)}$ respectively. For example, taking care of the definition of the column vectors $b_{j,u,i_u}^{(2,v)}$ which constitute the columns of $[B^{(2,1)}, \dots, B^{(2,m-2)}]$, the subsystem of equations (2.6.38b) corresponding to the columns of $[B^{(2,1)}, \dots, B^{(2,m-2)}]$ is given by

$$\begin{aligned} \theta_0 + \sum_{\substack{w=1 \\ w \neq u}}^v p'_{w0} \theta_w + p'_{ui_u} \theta_u + \sum_{w=v+1}^m p'_{wj} \theta_w + \sum_{w_1=1}^v \sum_{\substack{w_2=1 \\ w_1 < w_2 \neq u}}^v (p'_{w_1 0} \times p'_{w_2 0}) \theta_{w_1 w_2} + \sum_{w_1 < u} (p'_{w_1 0} \times p'_{ui_u}) \theta_{w_1 u} + \sum_{w_2 > u} (p'_{ui_u} \times p'_{w_2 0}) \theta_{u w_2} + \\ \sum_{w_1=1}^v \sum_{\substack{w_2=v+1 \\ w_1 \neq u}}^m (p'_{w_1 0} \times p'_{w_2 j}) \theta_{w_1 w_2} + \sum_{w_2=v+1}^m (p'_{ui_u} \times p'_{w_2 j}) \theta_{u w_2} + \\ \sum_{w_1=v+1}^m \sum_{\substack{w_2=v+1 \\ w_1 < w_2}}^m (p'_{w_1 j} \times p'_{w_2 j}) \theta_{w_1 w_2} + [\beta(v+1, q_1) \{ \beta(v+1, q_2) \{ \beta(v+1, q_3) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 0} \times p'_{q_3 0}) + \delta_{uq_2} (p'_{q_1 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 0}) + \\ \delta_{uq_3} (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_1} - \delta_{uq_2} - \delta_{uq_3}) (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 0}) \} + (1 - \beta(v+1, q_3)) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 0} \times p'_{q_3 j}) + \delta_{uq_2} (p'_{q_1 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + (1 - \delta_{uq_1} - \delta_{uq_2}) (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 j}) \} + (1 - \beta(v+1, q_2)) \{ \delta_{uq_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 j}) + (1 - \delta_{uq_1}) (p'_{q_1 0} \times p'_{q_2 j} \times p'_{q_3 j}) \} + \\ (1 - \beta(v+1, q_1)) (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 j}) \} \phi_1 + [\beta(v+1, q_4) \{ \beta(v+1, q_2) \{ \beta(v+1, \end{aligned}$$

$$\begin{aligned}
& q_3) \{ \delta_{uq_4} (p'_{q_4 i_{q_4}} \times p'_{q_2 0} \times p'_{q_3 0}) + \delta_{uq_2} (p'_{q_4 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 0}) + \\
& \delta_{uq_3} (p'_{q_4 0} \times p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{uq_4} - \delta_{uq_2} - \delta_{uq_3}) (p'_{q_4 0} \times \\
& p'_{q_2 0} \times p'_{q_3 0}) \} + (1 - \beta(v+1, q_3)) \{ \delta_{uq_4} (p'_{q_4 i_{q_4}} \times p'_{q_2 0} \times p'_{q_3 j}) + \\
& \delta_{uq_2} (p'_{q_4 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + (1 - \delta_{uq_4} - \delta_{uq_2}) (p'_{q_4 0} \times p'_{q_2 0} \times \\
& p'_{q_3 j}) \} + (1 - \beta(v+1, q_2)) \{ \delta_{uq_4} (p'_{q_4 i_{q_4}} \times p'_{q_2 j} \times p'_{q_3 j}) + (1 - \delta_{uq_4}) (p'_{q_4 0} \times \\
& p'_{q_2 j} \times p'_{q_3 j}) \} + (1 - \beta(v+1, q_4)) (p'_{q_4 j} \times p'_{q_2 j} \times p'_{q_3 j}) \} \Phi_2 = 0, \quad (2.6.38b)
\end{aligned}$$

where $s_v \leq j \leq s_{v+1} - 1$, $1 \leq u \leq v$, $1 \leq i_u \leq s_u - 1$ and $\beta(v_1, v_2) = 0$ if $v_1 \leq v_2$; $= 1$ if $v_1 > v_2$. Note that for given v , the equations given by (2.6.38b) are defined only when $s_v < s_{v+1}$.

Now for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$), if one subtracts equation (2.6.38a) with $v = 1$, $j = 0$ from equation (2.6.38c) with $v = 1$, $j = 0$, $u = t$, one obtains

$$\begin{aligned}
& (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - \\
& p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} = 0, \quad (2.6.39)
\end{aligned}$$

where $0 \leq i_t \leq s_t - 1$. For any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$), if one subtracts (2.6.38c) with $v = 1$, $j = 0$, $u = r$ from (2.6.38f) with $v = 1$, $j = 0$, $u_1 = r$, $u_2 = t$, one obtains

$$(p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t \neq r} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} -$$

$$p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} + [p'_{ri_r} \times (p'_{ti_t} - p'_{t0})] \theta_{rt} = 0, \quad (2.6.40)$$

where $1 \leq r < t$, $1 \leq i_t \leq s_t - 1$, $1 \leq i_r \leq s_r - 1$. Similarly, for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$), if one subtracts (2.6.38c) with $v = 1$, $j = 0$, $u = r$ from (2.6.38f) with $v = 1$, $j = 0$, $u_1 = t$, $u_2 = r$, one gets

$$(p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} + [(p'_{ti_t} - p'_{t0}) \times p'_{ri_r}] \theta_{tr} = 0, \quad (2.6.41)$$

where $t < r \leq m$, $1 \leq i_r \leq s_r - 1$, $1 \leq i_t \leq s_t - 1$. Summing the s_r equations that (2.6.39) and (2.6.40), one gets for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$),

$$(p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} = 0, \quad (2.6.42)$$

where $1 \leq i_t \leq s_t - 1$, $1 \leq r < t$. Similarly the equations (2.6.39) and (2.6.41), when added, give

$$(p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} = 0, \quad (2.6.43)$$

where $1 \leq i_t \leq s_t - 1$, $t < r \leq m$. Now if one subtracts (2.6.42) from the equations (2.6.39) and (2.6.40), one gets for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$)

$$[p'_{ri_r} \times (p'_{ti_t} - p'_{t0})] \theta_{rt} = 0, \quad (2.6.44)$$

where $1 \leq r < t$, $0 \leq i_r \leq s_r - 1$, $1 \leq i_t \leq s_t - 1$ and if one subtracts (2.6.43) from the equations (2.6.39) and (2.6.41), one obtains for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$),

$$[(p'_{ti_t} - p'_{t0}) \times p'_{ri_r}] \theta_{tr} = 0, \quad (2.6.45)$$

where $t < r \leq m$, $0 \leq i_r \leq s_r - 1$, $1 \leq i_t \leq s_t - 1$. Now (2.6.44) and (2.6.45) implies that for $1 \leq r < t \leq m$, $t \neq q_1, q_2, q_3, q_4$, $[P_r \times P_t^*] \theta_{rt} = 0$ and for $1 \leq t < r \leq m$, $t \neq q_1, q_2, q_3, q_4$, $[P_t^* \times P_r] \theta_{tr} = 0$, where P_t^* is an $((s_t - 1) \times (s_t - 1))$ matrix with rows $(p'_{ti_t} - p'_{t0})$, $1 \leq i_t \leq s_t - 1$. Since P_r and P_t^* are both of full column rank, one obtains for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$)

$$\theta_{rt} = 0, \quad 1 \leq r < t \leq m \quad \text{and} \quad \theta_{tr} = 0, \quad 1 \leq t < r \leq m, \quad (2.6.46)$$

Then from (2.6.39) and (2.6.46), one obtains

$$(p'_{ti_t} - p'_{t0}) \theta_t = 0, \quad (2.6.47)$$

where $1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4$ and $1 \leq i_t \leq s_t - 1$. From (2.6.47) and from **Lemma 1.8.1**, it follows that

$$\theta_t = 0, \quad 1 \leq t \leq m, \quad t \neq q_1, q_2, q_3, q_4. \quad (2.6.48)$$

Let $q_1 < q_4$ and $q_2 < m - 1$. For any fixed k_1 , $1 \leq k_1 \leq s_{q_1} - 1$, if one subtracts (2.6.38a) with $v = 1, j = 0$ from (2.6.38c) with $v = 1, j = 0, u_1 = q_1, i_{q_1} = k_1$ and if one uses (2.6.46), one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_4 0}] \theta_{q_1 q_4} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_2 0}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_3 0}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 0}] \phi_1 = 0. \end{aligned} \quad (2.6.49)$$

For any fixed k_1, k_4 , $1 \leq k_1 \leq s_{q_1} - 1$, $1 \leq k_4 \leq s_{q_4} - 1$, if one subtracts (2.6.38c) with $v = 1, j = 0, u = q_4, i_{q_4} = k_4$ from (2.6.38f) with $v = 1, j = 0, u_1 = q_1, i_{q_1} = k_1, u_2 = q_4, i_{q_4} = k_4$, one gets on using (2.6.46)

$$\begin{aligned}
& (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_4 k_4}] \theta_{q_1 q_4} + [(p'_{q_1 k_1} - \\
& p'_{q_1 0}) \times p'_{q_2 0}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_3 0}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - \\
& p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 0}] \phi_1 = 0.
\end{aligned} \tag{2.6.50}$$

Summing the s_{q_4} equations that (2.6.49) and (2.6.50), one gets for any fixed k_1 , $1 \leq k_1 \leq s_{q_1} - 1$,

$$\begin{aligned}
& (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\
& p'_{q_1 0}) \times p'_{q_3 0}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 0}] \phi_1 = 0.
\end{aligned} \tag{2.6.51}$$

If one subtracts (2.6.51) from (2.6.49) and (2.6.50), one obtains

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_4 k_4}] \theta_{q_1 q_4} = 0, \tag{2.6.52}$$

where $1 \leq k_1 \leq s_{q_1} - 1$, $0 \leq k_4 \leq s_{q_4} - 1$. Now (2.6.52) implies that $[P_{q_1}^* \times P_{q_4}] \theta_{q_1 q_4} = 0$, where $P_{q_1}^*$ is an $((s_{q_1} - 1) \times (s_{q_1} - 1))$ matrix with rows $(p'_{q_1 k_1} - p'_{q_1 0})$, $1 \leq k_1 \leq s_{q_1} - 1$. Since $P_{q_1}^*$ and P_{q_4} are both of full column rank, one obtains

$$\theta_{q_1 q_4} = 0. \tag{2.6.53a}$$

Similarly, if $q_4 < q_1$, it can be proved that

$$\theta_{q_4 q_1} = 0. \tag{2.6.53b}$$

Here, one may note that the equations (2.6.54-61) stated below are obtained on using (2.6.46) and (2.6.53a,b). For any fixed k_1, k_2 , $1 \leq k_1 \leq s_{q_1} - 1$, $1 \leq k_2 \leq s_{q_2} - 1$, if one subtracts (2.6.38c) with $v = 1, j = 0, u = q_2$ $i_{q_2} = k_2$ from

(2.6.38f) with $v = 1, j = 0, u_1 = q_1, i_{q_1} = k_1, u_2 = q_2, i_{q_2} = k_2$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 0}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 0}] \phi_1 = 0. \end{aligned} \quad (2.6.54a)$$

For any fixed $k_1, k_3, 1 \leq k_1 \leq s_{q_1} - 1, 1 \leq k_3 \leq s_{q_3} - 1$, if one subtracts equation (2.6.38c) with $v = 1, j = 0, u = q_3, i_{q_3} = k_3$ from the equation (2.6.38f) with $v = 1, j = 0, u_1 = q_1, i_{q_1} = k_1, u_2 = q_3, i_{q_3} = k_3$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 k_3}] \phi_1 = 0. \end{aligned} \quad (2.6.54b)$$

If $s_{v-1} < s_v$, then for any fixed $k_1, s_{v-1} \leq k_1 \leq s_v - 1, k_1 \neq 0$, if one subtracts (2.6.38c) with $j = k_1, u_1 = q_1, i_{q_1} = 0$ from (2.6.38a) with $j = k_1$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_1}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_1}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_1} \times p'_{q_3 k_1}] \phi_1 = 0, \end{aligned} \quad (2.6.55)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed $k_1, k_2, 1 \leq k_1 \leq s_{q_1} - 1, s_{v-1} \leq k_2 \leq s_v - 1, k_1 \neq k_2 \neq 0$, subtracting (2.6.38c) with $j = k_2, u = q_1, i_{q_1} = 0$ from (2.6.38c) with $j = k_2, u = q_1, i_{q_1} = k_1$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_2}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_2}] \phi_1 = 0, \end{aligned} \quad (2.6.56)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2 , $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $k_1 + 1 \leq k_2 \leq s_{q_2} - 1$, if one subtracts (2.6.38g) with $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$, $j = k_1$ from (2.6.38c) with $u = q_2$, $i_{q_2} = k_2$, $j = k_1$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_1}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_1}] \phi_1 = 0, \end{aligned} \quad (2.6.57a)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_3 , $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $k_1 + 1 \leq k_3 \leq s_{q_3} - 1$, if one subtracts (2.6.38g) with $j = k_1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$ from (2.6.38c) with $j = k_1$, $u = q_3$, $i_{q_3} = k_3$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_1}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_1} \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.57b)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2, k_3 , $1 \leq k_1 \leq k_2 - 1$, $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, $k_2 + 1 \leq k_3 \leq s_{q_3} - 1$, subtracting (2.6.38g) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$ from (2.6.38g) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = k_3$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.58a)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2, k_3 , $1 \leq k_1 \leq k_3 - 1$, $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_2 \leq s_{q_2} - 1$, subtracting (2.6.38g) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$ from (2.6.38g) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = k_1$,

$u_2 = q_2$, $i_{q_2} = k_2$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.58b)$$

where $1 \leq v \leq q_1$. Note that for given v , $1 \leq v \leq q_1$, the equations (2.6.55-58) are defined only when $s_{v-1} < s_v$. For any fixed k_1 , $1 \leq k_1 \leq s_{q_1} - 1$, if one subtracts (2.6.38a) with $j = k_2$, for $s_{v-1} \leq k_2 \leq s_v - 1$, $q_1 + 1 \leq v \leq q_2$ from (2.6.38b) with $u = q_1$, $i_{q_1} = k_1$, $j = k_2$ for $s_v \leq k_2 \leq s_{v+1} - 1$, $q_1 \leq v \leq q_2 - 1$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_2}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_2}] \phi_1 = 0. \end{aligned} \quad (2.6.59)$$

It is to be noted that for given v , $q_1 \leq v \leq q_2 - 1$, the equations given by (2.6.59) are defined only when $s_v < s_{v+1}$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2, k_3 , $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_1 \leq s_{q_1} - 1$, $k_3 + 1 \leq k_2 \leq s_{q_2} - 1$, if one subtracts (2.6.38g) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$ from (2.6.38f) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = k_2$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.60a)$$

where $1 \leq v \leq q_1$. Similarly if $s_{v-1} < s_v$, then for any fixed k_1, k_2, k_3 , $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, $k_2 + 1 \leq k_1 \leq s_{q_1} - 1$, $k_2 + 1 \leq k_3 \leq s_{q_3} - 1$, if one subtracts (2.6.38g) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$, from (2.6.38f) with $j = k_2$,

$u_1 = q_1, i_{q_1} = k_1, u_2 = q_3, i_{q_3} = k_3$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.60b)$$

where $1 \leq v \leq q_1$. Note that for given v , $1 \leq v \leq q_1$, the equations (2.6.60a) and (2.6.60b) are defined only when $s_{v-1} < s_v$. For any fixed k_1 , $1 \leq k_1 \leq s_{q_1} - 1$, if one subtracts (2.6.38c) with $j = k_2, u = q_3, i_{q_3} = k_3$ for $s_{v-1} \leq k_2 \leq s_v - 1, q_1 + 1 \leq v \leq q_2, k_2 + 1 \leq k_3 \leq s_{q_3} - 1$ from (2.6.38e) with $j = k_2, u_1 = q_1, i_{q_1} = k_1, u_2 = q_3, i_{q_3} = k_3$ for $s_v \leq k_2 \leq s_{v+1} - 1, q_1 \leq v \leq q_2 - 1, k_2 + 1 \leq k_3 \leq s_{q_3} - 1$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0. \end{aligned} \quad (2.6.61a)$$

Finally for any fixed k_1 , $1 \leq k_1 \leq s_{q_1} - 1$, if one subtracts (2.6.38c) with $j = k_3, u = q_2, i_{q_2} = k_2$ for $s_{v-1} \leq k_3 \leq s_v - 1, q_1 + 1 \leq v \leq q_2, k_3 + 1 \leq k_2 \leq s_{q_2} - 1$ from (2.6.38e) with $j = k_3, u_1 = q_1, i_{q_1} = k_1, u_2 = q_2, i_{q_2} = k_2$ for $s_v \leq k_3 \leq s_{v+1} - 1, q_1 \leq v \leq q_2 - 1, k_3 + 1 \leq k_2 \leq s_{q_2} - 1$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} - \\ & p'_{q_1 0}) \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0. \end{aligned} \quad (2.6.61b)$$

Now combining the equations (2.6.49), (2.6.54-61), one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2}] \theta_{q_1 q_2} + [(p'_{q_1 k_1} -$$

$$p'_{q_1 0} \times p'_{q_3 k_3}] \theta_{q_1 q_3} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (2.6.62)$$

where $1 \leq k_1 \leq s_{q_1} - 1$, $0 \leq k_2 \leq s_{q_2} - 1$, $0 \leq k_3 \leq s_{q_3} - 1$. Analogously to the derivations of (2.6.46) and (2.6.48), one can obtain from (2.6.62)

$$\theta_{q_1} = 0, \theta_{q_1 q_2} = 0, \theta_{q_1 q_3} = 0 \text{ and } \phi_1 = 0.$$

Exactly in a similar way one can obtain from the equations (2.6.38a-c, e-g), $\theta_0 = 0$,

$$\theta_{q_2} = 0, \theta_{q_3} = 0, \theta_{q_4} = 0, \theta_{q_4 q_2} = 0, \theta_{q_4 q_3} = 0, \theta_{q_2 q_3} = 0, \phi_2 = 0 \text{ and hence } \theta^* = 0.$$

The proof of $\theta^* = 0$, when $q_3 = m$, follows analogously through the equations (2.6.38 a-h).

Next suppose $q_1 = q_4$ and the remaining q_i 's are all distinct satisfying the conditions stated in (2.6.36). To prove the linear independence of the columns of $[\epsilon$,

$L_1, L_2, \dots, L_m, G_{12}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, V_{q_1 q_5 q_6}]$, consider

$$[\epsilon, L_1, L_2, \dots, L_m, G_{12}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, V_{q_1 q_5 q_6}] \theta^* = 0, \quad (2.6.63)$$

where θ^* is defined as earlier. Following the definitions of $L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, V_{q_1 q_5 q_6}$, (2.4.1) and (2.6.63) give, for a typical level combination $(i_1, i_2, \dots, i_m)$, the following equation:

$$\theta_0 + \sum_{w=1}^m p'_{w i_w} \theta_w + \sum_{w_1=1}^m \sum_{\substack{w_2=1 \\ w_1 < w_2}}^m (p'_{w_1 i_{w_1}} \times p'_{w_2 i_{w_2}}) \theta_{w_1 w_2} + (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) \phi_1 + (p'_{q_1 i_{q_1}} \times p'_{q_5 i_{q_5}} \times p'_{q_6 i_{q_6}}) \phi_2 = 0, \quad (i_1, \dots,$$

$$i_m) \in S(Q), \quad (2.6.64)$$

where $S(Q)$ is the set of level combinations given by the columns of the array Q . As

earlier, it is evident that the system of equations (2.6.64) can be split up into sub-systems (2.6.64a), (2.6.64b), . . . , (2.6.64h) exactly as was done with system (2.6.38).

For example, the column vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ which constitute the columns of $[B^{(8,1)}, \dots, B^{(8,m-3)}]$, give the subsystem of equations (2.6.64e) corresponding to the columns of $[B^{(8,1)}, \dots, B^{(8,m-3)}]$ as

$$\begin{aligned}
 & \theta_0 + \sum_{w=1}^v \sum_{w \neq u_1} p'_{w0} \theta_w + p'_{u_1 i_{u_1}} \theta_{u_1} + \sum_{w=v+1}^m \sum_{w \neq u_2} p'_{wj} \theta_w + p'_{u_2 i_{u_2}} \theta_{u_2} + \\
 & \sum_{w_1=1}^v \sum_{w_2=1}^v (p'_{w_1 0} \times p'_{w_2 0}) \theta_{w_1 w_2} + \sum_{w_1 < u_1} (p'_{w_1 0} \times p'_{u_1 i_{u_1}}) \theta_{w_1 u_1} + \\
 & \quad w_1 < w_2 \neq u_1 \\
 & \sum_{w_2 > u_2} (p'_{u_1 i_{u_1}} \times p'_{w_2 0}) \theta_{u_1 w_2} + \sum_{w_1=1}^v \sum_{w_2=v+1}^m \sum_{w_1 \neq u_1} \sum_{w_2 \neq u_2} (p'_{w_1 0} \times p'_{w_2 j}) \theta_{w_1 w_2} + \\
 & \sum_{w_2=v+1}^m \sum_{w_2 \neq u_2} (p'_{u_1 i_{u_1}} \times p'_{w_2 j}) \theta_{u_1 w_2} + \sum_{w_1=1}^v \sum_{w_1 \neq u_1} (p'_{w_1 0} \times p'_{u_2 i_{u_2}}) \theta_{w_1 u_2} + (p'_{u_1 i_{u_1}} \times \\
 & p'_{u_2 i_{u_2}}) \theta_{u_1 u_2} + \sum_{w_1=v+1}^m \sum_{w_2=v+1}^m \sum_{w_1 < w_2 \neq u_2} (p'_{w_1 j} \times p'_{w_2 j}) \theta_{w_1 w_2} + \sum_{w_1=v+1}^{u_2-1} (p'_{w_1 j} \times \\
 & p'_{u_2 i_{u_2}}) \theta_{w_1 u_2} + \sum_{w_2 > u_2}^m (p'_{u_2 i_{u_2}} \times p'_{w_2 j}) \theta_{u_2 w_2} + [\beta(v+1, q_1) \{\beta(v+1, \\
 & q_2) \{\beta(v+1, q_3) \{\delta_{u_1 q_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 0} \times p'_{q_3 0}) + \delta_{u_1 q_2} (p'_{q_1 0} \times p'_{q_2 i_{q_2}} \times \\
 & p'_{q_3 0}) + \delta_{u_1 q_3} (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_1 q_1} - \delta_{u_1 q_2} - \delta_{u_1 q_3}) (p'_{q_1 0} \times
 \end{aligned}$$

$$\begin{aligned}
& p'_{q_2 0} \times p'_{q_3 0} \} + (1 - \beta(v+1, q_3)) \{ \delta_{u_1 q_1} \{ \delta_{u_2 q_3} (p'_{q_1 i_{q_1}} \times p'_{q_2 0} \times \\
& p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_3}) (p'_{q_1 i_{q_1}} \times p'_{q_2 0} \times p'_{q_3 j}) \} + \delta_{u_1 q_2} \{ \delta_{u_2 q_3} (p'_{q_1 0} \times \\
& p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_3}) (p'_{q_1 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) \} + (1 - \delta_{u_1 q_1} - \\
& \delta_{u_1 q_2}) \{ \delta_{u_2 q_3} (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_3}) (p'_{q_1 0} \times p'_{q_2 0} \times \\
& p'_{q_3 j}) \} \} + (1 - \beta(v+1, q_2)) \{ \delta_{u_1 q_1} \{ \delta_{u_2 q_2} (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + \\
& \delta_{u_2 q_3} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_2} - \delta_{u_2 q_3}) (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times \\
& p'_{q_3 j}) \} + (1 - \delta_{u_1 q_1}) \{ \delta_{u_2 q_2} (p'_{q_1 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + \delta_{u_2 q_3} (p'_{q_1 0} \times \\
& p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_2} - \delta_{u_2 q_3}) (p'_{q_1 0} \times p'_{q_2 j} \times p'_{q_3 j}) \} \} + (1 - \beta(v+1, \\
& q_1)) \{ \delta_{u_2 q_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 j}) + \delta_{u_2 q_2} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + \\
& \delta_{u_2 q_3} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_1} - \delta_{u_2 q_2} - \delta_{u_2 q_3}) (p'_{q_1 j} \times \\
& p'_{q_2 j} \times p'_{q_3 j}) \} \} \Phi_1 + [\beta(v+1, q_1) \{ \beta(v+1, q_5) \{ \beta(v+1, q_6) \{ \delta_{u_1 q_1} (p'_{q_1 i_{q_1}} \times \\
& p'_{q_5 0} \times p'_{q_6 0}) + \delta_{u_1 q_5} (p'_{q_1 0} \times p'_{q_5 i_{q_5}} \times p'_{q_6 0}) + \delta_{u_1 q_6} (p'_{q_1 0} \times p'_{q_5 0} \times \\
& p'_{q_6 i_{q_6}}) + (1 - \delta_{u_1 q_1} - \delta_{u_1 q_5} - \delta_{u_1 q_6}) (p'_{q_1 0} \times p'_{q_5 0} \times p'_{q_6 0}) \} + (1 - \\
& \beta(v+1, q_6)) \{ \delta_{u_1 q_1} \{ \delta_{u_2 q_6} (p'_{q_1 i_{q_1}} \times p'_{q_5 0} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_6}) (p'_{q_1 i_{q_1}} \times
\end{aligned}$$

$$\begin{aligned}
& p'_{q_5 0} \times p'_{q_6 j} \} + \delta_{u_1 q_5} \{ \delta_{u_2 q_6} (p'_{q_1 0} \times p'_{q_5 i_{q_5}} \times p'_{q_6 i_{q_6}}) + (1 - \\
& \delta_{u_2 q_6}) (p'_{q_1 0} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) \} + (1 - \delta_{u_1 q_1} - \delta_{u_1 q_5}) \{ \delta_{u_2 q_6} (p'_{q_1 0} \times \\
& p'_{q_5 0} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_6}) (p'_{q_1 0} \times p'_{q_5 0} \times p'_{q_6 j}) \} \} + (1 - \beta(v+1, \\
& q_5)) \{ \delta_{u_1 q_1} \{ \delta_{u_2 q_5} (p'_{q_1 i_{q_1}} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) + \delta_{u_2 q_6} (p'_{q_1 i_{q_1}} \times p'_{q_5 j} \times \\
& p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_5} - \delta_{u_2 q_6}) (p'_{q_1 i_{q_1}} \times p'_{q_5 j} \times p'_{q_6 j}) \} + (1 - \\
& \delta_{u_1 q_1}) \{ \delta_{u_2 q_5} (p'_{q_1 0} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) + \delta_{u_2 q_6} (p'_{q_1 0} \times p'_{q_5 j} \times \\
& p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_5} - \delta_{u_2 q_6}) (p'_{q_1 0} \times p'_{q_5 j} \times p'_{q_6 j}) \} \} + (1 - \beta(v+1, \\
& q_1)) \{ \delta_{u_2 q_1} (p'_{q_1 i_{q_1}} \times p'_{q_5 j} \times p'_{q_6 j}) + \delta_{u_2 q_5} (p'_{q_1 j} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) + \\
& \delta_{u_2 q_6} (p'_{q_1 j} \times p'_{q_5 j} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_1} - \delta_{u_2 q_5} - \delta_{u_2 q_6}) (p'_{q_1 j} \times \\
& p'_{q_5 j} \times p'_{q_6 j}) \} \} \Phi_2 = 0, \quad (2.6.64e)
\end{aligned}$$

where $1 \leq v \leq m-3$, $s_v \leq j \leq s_{v+1}-1$, $1 \leq i_{u_1} \leq s_{u_1}-1$, $v+1 \leq u_2 \leq m$, $j+1 \leq i_{u_2} \leq s_{u_2}-1$. Note that for given v , the equations given by (2.6.64e) are defined only when $s_v < s_{v+1}$.

Now for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_5, q_6$), if one subtracts the equation (2.6.64a) with $v = 1$, $j = 0$ from the equation (2.6.64c) with $v = 1$, $j = 0$, $u = t$, one obtains

$$\begin{aligned}
& (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - \\
& p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} = 0,
\end{aligned} \tag{2.6.65}$$

where $0 \leq i_t \leq s_t - 1$. For any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_5, q_6$), if one subtracts (2.6.64c) with $v = 1$, $j = 0$, $u = r$ from (2.6.64f) with $v = 1$, $j = 0$, $u_1 = r$, $u_2 = t$, one obtains

$$\begin{aligned}
& (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - \\
& p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} + [p'_{ri_r} \times (p'_{ti_t} - p'_{t0})] \theta_{rt} = 0,
\end{aligned} \tag{2.6.66}$$

where $1 \leq r < t$, $1 \leq i_t \leq s_t - 1$, $1 \leq i_r \leq s_r - 1$. Similarly, for any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_5, q_6$), if one subtracts (2.6.64c) with $v = 1$, $j = 0$, $u = r$ from (2.6.64f) with $v = 1$, $j = 0$, $u_1 = t$, $u_2 = r$, one obtains

$$\begin{aligned}
& (p'_{ti_t} - p'_{t0}) \theta_t + \sum_{w_1 < t} [p'_{w_1 0} \times (p'_{ti_t} - p'_{t0})] \theta_{w_1 t} + \sum_{w_2 > t} [(p'_{ti_t} - \\
& p'_{t0}) \times p'_{w_2 0}] \theta_{tw_2} + [(p'_{ti_t} - p'_{t0}) \times p'_{ri_r}] \theta_{tr} = 0,
\end{aligned} \tag{2.6.67}$$

where $t < r \leq m$, $1 \leq i_r \leq s_r - 1$, $1 \leq i_t \leq s_t - 1$. Analogously to the derivations of (2.6.46) and (2.6.48) from (2.6.45), one can obtain from the equations (2.6.65-67),

$$\theta_{rt} = 0, 1 \leq r < t \leq m, \theta_{tr} = 0, 1 \leq t < r \leq m \text{ and } \theta_t = 0, \tag{2.6.68}$$

where $1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_5, q_6$.

Let $q_2 < q_3 < q_5 < q_6 \leq m-1$. For any fixed k_5 , $1 \leq k_5 \leq s_{q_5} - 1$, if one subtracts (2.6.64a) with $v = 1$, $j = 0$ from (2.6.64c) with $v = 1$, $j = 0$, $u = q_5$, $i_{q_5} = k_5$ and if one uses (2.6.68), one obtains

$$\begin{aligned}
& (p'_{q_5 k_5} - p'_{q_5 0}) \theta_{q_5} + [p'_{q_1 0} \times (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_1 q_5} + [p'_{q_2 0} \times \\
& (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_2 q_5} + [p'_{q_3 0} \times (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_3 q_5} + [(p'_{q_5 k_5} - \\
& p'_{q_5 0}) \times p'_{q_6 0}] \theta_{q_5 q_6} + [p'_{q_1 0} \times (p'_{q_5 k_5} - p'_{q_5 0}) \times p'_{q_6 0}] \phi_2 = 0. \quad (2.6.69)
\end{aligned}$$

For any fixed k_2 , k_5 , $1 \leq k_2 \leq s_{q_2} - 1$, $1 \leq k_5 \leq s_{q_5} - 1$, if one subtracts (2.6.64c) with $v = 1$, $j = 0$, $u = q_2$, $i_{q_2} = k_2$ from (2.6.64f) with $v = 1$, $j = 0$, $u_1 = q_2$, $i_{q_2} = k_2$, $u_2 = q_5$, $i_{q_5} = k_5$, one gets on using (2.6.68)

$$\begin{aligned}
& (p'_{q_5 k_5} - p'_{q_5 0}) \theta_{q_5} + [p'_{q_1 0} \times (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_1 q_5} + [p'_{q_2 k_2} \times \\
& (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_2 q_5} + [p'_{q_3 0} \times (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_3 q_5} + [(p'_{q_5 k_5} - \\
& p'_{q_5 0}) \times p'_{q_6 0}] \theta_{q_5 q_6} + [p'_{q_1 0} \times (p'_{q_5 k_5} - p'_{q_5 0}) \times p'_{q_6 0}] \phi_2 = 0. \quad (2.6.70)
\end{aligned}$$

The s_{q_2} equations that (2.6.69) and (2.6.70), when added, gives for any fixed k_5 , $1 \leq k_5 \leq s_{q_5} - 1$,

$$\begin{aligned}
& (p'_{q_5 k_5} - p'_{q_5 0}) \theta_{q_5} + [p'_{q_1 0} \times (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_1 q_5} + [p'_{q_3 0} \times \\
& (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_3 q_5} + [(p'_{q_5 k_5} - p'_{q_5 0}) \times p'_{q_6 0}] \theta_{q_5 q_6} + [p'_{q_1 0} \times \\
& (p'_{q_5 k_5} - p'_{q_5 0}) \times p'_{q_6 0}] \phi_2 = 0. \quad (2.6.71)
\end{aligned}$$

Now, if one subtracts (2.6.71) from (2.6.69) and (2.6.70), one gets

$$[p'_{q_2 k_2} \times (p'_{q_5 k_5} - p'_{q_5 0})] \theta_{q_2 q_5} = 0, \quad (2.6.72)$$

where $0 \leq k_2 \leq s_{q_2} - 1$, $1 \leq k_5 \leq s_{q_5} - 1$. Now (2.6.72) implies that $[P_{q_2} \times P_{q_5}^*] \theta_{q_2 q_5} = 0$, where $P_{q_5}^*$ is an $((s_{q_5} - 1) \times (s_{q_5} - 1))$ matrix with rows $(p_{q_5 k_5}^* - p_{q_5 0}^*)$, $1 \leq k_5 \leq s_{q_5} - 1$. Since P_{q_2} and $P_{q_5}^*$ are both of full column rank, one obtains

$$\theta_{q_2 q_5} = 0. \quad (2.6.73a)$$

Proceeding in a similar manner, one can obtain through the equations (2.6.64a, c, f)

$$\theta_{q_3 q_5} = 0, \theta_{q_2 q_6} = 0, \theta_{q_3 q_6} = 0. \quad (2.6.73b)$$

For any fixed k_2 , $1 \leq k_2 \leq s_{q_2} - 1$, if one subtracts (2.6.64a) with $v = 1$, $j = 0$ from (2.6.64c) with $v = 1$, $j = 0$, $u = q_2$, $i_{q_2} = k_2$, one obtains on using (2.6.68), (2.6.73a) and (2.6.73b)

$$\begin{aligned} & (p_{q_2 k_2}' - p_{q_2 0}') \theta_{q_2} + [p_{q_1 0}' \times (p_{q_2 k_2}' - p_{q_2 0}')] \theta_{q_1 q_2} + [(p_{q_2 k_2}' - p_{q_2 0}') \times p_{q_3 0}'] \theta_{q_2 q_3} \\ & + [p_{q_1 0}' \times (p_{q_2 k_2}' - p_{q_2 0}') \times p_{q_3 0}'] \phi_1 = 0. \end{aligned} \quad (2.6.74)$$

Here, one may note that the equations (2.6.75-80) stated below are obtained on using (2.6.68) and (2.6.73a,b). For any fixed k_2 , $1 \leq k_2 \leq s_{q_2} - 1$, if one subtracts (2.6.64c) with $v = 1$, $j = 0$, $u = q_1$, $i_{q_1} = k_1$ from (2.6.64f) with $v = 1$, $j = 0$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = k_2$, one then obtains

$$\begin{aligned} & (p_{q_2 k_2}' - p_{q_2 0}') \theta_{q_2} + [p_{q_1 k_1}' \times (p_{q_2 k_2}' - p_{q_2 0}')] \theta_{q_1 q_2} + [(p_{q_2 k_2}' - p_{q_2 0}') \times p_{q_3 0}'] \theta_{q_2 q_3} \\ & + [p_{q_1 k_1}' \times (p_{q_2 k_2}' - p_{q_2 0}') \times p_{q_3 0}'] \phi_1 = 0, \end{aligned} \quad (2.6.75a)$$

where $1 \leq k_1 \leq s_{q_1} - 1$, and for any fixed k_2 , $1 \leq k_2 \leq s_{q_2} - 1$, subtracting (2.6.64c) with $v = 1$, $j = 0$, $u = q_3$, $i_{q_3} = k_3$ from (2.6.64f) with $v = 1$, $j = 0$, $u_1 = q_2$, $i_{q_2} = k_2$,

$u_2 = q_3, i_{q_3} = k_3$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 0} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_3}] \theta_{q_2 q_3} + [p'_{q_1 0} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.75b)$$

where $1 \leq k_3 \leq s_{q_3} - 1$. If $s_{v-1} < s_v$, then for any fixed k_2 , $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, if one subtracts (2.6.64c) with $j = k_2$, $u_1 = q_2$, $i_{q_2} = 0$ from (2.6.64a) with $j = k_2$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_2} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_2}] \theta_{q_2 q_3} + [p'_{q_1 k_2} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_2}] \phi_1 = 0, \end{aligned} \quad (2.6.76)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2 , subtracting (2.6.64c) with $j = k_1$, $u = q_2$, $i_{q_2} = 0$ from (2.6.64c) with $j = k_1$, $u = q_2$, $i_{q_2} = k_2$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_1}] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_1}] \phi_1 = 0, \end{aligned} \quad (2.6.77)$$

where $1 \leq v \leq q_1$, $s_{v-1} \leq k_1 \leq s_v - 1$, $1 \leq k_2 \leq s_{q_2} - 1$, $k_1 \neq k_2 \neq 0$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2 , $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, $k_2 + 1 \leq k_1 \leq s_{q_1} - 1$, if one subtracts (2.6.64g) with $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = 0$, $j = k_2$ from (2.6.64c) with $u = q_1$, $i_{q_1} = k_1$, $j = k_2$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_2}] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_2}] \phi_1 = 0, \end{aligned} \quad (2.6.78a)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_2, k_3 , $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq s_v - 1$, $k_2 + 1 \leq k_3 \leq s_{q_3} - 1$, if one subtracts (2.6.64g) with $u_1 = q_2$, $i_{q_2} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$, $j = k_2$ from (2.6.64c) with $u = q_3$, $i_{q_3} = k_3$, $j = k_2$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_2} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_3}] \theta_{q_2 q_3} + [p'_{q_1 k_2} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.78b)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2, k_3 , $1 \leq k_2 \leq k_1 - 1$, $s_{v-1} \leq k_2 \leq s_v - 1$, $k_1 \neq 0$, $k_1 + 1 \leq k_3 \leq s_{q_3} - 1$, if one subtracts (2.6.64g) with $j = k_1$, $u_1 = q_2$, $i_{q_2} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$ from (2.6.64g) with $j = k_1$, $u_1 = q_2$, $i_{q_2} = k_2$, $u_2 = q_3$, $i_{q_3} = k_3$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_3}] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.79a)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2, k_3 , $1 \leq k_2 \leq k_3 - 1$, $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_1 \leq s_{q_1} - 1$, subtracting (2.6.64g) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = 0$ from (2.6.64g) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = k_2$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_3}] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.79b)$$

where $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2, k_3 , $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_1 \leq s_{q_1} - 1$, $k_3 + 1 \leq k_2 \leq s_{q_2} - 1$, if one subtracts (2.6.64g) with $j = k_3$,

$u_1 = q_1, i_{q_1} = k_1, u_2 = q_2, i_{q_2} = 0$ from (2.6.64f) with $j = k_3, u_1 = q_1, i_{q_1} = k_1, u_2 = q_2, i_{q_2} = k_2$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_3}] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.80a)$$

where $1 \leq v' \leq q_1$. Finally, if $s_{v-1} < s_v$, then for any fixed $k_1, k_2, k_3, s_{v-1} \leq k_1 \leq s_v - 1, k_1 \neq 0, k_1 + 1 \leq k_2 \leq s_{q_2} - 1, k_1 + 1 \leq k_3 \leq s_{q_3} - 1$, if one subtracts (2.6.64g) with $j = k_1, u_1 = q_2, i_{q_2} = 0, u_2 = q_3, i_{q_3} = k_3$ from (2.6.64f) with $j = k_1, u_1 = q_2, i_{q_2} = k_2, u_2 = q_3, i_{q_3} = k_3$, one obtains

$$\begin{aligned} & (p'_{q_2 k_2} - p'_{q_2 0}) \theta_{q_2} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0})] \theta_{q_1 q_2} + [(p'_{q_2 k_2} - \\ & p'_{q_2 0}) \times p'_{q_3 k_3}] \theta_{q_2 q_3} + [p'_{q_1 k_1} \times (p'_{q_2 k_2} - p'_{q_2 0}) \times p'_{q_3 k_3}] \phi_1 = 0, \end{aligned} \quad (2.6.80b)$$

where $1 \leq v \leq q_1$. Note that for given v , the equations (2.6.76-80) are defined only when $s_{v-1} < s_v$. From the equations (2.6.74-80), one can obtain, exactly in a similar manner as was done earlier, $\theta^* = 0$.

The proof of the linear independence of the columns of $[\epsilon, L_1, L_2, \dots, L_m, G_{12}, G_{13}, \dots, G_{m-1 m}, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}]$ for all other cases follow analogously to the earlier cases.

Chapter 3

DESIGNS FOR SEARCHING ONE AMONG THE TWO AND THREE-FACTOR INTERACTION EFFECTS IN THE GENERAL SYMMETRIC AND ASYMMETRIC FACTORIAL SET-UP

3.1 Introduction

We define a fractional factorial design as search design for searching one among the two and three-factor interactions if it is capable of estimating the general mean, all the main-effects and permits the detection and estimation of the possibly present at best one among the two and three-factor interactions. This chapter considers search designs of this type.

This search design problem was earlier considered among others by Srivastava (1975, 1977), Srivastava and Gupta (1979), Ghosh (1980, 1981), Gupta and Carvajal (1984a, b, 1987), Ohnishi and Shirakura (1985) and Gupta (1988). All of them considered the case of 2^m factorials.

This chapter deals with search designs for searching one among the two and three-factor interactions for the general asymmetric factorial set-up. As in the previous chapters, to motivate we begin with the general symmetric factorial set-up. In Section 3.3, search designs for the general symmetric factorials have been presented. Section 3.4 of this chapter describes the construction of search designs of the above type in the general asymmetric factorial set-up. A brief discussion has been presented in Section 2.5. Finally, in Section 2.6 the proofs of some of the theorems have been given.

3.2 Notations and Preliminaries

Consider a factorial set-up involving $m(m \geq 3)$ factors $F_1, F_2, \dots, F_m$ at $s_1, s_2, \dots, s_m$ levels respectively ($s_v \geq 2, 1 \leq v \leq m$), where without loss of generality, suppose

$$s_1 \leq s_2 \leq \dots \leq s_m \quad (3.2.1)$$

Suppose prior information is available regarding the absence of all interactions involving four or more factors. Throughout this chapter these interactions will be assumed to be absent. Now, one may observe that if one observation is made for each of the $\pi_{s_v}^m$ level combinations of $F_1, F_2, \dots, F_m$, then with reference to the notations given in the previous chapters and also analogously to (2.2.2), the design matrix will be of the form

$$\begin{bmatrix} 1, Z_1, Z_2, \dots, Z_m, Z_{12}, Z_{13}, \dots, Z_{m-1 m}, Z_{123}, Z_{124}, \dots, \\ Z_{m-2 m-1 m} \end{bmatrix} \quad (3.2.2)$$

where 1 corresponds to the general mean and the columns of $Z_v, Z_{v_1 v_2}, Z_{v_1 v_2 v_3}$ correspond to the main-effect F_v ($1 \leq v \leq m$), the two-factor interaction $F_{v_1} F_{v_2}$ ($1 \leq v_1 < v_2 \leq m$) and the three-factor interaction $F_{v_1} F_{v_2} F_{v_3}$ ($1 \leq v_1 < v_2 < v_3 \leq m$) respectively. Of course (3.2.2) incorporates $Z_{v_1 v_2}$ or $Z_{v_1 v_2 v_3}$ only when the associated interaction $F_{v_1} F_{v_2}$ or $F_{v_1} F_{v_2} F_{v_3}$ is included in the model. Let

$$H_1^{**} = [1, Z_1, Z_2, \dots, Z_m] \text{ and}$$

$$H_2^{**} = [Z_{12}, Z_{13}, \dots, Z_{m-1 m}, Z_{123}, Z_{124}, \dots, Z_{m-2 m-1 m}].$$

Further let S be a subset of N level combinations of $F_1, F_2, \dots, F_m$ with the corresponding sub-matrices of H_1^{**} and H_2^{**} as, say, $T_1^{**} = [e, L_1, L_2, \dots, L_m]$ and $T_2^{**} = [G_{12}, G_{13}, \dots, G_{m-1 m}, V_{123}, V_{124}, \dots, V_{m-2 m-1 m}]$ respectively. From **Theorem 1.2.1** of Chapter 1, one observes that S represents a search design of the present type if and only if the following matrices

$$\text{i) } [T_1^{**}, G_{q_1 q_2}, G_{q_3 q_4}], \text{ where } 1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 \leq m,$$

$$(q_1, q_2) \notin (q_3, q_4),$$

$$\text{ii) } [T_1^{**}, G_{q_1 q_2}, V_{q_3 q_4 q_5}], \text{ where } 1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 < q_5 \leq m \text{ and}$$

- iii) $[T_1^{**}, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}]$, where $1 \leq q_1 < q_2 < q_3 \leq m$, $1 \leq q_4 < q_5 < q_6 \leq m$, $(q_1, q_2, q_3) \neq (q_4, q_5, q_6)$,

have full column rank.

3.3 Main Result in the Symmetric Case

To motivate, we start with the general symmetric factorial set up. This section considers the construction of search designs that allow the estimation of the general mean, all main-effects and permit the search and estimation of possibly present at best one among the two and three-factor interactions. Here we consider $s_1 = s_2 = \dots = s_m = s$, say. The following notations, some of which have already been used in the previous chapters, will be helpful in describing the search designs considered here.

- (i) Define the $(m \times 1)$ vectors $a_j^{(1)}$ ($0 \leq j \leq s-1$) such that in $a_j^{(1)}$, all the elements are j ; let $A^{(1)}$ be an $(m \times s)$ array whose columns are given by $a_j^{(1)}$ for all j .
- (ii) For $0 \leq j \leq s-1$, define the $(m \times 1)$ vectors $a_{j,u,i_u}^{(2)}$ ($1 \leq u \leq m$, $0 \leq i_u \leq s-1$, $i_u \neq j$) such that in $a_{j,u,i_u}^{(2)}$, the u th element is i_u and the remaining elements all equal j ; let $A_j^{(2)}$ be an $(m \times m(s-1))$ array whose columns are $a_{j,u,i_u}^{(2)}$ for all u and i_u .
- (iii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ ($1 \leq u_1 < u_2 \leq m$, $0 \leq \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}) \leq s-1$) such that in $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} respectively and the remaining elements all equal j ; let $A_j^{(3)}$ be an $(m \times [m(m-1)(s-j-1)])$ array whose columns are given by $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(3)}$ is defined only when $s \geq 3$.
- (iv) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ ($1 \leq u_1 < u_2 \leq m$,

$j+1 \leq i_{u_1}, i_{u_2} \leq s-1$) such that in $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} respectively and the remaining elements all equal j ; let $A_j^{(4)}$ be an $(m \times [m(m-1)(s-j-1)^2/2])$ array whose columns are $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(4)}$ is defined only when $s \geq 3$.

(v) Define the $(m \times 1)$ vectors $a_{j,u_1,u_2}^{(5)}$ ($1 \leq j \leq s-1, 1 \leq u_1 \leq u_2 \leq m$) such that in $a_{j,u_1,u_2}^{(5)}$, the u_1 th and u_2 th elements are zero and the remaining elements all equal j ; let $A^{(5)}$ be an $(m \times [m(m-1)(s-1)/2])$ array whose columns are given by $a_{j,u_1,u_2}^{(5)}$ for all j, u_1 and u_2 .

Let

$$N = s + ms(s-1) + m(m-1)(s-1)/2 + m(m-1)s(s-1)(s-2)/6 + \\ m(m-1)(s-1)(s-2)(2s-3)/12$$

and define the following $(m \times N)$ array :

$$Q = [A^{(1)}, A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}, A_1^{(3)}, A_2^{(3)}, \dots, A_{s-1}^{(3)}, A_1^{(4)}, \\ A_2^{(4)}, \dots, A_{s-2}^{(4)}, A^{(5)}] \quad (3.3.1)$$

Then we have the following theorem :

Theorem 3.3.1. The columns of the matrix Q , given in (3.3.1), interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N runs.

The following example will illustrate the plan stated in this section.

Example 3.3.1. Let $m = 7$ and $s = 3$. Then according to **Theorem 3.3.1**, the following level combinations, written as columns, give a search design of the present type in 150 runs while the corresponding regular plan requires a total of 379 runs.

at best one non-zero effect among the two and three-factor interactions and estimating it along with the general mean and all main-effects. The following notations will be helpful in describing the search designs considered here.

Let $s_0 = 0$ and recall (3.2.1). We define :

- (i) For $1 \leq v \leq m-1$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_j^{(1,v)} (s_{v-1} \leq j \leq s_v-1)$ such that in $b_j^{(1,v)}$, the first $(v-1)$ elements equal 0 and the remaining elements all equal j ; let $B^{(1,v)}$ be an $(m \times (s_v - s_{v-1}))$ array whose columns are given by $b_j^{(1,v)}$ for all j . Note that $B^{(1,v)}$ is defined only when $s_{v-1} < s_v$.
- (ii) For $1 \leq v \leq m-2$, if $s_v < s_{v+1}$, then define the $(m \times 1)$ vectors $b_{j,u,i_u}^{(2,v)} (s_v \leq j \leq s_{v+1}-1, 1 \leq u \leq v, 1 \leq i_u \leq s_u-1)$ such that in $b_{j,u,i_u}^{(2,v)}$, the u th element is i_u , the last $(m-v)$ elements all equal j and the remaining elements all equal 0; let $B^{(2,v)}$ be an $(m \times [\sum_{u=1}^v (s_u-1)(s_{v+1} - s_v)])$ array whose columns are given by $b_{j,u,i_u}^{(2,v)}$ for all j, u and i_u . Note that $B^{(2,v)}$ is defined only when $s_v < s_{v+1}$.
- (iii) For $1 \leq v \leq m-2$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u,i_u}^{(3,v)} (s_{v-1} \leq j \leq s_v-1, v \leq u \leq m, 0 \leq i_u \leq s_u-1, i_u \neq j)$ such that in $b_{j,u,i_u}^{(3,v)}$, the first $(v-1)$ elements equal 0, the u th element is i_u and the remaining elements all equal j ; let $B^{(3,v)}$ be an $(m \times [\sum_{u=v}^m (s_v - s_{v-1})(s_u-1)])$ array whose columns are given by $b_{j,u,i_u}^{(3,v)}$ for all j, u and i_u . Note that $B^{(3,v)}$ is defined only when $s_{v-1} < s_v$.
- (iv) If $s_{m-2} < s_{m-1}-1$, then define the $(m \times 1)$ vectors $b_{jk}^{(4)} (s_{m-2} \leq j < s_{m-1}-1, s_{m-2} \leq k \leq s_{m-1}-1, j \neq k)$ such that in $b_{jk}^{(4)}$, the last two elements are j and k and all other elements equal 0; let $B^{(4)}$ be an $(m \times [(s_{m-1} - s_{m-2} - 1)(s_{m-1} - s_{m-2} - 1)])$ array

whose columns are given by $b_{jk}^{(4)}$ for all j and k . Note that $B^{(4)}$ is defined only

when $s_{m-2} < s_{m-1} - 1$.

(v) If $s_{m-2} < s_m$, then define the $(m \times 1)$ vectors $b_k^{(5)}$ ($s_{m-2} \leq k \leq s_{m-1}$, $k \neq s_{m-1} - 1$) such that in $b_k^{(5)}$, the last two elements are $s_{m-1} - 1$ and k and the remaining elements all equal 0; let $B^{(5)}$ be an $(m \times (s_m - s_{m-2} - 1 + \delta_{s_{m-2}s_{m-1}}))$ array, where $\delta_{s_{m-2}s_{m-1}}$ is the Kronecker delta, whose columns are $b_k^{(5)}$ for all k . Note that $B^{(5)}$ is defined only for $s_{m-2} < s_m$.

(vi) For $1 \leq v \leq m-3$, if $s_v < s_{v+1}$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ ($s_v \leq j \leq s_{v+1} - 1$, $1 \leq u_1 \leq v$, $1 \leq i_{u_1} \leq s_{u_1} - 1$, $v+1 \leq u_2 \leq m$, $j+1 \leq i_{u_2} \leq s_{u_2} - 1$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$, the u_1 th element is i_{u_1} , the remaining of the first v elements equal 0, the u_2 th element is i_{u_2} and the remaining elements all equal j ;

let $B^{(8,v)}$ be an $(m \times [\sum_{u_1=1}^v (s_{u_1} - 1)] [\sum_{j=s_v}^{s_{v+1}-1} \sum_{u_2=v+1}^m (s_{u_2} - j - 1)])$ array whose columns

are given by $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(8,v)}$ is defined only when $s_v < s_{v+1}$ and when $m \geq 4$.

(vii) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$ ($s_{v-1} \leq j \leq s_v - 1$, $v \leq u_1 < u_2 \leq m$, $j+1 \leq i_{u_1} \leq s_{u_1} - 1$, $j+1 \leq i_{u_2} \leq s_{u_2} - 1$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$, the first $(v-1)$ elements equal 0, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} and the remaining elements all equal j ; let $B^{(9,v)}$ be an $(m \times$

$[\sum_{u_1=v}^v \sum_{u_2=v}^m (s_{u_1} - j - 1)(s_{u_2} - j - 1)])$ array whose columns are given by

$b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,v)}$ for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(9,v)}$ is defined only when

$s_{v-1} < s_v$ and $m \geq 4$. For $v = 1$, we further define the $(m \times [\sum_{j=1}^{s_1-1} \sum_{u_1=1}^m \sum_{u_2=1}^m (s_{u_1} - u_1 < u_2$

$j - 1)(s_{u_2} - j - 1)])$ array $\bar{B}^{(9,1)}$ which we obtain from $B^{(9,1)}$ after deleting the $(m \times 1)$

vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(9,1)}$ with $j = 0$.

(viii) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$

$(s_{v-1} \leq j \leq s_v-1, j \neq 0, v \leq u_1 < u_2 \leq m, 0 \leq i_{u_1} \leq s_{u_1}-1, 0 \leq i_{u_2} \leq s_{u_2}-1, \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}))$ such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$, the first $(v-1)$ elements equal 0, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} and the remaining elements

all equal j ; let $B^{(10,v)}$ be an $(m \times [\sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m j(s_{u_1} + s_{u_2} - 2j - 2)])$ array whose

columns are $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$ for all j, u_1, u_2, i_{u_1} and i_{u_2} . Note that $B^{(10,v)}$ is defined

only when $s_{v-1} < s_v$ and $m \geq 4$.

(ix) If $s_{m-2} < s_{m-1}$, then define the $(m \times 1)$ vectors $b_{j,k,u,i_u}^{(11)}$ ($s_{m-2} \leq j \leq s_{m-1}-1$,

$s_{m-2} \leq k \leq s_{m-1}, j \neq k, 1 \leq u \leq m-2, 1 \leq i_u \leq s_u - 1$) such that in $b_{j,k,u,i_u}^{(11)}$, the

u th element is i_u , the last two elements are j and k and the remaining elements all

equal 0; let $B^{(11)}$ be an $(m \times [(\sum_{u=1}^{m-2} (s_u - 1)(s_{m-1} - s_{m-2})(s_m - s_{m-2} - 1)])$ array whose

columns are given by $b_{j,k,u,i_u}^{(11)}$ for all j, k, u and i_u . Note that $B^{(11)}$ is defined only

when $s_{m-2} < s_{m-1}$.

(x) For $1 \leq v \leq m-3$, if $s_{v-1} < s_v$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2}^{(12,v)}$ ($s_{v-1} \leq$

$j \leq s_v - 1$, $j \neq 0$, $v \leq u_1 < u_2 \leq m$ such that in $b_{j,u_1,u_2}^{(12,v)}$, the first $(v-1)$ elements,

the u_1 th and u_2 th elements all equal 0 and the remaining elements all equal j ; let

$B^{(12,v)}$ be an $(m \times [((m-v+1)(m-v)/2)(s_v - s_{v-1} - \delta_{1v}))]$ array whose columns are

$b_{j,u_1,u_2}^{(12,v)}$ for all possible j , u_1 and u_2 . Note that $B^{(12,v)}$ is defined only when

$s_{v-1} < s_v$ and $m \geq 4$.

(xi) If $s_{m-3} < s_{m-2}$, then define the $(m \times 1)$ vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(13)}$ ($s_{m-3} \leq j \leq s_{m-2} - 1$, $m-2 \leq u_1 < u_2 \leq m$, $0 \leq i_{u_1} \leq s_{u_1} - 1$, $0 \leq i_{u_2} \leq s_{u_2} - 1$, $\min(i_{u_1}, i_{u_2}) = 0$, $\max(i_{u_1}, i_{u_2}) > j$) such that in $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(13)}$, the first $(m-3)$ elements equal 0, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} and the remaining element equals j ;

let $B^{(13)}$ be an $(m \times [\sum_{j=s_{m-3}}^{s_{m-2}-1} \sum_{u_1=m-2}^m \sum_{u_2=m-2}^m (s_{u_1} + s_{u_2} - 2j - 2)])$ array whose columns

are $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(13)}$ for all possible j , u_1 , u_2 , i_{u_1} and i_{u_2} . Note that $B^{(13)}$ is defined only when $s_{m-3} < s_{m-2}$.

Let

$$N = \sum_{v=1}^{m-1} (s_v - s_{v-1}) + \sum_{v=1}^{m-2} \sum_{u=1}^v (s_u - 1) (s_{v+1} - s_v) + \sum_{v=1}^{m-2} \sum_{u=v}^m (s_v -$$

$$s_{v-1}) (s_u - 1) + (s_{m-1} - s_{m-2} - 1) (s_m - s_{m-2} - 1) + (s_m - s_{m-2} -$$

$$1 + \delta_{s_{m-1} s_{m-2}}) + \sum_{v=1}^{m-3} \sum_{u_1=1}^v (s_{u_1} - 1) \left(\sum_{j=s_v}^{s_{v+1}-1} \sum_{u_2=v+1}^m (s_{u_2} - j -$$

$$1) \right) + \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m (s_{u_1} - j - 1) (s_{u_2} - j - 1) +$$

$j \neq 0 \quad u_1 < u_2$

$$\begin{aligned}
& \sum_{v=1}^{m-3} \sum_{j=s_{v-1}}^{s_v-1} \sum_{u_1=v}^m \sum_{u_2=v}^m j(s_{u_1} + s_{u_2} - 2j - 2) + (s_{m-1} - s_{m-2})(s_m - \\
& \quad u_1 < u_2 \\
& s_{m-2}^{-1}) \sum_{u=1}^{m-2} (s_u - 1) + \sum_{v=1}^{m-3} [(m-v+1)(m-v)(s_v - s_{v-1} - \delta_{1v})/2] + \\
& \sum_{j=s_{m-3}}^{s_{m-2}-1} \sum_{u_1=m-2}^m \sum_{u_2=m-2}^m (s_{u_1} + s_{u_2} - 2j - 2), \\
& \quad u_1 < u_2
\end{aligned}$$

and the following $(m \times N)$ array :

$$\begin{aligned}
Q = [& B^{(1,1)}, \dots, B^{(1,m-1)}, B^{(2,1)}, \dots, B^{(2,m-2)}, B^{(3,1)}, \dots, \\
& B^{(3,m-2)}, B^{(4)}, B^{(5)}, B^{(8,1)}, \dots, B^{(8,m-3)}, \bar{B}^{(9,1)}, \dots, \\
& B^{(9,m-3)}, B^{(10,1)}, \dots, B^{(10,m-3)}, B^{(11)}, B^{(12,1)}, \dots, \\
& B^{(12,m-3)}, B^{(13)}] \quad (3.4.1)
\end{aligned}$$

where in Q any subarray of the form $B^{(u,v)}$ or $B^{(u)}$ is included provided it is defined.

Then the array Q provides search designs of the type considered in this chapter. We have the following theorems in this connection. Theorem 3.4.4 is the main result of this section. Theorems 3.4.1-3.4.3 lead to Theorem 3.4.4 (see e.g. the discussion at the end of Section 3.2).

Theorem 3.4.1. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for which the matrix $[T_1^{**}, G_{q_1 q_2}, G_{q_3 q_4}]$, where $1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 \leq m, (q_1, q_2) \neq (q_3, q_4)$, has full column rank.

Theorem 3.4.2. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for

which the matrix $[T_1^{**}, G_{q_1 q_2}, V_{q_3 q_4 q_5}]$, where $1 \leq q_1 < q_2 \leq m$, $1 \leq q_3 < q_4 < q_5 \leq m$, has full column rank.

Theorem 3.4.3. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for which the matrix $[T_1^{**}, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}]$, where $1 \leq q_1 < q_2 < q_3 \leq m$, $1 \leq q_4 < q_5 < q_6 \leq m$, $(q_1, q_2, q_3) \neq (q_4, q_5, q_6)$, has full column rank.

Theorem 3.4.4. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N runs.

The example, cited below, will illustrate the plans discussed in this section.

Example 3.4.1. Let $m = 7$, $s_1 = 3$, $s_2 = 3$, $s_3 = 3$, $s_4 = 3$, $s_5 = 3$, $s_6 = 4$, $s_7 = 4$. Then the following level combinations, written as columns, give the required search design in 217 runs. The corresponding regular resolution VII plan requires 536 runs.

012	0	1200000000
012	0	0012000000
012	0	0000120000
$B^{(1,1)} = 012$; $B^{(1,6)} = 0$; $B^{(2,5)} = 0000001200$;		
012	0	0000000012
012	3	3333333333
012	3	3333333333
1200000000000000002111111111111012222222222222		
001200000000000011021111111111112201222222222222		
00001200000000001111021111111112222012222222222		
$B^{(3,1)} = 000000120000000011111102111111122222012222222$;		
0000000012000000111111102111111222222201222222		
0000000000123000111111111102311222222222013222		
000000000000123111111111111023222222222222013		

$B^{(8,4)}$, $B^{(9,2)}$, $B^{(9,3)}$, $B^{(9,4)}$, $B^{(10,2)}$, $B^{(10,3)}$, $B^{(10,4)}$, $B^{(11)}$, $B^{(12,2)}$, $B^{(12,3)}$, $B^{(12,4)}$, and $B^{(13)}$ are not defined.

3.5 Discussion and Concluding Remarks

In the literature, methods for the construction of search designs for searching one among the two and three-factor interactions for the 2^m factorial set-up have been proposed by various authors. Although for 2^m factorials, our plans are not the smallest ones (see e.g. Srivastava (1975, 1977), Ghosh (1980, 1981), Gupta and Carvajal (1987) and Ohnishi and Shirakura (1985) for general s^m factorial set-up our plans are the only ones available in the literature. This advantage of our method is accentuated by the fact that apparently there is no obvious way of extending the plans for 2^m factorials, as available in the literature, to the general s^m factorial set-up.

Turning to asymmetric factorials we may mention that here also no plan other than ours is available in the literature. One may note that the plans proposed here are worthwhile when $m \geq 5$ where m is the number of factors. These plans also usually require a small numbers of runs compared to the corresponding regular fractions. These regular plans are, indeed, capable of estimating many more parameters than those considered in this chapter. Anyway, as mentioned earlier, if prior knowledge is available that at most one among two and three-factor interaction is present, then such elaborate estimation is not required and, in such situation, the designs presented in this chapter appear to be economic (vide Table 3.5.1).

TABLE 3.5.1

SOME EXAMPLES ILLUSTRATING THE NUMBER OF RUNS REQUIRED
FOR THE PLANS PRESENTED IN SECTIONS 3.3 AND 3.4
AND THE CORRESPONDING REGULAR RESOLUTION
VII PLAN

Serial no.	Set-up	Number of runs	
		Our plan	Regular plan
1	2^8	46	93
2	2^{10}	67	176
3	$2^4 \times 3^2 \times 5$	93	184
4	2^{11}	79	232
5	$2^8 \times 3^2$	99	277
6	$2^{10} \times 3$	92	288
7	$2^3 \times 3^5$	125	318
8	$2^6 \times 3^2 \times 4$	124	320
9	$2^5 \times 3^4$	122	330
10	$2^7 \times 3^3$	117	342
11	$2^9 \times 3^2$	114	354
12	$2^{11} \times 3$	106	366
13	$2^2 \times 3^6$	145	392
14	$2^5 \times 3^3 \times 4$	148	394
15	$2^6 \times 3^4$	138	418
16	$2^{10} \times 3^2$	130	444
17	2^{14}	121	470
18	$2^7 \times 3^2 \times 5$	165	472
19	$2^3 \times 3^6$	162	492
20	$2^8 \times 3^2 \times 4$	159	508
21	$2^7 \times 3^4$	155	520

(Contd. Table 3.5.1)

Serial no.	Set-up	Number of runs	
		Our plan	Regular plan
22	$2^9 \times 3^3$	150	534
23	$2^{13} \times 3$	137	562
24	$2^8 \times 3^2 \times 5$	185	585
25	$2^4 \times 3^6$	180	607
26	$2^8 \times 3^4$	173	637
27	$2^{12} \times 3^2$	165	667
28	$2^9 \times 3^4$	192	770
29	$2^{13} \times 3^2$	184	802
30	$2^{14} \times 3^3$	207	937
31	2^{18}	191	988
32	$2^7 \times 3^3 \times 4^2$	277	1002
33	$2^{15} \times 3^2$	225	1124
34	$2^7 \times 3^3 \times 4 \times 5$	326	1133
35	$2^{17} \times 3$	211	1142

3.6 Appendix

The proofs of Theorem 3.4.2, 3.4.3 are indicated below. The proof of Theorem 3.4.1 follows similarly. Theorem 3.4.4 follows as an immediate consequence of Theorem 3.4.1 - 3.4.3. Theorem 3.3.1 follows as a special case of Theorem 3.4.4.

Proof of Theorem 3.4.2. Under the notation of Sections 1.3, 2.2 and 3.2, consider the matrix $[T_1^{**}, G_{q_1 q_2}, V_{q_3 q_4 q_5}] = [\epsilon, L_1, L_2, \dots, L_m, G_{q_1 q_2}, V_{q_3 q_4 q_5}]$, where without loss of generality, suppose

$$1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 < q_5 \leq m. \quad (3.6.1)$$

According to the **Theorem 1.2.1**, here it is enough to prove the linear independence of the columns of the above matrix.

We first suppose that q_i 's ($i = 1, 2, \dots, 5$) are all distinct satisfying the conditions stated in (3.6.1). Now consider

$$[\epsilon, L_1, L_2, \dots, L_m, G_{q_1 q_2}, V_{q_3 q_4 q_5}] \theta = 0, \quad (3.6.2)$$

where $\theta = (\theta_0, \theta_1', \dots, \theta_m', \phi_1', \phi_2')$, say. Let p_{vj}' ($1 \leq v \leq m, 0 \leq j \leq s_v - 1$) denote the j th row of P_v . Recalling the definitions of $L_1, L_2, \dots, L_m, G_{q_1 q_2}, V_{q_3 q_4 q_5}$, one gets from (3.4.1) and (3.6.2), for a typical level combination $(i_1, i_2, \dots, i_m)$, the following equation

$$\begin{aligned} \theta_0 + \sum_{w=1}^m p_{wi}' \theta_w + (p_{q_1 i_{q_1}}' \times p_{q_2 i_{q_2}}') \phi_1' + (p_{q_3 i_{q_3}}' \times p_{q_4 i_{q_4}}' \times \\ p_{q_5 i_{q_5}}') \phi_2 = 0, (i_1, i_2, \dots, i_m) \in S(Q), \end{aligned} \quad (3.6.3)$$

where $S(Q)$ is the set of level combinations given by the array Q as stated in (3.4.1). Recalling the structure of Q , it is evident that the system of equations (3.6.3) can be split up into subsystems (3.6.3a), (3.6.3b), $\dots$, (3.6.3k) which arise from the level combinations given by the columns of $[B^{(1,1)}, \dots, B^{(1,m-1)}], [B^{(2,1)}, \dots, B^{(2,m-2)}], \dots, B^{(5)}, [B^{(8,1)}, \dots, B^{(8,m-3)}], \dots, B^{(13)}$ respectively. For example, taking care of the definition of the column vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(8,v)}$ which constitute the columns of $[B^{(8,1)}, \dots, B^{(8,m-3)}]$, the subsystem of equations (3.6.3f) corresponding to the columns of $[B^{(8,1)}, \dots, B^{(8,m-3)}]$ is given by

$$\theta_0 + \sum_{\substack{w=1 \\ w \neq u_1}}^v p_{w0}' \theta_w + p_{u_1 i_{u_1}}' \theta_{u_1} + \sum_{\substack{w=v+1 \\ w \neq u_2}}^m p_{wj}' \theta_w + p_{u_2 i_{u_2}}' \theta_{u_2} + [\beta(v+1,$$

$$\begin{aligned}
& q_1) \{ \beta(v+1, q_2) \{ \delta_{u_1 q_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 0}) + \delta_{u_1 q_2} (p'_{q_1 0} \times p'_{q_2 i_{q_2}}) + \\
& (1 - \delta_{u_1 q_1} - \delta_{u_1 q_2}) (p'_{q_1 0} \times p'_{q_2 0}) \} + (1 - \beta(v+1, q_2)) \{ \delta_{u_1 q_1} \{ \delta_{u_2 q_2} (p'_{q_1 i_{q_1}} \times \\
& p'_{q_2 i_{q_2}}) + (1 - \delta_{u_2 q_2}) (p'_{q_1 i_{q_1}} \times p'_{q_2 j}) \} + (1 - \delta_{u_1 q_1}) \{ \delta_{u_2 q_2} (p'_{q_1 0} \times \\
& p'_{q_2 i_{q_2}}) + (1 - \delta_{u_2 q_2}) (p'_{q_1 0} \times p'_{q_2 j}) \} \} \} + (1 - \beta(v+1, q_1)) \{ \delta_{u_2 q_1} (p'_{q_1 i_{q_1}} \times \\
& p'_{q_2 j}) + \delta_{u_2 q_2} (p'_{q_1 j} \times p'_{q_2 i_{q_2}}) + (1 - \delta_{u_2 q_1} - \delta_{u_2 q_2}) (p'_{q_1 j} \times p'_{q_2 j}) \} \} \phi_1 + \\
& [\beta(v+1, q_3) \{ \beta(v+1, q_4) \{ \beta(v+1, q_5) \{ \delta_{u_1 q_3} (p'_{q_3 i_{q_3}} \times p'_{q_4 0} \times p'_{q_5 0}) + \\
& \delta_{u_1 q_4} (p'_{q_3 0} \times p'_{q_4 i_{q_4}} \times p'_{q_5 0}) + \delta_{u_1 q_5} (p'_{q_3 0} \times p'_{q_4 0} \times p'_{q_5 i_{q_5}}) + (1 - \\
& \delta_{u_1 q_3} - \delta_{u_1 q_4} - \delta_{u_1 q_5}) (p'_{q_3 0} \times p'_{q_4 0} \times p'_{q_5 0}) \} + (1 - \beta(v+1, \\
& q_5)) \{ \delta_{u_1 q_3} \{ \delta_{u_2 q_5} (p'_{q_3 i_{q_3}} \times p'_{q_4 0} \times p'_{q_5 i_{q_5}}) + (1 - \delta_{u_2 q_5}) (p'_{q_3 i_{q_3}} \times \\
& p'_{q_4 0} \times p'_{q_5 j}) \} + \delta_{u_1 q_4} \{ \delta_{u_2 q_5} (p'_{q_3 0} \times p'_{q_4 i_{q_4}} \times p'_{q_5 i_{q_5}}) + (1 - \\
& \delta_{u_2 q_5}) (p'_{q_3 0} \times p'_{q_4 i_{q_4}} \times p'_{q_5 j}) \} + (1 - \delta_{u_1 q_3} - \delta_{u_1 q_4}) \{ \delta_{u_2 q_5} (p'_{q_3 0} \times \\
& p'_{q_4 0} \times p'_{q_5 i_{q_5}}) + (1 - \delta_{u_2 q_5}) (p'_{q_3 0} \times p'_{q_4 0} \times p'_{q_5 j}) \} \} \} + (1 - \beta(v+1, \\
& q_4)) \{ \delta_{u_1 q_3} \{ \delta_{u_2 q_4} (p'_{q_3 i_{q_3}} \times p'_{q_4 i_{q_4}} \times p'_{q_5 j}) + \delta_{u_2 q_5} (p'_{q_3 i_{q_3}} \times
\end{aligned}$$

$$\begin{aligned}
& p'_{q_4 j} \times p'_{q_5 i_{q_5}} + (1 - \delta_{u_2 q_4} - \delta_{u_2 q_5}) (p'_{q_3 i_{q_3}} \times p'_{q_4 j} \times p'_{q_5 j}) + \\
& (1 - \delta_{u_1 q_3}) \{ \delta_{u_2 q_4} (p'_{q_3 0} \times p'_{q_4 i_{q_4}} \times p'_{q_5 j}) + \delta_{u_2 q_5} (p'_{q_3 0} \times p'_{q_4 j} \times \\
& p'_{q_5 i_{q_5}}) + (1 - \delta_{u_2 q_4} - \delta_{u_2 q_5}) (p'_{q_3 0} \times p'_{q_4 j} \times p'_{q_5 j}) \} + (1 - \beta(v+1, \\
& q_3)) \{ \delta_{u_2 q_3} (p'_{q_3 i_{q_3}} \times p'_{q_4 j} \times p'_{q_5 j}) + \delta_{u_2 q_4} (p'_{q_3 j} \times p'_{q_4 i_{q_4}} \times \\
& p'_{q_5 j}) + \delta_{u_2 q_5} (p'_{q_3 j} \times p'_{q_4 j} \times p'_{q_5 i_{q_5}}) + (1 - \delta_{u_2 q_3} - \delta_{u_2 q_4} - \\
& \delta_{u_2 q_5}) (p'_{q_3 j} \times p'_{q_4 j} \times p'_{q_5 j}) \} \phi_2 = 0, \quad (3.6.3f)
\end{aligned}$$

where $1 \leq v \leq m-3$, $s_v \leq j \leq s_{v+1}-1$, $1 \leq u_1 \leq v$, $1 \leq i_{u_1} \leq s_{u_1}-1$, $v+1 \leq u_2 \leq m$, $j+1 \leq i_{u_2} \leq s_{u_2}-1$. Note that for given v , the equations (3.6.3f) are defined only when $s_v < s_{v+1}$.

For any fixed t ($1 \leq t \leq m$, $t \neq q_i$, $i = 1, 2, \dots, 5$), if one subtracts (3.6.3a) with $v = 1$, $j = 0$ from (3.6.3c) with $v = 1$, $j = 0$, $u = t$, one gets

$$(p'_{ti_t} - p'_{t0}) \theta_t = 0, \quad (3.6.4)$$

where $1 \leq i_t \leq s_t - 1$. From Lemma 1.8.1, it follows from (3.6.4) that

$$\theta_t = 0, \quad 1 \leq t \leq m, \quad t \neq q_i, \quad i = 1, 2, \dots, 5. \quad (3.6.5)$$

For any fixed k_3 , $1 \leq k_3 \leq s_{q_3} - 1$, if one subtracts (3.6.3a) with $v = 1$, $j = 0$ from (3.6.3c) with $v = 1$, $j = 0$, $u = q_3$, $i_{q_3} = k_3$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 0} \times p'_{q_5 0}] \phi_2 = 0. \quad (3.6.6)$$

If $s_{v-1} < s_v$, then for any fixed k_3 , $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, if one subtracts (3.6.3j) with $j = k_3$, $u_1 = q_3$, $u_2 = q_5$ from (3.6.3c) with $j = k_3$, $u_1 = q_5$, $i_{q_5} = 0$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_3} \times p'_{q_5 0}] \phi_2 = 0, \quad (3.6.7a)$$

where $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, subtracting (3.6.3j) with $j = k_3$, $u_1 = q_3$, $u_2 = q_4$ from (3.6.3c) with $j = k_3$, $u_1 = q_4$, $i_{q_4} = 0$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 0} \times p'_{q_5 k_3}] \phi_2 = 0, \quad (3.6.7b)$$

where $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , if one subtracts (3.6.3j) with $j = k_4$, $u_1 = q_3$, $u_2 = q_5$ from (3.6.3h) with $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_5$, $i_{q_5} = 0$, $j = k_4$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 0}] \phi_2 = 0, \quad (3.6.8a)$$

where $s_{v-1} \leq k_4 \leq s_v - 1$, $k_4 \neq 0$, $k_4 + 1 \leq k_3 \leq s_{q_3} - 1$, $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_5 , subtracting (3.6.3j) with $j = k_5$, $u_1 = q_3$, $u_2 = q_4$ from (3.6.3h) with $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_4$, $i_{q_4} = 0$, $j = k_5$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 0} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.8b)$$

where $s_{v-1} \leq k_5 \leq s_v - 1$, $k_5 \neq 0$, $k_5 + 1 \leq k_3 \leq s_{q_3} - 1$, $1 \leq v \leq q_3$. Now suppose $q_1 < q_2 < q_3 < q_4 < q_5 \leq m-1$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , if one subtracts (3.6.3c) with $j = 0$, $u = q_4$, $i_{q_4} = k_4$ from (3.6.3h) with $u_1 = q_4$, $i_{q_4} = k_4$, $u_2 = q_5$, $i_{q_5} = 0$, $j = k_3$, one gets

$$\begin{aligned}
& (p'_{q_1 k_3} - p'_{q_1 0}) \theta_{q_1} + (p'_{q_2 k_3} - p'_{q_2 0}) \theta_{q_2} + (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + \\
& [p'_{q_1 k_3} \times p'_{q_2 k_3} - p'_{q_1 0} \times p'_{q_2 0}] \phi_1 + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times \\
& p'_{q_5 0}] \phi_2 = 0, \quad (3.6.9a)
\end{aligned}$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_4 \leq s_{q_4} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , subtracting (3.6.3c) with $j = 0$, $u = q_4$, $i_{q_4} = k_4$ from (3.6.3h) with $u_1 = q_4$, $i_{q_4} = k_4$, $u_2 = q_5$, $i_{q_5} = 0$, $j = k_3$, one gets

$$\begin{aligned}
& (p'_{q_2 k_3} - p'_{q_2 0}) \theta_{q_2} + (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 0} \times p'_{q_2 k_3} - \\
& p'_{q_1 0} \times p'_{q_2 0}] \phi_1 + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 0}] \phi_2 = 0, \quad (3.6.9b)
\end{aligned}$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 + 1 \leq k_4 \leq s_{q_4} - 1$, $q_1 + 1 \leq v \leq q_2$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , if one subtracts (3.6.3c) with $j = 0$, $u = q_4$, $i_{q_4} = k_4$ from (3.6.3h) with $u_1 = q_4$, $i_{q_4} = k_4$, $u_2 = q_5$, $i_{q_5} = 0$, $j = k_3$, one obtains

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 0}] \phi_2 = 0, \quad (3.6.9c)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 + 1 \leq k_4 \leq s_{q_4} - 1$, $q_2 + 1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_5 , subtracting (3.6.3c) with $j = 0$, $u = q_5$, $i_{q_5} = k_5$ from (3.6.3h) with $u_1 = q_4$, $i_{q_4} = 0$, $u_2 = q_5$, $i_{q_5} = k_5$, $j = k_3$, one gets

$$\begin{aligned}
& (p'_{q_1 k_3} - p'_{q_1 0}) \theta_{q_1} + (p'_{q_2 k_3} - p'_{q_2 0}) \theta_{q_2} + (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + \\
& [p'_{q_1 k_3} \times p'_{q_2 k_3} - p'_{q_1 0} \times p'_{q_2 0}] \phi_1 + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 0} \times \\
& p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.10a)
\end{aligned}$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_5 \leq s_{q_5} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_5 , if one subtracts (3.6.3c) with $j = 0$, $u = q_5$, $i_{q_5} = k_5$ from (3.6.3h) with $u_1 = q_4$, $i_{q_4} = 0$, $u_2 = q_5$, $i_{q_5} = k_5$, $j = k_3$, one obtains

$$\begin{aligned} & (p'_{q_2 k_3} - p'_{q_2 0}) \theta_{q_2} + (p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [p'_{q_1 0} \times p'_{q_2 k_3} - \\ & p'_{q_1 0} \times p'_{q_2 0}] \phi_1 + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 0} \times p'_{q_5 k_5}] \phi_2 = 0, \end{aligned} \quad (3.6.10b)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 + 1 \leq k_5 \leq s_{q_5} - 1$, $q_1 + 1 \leq v \leq q_2$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_5 , if one subtracts (3.6.3c) with $j = 0$, $u = q_5$, $i_{q_5} = k_5$ from (3.6.3h) with $u_1 = q_4$, $i_{q_4} = 0$, $u_2 = q_5$, $i_{q_5} = k_5$, $j = k_3$, one obtains

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 0} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.10c)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 + 1 \leq k_5 \leq s_{q_5} - 1$, $q_2 + 1 \leq v \leq q_3$. Here one may note that the equations given by (3.6.9a,b,c) and (3.6.10a,b,c) are obtained on using (3.6.5). If $s_{v-1} < s_v$, then for any fixed k_3 , $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, subtracting (3.6.3c) with $j = k_3$, $u = q_3$, $i_{q_3} = 0$ from (3.6.3a) with $j = k_3$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_3} \times p'_{q_5 k_3}] \phi_2 = 0, \quad (3.6.11)$$

where $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , subtracting (3.6.3c) with $j = k_4$, $u = q_3$, $i_{q_3} = 0$ from (3.6.3c) with $j = k_4$, $u = q_3$, $i_{q_3} = k_3$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_4}] \phi_2 = 0, \quad (3.6.12)$$

where $s_{v-1} \leq k_4 \leq s_v - 1$, $k_4 \neq 0$, $1 \leq k_3 \leq s_{q_3} - 1$, $k_4 \neq k_3$, $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , subtracting (3.6.3h) with $j = k_3$, $u_1 = q_3$, $i_{q_3} = 0$,

$u_2 = q_4$, $i_{q_4} = k_4$ from (3.6.3c) with $j = k_3$, $u = q_4$, $i_{q_4} = k_4$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_3}] \phi_2 = 0, \quad (3.6.13a)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_4 \leq s_{q_4} - 1$, $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_5 , if one subtracts (3.6.3h) with $j = k_3$, $u_1 = q_3$, $i_{q_3} = 0$, $u_2 = q_5$, $i_{q_5} = k_5$ from (3.6.3c) with $j = k_3$, $u = q_5$, $i_{q_5} = k_5$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_3} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.13b)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_5 \leq s_{q_5} - 1$, $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , k_5 , if one subtracts (3.6.3h) with $j = k_4$, $u_1 = q_3$, $i_{q_3} = 0$, $u_2 = q_5$, $i_{q_5} = k_5$ from (3.6.3h) with $j = k_4$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_5$, $i_{q_5} = k_5$, one obtains

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.14a)$$

where $s_{v-1} \leq k_4 \leq s_v - 1$, $1 \leq k_3 \leq k_4 - 1$, $k_4 + 1 \leq k_5 \leq s_{q_5} - 1$, $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , k_5 , subtracting (3.6.3h) with $j = k_5$, $u_1 = q_3$, $i_{q_3} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$ from (3.6.3h) with $j = k_5$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_4$, $i_{q_4} = k_4$, one obtains

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.14b)$$

where $s_{v-1} \leq k_5 \leq s_v - 1$, $1 \leq k_3 \leq k_5 - 1$, $k_5 + 1 \leq k_4 \leq s_{q_4} - 1$, $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , k_5 , if one subtracts (3.6.3h) with $j = k_4$, $u_1 = q_3$, $i_{q_3} = 0$, $u_2 = q_5$, $i_{q_5} = k_5$ from (3.6.3g) with $j = k_4$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_5$, $i_{q_5} = k_5$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.15a)$$

where $s_{v-1} \leq k_4 \leq s_v - 1$, $k_4 \neq 0$, $k_4 + 1 \leq k_3 \leq s_{q_3} - 1$, $k_4 + 1 \leq k_5 \leq s_{q_5} - 1$, $1 \leq v \leq q_3$. If $s_{v-1} < s_v$, then for any fixed k_3 , k_4 , k_5 , if one subtracts (3.6.3h) with $j = k_5$, $u_1 = q_3$, $i_{q_3} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$ from (3.6.3g) with $j = k_5$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_4$, $i_{q_4} = k_4$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.15b)$$

where $s_{v-1} \leq k_5 \leq s_v - 1$, $k_5 \neq 0$, $k_5 + 1 \leq k_3 \leq s_{q_3} - 1$, $k_5 + 1 \leq k_4 \leq s_{q_4} - 1$, $1 \leq v \leq q_3$. Note that for given v , the equations (3.6.7-15) are defined only when $s_{v-1} < s_v$. For any fixed k_3 , k_4 , subtracting (3.6.3a) with $j = k_4$ for $s_{v-1} \leq k_4 \leq s_v - 1$, $q_3 + 1 \leq v \leq q_4$ from (3.6.3b) with $j = k_4$, $u = q_3$, $i_{q_3} = k_3$ for $1 \leq k_3 \leq s_{q_3} - 1$, $s_v \leq k_4 \leq s_{v+1} - 1$, $q_3 \leq v \leq q_4 - 1$, one gets

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_4}] \phi_2 = 0. \quad (3.6.16a)$$

For any fixed k_3 , k_4 , k_5 , if one subtracts (3.6.3c) with $j = k_4$, $u = q_5$, $i_{q_5} = k_5$ for $s_{v-1} \leq k_4 \leq s_v - 1$, $k_4 + 1 \leq k_5 \leq s_{q_5} - 1$, $q_3 + 1 \leq v \leq q_4$ from (3.6.3f) with $j = k_4$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_5$, $i_{q_5} = k_5$ for $s_v \leq k_4 \leq s_{v+1} - 1$, $k_4 + 1 \leq k_5 \leq s_{q_5} - 1$, $q_3 \leq v \leq q_4 - 1$, $1 \leq k_3 \leq s_{q_3} - 1$, one obtains

$$(p'_{q_3 k_3} - p'_{q_3 0}) \theta_{q_3} + [(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_5}] \phi_2 = 0. \quad (3.6.16b)$$

Finally, for any fixed k_3 , k_4 , k_5 , if one subtracts (3.6.3c) with $j = k_5$, $u = q_4$, $i_{q_4} = k_4$ for $s_{v-1} \leq k_5 \leq s_v - 1$, $k_5 + 1 \leq k_4 \leq s_{q_4} - 1$, $q_3 + 1 \leq v \leq q_4$ from (3.6.3f) with $j = k_5$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_4$, $i_{q_4} = k_4$ for $s_v \leq k_5 \leq s_{v+1} - 1$, $k_5 + 1 \leq k_4 \leq s_{q_4} - 1$, $q_3 \leq v \leq q_4 - 1$, $1 \leq k_3 \leq s_{q_3} - 1$, one gets

$$(\mathbf{p}'_{q_3 k_3} - \mathbf{p}'_{q_3 0}) \theta_{q_3} + [(\mathbf{p}'_{q_3 k_3} - \mathbf{p}'_{q_3 0}) \times \mathbf{p}'_{q_4 k_4} \times \mathbf{p}'_{q_5 k_5}] \phi_2 = 0. \quad (3.6.16c)$$

Note that for given v , $q_3+1 \leq v \leq q_4$, the equations (3.6.16a,b,c) are defined only when $s_{v-1} < s_v$. For any fixed k_3 , summing the s_{q_5} equations that (3.6.7a), (3.6.11), (3.6.13b) and (3.6.15b), one obtains $(\mathbf{p}'_{q_3 k_3} - \mathbf{p}'_{q_3 0}) \theta_{q_3} = 0$, $1 \leq k_3 \leq s_{q_3} - 1$. Then from Lemma 1.8.1, it follows that

$$\theta_{q_3} = 0. \quad (3.6.17)$$

Here one may note that the equations (3.6.18-20), which are stated below, are obtained on using (3.6.17). For any fixed k_3, k_4 , $1 \leq k_3 \leq s_{q_1} - 1$, $k_3+1 \leq k_4 \leq s_{q_4} - 1$, summing the s_{q_5} equations that (3.6.9a), (3.6.12), (3.6.13a), (3.6.14a,b), (3.6.15b) and (3.6.16a, b,c) appropriately, one obtains

$$(\mathbf{p}'_{q_1 k_3} - \mathbf{p}'_{q_1 0}) \theta_{q_1} + (\mathbf{p}'_{q_2 k_3} - \mathbf{p}'_{q_2 0}) \theta_{q_2} + [\mathbf{p}'_{q_1 k_3} \times \mathbf{p}'_{q_2 k_3} - \mathbf{p}'_{q_1 0} \times \mathbf{p}'_{q_2 0}] \phi_1 = 0. \quad (3.6.18)$$

If one subtracts (3.6.18) from (3.6.9a), one obtains, for any fixed k_3, k_4 ,

$$[(\mathbf{p}'_{q_3 k_3} - \mathbf{p}'_{q_3 0}) \times \mathbf{p}'_{q_4 k_4} \times \mathbf{p}'_{q_5 0}] \phi_2 = 0, \quad (3.6.19a)$$

where $1 \leq k_3 \leq s_{q_1} - 1$, $k_3+1 \leq k_4 \leq s_{q_4} - 1$. Again for any fixed k_3, k_4 , one obtains from the equations (3.6.9b), (3.6.12), (3.6.13a), (3.6.14a,b), (3.6.15b) and (3.6.16a,b,c),

$$[(\mathbf{p}'_{q_3 k_3} - \mathbf{p}'_{q_3 0}) \times \mathbf{p}'_{q_4 k_4} \times \mathbf{p}'_{q_5 0}] \phi_2 = 0, \quad (3.6.19b)$$

where $s_{q_1} \leq k_3 \leq s_{q_2} - 1$, $k_3+1 \leq k_4 \leq s_{q_4} - 1$. Similarly, for any k_3, k_5 , one obtains from the equations (3.6.10a,b), (3.6.12), (3.6.13b), (3.6.14a,b), (3.6.15a) and (3.6.16a,b,c),

$$[(\mathbf{p}'_{q_3 k_3} - \mathbf{p}'_{q_3 0}) \times \mathbf{p}'_{q_4 0} \times \mathbf{p}'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.20)$$

where $1 \leq k_3 \leq s_{q_2} - 1$, $k_3 + 1 \leq k_5 \leq s_{q_5} - 1$. Now, combining the equations (3.6.6), (3.6.7a,b), (3.6.8a,b), (3.6.9c), (3.6.10c), (3.6.19), (3.6.20) and using (3.6.17), one obtains for any fixed k_3 , k_4 and k_5 ,

$$[(p'_{q_3 k_3} - p'_{q_3 0}) \times p'_{q_4 k_4} \times p'_{q_5 k_5}] \phi_2 = 0, \quad (3.6.21)$$

where $1 \leq k_3 \leq s_{q_3} - 1$, $0 \leq k_4 \leq s_{q_4} - 1$, $0 \leq k_5 \leq s_{q_5} - 1$. Now (3.6.21) implies that $[P'_{q_3} \times P'_{q_4} \times P'_{q_5}] \phi_2 = 0$, where P'_{q_3} is an $((s_{q_3} - 1) \times (s_{q_3} - 1))$ matrix with rows $(p'_{q_3 k_3} - p'_{q_3 0})$, $1 \leq k_3 \leq s_{q_3} - 1$. Since P'_{q_3} , P'_{q_4} and P'_{q_5} are of full column rank, one gets

$$\phi_2 = 0. \quad (3.6.22)$$

Then it is easy to prove, from (3.6.3a) and (3.6.3c), that

$$\theta_{q_1} = 0, \theta_{q_2} = 0, \theta_{q_4} = 0, \theta_{q_5} = 0, \phi_1 = 0, \theta_0 = 0$$

and hence $\theta = 0$. The proof of $\theta = 0$, for all other cases, follows exactly along the same line from the equations (3.6.3a-k).

Proof of Theorem 3.4.3 : Under the notation of Section 1.3., 2.2 and 3.2, consider

the matrix $[T_1^{**}, v_{q_1 q_2 q_3}, v_{q_4 q_5 q_6}] = [\epsilon, L_1, L_2, \dots, L_m, v_{q_1 q_2 q_3}, v_{q_4 q_5 q_6}]$,

where without loss of generality, suppose

$$1 \leq q_1 < q_2 < q_3 \leq m, 1 \leq q_4 < q_5 < q_6 \leq m, q_3 \leq q_6, (q_1, q_2, q_3) \neq (q_4, q_5, q_6). \quad (3.6.23)$$

According to the Theorem 1.2.1, here it is enough to prove the linear independence of the columns of the above matrix. First suppose that q_i 's ($i = 1, 2, \dots, 6$) are all distinct satisfying the conditions stated in (3.6.23) and also $q_3 < q_4$. Consider

$$[\epsilon, L_1, L_2, \dots, L_m, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}] \theta = 0, \quad (3.6.24)$$

where θ is as defined earlier. Recalling the definitions of $L_1, L_2, \dots, L_m, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}$, one gets from (3.4.1) and (3.6.24), for a typical level combination $(i_1, i_2, \dots, i_m)$, the following equation

$$\begin{aligned} \theta_0 + \sum_{w=1}^m p'_{wi_w} \theta_w + (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) \phi_1 + (p'_{q_4 i_{q_4}} \times p'_{q_5 i_{q_5}} \times \\ p'_{q_6 i_{q_6}}) \phi_2 = 0, (i_1, i_2, \dots, i_m) \in S(Q), \end{aligned} \quad (3.6.25)$$

where $S(Q)$ is the set of level combinations given by the array Q as stated in (3.4.1). Following the structure of Q , it is evident that the system of equations (3.6.25) can be split up into subsystems (3.6.25a), (3.6.25b), $\dots$, (3.6.25k) which arise from the level combinations given by the columns of $[B^{(1,1)}, \dots, B^{(1,m-1)}], [B^{(2,1)}, \dots, B^{(2,m-2)}], \dots, B^{(5)}, [B^{(8,1)}, \dots, B^{(8,m-3)}], \dots, B^{(13)}$ respectively. For example, taking care of the definition of the column vectors $b_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(10,v)}$ which constitute the columns of $[B^{(10,1)}, B^{(10,2)}, \dots, B^{(10,m-3)}]$, the subsystem of equations (3.6.25h) corresponding to the columns of $[B^{(10,1)}, B^{(10,2)}, \dots, B^{(10,m-3)}]$ is given by

$$\begin{aligned} \theta_0 + \sum_{w=1}^{v-1} p'_{w0} \theta_w + \sum_{\substack{w=v \\ w \neq u_1, u_2}}^m p'_{wj} \theta_w + p'_{u_1 i_{u_1}} \theta_{u_1} + p'_{u_2 i_{u_2}} \theta_{u_2} + \{\beta(v, \\ q_1) \{\beta(v, q_2) \{\beta(v, q_3) (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 0}) + (1 - \beta(v, q_3)) \{\delta_{u_1 q_3} (p'_{q_1 0} \times \\ p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + \delta_{u_2 q_3} (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_1 q_3} - \\ \delta_{u_2 q_3}) (p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 j})\}\} + (1 - \beta(v, q_2)) \{\delta_{u_1 q_2} \{\delta_{u_2 q_3} (p'_{q_1 0} \times \end{aligned}$$

$$\begin{aligned}
& p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_3}) (p'_{q_1 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) \} + \delta_{u_1 q_3} (p'_{q_1 0} \times \\
& p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_1 q_2} - \delta_{u_1 q_3}) \{ \delta_{u_2 q_2} (p'_{q_1 0} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + \\
& \delta_{u_2 q_3} (p'_{q_1 0} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_2} - \delta_{u_2 q_3}) (p'_{q_1 0} \times p'_{q_2 j} \times \\
& p'_{q_3 j}) \} \} + (1 - \beta(v, q_1)) \{ \delta_{u_1 q_1} \{ \delta_{u_2 q_2} (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j}) + \\
& \delta_{u_2 q_3} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_2} - \delta_{u_2 q_3}) (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times \\
& p'_{q_3 j}) \} + \delta_{u_1 q_2} \{ \delta_{u_2 q_3} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_3}) (p'_{q_1 j} \times \\
& p'_{q_2 i_{q_2}} \times p'_{q_3 j}) \} + \delta_{u_1 q_3} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_1 q_1} - \delta_{u_1 q_2} - \\
& \delta_{u_1 q_3}) \{ \delta_{u_2 q_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 j}) + \delta_{u_2 q_2} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times \\
& p'_{q_3 j}) + \delta_{u_2 q_3} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}}) + (1 - \delta_{u_2 q_1} - \delta_{u_2 q_2} - \delta_{u_2 q_3}) (p'_{q_1 j} \times \\
& p'_{q_2 j} \times p'_{q_3 j}) \} \} \phi_1 + [\beta(v, q_4) \{ \beta(v, q_5) \{ \beta(v, q_6) (p'_{q_4 0} \times p'_{q_5 0} \times p'_{q_6 0}) + \\
& (1 - \beta(v, q_6)) \{ \delta_{u_1 q_6} (p'_{q_4 0} \times p'_{q_5 0} \times p'_{q_6 i_{q_6}}) + \delta_{u_2 q_6} (p'_{q_4 0} \times p'_{q_5 0} \times \\
& p'_{q_6 i_{q_6}}) + (1 - \delta_{u_1 q_6} - \delta_{u_2 q_6}) (p'_{q_4 0} \times p'_{q_5 0} \times p'_{q_6 j}) \} \} + (1 - \beta(v, \\
& q_5)) \{ \delta_{u_1 q_5} \{ \delta_{u_2 q_6} (p'_{q_4 0} \times p'_{q_5 i_{q_5}} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_6}) (p'_{q_4 0} \times p'_{q_5 i_{q_5}} \times
\end{aligned}$$

$$\begin{aligned}
& p'_{q_6 j} \} + \delta_{u_1 q_6} (p'_{q_4 0} \times p'_{q_5 j} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_1 q_5} - \\
& \delta_{u_1 q_6}) \{ \delta_{u_2 q_5} (p'_{q_4 0} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) + \delta_{u_2 q_6} (p'_{q_4 0} \times p'_{q_5 j} \times \\
& p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_5} - \delta_{u_2 q_6}) (p'_{q_4 0} \times p'_{q_5 j} \times p'_{q_6 j}) \} + (1 - \beta(v, \\
& q_4)) \{ \delta_{u_1 q_4} \{ \delta_{u_2 q_5} (p'_{q_4 i_{q_4}} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) + \delta_{u_2 q_6} (p'_{q_4 i_{q_4}} \times p'_{q_5 j} \times \\
& p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_5} - \delta_{u_2 q_6}) (p'_{q_4 i_{q_4}} \times p'_{q_5 j} \times p'_{q_6 j}) \} + \delta_{u_1 q_5} \{ \delta_{u_2 q_6} \\
& (p'_{q_4 j} \times p'_{q_5 i_{q_5}} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_6}) (p'_{q_4 j} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) \} + \\
& \delta_{u_1 q_6} (p'_{q_4 j} \times p'_{q_5 j} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_1 q_4} - \delta_{u_1 q_5} - \delta_{u_1 q_6}) \{ \delta_{u_2 q_4} \\
& (p'_{q_4 i_{q_4}} \times p'_{q_5 j} \times p'_{q_6 j}) + \delta_{u_2 q_5} (p'_{q_4 j} \times p'_{q_5 i_{q_5}} \times p'_{q_6 j}) + \delta_{u_2 q_6} (p'_{q_4 j} \times \\
& p'_{q_5 j} \times p'_{q_6 i_{q_6}}) + (1 - \delta_{u_2 q_4} - \delta_{u_2 q_5} - \delta_{u_2 q_6}) (p'_{q_4 j} \times p'_{q_5 j} \times \\
& p'_{q_6 j}) \} \} \Phi_2 = 0, \quad (3.6.25h)
\end{aligned}$$

where $1 \leq v \leq m-3$, $s_{v-1} \leq j \leq s_v-1$, $j \neq 0$, $v \leq u_1 < u_2 \leq m$, $0 \leq i_{u_1} \leq s_{u_1}-1$, $0 \leq i_{u_2} \leq s_{u_2}-1$, $\min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2})$.

For any fixed t ($1 \leq t \leq m$, $t \neq q_i$, $i = 1, 2, \dots, 6$), if one subtracts (3.6.25a) with $v = 1$, $j = 0$ from (3.6.25c) with $v = 1$, $j = 0$, $u = t$, one gets

$$(p'_{ti_t} - p'_{t0}) \theta_t = 0, \quad (3.6.26)$$

where $1 \leq i_t \leq s_t - 1$. From Lemma 1.8.1 and from (3.6.26), it follows that

$$\theta_t = 0, \quad 1 \leq t \leq m, \quad t \neq q_i, \quad i = 1, 2, \dots, 6. \quad (3.6.27)$$

For any fixed k_1 , $1 \leq k_1 \leq s_{q_1} - 1$, if one subtracts (3.6.25a) with $v = 1$, $j = 0$ from (3.6.25c) with $v = 1$, $j = 0$, $u = q_1$, $i_{q_1} = k_1$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 0}] \phi_1 = 0. \quad (3.6.28)$$

If $s_{v-1} < s_v$, then for any fixed k_1 , if one subtracts (3.6.25j) with $j = k_1$, $u_1 = q_1$, $u_2 = q_2$ from (3.6.25c) with $j = k_1$, $u = q_2$, $i_{q_2} = 0$, one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 k_1}] \phi_1 = 0, \quad (3.6.29a)$$

where $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1 , subtracting (3.6.25j) with $j = k_1$, $u_1 = q_1$, $u_2 = q_3$ from (3.6.25c) with $j = k_1$, $u = q_3$, $i_{q_3} = 0$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_1} \times p'_{q_3 0}] \phi_1 = 0, \quad (3.6.29b)$$

where $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1 , k_3 , if one subtracts (3.6.25j) with $j = k_3$, $u_1 = q_1$, $u_2 = q_2$ from (3.6.25h) with $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = 0$, $j = k_3$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (3.6.30a)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_1 \leq s_{q_1} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1 , k_2 , subtracting (3.6.25j) with $j = k_2$, $u_1 = q_1$, $u_2 = q_3$ from (3.6.25h) with $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = 0$, $j = k_2$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 0}] \phi_1 = 0, \quad (3.6.30b)$$

where $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, $k_2 + 1 \leq k_1 \leq s_{q_1} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_3 , if one subtracts (3.6.25c) with $j = 0$, $u = q_3$, $i_{q_3} = k_3$ from (3.6.25h) with $u_1 = q_2$, $i_{q_2} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$, $j = k_1$, one gets on using (3.6.27)

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + (p'_{q_4 k_1} - p'_{q_4 0}) \theta_{q_4} + (p'_{q_5 k_1} - p'_{q_5 0}) \theta_{q_5} + (p'_{q_6 k_1} - \\ & p'_{q_6 0}) \theta_{q_6} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 k_3}] \phi_1 + [p'_{q_4 k_1} \times p'_{q_5 k_1} \times \\ & p'_{q_6 k_1} - p'_{q_4 0} \times p'_{q_5 0} \times p'_{q_6 0}] \phi_2 = 0, \end{aligned} \quad (3.6.31a)$$

where $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $k_1 + 1 \leq k_3 \leq s_{q_3} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2 , subtracting (3.6.25c) with $j = 0$, $u = q_2$, $i_{q_2} = k_2$ from (3.6.25h) with $u_1 = q_2$, $i_{q_2} = k_2$, $u_2 = q_3$, $i_{q_3} = 0$, $j = k_1$, one gets on using (3.6.27)

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + (p'_{q_4 k_1} - p'_{q_4 0}) \theta_{q_4} + (p'_{q_5 k_1} - p'_{q_5 0}) \theta_{q_5} + (p'_{q_6 k_1} - \\ & p'_{q_6 0}) \theta_{q_6} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 0}] \phi_1 + [p'_{q_4 k_1} \times p'_{q_5 k_1} \times \\ & p'_{q_6 k_1} - p'_{q_4 0} \times p'_{q_5 0} \times p'_{q_6 0}] \phi_2 = 0, \end{aligned} \quad (3.6.31b)$$

where $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $k_1 + 1 \leq k_2 \leq s_{q_2} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1 , if one subtracts (3.6.25c) with $j = k_1$, $u = q_1$, $i_{q_1} = 0$ from (3.6.25c) with $j = k_1$, $u = q_1$, $i_{q_1} = k_1$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_1} \times p'_{q_3 k_1}] \phi_1 = 0, \quad (3.6.32)$$

where $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2 , if one subtracts (3.6.25c) with $j = k_2$, $u = q_1$, $i_{q_1} = 0$ from (3.6.25c) with $j = k_2$, $u = q_1$, $i_{q_1} = k_1$, one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_2}] \phi_1 = 0, \quad (3.6.33)$$

where $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, $1 \leq v \leq q_1$, $1 \leq k_1 \leq s_{q_1} - 1$, $k_1 \neq k_2$. If $s_{v-1} < s_v$, then for any fixed k_1, k_2 , subtracting (3.6.25h) with $j = k_1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$ from (3.6.25c) with $j = k_1$, $u = q_2$, $i_{q_2} = k_2$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_1}] \phi_1 = 0, \quad (3.6.34a)$$

where $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $k_1 + 1 \leq k_2 \leq s_{q_2} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1, k_3 , if one subtracts (3.6.25h) with $j = k_1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$ from (3.6.25c) with $j = k_1$, $u = q_3$, $i_{q_3} = k_3$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_1} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (3.6.34b)$$

where $s_{v-1} \leq k_1 \leq s_v - 1$, $k_1 \neq 0$, $k_1 + 1 \leq k_3 \leq s_{q_3} - 1$, $1 \leq v \leq q_1$. Note that for given v , the equations (3.6.29-34) are defined only when $s_{v-1} < s_v$. For any fixed k_1, k_2 , subtracting (3.6.25a) with $j = k_2$ for $s_{v-1} \leq k_2 \leq s_v - 1$, $q_1 + 1 \leq v \leq q_2$ from (3.6.25b) with $j = k_2$, $u = q_1$, $i_{q_1} = k_1$ for $1 \leq k_1 \leq s_{q_1} - 1$, $s_v \leq k_2 \leq s_{v+1} - 1$, $q_1 \leq v \leq q_2 - 1$, one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_2}] \phi_2 = 0. \quad (3.6.35)$$

Note that for given v , $q_1 + 1 \leq v \leq q_2$, the equations given by (3.6.35) are defined only when $s_{v-1} < s_v$. If $s_{v-1} < s_v$ then for any fixed k_1, k_2, k_3 , if one subtracts

(3.6.25h) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$ from (3.6.25h) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = k_3$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_2 = 0, \quad (3.6.36a)$$

where $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, $1 \leq k_1 \leq k_2 - 1$, $k_2 + 1 \leq k_3 \leq s_{q_3} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1 , k_2 , k_3 , if one subtracts (3.6.25h) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$ from (3.6.25h) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = k_2$, one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (3.6.36b)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $1 \leq k_1 \leq k_3 - 1$, $k_3 + 1 \leq k_2 \leq s_{q_2} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1 , k_2 , k_3 , if one subtracts (3.6.25h) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$ from (3.6.25g) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_3$, $i_{q_3} = k_3$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (3.6.37a)$$

where $s_{v-1} \leq k_2 \leq s_v - 1$, $k_2 \neq 0$, $k_2 + 1 \leq k_1 \leq s_{q_1} - 1$, $k_2 + 1 \leq k_3 \leq s_{q_3} - 1$, $1 \leq v \leq q_1$. If $s_{v-1} < s_v$, then for any fixed k_1 , k_2 , k_3 , subtracting (3.6.25h) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$ from (3.6.25g) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = k_2$, one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (3.6.37b)$$

where $s_{v-1} \leq k_3 \leq s_v - 1$, $k_3 \neq 0$, $k_3 + 1 \leq k_1 \leq s_{q_1} - 1$, $k_3 + 1 \leq k_2 \leq s_{q_2} - 1$, $1 \leq v \leq q_1$. Note that for given v , $1 \leq v \leq q_1$, the equations given by (3.6.36 - 37)

are defined only when $s_{v-1} < s_v$. For any fixed k_1, k_2, k_3 , if one subtracts (3.6.25c) with $j = k_2, u = q_3, i_{q_3} = k_3$ for $s_{v-1} \leq k_2 \leq s_v - 1, k_2 + 1 \leq k_3 \leq s_{q_3} - 1, q_1 + 1 \leq v \leq q_2$ from (3.6.25f) with $j = k_2, u_1 = q_1, i_{q_1} = k_1, u_2 = q_3, i_{q_3} = k_3$, for $s_v \leq k_2 \leq s_{v+1} - 1, k_2 + 1 \leq k_3 \leq s_{q_3} - 1, q_1 \leq v \leq q_2 - 1, 1 \leq k_1 \leq s_{q_1} - 1$, one obtains

$$(\dot{p}_{q_1 k_1} - \dot{p}_{q_1 0}) \theta_{q_1} + [(\dot{p}_{q_1 k_1} - \dot{p}_{q_1 0}) \times \dot{p}_{q_2 k_2} \times \dot{p}_{q_3 k_3}] \phi_1 = 0. \quad (3.6.38a)$$

Finally, for any fixed k_1, k_2, k_3 , if one subtracts (3.6.25c) with $j = k_3, u = q_2, i_{q_2} = k_2$ for $s_{v-1} \leq k_3 \leq s_v - 1, k_3 + 1 \leq k_2 \leq s_{q_2} - 1, q_1 + 1 \leq v \leq q_2$ from (3.6.25f) with $j = k_3, u_1 = q_1, i_{q_1} = k_1, u_2 = q_2, i_{q_2} = k_2$ for $s_v \leq k_3 \leq s_{v+1} - 1, k_3 + 1 \leq k_2 \leq s_{q_2} - 1, q_1 \leq v \leq q_2 - 1, 1 \leq k_1 \leq s_{q_1} - 1$, one gets

$$(\dot{p}_{q_1 k_1} - \dot{p}_{q_1 0}) \theta_{q_1} + [(\dot{p}_{q_1 k_1} - \dot{p}_{q_1 0}) \times \dot{p}_{q_2 k_2} \times \dot{p}_{q_3 k_3}] \phi_1 = 0. \quad (3.6.38b)$$

Note that for given $v, q_1 + 1 \leq v \leq q_2$, the equations given by (3.6.38a,b) are defined only when $s_{v-1} < s_v$. Now combining the equations (3.6.28-30) and (3.6.32-38), one gets for any fixed $k_1, k_2, k_3, 1 \leq k_1 \leq s_{q_1} - 1, 0 \leq k_2 \leq s_{q_2} - 1, 0 \leq k_3 \leq s_{q_3} - 1, (k_1, k_2, k_3) \neq (k_1, 0, k_3)$ for $1 \leq k_1 \leq s_{q_1} - 1, k_1 + 1 \leq k_3 \leq s_{q_3} - 1$ and $(k_1, k_2, k_3) \neq (k_1, k_2, 0)$ for $1 \leq k_1 \leq s_{q_1} - 1, k_1 + 1 \leq k_2 \leq s_{q_2} - 1$,

$$(\dot{p}_{q_1 k_1} - \dot{p}_{q_1 0}) \theta_{q_1} + [(\dot{p}_{q_1 k_1} - \dot{p}_{q_1 0}) \times \dot{p}_{q_2 k_2} \times \dot{p}_{q_3 k_3}] \phi_1 = 0. \quad (3.6.39)$$

The s_{q_2} equations that (3.6.39), when added gives, for any fixed $k_1, k_3, 1 \leq k_1 \leq s_{q_1} - 1, 1 \leq k_3 \leq k_1$,

$$(\dot{p}_{q_1 k_1} - \dot{p}_{q_1 0}) \theta_{q_1} = 0. \quad (3.6.40)$$

Then from Lemma 1.8.1 and from (3.6.40), one obtains

$$\theta_{q_1} = 0. \quad (3.6.41)$$

Here one may note that the equations (3.6.42-43) stated below are obtained on using (3.6.41). For any fixed $k_1, k_3, 1 \leq k_1 \leq s_{q_1} - 1, k_1 + 1 \leq k_3 \leq s_{q_3} - 1$, summing the s_{q_2} equations (3.6.31a), (3.6.33), (3.6.34b), (3.6.35), (3.6.36a,b), (3.6.37a), (3.6.38a,b), one gets

$$\begin{aligned} & (p'_{q_4 k_1} - p'_{q_4 0}) \theta_{q_4} + (p'_{q_5 k_1} - p'_{q_5 0}) \theta_{q_5} + (p'_{q_6 k_1} - p'_{q_6 0}) \theta_{q_6} + \\ & [p'_{q_4 k_1} \times p'_{q_5 k_1} \times p'_{q_6 k_1} - p'_{q_4 0} \times p'_{q_5 0} \times p'_{q_6 0}] \phi_2 = 0. \end{aligned} \quad (3.6.42)$$

If one subtracts (3.6.42) from (3.6.31a), one gets for any fixed k_1, k_3 ,

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 0} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (3.6.43a)$$

where $1 \leq k_1 \leq s_{q_1} - 1, k_1 + 1 \leq k_3 \leq s_{q_3} - 1$. Similarly from the equations (3.6.31b), (3.6.33), (3.6.34a), (3.6.35), (3.6.36a,b), (3.6.37b), (3.6.38a,b), one obtains for any fixed k_1, k_2 ,

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 0}] \phi_1 = 0, \quad (3.6.43b)$$

where $1 \leq k_1 \leq s_{q_1} - 1, k_1 + 1 \leq k_2 \leq s_{q_2} - 1$. Now combining the equations (3.6.39), (3.6.43a) and (3.6.43b) and using (3.6.41), one obtains for any fixed k_1, k_2, k_3 ,

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 k_2} \times p'_{q_3 k_3}] \phi_1 = 0, \quad (3.6.44)$$

where $1 \leq k_1 \leq s_{q_1} - 1, 0 \leq k_2 \leq s_{q_2} - 1, 0 \leq k_3 \leq s_{q_3} - 1$. Again (3.6.44) implies that $[P_{q_1}^* \times P_{q_2} \times P_{q_3}] \phi_1 = 0$, where $P_{q_1}^*$ is an $((s_{q_1} - 1) \times (s_{q_1} - 1))$ matrix with rows $(p'_{q_1 k_1} - p'_{q_1 0}), 1 \leq k_1 \leq s_{q_1} - 1$. Since $P_{q_1}^*, P_{q_2}$ and P_{q_3} are all of full column rank, one obtains

$$\phi_1 = 0.$$

Then, from (3.6.25a-k), it is easy to prove that

$$\theta_{q_1} = 0, \theta_{q_2} = 0, \theta_{q_4} = 0, \theta_{q_5} = 0, \theta_{q_6} = 0, \phi_2 = 0, \theta_0 = 0$$

and hence $\theta = 0$. The proof of $\theta = 0$, for all other cases, follows exactly along the same line from the equations (3.6.25a-k).

Chapter 4

SEARCH DESIGNS FOR SEARCHING ONE AMONG THE TWO, THREE AND FOUR-FACTOR INTERACTION EFFECTS IN THE s^m FACTORIAL SET-UP

4.1 Introduction

A fractional factorial design will be said to be a search design for searching one among the two, three and four-factor interactions if it is capable of estimating the general mean, all the main-effects and permits the detection and estimation of the possibly present at best one among the two, three and four-factor interactions. The available literature on this search design problem relates to 2^m factorials. Srivastava (1975) gave the basic mathematical formulation of the problem. Srivastava and Gupta (1979) gave a necessary and sufficient condition for the existence of the above mentioned search designs for the 2^m factorial set-up. Ghosh (1980, 1981) considered the construction procedures for the 2^m factorial set-up. Gupta and Carvajal (1984a, b, 1987) also considered search designs of the above type for the 2^m factorial set-up.

This chapter deals with search designs for searching one among the two, three and four-factor interactions for the s^m factorial set-up where s is not necessarily a prime or a prime power. In Section 4.3, the construction procedure for the s^m factorial set-up is given. Section 4.4 deals with the proofs of theorems.

As in the previous chapters, for 2^m factorials, our plans are less economic than the existing ones (see e.g. Srivastava (1975), Ghosh (1980, 1981)), whereas for more general symmetric factorials, no plans other than ours have so far been reported in the present context.

4.2 Notations and Preliminaries

Consider a factorial set-up involving $m(m \geq 4)$ factors $F_1, F_2, \dots, F_m$ each at s levels ($s \geq 2$). In addition to the notations given in Sections 1.2 and 2.2 of the earlier chapters, we further define

$$Z_{v_1 v_2 v_3 v_4} = \times_{w=1}^m P_w (\delta_{wv_1} + \delta_{wv_2} + \delta_{wv_3} + \delta_{wv_4}), \quad 1 \leq v_1 < v_2 < v_3 < v_4 \leq m,$$

where δ_{wv_i} 's ($i = 1, 2, 3, 4$) are Kronecker deltas and $\times$ is the Kronecker product. Suppose prior information is available regarding the absence of all interactions involving five or more factors. Throughout this chapter these interactions will be assumed to be absent. Now one may observe that if one observation is made for each of the s^m level combinations of $F_1, F_2, \dots, F_m$, the resulting design matrix will be of the form

$$\begin{aligned} &[1, Z_1, Z_2, \dots, Z_m, Z_{12}, Z_{13}, \dots, Z_{m-1 m}, Z_{123}, Z_{124}, \dots, \\ &Z_{m-2 m-1 m}, Z_{1234}, Z_{1235}, \dots, Z_{m-3 m-2 m-1 m}] \end{aligned} \quad (4.2.1)$$

where 1 corresponds to the general mean and the columns of $Z_v, Z_{v_1 v_2}, Z_{v_1 v_2 v_3}$, and $Z_{v_1 v_2 v_3 v_4}$ correspond to the main-effect F_v ($1 \leq v \leq m$), the two-factor interaction $F_{v_1} F_{v_2}$ ($1 \leq v_1 < v_2 \leq m$), the three-factor interaction $F_{v_1} F_{v_2} F_{v_3}$ ($1 \leq v_1 < v_2 < v_3 \leq m$) and the four-factor interaction $F_{v_1} F_{v_2} F_{v_3} F_{v_4}$ ($1 \leq v_1 < v_2 < v_3 < v_4 \leq m$) respectively. Of course (4.2.1) incorporates $Z_{v_1 v_2}, Z_{v_1 v_2 v_3}$ or $Z_{v_1 v_2 v_3 v_4}$ only when the associated interaction $F_{v_1} F_{v_2}, F_{v_1} F_{v_2} F_{v_3}$ or $F_{v_1} F_{v_2} F_{v_3} F_{v_4}$ is included in the model. Let

$$H_1^{***} = [1, Z_1, Z_2, \dots, Z_m] \text{ and}$$

$$H_2^{***} = [Z_{12}, Z_{13}, \dots, Z_{m-1 m}, Z_{123}, Z_{124}, \dots, \\ Z_{m-2 m-1 m}, Z_{1234}, \dots, Z_{m-3 m-2 m-1 m}]$$

Further let S be a subset of N level combinations of $F_1, F_2, \dots, F_m$ with the corresponding sub-matrices of H_1^{***} and H_2^{***} as, say, $T_1^{***} = [\epsilon, L_1, L_2, \dots, L_m]$ and $T_2^{***} = [G_{12}, G_{13}, \dots, G_{m-1 m}, V_{123}, V_{124}, \dots, V_{m-2 m-1 m}, W_{1234}, W_{1235}, \dots, W_{m-3 m-2 m-1 m}]$ respectively. From Theorem 1.2.1 of Chapter 1, one observes that S represents a search design of the present type if and only if the following matrices

- i) $[T_1^{***}, G_{q_1 q_2}, G_{q_3 q_4}]$, where $1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 \leq m, (q_1, q_2) \neq (q_3, q_4)$,
 - ii) $[T_1^{***}, G_{q_1 q_2}, V_{q_3 q_4 q_5}]$, where $1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 < q_5 \leq m$,
 - iii) $[T_1^{***}, G_{q_1 q_2}, W_{q_3 q_4 q_5 q_6}]$, where $1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 < q_5 < q_6 \leq m$,
 - iv) $[T_1^{***}, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}]$, where $1 \leq q_1 < q_2 < q_3 \leq m, 1 \leq q_4 < q_5 < q_6 \leq m, (q_1, q_2, q_3) \neq (q_4, q_5, q_6)$,
 - v) $[T_1^{***}, V_{q_1 q_2 q_3}, W_{q_4 q_5 q_6 q_7}]$, where $1 \leq q_1 < q_2 < q_3 \leq m, 1 \leq q_4 < q_5 < q_6 < q_7 \leq m$, and
 - vi) $[T_1^{***}, W_{q_1 q_2 q_3 q_4}, W_{q_5 q_6 q_7 q_8}]$, where $1 \leq q_1 < q_2 < q_3 < q_4 \leq m, 1 \leq q_5 < q_6 < q_7 < q_8 \leq m, (q_1, q_2, q_3, q_4) \neq (q_5, q_6, q_7, q_8)$
- have full column rank.

4.3 Main Result

The following notations, which have been partly used in the previous chapters, will be helpful in describing the search designs considered here.

- (i) Define the $(m \times 1)$ vectors $a_j^{(1)}$ ($0 \leq j \leq s-1$) such that in $a_j^{(1)}$, all the elements

are j ; let $A_j^{(1)}$ be an $(m \times s)$ array whose columns are given by $a_j^{(1)}$ for all j .

(ii) For $0 \leq j \leq s-1$, define the $(m \times 1)$ vectors $a_{j,u,i_u}^{(2)}$ ($1 \leq u \leq m$, $0 \leq i_u \leq s-1$, $i_u \neq j$) such that in $a_{j,u,i_u}^{(2)}$, the u th element is i_u and the remaining elements all equal j ; let $A_j^{(2)}$ be an $(m \times m(s-1))$ array whose columns are $a_{j,u,i_u}^{(2)}$ for all u and i_u .

(iii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ ($1 \leq u_1 < u_2 \leq m$, $0 \leq \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}) \leq s-1$) such that the u_1 th and u_2 th elements of $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ are i_{u_1} and i_{u_2} respectively and the remaining elements of the same all equal j ; let $A_j^{(3)}$ be an $(m \times [m(m-1)j(s-j-1)])$ array whose columns are given by $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(3)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(3)}$ is defined only when $s \geq 3$.

(iv) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ ($1 \leq u_1 < u_2 \leq m$, $j+1 \leq i_{u_1}, i_{u_2} \leq s-1$) such that in $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$, the u_1 th and u_2 th elements are i_{u_1} and i_{u_2} respectively and the remaining elements all equal j ; let $A_j^{(4)}$ be an $(m \times [m(m-1)(s-j-1)^2/2])$ array whose columns are $a_{j,u_1,u_2,i_{u_1},i_{u_2}}^{(4)}$ for all u_1, u_2, i_{u_1} and i_{u_2} . Note that $A_j^{(4)}$ is defined only when $s \geq 3$.

(v) Define the $(m \times 1)$ vectors $a_{j,u_1,u_2}^{(5)}$ ($1 \leq j \leq s-1$, $1 \leq u_1 < u_2 \leq m$) such that in $a_{j,u_1,u_2}^{(5)}$, the u_1 th and u_2 th elements are zero and the remaining elements all equal j ; let $A^{(5)}$ be an $(m \times [m(m-1)(s-1)/2])$ array whose columns are given by $a_{j,u_1,u_2}^{(5)}$ for all j, u_1 and u_2 .

(vi) Define the $(m \times 1)$ vectors $a_{j,u_1,u_2,u_3}^{(6)}$ ($1 \leq j \leq s-1$, $1 \leq u_1 < u_2 < u_3 \leq m$) such

that in $a_{j,u_1,u_2,u_3}^{(6)}$, the u_1 th, u_2 th and u_3 th elements are zero and the remaining elements all equal j ; let $A_j^{(6)}$ be an $(m \times [m(m-1)(m-2)(s-1)/6])$ array whose columns are given by $a_{j,u_1,u_2,u_3}^{(6)}$ for all j, u_1, u_2 and u_3 .

(vii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(7)}$ ($1 \leq u_1 < u_2 < u_3 \leq m$, $i_{(u_1)} = 0$, $i_{(u_2)} = 0$, $j+1 \leq i_{(u_3)} \leq s-1$, $i_{(u_1)}$, $i_{(u_2)}$ and $i_{(u_3)}$ represent the non-decreasingly ordered arrangement of i_{u_1} , i_{u_2} and i_{u_3} , e.g. if $i_{u_1} = 0$, $i_{u_2} = 2$, $i_{u_3} = 0$, the $i_{(u_1)} = i_{(u_2)} = 0$, $i_{(u_3)} = 2$) such that in

$a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(7)}$, the u_1 th, u_2 th and u_3 th elements are i_{u_1} , i_{u_2} and i_{u_3} respectively and the remaining elements all equal j ; let $A_j^{(7)}$ be an $(m \times [m(m-1)(m-2)(s-j-1)/2])$ array whose columns are given by $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(7)}$ for all $j, u_1, u_2, u_3, i_{u_1}, i_{u_2}$ and i_{u_3} . Note that $A_j^{(7)}$ is defined only when $s \geq 3$.

(viii) For $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(8)}$ ($1 \leq u_1 < u_2 < u_3 \leq m$, $0 \leq i_{(u_1)} \leq j-1$, $j+1 \leq i_{(u_2)}$, $i_{(u_3)} \leq s-1$, $i_{(u_1)}$, $i_{(u_2)}$ and $i_{(u_3)}$ are as defined above) such that in $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(8)}$, the u_1 th, u_2 th and u_3 th elements are i_{u_1} , i_{u_2} and i_{u_3} respectively and the remaining elements all equal j ; let $A_j^{(8)}$ be an $(m \times [m(m-1)(m-2)j(s-j-1)^2/2])$ array whose columns are given by $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(8)}$ for all $u_1, u_2, u_3, i_{u_1}, i_{u_2}$ and i_{u_3} . Note that $A_j^{(8)}$ is defined only when $s \geq 3$.

(ix) For $2 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(9)}$ ($1 \leq u_1 < u_2 < u_3 \leq m$, $i_{(u_1)} = 0$, $1 \leq i_{(u_2)} \leq j-1$, $j+1 \leq i_{(u_3)} \leq s-1$, $i_{(u_1)}$, $i_{(u_2)}$ and $i_{(u_3)}$ are as defined above) such that in $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(9)}$, the u_1 th, u_2 th and u_3 th

elements are i_{u_1} , i_{u_2} and i_{u_3} respectively and the remaining elements all equal

j ; let $A_j^{(9)}$ be an $(m \times [m(m-1)(m-2)(j-1)(s-j-1)])$ array whose columns are given by

$a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(9)}$ for all $u_1, u_2, u_3, i_{u_1}, i_{u_2}$ and i_{u_3} . Note that $A_j^{(9)}$ is defined only when $s \geq 4$.

(x) Finally, for $1 \leq j \leq s-2$, define the $(m \times 1)$ vectors $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(10)}$ ($1 \leq u_1 < u_2 < u_3 \leq m, j+1 \leq i_{u_1}, i_{u_2}, i_{u_3} \leq s-1$) such that in $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(10)}$, the u_1 th, u_2 th and u_3 th elements are i_{u_1} , i_{u_2} and i_{u_3} respectively and the remaining elements all equal j ; let $A_j^{(10)}$ be an $(m \times [m(m-1)(m-2)(s-j-1)^3/6])$ array whose columns are given by $a_{j,u_1,u_2,u_3,i_{u_1},i_{u_2},i_{u_3}}^{(10)}$ for all $u_1, u_2, u_3, i_{u_1}, i_{u_2}$ and i_{u_3} . Note that $A_j^{(10)}$ is defined only when $s \geq 3$.

Let

$$N = s + ms(s-1) + m(m-1)(s-1)/2 + m(m-1)s(s-1)(s-2)/6 + m(m-1)(s-1)(s-2)(2s-3)/12 + m(m-1)(m-2)(s-1)/6 + m(m-1)(m-2)(s-1)(s-2)(s^2 - 2)/12,$$

and define the following $(m \times N)$ array :

$$Q = [A^{(1)}, A_0^{(2)}, A_1^{(2)}, \dots, A_{s-1}^{(2)}, A_1^{(3)}, A_2^{(3)}, \dots, A_{s-2}^{(3)}, A_1^{(4)}, A_2^{(4)}, \dots, A_{s-2}^{(4)}, A^{(5)}, A^{(6)}, A_1^{(7)}, A_2^{(7)}, \dots, A_{s-2}^{(7)}, A_1^{(8)}, A_2^{(8)}, \dots, A_{s-2}^{(8)}, A_2^{(9)}, A_3^{(9)}, \dots, A_{s-2}^{(9)}, A_1^{(10)}, A_2^{(10)}, \dots, A_{s-2}^{(10)}] \quad (4.3.1)$$

Then we have the following theorems :

Theorem 4.3.1. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for which the matrix $[T_1^{***}, G_{q_1 q_2}, G_{q_3 q_4}]$, where $1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 \leq m, (q_1, q_2) \neq (q_3, q_4)$, has full column rank.

Theorem 4.3.2. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for which the matrix $[T_1^{***}, G_{q_1 q_2}, V_{q_3 q_4 q_5}]$, where $1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 < q_5 \leq m$, has full column rank.

Theorem 4.3.3. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for which the matrix $[T_1^{***}, G_{q_1 q_2}, W_{q_3 q_4 q_5 q_6}]$, where $1 \leq q_1 < q_2 \leq m, 1 \leq q_3 < q_4 < q_5 < q_6 \leq m$, has full column rank.

Theorem 4.3.4. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for which the matrix $[T_1^{***}, V_{q_1 q_2 q_3}, V_{q_4 q_5 q_6}]$, where $1 \leq q_1 < q_2 < q_3 \leq m, 1 \leq q_4 < q_5 < q_6 \leq m, (q_1, q_2, q_3) \neq (q_4, q_5, q_6)$, has full column rank.

Theorem 4.3.5. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, be a subset of N level combinations of $F_1, F_2, \dots, F_m$ for which the matrix $[T_1^{***}, V_{q_1 q_2 q_3}, W_{q_4 q_5 q_6 q_7}]$, where $1 \leq q_1 < q_2 < q_3 \leq m, 1 \leq q_4 < q_5 < q_6 < q_7 \leq m$, has full column rank.

Theorem 4.3.6. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, constitute a subset of N level combinations of $F_1, F_2, \dots, F_m$.

F_m for which the matrix $[T_1^{***}, W_{q_1 q_2 q_3 q_4}, W_{q_5 q_6 q_7 q_8}]$, where $1 \leq q_1 < q_2 < q_3 < q_4 \leq m$, $1 \leq q_5 < q_6 < q_7 < q_8 \leq m$, $(q_1, q_2, q_3, q_4) \neq (q_5, q_6, q_7, q_8)$, has full column rank.

Now we give our main theorem of this chapter.

Theorem 4.3.7. The columns of the matrix Q , interpreted as the level combinations of $F_1, F_2, \dots, F_m$, give the required search design in N runs.

4.4 Appendix

We present below the proof of Theorem 4.3.6. The proofs of all other theorems are similar. Theorem 4.3.7 follows from Theorems 4.3.1 - 4.3.6.

Proof of Theorem 4.3.6. Under the notation of Sections 1.2, 2.2 and 4.2, consider the matrix $[T_1^{***}, W_{q_1 q_2 q_3 q_4}, W_{q_5 q_6 q_7 q_8}] = [\epsilon, L_1, L_2, \dots, L_m, W_{q_1 q_2 q_3 q_4}, W_{q_5 q_6 q_7 q_8}]$, where

$$1 \leq q_1 < q_2 < q_3 < q_4 \leq m, 1 \leq q_5 < q_6 < q_7 < q_8 \leq m, \\ q_4 \leq q_8 \text{ and } (q_1, q_2, q_3, q_4) \neq (q_5, q_6, q_7, q_8) \quad (4.4.1)$$

Here it is enough to prove that the columns of the above matrix are linearly independent. To prove this, we first suppose $q_1 = q_5, q_2 = q_6, q_3 = q_7$ and $q_4 < q_8$ in (4.4.1). Consider

$$[\epsilon, L_1, L_2, \dots, L_m, W_{q_1 q_2 q_3 q_4}, W_{q_1 q_2 q_3 q_8}] \theta = 0, \quad (4.4.2)$$

where $\theta = (\theta_0, \theta_1', \theta_2', \dots, \theta_m', \phi_1', \phi_2')'$, say. If one recalls the definitions of $L_1, L_2, \dots, L_m, W_{q_1 q_2 q_3 q_4}, W_{q_1 q_2 q_3 q_8}$, one gets from (4.3.1) and (4.4.2),

the following equation for a typical level combination $(i_1, i_2, \dots, i_m)$

$$\begin{aligned} \theta_0 + \sum_{w=1}^m p'_{wi} \theta_w + (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}} \times p'_{q_4 i_{q_4}}) \phi_1 + \\ (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}} \times p'_{q_8 i_{q_8}}) \phi_2 = 0, \quad (i_1, i_2, \dots, \\ i_m) \in S(Q), \end{aligned} \quad (4.4.3)$$

where $S(Q)$ is the set of level combinations represented by the columns of the array Q . Recalling the structure of Q , as given by (4.3.1), it is evident that the system of equations (4.4.3) can be split up into subsystems of equations (4.4.3a), (4.4.3b), ..., (4.4.3j) which arise from the level combinations given by the columns of $A^{(1)}$, $[A^{(2)}_0, A^{(2)}_1, \dots, A^{(2)}_{s-1}]$, ..., $[A^{(10)}_1, A^{(10)}_2, \dots, A^{(10)}_{s-2}]$ respectively. For example, taking care of the definitions of the column vectors $a^{(3)}_{j, u_1, u_2, i_{u_1}, i_{u_2}}$ which constitute the columns of $[A^{(3)}_1, A^{(3)}_2, \dots, A^{(3)}_{s-2}]$, the subsystems of equations (4.4.3c) corresponding to the columns of $[A^{(3)}_1, A^{(3)}_2, \dots, A^{(3)}_{s-2}]$ is given by

$$\begin{aligned} \theta_0 + \sum_{w=1}^m p'_{wj} \theta_w + p'_{u_1 i_{u_1}} \theta_{u_1} + p'_{u_2 i_{u_2}} \theta_{u_2} + \\ [\delta_{u_1 q_1} \{ \delta_{u_2 q_2} (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j} \times p'_{q_4 j}) + \delta_{u_2 q_3} (p'_{q_1 i_{q_1}} \times \\ p'_{q_2 j} \times p'_{q_3 i_{q_3}} \times p'_{q_4 j}) + \delta_{u_2 q_4} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 j} \times p'_{q_4 i_{q_4}}) + (1 - \\ \delta_{u_2 q_2} - \delta_{u_2 q_3} - \delta_{u_2 q_4}) (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 j} \times p'_{q_4 j}) \} + \\ \delta_{u_1 q_2} \{ \delta_{u_2 q_3} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}} \times p'_{q_4 j}) + \delta_{u_2 q_4} (p'_{q_1 j} \times \end{aligned}$$

$$\begin{aligned}
& p'_{q_2 i_{q_2}} \times p'_{q_3 j} \times p'_{q_4 i_{q_4}}) + (1 - \delta_{u_2 q_3} - \delta_{u_2 q_4}) (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j} \times \\
& p'_{q_4 j}) \} + \delta_{u_1 q_3} \{ \delta_{u_2 q_4} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}} \times p'_{q_4 i_{q_4}}) + (1 - \delta_{u_2 q_4}) (p'_{q_1 j} \times \\
& p'_{q_2 j} \times p'_{q_3 i_{q_3}} \times p'_{q_4 j}) \} + \delta_{u_1 q_4} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 j} \times p'_{q_4 i_{q_4}}) + (1 - \\
& \delta_{u_1 q_1} - \delta_{u_1 q_2} - \delta_{u_1 q_3} - \delta_{u_1 q_4}) \{ \delta_{u_2 q_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 j} \times p'_{q_4 j}) + \\
& \delta_{u_2 q_2} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j} \times p'_{q_4 j}) + \delta_{u_2 q_3} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}} \times \\
& p'_{q_4 j}) + \delta_{u_2 q_4} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 j} \times p'_{q_4 i_{q_4}}) + (1 - \delta_{u_2 q_1} - \delta_{u_2 q_2} - \delta_{u_2 q_3} - \\
& \delta_{u_2 q_4}) (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 j} \times p'_{q_4 j}) \} \Phi_1 + [\delta_{u_1 q_1} \{ \delta_{u_2 q_2} (p'_{q_1 i_{q_1}} \times p'_{q_2 i_{q_2}} \times \\
& p'_{q_3 j} \times p'_{q_8 j}) + \delta_{u_2 q_3} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}} \times p'_{q_8 j}) + \delta_{u_2 q_8} (p'_{q_1 i_{q_1}} \times \\
& p'_{q_2 j} \times p'_{q_3 j} \times p'_{q_8 i_{q_8}}) + (1 - \delta_{u_2 q_2} - \delta_{u_2 q_3} - \delta_{u_2 q_8}) (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times \\
& p'_{q_3 j} \times p'_{q_8 j}) \} + \delta_{u_1 q_2} \{ \delta_{u_2 q_3} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 i_{q_3}} \times p'_{q_8 j}) + \\
& \delta_{u_2 q_8} (p'_{q_1 j} \times p'_{q_2 i_{q_2}} \times p'_{q_3 j} \times p'_{q_8 i_{q_8}}) + (1 - \delta_{u_2 q_3} - \delta_{u_2 q_8}) (p'_{q_1 j} \times \\
& p'_{q_2 i_{q_2}} \times p'_{q_3 j} \times p'_{q_8 j}) \} + \delta_{u_1 q_3} \{ \delta_{u_2 q_8} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}} \times p'_{q_8 i_{q_8}}) + \\
& (1 - \delta_{u_2 q_8}) (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 i_{q_3}} \times p'_{q_8 j}) \} + \delta_{u_1 q_8} (p'_{q_1 j} \times p'_{q_2 j} \times p'_{q_3 j} \times \\
& p'_{q_8 i_{q_8}}) + (1 - \delta_{u_1 q_1} - \delta_{u_1 q_2} - \delta_{u_1 q_3} - \delta_{u_1 q_8}) \{ \delta_{u_2 q_1} (p'_{q_1 i_{q_1}} \times p'_{q_2 j} \times
\end{aligned}$$

$$\begin{aligned}
& p'_{q_3j} \times p'_{q_8j}) + \delta_{u_2q_2} (p'_{q_1j} \times p'_{q_2i_{q_2}} \times p'_{q_3j} \times p'_{q_8j}) + \delta_{u_2q_3} (p'_{q_1j} \times \\
& p'_{q_2j} \times p'_{q_3i_{q_3}} \times p'_{q_8j}) + \delta_{u_2q_8} (p'_{q_1j} \times p'_{q_2j} \times p'_{q_3j} \times p'_{q_8i_{q_8}}) + (1 - \\
& \delta_{u_2q_1} - \delta_{u_2q_2} - \delta_{u_2q_3} - \delta_{u_2q_8}) (p'_{q_1j} \times p'_{q_2j} \times p'_{q_3j} \times p'_{q_8j})] \phi_2 = 0,
\end{aligned}
\tag{4.4.3c}$$

where $1 \leq u_1 < u_2 \leq m$, $0 \leq \min(i_{u_1}, i_{u_2}) < j < \max(i_{u_1}, i_{u_2}) \leq s-1$, $1 \leq j \leq s-2$.

For any fixed t ($1 \leq t \leq m$, $t \neq q_1, q_2, q_3, q_4, q_8$), if one subtracts equation (4.4.3a) with $j = 0$ from (4.4.3b) with $u = t$, $j = 0$, one gets

$$(p'_{ti_t} - p'_{t0}) \theta_t = 0, \tag{4.4.4}$$

where $1 \leq i_t \leq s_t - 1$. From (4.4.4) and Lemma 1.2.1, it follows that

$$\theta_t = 0, \quad 1 \leq t \leq m, \quad t \neq q_1, q_2, q_3, q_4, q_8. \tag{4.4.5}$$

Subtracting (4.4.3f) with $j = 1$, $u_1 = q_1$, $u_2 = q_4$, $u_3 = q_8$ from (4.4.3e) with $j = 1$, $u_1 = q_4$, $u_2 = q_8$, one obtains

$$\begin{aligned}
& (p'_{q_11} - p'_{q_10}) \theta_{q_1} + [(p'_{q_11} - p'_{q_10}) \times p'_{q_21} \times p'_{q_31} \times p'_{q_40}] \phi_1 + \\
& [(p'_{q_11} - p'_{q_10}) \times p'_{q_21} \times p'_{q_31} \times p'_{q_80}] \phi_2 = 0.
\end{aligned}
\tag{4.4.6}$$

If one subtracts (4.4.3g) with $j = 1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$, $u_3 = q_8$, $i_{q_8} = 0$ from (4.4.3c) with $j = 1$, $u_1 = q_4$, $i_{q_4} = k_4$, $u_2 = q_8$, $i_{q_8} = 0$, one gets

$$\begin{aligned}
& (p'_{q_11} - p'_{q_10}) \theta_{q_1} + [(p'_{q_11} - p'_{q_10}) \times p'_{q_21} \times p'_{q_31} \times p'_{q_4k_4}] \phi_1 + \\
& [(p'_{q_11} - p'_{q_10}) \times p'_{q_21} \times p'_{q_31} \times p'_{q_80}] \phi_2 = 0,
\end{aligned}
\tag{4.4.7a}$$

where $2 \leq k_4 \leq s-1$. Similarly, subtracting (4.4.3g) with $j = 1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = 0$, $u_3 = q_8$, $i_{q_8} = k_8$ from (4.4.3c) with $j = 1$, $u_1 = q_4$, $i_{q_4} = 0$, $u_2 = q_8$, $i_{q_8} = k_8$, one obtains

$$\begin{aligned} & (p'_{q_1 1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 0}] \phi_1 + \\ & [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 k_8}] \phi_2 = 0, \end{aligned} \quad (4.4.7b)$$

where $2 \leq k_8 \leq s-1$. If one subtracts (4.4.3e) with $j = 1$, $u_1 = q_1$, $u_2 = q_8$ from (4.4.3b) with $j = 1$, $u = q_8$, $i_{q_8} = 0$, one obtains

$$\begin{aligned} & (p'_{q_1 1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 1}] \phi_1 + \\ & [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 0}] \phi_2 = 0. \end{aligned} \quad (4.4.8a)$$

Similarly, if one subtracts (4.4.3e) with $j = 1$, $u_1 = q_1$, $u_2 = q_4$ from (4.4.3b) with $j = 1$, $u = q_4$, $i_{q_4} = 0$, one gets

$$\begin{aligned} & (p'_{q_1 1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 0}] \phi_1 + \\ & [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 1}] \phi_2 = 0. \end{aligned} \quad (4.4.8b)$$

Subtracting (4.4.3b) with $u = q_1$, $i_{q_1} = 0$, $j = 1$ from (4.4.3a) with $j = 1$, one gets

$$\begin{aligned} & (p'_{q_1 1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 1}] \phi_1 + \\ & [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 1}] \phi_2 = 0. \end{aligned} \quad (4.4.9)$$

If one subtracts (4.4.3c) with $j = 1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$ from

(4.4.3b) with $j = 1$, $u = q_4$, $i_{q_4} = k_4$, one obtains

$$\begin{aligned} & (p'_{q_1 1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 k_4}] \phi_1 + \\ & [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 1}] \phi_2 = 0, \end{aligned} \quad (4.4.10a)$$

where $2 \leq k_4 \leq s-1$. Similarly, subtracting (4.4.3c) with $j = 1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_8$, $i_{q_8} = k_8$ from (4.4.3b) with $j = 1$, $u = q_8$, $i_{q_8} = k_8$, one gets

$$\begin{aligned} & (p'_{q_1 1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 1}] \phi_1 + \\ & [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 k_8}] \phi_2 = 0, \end{aligned} \quad (4.4.10b)$$

where $2 \leq k_8 \leq s-1$. If one subtracts (4.4.3h) with $j = 1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$, $u_3 = q_8$, $i_{q_8} = k_8$ from (4.4.3d) with $j = 1$, $u_1 = q_4$, $i_{q_4} = k_4$, $u_2 = q_8$, $i_{q_8} = k_8$, one obtains

$$\begin{aligned} & (p'_{q_1 1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 k_4}] \phi_1 + \\ & [(p'_{q_1 1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 k_8}] \phi_2 = 0, \end{aligned} \quad (4.4.11)$$

where $2 \leq k_4, k_8 \leq s-1$. For any fixed k_1 , if one subtracts (4.4.3f) with $u_1 = q_1$, $u_2 = q_4$, $u_3 = q_8$, $j = 1$, from (4.4.3g) with $j = 1$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_4$, $i_{q_4} = 0$, $u_3 = q_8$, $i_{q_8} = 0$, one gets

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 0}] \phi_1 + \\ & [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 0}] \phi_2 = 0, \end{aligned} \quad (4.4.12)$$

where $2 \leq k_1 \leq s-1$. For any fixed k_1 , subtracting (4.4.3e) with $j = 1$, $u_1 = q_1$, $u_2 = q_8$, from (4.4.3c) with $j = 1$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_8$, $i_{q_8} = 0$, one gets

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 1}] \phi_1 + \\ & [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 0}] \phi_2 = 0, \end{aligned} \quad (4.4.13a)$$

where $2 \leq k_1 \leq s-1$. Similarly, for any fixed k_1 , if one subtracts (4.4.3e) with $j = 1$, $u_1 = q_1$, $u_2 = q_4$ from (4.4.3c) with $j = 1$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_4$, $i_{q_4} = 0$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 0}] \phi_1 + \\ & [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 1}] \phi_2 = 0, \end{aligned} \quad (4.4.23b)$$

where $2 \leq k_1 \leq s-1$. For any fixed k_1 , subtracting (4.4.3g) with $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$, $u_3 = q_8$, $i_{q_8} = 0$, $j = 1$ from (4.4.3h) with $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_4$, $i_{q_4} = k_4$, $u_3 = q_8$, $i_{q_8} = 0$, $j = 1$, one obtains

$$\begin{aligned} & (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 k_4}] \phi_1 + \\ & [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 0}] \phi_2 = 0, \end{aligned} \quad (4.4.14a)$$

where $2 \leq k_1, k_4 \leq s-1$. Similarly, for any fixed k_1 , if one subtracts (4.4.3g) with $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = 0$, $u_3 = q_8$, $i_{q_8} = k_8$, $j = 1$ from (4.4.3h) with $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_4$, $i_{q_4} = 0$, $u_3 = q_8$, $i_{q_8} = k_8$, $j = 1$, one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 0}] \phi_1 +$$

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 k_8}] \phi_2 = 0, \quad (4.4.14b)$$

where $2 \leq k_1, k_8 \leq s-1$. For any fixed k_1 , if one subtracts (4.4.3b) with $j = 1$, $u = q_1$, $i_{q_1} = 0$ from (4.4.3b) with $j = 1$, $u = q_1$, $i_{q_1} = k_1$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 1}] \phi_1 +$$

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 1}] \phi_2 = 0, \quad (4.4.15)$$

where $2 \leq k_1 \leq s-1$. For any fixed k_1 , subtracting (4.4.3c) with $j = 1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$ from (4.4.3d) with $j = 1$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_4$, $i_{q_4} = k_4$, one gets

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 k_4}] \phi_1 +$$

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 1}] \phi_2 = 0, \quad (4.4.16a)$$

where $2 \leq k_1, k_4 \leq s-1$. Similarly, for any fixed k_1 , if one subtracts (4.4.3c) with $j = 1$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_8$, $i_{q_8} = k_8$ from (4.4.3d) with $j = 1$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_8$, $i_{q_8} = k_8$, one obtains

$$(p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 1}] \phi_1 +$$

$$[(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 k_8}] \phi_2 = 0, \quad (4.4.16b)$$

where $2 \leq k_1, k_8 \leq s-1$. For any fixed k_1 , subtracting (4.4.3h) with $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$, $u_3 = q_8$, $i_{q_8} = k_8$, $j = 1$ from (4.4.3j) with $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_4$, $i_{q_4} = k_4$, $u_3 = q_8$, $i_{q_8} = k_8$, $j = 1$, one gets

$$\begin{aligned}
& (p'_{q_1 k_1} - p'_{q_1 0}) \theta_{q_1} + [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_4 k_4}] \phi_1 + \\
& [(p'_{q_1 k_1} - p'_{q_1 0}) \times p'_{q_2 1} \times p'_{q_3 1} \times p'_{q_8 k_8}] \phi_2 = 0, \quad (4.4.17)
\end{aligned}$$

where $2 \leq k_1, k_4, k_8 \leq s-1$. Now from the equations (4.4.6-17) and from **Lemma 1.8.1**, one observes that

$$\theta_{q_1} = 0. \quad (4.4.18a)$$

Exactly in a similar manner, it can be shown that

$$\theta_{q_2} = 0, \theta_{q_3} = 0, \theta_{q_4} = 0, \theta_{q_8} = 0. \quad (4.4.18b)$$

Here one may note that the equations (4.4.19-21), (4.4.23-24), (4.4.26-36) are obtained on using (4.4.18a,b). Now for any fixed k_4 , if one subtracts (4.4.3a) with $j = 0$ from (4.4.3b) with $u = q_4$, $i_{q_4} = k_4$, $j = 0$, one obtains

$$[p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.19)$$

where $1 \leq k_4 \leq s-1$. For any fixed k_4 , subtracting (4.4.3f) with $j = k_4$, $u_1 = q_2$, $u_2 = q_3$, $u_3 = q_4$ from (4.4.3e) with $j = k_4$, $u_1 = q_2$, $u_2 = q_3$, one obtains

$$[p'_{q_1 k_4} \times p'_{q_2 0} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.20a)$$

where $1 \leq k_4 \leq s-1$. Similarly for any fixed k_4 , if one subtracts (4.4.3f) with suitable choices of u_1, u_2, u_3 and j from (4.4.3e) with suitable choices of u_1, u_2 and j , one obtains

$$[p'_{q_1 0} \times p'_{q_2 k_4} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0 \quad (4.4.20b)$$

and

$$[p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.20c)$$

where $1 \leq k_4 \leq s-1$. For any fixed k_4 , subtracting (4.4.3f) with $j = k_1$, $u_1 = q_2$, $u_2 = q_3$, $u_3 = q_4$ from (4.4.3g) with $j = k_1$, $u_1 = q_2$, $i_{q_2} = 0$, $u_2 = q_3$, $i_{q_3} = 0$, $u_3 = q_4$, $i_{q_4} = k_4$, one gets

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.21a)$$

where $1 \leq k_1 \leq k_4-1$, $2 \leq k_4 \leq s-1$. Similarly, for any fixed k_4 ($2 \leq k_4 \leq s-1$), if one subtracts (4.4.3f) with suitable choices of u_1 , u_2 , u_3 and j from (4.4.3g) with suitable choices of u_1 , i_{u_1} , u_2 , i_{u_2} , u_3 , i_{u_3} and j , one obtains

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.21b)$$

where $1 \leq k_2 \leq k_4-1$, and

$$[p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.21c)$$

where $1 \leq k_3 \leq k_4-1$. For any fixed k_4 , subtracting (4.4.3b) with $j = 0$, $u = q_1$, $i_{q_1} = k_1$ from (4.4.3g) with $j = k_4$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = 0$, $u_3 = q_3$, $i_{q_3} = 0$ and using (4.4.5) and (4.4.18b), one obtains

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 + [p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 0} \times (p'_{q_8 k_4} - p'_{q_8 0})] \phi_2 = 0, \quad (4.4.22a)$$

where $k_4+1 \leq k_1 \leq s-1$. Similarly, for any fixed k_4 ($1 \leq k_4 \leq s-2$), if one subtracts (4.4.3b) with suitable choices of u , i_u and j from (4.4.3g) with suitable choices of u_1 , i_{u_1} , u_2 , i_{u_2} , u_3 , i_{u_3} and j and if one uses (4.4.5) and (4.4.18b), one obtains

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 + [p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_8 k_4} - p'_{q_8 0})] \phi_2 = 0 , \quad (4.4.22b)$$

where $k_4 + 1 \leq k_2 \leq s-1$, and

$$[p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 + [p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_8 k_4} - p'_{q_8 0})] \phi_2 = 0 , \quad (4.4.22c)$$

where $k_4 + 1 \leq k_3 \leq s-1$. For any fixed k_4 ($1 \leq k_4 \leq s-1$) , if one subtracts (4.4.3e) with $j = k_4$, $u_1 = q_1$, $u_2 = q_4$ from (4.4.3b) with $j = k_4$, $u = q_1$, $i_{q_1} = 0$, one gets

$$[p'_{q_1 0} \times p'_{q_2 k_4} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0 . \quad (4.4.23a)$$

Similarly, for any fixed k_4 ($1 \leq k_4 \leq s-1$) , if one subtracts (4.4.3e) with suitable choices of u_1 , u_2 and j from (4.4.3b) with suitable choices of u , i_u and j , one obtains the following equations

$$[p'_{q_1 k_4} \times p'_{q_2 0} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0 \quad (4.4.23b)$$

and

$$[p'_{q_1 k_4} \times p'_{q_2 k_4} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0 . \quad (4.4.23c)$$

For any fixed k_4 , subtracting (4.4.3e) with $j = k_2$, $u_1 = q_1$, $u_2 = q_4$ from (4.4.3c) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_4$, $i_{q_4} = k_4$, one gets

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_2} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0 , \quad (4.4.24a)$$

where $2 \leq k_4 \leq s-1$, $1 \leq k_2 \leq k_4-1$. Similarly, for any fixed k_4 , if one subtracts

(4.4.3e) suitably choosing u_1 , u_2 and j from (4.5.3c) with suitable choices of u_1 , i_{u_1} , u_2 , i_{u_2} and j , one obtains the following equations

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_1} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.24b)$$

where $2 \leq k_4 \leq s-1$, $1 \leq k_1 \leq k_4-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_1} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.24c)$$

where $2 \leq k_4 \leq s-1$, $1 \leq k_1 \leq k_4-1$. For any fixed k_4 , subtracting (4.4.3f) with $j = k_2$, $u_1 = q_1$, $u_2 = q_4$, $u_3 = q_8$ from (4.4.3h) with $j = k_4$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$, $u_2 = q_3$, $i_{q_3} = k_2$ and using (4.4.5) and (4.4.18b), one gets

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_2} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 + [p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_2} \times (p'_{q_8 k_4} - p'_{q_8 0})] \phi_2 = 0, \quad (4.4.25a)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_2 \leq s-1$. Similarly, for any fixed k_4 , if one subtracts (4.4.3f) with suitable choices of u_1 , u_2 , u_3 and j from (4.4.3h) with suitable choices of u_1 , i_{u_1} , u_2 , i_{u_2} , u_3 , i_{u_3} and j and if one uses (4.4.5) and (4.4.18b), one obtains the following equations

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_1} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 + (p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_1} \times (p'_{q_8 k_4} - p'_{q_8 0})) \phi_2 = 0, \quad (4.4.25b)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_1 \leq s-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_1} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 + [p'_{q_1 k_1} \times p'_{q_2 k_1} \times$$

$$p'_{q_3 0} \times (p'_{q_8 k_4} - p'_{q_8 0}) \phi_2 = 0, \quad (4.4.25c)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_1 \leq s-1$. For any fixed k_4 , subtracting (4.4.3g) with $j = k_4$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = 0$, $u_3 = q_4$, $i_{q_4} = 0$ from (4.4.3c) with $j = k_4$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = 0$, one gets

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.26a)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_1 \leq s-1$. Similarly, for any fixed k_4 , if one subtracts (4.4.3g) with suitable choices of u_1 , i_{u_1} , u_2 , i_{u_2} , u_3 , i_{u_3} and j from (4.4.3c) with suitable choices of u_1 , i_{u_1} , u_2 , i_{u_2} and j , one obtains the following equations

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.26b)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_2 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 k_4} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.26c)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_1 \leq s-1$,

$$[p'_{q_1 0} \times p'_{q_2 k_4} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.26d)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_3 \leq s-1$,

$$[p'_{q_1 k_4} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.26e)$$

where $1 \leq k_4 \leq s-2$, $k_4+1 \leq k_2 \leq s-1$, and

$$[p'_{q_1 k_4} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.26f)$$

where $1 \leq k_4 \leq s-2$, $k_4 + 1 \leq k_3 \leq s-1$. For any fixed k_4 ($2 \leq k_4 \leq s-1$), subtracting (4.4.3g) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$, $u_3 = q_4$, $i_{q_4} = 0$ from (4.4.3h) with $j = k_3$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_2$, $i_{q_2} = k_2$, $u_3 = q_4$, $i_{q_4} = k_4$, one gets

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.27a)$$

where $1 \leq k_3 \leq s-2$, $k_3+1 \leq k_2$, $k_4 \leq s-1$. Similarly, for any fixed k_4 ($2 \leq k_4 \leq s-1$), if one subtracts, choosing suitably the values of u_1 , i_{u_1} , u_2 , i_{u_2} , u_3 , i_{u_3} and j , the equation (4.4.3g) from (4.4.3h), one obtains the following equations

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.27b)$$

where $1 \leq k_2 \leq s-2$, $k_2+1 \leq k_3$, $k_4 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.27c)$$

where $1 \leq k_1 \leq s-2$, $k_1+1 \leq k_3$, $k_4 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.27d)$$

where $1 \leq k_3 \leq s-2$, $k_3+1 \leq k_1$, $k_4 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.27e)$$

where $1 \leq k_1 \leq s-2$, $k_1+1 \leq k_2$, $k_4 \leq s-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.27f)$$

where $1 \leq k_2 \leq s-2$, $k_2+1 \leq k_1$, $k_4 \leq s-1$. For any fixed k_4 ($1 \leq k_4 \leq s-3$), subtracting (4.4.3g) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$, $u_3 = q_4$, $i_{q_4} = 0$

from (4.4.3i) with $j = k_2$, $u_1 = q_1$, $i_{q_1} = 0$, $u_2 = q_3$, $i_{q_3} = k_3$, $u_3 = q_4$, $i_{q_4} = k_4$, one obtains

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.28a)$$

where $k_4+1 \leq k_2 \leq s-2$, $k_2+1 \leq k_3 \leq s-1$. Similarly, for any fixed k_4 ($1 \leq k_4 \leq s-3$), if one subtracts (4.4.3g) from (4.4.3i) with suitable choices of u_1 , i_{u_1} , u_2 , i_{u_2} , u_3 , i_{u_3} and j , one obtains the following equations

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.28b)$$

where $k_4+1 \leq k_3 \leq s-2$, $k_3+1 \leq k_2 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.28c)$$

where $k_4+1 \leq k_1 \leq s-2$, $k_1+1 \leq k_3 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.28d)$$

where $k_4+1 \leq k_3 \leq s-2$, $k_3+1 \leq k_1 \leq s-1$,

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.28e)$$

where $k_4+1 \leq k_1 \leq s-2$, $k_1+1 \leq k_2 \leq s-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.28f)$$

where $k_4+1 \leq k_2 \leq s-2$, $k_2+1 \leq k_1 \leq s-1$. For any fixed k_4 ($1 \leq k_4 \leq s-1$), if one subtracts (4.4.3b) with $u = q_4$, $i_{q_4} = 0$, $j = k_4$ from (4.4.3a) with $j = k_4$, one gets

$$[p'_{q_1 k_4} \times p'_{q_2 k_4} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0. \quad (4.4.29)$$

(4.4.3c) with suitable choices of $u_1, i_{u_1}, u_2, i_{u_2}$ and j from (4.4.3b) with suitable choices of u, i_u and j , one obtains the following equations

$$[p'_{q_1 k_4} \times p'_{q_2 k_2} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.32b)$$

where $k_4+1 \leq k_2 \leq s-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_4} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.32c)$$

where $k_4+1 \leq k_1 \leq s-1$. For any fixed k_4 ($1 \leq k_4 \leq s-3$), if one subtracts (4.4.3h) with $j = k_1, u_1 = q_2, i_{q_2} = k_2, u_2 = q_3, i_{q_3} = k_3, u_3 = q_4, i_{q_4} = 0$ from (4.4.3h) with $j = k_1, u_1 = q_2, i_{q_2} = k_2, u_2 = q_3, i_{q_3} = k_3, u_3 = q_4, i_{q_4} = k_4$, one gets

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.33a)$$

where $k_4+1 \leq k_1 \leq s-2, k_1+1 \leq k_2, k_3 \leq s-1$. Similarly, for any fixed k_4 ($1 \leq k_4 \leq s-3$), subtracting (4.4.3h) from itself with suitable choices of $u_1, i_{u_1}, u_2, i_{u_2}, u_3, i_{u_3}$ and j , one obtains the following equations

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.33b)$$

where $k_4+1 \leq k_2 \leq s-2, k_2+1 \leq k_1, k_3 \leq s-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.33c)$$

where $k_4+1 \leq k_3 \leq s-2, k_3+1 \leq k_1, k_2 \leq s-1$. For any fixed k_4 ($1 \leq k_4 \leq s-3$), if one subtracts (4.4.3c) with $j = k_1, u_1 = q_3, i_{q_3} = k_3, u_2 = q_4, i_{q_4} = 0$ from (4.4.3c) with $j = k_1, u_1 = q_3, i_{q_3} = k_3, u_2 = q_4, i_{q_4} = k_4$, one obtains

$$[p'_{q_1 k_1} \times p'_{q_2 k_1} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.34a)$$

where $k_4+1 \leq k_1 \leq s-2$, $k_1+1 \leq k_3 \leq s-1$. Similarly, for any fixed k_4 ($1 \leq k_4 \leq s-3$), subtracting (4.4.3c) from itself with suitable choices of $u_1, i_{u_1}, u_2, i_{u_2}$ and j , one obtains the following equations

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_1} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.34b)$$

where $k_4+1 \leq k_1 \leq s-2$, $k_1+1 \leq k_2 \leq s-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_2} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.34c)$$

where $k_4+1 \leq k_2 \leq s-2$, $k_2+1 \leq k_1 \leq s-1$. For any fixed k_4 ($2 \leq k_4 \leq s-1$), if one subtracts (4.4.3c) with $j = k_1$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_4$, $i_{q_4} = 0$ from (4.4.3d) with $j = k_1$, $u_1 = q_3$, $i_{q_3} = k_3$, $u_2 = q_4$, $i_{q_4} = k_4$, one gets

$$[p'_{q_1 k_1} \times p'_{q_2 k_1} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.35a)$$

where $1 \leq k_1 \leq s-2$, $k_1+1 \leq k_3$, $k_4 \leq s-1$. Similarly, for any fixed k_4 ($2 \leq k_4 \leq s-1$), subtracting (4.4.3c) from (4.4.3d) with suitable choices of $u_1, i_{u_1}, u_2, i_{u_2}$ and j , one obtains the following equations

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_1} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.35b)$$

where $1 \leq k_1 \leq s-2$, $k_1+1 \leq k_2$, $k_4 \leq s-1$, and

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_2} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.35c)$$

where $1 \leq k_2 \leq s-2$, $k_2+1 \leq k_1$, $k_4 \leq s-1$. Finally, for any fixed k_4 ($1 \leq k_4 \leq s-2$), if one subtracts (4.4.3h) with $j = k_4$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = k_2$, $u_3 = q_4$, $i_{q_4} = 0$ from (4.4.3d) with $j = k_4$, $u_1 = q_1$, $i_{q_1} = k_1$, $u_2 = q_2$, $i_{q_2} = k_2$, one gets

$$[p'_{q_1 k_1} \times p'_{q_2 k_2} \times p'_{q_3 k_4} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.36a)$$

where $k_4 + 1 \leq k_1, k_2 \leq s-1$. Similarly, for any fixed k_4 ($1 \leq k_4 \leq s-2$), subtracting (4.4.3h) with suitable choices of $u_1, i_{u_1}, u_2, i_{u_2}, u_3, i_{u_3}$ and j from (4.4.3d) with suitable choices of $u_1, i_{u_1}, u_2, i_{u_2}$ and j , one obtains the following equations

$$[p'_{q_1 k_1} \times p'_{q_2 k_4} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.36b)$$

where $k_4 + 1 \leq k_1, k_3 \leq s-1$, and

$$[p'_{q_1 k_4} \times p'_{q_2 k_2} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.36c)$$

where $k_4 + 1 \leq k_2, k_3 \leq s-1$. From the equations (4.4.25b), (4.4.30b), (4.4.31), (4.4.33b), (4.4.34b) and (4.4.36b), one can obtain for any fixed k_1, k_4 ,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_1} \times (p'_{q_8 k_4} - p'_{q_8 0})] \phi_2 = 0, \quad (4.4.37)$$

where $1 \leq k_4 < k_1 \leq s-1$. Now, if one subtracts (4.4.37) from (4.4.25b), one obtains for any fixed k_1, k_4 ,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 k_1} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.38a)$$

where $1 \leq k_4 < k_1 \leq s-1$. Exactly in a similar manner, one can obtain, for any fixed k_2, k_4 , from the equations (4.4.25a), (4.4.30a), (4.4.31), (4.4.33a), (4.4.34c) and (4.4.36c)

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 k_2} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.38b)$$

where $1 \leq k_4 < k_2 \leq s-1$ and for any fixed k_1, k_4 , one gets from the equations (4.4.25c), (4.4.30c), (4.4.31), (4.4.33c), (4.4.34a) and (4.4.36a)

$$[p'_{q_1 k_1} \times p'_{q_2 k_1} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.38c)$$

where $1 \leq k_4 < k_1 \leq s-1$. From equations (4.4.22a), (4.4.26a), (4.4.27a), (4.4.28c,d) and (4.4.38a), one obtains for any fixed k_1, k_4 ,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_2 = 0, \quad (4.4.39)$$

where $1 \leq k_4 < k_1 \leq s-1$. If one subtracts (4.4.39) from (4.4.22a), one obtains for any fixed k_1, k_4 ,

$$[p'_{q_1 k_1} \times p'_{q_2 0} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.40a)$$

where $1 \leq k_4 < k_1 \leq s-1$. Exactly in a similar way, one can obtain, for any fixed k_2, k_4 , from the equations (4.4.22b), (4.4.26b), (4.4.27a), (4.4.28a,b) and (4.4.38b),

$$[p'_{q_1 0} \times p'_{q_2 k_2} \times p'_{q_3 0} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.40b)$$

where $1 \leq k_4 < k_2 \leq s-1$ and from the equations (4.4.22c), (4.4.26d), (4.4.27b), (4.4.28 a,b) and (4.4.38b), one obtains for any fixed k_3, k_4 ,

$$[p'_{q_1 0} \times p'_{q_2 0} \times p'_{q_3 k_3} \times (p'_{q_4 k_4} - p'_{q_4 0})] \phi_1 = 0, \quad (4.4.40c)$$

where $1 \leq k_4 < k_3 \leq s-1$. Now from the equations (4.4.19-21), (4.4.23-24), (4.4.26-36), (4.4.38) and (4.4.40) it follows that $[P_{q_1} \times P_{q_2} \times P_{q_3} \times P_{q_4}^*] \phi_1 = 0$, where $P_{q_4}^*$ is an $((s-1) \times (s-1))$ matrix with rows $(p'_{q_4 k_4} - p'_{q_4 0})$, $1 \leq k_4 \leq s-1$. From **Lemma 1.8.1**, now it follows that

$$\phi_1 = 0. \quad (4.4.41)$$

From the equations (4.4.3a-j), it is now easy to verify that $\phi_2 = 0$. The proofs of the other cases follow exactly in a similar manner.

Chapter 5A

LINEAR-TREND-FREE ORTHOGONAL MAIN-EFFECT PLANS

5A.1 Introduction and Preliminaries

This chapter considers the construction of orthogonal main-effect plans for factorial experiments under the presence of a linear time trend. In industrial and agricultural experimentation, when the observations are taken over a period of time, it happens in many practical situations that the observations are affected by a time trend in addition to the effects of the treatments under consideration. The problem of factorial experimentation in the presence of such time trend has been considered among others by Hill (1960), Daniel and Wilcoxon (1966), Draper and Stoneman (1968), Phillips (1968) and Joiner and Campbell (1976) (see also Dey (1985) in this context) and more recently by Cheng (1985), Coster and Cheng (1988) and Cheng and Jacroux (1988). These authors discussed in detail the philosophical basis of the problem, cited numerous practical applications and suggested relevant experimental plans mainly with reference to 2^m factorials. In addition to trend-free fractional factorials, there is a vast literature on trend-free block designs for varietal trials. This is, however, beyond the scope of this chapter and we only refer to Bradley and Yeh (1980), Yeh and Bradley (1983) and Stufken (1988) and the references therein. It, however, appears that much work yet remains to be done on fractions of more complex factorials in the presence of a time trend. In an attempt to fill this gap to some extent, the present chapter considers the development of orthogonal main-effect plans for general factorials in the presence of a linear time trend.

Consider an $s_1 \times \dots \times s_m$ factorial, a typical level combination being denoted by $(i_1, \dots, i_m)$, $0 \leq i_v \leq s_v - 1$, $1 \leq v \leq m$. Let $S = \{(i_1^{(t)}, \dots, i_m^{(t)}) : 0 \leq t \leq N-1\}$ be a subset of level combinations such that the combination $(i_1^{(t)}, \dots, i_m^{(t)})$ is applied at time point t . Throughout this chapter, the N time points are assumed to

be equispaced as happens in most practical situations. Let y_t denote the observation corresponding to the combination $(i_1^{(t)}, \dots, i_m^{(t)})$. Then assuming a linear time trend and the absence of all interactions, we take the following fixed effects linear model :

$$E(y_t) = \alpha_0 + \alpha_1 t + \gamma_{1i_1}(t) + \gamma_{2i_2}(t) + \dots + \gamma_{mi_m}(t), \quad (5A.1.1)$$

where $\alpha_0, \alpha_1, \gamma_{vi_v}(t)$ ($1 \leq v \leq m$) are unknown parameters. For $0 \leq j \leq s_v - 1$, γ_{vj} represents the fixed effect due to the level j of the v th factor ($1 \leq v \leq m$) while α_0 and α_1 are parameters associated with the linear time trend. As usual $\sum_{j=0}^{s_v-1} \gamma_{vj} = 0$ for each v ($1 \leq v \leq m$) and it is assumed that the N observations are independent and homoscedastic.

A fraction, involving N observations or runs, of an $s_1 \times \dots \times s_m$ factorial will be called a **Trend-Free Orthogonal Main Effect** plan (TFOME plan) if under the model (5A.1.1) it allows estimation of the main effect contrasts on γ_{vj} orthogonally among themselves and also orthogonally to $\hat{\alpha}_0$ and $\hat{\alpha}_1$, the best linear unbiased estimators of α_0 and α_1 respectively. In the absence of a time trend (i.e., if $\alpha_1 = 0$ in (5A.1.1)) it is well-known (cf. Rao (1947, 1973)) that an orthogonal array of strength 2, with its columns interpreted as level combinations, represents an orthogonal main effect plan and such plans are universally optimal for the estimation of the main effects (see e.g., Cheng (1980), Mukerjee (1982)). It is, therefore, natural to examine the conditions under which such an array represents a TFOME plan under the model (5A.1.1). In this connection, we have the following theorem, where the notations relating to the subset S are the same as in the preceding paragraph. Further, for $0 \leq j \leq s_v - 1$, $1 \leq v \leq m$, let R_{vj} denote the sum of the time points in which the level j of the v th factor

is applied, i.e., $R_{vj} = \sum t$, the summation being taken over those t for which $i_v^{(t)}$ equals j .

Theorem 5A.1.1. Assume that the N level combinations included in S form an orthogonal array of strength 2 and further that

$$R_{v0} = R_{v1} = \dots = R_{vs_v-1} \quad \text{for each } v \ (1 \leq v \leq m). \quad (5A.1.2)$$

Then S represents a TFOME plan for an $s_1 \times \dots \times s_m$ factorial.

The proof of Theorem 5A.1.1 follows essentially along the line of Phillips (1968) and is hence omitted here. The plans satisfying the conditions of Theorem 5A.1.1 are not only simple to analyze but also may be seen (cf. Cheng (1980)) to be universally optimal for the estimation of the main effects under the presence of a linear trend. Theorem 5A.1.1 serves as a very powerful tool in identifying the orthogonal arrays of strength 2 which represent TFOME plans. In fact orthogonal arrays of strength 2 are available in abundance and one has only to investigate under what permutation of the columns (which dictates the sequence in which the corresponding level combinations have to be applied) such an array satisfies the condition (5A.1.2) in Theorem 5A.1.1. The problem is highly non-trivial since there are orthogonal arrays for which no permutation of the columns exists satisfying these conditions. For example, for the orthogonal array of strength 2

$$\begin{array}{cccc} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1, \\ 0 & 1 & 1 & 0 \end{array}$$

which represents a four-run fraction of a 2^3 factorial, it may be seen that none of the 4! permutations of the columns satisfies (5A.1.2) and hence the array is incapable of generating a TFOME plan.

A comprehensive discussion on the available methods for the construction of orthogonal arrays of strength 2 is available in Raghavarao (1971). In the rest of the chapter, we shall investigate the appropriate modifications of these methods so as to generate TFOME plans. As usual, an orthogonal array (with possibly variable symbols - see e.g., Rao (1973)) with N assemblies, m constraints, $s_1, \dots, s_m$ symbols and strength 2 will be denoted by an $OA[N, m, s_1, \dots, s_m, 2]$. In particular, if $s_1 = s_2 = \dots = s_m = s$, then it will be denoted by an $OA[N, m, s, 2]$.

5A.2 Use of Projective Geometry

Let B_1 be an $OA[s^r, m+r, s, 2]$ where $m, r \geq 2$. Suppose B_1 can be partitioned as $B_1 = [B_2', B_3']$, where B_2 is $(r \times s^r)$ and B_3 is $(m \times s^r)$ such that for $0 \leq l_1, l_2, \dots, l_r \leq s-1$, the $(l_1 s^{r-1} + l_2 s^{r-2} + \dots + l_r)$ th column of B_2 is given by $(l_1, l_2, \dots, l_r)'$. For example, with $s = 3, r = 3$, B_2 should be of the form

$$B_2: \begin{array}{l} 000000000111111111222222222 \\ 000111222000111222000111222 \\ 012012012012012012012012012 \end{array}$$

Then we have the following lemma which will be helpful in the sequel:

Lemma 5A.2.1. The columns of B_3 , interpreted as level combinations, represent a TFOME plan in s^r runs for an s^m factorial.

Proof. Clearly B_3 is an $OA[s^r, m, s, 2]$. It remains to verify the condition (5A.1.2) in Theorem 5A.1.1. Take any row of B_3 (say, the v th row, $1 \leq v \leq m$) and note that for a fixed j ($0 \leq j \leq s-1$) the symbol j occurs in exactly s^{r-1} positions in that row. Let a typical position occupied by j in this row be the $(\xi_1 s^{r-1} + \xi_2 s^{r-2} + \dots + \xi_r)$ th position where $0 \leq \xi_1, \xi_2, \dots, \xi_r \leq s-1$. Since B_1 is an orthogonal array of strength 2, in the two-rowed submatrix of B_1 given by the first row of B_2 and the v th row of B_3 every ordered pair of symbols

occurs as a column vector s^{r-2} times. This phenomenon, together with the special structure of B_2 , implies that ξ_1 must take each of the values $0, 1, \dots, s-1$ exactly s^{r-2} times. Similarly, it follows that each of $\xi_1, \dots, \xi_r$ must take each of the values $0, 1, \dots, s-1$ exactly s^{r-2} times. Hence with the notations introduced earlier and with reference to the array B_3 , one has

$$R_{vj} = s^{r-2} (0+1 + \dots + s-1) (s^{r-1} + s^{r-2} + \dots + 1) = \frac{s^{r-1}(s^r-1)}{2},$$

$$(0 \leq j \leq s-1; 1 \leq v \leq m). \quad (5A.2.1)$$

Since the right-hand member of (5A.2.1) does not depend on j , the lemma follows.

Remark : If $r=2$ in particular, then B_1 is an orthogonal array of strength 2 and index unity and, by a suitable permutation of the columns, B_1 can always be so written that its first two rows are of the form:

$$\begin{array}{cccccccccccc} 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots & s-1 & s-1 & \dots & s-1 \\ 0 & 1 & \dots & s-1 & 0 & 1 & \dots & s-1 & \dots & 0 & 1 & \dots & s-1 \end{array}$$

It is then immediate from **Lemma 5A.2.1** that the existence of an $OA[s^2, m+2, s, 2]$ always implies the existence of a TFOME plan in s^2 runs for an s^m factorial.

Lemma 5A.2.1 can be applied to construct TFOME plans for s^m factorials using finite geometries by suitably modifying the methods described in Sections 2.3, 2.4 in Raghavarao (1971). Let s be a prime or prime power, $\rho_0, \rho_1, \dots, \rho_{s-1}$ be the elements of $GF(s)$ and $r (\geq 2)$ be a positive integer. Let U be the set of all points in the projective geometry $PG(r-1, s)$. There are $(s^r-1)/(s-1)$ members in U , a typical member being denoted by $(b_1, \dots, b_r)$. Suppose the first r members of U are $\tau_1 = (1, 0, \dots, 0)$, $\tau_2 = (0, 1, \dots, 0)$, $\dots$, $\tau_r = (0, 0, \dots, 1)$. Form an $((s^r-1)/(s-1)) \times s^r$ array B_1 such that

- (i) the rows of B_1 are identified with the members of U ,
- (ii) for any $\ell_1, \ell_2, \dots, \ell_r$ ($0 \leq \ell_1, \ell_2, \dots, \ell_r \leq s-1$), in the $(b_1, \dots,$

b_r)th row of B_1 the symbol j occurs in the $(\ell_1 s^{r-1} + \ell_2 s^{r-2} + \dots + \ell_r)$ th position if $b_1 \rho_{\ell_1} + b_2 \rho_{\ell_2} + \dots + b_r \rho_{\ell_r} = \rho_j$.

Then following Raghavarao (1971), B_1 is an $OA[s^r, (s^r-1)/(s-1), s, 2]$. Further, from the special pattern of the first r members of U , it may be seen that B_1 admits a partitioning $B_1 = [B_2', B_3']'$, where B_2 is $(r \times s^r)$ and satisfies the conditions stated in the paragraph preceding **Lemma 5A.2.1**. Hence by **Lemma 5A.2.1**, the array B_3 , which is an $OA[s^r, m, s, 2]$ where $m = (s^r-1)/(s-1) - r$, represents a TFOME plan for an s^m factorial. The following example serves as an illustration.

Example 5A.2.1. Let $s = 4$, $r = 2$. Then the elements of $GF(s)$ are $\rho_0 = 0$, $\rho_1 = 1$, $\rho_2 = x$, $\rho_3 = x+1$ (with $x^2 \equiv x+1$) and taking $U = \{(1,0), (0,1), (1,1), (1,x), (1,x+1)\}$, one obtains

```

0000111122223333
0123012301230123
B1: 0123103223013210
      0231132020133102
      0312120321303021

```

The subarray B_3 , given by the last three rows of B_1 represents a TFOME plan for a 4^3 factorial in 16 runs.

5A.3 The Method of Differences

Here we consider a modification of a method described in Section 2.5 in Raghavarao (1971) so as to obtain TFOME plans. Let V be a module with s elements $u_0, u_1, \dots, u_{s-1}$, the zero element being given by u_0 . Let A_0 be an $((m+1) \times r)$ array with entries given by the members of V such that among the differences arising out of the corresponding elements in any two distinct rows of A_0 , every element of V occurs exactly λ times (i.e., $r = \lambda s$). Note that

this property of A_0 remains unaltered if a constant is added to each column of A_0 (the constants added to the different columns of A_0 being possibly different). Hence, without loss of generality, it may be assumed that all the entries in the top row of A_0 equal u_0 . For $1 \leq j \leq s-1$, let A_j be an $((m+1) \times r)$ array obtained by adding u_j to each element of A_0 . Let $A = [A_0, A_1, \dots, A_{s-1}]$, and $\bar{A}$ be an $(m \times rs)$ array obtained by deleting the top row of A . Then the following holds:

Theorem 5A.3.1. The columns of $\bar{A}$, interpreted as level combinations, represent a TFOME plan in $rs(= \lambda s^2)$ runs for an s^m factorial.

Proof: From Raghavarao (1971, p. 23) $\bar{A}$ is an $OA[rs, m, s, 2]$. In order to verify the condition (5A.1.2) in Theorem 5A.1.1, suppose $\bar{A}$ is partitioned as $\bar{A} = [\bar{A}_0, \bar{A}_1, \dots, \bar{A}_{s-1}]$ where each $\bar{A}_j$ ($0 \leq j \leq s-1$) is of size $(m \times r)$. Consider any row (say, the v th row, $1 \leq v \leq m$) of $\bar{A}$. In view of the special structure of A_0 , the following may be observed:

- (i) in the j th row of $\bar{A}_0$, each element of V occurs exactly λ times,
- (ii) for any fixed j, j' ($0 \leq j, j' \leq s-1$), if $u_j - u_{j'} = u_{\ell}$, then the element u_j occurs in the v th row of $\bar{A}_{\ell}$ in precisely the same positions as $u_{j'}$ occurs in the v th row of $\bar{A}_0$.

From these considerations, it follows that the condition (5A.1.2) holds for the array A , completing the proof.

5A.4 The Modified Addelman-Kempthorne Plans

First we prove a lemma which will be helpful in the subsequent development. Let L_0 be an $OA[2s^r, m+r, s, 2]$, where $m, r \geq 2$. Suppose L_0 can be partitioned as

$$L_0 = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \quad (5A.4.1)$$

where L_1, L_2, L_3, L_4 are of orders $(r \times s^r), (r \times s^r), (m \times s^r), (m \times s^r)$ respectively, such that the following conditions hold :

- (a) for $0 \leq l_1, l_2, \dots, l_r \leq s-1$, the $(l_1 s^{r-1} + l_2 s^{r-2} + \dots + l_r)$ th columns of both L_1 and L_2 are both given by $(l_1, l_2, \dots, l_r)'$ (cf. the discussion preceding **Lemma 5A.2.1**),
- (b) in each row of L_3 , each of the s symbols occurs s^{r-1} times.

Then the following holds :

Lemma 5A.4.1. Under the above assumptions, the columns of $\bar{L}_0 = [L_3 \ L_4]$, interpreted as level combinations, represent a TFOME plan in $2s^r$ runs for an s^m factorial.

Proof. The proof proceeds along the line of proof of **Lemma 5A.2.1**. Clearly, $\bar{L}_0$ is an $OA[2s^r, m, s, 2]$. To verify the condition (5A.1.2) in Theorem 5A.1.1, consider any row of $\bar{L}_0$ (say, the v th row, $1 \leq v \leq m$) and note that by (b) above, for any fixed j ($0 \leq j \leq s-1$), the symbol j occurs s^{r-1} times in the v th row of L_3 and s^{r-1} times in the v th row of L_4 . Suppose a typical position in the v th row of L_3 where j occurs is the $(\xi_1 s^{r-1} + \xi_2 s^{r-2} + \dots + \xi_r)$ th position and a typical position in the v th row of L_4 where j occurs is the $(\zeta_1 s^{r-1} + \zeta_2 s^{r-2} + \dots + \zeta_r)$ th position, where $0 \leq \xi_1, \xi_2, \dots, \xi_r, \zeta_1, \zeta_2, \dots, \zeta_r \leq s-1$. For $0 \leq w \leq s-1$, $1 \leq i \leq r$, let ϕ_{iw} be the number of times ξ_i equals w , and ψ_{iw} be the number of times ζ_i equals w .

Now, since L_0 is an orthogonal array of strength 2, in the two-rowed sub-matrix of L_0 given by the top row of $[L_1 \ L_2]$ and the v th row of $[L_3 \ L_4]$, every ordered pair of symbols occurs as a column vector $2s^{r-2}$ times. Therefore,

by the special structure of L_1, L_2 described in (a) above, it follows that $\phi_{1w} + \psi_{1w} = 2s^{r-2} (0 \leq w \leq s-1)$. A similar argument shows that

$$\phi_{iw} + \psi_{iw} = 2s^{r-2} (0 \leq w \leq s-1, 1 \leq i \leq r). \quad (5A.4.2)$$

Hence under the notations introduced in Section 5A.1 and with reference to the array $\bar{L}_0$, one has

$$R_{vj} = \sum_{i=1}^r \sum_{w=0}^{s-1} w\phi_{iw} s^{r-i} + \sum_{i=1}^r \sum_{w=0}^{s-1} w\psi_{iw} s^{r-i} + s^{r-1} \times s^r = s^{r-1}(2s^r-1), \quad (5A.4.3)$$

by (5A.4.2), on simplification. Thus R_{vj} does not depend on j and the lemma follows by Theorem 5A.1.1. It may be remarked that the term $s^{r-1} \times s^r$ in the right-hand member of (5A.4.3) arises because (i) for any $\ell (0 \leq \ell \leq s^r-1)$ the occurrence of j in the ℓ th position in the v th row of L_4 implies the occurrence of j in the $(s^r + \ell)$ th position in the v th row of $\bar{L}_0$ and (ii) by (b) above, the symbol j occurs s^{r-1} times in the v th row of L_4 .

The above lemma may be employed for deriving TFOME plans from the well-known Addelman-Kempthorne plans. Addelman and Kempthorne (1961) developed an ingenious method for constructing an $OA[2s^r, \{2(s^r-1)/(s-1)\}-1, s, 2]$ where s is a prime or prime power and $r \geq 2$. In order to save space, we do not describe here the details of their procedure which are available in the original paper and also in Raghavarao (1971) and Dey (1985). Let L_0 be an orthogonal array obtained by the approach of Addelman and Kempthorne. Then an inspection of their procedure readily reveals that L_0 can be partitioned as in (5A.4.1) such that the conditions (a) and (b) are satisfied. Hence, by **Lemma 5A.4.1**, the array $\bar{L}_0$, obtained by deleting the first rows of L_0 , will give a TFOME plan in $2s^r$ runs for an s^m factorial, where $m = \{2(s^r-1)/(s-1)\}-1-r$.

5A.5 The Method of Direct Product

In the following, the (symbolic) direct product of two row vectors $\mathbf{a}_1 = (a_{10}, a_{11}, \dots, a_{1N_1-1})$ and $\mathbf{a}_2 = (a_{20}, a_{21}, \dots, a_{2N_2-1})$ is defined as (cf. Section 1.3 of Chapter 1)

$$\begin{aligned} \mathbf{a}_1 \otimes \mathbf{a}_2 = & [(a_{10}, a_{20}), (a_{10}, a_{21}), \dots, (a_{10}, a_{2N_2-1}), (a_{11}, a_{20}), \\ & (a_{11}, a_{21}), \dots, (a_{11}, a_{2N_2-1}), \dots, (a_{1N_1-1}, a_{20}), \\ & (a_{1N_1-1}, a_{21}), \dots, (a_{1N_1-1}, a_{2N_2-1})]. \end{aligned}$$

Theorem 5A.5.1. For $i = 1, 2$, let C_i be an $OA[N_i, m_i, s_i, 2]$ satisfying the condition (5A.1.2) in Theorem 5A.1.1. Then there exists a TFOME plan in $N_1 N_2$ runs for an s^m factorial where $s = s_1 s_2$ and $m = \min \{m_1, m_2\}$.

Proof. Form an $(m \times N_1 N_2)$ array C such that for $1 \leq v \leq m$, the v th row of C is given by the (symbolic) direct product of the v th rows of C_1 and C_2 . For $i = 1, 2$, if the symbols in C_i be $0, 1, \dots, s_i - 1$, then the symbols in C are the ordered pairs (j_1, j_2) , $0 \leq j_1 \leq s_1 - 1$, $0 \leq j_2 \leq s_2 - 1$. Let C_0 be an $(m \times N_1 N_2)$ array obtained from C by replacing for each j_1, j_2 , the ordered pair (j_1, j_2) in C by $s_2 j_1 + j_2$. It is readily seen that C_0 is an $OA[N_1 N_2, m, s, 2]$, where $s = s_1 s_2$, the s symbols in C_0 being given by $0, 1, \dots, s - 1$. From the method of construction of C_0 and also from the fact that both C_1 and C_2 satisfy the condition (5A.1.2) in Theorem 5A.1.1, it is not hard to see that C_0 also satisfies the same condition. Thus the columns of C_0 , interpreted as level combinations, give a TFOME plan for an s^m factorial in $N_1 N_2$ runs. This completes the proof.

Remark: A repeated application of the above theorem shows that for $i = 1, 2$,

$\dots, k$, if C_i be an $OA[N_i, m_i, s_i, 2]$ satisfying the condition (5A.1.2) in Theorem 5A.1.1, then there exists a TFOME plan in $\prod_{i=1}^k N_i$ runs for an s^m factorial, where $s = \prod_{i=1}^k s_i$ and $m = \min_{1 \leq i \leq k} m_i$.

5A.6 Concluding Remarks

It is evident from the preceding sections that corresponding to many of the well-known systematic methods for the construction of orthogonal arrays of strength 2 (cf. Raghavarao (1971), Dey (1985)) there exist suitable modifications which yield TFOME plans. The results described so far, however, all relate to trend free fractions for symmetric factorials. As for asymmetric factorials the problem appears to be more involved - this is more so because not much is available in the literature on systematic procedures for the construction of orthogonal arrays with variable symbols. Anyway, the following general result is easy to prove :

Theorem 5A.6.1. Let D be an orthogonal array (with variable symbols) having N assemblies, m constraints, $s_1, s_2, \dots, s_m$ symbols and strength 2. Let $\bar{D}$ be an $(m \times N)$ array obtained from D by writing the columns of D in the reversed order. Then the $(m \times 2N)$ array $D_0 = [D, \bar{D}]$, with its columns interpreted as level combinations, gives a TFOME plan in $2N$ runs of an $s_1 \times s_2 \times \dots \times s_m$ factorial.

Chapter 5B

ESTIMABILITY AND EFFICIENCY IN PROPORTIONAL FREQUENCY PLANS

5B.1 Introduction

Plackett (1946) introduced the criterion of proportional frequency in a multifactor setting. It was contended by Addelman (1962) that fractional factorial plans based on this criterion lead to orthogonal estimation of lower order factorial effects when higher order effects are absent. Lewis and John (1976) and later, in more detail, Kuwada and Nishii (1979), Mukerjee (1980) had shown that the contention of Addelman regarding orthogonality in proportional frequency plans is not in general true and established, in the general case, the criterion of equal frequency (in terms of orthogonal arrays) as a necessary and sufficient condition for orthogonal fractional factorial plans.

Unfortunately, however, as shown in Mukerjee (1980), especially in the asymmetric case, there are situations where orthogonal plans based on the equal frequency criterion are nonexistent. Hence in fractional experimentation under such circumstances one has no other alternative than to forgo orthogonality and search for possibly nonorthogonal plans. Since it is comparatively simpler to obtain proportional frequency plans (cf. Addelman (1963, 1972)) such plans may be considered in this context. But before the proportional frequency plans are used, their worth with regard to estimability and efficiency should be critically reexamined. This chapter makes a study in this line by extending the calculus of factorial experiments (cf. Kurkjian and Zelen (1963)) to a fractional factorial set-up. Though the main emphasis will be on proportional frequency plans it is anticipated that the methods developed will be helpful in dealing with other problems in fractional replication as well.

5B.2. Notations and Preliminaries

Consider a factorial experiment involving m factors $F_1, \dots, F_m$ at $s_1, \dots, s_m$ levels respectively ($s_v \geq 2, 1 \leq v \leq m$), there being $v^* = \prod_{v=1}^m s_v$ level combinations in all. Throughout this chapter the v^* level combinations will be assumed to be lexicographically ordered (cf. Kurkjian and Zelen (1963)). Let τ be the vector of treatment effects arranged lexicographically. In analogy with Chapter 1, we introduce the following notations.

Let for $1 \leq v \leq m$, $\mathbf{1}_v^{(s_v \times 1)} = (1, 1, \dots, 1)'$ and P_v be an $(s_v \times (s_v - 1))$ matrix such that $[s_v^{-1/2} \mathbf{1}_v, P_v]$ is orthogonal. Let

$$P_v^{x_v} = s_v^{-1/2} \mathbf{1}_v \text{ if } x_v = 0; = P_v \text{ if } x_v = 1. \quad (5B.2.1)$$

Further let $\Omega = \{(x_1, x_2, \dots, x_m) : x_v = 0, 1 \forall v, \text{ not all } x_v \text{'s are zero}\}$. For any $\mathbf{x} = (x_1, x_2, \dots, x_m) \in \Omega$, writing

$$P^{\mathbf{x}} = \bigotimes_{v=1}^m P_v^{x_v} = P_1^{x_1} \times \dots \times P_m^{x_m} \quad (5B.2.2)$$

where $\times$ denotes Kronecker product, it is easy to check that the $\prod_{v=1}^m (s_v - 1)^{x_v}$ linear functions $P^{\mathbf{x}'} \tau$ represent a complete set of orthonormal treatment contrasts belonging to the factorial effect $F_1^{x_1} F_2^{x_2} \dots F_m^{x_m}$ (cf. Mukerjee (1980)). At this stage we differ slightly from our approach in Chapters 1-4. Here for each factorial effect we deal with a complete set of **orthonormal** contrasts while in Chapters 1-4, we had considered any complete set of contrasts. The consideration of orthogonal contrasts is helpful, in particular, in obtaining results on efficiency (vide Section 5B.4) which is important in the context of the present chapter since the error variance is no longer assumed to be zero. Let for $1 \leq u \leq m$,

$$J_u = \{(x_1, \dots, x_m) : (x_1, \dots, x_m) \in \Omega, \text{ at most } u \text{ of } x_1, \dots, x_m \text{ are } 1\},$$

$$P^{(u)} = (\dots, P^x, \dots), \bar{P}^{(u)} = (\dots, P^x, \dots), \quad (5B.2.3)$$

P^x being included in $P^{(u)}$ ($\bar{P}^{(u)}$) if and only if $x \in J_u$ ($\Omega - J_u$).

In the factorial context it often happens that the higher order interactions, say those involving more than t factors, are absent i.e.

$$\bar{P}^{(t)'} \tau = 0. \quad (5B.2.4)$$

Then imposing the side restriction $\epsilon' \tau = 0$, where $\epsilon^{(v^* \times 1)} = (1, 1, \dots, 1)'$ and following the line of Mukerjee (1980), we have

$$\tau = P^{(t)} \tau^*, \quad (5B.2.5)$$

where τ^* is a vector of new parameters representing treatment effects under (5B.2.4).

Consider an arrangement of v^* level combinations in an (unblocked) completely randomised design such that for $1 \leq i \leq v^*$, the i th level combination is replicated r_i times.

Let $r^{(v^* \times 1)} = (r_1, r_2, \dots, r_{v^*})'$, $r^\delta = \text{Diag}(r_1, \dots, r_{v^*})$, $n = \sum_{i=1}^{v^*} r_i (> 0)$
 $T^{(v^* \times 1)}$ = vector of observed treatment totals, $G = \epsilon' T$

$$\left. \begin{aligned} C^* &= r^\delta - n^{-1} r r', \\ Q &= T - n^{-1} G r \end{aligned} \right\} \quad (5B.2.6)$$

Then following Mukerjee (1980), assuming the usual fixed effects linear model with a constant error variance (σ^2 , say) under the reparametrized set-up (5B.2.5), the reduced normal equations are found to be

$$P^{(t)'} C^* P^{(t)} P^{(t)'} \tau = P^{(t)'} Q. \quad (5B.2.7)$$

5B.3 Estimability in Proportional Frequency Plans

Consider an unblocked m -factor design in which the i th level combination

is replicated $r_i (\geq 0)$ times, $1 \leq i \leq v^*$. Let $1 \leq v_1 < v_2 < \dots < v_q \leq m$, $1 \leq q \leq m$, $0 \leq i_{v_\ell} \leq s_{v_\ell} - 1$ ($1 \leq \ell \leq q$), $n_{i_{v_1}, \dots, i_{v_q}}$ represents the number of times the level combination $(i_{v_1}, i_{v_2}, \dots, i_{v_q})$ of the factors $F_{v_1}, F_{v_2}, \dots, F_{v_q}$ occur among the level combinations included in the experiment. The chosen level combinations will be defined to represent a proportional frequency plan of strength $d (\leq m)$ if for every $v_1, v_2, \dots, v_d$ and every $i_{v_1}, i_{v_2}, \dots, i_{v_d}$ ($1 \leq v_1 < \dots < v_d \leq m$; $0 \leq i_{v_\ell} \leq s_{v_\ell} - 1$ ($1 \leq \ell \leq d$))

$$n_{i_{v_1}, \dots, i_{v_d}}^{(v_1, \dots, v_d)} = n^{-(d-1)} \prod_{\ell=1}^d n_{i_{v_\ell}}^{(v_\ell)} > 0 \quad (5B.3.1)$$

where $n = \sum r_i (> 0)$. In this section we shall study the worth of such plans from the point of view of estimability.

For $1 \leq v \leq m$, let

$$\left. \begin{aligned} D_v^* &= \text{Diag } (n_0^{(v)}, \dots, n_{s_v-1}^{(v)}), \\ n^{(v)} &= (n_0^{(v)}, \dots, n_{s_v-1}^{(v)})', \end{aligned} \right\} \quad (5B.3.2a)$$

$$V_v = n^{-1} D_v^*, T_v = n^{-2} n^{(v)} n^{(v)'}, I_v = I_{(s_v \times s_v)}$$

$$\left. \begin{aligned} e_v^{x_v} &= \mathbf{1}_v \text{ if } x_v = 0; = I_v \text{ if } x_v = 1, \\ V_v^{(x_v, y_v)} &= n^{-1} D_v^* \text{ if } x_v = y_v = 1 \\ &= n^{-1} n^{(v)} \text{ if } x_v = 1, y_v = 0 \\ &= n^{-1} n^{(v)'} \text{ if } x_v = 0, y_v = 1 \\ &= 1 \text{ if } x_v = y_v = 0 \end{aligned} \right\} \quad (5B.3.2b)$$

$$\left. \begin{aligned} T_v^{(x_v, y_v)} &= n^{-2} n^{(v)} n^{(v)'} \quad \text{if } x_v = y_v = 1 \\ &= V_v^{(x_v, y_v)} \quad \text{otherwise.} \end{aligned} \right\}$$

It is then easy to check that for $1 \leq v \leq m$; $x_v = 0, 1$; $y_v = 0, 1$,

$$\epsilon_v^{x_v'} V_v \epsilon_v^{y_v} = V_v^{(x_v, y_v)}, \quad (5B.3.3)$$

$$\epsilon_v^{x_v'} T_v \epsilon_v^{y_v} = T_v^{(x_v, y_v)}.$$

For $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_m)$, $\mathbf{x}, \mathbf{y} \in \Omega$, define

$$\epsilon^{\mathbf{x}} = \prod_{v=1}^m \epsilon_v^{x_v}, \quad V(\mathbf{x}, \mathbf{y}) = \prod_{v=1}^m V_v^{(x_v, y_v)}, \quad T(\mathbf{x}, \mathbf{y}) = \prod_{v=1}^m T_v^{(x_v, y_v)} \quad (5B.3.4)$$

By (5B.3.2a,b), (5B.3.3) and (5B.3.4), it is immediate that for $\mathbf{x}, \mathbf{y} \in \Omega$,

$$\epsilon^{\mathbf{x}'} \left(\prod_{v=1}^m V_v \right) \epsilon^{\mathbf{y}} = V(\mathbf{x}, \mathbf{y}), \quad \epsilon^{\mathbf{x}'} \left(\prod_{v=1}^m T_v \right) \epsilon^{\mathbf{y}} = T(\mathbf{x}, \mathbf{y}) \quad (5B.3.5)$$

Theorem 5B.3.1. If the chosen level combinations form a proportional frequency plan of strength $2g$ ($\leq m$) then all contrasts belonging to effects involving g or less factors are estimable when all effects involving more than g factors are absent.

Proof. Because of the reduced normal equations (5B.2.7) (here $g = t$) we only have to show that $P^{(g)'} C P^{(g)}$ is positive definite under the given condition. Now if the chosen level combinations form a proportional frequency plan (i.e. if (5B.3.1) holds with $d = 2g$) proceeding as in the proof of Lemma 2.2 of Mukerjee (1980) by showing that the corresponding elements of the two sides are equal, we have for any $\mathbf{x}, \mathbf{y} \in J_g$

$$\epsilon^{x'} C^* \epsilon^y = n [V^{(x,y)} - T^{(x,y)}],$$

i.e. by (5B.3.5)

$$\epsilon^{x'} C^* \epsilon^y = n \epsilon^{x'} C^{**} \epsilon^y, \quad (5B.3.6)$$

where by (5B.3.2)

$$C^{**} = \sum_{v=1}^m (n^{-1} D_v^*) - \left\{ \sum_{v=1}^m (n^{-1} n^{(v)}) \right\} \left\{ \sum_{v=1}^m (n^{-1} n^{(v)})' \right\} \text{ i.e.}$$

$$C^{**} = \text{Diag}(a_1, \dots, a_{v^*}) - aa', \quad (5B.4.7)$$

where $a = (a_1, a_2, \dots, a_{v^*})' = \sum_{v=1}^m (n^{-1} n^{(v)})$. Since the chosen level combinations form a proportional frequency plan of strength $2g$, by (5B.3.1), $n_{iv}^{(v)} > 0 \forall i_v, v$. Hence $a_v > 0, v = 1, 2, \dots, v^*$.

By (5B.2.1), (5B.2.2), (5B.3.4), for any $x \in \Omega$, there exists matrix L_x^* such that $P^x = \epsilon^x L_x^*$. Hence by (5B.3.6), under the given conditions for $x, y \in J_g$, $P^{x'} C^* P^y = n P^{x'} C^{**} P^y$, so that by (5B.2.3)

$$P^{(g)'} C^* P^{(g)} = n P^{(g)'} C^{**} P^{(g)}. \quad (5B.3.8)$$

By (5B.2.1), (5B.2.2), (5B.2.3), $P^{(g)'} \left(\sum_{v=1}^m 1_v \right) = 0$ and $P^{(g)'} P^{(g)} = I$. Further $a_v > 0, v = 1, 2, \dots, v^*$ and $\left(\sum_{v=1}^m 1_v \right)' a = 1$. Hence by (5B.3.7) and some standard steps in matrix algebra, it is evident that $P^{(g)'} C^{**} P^{(g)}$ (and hence $P^{(g)'} C^* P^{(g)}$, by (5B.3.8)) is positive definite.

One would naturally like to investigate the possible extension of the above theorem to the case $g < t$ i.e. to examine whether all contrasts belonging to effects involving g or less factors are estimable when all effects involving more than t factors are absent given that the chosen level combinations form a proportional frequency plan of strength $(g+t)$.

If $g < t$, by (5B.2.3), (5B.2.5), the reduced normal equations (5B.2.7) can be written as

$$\begin{bmatrix} P^{(g)'} C^* P^{(g)} & P^{(g)'} C^* \bar{P}^{(g,t)} \\ \bar{P}^{(g,t)'} C^* P^{(g)} & \bar{P}^{(g,t)'} C^* \bar{P}^{(g,t)} \end{bmatrix} \begin{bmatrix} P^{(g)'} \tau \\ \bar{P}^{(g,t)'} \tau \end{bmatrix} = \begin{bmatrix} P^{(g)'} Q \\ \bar{P}^{(g,t)'} Q \end{bmatrix} \quad (5B.3.9)$$

where $\bar{P}^{(g,t)} = (\dots, P^x, \dots)$, P^x being included in $\bar{P}^{(g,t)}$ if and only if $x \in J_t - J_g$. Since here we are interested in estimating the elements of $P^{(g)'} \tau$ the problem reduces to examine whether, given that the chosen level combinations form a proportional frequency plan of strength $(g+t)$, it is necessarily true that

$$\text{row space } [I^*, O] \subset \text{row space } [P^{(t)'} C^* P^{(t)}], \quad (5B.3.10)$$

where I^* is identity matrix of order v_1 , O is null matrix of order $v_1 \times (v_2 - v_1)$ and v_1, v_2 denote respectively the number of rows in $P^{(g)'}$, $P^{(t)'}$.

The answer, however, will, in general, be in the negative as the following example illustrates.

Example 5B.3.1. Consider the following plan of a $3^2 \times 2^2$ factorial (the level combinations are written as columns) which is easily seen to be a proportional frequency plan of strength 3.

```
00010111000101110002022200020222
00101011002020220010101100202022
01001101010011010100110101001101
01110001011100010111000101110001
```

with $g = 1$, $t = 2$, it may be checked that (5B.3.10) does not hold. The process which involves actual computation of $P^{(t)'} C^* P^{(t)}$ is lengthy and is not shown here. Thus though the chosen level combinations form a proportional frequency plan of strength 3, it is not possible to estimate all main effect contrasts even when all interactions involving more than two factors are absent.

Thus it is seen that further extension of Theorem 5B.3.1 in the line contemplated above is not possible. As such the use of proportional frequency plans should be as envisaged in Theorem 5B.3.1.

5B.4 Analysis and Efficiency

We now consider in brief the analysis of such plans. Consider a proportional frequency plan of strength $2g$. Then by (5B.2.7) (with $g = t$), (5B.3.8) and Theorem 5B.3.1, assuming the absence of all interactions involving more than g factors

$$P^{(g)'} \hat{\tau} = (P^{(g)'} C^* P^{(g)})^{-1} P^{(g)'} Q \quad (5B.4.1)$$

so that

$$\text{Disp}(P^{(g)'} \hat{\tau}) = \sigma^2 (P^{(g)'} C^* P^{(g)})^{-1} = \sigma^2 n^{-1} (P^{(g)'} C^{**} P^{(g)})^{-1} \quad (5B.4.2)$$

As such the sum of squares (SS) due to $P^{(g)'} \hat{\tau}$ is given by

$$n^{-1} Q' P^{(g)} (P^{(g)'} C^{**} P^{(g)})^{-1} P^{(g)'} Q$$

The error SS is obtained by subtracting this from the total SS. For any $x \in J_g$, $P^{x'} \hat{\tau}$, $\text{Disp}(P^{x'} \hat{\tau})$ may be obtained from appropriate subvector and submatrix of (5B.4.1), (5B.4.2) respectively. It should be noted that the plan under study is non-orthogonal so that separate ANOVA tables would be required for various factorial effects.

We can derive a lower bound for the average efficiency of such plans.

It is easy to check that $\sum_{v=1}^m (n D_v^{*-1}) = E$, say, is a generalised inverse of C^{**} .

Hence by (5B.4.2) after some simplification, we have

$$\text{Disp}(P^{(g)'} \hat{\tau}) = \sigma^2 n^{-1} [P^{(g)'} E P^{(g)} - P^{(g)'} E P^{(g)} (P^{(g)'} E P^{(g)})^{-1} P^{(g)'} E P^{(g)}] \quad (5B.4.3)$$

It is difficult to have a compact expression for $\text{Trace} [\text{Disp}(P^{(g)'t})]$. However, noting that the second term within the square brackets on the right hand side of (5B.4.3) is nonnegative definite, by (5B.2.3).

$$\begin{aligned} \text{Trace} [\text{Disp}(P^{(g)'t})] &\leq \sigma^2 n^{-1} \text{Trace} [P^{(g)'} E P^{(g)}] \\ &= \sigma^2 n^{-1} \text{Trace} [E P^{(g)} P^{(g)'}] \\ &= \sigma^2 n^{-1} \sum_{x \in J_g} \text{Trace} [E P^x P^{x'}] \end{aligned} \quad (5B.4.4)$$

By (5B.2.1), (5B.2.2), for any $x \in \Omega$, $P^x P^{x'} = x \sum_{v=1}^m W_v^x W_v^{x'}$, where for $1 \leq v \leq m$,

$$\begin{aligned} W_v^x &= I_v - s_v^{-1} \mathbf{1}_v \mathbf{1}_v' \quad \text{if } x_v = 1 \\ &= s_v^{-1} \mathbf{1}_v \mathbf{1}_v' \quad \text{if } x_v = 0. \end{aligned}$$

Hence writing for $x \in \Omega$, $f(x) = \prod_{v=1}^m (s_v^{-1})^{x_v}$

$$\begin{aligned} \text{Trace} [E P^x P^{x'}] &= \text{Trace} \left[\left\{ \sum_{v=1}^m (n D_v^* - 1) \right\} P^x P^{x'} \right] \\ &= n^m f(x)(v^*)^{-1} \sum_{i_1=0}^{s_1-1} \dots \sum_{i_m=0}^{s_m-1} (n_{i_1}^{(1)} \dots n_{i_m}^{(m)})^{-1} \\ &= n^m f(x)(v^*)^{-1} \prod_{v=1}^m \left\{ \sum_{i_v=0}^{s_v-1} (n_{i_v}^{(v)})^{-1} \right\} \end{aligned}$$

Thus by (5B.4.4),

$$\text{Trace} [\text{Disp}(P^{(g)'t})] \leq \sigma^2 n^{m-1} \left[\prod_{v=1}^m \left\{ \sum_{i_v=0}^{s_v-1} (n_{i_v}^{(v)})^{-1} \right\} \right] \sum_{x \in J_g} f(x)(v^*)^{-1} \quad (5B.4.5)$$

Since $\text{Disp}(P^{(g)'t})$ is a square matrix of order $\sum_{x \in J_g} f(x)$, by (5B.4.5), average variance of the elements of $P^{(g)'t}$

$$\leq \sigma^2 n^{m-1} (v^*)^{-1} \prod_{v=1}^m \left\{ \sum_{i_v=0}^{s_v-1} (n_{i_v}^{(v)})^{-1} \right\}. \quad (5B.4.6)$$

As shown in Mukerjee (1982) for any given n , an equal frequency plan (based on orthogonal array with variable symbols) if it exists is universally optimal with

$$\text{Disp}(P^{(g)}_{\mathbf{t}}) = \sigma^2 n^{-1} \mathbf{v}^* \mathbf{I} \quad (5B.4.7)$$

where $\mathbf{I}$ is identity matrix of appropriate order. For any particular n , even if an equal frequency does not exist we shall define the relative efficiency of any other plan relative to a hypothetical equal frequency plan for the sake of definiteness. Thus by (5B.4.6), (5B.4.7), the average relative efficiency of a plan based on a proportional frequency criterion (of strength $2g$) is at least

$$(\mathbf{v}^*)^2 n^{-m} \prod_{v=1}^m \left\{ \sum_{i_v=0}^{s_v-1} (n_{i_v}^{(v)})^{-1} \right\}^{-1} \quad (5B.4.8)$$

The smaller the scatter of $n_{i_v}^{(v)}$, $i_v = 0, 1, 2, \dots, s_v-1$, for different v , the greater the value of (5B.4.8). Hence in the construction of proportional frequency plans, $n_0^{(v)}, \dots, n_{s_v-1}^{(v)}$ should be as nearly equal as possible for each v .

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