Measure Theoretic Probability

M. Stat. 1st year
Final exam
3 hours 30 minutes
Full marks: 60

5 May 2025

Answer **any four** questions. Each question carries 15 marks. Results proved in the class may be quoted and used.

1. Suppose $\mathcal{F}$ is a field on a non-empty set Ω. Assume $\mathcal{G} \subset \mathcal{F}$ and that $\mathcal{G}$ has the following property: if $K_1, K_2, \ldots \in \mathcal{G}$ are such that

$$K_1 \cap \ldots \cap K_n \neq \emptyset$$

for all n, then $K_1 \cap K_2 \cap K_3 \cap \ldots \neq \emptyset$. Suppose $\mu : \mathcal{F} \to [0, \infty)$ is a finitely additive function satisfying

$$\mu(A) = \sup\{\mu(K) : K \subset A, K \in \mathcal{G}\} \text{ for all } A \in \mathcal{F}.$$

   (a) [**12 marks**] If $A_n \in \mathcal{F}$ are such that $A_n \downarrow \emptyset$, then show that $\mu(A_n) \downarrow 0$.

   (b) [**3 marks**] Hence or otherwise, prove that μ can be extended to a measure on $\big(\Omega, \sigma(\mathcal{F})\big)$.

2. Suppose that $(\Omega, \mathcal{A}, \mu)$ is a measure space and $f, f_1, f_2, \ldots$ are non-negative integrable functions on Ω. Prove the following.

   (a) [**8 marks**] As $n \to \infty$, $f_n \to f$ in L^1 if and only if

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$$

   and every subsequence of $\{f_n\}$ has a further subsequence which converges to f a.e.

   (b) [**7 marks**] As $n \to \infty$, $f_n \to f$ in L^1 if and only if

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$$

   and

$$\lim_{n\to\infty} \mu\left(|f_n - f| > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0.$$

3. Suppose that $(\Omega, \mathcal{A}, P)$ is a probability space on which X is a non-negative integrable random variable. Define a measure μ on $(\Omega, \mathcal{A})$ by

$$\mu(A) = \int_A X \, dP, \; A \in \mathcal{A}.$$

Suppose that $\mathcal{F} \subset \mathcal{A}$ is also a σ-field.

   (a) **[3 marks]** Show that there exists a non-negative $Y \in L^1(\Omega, \mathcal{F}, P)$ such that

   $$\int_A Y \, dP = \mu(A) \text{ for all } A \in \mathcal{F}.$$

   (b) **[6 marks]** Suppose $Z \in L^1(\Omega, \mathcal{F}, \mu)$. Show that

   $$\int_\Omega Z \, d\mu = \int_\Omega YZ \, dP.$$

   (c) **[6 marks]** Prove that for all Z as in (b),

   $$\int_\Omega XZ \, dP = \int_\Omega YZ \, dP.$$

4. Suppose U, V, V' are independent random variables. Assume V and V' have finite third moments. Further, the mean and variance of V are same as those of V', respectively.

   (a) **[9 marks]** Show that for all $t \in \mathbb{R}$,

   $$|\phi_{U+V}(t) - \phi_{U+V'}(t)| \le |t|^3 \mathrm{E}\left(|V|^3 + |V'|^3\right),$$

   where ϕ_{U+V} and $\phi_{U+V'}$ are the respective characteristic functions of $U + V$ and $U + V'$.

   (b) **[6 marks]** Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with mean zero, variance one and finite third moment. Using (a) or otherwise, show that there exists $0 < C < \infty$ such that

   $$\left|\phi_n(t) - e^{-t^2/2}\right| \le C n^{-1/2} |t|^3 \text{ for all } t \in \mathbb{R},$$

   where ϕ_n is the characteristic function of $(X_1 + \ldots + X_n)/\sqrt{n}$.

5. Suppose that for all $n \in \mathbb{N}$, $X_{n1}, \ldots, X_{nn}$ are i.i.d. random variables taking values $-1, 0, 1$ with probabilities $p_n/2$, $1 - p_n$, $p_n/2$, respectively, for some $0 < p_n < 1$.

   (a) **[5 marks]** If $np_n \to \infty$, show that

   $$(np_n)^{-1/2} \sum_{i=1}^{n} X_{ni} \Rightarrow Z, \; n \to \infty,$$

   where Z follows standard normal.

(b) [**8 marks**] If $np_n \to \lambda \in (0, \infty)$, show that

$$\sum_{i=1}^{n} X_{ni} \Rightarrow Y_1 - Y_2,\ n \to \infty,$$

where Y_1 and Y_2 are i.i.d. from Poisson$(\lambda/2)$.

(c) [**2 marks**] If $np_n \to 0$, then show that

$$\lim_{n\to\infty} P\left(\sum_{i=1}^{n} X_{ni} = 0\right) = 1\,.$$