On Algebraic Aspects and Functoriality of the Set of Unbounded Operators Affiliated with a von Neumann Algebra

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Abstract

In quantum theory, physically meaningful observables such as position and momentum are modeled by unbounded self-adjoint operators on Hilbert spaces. To study such operators in a rigorous mathematical framework, Murray and von Neumann introduced the concept of affiliation with a von Neumann algebra, giving rise to what are now known as *Murray–von Neumann affiliated operators*. For a single (unbounded) self-adjoint operator, its spectral projections span an abelian von Neumann algebra to which the operator is affiliated. If we want a common framework to study families of (unbounded) self-adjoint operators which do not necessarily commute with each other, it is instructive to study the notion of affiliation for general von Neumann algebras. Unfortunately, the set of affiliated operators has so far not been realized as a familiar algebraic structure that would allow for a systematic study. For example, they are not closed under natural algebraic operations such as addition and multiplication. Moreover, their behavior under morphisms of von Neumann algebras is not immediately evident from the classical definition.

This thesis addresses these limitations by introducing a new, intrinsically defined class of unbounded operators affiliated to a von Neumann algebra, which generalizes the classical notion while retaining its essential features. The proposed class forms a near-semiring, is functorial with respect to unital normal $*$-homomorphisms of von Neumann algebras, and contains all MvN-affiliated operators. As a key application, we show that the classical construction of MvN-affiliated operators itself becomes functorial within this broader framework.

We also explore connections with measurable and locally measurable operator algebras, and clarify the subtleties in their functorial behavior. In the later chapters, we revisit the Krein–von Neumann extension theory and establish the functoriality of both the Krein and Friedrich extensions for positive symmetric operators affiliated with a given von Neumann algebra. Additionally, we extend our affiliation framework beyond von Neumann algebras to monotone σ-complete C^*-algebras, enabling a more general theory with potential applications to noncommutative geometry and mathematical physics.

The results developed in this thesis provide a robust algebraic and categorical foundation for studying unbounded affiliated operators in operator algebras, and open the door to new directions in the analysis of infinite quantum systems and beyond.

Dedication

To Taniya — for walking beside me when the road was uncertain, for believing in me when I doubted myself, and for giving more than I ever knew to ask, making this journey to my PhD possible.

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I am also thankful to the research scholars at ISI Bangalore, whose support and intellectual exchanges have been both inspiring and fruitful. The discussions with my peers have significantly enriched my research experience, and the encouragement and guidance from our seniors have been invaluable throughout this journey.

I am profoundly grateful to my parents for the life they have given me and the unwavering support they have provided. Thanks to them, I have been able to focus solely on my studies without any worries. Their love, sacrifices, and belief in me have been my greatest sources of strength.

To my friends at ISI, a special thanks for making my life here as unpredictable as a divergent series. You all somehow managed to keep me both sane and entertained, with just the right balance of chaos and calm. From spontaneous tea breaks that turned into philosophical debates to random late-night discussions about everything except research, you ensured that my PhD life was never boring. For those who were a part of my life at some point but aren't around anymore: thank you too! Whether you stayed for a season or a lifetime, you left behind lessons, memories, and maybe even a few tips on how to navigate the maze of life—or at least how to survive it with my sanity mostly intact. I've truly learned something valuable from each of you.

Finally, I would like to thank Taniya, who stood by me during a crucial phase of my life—before my PhD began. Her belief in me, at a time when I doubted myself, and the quiet strength of her support played a vital role in helping me step into the world of research. Though our journeys have since diverged, I remain deeply grateful for all that she gave and all that she meant to me.

— *Indrajit Ghosh*

Contents

List of Symbols

$\mathbb{N}$	Set of all Natural numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{R}$	Set of all Real numbers
$\mathbb{R}$	The field of complex numbers
$\mathbb{M}_n(\mathbb{C})$	$n \times n$ matrix algebra over $\mathbb{C}$
$\Re z, \Im z$	Real and Imaginary part of the complex number z
$\mathcal{B}(\mathcal{H})$	Algebra of all bounded operators on the Hilbert space $\mathcal{H}$
$\mathcal{C}(\mathcal{H})$	Closed unbounded operators on the Hilbert space $\mathcal{H}$
$\mathrm{Gr}\,(T)$	Graph of an Operator T
$T^{\dagger}$	Kaufman inverse of the unbounded operator T
$\chi(T)$	Characteristic projection of the closed (unbounded) operator T
$\mathbb{1}_{\Omega}$	Indicator function of the set Ω
$\mathscr{M}_{\mathrm{sa}}$	Self-adjoint part of the von Neumann algebra $\mathscr{M}$
$\mathscr{M}_{\mathrm{proj}}$	Lattice of projections in the von Neumann algebra $\mathscr{M}$
$\mathrm{Aff}_s\,(\mathscr{M})$	Lattice of affiliated subspaces associated with the von Neumann algebra $\mathscr{M}$
$\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$	Murray von Neumann affiliated operators associated with the von Neumann algebra $\mathscr{M}$
$\mathscr{M}_{\mathrm{aff}}^{dd-c}$	The set of all densely-defined Murray von Neumann affiliated operators associated with $\mathscr{M}$
$\mathscr{M}_{\mathrm{aff}}$	The set of affiliated operators associated with the von Neumann algebra $\mathscr{M}$
$\mathscr{M}_{\mathrm{aff}}^{c}$	$\mathscr{M}_{\mathrm{aff}} \cap \mathcal{C}(\mathcal{H})$
$\mathfrak{A}_{\mathrm{aff}}$	The set of affiliated operators associated with the C^*-algebra $\mathfrak{A}$
$\mathbb{M}_n\,(\mathfrak{A})$	$n \times n$ matrix algebra over the C^*-algebra $\mathfrak{A}$
$\mathbf{C^*Alg}$	Category of unital C^*-algebras with morphisms unital $*$-homomorphisms
$\mathbf{W^*Alg}$	Category of von Neumann algebras with morphisms normal $*$-homomorphisms
$S^{\mathbf{Kr}}$	Krein extension of the operator S
$S^{\mathbf{Fr}}$	Friedrich extension of the operator S

Introduction

The mathematical formalism of quantum mechanics, a cornerstone of modern physics, inherently involves the study of linear operators on Hilbert spaces. While bounded operators play a crucial role, many fundamental physical observables, such as position, momentum, and Hamiltonian operators representing energy, are naturally described by unbounded operators [Neu32; RS80]. Furthermore, the time evolution of a closed quantum system is governed by a one-parameter unitary group, which, according to Stone's theorem, is generated by a self-adjoint, generally unbounded, operator – the Hamiltonian [Sto32]. This fundamental connection underscores the indispensable nature of unbounded operators for capturing the dynamical aspects of quantum phenomena. For instance, the Heisenberg uncertainty principle, a foundational concept stating that certain pairs of physical properties cannot be simultaneously known with arbitrary precision, finds its rigorous mathematical formulation in terms of the non-commutativity of unbounded operators [KL14].

Let $\mathcal{H}$ be a Hilbert space. By an *unbounded operator on* $\mathcal{H}$, we mean a "not-necessarily-bounded" linear operator whose domain and range are both contained in $\mathcal{H}$. The *Kaufman inverse* of such an operator (see Definition 1.6) is denoted by $(-)^\dagger$ and plays a central role in this thesis. We denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded, everywhere-defined linear operators on $\mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, the closed graph theorem implies that A is a closed operator. Moreover, if $\mathcal{V} \subseteq \mathcal{H}$ is a closed subspace, then the restriction $A|_{\mathcal{V}}$ is again a closed (and bounded) operator. From Remark 1.3, we have the identity

$$A|_{\mathcal{V}} = AE_{\mathcal{V}}^\dagger,$$

where $E_{\mathcal{V}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{V}$. According to Lemma 1.2, the Kaufman inverse $A^\dagger$ of a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is itself a closed operator. Furthermore, it is a basic fact (see Lemma 1.1) that if T and A' are closed operators and A' is also bounded, then the composition TA' is closed. Therefore, for bounded operators $A, B, E \in \mathcal{B}(\mathcal{H})$ with E a projection, setting $T = B^\dagger$ and $A' = AE^\dagger$, we deduce that the composition $B^\dagger AE^\dagger$ is a closed operator on $\mathcal{H}$. In Theorem 4.7 (ii), we prove the converse: every closed operator on $\mathcal{H}$ can be expressed in the form $P^{-1}AE^\dagger$, where $A, P, E \in \mathcal{B}(\mathcal{H})$, P is a positive operator with dense range (and hence trivial kernel), and E is a projection.

Within the realm of operator algebras, von Neumann algebras stand out as a powerful framework for studying bounded linear operators on a Hilbert space that are closed under the adjoint operation and the weak operator topology (see [Neu36] and §1.11). These algebras, initially conceived by Murray and von Neumann in their seminal series of papers [MN36; MN37; Neu40a; MN43], have found profound applications not only in the mathematical foundations of quantum mechanics, where they provide a natural setting for describing algebras of observables, but also in diverse areas such as representation theory, ergodic theory, and non-commutative geometry [Con94; Jon83].

A crucial concept linking von Neumann algebras with unbounded operators is that of "affiliation". Murray and von Neumann introduced certain *closed* unbounded operators associated to a given von Neumann algebra $\mathscr{M}$ acting on $\mathcal{H}$. The class of such operators — known as *Murray–von Neumann affiliated operators* or *MvN-affiliated operators* — has played a foundational role in various aspects of operator algebra theory and its applications to quantum physics. For $\mathscr{M}$, this class of unbounded operators will be denoted as $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ throughout the thesis.

In the algebraic formulation of quantum field theory (AQFT), local observable algebras $\mathcal{A}(\mathcal{O})$, which are von Neumann algebras, are associated with bounded regions of spacetime $\mathcal{O}$. These algebras satisfy crucial axioms like locality and Poincaré covariance. While the Hamiltonian operator H, the generator of time translations, is an unbounded operator and thus not an element of any local $\mathcal{A}(\mathcal{O})$, it is considered to be MvN-affiliated with the von Neumann algebra of all observables in the entire spacetime, denoted by $\mathcal{A}$. It can be proved that an unbounded operator T is MvN-affiliated with a von Neumann algebra $\mathscr{M}$ if all of its spectral projections belong to $\mathscr{M}$. In AQFT, it is generally assumed that the spectral projections of the self-adjoint Hamiltonian H belong to $\mathcal{A}$, signifying that the energy of the quantum field system is fundamentally linked to the algebra of its observables [Haa92].

Despite their foundational role, MvN-affiliated operators exhibit certain structural limitations. Most notably, the collection of such operators does not, in general, fit into any familiar algebraic framework. For instance, the sum or product of two MvN-affiliated operators may fail to be closed, let alone MvN-affiliated with the same von Neumann algebra. This raises an important conceptual question. As we have seen, MvN-affiliated operators capture the unbounded observables that arise in a wide range of physical settings, and in this sense, the MvN-affiliation relation can be thought of as a way of relating certain unbounded operators to a given von Neumann algebra. But suppose $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$ — what about T^2? Or given another operator $S \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$, what about $S + T$ or ST? For a general von Neumann algebra $\mathscr{M}$, these expressions need not even be pre-closed (see Definition 1.2), and hence need not be MvN-affiliated with $\mathscr{M}$. Nevertheless, one naturally wonders: can we say something canonical about the relationship of these algebraic expressions — which are ubiquitous in physics — with the underlying von Neumann algebra?

This is the first of two motivations for our search for a more structured notion of affiliation: a framework that canonically relates a broader class of unbounded operators — including those built algebraically from MvN-affiliated ones — to a given von Neumann algebra. The second motivation stems from the question of functoriality: is the set $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ intrinsic to the von Neumann algebra $\mathscr{M}$? In other words, how does the class of MvN-affiliated operators transform when we change representations of $\mathscr{M}$ via normal $*$-homomorphisms, which are not necessarily isomorphisms? This is not immediately clear, as the classical definition of affiliation depends explicitly on the commutant $\mathscr{M}'$, which may not behave well under such morphisms.

In [Ska79, pg. 189], it is observed that if $\Phi : \mathscr{M} \to \mathscr{N}$ is a $*$-isomorphism between von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$, then there exists a one-to-one correspondence between $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ and $\mathscr{N}_{\text{aff}}^{\text{MvN}}$. This identification is explicitly described using the polar decomposition theorem for MvN-affiliated operators, together with the spectral theorem for unbounded self-adjoint operators. While this provides a heuristic indication that the class $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ may be intrinsic to the von Neumann algebra $\mathscr{M}$, it leaves open several fundamental questions that must be addressed to establish a complete and satisfactory picture:

1. Is the identification described in [Ska79, pg. 189] canonically determined by the underlying isomorphism Φ?

2. More generally, if $\Phi : \mathscr{M} \to \mathscr{N}$ is a unital normal $*$-homomorphism (not necessarily an isomorphism), does Φ induce a canonical map from $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ to $\mathscr{N}_{\text{aff}}^{\text{MvN}}$? That is, is the assignment $\mathscr{M} \mapsto \mathscr{M}_{\text{aff}}^{\text{MvN}}$ functorial in an appropriate sense?

A key contribution of this work is to answer both of these questions in the affirmative (see Definition 3.2 and Corollary 4.9). This shows that, despite its traditional formulation in terms of commutation with the commutant $\mathscr{M}'$, the MvN-affiliation relation is in fact intrinsic to the algebra $\mathscr{M}$ itself. Before outlining our approach, we first highlight some subtleties underlying the above questions, and explain why existing claims of "functoriality" in the literature — which typically involve only $*$-isomorphisms — may be misleading or incomplete.

Let $\mathbf{W^*Alg}$ denote the category of von Neumann algebras with unital normal $*$-homomorphisms as morphisms. Let $\mathcal{S}(\mathscr{M})$ (resp. $\mathcal{L}(\mathscr{M})$) denote the $*$-algebra of measurable (resp. locally measurable) operators affiliated with $\mathscr{M}$, both of which are subsets of $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ (see Definitions 4.1 and 4.2). As we observe in Remark 4.2, the assignments $\mathscr{M} \mapsto \mathcal{S}(\mathscr{M})$ and $\mathscr{M} \mapsto \mathcal{L}(\mathscr{M})$ are *not* functorial on $\mathbf{W^*Alg}$.

However, when restricted to $*$-isomorphisms $\mathscr{M} \cong \mathscr{N}$, these constructions behave well: there exist canonical $*$-algebra isomorphisms $\mathcal{S}(\mathscr{M}) \cong \mathcal{S}(\mathscr{N})$ and $\mathcal{L}(\mathscr{M}) \cong \mathcal{L}(\mathscr{N})$. In other words, the measurable and locally measurable operator algebras associated with a von Neumann algebra $\mathscr{M}$ are well-behaved under *faithful* normal $*$-representations, but not necessarily under arbitrary normal $*$-homomorphisms. This distinction is crucial when seeking a genuinely functorial extension of the classical notion of affiliation.

When $\mathscr{M}$ is a finite von Neumann algebra, the set of densely defined operators in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ has been extensively studied and is known to possess a rich algebraic structure. In [Ber82; Roo68], this set is identified as the *maximal quotient ring* of $\mathscr{M}$. More precisely, it can be realized as the Ore localization of $\mathscr{M}$ with respect to the multiplicative subset of non-zero-divisors, so that every non-zero-divisor in $\mathscr{M}$ admits an inverse in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ (see [Han83]).

Furthermore, as observed in [MN36], the operations of strong sum and strong product (see Definition 1.4) endow the set of densely defined operators in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ with a ring structure, which becomes a $*$-algebra when equipped with the adjoint operation. These structures are commonly referred to in the literature as *Murray–von Neumann algebras* (see [KL14; Nay21a]). In [Nay21a], additional topological and order-theoretic aspects of these algebras are investigated, and it is shown that the $*$-algebra of densely defined MvN-affiliated operators for a finite von Neumann algebra $\mathscr{M}$ yields a functorial construction from the category of finite von Neumann algebras to the category of unital ordered complex topological $*$-algebras.

As this body of work illustrates, in the case where $\mathscr{M}$ is finite, the set of densely defined operators in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ is not only well-behaved but also equipped with a robust and well-understood algebraic and topological structure.

Returning to the first issue concerning the algebraic structure of $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ for a general von Neumann algebra $\mathscr{M}$, the lack of closure under natural algebraic operations presents serious challenges—particularly in the context of mathematical physics, where expressions involving sums and products of unbounded operators are pervasive. For instance, in quantum field theory and statistical mechanics, Hamiltonians or generators of dynamics frequently arise as formal algebraic combinations of such operators. In these settings, the absence of a stable and well-defined algebraic framework for handling these expressions can significantly hinder both conceptual clarity and technical progress. Thus, a more robust and structured notion of affiliation, one that accommodates these natural algebraic operations, is highly desirable.

In this thesis, we introduce a new, intrinsically defined class of unbounded operators associated with a von Neumann algebra, which we term *affiliated operators* (in a generalized sense; see §3). Our construction satisfies the following key properties:

- The class is closed under operator sum and product and infact forms a algebraic structure of *near-semiring* (see Definition 1.16; Theorem 3.2).
- The construction is functorial: morphisms of von Neumann algebras induce well-defined morphisms between the associated classes of affiliated operators (see Definition 3.2)
- As a main result, we show that every MvN-affiliated operator is affiliated to the underlying von Neumann algebra in our generalized sense (see Theorem 4.7). Thus, our framework provides a genuine extension of the classical notion, retaining its strengths while addressing its limitations.
- Furthermore, by leveraging the functoriality of this newly defined class of affiliated operators, we demonstrate that the classical class $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ itself admits a functorial construction (see Corollary 4.9).

Equipping a suitable class of unbounded operators—associated in a precise and canonical way to a given von Neumann algebra—with an enhanced algebraic structure not only resolves various technical challenges in operator algebra but also paves the way for new applications in quantum theory and noncommutative geometry. In the chapters that follow, we systematically develop this framework and establish its foundational properties.

Notably, we are able to extend this theory of affiliation beyond von Neumann algebras to a broader class of operator algebras—namely, the monotone σ-complete C^*-algebras (see § 1.3.3)—which generalize von Neumann algebras. As a result, most of our main results are formulated in this more general setting, with the corresponding statements for von Neumann algebras either following directly or appearing as special cases and corollaries.

Organization of the Thesis. The remainder of this thesis is structured as follows.

- **Chapter 1** gathers the foundational material required for the subsequent chapters. We begin with a concise overview of unbounded operators on Hilbert spaces, covering notions such as adjoints, closures, and MvN-affiliated operators. A key point to note is that the standard definition of the adjoint of an unbounded operator, as commonly found in the literature, assumes the operator is densely defined. One of the distinguishing features of this thesis is the systematic use of general (not necessarily densely defined) unbounded operators. As a result, we extend the notion of the adjoint to accommodate this broader setting in a meaningful way; see Definition 1.7 for details. This generalized definition agrees with the classical one in the densely defined case. Another important concept introduced in this chapter is the *Kaufman inverse* (see Definition 1.6), which plays a central role in the *quotient-based* approach to representing affiliated operators.

  The chapter then proceeds to survey the relevant aspects of operator algebras, including C^*-algebras, von Neumann algebras, and monotone σ-complete C^*-algebras. We conclude with a discussion of near-semirings, an algebraic structure that provides the foundation for the algebraic framework developed later in the thesis.

- **Chapter 2** develops a new perspective on affiliation by working at the level of Hilbert space subspaces. The theory of unbounded operators is inherently subtle, particularly with respect to their domains. For instance, the domain of the sum or product of two densely defined operators may be trivial. Therefore, any attempt to impose an algebraic structure on such operators must carefully account for the behavior of their domains.

To address this, we employ a variant of the Douglas factorization lemma (see Theorem 2.1) within the setting of certain operator algebras. This lemma establishes a connection between operators in the algebra and specific subspaces through the structure of their ranges. The chapter begins with a detailed exposition of this lemma in the context of unital monotone σ-complete C^*-algebras. This tool allows us to isolate certain canonical subspaces associated with a given algebra, which we term *affiliated subspaces* (see Definition 2.1). For such a C^*-algebra $\mathfrak{A}$, we denote this collection of such subspaces by $\mathrm{Aff}_s(\mathfrak{A})$.

The analysis of $\mathrm{Aff}_s(\mathfrak{A})$ lays the groundwork for identifying the affiliated operators associated with $\mathfrak{A}$ in the next chapter. We also show that $\mathrm{Aff}_s(\mathfrak{A})$ admits a natural lattice structure (see Lemma 2.5) and that the assignment $\mathfrak{A} \mapsto \mathrm{Aff}_s(\mathfrak{A})$ is functorial (see Theorem 2.7).

- **Chapter 3** develops the central algebraic framework for affiliated operators. Building on the affiliated subspace theory introduced in the previous chapter, we construct a generalized class of affiliated operators for any unital monotone σ-complete C^*-algebra $\mathfrak{A}$, denoted by $\mathfrak{A}_{\mathrm{aff}}$ (see Definition 3.1). Elements of $\mathfrak{A}_{\mathrm{aff}}$ include the algebra $\mathfrak{A}$ itself and consist of (possibly unbounded) operators acting on the Hilbert space $\mathcal{H}$ on which $\mathfrak{A}$ acts. Notably, these operators are **not** required to be densely defined or closed.

  Each operator $T \in \mathfrak{A}_{\mathrm{aff}}$ can be expressed as a *quotient* of the form $AB^\dagger$, where $A, B \in \mathfrak{A}$ and $(\cdot)^\dagger$ denotes the Kaufmann inverse introduced earlier. We show that this class of operators is closed under sum, product, taking adjoints, and Kaufmann inversion (see Theorems 3.4, 3.5). As a result, $\mathfrak{A}_{\mathrm{aff}}$ carries the structure of a near-semiring (see Theorem 3.2). Moreover, the assignment $\mathfrak{A} \mapsto \mathfrak{A}_{\mathrm{aff}}$ is functorial: any morphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ between unital monotone σ-complete C^*-algebras induces a corresponding near-semiring homomorphism $\Phi_{\mathrm{aff}} : \mathfrak{A}_{\mathrm{aff}} \to \mathfrak{B}_{\mathrm{aff}}$ (see Definition 3.2, Theorem 3.6).

  While the quotient representation of affiliated operators is not unique in general, we show that for a von Neumann algebra $\mathscr{M}$, every operator in $\mathscr{M}_{\mathrm{aff}}$ admits an *essentially unique* expression of the form $AB^\dagger$. Specifically, the denominators (and respectively, the numerators) of any two such representations must be related via central projections in $\mathscr{M}$. This result relies on the comparison theory of projections in $\mathscr{M}$. Throughout the chapter, we extend several classical results from operator theory to the broader context of von Neumann algebras and monotone σ-complete C^*-algebras (see Lemma 3.1, Theorems 3.2, 3.11 etc).

- **Chapter 4** focuses on the subclass of *closed* affiliated operators within $\mathscr{M}_{\mathrm{aff}}$, denoted by $\mathscr{M}_{\mathrm{aff}}^c$, where $\mathscr{M}$ is a von Neumann algebra. The main goal of this chapter is to show that our generalized notion of affiliation recovers the classical Murray–von Neumann affiliated operators. We begin by analyzing the domains of operators in $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$. In Theorem 4.6, we prove that the domains of MvN-affiliated operators are themselves affiliated to $\mathscr{M}$; more precisely, the collection of all such domains coincides with the lattice $\mathrm{Aff}_s(\mathscr{M})$. This result generalizes a theorem of Fillmore and Williams to arbitrary von Neumann algebras and simultaneously demonstrates the functorial nature of domains of operators in $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$.

  Using this characterization of domains, we establish the main result of the chapter:

$$\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}} = \mathscr{M}_{\mathrm{aff}}^c.$$

  That is, the classical Murray–von Neumann affiliated operators coincide precisely with the closed operators in our generalized framework. Moreover, we show that any operator

$T \in \mathscr{M}_{\text{aff}}^{c}$ admits a representation of the form

$$T = P^{-1} A E^{\dagger},$$

where P is a positive, injective operator with dense range, E is a projection, and $P, A, E \in \mathscr{M}$ (see Theorem 4.7-(i, ii)).

Leveraging the functoriality of the construction $\mathscr{M}_{\text{aff}}$, we prove that both $\mathscr{M} \mapsto \mathscr{M}_{\text{aff}}^{\text{MvN}}$ and $\mathscr{M} \mapsto \mathscr{M}_{\text{aff}}^{dd-c}$ are functorial for arbitrary von Neumann algebras, where $\mathscr{M}_{\text{aff}}^{dd-c}$ denotes the class of densely defined, closed affiliated operators (see Corollary 4.9). Additionally, we show that for any normal $*$-homomorphism $\Phi : \mathscr{M} \to \mathscr{N}$, the induced map Φ_{aff} preserves strong sums and products on $\mathscr{M}_{\text{aff}}^{dd-c}$ (see Corollary 4.10).

The remainder of the chapter is devoted to justifying the canonical nature of the extension Φ_{aff} (see Theorem 4.12). We conclude by demonstrating that Φ_{aff} preserves a wide range of natural operator-theoretic properties, including symmetry, positivity, self-adjointness, normality, accretivity, and sectoriality (see Theorem 4.14).

- **Chapter 5** investigates the Krein and Friedrichs extension theory for positive operators affiliated with a von Neumann algebra $\mathscr{M}$. The primary objective is to demonstrate that both of these canonical extensions behave functorially. Specifically, given a normal $*$-homomorphism $\Phi : \mathscr{M} \to \mathscr{N}$ and a positive operator S affiliated with $\mathscr{M}$, we establish that

$$\Phi_{\text{aff}}(S^{\mathbf{Fr}}) = \Phi_{\text{aff}}(S)^{\mathbf{Fr}} \quad \text{and} \quad \Phi_{\text{aff}}(S^{\mathbf{Kr}}) = \Phi_{\text{aff}}(S)^{\mathbf{Kr}},$$

where $(\cdot)^{\mathbf{Kr}}$ and $(\cdot)^{\mathbf{Fr}}$ are respectively the Krein and Friedrich extensions (see Theorem 5.3). To this end, we also provide a comprehensive treatment of Krein extensions, ensuring the necessary background and completeness of the theory.

- **Chapter 6** demonstrates an application of the functorial framework developed in the preceding chapters, using abstract methods that can aid a working operator algebraist in their day-to-day considerations.

When working with a properly infinite von Neumann algebra $\mathscr{M}$, it is often extremely challenging to compute its commutant, which in turn makes it difficult to explicitly describe the set of unbounded operators affiliated with $\mathscr{M}$ in the Murray–von Neumann sense—that is the set $\mathscr{M}_{\text{aff}}^{\text{MvN}}$. Consequently, answering questions concerning $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ can be particularly subtle. To illustrate, consider the following three problems for a properly infinite von Neumann algebra $\mathscr{M}$ acting on a Hilbert space $\mathcal{H}$:

Question 1. Is there a positive self-adjoint operator T in $\mathscr{M}_{\text{aff}}^{c}$ and a unitary operator $U \in \mathscr{M}$ such that $\text{dom}(T) \cap \text{dom}(UTU^*) = \{0\}_{\mathcal{H}}$?

Question 2. Is there a densely-defined closed symmetric operator A in $\mathscr{M}_{\text{aff}}^{c}$ such that $\text{dom}(A^2) = \{0\}_{\mathcal{H}}$?

Question 3. Are there non-singular positive self-adjoint operators $A, B \in \mathscr{M}_{\text{aff}}^{c}$ such that

$$\text{dom}(A) \cap \text{dom}(B) = \text{dom}(A^{-1}) \cap \text{dom}(B^{-1}) = \{0\}_{\mathcal{H}}?$$

At first glance, these questions may seem difficult to answer in the general setting of $\mathscr{M}$. However, they have been affirmatively resolved in the special case $\mathscr{M} = \mathcal{B}(\mathcal{H})$. Since it is a well-known result that $\mathcal{B}(\mathcal{H})$ can be embedded normally into any properly infinite von Neumann algebra $\mathscr{M}$, the functoriality of the affiliated operator framework allows us to transfer these solutions from $\mathcal{B}(\mathcal{H})$ to $\mathscr{M}$. Moreover, the theory established in earlier chapters guarantees that the domains of operators in $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ behave functorially under such embeddings.

This chapter discusses these questions in detail and illustrates how the functorial machinery developed earlier can be applied effectively in concrete problems involving domains of affiliated operators.

Taken together, these chapters provide a new and robust formalism for unbounded operators associated with von Neumann algebras and more general operator algebras, one that is functorial, algebraically rich, and capable of supporting future developments in mathematical physics and noncommutative geometry.

CHAPTER 1

Preliminaries

This chapter presents a collection of notations, fundamental concepts, well-known results, and elementary lemmas that will be used throughout this thesis. For unbounded operators, we primarily refer to [Sch12], and for the general theory of von Neumann algebras, we rely on [KR97; KR86]. A brief discussion of near-semirings is provided in §1.4.

Notations

All Hilbert spaces considered in this thesis are over the field $\mathbb{C}$. We use the following notations:

- Hilbert spaces will be denoted by $\mathcal{H}, \mathcal{K}, \ldots$, von Neumann algebras by $\mathscr{M}, \mathscr{N}, \ldots$, and C^*-algebras by $\mathfrak{A}, \mathfrak{B}, \ldots$.
- Operators on Hilbert spaces are denoted by $S, T, A, B, \ldots$, and vectors by $x, y, z, \ldots$.
- The zero-dimensional subspace of a Hilbert space $\mathcal{H}$ is denoted by $\{0\}_{\mathcal{H}}$.
- The set of bounded operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$, and the set of closed operators on $\mathcal{H}$ is $\mathcal{C}(\mathcal{H})$.
- For a represented von Neumann algebra $(\mathscr{M}; \mathcal{H})$, the lattice of projections in $\mathscr{M}$ is denoted by $\mathscr{M}_{\text{proj}}$, and the set of all self-adjoint operators in $\mathscr{M}$ is denoted by $\mathscr{M}_{\text{sa}}$.
- Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital $*$-homomorphism. Then for $k \in \mathbb{N}$, $\Phi_{(k)}$ denotes the unital $*$-homomorphism from $\mathbb{M}_k(\mathfrak{A})$ to $\mathbb{M}_k(\mathfrak{B})$ given by
$$[A_{ij}]_{1 \le i,j \le k} \mapsto [\Phi(A_{ij})]_{1 \le i,j \le k}.$$
- For a bounded operator T on a Hilbert space, $|T| := (T^*T)^{\frac{1}{2}}$.

For a Hilbert space $\mathcal{H}$, note that $\mathcal{B}(\oplus_{i=1}^k \mathcal{H})$ and $\mathbb{M}_k\big(\mathcal{B}(\mathcal{H})\big)$ are isomorphic as von Neumann algebras (see [KR97, §2.6] for more details). We often use this identification to view $\mathbb{M}_k\big(\mathcal{B}(\mathcal{H})\big)$ as acting on $\oplus_{i=1}^k \mathcal{H}$. For a represented von Neumann algebra $(\mathscr{M}; \mathcal{H})$, we usually consider $\mathbb{M}_k(\mathscr{M})$ as the represented von Neumann algebra $\big(\mathbb{M}_k(\mathscr{M}); \oplus_{i=1}^k \mathcal{H}\big)$ in the above sense. For bounded (everywhere-defined) operators A', B' on a Hilbert space $\mathcal{H}'$, we use the notation,
$$\mathfrak{T}_{A',B'} := \begin{bmatrix} A' & -B' \\ 0 & 0 \end{bmatrix}, \mathfrak{S}_{A',B'} := \begin{bmatrix} B' & 0 \\ A' & 0 \end{bmatrix} \in M_2\big(\mathcal{B}(\mathcal{H}')\big).$$

1.1 Unbounded Operators

Unbounded operators are a central theme in this thesis. Below, we recall some basic definitions and results.

Basic Definitions

Definition 1.1 (Unbounded Operator)**.** Any linear operator T with domain and range in $\mathcal{H}$ is called an *unbounded operator* . The domain of T is denoted by $\operatorname{dom}(T)$. By "unbounded", we mean "not necessarily bounded" rather than "not bounded". The *graph* of T is defined as

$$\operatorname{Gr}(T) := \Big\{(x, Tx) \ : \ x \in \operatorname{dom}(T)\Big\} \subseteq \mathcal{H} \oplus \mathcal{H}.$$

Definition 1.2 (Closed Operator)**.** An operator T is said to be *closed* if $\operatorname{Gr}(T)$ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$. If T is pre-closed, i.e., $\overline{\operatorname{Gr}(T)}$ is the graph of an operator, this operator is called the *closure* of T, denoted by $\overline{T}$.

Definition 1.3 (Extension)**.** For operators T, T_0 on $\mathcal{H}$, T_0 is an *extension* of T, written as $T \subseteq T_0$, if

1. $\operatorname{dom}(T) \subseteq \operatorname{dom}(T_0)$, and
2. $Tx = T_0 x$ for all $x \in \operatorname{dom}(T)$.

Operations on Unbounded Operators

Definition 1.4 (Sum and Product)**.** For operators S, T on $\mathcal{H}$, the sum $S + T$ and product ST are defined as:

$$\begin{aligned}
\operatorname{dom}(S+T) &:= \operatorname{dom}(S) \cap \operatorname{dom}(T), \\
(S+T)x &:= Sx + Tx, \ \text{for all } x \in \operatorname{dom}(S+T); \\
\operatorname{dom}(ST) &:= \{x \in \operatorname{dom}(T) \ : \ Tx \in \operatorname{dom}(S)\}, \\
(ST)x &:= S(Tx), \ \text{for all } x \in \operatorname{dom}(ST).
\end{aligned}$$

If S, T are closed operators on $\mathcal{H}$ such that $S + T$ (ST, respectively) is pre-closed, then its closure $\overline{S+T}$ ($\overline{ST}$, respectively) is called the *strong-sum* (*strong-product*, respectively) of S and T, and is denoted by $S \hat{+} T$ ($S \hat{\cdot} T$, respectively).

Remark 1.1. Let $\mathcal{H}$ be a Hilbert space, and R, S, T be unbounded operators on $\mathcal{H}$. Then it is easily verified that

$$R + (S + T) = (R + S) + T \ \text{ and } \ R(ST) = (RS)T.$$

In other words, the sum and product are associative binary operations on the set of unbounded operators on $\mathcal{H}$. But only right-distributivity holds in general:

$$(S + T)R = SR + TR, \ \ R(S + T) \supseteq RS + RT.$$

When $\mathcal{H}$ is infinite-dimensional, by a result of Neumark (see [Neu40b], [Sch83], [Che83]), it is known that there is a densely-defined closed symmetric operator T on $\mathcal{H}$ such that $\operatorname{dom}(T^2) = \{0\}_{\mathcal{H}}$. It is straightforward to see that $T^2 + (-T^2) = 0|_{\{0\}_{\mathcal{H}}}$, whereas $T\,(T + (-T)) = 0|_{\operatorname{dom}(T)}$. Thus left-distributivity does **not** hold in general.

Remark 1.2. Let S, S_0, T, T_0 be unbounded operators on $\mathcal{H}$ with $S \subseteq S_0, T \subseteq T_0$, then it is straightforward to verify that,

$$ST \subseteq S_0 T_0, \ TS \subseteq T_0 S_0, \ S + T \subseteq S_0 + T_0.$$

Other fundamental concepts such as range and null projections are introduced in Definition 1.5. Further details on specific operator classes will be explored in later chapters.

Definition 1.5. Let T be an unbounded operator acting on $\mathcal{H}$. We define,

$$\mathrm{null}(T) := \{x \in \mathrm{dom}(T) : Tx = 0\}\,, \ \mathrm{ran}(T) := \{Tx : x \in \mathrm{dom}(T)\}\,.$$

Since $\{0\}_{\mathcal{H}} \subseteq \mathrm{null}(T)$, note that the nullspace of T is non-empty. The projection onto the closed subspace $\overline{\mathrm{ran}(T)} \subseteq \mathcal{H}$ is said to be the *range projection* of T, and is denoted by $\mathbf{R}(T)$. The projection onto the closed subspace $\overline{\mathrm{null}(T)} \subseteq \mathcal{H}$ is said to be the *null projection* of T, and is denoted by $\mathbf{N}(T)$.

1.2 Kaufman Inverse

In [Kau78], in the course of his investigation of quotient representations of closed operators, Kaufman defines an 'inverse' for arbitrary bounded operators which need not be one-to-one. Below we extend this operation to arbitrary unbounded operators and call it the *Kaufman inverse* in his honour.

Definition 1.6 (Kaufman Inverse)**.** Let $\mathcal{H}$ be a Hilbert space and T be an unbounded operator on $\mathcal{H}$. Consider the unbounded operator, $T^\dagger$, on $\mathcal{H}$, with $\mathrm{dom}(T^\dagger) := \mathrm{ran}(T)$, which is defined as follows,

$$Tx \mapsto \big(I - \mathbf{N}(T)\big)x, \ \text{(for } x \in \mathrm{dom}(T)).$$

This is clearly well-defined and we call $T^\dagger$ the *Kaufman inverse* of T. When T is one-to-one, that is, $\mathrm{null}(T) = \{0\}_{\mathcal{H}}$, it is easy to see that $T^\dagger = T^{-1}$ on $\mathrm{ran}(T)$.

Clearly,

$$\mathrm{ran}(T^\dagger) = \big(I - \mathbf{N}(T)\big)\mathrm{dom}(T) \subseteq \overline{\mathrm{dom}(T)} \cap \mathrm{null}(T)^\perp.$$

It is left to the reader to verify that $\mathrm{ran}(T^\dagger)$ is a dense subspace of $\overline{\mathrm{dom}(T)} \cap \mathrm{null}(T)^\perp$. Thus $T^\dagger : \mathrm{ran}(T) \to \overline{\mathrm{dom}(T)} \cap \mathrm{null}(T)^\perp$ is a one-to-one mapping with dense range.

Remark 1.3. Since a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is everywhere-defined, that is, $\mathrm{dom}(A) = \mathcal{H}$, $A^\dagger$ gives a bijection between $\mathrm{ran}(A)$ and $\mathrm{null}(A)^\perp$. Furthermore for a projection $E \in \mathcal{B}(\mathcal{H})$, it is straightforward to see that $E^\dagger$ is the restriction of the identity operator to $\mathrm{ran}(E)$. It is helpful to keep these two observations in mind for later use in this paper (see Lemma 1.2, Proposition 1.2, Theorem 3.7).

The restriction of $0_{\mathcal{H}}$ to $\{0\}_{\mathcal{H}}$ is said to be the *trivial affiliated operator*, and is equal to $0_{\mathcal{H}}^\dagger$.

Although the following lemma about closed operators is well-known and standard, we include its proof with a view towards Lemma 1.9 which generalizes it to the context of affiliated operators.

Lemma 1.1. *Let $\mathcal{H}$ be a Hilbert space. Let T, A be closed operators on $\mathcal{H}$ such that $\mathrm{dom}(A)$ is a closed subspace of $\mathcal{H}$. Then TA is a closed operator on $\mathcal{H}$.*

Proof. By the closed graph theorem, the operator $A : \mathrm{dom}(A) \to \mathcal{H}$ is bounded. In order to show that TA is closed, it suffices to show that whenever $x_n \to 0$ in $\mathrm{dom}(TA)$ and $(TA)x_n \to y$, we have $y = 0$.

Let $x_n \to 0$ in $\mathrm{dom}(TA)$ with $(TA)x_n \to y$. Since A is bounded, we have $Ax_n \to 0$ in $\mathrm{dom}(T)$. As T is closed and $T(Ax_n) \to y$, we see that $y = 0$. Thus TA is a closed operator. ■

Lemma 1.2. *Let T be a closed operator on a Hilbert space $\mathcal{H}$.*

(i) Then

$$\operatorname{ran}(T^\dagger) = \operatorname{dom}(T) \cap \operatorname{null}(T)^\perp.$$

Thus $T^\dagger$ gives a bijection between $\operatorname{ran}(T)$ *and* $\operatorname{dom}(T) \cap \operatorname{null}(T)^\perp$*.*

(ii) Furthermore, $T^\dagger$ is closed and is the unique operator satisfying the following properties:

$$T^\dagger T = (I - \mathbf{N}(T))|_{\operatorname{dom}(T)}, \quad TT^\dagger = I_{\mathcal{H}}|_{\operatorname{ran}(T)}, \quad \operatorname{dom}(T^\dagger) = \operatorname{ran}(T). \tag{1.2.1}$$

Proof. (i) Since T is closed, $\operatorname{null}(T)$ is a closed subspace of $\mathcal{H}$ (see [KR97, Exercise 2.8.45]). Let x be a vector in $\operatorname{dom}(T)$. Note that $\mathbf{N}(T)x \in \overline{\operatorname{null}(T)} = \operatorname{null}(T) \subseteq \operatorname{dom}(T)$. Thus $\big(I - \mathbf{N}(T)\big)x \in \operatorname{dom}(T) \cap \operatorname{null}(T)^\perp$. We conclude that $\operatorname{ran}(T^\dagger) \subseteq \operatorname{dom}(T) \cap \operatorname{null}(T)^\perp$. For the reverse inclusion, note that $\operatorname{ran}\big(I - \mathbf{N}(T)\big) = \operatorname{null}(T)^\perp$. Hence if $x \in \operatorname{dom}(T) \cap \operatorname{null}(T)^\perp$, then $x = \big(I - \mathbf{N}(T)\big)x \in \operatorname{ran}(T^\dagger)$.

(ii) Let T_0 be the restriction of T to the subspace $\operatorname{ran}(T^\dagger) = \operatorname{dom}(T) \cap \operatorname{null}(T)^\perp$. Note that T_0 is a bijection from the space $\operatorname{ran}(T^\dagger)$ onto $\operatorname{dom}(T^\dagger) = \operatorname{ran}(T)$. It is then clear from the definition of $T^\dagger$ that $T_0^{-1} = T^\dagger$.

Since T is closed and $\operatorname{dom}\Big(\big(I - \mathbf{N}(T)\big)^\dagger\Big) = \operatorname{null}(T)^\perp$ is a closed subspace of $\mathcal{H}$, by Remark 1.3 and Lemma 1.1, we have that $T_0 = T(I - \mathbf{N}(T))^\dagger$ is a closed operator. Moreover, $\operatorname{ran}(T_0) = \operatorname{ran}(T)$. Since $\operatorname{Gr}\left(T_0^{-1}\right)$ is obtained by flipping coordinates of $\operatorname{Gr}\left(T_0\right) \subseteq \mathcal{H} \oplus \mathcal{H}$, we note that $\operatorname{Gr}\left(T_0^{-1}\right)$ is closed, that is, T_0^{-1} is a closed operator. In summary, $T^\dagger$ is closed whenever T is closed. The equations in (1.2.1) are a matter of unwrapping the definitions, and it is straightforward to see that they completely determine $T^\dagger$. ■

Remark 1.4. Let $A \in \mathcal{B}(\mathcal{H})$. For any unbounded operator T on $\mathcal{H}$ satisfying $TA = I - \mathbf{N}(A)$, note that T is an extension of $A^\dagger$. Indeed, $\{x \in \mathcal{H} : Ax \in \operatorname{dom}(T)\} = \operatorname{dom}(TA) = \mathcal{H}$, that is, $\operatorname{dom}(A^\dagger) = \operatorname{ran}(A) \subseteq \operatorname{dom}(T)$. Also, $T(Ax) = (I - \mathbf{N}(A))x = A^\dagger(Ax)$ for every $x \in \mathcal{H}$; in other words, $A^\dagger$ and T agree on $\operatorname{ran}(A) = \operatorname{dom}(A^\dagger)$. Thus $A^\dagger \subseteq T$. This algebraic observation will be helpful in the proof of Theorem 4.12.

Remark 1.5. An operator on $\mathcal{H}$ (not necessarily everywhere-defined on $\mathcal{H}$) is closed and bounded if and only if it is the restriction of an everywhere-defined bounded operator on $\mathcal{H}$, that is, an operator in $\mathcal{B}(\mathcal{H})$, to a closed subspace of $\mathcal{H}$. This follows from the well-known fact that a densely-defined closed operator is bounded if and only it is everywhere-defined. We denote the set of closed and bounded operators on $\mathcal{H}$ by $\mathcal{CB}(\mathcal{H})$. In view of Remark 1.3, every operator in $\mathcal{CB}(\mathcal{H})$ is of the form $AE^\dagger$ for $A, E \in \mathcal{B}(\mathcal{H})$ where E is a projection.

Definition 1.7 (Adjoint)**.** For an unbounded operator T acting on the Hilbert space $\mathcal{H}$, $T : \operatorname{dom}(T) \to \mathcal{H}$, we define an unbounded operator on $\mathcal{H}$, $T^* : \operatorname{dom}(T^*) \to \mathcal{H}$, called the *adjoint of* T, as follows. Its domain is given by:

$$\operatorname{dom}(T^*) = \{y \in \mathcal{H} : \text{ the mapping } x \mapsto \langle Tx, y\rangle \; \big(x \in \operatorname{dom}(T)\big) \text{ is continuous } \},$$

By the Riesz representation theorem (see [KR97, Theorem 2.3.1]) on the Hilbert space $\overline{\operatorname{dom}(T)}$, for every $y \in \operatorname{dom}(T^*)$ there is a *unique* $z \in \overline{\operatorname{dom}(T)}$ such that $\langle Tx, y\rangle = \langle x, z\rangle$ for all $x \in \operatorname{dom}(T)$. We define $T^*y = z$. Clearly, the range of T^* lies in $\overline{\operatorname{dom}(T)}$.

Caution. In the literature, the adjoint is typically defined for a densely-defined operator. As seen above, we may view T as a densely-defined operator from the Hilbert space $\overline{\operatorname{dom}(T)}$ to the Hilbert space $\mathcal{H}$ and make perfectly good sense of its adjoint. It is crucial to remember that in our definition, the codomain of T^* is $\mathcal{H}$ and **not** $\overline{\operatorname{dom}(T)}$, though we have that $\operatorname{ran}(T^*) \subseteq \overline{\operatorname{dom}(T)}$. The reader may verify that the standard definition of adjoint (see [KR97, Definition 2.7.4]) very much depends on the codomain Hilbert space as any vector in $\operatorname{ran}(T)^\perp$

must lie in the domain of T^*. We hope that this clarification will be helpful in interpreting the meaning of T^{**} as the adjoint of T^*, and in verifying that there is no sleight-of-hand involved when we invoke standard results from the literature about the adjoint in our proofs.

Using the same line of argument given in [KR97, Remark 2.7.6], note that for every unbounded operator T on $\mathcal{H}$, its adjoint T^* is closed on $\mathcal{H}$.

Proposition 1.1 (cf. [KR97, Theorem 2.7.8])**.** *If T is a densely-defined transformation from a Hilbert space $\mathcal{K}$ to a Hilbert space $\mathcal{H}$, then*

(i) T is pre-closed if and only if $\mathrm{dom}(T^*)$ is dense in $\mathcal{H}$,

(ii) if T is pre-closed, $\overline{T} = T^{**}$.

Proposition 1.2. *Let T be a pre-closed operator (not necessarily densely-defined) acting on the Hilbert space $\mathcal{H}$. Then T^* is densely-defined and we have $\overline{T} = T^{**}E^{\dagger}$ where E is the projection onto $\overline{\mathrm{dom}(T)}$.*

Proof. By Proposition 1.1-(i) in the context of the Hilbert space $\mathcal{K} = \overline{\mathrm{dom}(T)}$ and viewing T as a pre-closed densely-defined operator from $\mathcal{K}$ to $\mathcal{H}$, we see that T^* is densely-defined on $\mathcal{H}$.

For pre-closed operators T_1, T_2, on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively, it is straightforward to verify that $T_1 \oplus T_2$ is a pre-closed operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$, and $\overline{T_1 \oplus T_2} = \overline{T_1} \oplus \overline{T_2}$.

Let E denote the projection onto $\overline{\mathrm{dom}(T)}$. Clearly,

$$\mathrm{dom}(TE) = \mathrm{dom}(T) \oplus \mathrm{dom}(T)^{\perp}, \text{ and } TE = T \oplus 0|_{\mathrm{dom}(T)^{\perp}}.$$

Thus TE is a pre-closed operator on the Hilbert space $\overline{\mathrm{dom}(T)} \oplus \mathrm{dom}(T)^{\perp}$ with closure $\overline{TE} = \overline{T} \oplus 0|_{\mathrm{dom}(T)^{\perp}}$. It follows that,

$$\overline{T} = (\overline{T} \oplus 0|_{\mathrm{dom}(T)^{\perp}})E^{\dagger} = (\overline{TE})E^{\dagger}.$$

It may be directly verified from the definition of adjoint (Definition 1.7) that $(TE)^* = T^*$. Since TE is densely-defined on $\mathcal{H}$, using Proposition 1.1 we see that $\overline{TE} = (TE)^{**} = T^{**}$. Hence,

$$\overline{T} = (\overline{TE})E^{\dagger} = (TE)^{**}E^{\dagger} = T^{**}E^{\dagger}.$$ ■

Proposition 1.3 (Rank-nullity theorem)**.** *For a densely-defined closed operator T acting on $\mathcal{H}$ we have:*

$$\boldsymbol{R}(T) = I - \mathbf{N}(T^*), \;\; \mathbf{N}(T) = I - \boldsymbol{R}(T^*).$$

Proof. See [KR97, Exercise 2.8.45]. ■

Definition 1.8. For an unbounded linear operator T on $\mathcal{H}$, the *characteristic projection* of T, denoted $\chi(T)$, is defined to be the orthogonal projection from $\mathcal{H} \oplus \mathcal{H}$ onto the closure of the graph of T. We may view $\chi(T)$ as a 2×2 matrix of operators in $\mathcal{B}(\mathcal{H})$.

Remark 1.6. Let T be a closed operator acting on the Hilbert space $\mathcal{H}$ with characteristic projection given by,

$$\chi(T) = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \in \mathbb{M}_2\big(\mathcal{B}(\mathcal{H})\big),$$

where $E_{ij} \in \mathcal{B}(\mathcal{H})$ for $1 \le i, j \le 2$. Since T is a closed operator, $\mathrm{ran}\big(\chi(T)\big) = \mathrm{Gr}\,(T)$, so that $\mathrm{dom}(T)$ is the set of first coordinates of points in $\mathrm{Gr}\,(T)$. Note that,

$$\chi(T)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} E_{11}x + E_{12}y \\ E_{21}x + E_{22}y \end{bmatrix}, \text{ for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}.$$

Thus,

$$\mathrm{dom}(T) = \mathrm{ran}(E_{11}) + \mathrm{ran}(E_{12}). \tag{1.2.2}$$

1.3 General Overview of Operator Algebras

In this section, we provide a brief overview of foundational concepts in the theory of operator algebras, focusing on C^*-algebras, von Neumann algebras, and monotone σ-complete C^*-algebras. We also present key structure theorems without proof and introduce the categorical framework for these objects. A thorough exploration of the theory of operator algebras is provided in [KR97; KR86], [Mur90], [Tak02], and [Bla06].

1.3.1 C^*-Algebras

A C^*-algebra is a Banach algebra equipped with an involution, satisfying the C^*-identity which links the algebraic structure to the norm of its elements. This section introduces the essential properties of C^*-algebras, starting with the definition of a C^*-algebra and the concept of a $*$-homomorphism between such algebras. We then explore key results, such as the automatic continuity of $*$-homomorphisms and the Gelfand–Naimark Representation Theorem, which reveals the connection between abstract C^*-algebras and represented operator algebras on Hilbert spaces. Additionally, we discuss the structure of commutative C^*-algebras, with the Gelfand–Naimark Theorem for these algebras highlighting their equivalence to the algebra of continuous functions on compact Hausdorff spaces.

Definition 1.9 (C^*-Algebra)**.** A *C^*-algebra* is a Banach algebra $\mathfrak{A}$ over $\mathbb{C}$ equipped with an involution $* : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|A^*A\| = \|A\|^2 \quad \text{for all } A \in \mathfrak{A}.$$

Definition 1.10 ($*$-Homomorphism)**.** Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras. A linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ is called a *$*$-homomorphism* if

$$\Phi(AB) = \Phi(A)\Phi(B), \quad \Phi(A^*) = \Phi(A)^* \quad \text{for all } A, B \in \mathfrak{A}.$$

It is called *unital* if $\Phi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$, and *injective* if it is one-to-one.

Theorem 1.3 (Automatic Continuity of $*$-Homomorphisms [KR97, Theorem 4.1.8])**.** *Every $*$-homomorphism between C^*-algebras is contractive, i.e., for all $A \in \mathfrak{A}$,*

$$\|\Phi(A)\| \leq \|A\|.$$

In particular, all $*$-homomorphisms are continuous.

Remark 1.7. In fact, if Φ is injective, then it is isometric. This is an important feature unique to the category of C^*-algebras among Banach algebras.

Theorem 1.4 (Gelfand–Naimark Representation Theorem [KR97, Theorem 4.5.6])**.** *Every C^*-algebra $\mathfrak{A}$ is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra of bounded operators on a Hilbert space. That is, there exists a Hilbert space $\mathcal{H}$ and an injective $*$-homomorphism $\pi : \mathfrak{A} \hookrightarrow \mathcal{B}(\mathcal{H})$.*

Corollary 1.5. *Every C^*-algebra admits a faithful $*$-representation on a Hilbert space.*

Remark 1.8. This theorem allows us to view every abstract C^*-algebra represented as a subalgebra of $\mathcal{B}(\mathcal{H})$. In this thesis, such algebras will be referred to as *represented C^*-algebras.*

Theorem 1.6 (Gelfand–Naimark Theorem for Commutative C^*-Algebras [KR97, Theorem 4.4.3]). *Let $\mathfrak{A}$ be a commutative unital C^*-algebra. Then there exists a compact Hausdorff space X such that*

$$\mathfrak{A} \cong C(X),$$

where $C(X)$ is the algebra of continuous complex-valued functions on X, and the isomorphism is an isometric $*$-isomorphism.

Remark 1.9. This result shows that the theory of commutative C^*-algebras is equivalent to the theory of compact Hausdorff spaces via the contravariant functor $X \mapsto C(X)$. It forms the foundation of noncommutative geometry.

Examples:

1. The algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ is a noncommutative, unital C^*-algebra.
2. $C(X)$ for a compact Hausdorff space X is a commutative unital C^*-algebra.
3. The algebra $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$ is a non-unital, simple C^*-algebra.

1.3.2 von Neumann Algebras

Von Neumann algebras, also known as W^*-algebras, are a central class of operator algebras in functional analysis, distinguished by their closure properties and their deep connection to quantum mechanics and mathematical physics. These algebras are unital C^*-subalgebras of bounded operators on a Hilbert space, which are additionally closed in the weak operator topology (WOT). A key feature of von Neumann algebras is their characterization through the Double Commutant Theorem, which identifies them as those algebras that are equal to their double commutant.

Von Neumann algebras play a crucial role in mathematical physics, particularly in the study of quantum statistical mechanics, quantum field theory, and operator-theoretic formulations of quantum mechanics. This section introduces the fundamental concepts of von Neumann algebras, including their structure, normal homomorphisms, and their relationship with Murray-von Neumann affiliated operators. Key results such as Sakai's theorem and the notion of preduals further illuminate the rich structure of von Neumann algebras. In the latter part of this section, we will discuss *Murray-von Neumann algebras* – these are unital $*$-algebras of unbounded operators associated with a given von Neumann algebra, and explore their algebraic structure and significance.

Definition 1.11 (von Neumann Algebra). A *von Neumann algebra* $\mathscr{M} \subseteq \mathcal{B}(\mathcal{H})$ is a unital C^*-subalgebra that is closed in the weak operator topology (WOT) or equivalently, the ultraweak topology.

Theorem 1.7 (Double Commutant Theorem [KR97, Theorem 5.1.3]). *Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital $*$-subalgebra. Then the following are equivalent:*

(i) *$\mathfrak{A}$ is strongly (or ultraweakly) closed;*

(ii) *$\mathfrak{A}$ is equal to its double commutant: $\mathfrak{A} = \mathfrak{A}''$.*

Remark 1.10. The double commutant theorem provides an algebraic characterization of von Neumann algebras: they are precisely those unital $*$-subalgebras of $\mathcal{B}(\mathcal{H})$ that equal their bicommutant.

Theorem 1.8 (Sakai's Theorem [Sak71, Theorem 1.6.1]). *A C^*-algebra is a von Neumann algebra if and only if it has a unique predual.*

Remark 1.11. In this thesis, we work primarily with *represented* von Neumann algebras, that is, von Neumann algebras realized as subalgebras of $\mathcal{B}(\mathcal{H})$. The abstract characterization via the predual is included here for completeness.

Definition 1.12 (Normal $*$-Homomorphism)**.** A $*$-homomorphism $\Phi : \mathscr{M} \to \mathscr{N}$ between von Neumann algebras is said to be *normal* if for every increasing net $(A_\lambda)_{\lambda\in\Lambda}$ of self-adjoint elements in $\mathscr{M}$ with supremum $A = \sup_\lambda A_\lambda$ in $\mathscr{M}$, we have

$$\Phi(A) = \sup_\lambda \Phi(A_\lambda).$$

Examples:

1. The algebra $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.
2. The algebra $L^\infty(X, \mu)$ of essentially bounded measurable functions on a measure space (X, μ) is a commutative von Neumann algebra acting on $L^2(X, \mu)$ via multiplication.

MvN-Affiliated Operators

We recall the definition of Murray-von Neumann affiliated operators associated with a von Neumann algebra.

Definition 1.13 (MvN-Affiliated Operators, cf. [MN36])**.** Let $\mathscr{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A closed operator T on $\mathcal{H}$ is said to be *Murray–von Neumann affiliated* (or *MvN-affiliated*) with $\mathscr{M}$ if $U^*TU = T$ for all unitary operators $U \in \mathscr{M}'$, the commutant of $\mathscr{M}$.

Remark 1.12. Let T be a closed operator and U a unitary such that $UT \subseteq TU$. Then $U(\mathrm{dom}T) \subseteq \mathrm{dom}T$. The following are equivalent:

1. $TU = UT$ for all unitaries $U \in \mathscr{M}'$;
2. $U^*TU = T$ for all unitaries $U \in \mathscr{M}'$;
3. $UT \subseteq TU$ for all unitaries $U \in \mathscr{M}'$.

Lemma 1.9. *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. For $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$ and $A \in \mathscr{M}$, we have $T^\dagger, TA$ lie in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$. Moreover, if $\mathrm{ran}(A) \subseteq \mathrm{dom}(T)$, then $TA \in \mathscr{M}$.*

Proof. First note that by Lemma 1.2-(ii), $T^\dagger$ is closed and by Lemma 1.1, TA is closed.

Let U be any unitary in $\mathscr{M}'$, the commutant of $\mathscr{M}$. Since $UT = TU$ and $U^*T = TU^*$, it is clear that $U\mathrm{null}(T) = \mathrm{null}(T)$ so that $U(\mathbf{N}(T)) = \mathbf{N}(T)U$. For every $x \in \mathcal{H}$, we have,

$$(UT^\dagger)Tx = U\big(I - \mathbf{N}(T)\big)x = \big(I - \mathbf{N}(T)\big)Ux = T^\dagger TUx = (T^\dagger U)Tx.$$

Since $\mathrm{dom}(T^\dagger) = \mathrm{ran}(T)$, we conclude that $UT^\dagger = T^\dagger U$. Hence, $T^\dagger \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$.

Similarly, since $UT = TU$ and $UA = AU$, we have $U(TA) = TUA = (TA)U$. Thus $TA \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$.

If, in addition, $\mathrm{ran}(A) \subseteq \mathrm{dom}(T)$, then $\mathrm{dom}(TA) = \mathcal{H}$ so that TA is a bounded operator in $(\mathscr{M}')' = \mathscr{M}$. ■

Murray-von Neumann algebras

A *finite von Neumann algebra* is a von Neumann algebra with no proper isometries; that is, every isometry in the algebra is a unitary. Examples include $L^\infty(\Omega, \mu)$ for a finite measure space (Ω, μ), type I_n von Neumann algebras (for $n < \infty$), and type II_1 factors. A rich class of finite von Neumann algebras also arises through the construction of *group von Neumann algebras* $L(G)$ for discrete groups G.

Let $\mathscr{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Denote by $\mathscr{M}_{\text{aff}}^{dd-c}$ the set of all densely defined Murray–von Neumann affiliated operators with $\mathscr{M}$. This set admits a well-understood algebraic structure; specifically, it can be endowed with the structure of a unital $*$-algebra with respect to strong-sum $(\hat{+})$ and strong product $(\hat{\cdot})$ (see 1.4, [MN36, Theorem XV], [KL14, §6.2]). In [Nay21a], such $*$-algebras are referred to as *Murray–von Neumann algebras*. Furthermore, in this paper, the author defined a certain topology on a *countably decomposable* finite von Neumann algebra $\mathscr{M}$ as follows:

Definition 1.14. For $\epsilon, \delta > 0$ and a normal tracial state τ on $\mathscr{M}$, define

$$\mathcal{O}(\tau, \epsilon, \delta) := \{A \in \mathscr{M} : \exists E \in \mathscr{M}_{\text{proj}} \text{ such that } \tau(I - E) \leq \delta,\ \|AE\| \leq \epsilon\}.$$

The translation-invariant topology generated by the fundamental system of neighborhoods of 0 given by $\{\mathcal{O}(\tau, \epsilon, \delta)\}$ is called the $\mathfrak{m}$-topology on $\mathscr{M}$.

It is shown in [Nay21a, Theorem 4.3] that the completion of $\mathscr{M}$ with respect to the $\mathfrak{m}$-topology coincides with $\mathscr{M}_{\text{aff}}^{dd-c}$.

Thus, there is a well-developed theory concerning the algebraic structure of $\mathscr{M}_{\text{aff}}^{dd-c}$ when $\mathscr{M}$ is finite. However, for a general von Neumann algebra $\mathscr{M}$, the structure of $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ or even $\mathscr{M}_{\text{aff}}^{dd-c}$ remains largely unknown. One of the challenges in this broader setting is that the sum or product of two such MvN-affiliated operators may fail to be closed or densely defined.

In Chapter 3, we introduce a new class of unbounded operators, referred to as *affiliated operators*, which are defined in a more algebraic fashion. We show that this class has a nice algebraic structure and contains $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ as a subclass. Moreover these set of operators is intrinsically linked to the underlying von Neumann algebra, even when $\mathscr{M}$ is not finite. As such, they offer a robust framework for the study of unbounded operators affiliated with a general von Neumann algebra.

1.3.3 Monotone σ-Complete C^*-Algebras

A C^*-algebra $\mathfrak{A}$ is *monotone σ-complete* if each norm bounded, monotone increasing sequence in $\mathfrak{A}_{\text{sa}}$ has a least upper bound in $\mathfrak{A}_{\text{sa}}$. These algebras generalize von Neumann algebras while maintaining a structure that allows limit processes.

Note that $\mathfrak{A}$ is monotone σ-complete if each norm bounded monotone decreasing sequence in $\mathfrak{A}_{\text{sa}}$ has a greatest lower bound in $\mathfrak{A}_{\text{sa}}$. It is clear that if an increasing sequence in $\mathfrak{A}_{\text{sa}}$ is *upper bounded* w.r.to the ordering of $\mathfrak{A}_{\text{sa}}$, then the sequence is norm bounded. But the converse may not be true in general. For further reading, see [SW15; Bla06].

Example: Every von Neumann algebra is a monotone σ-complete C^*-algebra.

We now define the morphisms in the category of monotone σ-complete C^*-algebras namely the *σ-normal* $*$-homomorphisms:

Definition 1.15. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a positive map. Then Φ is said to be σ-normal if, whenever (A_n) is an increasing sequence in $\mathfrak{A}_{\text{sa}}$ with supremum A, then supremum of $(\Phi(A_n))$ is $\Phi(A)$.

The following lemma is a workhorse in this article in our pursuit of functoriality. We shall use the lemma in the particular case of represented von Neumann algebras.

Lemma 1.10. *Let $(\mathfrak{A};\mathcal{H}),(\mathfrak{B};\mathcal{K})$ be unital represented monotone σ-complete C^*-algebras and $\Phi:\mathfrak{A}\to\mathfrak{B}$ be a unital σ-normal $*$-homomorphism. For $A\in\mathfrak{A}$, we have*

$$\Phi\big(\boldsymbol{R}(A)\big)=\boldsymbol{R}\big(\Phi(A)\big),$$
$$\Phi\big(\mathbf{N}(A)\big)=\mathbf{N}\big(\Phi(A)\big).$$

Proof. Let H be a positive contraction in $\mathfrak{A}$, that is, $\|H\|\le 1$. Note that $\Phi(H)$ is a positive contraction in $\mathfrak{B}$. From [KR97, Lemma 5.1.5], we note that $H^{\frac{1}{n}}\uparrow\mathbf{R}(H)$ and $\Phi(H)^{\frac{1}{n}}\uparrow\mathbf{R}\big(\Phi(H)\big)$. Using σ-normality of Φ, we have

$$\Phi\big(\mathbf{R}(H)\big)=\Phi\big(\bigvee_n H^{\frac{1}{n}}\big)=\bigvee_n\Phi\big(H^{\frac{1}{n}}\big)=\bigvee_n\Phi(H)^{\frac{1}{n}}=\mathbf{R}\big(\Phi(H)\big).$$

Using the rank-nullity theorem (see Proposition 1.3), we have,

$$\Phi\big(\mathbf{N}(A)\big)=\mathbf{N}\big(\Phi(A)\big).$$ ∎

Below, we gather some results that will be used in the next chapter to prove the *Douglas Factorization Lemma* for monotone σ-complete C^*-algebras. Proofs of these results can be found in [SW15].

Proposition 1.4 ([SW15, Proposition 2.2.12])**.** *Let $\mathfrak{A}$ be a monotone σ-complete C^*-algebra. Let $A\in\mathfrak{A}$ and $B\in\mathfrak{A}_{\rm sa}$ be such that $A^*A\le\lambda B^2$ for some $\lambda>0$. Then there exists $X\in\mathfrak{A}$ such that $XB=A$.*

Corollary 1.11 (Polar Decomposition, [SW15, Theorem 2.2.15])**.** *Let $\mathfrak{A}$ be a monotone σ-complete C^*-algebra. For any $A\in\mathfrak{A}$ there is a partial isometry V such that $A=V|A|$. Also $V^*V=\boldsymbol{R}(A^*)$ and $VV^*=\boldsymbol{R}(A)$.*

Proposition 1.5. *Let $\mathfrak{A}$ be a monotone σ-complete C^*-algebra. Let $A,B\in\mathfrak{A}$. Then the following conditions are equivalent:*

(i) $XB=A$ for some $X\in\mathfrak{A}$;

(ii) $A^*A\le\lambda B^*B$ for some $\lambda>0$;

Proof. (i) $\Rightarrow$ (ii). Since $A^*A=B^*X^*XB\le\|X\|^2B^*B$, we may choose $\lambda>\|X\|^2$.

(ii) $\Rightarrow$ (i). Since $A^*A\le\lambda B^*B=\lambda|B|^2$, by Proposition 1.4 there is $Y\in\mathfrak{A}$ such that $A=Y|B|$. Now from Corollary 1.11 we know that $|B|=V^*B$ for some partial isometry $V\in\mathfrak{A}$. Hence for $X:=YV^*$ we have $XB=A$. ∎

Proposition 1.6 ([SW15, Theorem 2.4.6])**.** *Let $\mathfrak{A}$ be a monotone σ-complete C^*-algebra. Then for each $n\in\mathbb{N}$ the matrix algebra $\mathbb{M}_n(\mathfrak{A})$ is also monotone σ-complete.*

1.3.4 Categories of Operator Algebras

We now define three important categories that encapsulate these structures:

1. **C*Alg**: The category of unital C^*-algebras, with objects being unital C^*-algebras and morphisms being unital $*$-homomorphisms.

2. **MσC*Alg**: The category of unital monotone σ-complete C^*-algebras, with morphisms being unital σ-normal $*$-homomorphisms.
3. **W*Alg**: The category of von Neumann algebras, with morphisms being unital normal $*$-homomorphisms.

These categories provide a natural framework for studying operator algebras in a functorial setting.

1.4 Near-Semirings

The concept of near-semirings was introduced in [VR62, see §2] under the name *Fasthalbringe*. Unlike *near-rings*, for which a rich theory has already been developed, the study of near-semirings has not received significant attention. Even the definition of a near-semiring lacks consistency in the literature, although the various definitions are closely related. They are sometimes referred to as *seminear-rings*.

In this thesis, we adopt the definition used in [MS97, §Introduction], [Sam09, pg. 714], and [KK14, Definition 1.1], stated below.

Definition 1.16 (Near-Semiring)**.** An algebraic structure $(S, +, \circ)$ with binary operations $+$ and $\circ$ is said to be a *right near-semiring* if it satisfies the following axioms:

1. $(S, +)$ and $(S, \circ)$ are semigroups.
2. $(a + b) \circ c = a \circ c + b \circ c$ for all $a, b, c \in S$.

In this thesis, the term *near-semiring* will mean *right near-semiring*. If, instead of axiom (2), left-distributivity is assumed, the structure is called a *left near-semiring*.

Let S_1 and S_2 be near-semirings. A mapping $\varphi : S_1 \to S_2$ is said to be a *near-semiring homomorphism* if:

$$\varphi(a + b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(a \circ b) = \varphi(a) \circ \varphi(b) \quad \text{for all } a, b \in S_1.$$

Example 1.1. The set $\mathbb{N} \cup \{0\}$ of all non-negative integers with the usual addition and multiplication forms a near-semiring.

Example 1.2. Let $(\Gamma, +)$ be a semigroup, and let $M(\Gamma)$ denote the set of all mappings from Γ into itself. Define the operations $+$ and $\circ$ for $f, g \in M(\Gamma)$ by:

$$\begin{aligned}(f + g)(x) &:= f(x) + g(x),\\ (f \circ g)(x) &:= f(g(x)),\end{aligned}$$

for all $x \in \Gamma$. Then, $(M(\Gamma), +, \circ)$ is a near-semiring.

Note: Example 1.2 does **not** satisfy left-distributivity in general. In §**??**, we will see that for a von Neumann algebra $\mathscr{M}$, the set of affiliated operators (see Definition 3.1) forms a near-semiring under the sum and product of (unbounded) operators, and left-distributivity does **not** hold in general (see Remark 1.1).

As noted earlier, the literature on near-semirings can be confusing due to the lack of consensus on the core definitions. Readers who wish to delve deeper into the topic may refer to [VHVR67], [Kri05], and [KK14] for additional perspectives and motivations for the various defining axioms of near-semirings.

Chapter 2

Affiliated Subspaces

In this chapter, we introduce the concept of *affiliated subspaces* for a represented unital monotone σ-complete C^*-algebra. One of the key observations regarding the set of affiliated subspaces is that it forms a lattice under vector sum as *join* and set intersection as *meet*. Furthermore, this structure transforms functorially under unital σ-normal $*$-homomorphisms between represented monotone σ-complete C^*-algebras. A crucial role in our discussion is played by the Douglas factorization lemma in the context of monotone σ-complete C^*-algebras, whose proof basically follows from Proposition 1.5. To build up to the main results, we begin by analyzing this lemma in detail.

2.1 The Douglas Factorization Lemma

Theorem 2.1 (Douglas Lemma; Cf. [GN24, Theorem 3.1])**.** *Let $\mathfrak{A}$ be a monotone σ-complete C^*-algebra acting on a Hilbert space $\mathcal{H}$, and let $A, B \in \mathfrak{A}$. The following conditions are equivalent:*

(i) *There exists $X \in \mathfrak{A}$ such that $BX = A$;*

(ii) *There exists $\lambda > 0$ such that $AA^* \leq \lambda^2 BB^*$;*

(iii) $\mathrm{ran}(A) \subseteq \mathrm{ran}(B)$.

Moreover, if $\mathfrak{A}$ is a von Neumann algebra, then the above conditions are also equivalent to:

(iv) *$B^\dagger A$ is a bounded operator and belongs to $\mathfrak{A}$.*

Proof. (i) $\iff$ (ii). This follows from Proposition 1.5 in the context of A^* and B^*.

(i) $\Rightarrow$ (iii). This is trivial since $Ay = BXy$ for all $y \in \mathcal{H}$.

(iii) $\Rightarrow$ (ii). Assume $\mathrm{ran}(A) \subseteq \mathrm{ran}(B)$. If we now define $Y := B^\dagger A$ then $\mathrm{dom}(Y) = \mathcal{H}$ and since $B^\dagger$ is closed and A is bounded so Y is in $\mathcal{B}(\mathcal{H})$. Furthermore $B^\dagger B : \mathrm{ran}(B) \to \mathrm{ran}(B)$ is the identity mapping and $\mathrm{ran}(A) \subseteq \mathrm{ran}(B)$. We conclude that $BY = A$.

Therefore we get $AA^* = BXX^*B^* \leq \|X\|^2 BB^*$, which implies the inequality in (ii) for $\lambda > \|X\|$.

Now assume that $\mathfrak{A}$ is a von Neumann algebra. Then we can show that (iv) $\iff$ (iii):

(iii) $\Rightarrow$ (iv). Since $\mathrm{ran}(A) \subseteq \mathrm{ran}(B) \subseteq \mathrm{dom}(B^\dagger)$ and $B^\dagger \in \mathfrak{A}_{\mathrm{aff}}^{\mathrm{MvN}}$, by Lemma 1.9, $X = B^\dagger A$ is in $\mathfrak{A}$.

(iv) $\Rightarrow$ (iii). This direction is trivial. Assume $B^\dagger A \in \mathfrak{A}$ so that it is defined everywhere on $\mathcal{H}$. Hence we get $\mathcal{H} = \mathrm{dom}(B^\dagger A) = A^{-1}(\mathrm{ran}(B))$, which implies $\mathrm{ran}(A) = A(\mathcal{H}) = AA^{-1}(\mathrm{ran}(B)) \subseteq \mathrm{ran}(B)$. ∎

Corollary 2.2. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Then* $\mathrm{ran}(A) = \mathrm{ran}(|A^*|)$.

Proof. Consider the self-adjoint operator $B = |A^*|$. Clearly, $AA^* = BB^*$. The assertion follows from the equivalence "(i) $\iff$ (iv)" in the statement of Theorem 2.1. ■

Proposition 2.1. *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and let $A, B \in \mathscr{M}$ satisfy $\mathrm{ran}(A) \subseteq \mathrm{ran}(B)$. Then $B^\dagger A$ is the unique operator X in $\mathscr{M}$ satisfying $BX = A$ and $\mathbf{R}(X) \le \mathbf{R}(B^*)$.*

Proof. Note that the condition, $\mathbf{R}(X) \le \mathbf{R}(B^*)$, is equivalent to the condition, $\mathrm{ran}(X) \subseteq \mathrm{ran}(B^*)$. By taking the orthogonal complement of both sides, we see that

$$\mathrm{null}(B) \subseteq \mathrm{null}(X^*).$$

Let X_1, X_2 be two operators satisfying the given conditions. Then we get $X_1^* B^* = A^* = X_2^* B^*$. Thus $X_1^* = X_2^*$ on $\mathrm{ran}(B^*)$. If $x \in \mathrm{ran}(B^*)^\perp = \mathrm{null}(B) \subseteq \mathrm{null}(X_1^*) \cap \mathrm{null}(X_2^*)$, then $X_1^* x = 0 = X_2^* x$. Thus $X_1^* = X_2^*$, or equivalently, $X_1 = X_2$. This proves the uniqueness of such X (if one exists).

As seen in the proof of Theorem 2.1, clearly $B(B^\dagger A) = A$, and from the definition of the Kaufman-inverse, we have

$$\mathrm{ran}(B^\dagger A) \subseteq \mathrm{ran}(B^\dagger) \subseteq \mathrm{null}(B)^\perp = \overline{\mathrm{ran}(B^*)}.$$ ■

Lemma 2.3. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. Let $A, B \in \mathfrak{A}$ with $\mathrm{ran}(A) \subseteq \mathrm{ran}(B)$. Then we have:*

(i) $\mathrm{ran}\big(\Phi(A)\big) \subseteq \mathrm{ran}\big(\Phi(B)\big)$;

(ii) for any $X \in \mathfrak{A}$ satisfying $BX = A$, we have $\Phi(B)\Phi(X) = \Phi(A)$.

Proof. Straightforward to see (use Theorem 2.1)! ■

Theorem 2.4 (cf. [FW71, Theorem 2.2, Corollary 2])**.** *Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra acting on the Hilbert space $\mathcal{H}$. Let A, B be operators in $\mathfrak{A}$ and $\mathfrak{T}_{A,B}$ denote the operator in $\mathbb{M}_2(\mathfrak{A})$ as defined in §1. Then we have the following.*

(i) The range of the operator $|\mathfrak{T}_{A,B}^*|_{11} \in \mathfrak{A}$ is $\mathrm{ran}(A) + \mathrm{ran}(B)$.

(ii) The range of the operator $\sqrt{A\, \mathbf{N}(\mathfrak{T}_{A,B})_{11} A^*} \in \mathfrak{A}$ is $\mathrm{ran}(A) \cap \mathrm{ran}(B)$.

(iii) The range of the operator $\sqrt{\mathbf{N}(\mathfrak{T}_{A,B})_{11}} \in \mathfrak{A}$ is $A^{-1}\Big(\mathrm{ran}(B)\Big)$.

Proof. (i) By applying $\mathfrak{T}_{A,B}$ to vectors in $\mathcal{H} \oplus \mathcal{H}$, it is straightforward to see that

$$\mathrm{ran}(\mathfrak{T}_{A,B}) = \big(\mathrm{ran}(A) + \mathrm{ran}(B)\big) \oplus \{0\}_{\mathcal{H}}.$$

Note that,

$$|\mathfrak{T}_{A,B}^*| = \begin{bmatrix} \sqrt{AA^* + BB^*} & 0 \\ 0 & 0 \end{bmatrix}.$$

From Corollary 2.2 in the context of the algebra $\mathbb{M}_2(\mathfrak{A})$, we have

$$\begin{aligned}\big(\mathrm{ran}(A) + \mathrm{ran}(B)\big) \oplus \{0\}_{\mathcal{H}} = \mathrm{ran}(\mathfrak{T}_{A,B}) &= \mathrm{ran}(|\mathfrak{T}_{A,B}^*|) \\ &= \mathrm{ran}(\sqrt{AA^* + BB^*}) \oplus \{0\}_{\mathcal{H}}.\end{aligned}$$

(ii)
(iii) Note that for $x, y \in \mathcal{H}$, the vector $(x, y) \in \mathcal{H} \oplus \mathcal{H}$ is in $\mathrm{null}(\mathfrak{T}_{A,B})$ if and only if $Ax = By$. Thus x is the first coordinate of a vector in $\mathrm{null}(\mathfrak{T}_{A,B})$ if and only if $Ax \in \mathrm{ran}(B)$ or equivalently, $Ax \in \mathrm{ran}(A) \cap \mathrm{ran}(B)$. Thus,

$$\mathrm{ran}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}(\mathfrak{T}_{A,B})\right) = (\mathrm{ran}(A) \cap \mathrm{ran}(B)) \oplus \{0\}_{\mathcal{H}}$$
$$\mathrm{ran}\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}(\mathfrak{T}_{A,B})\right) = A^{-1}(\mathrm{ran}(B)) \oplus \{0\}_{\mathcal{H}}.$$

Since

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}(\mathfrak{T}_{A,B}) \begin{bmatrix} A^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A\mathbf{N}(\mathfrak{T}_{A,B})_{11}A^* & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}(\mathfrak{T}_{A,B}) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{N}(\mathfrak{T}_{A,B})_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

using Corollary 2.2 in the context of $\mathbb{M}_2(\mathfrak{A})$, we conclude that,

$$\Big(\mathrm{ran}(A) \cap \mathrm{ran}(B)\Big) \oplus \{0\}_{\mathcal{H}} = \mathrm{ran}\left(\sqrt{A\,\mathbf{N}(\mathfrak{T}_{A,B})_{11}A^*}\right) \oplus \{0\}_{\mathcal{H}},$$
$$A^{-1}\Big(\mathrm{ran}(B)\Big) \oplus \{0\}_{\mathcal{H}} = \mathrm{ran}\left(\sqrt{\mathbf{N}(\mathfrak{T}_{A,B})_{11}}\right) \oplus \{0\}_{\mathcal{H}}. \qquad \blacksquare$$

Remark 2.1. Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra acting on the Hilbert space $\mathcal{H}$ and $A, B \in \mathfrak{A}$. In the context of Theorem 2.4, we denote the operator in parts (i), (ii), (iii), respectively, by $A \boxplus B$, $A \boxdot B$, $A \overset{\mathsf{inv}}{\circ} B$, respectively. Thus the ranges of the operators $A \boxplus B, A \boxdot B$ and $A \overset{\mathsf{inv}}{\circ} B$ in $\mathscr{M}$ are given by:

(i) $\mathrm{ran}(A \boxplus B) = \mathrm{ran}(A) + \mathrm{ran}(B)$;

(ii) $\mathrm{ran}(A \boxdot B) = \mathrm{ran}(A) \cap \mathrm{ran}(B)$;

(iii) $\mathrm{ran}(A \overset{\mathsf{inv}}{\circ} B) = A^{-1}(\mathrm{ran}(B))$.

In fact, from the proof of Theorem 2.4, we have an explicit formula for $A \boxplus B$, given by $\sqrt{AA^* + BB^*}$.

2.2 Functoriality of the Construction $\mathfrak{A} \mapsto \mathrm{Aff}_s(\mathfrak{A})$

Definition 2.1. A subspace $\mathcal{V} \subseteq \mathcal{H}$ is said to be *affiliated* to $\mathfrak{A}$ if there is an operator in $\mathfrak{A}$ whose range is $\mathcal{V}$. We denote the set of affiliated subspaces for $\mathfrak{A}$, or equivalently, the set of operator ranges for $\mathfrak{A}$, by $\mathrm{Aff}_s(\mathfrak{A})$.

The reasoning behind this terminology will become evident in Theorem 4.6, where we establish that $\mathrm{Aff}_s(\mathfrak{A})$ precisely corresponds to the set of domains of affiliated operators. It is well known that the collection of operator ranges in $\mathcal{B}(\mathcal{H})$ forms a lattice, with vector addition serving as the join and set intersection as the meet (see [Mac46], [Dix49], [FW71]). In Lemma 2.5 below, we prove an analogous result within the framework of unital monotone σ-complete C^*-algebras.

Lemma 2.5. *Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra acting on the Hilbert space $\mathcal{H}$.*

(i) For affiliated subspaces $\mathcal{V}, \mathcal{W} \in \mathrm{Aff}_s(\mathfrak{A})$, we have $\mathcal{V} + \mathcal{W}, \mathcal{V} \cap \mathcal{W} \in \mathrm{Aff}_s(\mathfrak{A})$; in other words, $\mathrm{Aff}_s(\mathfrak{A})$ is a lattice under vector addition $(+)$ and set intersection $(\cap)$.

(ii) For an affiliated subspace $\mathcal{V} \in \mathrm{Aff}_s(\mathfrak{A})$, we have $\overline{\mathcal{V}}, \mathcal{V} + \mathcal{V}^{\perp}$ are also affiliated subspaces in $\mathrm{Aff}_s(\mathfrak{A})$.

Proof. (i) Let $A, B \in \mathfrak{A}$ be operators such that $\mathcal{V} = \mathrm{ran}(A), \mathcal{W} = \mathrm{ran}(B)$. From Remark 2.1, we have

$$\mathcal{V} + \mathcal{W} = \mathrm{ran}(A) + \mathrm{ran}(B) = \mathrm{ran}(A \boxplus B) \in \mathrm{Aff}_s(\mathfrak{A}),$$
$$\mathcal{V} \cap \mathcal{W} = \mathrm{ran}(A) \cap \mathrm{ran}(B) = \mathrm{ran}(A \boxdot B) \in \mathrm{Aff}_s(\mathfrak{A}).$$

(ii) Since $\mathbf{R}(A) \in \mathfrak{A}$ and $\overline{\mathcal{V}} = \mathrm{ran}\big(\mathbf{R}(A)\big)$, we have $\overline{\mathcal{V}} = \mathrm{Aff}_s(\mathfrak{A})$. Let E be the projection onto $\overline{\mathcal{V}}$. Then $\mathrm{ran}(I - E) = \mathcal{V}^{\perp}$ and $\mathcal{V} + \mathcal{V}^{\perp} = \mathrm{ran}(A) + \mathrm{ran}(I - E) = \mathrm{ran}\big(A \boxplus (I - E)\big)$ which lies in $\mathrm{Aff}_s(\mathfrak{A})$ by part (i). ■

Lemma 2.6. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be represented unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. For $A, B \in \mathfrak{A}$, we have*

$$\Phi(A \boxplus B) = \Phi(A) \boxplus \Phi(B),$$
$$\Phi(A \boxdot B) = \Phi(A) \boxdot \Phi(B),$$
$$\Phi(A \overset{\mathsf{inv}}{\circ} B) = \Phi(A) \overset{\mathsf{inv}}{\circ} \Phi(B).$$

Proof. Consider the unital σ-normal $*$-homomorphism $\Phi_{(2)} : \mathbb{M}_2(\mathfrak{A}) \to \mathbb{M}_2(\mathfrak{B})$. Recall the notation $\mathfrak{T}_{A',B'}$ from §1. It is straightforward to see that $\Phi_{(2)}(\mathfrak{T}_{A,B}) = \mathfrak{T}_{\Phi(A),\Phi(B)}$. Using Lemma 1.10, we get $\Phi_{(2)}\big(\mathbf{N}(\mathfrak{T}_{A,B})\big) = \mathbf{N}\big(\mathfrak{T}_{\Phi(A),\Phi(B)}\big)$. The desired result follows from Theorem 2.4 and the three equations below,

$$\Phi_{(2)}(\mathfrak{T}_{A,B}) = \mathfrak{T}_{\Phi(A),\Phi(B)};$$
$$\Phi_{(2)}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}(\mathfrak{T}_{A,B})\right) = \begin{bmatrix} \Phi(A) & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}\Big(\mathfrak{T}_{\Phi(A),\Phi(B)}\Big);$$
$$\Phi_{(2)}\left(\begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}(\mathfrak{T}_{A,B})\right) = \begin{bmatrix} I_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{N}\Big(\mathfrak{T}_{\Phi(A),\Phi(B)}\Big).$$ ■

Theorem 2.7. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be represented unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism.*

(i) The mapping $\Phi^s_{\mathsf{aff}} : \mathrm{Aff}_s(\mathfrak{A}) \to \mathrm{Aff}_s(\mathfrak{B})$ given by $\mathrm{ran}(A) \mapsto \mathrm{ran}\big(\Phi(A)\big)$ for $A \in \mathfrak{A}$, is a well-defined lattice homomorphism;

(ii) $\Phi^s_{\mathsf{aff}}(\{0\}_{\mathcal{H}}) = \{0\}_{\mathcal{K}}$, and $\Phi^s_{\mathsf{aff}}(\mathcal{H}) = \mathcal{K}$;

(iii) for all $\mathcal{V} \in \mathrm{Aff}_s(\mathfrak{A})$, $\Phi^s_{\mathsf{aff}}\left(\overline{\mathcal{V}}\right) = \overline{\Phi^s_{\mathsf{aff}}(\mathcal{V})}$;

(iv) if $\mathcal{V}$ dense in $\mathcal{H}$, then $\Phi^s_{\mathsf{aff}}(\mathcal{V})$ is dense in $\mathcal{K}$;

(v) if $\mathcal{V}, \mathcal{W} \in \mathrm{Aff}_s(\mathfrak{A})$ are such that $\mathcal{V} \subseteq \mathcal{W}$, then $\Phi^s_{\mathsf{aff}}(\mathcal{V}) \subseteq \Phi^s_{\mathsf{aff}}(\mathcal{W})$.

Proof. (i) Let $A, B \in \mathfrak{A}$ such that $\mathrm{ran}(A) = \mathrm{ran}(B) \subseteq \mathcal{H}$. By Theorem 2.3-(i), we have $\mathrm{ran}(\Phi(A)) = \mathrm{ran}(\Phi(B)) \subseteq \mathcal{K}$. Thus Φ^s_{aff} is well-defined.

Let $\mathcal{V}, \mathcal{W} \in \mathrm{Aff}_s(\mathfrak{A})$ so that there are operators $A, B \in \mathfrak{A}$ with $\mathcal{V} = \mathrm{ran}(A), \mathcal{W} = \mathrm{ran}(B)$. Using Lemma 2.5 and Remark 2.1, we note that,

$$\mathcal{V} + \mathcal{W} = \mathrm{ran}(A) + \mathrm{ran}(B) = \mathrm{ran}(A \boxplus B).$$

Thus

$$\begin{aligned}\Phi^s_{\text{aff}}(\mathcal{V}+\mathcal{W}) &= \Phi^s_{\text{aff}}\left(\operatorname{ran}(A \boxplus B)\right) = \operatorname{ran}\left(\Phi(A \boxplus B)\right)\\ &= \operatorname{ran}\left(\Phi(A) \boxplus \Phi(B)\right), \text{ by Lemma 2.6}\\ &= \operatorname{ran}(\Phi(A)) + \operatorname{ran}(\Phi(B)), \text{ by Remark 2.1}\\ &= \Phi^s_{\text{aff}}(\mathcal{V}) + \Phi^s_{\text{aff}}(\mathcal{W}).\end{aligned}$$

On the other hand using Lemma 2.5 and Theorem 2.4-(ii), we get,

$$\mathcal{V} \cap \mathcal{W} = \operatorname{ran}(A) \cap \operatorname{ran}(B) = \operatorname{ran}(A \boxdot B).$$

Hence,

$$\begin{aligned}\Phi^s_{\text{aff}}(\mathcal{V}\cap\mathcal{W}) &= \Phi^s_{\text{aff}}\left(\operatorname{ran}(A \boxdot B)\right) = \operatorname{ran}\left(\Phi(A \boxdot B)\right)\\ &= \operatorname{ran}\left(\Phi(A) \boxdot \Phi(B)\right), \text{ by Lemma 2.6}\\ &= \operatorname{ran}\left(\Phi(A)\right) \cap \operatorname{ran}\left(\Phi(B)\right), \text{ by Remark 2.1}\\ &= \Phi^s_{\text{aff}}(\mathcal{V}) \cap \Phi^s_{\text{aff}}(\mathcal{W}).\end{aligned}$$

(ii) Follows immediately from $\Phi(0_{\mathcal{H}}) = 0_{\mathcal{K}}$ and $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, respectively.

(iii) Let $A \in \mathfrak{A}$ be such that $\mathcal{V} = \operatorname{ran}(A)$. Then we have,

$$\begin{aligned}\Phi^s_{\text{aff}}(\overline{\mathcal{V}}) &= \Phi^s_{\text{aff}}\left(\overline{\operatorname{ran}(A)}\right) = \Phi^s_{\text{aff}}\big(\operatorname{ran}\mathbf{R}(A)\big)\\ &= \operatorname{ran}\big(\Phi\left(\mathbf{R}(A)\right)\big) = \operatorname{ran}\big(\mathbf{R}\left(\Phi(A)\right)\big)\\ &= \overline{\operatorname{ran}\Phi(A)} = \overline{\Phi^s_{\text{aff}}(\mathcal{V})}.\end{aligned}$$

(iv) If $\mathcal{V}$ is a dense affiliated subspace of $\mathcal{H}$, then $\overline{\Phi^s_{\text{aff}}(\mathcal{V})} = \Phi^s_{\text{aff}}(\overline{\mathcal{V}}) = \Phi^s_{\text{aff}}(\mathcal{H}) = \mathcal{K}$.

(v) Let $\mathcal{V} = \operatorname{ran}(A)$ and $\mathcal{W} = \operatorname{ran}(B)$ for $A, B \in \mathfrak{A}$. Then by Lemma 2.3-(i) we get, $\Phi^s_{\text{aff}}(\mathcal{V}) = \operatorname{ran}(\Phi(A)) \subseteq \operatorname{ran}(\Phi(B)) = \Phi^s_{\text{aff}}(\mathcal{W})$. ■

Remark 2.2. Thus $\mathfrak{A} \mapsto \mathrm{Aff}_s(\mathfrak{A})$, $\Phi \mapsto \Phi^s_{\text{aff}}$ gives a functor from the category of (represented) unital monotone σ-complete C^*-algebras to the category of lattices. In particular, we observe that the lattice of operator ranges is intrinsically associated with the algebra independent of its representation on Hilbert space.

Chapter 3

Affiliated Operators

Let $\mathcal{H}$ be a Hilbert space and $A, B \in \mathcal{B}(\mathcal{H})$. There is a unique (unbounded) linear operator $T : \operatorname{ran}(B) \to \mathcal{H}$ such that $TB = A$, if and only if $\operatorname{null}(B) \subseteq \operatorname{null}(A)$. Clearly, for every $x \in \mathcal{H}$, the mapping T must send Bx to Ax, and such a mapping makes sense if and only if $\operatorname{null}(B) \subseteq \operatorname{null}(A)$. If $\operatorname{null}(B) \subseteq \operatorname{null}(A)$, we use the notation A/B to denote the operator T and call it the *quotient* of A and B, or a *quotient representation* of the operator T; note that $A/B = AB^{\dagger}$.

We may draw the analogy that for integers a, b, there is a unique rational number a/b such that $(a/b)b = a$ if and only if $b \neq 0$. The nullspace condition assists in the 'cancellation' of the singular parts of A and B, thereby making the expression A/B sensible. If the nullspace condition is not met, the expression A/B should be regarded as invalid syntax, much like how $a/0$ is deemed invalid syntax. For instance, the notation I/B makes sense only if $\mathbf{N}(B) = 0$. Throughout this section, $\mathscr{M}$ represents a von Neumann algebra, while $\mathfrak{A}$ denotes a unital monotone σ-complete C^*-algebra acting on the Hilbert space $\mathcal{H}$.

Definition 3.1. In the above notation, we define the set of *affiliated operators* for the unital monotone σ-complete C^*-algebra $\mathfrak{A}$ to be the set of quotients of operators in $\mathfrak{A}$, that is,

$$\mathfrak{A}_{\text{aff}} := \{A/B : A, B \in \mathfrak{A} \text{ with } \operatorname{null}(B) \subseteq \operatorname{null}(A)\}.$$

Our main goal here is to establish the basic algebraic structure of the set of affiliated operators and show that it transforms functorially under unital σ-normal $*$-homomorphisms between represented monotone σ-complete C^*-algebras. To this end, it becomes essential to gain a thorough understanding of the quotient representation of affiliated operators.

3.1 Algebraic Structure of $\mathfrak{A}_{\text{aff}}$

In this section, we equip the set of affiliated operators with the algebraic structure given by two binary operations, the *sum* and the *product*, and two unary operations, the *Kaufman-inverse* and the *adjoint*. Thus from Remark 1.1, we note that $(\mathfrak{A}_{\text{aff}}; +, \cdot)$ is a near-semiring (see Definition 1.16).

It is important to note that the quotient representation of two operators $A, B \in \mathcal{B}(\mathcal{H})$ is not necessarily unique. In fact, for any right-invertible operator $C \in \mathcal{B}(\mathcal{H})$, we have $A/B = AC/BC$ (see §3.3). Below we generalize [Izu89, Lemma 2.2] to the framework of monotone σ-complete C^*-algebras. This result is fundamental in demonstrating that the set of affiliated operators constitutes a functorial construction (see Definition 3.2). From this, a weaker form of the uniqueness of quotient representation can be derived for monotone σ-complete C^*-algebras. In the next section, we establish a stronger result for von Neumann algebras (see Theorem 3.11).

Lemma 3.1 (cf. [Izu89, Lemma 2.2]). *Let $\mathfrak{A}$ be a monotone σ-complete C^*-algebra acting on the Hilbert space $\mathcal{H}$. For $i = 1, 2$, let A_i, B_i be bounded operators in $\mathfrak{A}$ with $\mathrm{null}(B_i) \subseteq \mathrm{null}(A_i)$. Then A_2/B_2 is an extension of A_1/B_1 if and only if there is an operator $C \in \mathfrak{A}$ such that, $A_1 = A_2C,\ B_1 = B_2C$.*

Proof. For bounded (everywhere-defined) operators A', B' acting on a Hilbert space $\mathcal{H}'$, recall the notation $\mathfrak{S}_{A',B'}$ from §1. If $\mathrm{null}(B') \subseteq \mathrm{null}(A')$, it is straightforward to see that,

$$\mathrm{Gr}\left(A'/B'\right) = \left\{(B'x, A'x) : x \in \mathcal{H}'\right\} = \mathrm{ran}(\mathfrak{S}_{A',B'}). \tag{3.1.1}$$

Thus we have the following equivalences,

$$\begin{aligned} A_1/B_1 \subseteq A_2/B_2 &\iff \mathrm{Gr}\left(A_1/B_1\right) \subseteq \mathrm{Gr}\left(A_2/B_2\right) \\ &\iff \mathrm{ran}(\mathfrak{S}_{A_1,B_1}) \subseteq \mathrm{ran}(\mathfrak{S}_{A_2,B_2}). \end{aligned} \tag{3.1.2}$$

($\implies$) Assume that $A_1/B_1 \subseteq A_2/B_2$. From Theorem 2.1 in the context of the monotone σ-complete C^*-algebra $\mathbb{M}_2\left(\mathfrak{A}\right)$ (see Proposition 1.6) and equation (3.1.2), there is a matrix $\mathfrak{X} \in \mathbb{M}_2\left(\mathfrak{A}\right)$ such that

$$\mathfrak{S}_{A_1,B_1} = \mathfrak{S}_{A_2,B_2}\mathfrak{X}.$$

Choosing $C = \mathfrak{X}_{11}$ which lies in $\mathfrak{A}$, the forward implication in the assertion follows.

($\impliedby$) Conversely, if $A_1 = A_2C, B_1 = B_2C$, for some $C \in \mathfrak{A}$, then

$$\mathfrak{S}_{A_1,B_1} = \mathfrak{S}_{A_2,B_2}(C \oplus 0_{\mathcal{H}}).$$

Again by using Theorem 2.1 in the context of $\mathbb{M}_2\left(\mathfrak{A}\right)$ and equation (3.1.2), we conclude that $A_1/B_1 \subseteq A_2/B_2$. ■

Towards the algebraic structure of $\mathfrak{A}_{\mathrm{aff}}$ the first step is to note that $\mathfrak{A}_{\mathrm{aff}}$ is closed under sums and products of operators as defined in Definition 1.4. This generalizes [Izu89, Theorem 3.1, 3.2] to the setting of monotone σ-complete C^*-algebras.

Theorem 3.2. *Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra acting on some Hilbert space $\mathcal{H}$ and $T_1, T_2 \in \mathfrak{A}_{\mathrm{aff}}$. Let $A_1, B_1, A_2, B_2 \in \mathfrak{A}$ be such that $T_1 = A_1/B_1, T_2 = A_2/B_2$. Then*

(i) $T_1 + T_2$ is in $\mathfrak{A}_{\mathrm{aff}}$ with quotient representation A/B, where

$$A = A_1X_1 + A_2X_2; \quad B = B_1 \boxdot B_2,$$

and X_1, X_2 are elements in $\mathfrak{A}$ satisfying $B_1X_1 = B_1 \boxdot B_2$ and $B_2X_2 = B_1 \boxdot B_2$ respectively.

(ii) T_1T_2 is in $\mathfrak{A}_{\mathrm{aff}}$ with quotient representation A/B, where

$$A = A_1X; \quad B = B_2(A_2 \overset{\mathsf{inv}}{\circ} B_1),$$

and X is an element in $\mathfrak{A}$ satisfying $B_1X = (A_2(A_2 \overset{\mathsf{inv}}{\circ} B_1))$.

Proof. (i) Clearly the operators A, B lie in $\mathfrak{A}$. Furthermore from the following calculation it is clear that $\mathrm{null}(B) \subseteq \mathrm{null}(A)$:

$$\begin{aligned} \mathrm{null}(B) &\subseteq \mathrm{null}(B_1X_1) \cap \mathrm{null}(B_2X_2); \text{ since } B_1X_1 = B = B_2X_2 \\ &\subseteq \mathrm{null}(A_1X_1) \cap \mathrm{null}(A_2X_2); \text{ since } \mathrm{null}(B_i) \subseteq \mathrm{null}(A_i) \\ &\subseteq \mathrm{null}(A). \end{aligned}$$

Now note that,

$$\operatorname{dom}(T_1+T_2)=\operatorname{dom}(T_1)\cap\operatorname{dom}(T_2)=\operatorname{ran}(B_1)\cap\operatorname{ran}(B_2)=\operatorname{ran}(B_1\boxdot B_2)=\operatorname{ran}(B).$$

Since $T_iB_i=A_i\,(i=1,2)$ we have

$$\begin{aligned}(T_1+T_2)B&=(T_1+T_2)(B_1\boxdot B_2)=T_1(B_1\boxdot B_2)+T_2(B_1\boxdot B_2)\\&=T_1B_1X_1+T_2B_2X_2=A_1X_1+A_2X_2\\&=A.\end{aligned}$$

Thus $T_1+T_2=A/B$.

(ii) From Remark 2.1, note that $A_2\overset{\text{inv}}{\circ}B_1\in\mathscr{M}$, and,

$$A_2^{-1}\big(\operatorname{ran}(B_1)\big)=\operatorname{ran}(A_2\overset{\text{inv}}{\circ}B_1),$$

which implies that

$$\operatorname{ran}\Big(A_2(A_2\overset{\text{inv}}{\circ}B_1)\Big)=A_2\operatorname{ran}(A_2\overset{\text{inv}}{\circ}B_1)\subseteq\operatorname{ran}(B_1).$$

Therefore $A,B\in\mathfrak{A}$ and from the following calculation we can see that $\operatorname{null}(B)\subseteq\operatorname{null}(A)$:

$$\operatorname{null}(B)=\operatorname{null}\Big(B_2(A_2\overset{\text{inv}}{\circ}B_1))\Big)\subseteq\operatorname{null}\Big(A_2(A_2\overset{\text{inv}}{\circ}B_1\Big)=\operatorname{null}(B_1X)\subseteq\operatorname{null}(A_1X)=\operatorname{null}(A)$$

Furthermore,

$$\begin{aligned}\operatorname{dom}(T_1T_2)&=\{x\in\operatorname{dom}(T_2):T_2x\in\operatorname{dom}(T_1)\}=\{B_2x:A_2x\in\operatorname{ran}(B_1),x\in\mathcal{H}\}\\&=B_2\left(A_2^{-1}\big(\operatorname{ran}(B_1)\big)\right)=B_2\Big(\operatorname{ran}(A_2\overset{\text{inv}}{\circ}B_1)\Big)=\operatorname{ran}\Big(B_2(A_2\overset{\text{inv}}{\circ}B_1)\Big)\\&=\operatorname{ran}(B).\end{aligned}$$

A straightforward algebraic computation shows that,

$$(T_1T_2)B=T_1T_2B_2(A_2\overset{\text{inv}}{\circ}B_1)=T_1A_2(A_2\overset{\text{inv}}{\circ}B_1)=T_1B_1X=A_1X=A,$$

whence $T_1T_2=A/B$. ■

Lemma 3.3. *Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra. For $A,B\in\mathfrak{A}$, $AB^\dagger$ is in $\mathfrak{A}_{\text{aff}}$.*

Proof. Note that $\operatorname{null}(B)=\operatorname{null}\big(I-\mathbf{N}(B)\big)\subseteq\operatorname{null}\big(A\big(I-\mathbf{N}(B)\big)\big)$ and $\operatorname{ran}(B^\dagger)\subseteq\operatorname{null}(B)^\perp$. Thus,

$$AB^\dagger=A\big(I-\mathbf{N}(B)\big)B^\dagger=\big(A(I-\mathbf{N}(B))\big)/B\in\mathfrak{A}_{\text{aff}}.$$

■

Below we show that $\mathfrak{A}_{\text{aff}}$ is closed under the unary operation, Kaufman-inverse, and give a quotient representation of $T^\dagger$ for $T\in\mathfrak{A}_{\text{aff}}$.

Theorem 3.4. *Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra acting on the Hilbert space $\mathcal{H}$, and let $T\in\mathfrak{A}_{\text{aff}}$. Let A/B be a quotient representation of T with $A,B\in\mathfrak{A}$ satisfying* $\operatorname{null}(A)\subseteq\operatorname{null}(B)$*. Then* *(i)* $\mathbf{R}(T), \mathbf{N}(T)$ *are both contained in* $\mathfrak{A}$*. In fact,* $\mathbf{N}(T) = \mathbf{R}\big(B\mathbf{N}(A)\big)$.

(ii) $T^\dagger$ *is contained in* $\mathfrak{A}_{\text{aff}}$ *with quotient representation given by*

$$T^\dagger = \Big(\big(I - \mathbf{N}(T)\big)\, B\Big)/A.$$

(ii) T *is one-to-one if and only if* $\text{null}(A) = \text{null}(B)$*, in case of which* T^{-1} *lies in* $\mathfrak{A}_{\text{aff}}$ *with quotient representation* B/A.

Proof. (i) Clearly $\mathbf{R}(T) = \mathbf{R}(A)$, which lies in $\mathfrak{A}$. Since T is given by the mapping $Bx \mapsto Ax$ $(x \in \mathcal{H})$, note that

$$\text{null}(T) = \{Bx : Ax = 0, x \in \mathcal{H}\} = B\Big(\text{null}(A)\Big). \tag{3.1.3}$$

Thus $\mathbf{N}(T) = \mathbf{R}\big(B\mathbf{N}(A)\big)$, which lies in $\mathscr{M}$.

(ii) From equation (3.1.3), note that the mapping $Ax \mapsto \big(I - \mathbf{N}(T)\big)Bx$ $(x \in \mathcal{H})$ is well-defined and in fact, gives the Kaufman inverse of T (see Definition 1.6). Thus,

$$T^\dagger = \Big(\big(I - \mathbf{N}(T)\big)\, B\Big)/A.$$

From this expression it is also clear that if $T \in \mathfrak{A}_{\text{aff}}$, then we also have $T^\dagger \in \mathfrak{A}_{\text{aff}}$, that is, $\mathfrak{A}_{\text{aff}}$ is closed under Kaufman-inverse.

(iii) Keeping in mind the nullspace condition $\text{null}(B) \subseteq \text{null}(A)$ and equation (3.1.3), we have the following equivalences,

$$\begin{aligned} T \text{ is one-to-one } &\iff \text{null}(T) = \{0\}_{\mathcal{H}} \iff B\big(\text{null}(A)\big) = \{0\}_{\mathcal{H}} \\ &\iff \text{null}(A) \subseteq \text{null}(B) \iff \text{null}(A) = \text{null}(B). \end{aligned}$$

When T is one-to-one, it is clear that T^{-1} is given by the mapping $Ax \mapsto Bx$ $(x \in \mathcal{H})$ so that $T^{-1} = B/A$. ∎

The adjoint of an unbounded operator (not necessarily densely-defined) is defined in Definition 1.7. Below we show that $\mathfrak{A}_{\text{aff}}$ is closed under adjoint.

Theorem 3.5. *Let* $\mathfrak{A}$ *is a unital monotone* σ*-complete* C^**-algebra and* $A, B \in \mathfrak{A}$ *with* $\text{null}(B) \subseteq \text{null}(A)$*. Then*

$$(A/B)^* = (B^*)^\dagger A^*.$$

Thus for $T \in \mathfrak{A}_{\text{aff}}$*, its adjoint,* T^**, also lies in* $\mathfrak{A}_{\text{aff}}$*. In other words,* $\mathfrak{A}_{\text{aff}}$ *is* $*$*-closed.*

Proof. We prove the assertion in two steps below.

Claim 1. $\text{dom}\left((A/B)^*\right) = (A^*)^{-1}\Big(\text{ran}(B^*)\Big) = \text{dom}\left((B^*)^\dagger A^*\right).$

Proof of Claim 1: Note that $\text{dom}\left((B^*)^\dagger A^*\right) = (A^*)^{-1}\Big(\text{ran}(B^*)\Big)$, and from Definition 1.7,

$$\begin{aligned} \text{dom}((A/B)^*) &= \big\{y \in \mathcal{H} : x \mapsto \Big\langle AB^\dagger x, y\Big\rangle \in \mathbb{C} \text{ is continuous on } \text{dom}\big(AB^\dagger\big)\big\} \\ &= \big\{y \in \mathcal{H} : Bx \mapsto \langle Ax, y\rangle \in \mathbb{C} \text{ is continuous on } \text{ran}(B)\big\}. \end{aligned} \tag{3.1.4}$$

Let $y \in \operatorname{dom}\big((B^*)^\dagger A^*\big)$. Then there exists $z \in \mathcal{H}$ such that $A^*y = B^*z$. For all $x \in \mathcal{H}$, we have,

$$\langle Ax, y\rangle = \langle x, A^*y\rangle = \langle x, B^*z\rangle = \langle Bx, z\rangle .$$

Since the mapping, $Bx \mapsto \langle Bx, z\rangle \, (= \langle Ax, y\rangle)$, is continuous on $\operatorname{ran}(B)$, by (3.1.4) we have $y \in \operatorname{dom}((AB^\dagger)^*)$. Thus,

$$\operatorname{dom}\Big((B^*)^\dagger A^*\Big) \subseteq \operatorname{dom}\Big((AB^\dagger)^*\Big). \tag{3.1.5}$$

For the converse, suppose $y \in \operatorname{dom}((AB^\dagger)^*)$ so that the mapping, $Bx \mapsto \langle Ax, y\rangle$, is continuous. By the Riesz representation theorem, there exists a $z \in \overline{\operatorname{ran}(B)}$ such that $\langle Ax, y\rangle = \langle Bx, z\rangle$ for all $x \in \mathcal{H}$. Thus $A^*y = B^*z$ and $y \in (A^*)^{-1}\Big(\operatorname{ran}(B^*)\Big)$. Thus,

$$\operatorname{dom}\Big((AB^\dagger)^*\Big) \subseteq \operatorname{dom}\Big((B^*)^\dagger A^*\Big). \tag{3.1.6}$$

From (3.1.5) and (3.1.6), we conclude that $\operatorname{dom}\big((AB^\dagger)^*\big) = \operatorname{dom}\big((B^*)^\dagger A^*\big)$.

Claim 2. $(A/B)^*y = (B^*)^\dagger A^*y$ for every vector y in $(A^*)^{-1}\Big(\operatorname{ran}(B^*)\Big)$.

Proof of Claim 2. Let $z \in \mathcal{H}$ be such that $A^*y = B^*z$. Then,

$$(B^*)^\dagger A^*y = (B^*)^\dagger B^*z = \big(I - \mathbf{N}(B^*)\big)z = \mathbf{R}(B)z \in \overline{\operatorname{ran}(B)}.$$

From the definition of T^*, it is clear that $\operatorname{ran}(T^*) \subseteq \overline{\operatorname{dom}(T)}$ whence

$$(AB^\dagger)^*y \in \operatorname{ran}\Big((AB^\dagger)^*\Big) \subseteq \overline{\operatorname{ran}(B)}.$$

Thus the vector $v := (AB^\dagger)^*y - (B^*)^\dagger A^*y$ is in $\overline{\operatorname{ran}(B)}$. For every vector $x \in \mathcal{H}$, we have

$$\begin{aligned}\Big\langle (AB^\dagger)^*y, Bx\Big\rangle &= \Big\langle y, AB^\dagger Bx\Big\rangle = \langle y, Ax\rangle = \langle A^*y, x\rangle = \langle B^*z, x\rangle \\ &= \langle z, Bx\rangle = \langle z, \mathbf{R}(B)Bx\rangle = \langle \mathbf{R}(B)z, Bx\rangle \\ &= \Big\langle (B^*)^\dagger A^*y, Bx\Big\rangle,\end{aligned}$$

which implies that v is orthogonal to $\overline{\operatorname{ran}(B)}$. Thus $v = 0$, that is, $(AB^\dagger)^*y = (B^*)^\dagger A^*y$.

Since $(B^*)^\dagger, A^* \in \mathfrak{A}_{\text{aff}}$, by Theorem 3.2-(ii), we conclude that $(A/B)^* = (B^*)^\dagger A^*$ is in $\mathfrak{A}_{\text{aff}}$. ■

3.2 Functoriality of the Construction $\mathfrak{A} \mapsto \mathfrak{A}_{\text{aff}}$

For a unital σ-normal $*$-homomorphism Φ between represented unital monotone complete C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, we define a mapping Φ_{aff} between their respective near-semirings of affiliated operators, $\mathfrak{A}_{\text{aff}}$ and $\mathfrak{B}_{\text{aff}}$, and show that it respects the four algebraic operations discussed in §3.1; In particular, it is a near-semiring homomorphism. Thus the construction

$$\mathfrak{A} \mapsto \mathfrak{A}_{\text{aff}}, \Phi \mapsto \Phi_{\text{aff}},$$

is functorial. In Theorem 3.7, we argue that Φ_{aff} is the canonical extension of Φ to $\mathfrak{A}_{\text{aff}}$ (in an algebraic sense) by showing that it is the unique extension of Φ which respects product and Kaufman-inverse; in Theorem 3.6, we note that Φ_{aff} respects sum and adjoint.

Proposition 3.1. *Let $(\mathfrak{A};\mathcal{H})$ and $(\mathfrak{B};\mathcal{K})$ be two unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. For $i = 1, 2$, let A_i, B_i be operators in $\mathfrak{A}$ with $\mathrm{null}(B_i) \subseteq \mathrm{null}(A_i)$. Then we have the following:*

(i) $\mathrm{null}\big(\Phi(B_i)\big) \subseteq \mathrm{null}\big(\Phi(A_i)\big)$;

(ii) If $A_1/B_1 \subseteq A_2/B_2$, then $\Phi(A_1)/\Phi(B_1) \subseteq \Phi(A_2)/\Phi(B_2)$.

(iii) If $A_1/B_1 = A_2/B_2$, then $\Phi(A_1)/\Phi(B_1) = \Phi(A_2)/\Phi(B_2)$.

Proof. (i) Follows from Lemma 1.10. Thus the quotients in parts (ii) and (iii) make sense.

(ii) By Lemma 3.1, there is an operator $C \in \mathfrak{A}$ such that $A_1 = A_2C, B_1 = B_2C$. Thus $\Phi(A_1) = \Phi(A_2)\Phi(C), \Phi(B_1) = \Phi(B_2)\Phi(C)$. Again from Lemma 3.1 in the context of $\mathfrak{B}$, the desired conclusion follows.

(iii) Immediately follows from part (ii). ■

Definition 3.2. Let $(\mathfrak{A};\mathcal{H}), (\mathfrak{B};\mathcal{K})$ be two represented unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal homomorphism. Let $\Phi_{\mathrm{aff}} : \mathfrak{A}_{\mathrm{aff}} \to \mathfrak{B}_{\mathrm{aff}}$ be the mapping, which for every pair of operators $A, B \in \mathfrak{A}$ with $\mathrm{null}(B) \subseteq \mathrm{null}(A)$, sends A/B to $\Phi(A)/\Phi(B)$; by Proposition 3.1, this mapping makes sense and is well-defined. In short,

$$\Phi_{\mathrm{aff}}(A/B) := \Phi(A)/\Phi(B).$$

Theorem 3.6. *Let $(\mathfrak{A};\mathcal{H}), (\mathfrak{B};\mathcal{K})$ be unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathfrak{A}_{\mathrm{aff}} \to \mathfrak{B}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Then for $T_1, T_2 \in \mathfrak{A}_{\mathrm{aff}}$, we have,*

(i) $\Phi_{\mathrm{aff}}(T_1 + T_2) = \Phi_{\mathrm{aff}}(T_1) + \Phi_{\mathrm{aff}}(T_2)$;

(ii) $\Phi_{\mathrm{aff}}(T_1T_2) = \Phi_{\mathrm{aff}}(T_1)\ \Phi_{\mathrm{aff}}(T_2)$;

(iii) $\Phi_{\mathrm{aff}}(T_1^\dagger) = \Phi(T_1)^\dagger$*, and in particular, if $T \in \mathfrak{A}_{\mathrm{aff}}$ is non-singular, then $\Phi_{\mathrm{aff}}(T)$ is non-singular with $\Phi_{\mathrm{aff}}(T^{-1}) = \Phi_{\mathrm{aff}}(T)^{-1}$;*

(iv) $\Phi_{\mathrm{aff}}(T_1^*) = \Phi_{\mathrm{aff}}(T_1)^*$.

In other words, Φ_{aff} preserves sum, product, Kaufman-inverse, adjoint.

Proof. Let $T_i = A_i/B_i$, where $A_i, B_i \in \mathfrak{A}$ for $i = 1, 2$. Then we have

$$\Phi_{\mathrm{aff}}(T_i) = \Phi(A_i)/\Phi(B_i),\ (i = 1, 2).$$

(i) From Theorem 3.2-(i), $T_1 + T_2 = A/B$, where

$$A := A_1X_1 + A_2X_2;\ B := B_1 \boxdot B_2;\ \text{and } B_iX_i = B\ (i = 1, 2).$$

Since $X_1, X_2 \in \mathfrak{A}$ and by Lemma 2.6 Φ preserves $\boxdot$, we get,

$$\begin{aligned}
\Phi(A) &= \Phi(A_1)\Phi(X_1) + \Phi(A_2)\Phi(X_2),\\
\Phi(B) &= \Phi(B_1) \boxdot \Phi(B_2) \text{ and}\\
\Phi(B) &= \Phi(B_i)\Phi(X_i),\ (i = 1, 2).
\end{aligned}$$

Now applying Theorem 3.2-(i) on the two quotients $\Phi(A_i)/\Phi(B_i)$, for $i = 1, 2$ we get,

$$\Phi_{\mathrm{aff}}(T_1) + \Phi_{\mathrm{aff}}(T_2) = \Big(\Phi(A_1)/\Phi(B_1)\Big) + \Big(\Phi(A_2)/\Phi(B_2)\Big) = \Phi(A)/\Phi(B).$$

Therefore we have,

$$\Phi_{\mathrm{aff}}(T_1 + T_2) = \Phi_{\mathrm{aff}}(A/B) = \Phi(A)/\Phi(B) = \Phi_{\mathrm{aff}}(T_1) + \Phi_{\mathrm{aff}}(T_2).$$

(ii) The proof is almost identical to that of part (i) using Theorem 3.2-(ii) in lieu of Theorem 3.2-(i).

(iii) From Theorem 3.4-(ii), we have the following quotient representation of $T_1^\dagger$,

$$T_1^\dagger = \Big((I - \mathbf{N}(T_1))B_1\Big)/A_1.$$

Since by Theorem 3.4-(i), we have $\mathbf{N}(T_1) = \mathbf{R}\big(B_1\mathbf{N}(A_1)\big)$, Lemma 1.10 tells us that $\Phi\big(\mathbf{N}(T_1)\big) = \mathbf{N}\big(\Phi_{\mathrm{aff}}(T_1)\big)$; thus,

$$\Phi_{\mathrm{aff}}(T_1^\dagger) = \Big(\,\big(I - \mathbf{N}\big(\Phi_{\mathrm{aff}}(T_1)\big)\big)\,\Phi(B_1)\Big)/\Phi(A_1) = \Phi_{\mathrm{aff}}(T_1)^\dagger.$$

For a non-singular (that is, one-to-one) affiliated operator T', clearly $T'^{-1} = T'^\dagger$. From Theorem 3.4-(iii) and Lemma 1.10, we have,

$$\begin{aligned} T_1 \text{ is one-to-one } &\implies \mathbf{N}(A_1) = \mathbf{N}(B_1) \implies \mathbf{N}\big(\Phi(A_1)\big) = \mathbf{N}\big(\Phi(B_1)\big) \\ &\implies \Phi_{\mathrm{aff}}(T_1) = \Phi(A_1)/\Phi(B_1) \text{ is one-to-one.} \end{aligned}$$

Thus, if T_1 is one-to-one, then $\Phi_{\mathrm{aff}}(T_1)$ is one-to-one and we have $\Phi_{\mathrm{aff}}(T_1^{-1}) = \Phi_{\mathrm{aff}}(T_1)^{-1}$.

(iv) By Theorem 3.5, $T_1^* = (B_1^*)^\dagger A_1^*$. Hence using part (ii) and (iii), we get,

$$\begin{aligned} \Phi_{\mathrm{aff}}(T_1^*) = \Phi_{\mathrm{aff}}\left((B_1^*)^\dagger A_1^*\right) &\overset{\text{(ii)}}{=} \Phi_{\mathrm{aff}}\left((B_1^*)^\dagger\right)\Phi_{\mathrm{aff}}(A_1^*) \\ &\overset{\text{(iii)}}{=} \left(\Phi(B_1)^*\right)^\dagger \Phi(A_1)^* = \Phi_{\mathrm{aff}}(T_1)^*. \end{aligned}$$ ■

In the theorem below, we note that Φ_{aff} is the canonical extension of Φ in an algebraic sense.

Theorem 3.7. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be represented monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathfrak{A}_{\mathrm{aff}} \to \mathfrak{B}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Then Φ_{aff} is the unique mapping from $\mathfrak{A}_{\mathrm{aff}}$ to $\mathfrak{B}_{\mathrm{aff}}$ which extends Φ, and respects product and Kaufman-inverse.*

Proof. Let Ψ be another such extension of Φ. Then for $T = AB^\dagger$ in $\mathfrak{A}_{\mathrm{aff}}$,

$$\Psi(T) = \Psi(A)\Psi(B^\dagger) = \Psi(A)\Psi(B)^\dagger = \Phi(A)\Phi(B)^\dagger = \Phi_{\mathrm{aff}}(AB^\dagger) = \Phi_{\mathrm{aff}}(T).$$ ■

We rephrase Proposition 3.1-(ii) in terms of Φ_{aff} and note that Φ_{aff} is well-behaved with respect to extensions of operators. In §4.3 and §5, we will see other algebraic properties that are preserved by Φ_{aff}.

Theorem 3.8. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathfrak{A}_{\mathrm{aff}} \to \mathfrak{B}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Let T, T_0 be operators in $\mathfrak{A}_{\mathrm{aff}}$ with T_0 being an extension of T. Then $\Phi_{\mathrm{aff}}(T_0)$ is an extension of $\Phi_{\mathrm{aff}}(T)$.*

Proof. Immediate from Proposition 3.1-(ii). ■

In our pursuit of functoriality, we have taken primarily an algebraic viewpoint of the set of affiliated operators. An alternative geometric viewpoint becomes available by looking at the graphs of affiliated operators. Let $T \in \mathfrak{A}_{\mathrm{aff}}$. From equation (3.1.1), clearly $\mathrm{Gr}\,(T)$ is in $\mathrm{Aff}_s\,(\mathbb{M}_2\,(\mathfrak{A}))$. Theorem 3.9 below shows that it is possible to provide a geometric definition of Φ_{aff} by considering the mapping induced by Φ (more precisely, $\Phi_{(2)}$) on the affiliated subspaces for $\mathbb{M}_2\,(\mathfrak{A})$.

Theorem 3.9. *Let $(\mathfrak{A};\mathcal{H}), (\mathfrak{B};\mathcal{K})$ be represented von Neumann algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathfrak{A}_{\mathrm{aff}} \to \mathfrak{B}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathfrak{A}_{\mathrm{aff}} \arrow[r, "\mathrm{Gr}"] \arrow[d, "\Phi_{\mathrm{aff}}"'] & \mathrm{Aff}_s\Big(\mathbb{M}_2\,(\mathfrak{A})\Big) \arrow[d, "\left(\Phi_{(2)}\right)^s_{\mathrm{aff}}"] \\
\mathfrak{B}_{\mathrm{aff}} \arrow[r, "\mathrm{Gr}"] & \mathrm{Aff}_s\Big(\mathbb{M}_2\,(\mathfrak{B})\Big)
\end{tikzcd}$$

In other words, $\Phi_{\mathrm{aff}}(T)$ is the unique operator in $\mathfrak{B}_{\mathrm{aff}}$ which satisfies,

$$Gr\,(\Phi_{\mathrm{aff}}(T)) = (\Phi_{(2)})^s_{\mathrm{aff}}\,(Gr\,(T))\,.$$

Proof. Recall the notation $\mathfrak{S}_{A',B'}$ from §1. Let $T \in \mathfrak{A}_{\mathrm{aff}}$ and A/B be a quotient representation of T for $A, B \in \mathfrak{A}$ with $\mathrm{null}(B) \subseteq \mathrm{null}(A)$. As observed before in Proposition 3.1, $\Phi(A)/\Phi(B)$ is a quotient representation of $\Phi_{\mathrm{aff}}(T)$. Since $\mathfrak{S}_{\Phi(A),\Phi(B)} = \Phi_{(2)}(\mathfrak{S}_{A,B})$, using equation (3.1.1) we have,

$$\begin{aligned}
\mathrm{Gr}\,(\Phi_{\mathrm{aff}}(T)) &\overset{(3.1.1)}{=} \mathrm{ran}\,\big(\mathfrak{S}_{\Phi(A),\Phi(B)}\big) = \mathrm{ran}\,\big(\Phi_{(2)}(\mathfrak{S}_{A,B})\big) \\
&= (\Phi_{(2)})^s_{\mathrm{aff}}\big(\mathrm{ran}\,(\mathfrak{S}_{A,B})\,\big) \overset{(3.1.1)}{=} (\Phi_{(2)})^s_{\mathrm{aff}}\big(\mathrm{Gr}\,(T)\,\big).
\end{aligned}$$

■

3.3 Equi-range Operators and Uniqueness of Quotient Representation

This section primarily focuses on represented von Neumann algebras. First, note that any right-invertible operator $C \in \mathcal{B}(\mathcal{H})$ must be surjective. Consequently, for $A, B \in \mathcal{B}(\mathcal{H})$ satisfying $\mathrm{null}(B) \subseteq \mathrm{null}(A)$, it is straightforward to verify that $A/B = AC/BC$. The main goal of this section is to establish a converse result for operator quotients within a represented von Neumann algebra, thereby demonstrating that the quotient representation of an affiliated operator associated with von Neumann algebras is essentially unique (see Theorem 3.11).

Proposition 3.2. *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Let A, B be two operators in $\mathscr{M}$. Then $\mathrm{ran}(A) = \mathrm{ran}(B)$ if and only if there is a right-invertible element $C \in \mathscr{M}$ and projections P, Q in the centre of $\mathscr{M}$ with $P+Q = I$ such that $AP = BCP$ and $BQ = ACQ$.*

Proof. ($\Longrightarrow$). Since $\mathrm{ran}(A) = \mathrm{ran}(B)$, the operators $B^\dagger A, A^\dagger B$ are bounded operators in $\mathscr{M}$. Note that

$$\begin{aligned}
(B^\dagger A)(A^\dagger B) &= B^\dagger B = I - \mathbf{N}(B), \\
(A^\dagger B)(B^\dagger A) &= A^\dagger A = I - \mathbf{N}(A).
\end{aligned}$$

To begin with, let us assume that the projections $\mathbf{N}(A)$ and $\mathbf{N}(B)$ are comparable (say, $\mathbf{N}(B) \precsim \mathbf{N}(A)$); for instance, this will be the case when $\mathscr{M}$ is a factor.

Let V be a partial isometry in $\mathscr{M}$ such that $VV^* = \mathbf{N}(B)$ and $V^*V \le \mathbf{N}(A)$, so that $A(V^*V) = 0$ and $B(VV^*) = 0$. Keeping in mind the basic identities for partial isometries, $VV^*V = V, V^*VV^* = V^*$, we note that

$$\begin{aligned} AV^* &= A(V^*V)V^* = 0, \\ VA^\dagger &= V(V^*V)A^\dagger = 0, \\ BV &= B(VV^*)V = 0. \end{aligned}$$

Consider the operators $C := B^\dagger A + V, D := A^\dagger B + V^*$ in $\mathscr{M}$. Using the above equalities, we have

$$\begin{aligned} CD &= B^\dagger AA^\dagger B + B^\dagger AV^* + VA^\dagger B + VV^* \\ &= (I - \mathbf{N}(B)) + 0 + 0 + \mathbf{N}(B) \\ &= I, \text{ and} \\ BC &= BB^\dagger A + BV = A. \end{aligned}$$

Thus C is a right-invertible element in $\mathscr{M}$ with $A = BC$. In the case where $\mathbf{N}(B) \precsim \mathbf{N}(A)$, by a symmetric argument, we have a right-invertible element $C \in \mathscr{M}$ such that $B = AC$.

Next we proceed to the general case where $\mathbf{N}(A)$ and $\mathbf{N}(B)$ may not be comparable. By the comparison theorem, [KR86, Theorem 6.2.7], there are central projections P, Q with $P + Q = I$ such that $\mathbf{N}(A)P \precsim \mathbf{N}(B)P$ and $\mathbf{N}(B)Q \prec \mathbf{N}(A)Q$. Consider the von Neumann algebra $\mathscr{M}P$ acting on the Hilbert space $P(\mathcal{H})$, and view the operators AP, BP as acting on $P(\mathcal{H})$. In the context of $\mathscr{M}P, AP, BP$, the preceding discussion tells us that there is an operator $C_P \in \mathscr{M}P$ which is right-invertible relative to $\mathscr{M}P$ such that $AP = (BP)C_P$. Similarly there is a right-invertible operator $C_Q \in \mathscr{M}Q$ such that $BQ = (AQ)C_Q$. The operator $C := C_P \oplus C_Q$ is a right-invertible operator in $\mathscr{M}$ satisfying the desired properties.

($\Longleftarrow$). Since $AP = BCP = BPC$, we have $\mathrm{ran}(AP) \subseteq \mathrm{ran}(BP)$. Let D be the right-inverse of C. Note that $(AP)D = ADP = BCDP = BP$ whence $\mathrm{ran}(BP) \subseteq \mathrm{ran}(AP)$. Thus $\mathrm{ran}(AP) = \mathrm{ran}(BP)$ and similarly, $\mathrm{ran}(AQ) = \mathrm{ran}(BQ)$. By Remark 2.1,

$$\mathrm{ran}(AP) + \mathrm{ran}(AQ) = \mathrm{ran}(\sqrt{AA^*P + AA^*Q}) = \mathrm{ran}\left(\sqrt{AA^*}\right) = \mathrm{ran}(A),$$

and similarly, $\mathrm{ran}(BP) + \mathrm{ran}(BQ) = \mathrm{ran}(B)$. We conclude that $\mathrm{ran}(A) = \mathrm{ran}(B)$. ■

Corollary 3.10. *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, and A, B be two operators in $\mathscr{M}$.*

(i) If $\mathscr{M}$ is a factor, then $\mathrm{ran}(A) = \mathrm{ran}(B)$ if and only if there is a right-invertible element $C \in \mathscr{M}$ such that either $A = BC$ or $B = AC$.

(ii) If $\mathscr{M}$ is a finite von Neumann algebra, $\mathrm{ran}(A) = \mathrm{ran}(B)$ if and only if there is an invertible element $C \in \mathscr{M}$ such that $A = BC$.

Proof. Since the only central projections in a factor are 0 and I, part (i) follows from Proposition 3.2. Part (ii) follows from the observation that every right-invertible element in a finite von Neumann algebra is invertible, and the fact that $A = BC$ if and only if $A = BC^{-1}$. ■

In the theorem below, we establish that the quotient representation for affiliated operators is essentially unique.

Theorem 3.11 (Uniqueness of quotient representation)**.** *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. For $i = 1, 2$, let A_i, B_i be bounded operators in $\mathscr{M}$ with* $\mathrm{null}(B_i) \subseteq \mathrm{null}(A_i)$*. Then $A_1/B_1 = A_2/B_2$ if and only if there is a right-invertible element $C \in \mathscr{M}$ and projections P, Q in the centre of $\mathscr{M}$ with $P + Q = I$ such that*

$$A_1P = A_2CP, \ B_1P = B_2CP, \ \text{and}$$
$$A_2Q = A_1CQ, \ B_2Q = B_1CQ.$$

Proof. ($\Longrightarrow$) For $S = A_1/B_1 = A_2/B_2$, we have,

$$SB_ix = A_ix. \ (\forall x \in \mathcal{H} \text{ and } i = 1, 2). \tag{3.3.1}$$

Since $\mathrm{ran}(B_1) = \mathrm{ran}(B_2)(= \mathrm{dom}(S))$, using Proposition 3.2, we observe that there exists a right invertible element $C \in \mathscr{N}$ and two projections P, Q in the centre of $\mathscr{N}$ with $P+Q = I$ such that

$$B_1P = B_2CP \text{ and } B_2Q = B_1CQ.$$

Using equation (3.3.1), for every $x \in \mathcal{H}$, we have $SB_1(Px) = A_1(Px)$ and $SB_2(CPx) = A_2CPx$. Since $B_1P = B_2CP$, we note that $A_1Px = SB_1Px = SB_2CPx = A_2CPx$ for every $x \in \mathcal{H}$, whence $A_1P = A_2CP$. In a similar way from $B_2Q = B_1CQ$ we get $A_2Q = A_1CQ$. This completes the forward implication of the theorem.

($\Longleftarrow$) Assume that there are operators C, P, Q in $\mathscr{M}$ as described in the assertion. First we observe that the existence of such C, P, Q necessarily implies (by Proposition 3.2) that,

$$\mathrm{dom}(A_1/B_1) = \mathrm{ran}(B_1) = \mathrm{ran}(B_2) = \mathrm{dom}(A_2/B_2).$$

Now note that for any $x \in \mathcal{H}$,

$$(A_1/B_1)(B_1Px) = A_1Px = A_2CPx = (A_2/B_2)(B_2CPx) = (A_2/B_2)(B_1Px).$$

Hence $A_1/B_1 = A_2/B_2$ on $\mathrm{ran}(B_1P)$. A similar calculation shows that $(A_2/B_2)(B_2Qx) = (A_1/B_1)(B_2Qx)$ for all $x \in \mathcal{H}$, that is, $A_1/B_1 = A_2/B_2$ on $\mathrm{ran}(B_2Q)$. As C is surjective (being right-invertible), we have $\mathrm{ran}(B_2Q) = \mathrm{ran}(B_1CQ) = \mathrm{ran}(B_1QC) = \mathrm{ran}(B_1Q)$. Note that,

$$\mathrm{ran}(B_1P) + \mathrm{ran}(B_1Q) = \mathrm{ran}\left(\sqrt{B_1PB_1^* + B_1QB_1^*}\right) = \mathrm{ran}\left(\sqrt{B_1B_1^*}\right) = \mathrm{ran}(B_1).$$

Therefore we get,

$$A_1/B_1 = A_2/B_2 \text{ on } \mathrm{ran}(B_1).$$

Hence, $A_1/B_1 = A_2/B_2$. ■

Corollary 3.12. *Let $\mathscr{M}$ be a factor acting on $\mathcal{H}$ and $A_1, B_1, A_2, B_2 \in \mathscr{M}$. If $A_1/B_1 = A_2/B_2$, then there is a right-invertible element $C \in \mathscr{M}$ such that,*

$$\textit{either, } A_1 = A_2C, \ B_1 = B_2C,$$
$$\textit{or, } A_2 = A_1C, \ B_2 = B_1C.$$

Proof. Since a factor only has trivial central projections, the result immediately follows from Theorem 3.11. ■

Chapter 4

Closed Affiliated Operators

In this chapter, we focus on the concept of affiliation for von Neumann algebras. Our primary objective is to demonstrate that our definition of affiliation (Definition 3.1) broadens the traditional scope of the term in the context of von Neumann algebras. Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. We establish that a *closed* operator T on $\mathcal{H}$ is MvN-affiliated (see Definition 1.13) if and only if it satisfies our definition of affiliation. This shows that our approach preserves the classical usage of the term, as introduced by Murray and von Neumann, while offering notable algebraic simplicity. Thus, adopting our definition of $\mathscr{M}_{\text{aff}}$ enhances clarity and structure without compromising established interpretations.

4.1 Quotient Representation of Closed Affiliated Operators

For clarity, we introduce some notation that will be used throughout our discussion. As usual, $\mathscr{M}$ denotes a von Neumann algebra, and $\mathfrak{A}$ represents a unital monotone σ-complete C^*-algebra.

$$\begin{aligned}
\mathcal{C}(\mathcal{H}) &:= \text{the collection of all closed operators acting on the Hilbert space } \mathcal{H} \\
\mathcal{CB}(\mathcal{H}) &:= \{T \in \mathcal{C}(\mathcal{H}) \ : \ T \text{ is bounded on } \mathrm{dom}(T)\} \\
\mathfrak{A}_{\text{aff}} &:= \{A/B : A, B \in \mathfrak{A} \text{ satisfying } \mathrm{null}(B) \subseteq \mathrm{null}(A)\} \\
\mathfrak{A}_{\text{aff}}^{c} &:= \mathfrak{A}_{\text{aff}} \cap \mathcal{C}(\mathcal{H}) = \{T \in \mathfrak{A}_{\text{aff}} : T \text{ is closed}\} \\
\mathfrak{A}_{\text{aff}}^{cb} &:= \mathfrak{A}_{\text{aff}} \cap \mathcal{CB}(\mathcal{H}) = \{T \in \mathfrak{A}_{\text{aff}} : T \text{ is closed and bounded on } \mathrm{dom}(T)\} \\
\mathscr{M}_{\text{aff}}^{\text{MvN}} &:= \{T \in \mathcal{C}(\mathcal{H}) : U^*TU = T \text{ for all unitary } U \in \mathscr{M}'\}.
\end{aligned}$$

It is straightforward to verify that $\mathfrak{A} \subseteq \mathfrak{A}_{\text{aff}}^{cb} \subseteq \mathfrak{A}_{\text{aff}}^{c}$. As the main result of this chapter (see Theorem 4.7), we will establish that for a von Neumann algebra $\mathscr{M}$,

$$\mathscr{M}_{\text{aff}}^{c} = \mathscr{M}_{\text{aff}}^{\text{MvN}}.$$

Proposition 4.1. *Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra acting on the Hilbert space $\mathcal{H}$, and let $T \in \mathfrak{A}_{\text{aff}}$ be a closed operator that is bounded on its domain. Then* $\mathrm{dom}(T)$ *is a closed subspace in* $\mathrm{Aff}_s(\mathfrak{A})$, *and there exists an operator $X \in \mathfrak{A}$ such that $T = X|_{\mathrm{dom}(T)}$.*

Proof. First, observe that if S is any closed operator on $\mathcal{H}$ that is bounded on its domain, then by [Sch12, Proposition 1.4], it follows directly that $\mathrm{dom}(S)$ is a closed subspace of $\mathcal{H}$. Moreover, by defining 0 on $\mathrm{dom}(S)^{\perp}$, we can extend S to a bounded operator in $\mathcal{B}(\mathcal{H})$. In other words, there exists $\tilde{S} \in \mathcal{B}(\mathcal{H})$ such that $S = \tilde{S}|_{\mathrm{dom}(S)}$.

Now, suppose $T \in \mathfrak{A}_{\text{aff}}$ is a closed operator that is bounded on $\mathrm{dom}(T)$. Choose a quotient representation $T = A/B$ for some $A, B \in \mathfrak{A}$. By the first paragraph, there exists an operator

$C \in \mathcal{B}(\mathcal{H})$ such that $T = C$ on $\mathrm{dom}(T)$, meaning that $CB = A$. This implies $B^*C^* = A^*$ in $\mathcal{B}(\mathcal{H})$, leading to $\mathrm{ran}(A^*) \subseteq \mathrm{ran}(B^*)$.

By Douglas' lemma (Theorem 2.1) for $\mathfrak{A}$, there exists an operator $Y \in \mathfrak{A}$ such that $B^*Y = A^*$. Setting $X := Y^* \in \mathfrak{A}$, we obtain $XB = A$, which shows that $T = X|_{\mathrm{dom}(T)}$. This completes the proof. ■

Remark 4.1. It is straightforward to verify that the converse of Proposition 4.1 holds. Therefore, for a monotone σ-complete C^*-algebra $\mathfrak{A}$, an operator in $\mathfrak{A}_{\mathrm{aff}}$ is *closed and bounded on its domain* if and only if it is the restriction of an operator in $\mathfrak{A}$ to a closed affiliated subspace of $\mathfrak{A}$. As a result, we obtain

$$\mathfrak{A}_{\mathrm{aff}}^{cb} = \{AE^\dagger : A, E \in \mathfrak{A} \text{ with } E \text{ a projection in } \mathfrak{A}\}.$$

Next, we present a characterization of closed operators in $\mathfrak{A}_{\mathrm{aff}}$ in terms of certain *closed-range* operators in the algebra $\mathfrak{A}$. To this end, we first establish the following elementary lemma:

Lemma 4.1. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then A has closed range if and only if* 0 *is an isolated point of* $\mathrm{sp}\,(A)$.

Proof. ($\Leftarrow$) Suppose 0 is an isolated point of $\mathrm{sp}\,(A)$. Then $\mathbb{1}_{\{0\}} \in C(\mathrm{sp}\,(A))$ and we have:

$$\left(1 - \mathbb{1}_{\{0\}}(z)\right) z = z \text{ on } \mathrm{sp}\,(A)$$

Let $E := I - \mathbb{1}_{\{0\}}(A)$. Then E is a projection and the above equation implies by continuous functional calculus:

$$EA = A, \textit{i.e.}, \ \mathrm{ran}(A) \subseteq \mathrm{ran}(E)$$

For the reverse inclusion let us define the continuous function f on $\mathrm{sp}\,(A)$:

$$f(z) := \begin{cases} \frac{1 - \mathbb{1}_{\{0\}}(z)}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Then $zf(z) = 1 - \mathbb{1}_{\{0\}}(z)$ on $\mathrm{sp}\,(A)$. Again by continuous functional calculus we get

$$Af(A) = E, \textit{i.e.}, \ \mathrm{ran}(E) \subseteq \mathrm{ran}(A).$$

($\Rightarrow$) Let $A(\mathcal{H})$ is closed. To prove that 0 is an isolated point of the spectrum we need to show that for any λ near 0 the operator $A - \lambda$ is invertible.

In this case $\mathcal{H} = A(\mathcal{H}) \oplus \ker(A)$. Thus $A : A(\mathcal{H}) \to A(\mathcal{H})$ has a bounded inverse on the Hilbert space $A(\mathcal{H})$ so that $A - \lambda$ is also invertible with inverse, say, B on $A(\mathcal{H})$ for all λ near 0 (since resolvent set of an operator is open). Therefore for λ near 0 we can see that

$A - \lambda$ can be thought of as $A - \lambda \oplus -\lambda$ on $A(\mathcal{H}) \oplus \ker(A)$ so that $(A - \lambda)^{-1}$ should be $B \oplus -\frac{1}{\lambda}$ on $A(\mathcal{H}) \oplus \ker(A)$. Hence $A - \lambda$ is invertible in $\mathcal{B}(\mathcal{H})$. ■

Corollary 4.2. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Then A has closed range if and only if* 0 *is an isolated point of* $\mathrm{sp}\,(|A^*|)$.

Proof. By corollary 2.2, we have $\mathrm{ran}(A) = \mathrm{ran}(|A^*|)$ and by Lemma 4.1, this range is closed if and only if 0 is an isolated point in $\mathrm{sp}\,(|A^*|)$. ■

Corollary 4.3. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Then* $\operatorname{ran}(A)$ *is closed if and only if* $\operatorname{ran}(A^*)$ *is closed.*

Proof. It follows from the Corollary 4.2 and Douglas lemma (Theorem 2.1). ∎

Theorem 4.4 (cf. [Kol14, Theorem 4])**.** *Let $\mathfrak{A}$ be a unital monotone σ-complete C^*-algebra acting on $\mathcal{H}$, and let $A, B \in \mathfrak{A}$ satisfy* $\operatorname{null}(B) \subseteq \operatorname{null}(A)$*. Then $A/B \in \mathfrak{A}_{\text{aff}}$ is closed if and only if $A^* \boxplus B^* \in \mathfrak{A}$ has a closed range in $\mathcal{H}$.*

Proof. The operator A/B is closed if and only if the range of the matrix

$$\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \in \mathbb{M}_2\left(\mathfrak{A}\right)$$

(which represents the graph of A/B) is closed in $\mathcal{H} \oplus \mathcal{H}$. By Corollary 4.3, this occurs if and only if the range of the matrix

$$\begin{bmatrix} A^* & B^* \\ 0 & 0 \end{bmatrix}$$

(which is given by $\operatorname{ran}(A^*) + \operatorname{ran}(B^*) = \operatorname{ran}(A^* \boxplus B^*)$, see Remark 2.1) is closed in $\mathcal{H}$. ∎

In Theorem 4.7-(ii) below, we provide a more detailed characterization of closed operators affiliated with a von Neumann algebra.

Lemma 4.5. *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. A closed operator T on $\mathcal{H}$ is in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ if and only if its characteristic projection, $\chi(T)$, is in $\mathbb{M}_2\left(\mathscr{M}\right)$.*

Proof. Note that every unitary operator in $\mathbb{M}_2\left(\mathscr{M}\right)'$ is of the form $\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$ for some unitary $U \in \mathscr{M}'$.

Let $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$. For a vector $x \in \operatorname{dom}(T)$ and $U \in \mathscr{M}'$, note that $Ux \in \operatorname{dom}(T)$, and

$$\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} x \\ Tx \end{bmatrix} = \begin{bmatrix} Ux \\ UTx \end{bmatrix} = \begin{bmatrix} Ux \\ TUx \end{bmatrix}.$$

Thus $\operatorname{Gr}(T)$ is invariant under the action of any unitary operator in $\mathbb{M}_2\left(\mathscr{M}\right)'$ and by the double commutant theorem $\chi(T) \in \mathbb{M}_2\left(\mathscr{M}\right)$.

Conversely, assume that T is a closed operator on $\mathcal{H}$ with $\chi(T) \in \mathbb{M}_2\left(\mathscr{M}\right)$. Let U be a unitary in $\mathscr{M}'$. Then $\chi(T)$ commutes with $\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$ so that $\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$ leaves $\operatorname{Gr}(T)$ invariant, that is,

$$\begin{bmatrix} Ux \\ UTx \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} x \\ Tx \end{bmatrix} \in \operatorname{Gr}(T) \quad \text{for all } x \in \operatorname{dom}(T).$$

Thus the second coordinate of $\begin{bmatrix} Ux \\ UTx \end{bmatrix}$ must be $T(Ux)$ so that $UTx = TUx$ for all $x \in \operatorname{dom}(T)$. Hence T commutes with every unitary U in $\mathscr{M}'$, that is, $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$. ∎

The following theorem generalizes [FW71, Theorem 1.1] and shows that the lattice of affiliated subspaces, $\operatorname{Aff}_s\left(\mathscr{M}\right)$, is precisely the set of domains of MvN-affiliated operators.

Theorem 4.6. *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Let $\mathcal{V}$ be a linear subspace of $\mathcal{H}$. Then the following are equivalent:*

(i) $\mathcal{V}$ is the range of an operator in $\mathscr{M}$;

(ii) $\mathcal{V}$ is the range of an operator in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$;

(iii) $\mathcal{V}$ is the domain of an operator in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$;

(iv) $\mathcal{V}$ is the domain of an operator in $\mathscr{M}_{\text{aff}}$.

Proof. (i) $\Rightarrow$ (ii). Straightforward to see.

(ii) $\Rightarrow$ (iii). Let T be a closed operator in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$. Then $T^\dagger$ is a closed operator in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ and $\text{dom}(T^\dagger) = \text{ran}(T)$.

(iii) $\Rightarrow$ (i). Let $\mathcal{V}$ be the domain of an operator $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$, that is, $\mathcal{V} = \text{dom}(T)$. From Lemma 4.5, the characteristic projection of T,

$$\chi(T) = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

is contained in $\mathbb{M}_2(\mathscr{M})$. By equation (1.2.2) and Remark 2.1, we have

$$\mathcal{V} = \text{dom}(T) = \text{ran}(E_{11}) + \text{ran}(E_{12}) = \text{ran}(E_{11} \boxplus E_{12}),$$

and $E_{11} \boxplus E_{12} \in \mathscr{M}$ as $E_{11}, E_{12} \in \mathscr{M}$.

(i) $\Leftrightarrow$ (iv). If a subspace $\mathcal{V} \subseteq \mathcal{H}$ is the range of $B \in \mathscr{M}$, since $\text{dom}(B^\dagger) = \text{ran}(B)$, we have $\mathcal{V}$ is the domain of $B^\dagger \in \mathscr{M}_{\text{aff}}$. Conversely, if $\mathcal{V}$ is the domain of an operator T in $\mathscr{M}_{\text{aff}}$ and A/B is a quotient representation of T for $A, B \in \mathscr{M}$ with $\text{null}(B) \subseteq \text{null}(A)$, then $\mathcal{V} = \text{dom}(T) = \text{ran}(B)$. ∎

Theorem 4.7 (Quotient representation of closed affiliated operators)**.** *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Let T be a closed operator on $\mathcal{H}$.*

(i) Then T is affiliated with $\mathscr{M}$ in the sense of Murray and von Neumann if and only if $T \in \mathscr{M}_{\text{aff}}$, that is,

$$\mathscr{M}_{\text{aff}}^{\text{MvN}} = \mathscr{M}_{\text{aff}}^{c}.$$

(ii) We have $T \in \mathscr{M}_{\text{aff}}^{c}$ if and only if $T = P^{-1}AE^\dagger$ for operators $P, A, E \in \mathscr{M}$ with P being a positive one-to-one operator with dense range, and E a projection.

Proof. (i) Let $T = AB^\dagger$ for $A, B \in \mathscr{M}$, and U be a unitary in $\mathscr{M}'$. By Lemma 1.9 $B^\dagger \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$ and thus $UT = UAB^\dagger = AUB^\dagger = AB^\dagger U = TU$. Hence $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$, and we have,

$$\mathscr{M}_{\text{aff}}^{c} \subseteq \mathscr{M}_{\text{aff}}^{\text{MvN}}.$$

Next suppose $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$. By Theorem 4.6, there is an operator B' in $\mathscr{M}$ such that $\text{ran}(B') = \text{dom}(T)$. Then by Lemma 1.9, TB' is a bounded operator in $\mathscr{M}$ with $\text{ran}(T) = \text{ran}(TB')$ and $\text{null}(B') \subseteq \text{null}(TB')$. Defining $A' := TB'$, we have $T = (TB')B'^\dagger = A'B'^\dagger$ with $\text{dom}(T) = \text{ran}(B')$, $\text{ran}(T) = \text{ran}(A')$, and $\text{null}(B') \subseteq \text{null}(A')$. Thus $T = A'/B'$ is in $\mathscr{M}_{\text{aff}}^{c}$, and we conclude that

$$\mathscr{M}_{\text{aff}}^{\text{MvN}} \subseteq \mathscr{M}_{\text{aff}}^{c}.$$

(ii) ($\Longleftarrow$) Let A, P, E be as given in the assertion. Note that P^{-1} is a closed operator and $AE^\dagger = A|_{\text{ran}(E)}$ is closed and bounded. Thus the operator $P^{-1}AE^\dagger$ is closed. Since

$P^{-1}, AE^\dagger \in \mathscr{M}_{\text{aff}}$ and $\mathscr{M}_{\text{aff}}$ is closed under product by Theorem **??**, $P^{-1}AE^\dagger$ lies in $\mathscr{M}_{\text{aff}}$. In summary, $P^{-1}AE^\dagger \in \mathscr{M}_{\text{aff}}^c$.

($\Longrightarrow$) Let $T \in \mathscr{M}_{\text{aff}}$ be closed. By Theorem 3.5, $T^* \in \mathscr{M}_{\text{aff}}$ and by Proposition 1.2, it is closed and densely-defined. Since $\text{dom}(T^*) \in \text{Aff}_s(\mathscr{M})$, using Corollary 2.2, we get a positive operator $P \in \mathscr{M}$ such that $\text{dom}(T^*) = \text{ran}(P)$. Since T^* is densely-defined, P has dense range and by Proposition 1.3, it has trivial nullspace. Since $T^* \in \mathscr{M}_{\text{aff}}^c$, part (i) tells us that T^* is in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$. By Lemma 1.9, the operator $A' := T^*P$ lies in $\mathscr{M}$, and we see that $T^* = A'P^{-1} = A'/P$. Let E be the projection onto $\overline{\text{dom}(T)}$. Using Proposition 1.2, Theorem 3.5, and our hypothesis that T is closed, we see that,

$$T = \overline{T} = T^{**}E^\dagger = (A'/P)^*E^\dagger = P^{-1}A'^*E^\dagger. \qquad \blacksquare$$

4.2 Functoriality of the Construction $\mathscr{M} \mapsto \mathscr{M}_{\text{aff}}^c$

Since the existing literature on affiliation primarily concerns MvN-affiliated operators, it is essential to examine how our theory applies in this setting. Throughout this section, $\mathscr{M}$ and $\mathscr{N}$ denote von Neumann algebras, and $\Phi : \mathscr{M} \to \mathscr{N}$ is a unital normal $*$-homomorphism. We first establish that the restriction of Φ_{aff} (see Definition 3.2) to $\mathscr{M}_{\text{aff}}^{\text{MvN}}$ maps into $\mathscr{N}_{\text{aff}}^{\text{MvN}}$ while preserving densely defined operators. We then shift our focus to demonstrating, from various perspectives, that Φ_{aff} serves as the natural extension of Φ to $\mathscr{M}_{\text{aff}}$.

Theorem 4.8. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be two unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ- normal $*$-homomorphism. Let $\Phi_{\text{aff}} : \mathfrak{A}_{\text{aff}} \to \mathfrak{B}_{\text{aff}}$ be the extension of Φ as given in Definition 3.2. Let $T \in \mathfrak{A}_{\text{aff}}$.*

(i) Then $\text{dom}\big(\Phi_{\text{aff}}(T)\big) = \Phi_{\text{aff}}^s\big(\text{dom}(T)\big)$. If T is densely-defined, then so is $\Phi_{\text{aff}}(T)$.

(ii) If T is closed and bounded (on its domain), then so is $\Phi_{\text{aff}}(T)$.

(iii) If T is closed, then so is $\Phi_{\text{aff}}(T)$.

(iv) If T is pre-closed, then so is $\Phi_{\text{aff}}(T)$. Furthermore, $\overline{T} \in \mathfrak{A}_{\text{aff}}$ and $\Phi_{\text{aff}}(\overline{T}) = \overline{\Phi_{\text{aff}}(T)}$.

Proof. (i) Let $T = A/B$ for $A, B \in \mathfrak{A}$ with $\text{null}(B) \subseteq \text{null}(A)$, so that $\Phi_{\text{aff}}(T) = \Phi(A)/\Phi(B)$. From Theorem 2.7-(i), we have

$$\text{dom}\big(\Phi_{\text{aff}}(T)\big) = \text{ran}\ \Phi(B) = \Phi_{\text{aff}}^s\big(\text{ran}(B)\big) = \Phi_{\text{aff}}^s\big(\text{dom}(T)\big). \tag{4.2.1}$$

Let T be densely-defined, that is, $\text{dom}(T)$ is dense in $\mathcal{H}$. From Theorem 2.7-(iv), note that $\Phi_{\text{aff}}^s\big(\text{dom}(T)\big)$ is dense in $\mathcal{K}$. Equation (4.2.1) tells us that $\Phi_{\text{aff}}(T)$ is densely-defined.

(ii) If T is closed and bounded, then by Remark 4.1, $T = AE^\dagger$ for $A \in \mathfrak{A}$ and E a projection in $\mathfrak{A}$. Thus $\Phi_{\text{aff}}(T) = \Phi(A)\Phi(E)^\dagger$ where $\Phi(A) \in \mathscr{N}$ and $\Phi(E)$ is a projection in $\mathscr{N}$. By Remark 4.1, the assertion follows.

(iii) Let $T = A/B$ for $A, B \in \mathfrak{A}$ with $\text{null}(B) \subseteq \text{null}(A)$, so that $\Phi_{\text{aff}}(T) = \Phi(A)/\Phi(B)$. Then we get:

$$\begin{aligned} &A/B \text{ is closed} \\ &\iff \text{ran}(A^* \boxplus B^*) \text{ is closed (by Theorem 4.4)} \\ &\implies \text{ran}(\Phi(A)^* \boxplus \Phi(B)^*) = \text{ran}(\Phi(A^* \boxplus B^*)) = \Phi_{\text{aff}}^s\left(\text{ran}(A^* \boxplus B^*)\right) \text{ is closed} \\ &\iff \Phi_{\text{aff}}(A/B) = \Phi(A)/\Phi(B) \text{ is closed.} \end{aligned}$$

The second implication is true by Lemma 2.6 and Theorem 2.7-(iii).

(iv) Let T be pre-closed. By Proposition 1.1, $\mathrm{dom}(T^*)$ is dense in $\mathcal{H}$. By Theorem 3.6-(iv) and part (i) above, $\Phi_{\mathrm{aff}}(T^*) = \Phi_{\mathrm{aff}}(T)^*$ is densely-defined on $\mathcal{K}$. Again using Proposition 1.1-(i), we conclude that $\Phi_{\mathrm{aff}}(T)$ is pre-closed.

Let A/B be a quotient representation of T for $A, B \in \mathfrak{A}$ with $\mathrm{null}(B) \subseteq \mathrm{null}(A)$. Let E denote the projection onto $\overline{\mathrm{dom}(T)}$. Since $\mathrm{dom}(T) = \mathrm{ran}(B)$, clearly $E = \mathbf{R}(B)$. Since $\Phi(A)/\Phi(B)$ is a quotient representation of $\Phi_{\mathrm{aff}}(T)$ and $\Phi(E) = \mathbf{R}\big(\Phi(B)\big)$ (by Lemma 1.10), we note that $\Phi(E)$ is the projection onto $\overline{\mathrm{dom}(\Phi_{\mathrm{aff}}(T))}$. From Proposition 1.2 and Theorem 3.6, we have

$$\Phi_{\mathrm{aff}}(\overline{T}) = \Phi_{\mathrm{aff}}(T^{**}E^{\dagger}) = \Phi_{\mathrm{aff}}(T)^{**}\Phi(E)^{\dagger} = \overline{\Phi_{\mathrm{aff}}(T)}. \qquad \blacksquare$$

As a direct consequence of the previous theorem in the context of von Neumann algebras, we obtain the following result:

Corollary 4.9. *Let $(\mathscr{M};\mathcal{H}), (\mathscr{N};\mathcal{K})$ be represented von Neumann algebras and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathscr{M}_{\mathrm{aff}} \to \mathscr{N}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Then Φ_{aff} maps $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ into $\mathscr{N}_{\mathrm{aff}}^{\mathrm{MvN}}$, and in fact, sends densely-defined closed affiliated operators for $\mathscr{M}$ to densely-defined closed affiliated operators for $\mathscr{N}$.*

Let $\mathscr{M}$ be a finite von Neumann algebra. The set of densely-defined operators in $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ is a $*$-algebra under strong-sum, strong-product, and adjoint. For the sake of brevity, we denote this $*$-algebra by $\mathscr{M}_{\mathrm{aff}}^{dd-c}$. In fact, it is a monotone-complete partially-ordered $*$-algebra under the order structure given by the cone of positive operators in $\mathscr{M}_{\mathrm{aff}}^{dd-c}$ (see [Nay21a, Proposition 4.21]). Let $\mathscr{N}$ be another finite von Neumann algebra and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. In [Nay21a, Theorems 4.9, 4.24] and [AM12, Proposition 5.14], it is shown that Φ has a unique extension to a unital normal $*$-homomorphism between $\mathscr{M}_{\mathrm{aff}}^{dd-c}$ and $\mathscr{N}_{\mathrm{aff}}^{dd-c}$. From the corollary below, we observe that this extension is nothing but the restriction of our Φ_{aff} to $\mathscr{M}_{\mathrm{aff}}^{dd-c}$. This shows the compatibility of Φ_{aff} with the canonical extension of Φ in the context of finite von Neumann algebras.

Corollary 4.10. *Let $(\mathscr{M};\mathcal{H}), (\mathscr{N};\mathcal{K})$ be represented von Neumann algebras and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathscr{M}_{\mathrm{aff}} \to \mathscr{N}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Let T_1, T_2 be closed operators affiliated with $\mathscr{M}$.*

(i) If $T_1 + T_2$ is pre-closed, then $\Phi_{\mathrm{aff}}(T_1) + \Phi_{\mathrm{aff}}(T_2)$ is pre-closed, and

$$\Phi_{\mathrm{aff}}(T_1 \hat{+} T_2) = \Phi_{\mathrm{aff}}(T_1) \hat{+} \Phi_{\mathrm{aff}}(T_2).$$

(ii) If T_1T_2 is pre-closed, then $\Phi_{\mathrm{aff}}(T_1)\Phi_{\mathrm{aff}}(T_2)$ is pre-closed, and

$$\Phi_{\mathrm{aff}}(T_1 \,\hat{\cdot}\, T_2) = \Phi_{\mathrm{aff}}(T_1) \,\hat{\cdot}\, \Phi_{\mathrm{aff}}(T_2).$$

In other words, Φ_{aff} preserves strong-sum and strong-product.

Proof. Since $T_1, T_2, \in \mathscr{M}_{\mathrm{aff}}$, from Theorem 3.6 we have,

$$\Phi_{\mathrm{aff}}(T_1 + T_2) = \Phi_{\mathrm{aff}}(T_1) + \Phi_{\mathrm{aff}}(T_2), \Phi_{\mathrm{aff}}(T_1T_2) = \Phi_{\mathrm{aff}}(T_1)\Phi_{\mathrm{aff}}(T_2).$$

Then the assertion is an immediate consequence of Theorem 4.8-(iv). $\blacksquare$

A geometric way to build an extension of Φ to $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ is via Stone's idea of characteristic projection matrices (see [Sto51]). Note that there is a one-to-one correspondence between a closed operator on $\mathcal{H}$ and its characteristic projection matrix acting on $\mathcal{H} \oplus \mathcal{H}$. In Lemma 4.5, we have shown that if $T \in \mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$, then its characteristic projection, $\chi(T)$, lies in $M_2(\mathscr{M})$. The mapping Φ induces a mapping via $\Phi_{(2)}$ between $M_2(\mathscr{M})$ to $M_2(\mathscr{N})$ and clearly, sends projection matrices to projection matrices. In the theorem below, we show that this strategy for obtaining an extension of Φ to $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ also begets our now-familiar extension Φ_{aff}.

Theorem 4.11. *Let $(\mathscr{M};\mathcal{H}),(\mathscr{N};\mathcal{K})$ be represented von Neumann algebras and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathscr{M}_{\mathrm{aff}} \to \mathscr{N}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}} & \xrightarrow{\chi} & \mathbb{M}_2(\mathscr{M}) \\ {\scriptstyle \Phi_{\mathrm{aff}}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{(2)}} \\ \mathscr{N}_{\mathrm{aff}}^{\mathrm{MvN}} & \xrightarrow{\chi} & M_2(\mathscr{N}) \end{array}$$

In other words, the extension $\Phi_{\mathrm{aff}} : \mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}} \to \mathscr{N}_{\mathrm{aff}}^{\mathrm{MvN}}$ of Φ, may be defined in a geometric manner via characteristic projections as the mapping,

$$T \mapsto (\chi^{-1} \circ \Phi_{(2)} \circ \chi)(T), \text{ for } T \in \mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}.$$

Proof. Recall the notation $\mathfrak{S}_{A',B'}$ from §1. Let $T \in \mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ and A/B be a quotient representation of T for $A, B \in \mathscr{M}$ with $\mathrm{null}(B) \subseteq \mathrm{null}(A)$. By Proposition 3.1, $\Phi(A)/\Phi(B)$ is a quotient representation of $\Phi_{\mathrm{aff}}(T)$. Since T is a closed operator, Theorem 4.8-(iii) tells us that $\Phi_{\mathrm{aff}}(T)$ must be a closed operator, and therefore its graph, $\mathrm{Gr}(\Phi_{\mathrm{aff}}(T))$, is a closed subspace of $\mathcal{K} \oplus \mathcal{K}$. Thus,

$$\begin{aligned} \chi\big(\Phi_{\mathrm{aff}}(T)\big) &= \chi\big(\Phi(A)/\Phi(B)\big) \overset{(3.1.1)}{=} \mathbf{R}(\mathfrak{S}_{\Phi(A),\Phi(B)}) \\ &= \mathbf{R}\left(\Phi_{(2)}(\mathfrak{S}_{A,B})\right) = \Phi_{(2)}\Big(\mathbf{R}(\mathfrak{S}_{A,B})\Big), \text{ by Lemma 1.10} \\ &\overset{(3.1.1)}{=} \Phi_{(2)}\big(\chi(A/B)\big) = \Phi_{(2)}\big(\chi(T)\big). \end{aligned}$$

■

Traditionally, $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ is viewed as being equipped with the binary operations of strong-sum and strong-product (which are not necessarily defined for every pair of operators in $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$), and the unary operation of adjoint. In the theorem below, we show that there is a unique extension of Φ that respects these algebraic operations; yet again, this extension turns out to be the restriction of Φ_{aff} (see Definition 3.2) to $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$. At this point, we hope that we have been able to persuade the reader through various perspectives that Φ_{aff} indeed gives the canonical extension of Φ to $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$.

Theorem 4.12. *Let $(\mathscr{M};\mathcal{H}),(\mathscr{N};\mathcal{K})$ be represented von Neumann algebras and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. Let $\Phi_{\mathrm{aff}} : \mathscr{M}_{\mathrm{aff}} \to \mathscr{N}_{\mathrm{aff}}$ be the extension of Φ as given in Definition 3.2. Then the restriction of Φ_{aff} to $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ is the unique mapping from $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ to $\mathscr{N}_{\mathrm{aff}}^{\mathrm{MvN}}$ which extends Φ, and respects strong-sum, strong-product and adjoint.*

Proof. Let Ψ be a map from $\mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$ to $\mathscr{N}_{\mathrm{aff}}^{\mathrm{MvN}}$ that extends Φ, respects strong-sum, strong-product and adjoint. Before we proceed with the proof, here are a couple of things to keep in mind. From our hypothesis that Ψ maps into $\mathscr{N}_{\mathrm{aff}}^{\mathrm{MvN}}$, $\Psi(T)$ is a closed operator on $\mathcal{K}$ for all $T \in \mathscr{M}_{\mathrm{aff}}^{\mathrm{MvN}}$. If T_1, T_2 are closed operators on a Hilbert space such that T_1T_2 is closed, then by definition of the strong-product, $T_1 \hat{\cdot} T_2 = T_1T_2$.

Observation 1. For every $B \in \mathscr{M}$, we have $\Psi(B)^\dagger \subseteq \Psi(B^\dagger)$.

Proof of Observation 1. From equation (1.2.1) note that $B^\dagger B = I - \mathbf{N}(B)$ is closed. Since $\Psi(B^\dagger) \in \mathscr{N}_{\mathrm{aff}}^{\mathrm{MvN}}$ is closed and $\Psi(B) = \Phi(B) \in \mathscr{N}$, by Lemma 1.9 we observe that $\Psi(B^\dagger)\Psi(B)$

is closed. Thus from the hypothesis on Ψ and Lemma 1.10, we get,

$$\Psi(B^\dagger)\Psi(B) = \Psi(B^\dagger)\hat{\cdot}\Psi(B) = \Psi(B^\dagger\hat{\cdot}B) = \Psi\big(I - \mathbf{N}(B)\big) = I - \mathbf{N}\big(\Psi(B)\big).$$

From Remark 1.4 and the preceding equation, it follows that $\Psi(B)^\dagger \subseteq \Psi(B^\dagger)$.

Observation 2. For every $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$, we have $\Phi_{\text{aff}}(T) \subseteq \Psi(T)$.

Proof of Observation 2. Let $A, B \in \mathscr{M}$ such that $T = AB^\dagger$. From the hypothesis on Ψ, we have

$$\begin{aligned}\Psi(T) = \Psi(A)\hat{\cdot}\Psi(B^\dagger) &\supseteq \Psi(A)\hat{\cdot}\Psi(B)^\dagger, \quad \text{by Observation 1 and Remark 1.2}\\ &= \Phi(A)\hat{\cdot}\Phi(B)^\dagger = \Phi_{\text{aff}}(AB^\dagger), \quad \text{by Corollary 4.10-(ii)}\\ &= \Phi_{\text{aff}}(T).\end{aligned}$$

Observation 3. For every projection $E \in \mathscr{M}$, we have $\Psi(E^\dagger) = \Psi(E)^\dagger$.

Proof of Observation 3. For a closed operator T on $\mathcal{H}$ and a bounded operator B on $\mathcal{H}$, note that $T + B$ is closed so that $T \hat{+} B = T + B$. It is easy to verify that $E^\dagger + (I - E) = E^\dagger$. Applying Ψ on both sides, we get

$$\begin{aligned}\Psi(E^\dagger) = \Psi\left(E^\dagger + (I-E)\right) &= \Psi\left(E^\dagger \hat{+} (I-E)\right)\\ &= \Psi(E^\dagger)\hat{+}\Psi(I-E) = \Psi(E^\dagger)\hat{+}\left(I - \Psi(E)\right)\\ &= \Psi(E^\dagger) + \left(I - \Psi(E)\right).\end{aligned}$$

Note that $\Psi(E) = \Phi(E)$ is a projection in $\mathscr{N}$. Thus, we have

$$\operatorname{dom}\left(\Psi(E^\dagger)\right) \subseteq \operatorname{null}\left(I - \Psi(E)\right) = \operatorname{ran}\left(\Psi(E)\right) = \operatorname{dom}\left(\Psi(E)^\dagger\right).$$

Together with Observation 1, this tells us that $\Psi(E^\dagger) = \Psi(E)^\dagger$.

Observation 4. For every $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$, we have $\Psi(T) = \Phi_{\text{aff}}(T)$.

Proof of Observation 4. Let $S \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$ be densely-defined. From Theorem 4.8-(i), we see that $\Phi_{\text{aff}}(S)$ is densely-defined on $\mathcal{K}$. Since by Observation 2, $\Phi_{\text{aff}}(S) \subseteq \Psi(S)$, clearly $\Psi(S)$ is also densely-defined on $\mathcal{K}$. Recall that Φ_{aff} preserves adjoint (by Theorem 3.6-(iv)) and by the hypothesis, Ψ also preserves adjoint. Using [KR97, Remark 2.7.5] and Observation 2, we get

$$\Psi(S)^* \subseteq \Phi_{\text{aff}}(S)^* = \Phi_{\text{aff}}(S^*) \subseteq \Psi(S^*) = \Psi(S)^*.$$

From the sandwich of extensions, we have

$$\Phi_{\text{aff}}(S^*) = \Psi(S^*), \tag{4.2.2}$$

for all densely-defined S in $\mathscr{M}_{\text{aff}}^{\text{MvN}}$.

Let $T \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$. By Proposition 1.2, $T = T^{**}E^\dagger$ for some projection $E \in \mathscr{M}$. Since T^* is densely-defined and $\Psi(E^\dagger) = \Psi(E)^\dagger$ (by Observation 3) the result follows from equation (4.2.2) for $S = T^*$. Indeed,

$$\Psi(T) = \Psi(T^{**})\Psi(E)^\dagger = \Phi_{\text{aff}}(T^{**})\Phi(E)^\dagger = \Phi_{\text{aff}}(T^{**}E^\dagger) = \Phi_{\text{aff}}(T). \qquad \blacksquare$$

There are several special classes of affiliated operators studied in the literature. In fact, non-commutative function theory primarily deals with non-commutative analogues of classical function spaces (such as L^p-spaces, Sobolev spaces, etc.) which are often related to the set of affiliated operators for a von Neumann algebra. In Remark 4.2, we consider the $*$-algebra of measurable operators and the $*$-algebra of locally measurable operators associated with a von Neumann algebra, demonstrating that, unlike the near-semiring of affiliated operators, they are **not** functorial constructions.

Definition 4.1 (Measurable Operator, [Seg53; Yea73])**.** An operator $A \in \mathscr{M}^c_{\text{aff}}$ is called *measurable* iff $\text{dom}(A)$ is dense and $I - E_\lambda$ is *finite* (relative to $\mathscr{M}$) for some $\lambda > 0$ where $|A| = \int_0^\infty \lambda \mathrm{d}E_\lambda$ is the spectral resolution of $|A|$. The set of all measurable operators is denoted by $\mathcal{S}(\mathscr{M})$.

Definition 4.2 (Locally Measurable Operator, [Yea73])**.** An operator $A \in \mathscr{M}^c_{\text{aff}}$ is said to be *locally measurable* if there exist projections $Q_n \in \mathscr{C}$, the center of $\mathscr{M}$, and $E_n \in \mathscr{M}$ such that $Q_n \uparrow I, E_n \uparrow I, \text{ran}(E_n) \subseteq \text{dom}(A)$ and $Q_n(I - E_n)$ is finite for all n. The set of all locally measurable operators associated with $\mathscr{M}$ is denoted by $\mathcal{L}(\mathscr{M})$.

Remark 4.2. Suppose $\Phi : \mathscr{M} \to \mathscr{N}$ is a unital normal $*$-homomorphism between two von-Neumann algebras. Now it is natural to ask whether or not Φ_{aff} preserves *measurability* of affiliated operators. In general, it doesn't! Indeed, let T be a densely-defined self-adjoint operator acting on $\mathcal{H}$. Then T generates an abelian von-Neumann algebras $\mathscr{A}$ such that $T \in \mathscr{A}_{\text{aff}}$. Since $\mathscr{A}$ is finite so $T \in \mathcal{S}(\mathscr{A})$, the set of all measurable operators on $\mathscr{A}$. Now if we consider Φ to be the inclusion $\mathscr{A} \hookrightarrow \mathcal{B}(\mathcal{H})$, then $\Phi_{\text{aff}}(T)$ is a densely-defined self-adjoint operator on $\mathcal{H}$ which is not contained in $\mathcal{S}\big(\mathcal{B}(\mathcal{H})\big) = \mathcal{B}(\mathcal{H})$; that is, Φ_{aff} doesn't preserve the measurability of operators. Similarly, since $\mathcal{L}\big(\mathcal{B}(\mathcal{H})\big) = \mathcal{B}(\mathcal{H})$, Φ_{aff} doesn't preserve the local measurability either.

4.3 Some Operator Properties Preserved by Φ_{aff}

In this section, we show that Φ_{aff} preserves operator properties such as being *symmetric*, *positive*, *self-adjoint*, *normal*, *accretive*, *sectorial*. First we recall some definitions.

Definition 4.3. An unbounded operator T on the Hilbert space $\mathcal{H}$ is said to be

(i) *symmetric* if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \text{dom}(T)$;

(ii) *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in \text{dom}(T)$;

(iii) *accretive* if $\Re \langle Tx, x\rangle \geq 0$ for all $x \in \text{dom}(T)$;

(iv) *sectorial* if $\langle Tx, x\rangle \in S_{c,\theta} := \{\lambda \in \mathbb{C} \ : \ |\arg(\lambda - c)| \leq \theta\}$ for all $x \in \text{dom}(T)$, where $c \in \mathbb{C}$ and $\theta \in [0, \frac{\pi}{2})$.

The *numerical range* of T, denoted by $W(T)$, is defined by

$$W(T) := \{\langle Tx, x\rangle : x \in \text{dom}(T), \|x\| = 1\}.$$

Remark 4.3. Let T be an unbounded operator on the Hilbert space $\mathcal{H}$. We denote the closed right half-plane in $\mathbb{C}$ by,

$$\mathbb{H}_{0,0} := \{z \in \mathbb{C} \ : \ \Re(z) \geq 0\}.$$

Table 4.1 recasts the operator properties described in Definition 4.3 in terms of numerical ranges.

T is symmetric	$W(T) \subseteq \mathbb{R}$
T is positive	$W(T) \subseteq \mathbb{R}_{\geq 0}$
T is accretive	$W(T) \subseteq \mathbb{H}_{0,0}$
T is sectorial	$W(T) \subseteq S_{c,\theta}$

Table 4.1: Operator properties in terms of their numerical ranges

For a complex number $\alpha \in \mathbb{C}$ and an angle $\theta \in [-\pi, \pi)$ we will denote the affine transformation $\mathbb{C} \to \mathbb{C} : z \mapsto e^{i\theta}(z + \alpha)$ by $\mathfrak{L}_{\theta,\alpha}$. Every *closed half-space* in $\mathbb{C}$ is the image of $\mathbb{H}_{0,0}$ under some $\mathfrak{L}_{\theta,\alpha}$ and is denoted by $\mathbb{H}_{\theta,\alpha}$, that is,

$$\mathbb{H}_{\theta,\alpha} := \mathfrak{L}_{\theta,\alpha}(\mathbb{H}_{0,0}).$$

Note that the affine transformation $\mathfrak{L}_{\theta,\alpha}$ is invertible and its inverse $\mathfrak{L}_{\theta,\alpha}^{-1} = \mathfrak{L}_{-\theta,-e^{i\theta}\alpha}$ is an affine transformation. For the affine transformation, $\mathfrak{L}_{\theta,\alpha}$, we define $\mathfrak{L}_{\theta,\alpha}(T) := e^{i\theta}(T + \alpha I)$. Clearly, $\mathrm{dom}\big(\mathfrak{L}_{\theta,\alpha}(T)\big) = \mathrm{dom}(T)$.

Lemma 4.13. *For an operator T acting on the Hilbert space $\mathcal{H}$ we have,*

$$W\big(\mathfrak{L}_{\theta,\alpha}(T)\big) = \mathfrak{L}_{\theta,\alpha}\big(W(T)\big),$$

where $\alpha \in \mathbb{C}$ and $\theta \in [0, \frac{\pi}{2})$.

Proof. Just unwrapping the appropriate definitions, we get

$$W\big(\mathfrak{L}_{\theta,\alpha}(T)\big) = W\left(e^{i\theta}(T + \alpha I)\right) = e^{i\theta}(W(T) + \alpha) = \mathfrak{L}_{\theta,\alpha}\big(W(T)\big). \qquad \blacksquare$$

Proposition 4.2. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. For $T \in \mathfrak{A}_{\mathrm{aff}}$, if $W(T) \subseteq \mathbb{H}_{0,0}$, then $W\big(\Phi_{\mathrm{aff}}(T)\big) \subseteq \mathbb{H}_{0,0}$.*

Proof. Let A', B' be bounded operators on a Hilbert space with $\mathrm{null}(B') \subseteq \mathrm{null}(A')$, and let $T' = A'/B'$. Keeping in mind that $\mathrm{dom}(T') = \mathrm{ran}(B')$, we have,

$$\begin{aligned}
W(T') \subseteq \mathbb{H}_{0,0} &\iff \forall y \in \mathrm{dom}(T'), 2\Re\big\langle T'y, y\big\rangle = \big\langle T'y, y\big\rangle + \big\langle y, T'y\big\rangle \geq 0 \\
&\iff \forall x \in \mathcal{H}, \big\langle T'B'x, B'x\big\rangle + \big\langle B'x, T'B'x\big\rangle \geq 0 \\
&\iff \forall x \in \mathcal{H}, \big\langle A'x, B'x\big\rangle + \big\langle B'x, A'x\big\rangle \geq 0 \\
&\iff B'^*A' + A'^*B' \geq 0.
\end{aligned}$$

Let A/B be a quotient representation of T so that $\Phi(A)/\Phi(B)$ is a quotient representation of $\Phi_{\mathrm{aff}}(T)$. From the above discussion, we see that $B^*A + A^*B \geq 0$ and by positivity of Φ, clearly $\Phi(B)^*\Phi(A) + \Phi(A)^*\Phi(B) \geq 0$, whence $W(\Phi_{\mathrm{aff}}(T)) \subseteq \mathbb{H}_{0,0}$. $\blacksquare$

Proposition 4.3. *Let $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ be unital monotone σ-complete C^*-algebras and $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital σ-normal $*$-homomorphism. Let $T \in \mathfrak{A}_{\mathrm{aff}}$ and K be a closed convex set in $\mathbb{C}$. If $W(T) \subseteq K$, then $W\big(\Phi_{\mathrm{aff}}(T)\big) \subseteq K$.*

Proof. If $K = \mathbb{C}$, there is nothing to prove. Thus we may assume that K is a proper subset of $\mathbb{C}$. First, we prove the assertion for the closed half-plane $K = \mathbb{H}_{\theta,\alpha}$ where $\theta \in [-\pi, \pi)$ and $\alpha \in \mathbb{C}$. For $\theta' = -\theta, \alpha' = -e^{i\theta}\alpha$, note that $\mathfrak{L}_{\theta',\alpha'} = \mathfrak{L}_{\theta,\alpha}^{-1}$. We have,

$$\begin{aligned} W(T) \subseteq \mathbb{H}_{\theta,\alpha} &\implies W\left(\mathfrak{L}_{\theta',\alpha'}(T)\right) = \mathfrak{L}_{\theta',\alpha'}\big(W(T)\big) \subseteq \mathfrak{L}_{\theta',\alpha'}(H_{\theta,\alpha}) = \mathbb{H}_{0,0} \\ &\implies W\big(\Phi_{\mathrm{aff}}(\mathfrak{L}_{\theta',\alpha'}(T))\big) \subseteq \mathbb{H}_{0,0}, \quad \text{by Proposition 4.2} \\ &\implies W\Big(\mathfrak{L}_{\theta',\alpha'}(\Phi_{\mathrm{aff}}(T))\Big) = \mathfrak{L}_{\theta',\alpha'}\Big(W\big(\Phi_{\mathrm{aff}}(T)\big)\Big) \subseteq \mathbb{H}_{0,0} \\ &\implies W\big(\Phi_{\mathrm{aff}}(T)\big) \subseteq \mathfrak{L}_{\theta,\alpha}(\mathbb{H}_{0,0}) = \mathbb{H}_{\theta,\alpha}. \end{aligned}$$

From here, the assertion follows from a basic theorem in convex geometry which asserts that every closed convex proper subset of $\mathbb{C}$ (in fact, for any Euclidean space) is an intersection of closed half-spaces (see [Sch93, Corollary 1.3.5]). ■

Theorem 4.14. *Let* $(\mathfrak{A}; \mathcal{H}), (\mathfrak{B}; \mathcal{K})$ *be unital monotone* σ*-complete* C^**-algebras and* $\Phi : \mathfrak{A} \to \mathfrak{B}$ *be a unital* σ*-normal* $*$*-homomorphism. Let* $\Phi_{\mathrm{aff}} : \mathfrak{A}_{\mathrm{aff}} \to \mathfrak{B}_{\mathrm{aff}}$ *be the extension of* Φ *as given in Definition 3.2, and let* $T \in \mathfrak{A}_{\mathrm{aff}}$*.*

(i) If T *is symmetric, then so is* $\Phi_{\mathrm{aff}}(T)$*.*

(ii) If T *is positive, then so is* $\Phi_{\mathrm{aff}}(T)$*.*

(iii) If T *is accretive, then so is* $\Phi_{\mathrm{aff}}(T)$*.*

(iv) If T *is sectorial, then so is* $\Phi_{\mathrm{aff}}(T)$*.*

(v) If T *is self-adjoint, then so is* $\Phi_{\mathrm{aff}}(T)$*.*

(vi) If T *is normal, then so is* $\Phi_{\mathrm{aff}}(T)$*.*

Proof. Since $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{H}_{0,0}, S_{c,\theta}$ are closed convex sets in $\mathbb{C}$, the assertions in parts (i)-(iv) follow from Proposition 4.3 and the description of these operators in Table 4.1. Parts (v) and (vi) are immediate from Theorem 3.6. ■

Chapter 5

Krein Extension Theory for Positive Affiliated Operators

Consider the following classical question: Given a *positive (symmetric)* operator S acting on the Hilbert space $\mathcal{H}$, does there always exist a *positive self-adjoint* extension of S? Since self-adjoint operators are closed, for an affirmative answer, S must be pre-closed, and without loss of generality, we may assume that S is closed. When S is densely-defined, the answer to the above question is affirmative; two such classical examples are given by the Friedrichs extension (see [Fri34]) which we denote by $S^{\mathbf{Fr}}$, and the Krein-von Neumann extension (see [Kre47]) which we denote by $S^{\mathbf{Kr}}$. In fact, among all possible such extensions, $S^{\mathbf{Fr}}$ is the *maximal* one, and $S^{\mathbf{Kr}}$ is the *minimal* one.

Throughout this section, $\mathscr{M}$ denotes a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Let S be a densely-defined closed positive operator affiliated with $\mathscr{M}$. The Friedrichs and Krein-von Neumann extensions, $S^{\mathbf{Fr}}, S^{\mathbf{Kr}}$, are both affiliated with $\mathscr{M}$ (proofs can be found in [Kad98, Appendix], [Ska79, §4], respectively). Let $\mathscr{N}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{K}$ and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. By Theorem 4.8-(i) and Theorem 4.14, we note that $\Phi_{\mathrm{aff}}(S)$ is a densely-defined closed positive operator affiliated with $\mathscr{N}$. From Theorem 4.14 and Theorem 3.8, it is clear that $\Phi_{\mathrm{aff}}(S^{\mathbf{Fr}}), \Phi_{\mathrm{aff}}(S^{\mathbf{Kr}})$ are positive self-adjoint extensions of $\Phi_{\mathrm{aff}}(S)$ which are affiliated with $\mathscr{N}$. In this section, our primary objective is to show that these are in fact respectively the Friedrichs and Krein extensions of $\Phi_{\mathrm{aff}}(S)$, that is,

$$\Phi_{\mathrm{aff}}(S^{\mathbf{Fr}}) = \Phi_{\mathrm{aff}}(S)^{\mathbf{Fr}} \text{ and } \Phi_{\mathrm{aff}}(S^{\mathbf{Kr}}) = \Phi_{\mathrm{aff}}(S)^{\mathbf{Kr}}.$$

To this end, we need to discuss Krein's extension theory (see [Kre47]) for a positive operator on a Hilbert space. A rough sketch may be found in [Ska79] and a more comprehensive account can be found in [Sch12, Chapter 13]. Since Krein's extension theory has an algebraic flavour, we expect it to be compatible with morphisms in $\mathbf{W^*Alg}$. Below we give a reasonably self-contained account rephrasing the results from [Ska79, §4] in a language which makes functoriality evident.

Definition 5.1. Let $\mathcal{C}(\mathcal{H})^+$ denote the family of closed positive operators on $\mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ denote the family of closed and bounded (not necessarily everywhere-defined) symmetric operators B with $||B|| \le 1$ and $\mathbf{N}(I - B) = 0$.

For an operator $S \in \mathcal{C}(\mathcal{H})^+$, note that $\mathbf{N}(S + I) = 0$ so that $S + I$ is one-to-one. The *Krein transform* of S is defined as,

$$\mathfrak{K}(S) := (S - I)(S + I)^{-1};\ \mathrm{dom}\,(\mathfrak{K}(S)) = \mathrm{ran}(S + I).$$

For $B \in \mathcal{F}(\mathcal{H})$, the *inverse Krein transform* of B is defined as,

$$\mathfrak{K}^{-1}(B) := (I + B)(I - B)^{-1};\ \mathrm{dom}\left(\mathfrak{K}^{-1}(B)\right) = \mathrm{ran}(I - B).$$

Note that,

$$\begin{aligned} \mathfrak{K}(S) &= (S-I)(S+I)^{-1} = (S+I-2I)(S+I)^{-1} = I - 2(S+I)^{-1}, \\ \mathfrak{K}^{-1}(B) &= (I+B)(I-B)^{-1} = \big(2I-(I-B)\big)(I-B)^{-1} = 2(I-B)^{-1} - I. \end{aligned} \tag{5.0.1}$$

Proposition 5.1. *Let S be a closed positive operator on $\mathcal{H}$. Then we have the following.*

(i) $\mathrm{null}(S+I) = \{0\}_{\mathcal{H}}$, *and* $\mathrm{ran}(S+I)$ *is a closed subspace of* $\mathcal{H}$. *If* $\mathrm{null}(S) = \{0\}_{\mathcal{H}}$, *then* S^{-1} *is a closed positive operator on* $\mathcal{H}$.

(ii) $(S+I)^{-1}$ *is a closed bounded positive operator with norm less than or equal to* 1, *and has trivial nullspace.* $\big($*Note that* $\mathrm{dom}\,\big((S+I)^{-1}\big) = \mathrm{ran}(S+I)$.$\big)$

(iii) S *is self-adjoint if and only if* $\mathrm{ran}(S+I) = \mathcal{H}$;

(iv) S_0 *is a positive extension of* S *if and only if* $(S_0+I)^{-1}$ *is a positive extension of* $(S+I)^{-1}$.

Proof. Parts (i) and (ii) follows from [KR97, Lemma 2.7.9] and (iii) follows from [KR97, Proposition 2.7.10]. For (iv), note that $\mathrm{ran}(S+I) \subseteq \mathrm{ran}(S_0+I)$ or equivalently, $\mathrm{dom}\big((S+I)^{-1}\big) \subseteq \mathrm{dom}\big((S_0+I)^{-1}\big)$, and then use part (ii). ■

Using Proposition 5.1, we have the following result.

Proposition 5.2 (cf. [Sch12, Proposition 13.22])**.** *The mapping $\mathfrak{K}$ gives a bijection between $\mathcal{C}(\mathcal{H})^+$, and $\mathcal{F}(\mathcal{H})$ and its inverse mapping is the inverse Krein transform, $\mathfrak{K}^{-1}$. Furthermore, S_0 is a positive self-adjoint extension of S if and only if $\mathfrak{K}(S_0)$ is a positive self-adjoint extension of $\mathfrak{K}(S)$.*

Lemma 5.1. *Let $\mathscr{M}$ be a von-Neumann algebra acting on the Hilbert space $\mathcal{H}$ and $A \in \mathscr{M}^+, E \in \mathscr{M}_{\mathrm{proj}}$ be given. Then the set*

$$\{D \in \mathcal{B}(\mathcal{H})_{sa} : \mathbf{R}(D) \le E, D \le A\}$$

contains a largest operator $\mathcal{D}(A;E)$, which lies in $\mathscr{M}$ and is given by the following expression,

$$\mathcal{D}(A;E) = A^{\frac{1}{2}} \mathbf{N}\left((I-E)A^{\frac{1}{2}}\right) A^{\frac{1}{2}}. \tag{5.0.2}$$

Proof. Note that,

$$\begin{aligned} x \in \mathrm{null}((I-E)A^{\frac{1}{2}}) \iff (I-E)A^{\frac{1}{2}}x = 0 \iff E(A^{\frac{1}{2}}x) = A^{\frac{1}{2}}x \\ \iff A^{\frac{1}{2}}x \in \mathrm{ran}(E). \end{aligned}$$

Thus $\mathbf{N}\left((I-E)A^{\frac{1}{2}}\right)$ is the projection onto the set $\{x \in \mathcal{H} : A^{\frac{1}{2}}x \in \mathrm{ran}(E)\}$, and clearly it lies in $\mathscr{M}$ as $(I-E)A^{\frac{1}{2}} \in \mathscr{M}$. With this observation in mind, we just have to follow the line of argument in the (elementary) proof of the "Fundamental Lemma" in [Ska79, pg. 179] to reach our desired conclusion. ■

Let S be a closed positive (symmetric) operator affiliated with $\mathscr{M}$. When S is densely-defined, a positive self-adjoint extension of S always exists ([Sch12, Theorem 10.17]). But in general, this may not be true (see [Sch12, Example 13.2, Exercise 13.6.9(b)]). Nevertheless, from Proposition 5.2 and Lemma 5.2, it follows that there is a one-to-one correspondence between the set of positive self-adjoint extensions of S which are affiliated with $\mathscr{M}$, and the set of self-adjoint extensions of the Krein transform of S, $\mathfrak{K}(S)$, which lie in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$. Below we see how to find all such self-adjoint extensions of $\mathfrak{K}(S)$ using Lemma 5.1, starting with a particular example of such an extension.

Proposition 5.3. *Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and S be a positive (symmetric) operator in $\mathscr{M}_{\text{aff}}$. Let K be a self-adjoint extension of the Krein transform of S, $\mathfrak{K}(S)$, lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$, and F be the projection onto $\text{dom}(\mathfrak{K}(S))^{\perp}$. Let $\mathcal{D}(\cdot\,;\cdot)$ be as in Lemma 5.1.*

(i) Then $K - \mathcal{D}(I+K;F)$ is a self-adjoint extension of $\mathfrak{K}(S)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$. For every self-adjoint extension K' of $\mathfrak{K}(S)$, we have

$$K - \mathcal{D}(I+K;F) \leq K'.$$

In particular, for any two self-adjoint extensions K, K' of $\mathfrak{K}(S)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$, we have

$$K - \mathcal{D}(I+K;F) = K' - \mathcal{D}(I+K';F)(=: \mathfrak{K}(S)_{\min}). \tag{5.0.3}$$

(ii) In addition, if S is densely-defined, then $K + \mathcal{D}(I-K;F)$ is a self-adjoint extension of $\mathfrak{K}(S)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$. For any other self-adjoint extension K' of $\mathfrak{K}(S)$, we have

$$K' \leq K + \mathcal{D}(I-K;F).$$

Thus, for any two self-adjoint extensions K, K' of $\mathfrak{K}(S)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$, we have

$$K + \mathcal{D}(I-K;F) = K' + \mathcal{D}(I-K';F)(=: \mathfrak{K}(S)_{\max}). \tag{5.0.4}$$

Proof. Since $\text{dom}(\mathfrak{K}(T)) \in \text{Aff}_s(\mathscr{M})$, clearly $F \in \mathscr{M}$. For the sake of notational simplicity, we define,

$$K_m := K - \mathcal{D}(I+K;F),\ K_M := K + \mathcal{D}(I-K;F).$$

Since $K \in \mathscr{M} \cap \mathcal{F}(\mathcal{H})$, we have $\|K\| \leq 1$ so that $-I \leq K \leq I$. From Lemma 5.1, we know that

$$\mathcal{D}(I+K;F) \in \mathscr{M}^{+}, \qquad \mathcal{D}(I-K;F) \in \mathscr{M}^{+}, \tag{5.0.5}$$

$$\mathcal{D}(I+K;F) \leq I+K, \qquad \mathcal{D}(I-K;F) \leq I-K, \tag{5.0.6}$$

$$\mathbf{R}\big(\mathcal{D}(I+K;F)\big) \leq F, \qquad \mathbf{R}\big(\mathcal{D}(I-K;F)\big) \leq F. \tag{5.0.7}$$

Let K' be a self-adjoint extension of $\mathfrak{K}(T)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$ and $C := K' - K$, that is, $K' = K + C$. Then it is easy to see that $K' \in \mathcal{F}(\mathcal{H})$ if and only if C is self-adjoint, $\mathbf{R}(C) \leq F$ and $-(I+K) \leq C \leq I-K$. Hence applying Lemma 5.1, C satisfies

$$-\mathcal{D}(I+K,F) \leq C \leq \mathcal{D}(I-K,F).$$

Adding K to all terms of the inequality, we have

$$K_m \leq K' \leq K_M. \tag{5.0.8}$$

A key thing to verify is whether K_m, K_M lie in $\mathcal{F}(\mathcal{H})$.

Observation 1: K_m, K_M are self-adjoint extensions of $\mathfrak{K}(S)$ lying in $\mathscr{M}$.

Proof of Observation 1: Note that (5.0.7) is equivalent to,

$$I - F \leq \mathbf{N}\big(\mathcal{D}(I+K;F)\big),\ \ I - F \leq \mathbf{N}\big(\mathcal{D}(I-K;F)\big).$$

Since $\text{dom}\big(\mathfrak{K}(S)\big) \subseteq \text{ran}(I-F)$, for every $x \in \text{dom}\big(\mathfrak{K}(S)\big)$ we get,

$$\mathcal{D}(I+K;F)x = \mathcal{D}(I-K;F)x = 0,$$

whence $K_m = K_M = K$ on $\mathrm{dom}\big(\mathfrak{K}(S)\big)$.

Observation 2: $\|K_m\| \le 1, \|K_M\| \le 1$.

Proof of Observation 2: From (5.0.5) and (5.0.6), clearly

$$-I \le K_m \le K \le K_M \le I.$$

(i) Since $0 \le I - K \le I - K_m$, we have $\mathrm{null}(I - K_m) \subseteq \mathrm{null}(I - K) = \{0\}_{\mathcal{H}}$ so that $\mathrm{null}(I - K_m) = \{0\}_{\mathcal{H}}$. Together with Observations 1 and 2, this tells us that $K_m \in \mathscr{M} \cap \mathcal{F}(\mathcal{H})$. In other words, for every self-adjoint extension K of $\mathfrak{K}(S)$ lying in the $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$, the operator K_m is again such an extension of $\mathfrak{K}(T)$. From equation (5.0.8), we have

$$K_m \le K'.$$

In particular, the above inequality is also valid for $(K')_m$ so that $K_m \le (K')_m$ for every pair of self-adjoint extensions, K, K', of $\mathfrak{K}(S)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$. By symmetry, it is clear that

$$K_m = (K')_m.$$

(ii) If S is densely-defined, then $I - \mathfrak{K}(S)$ has dense range. Thus for any self-adjoint extension K of $\mathfrak{K}(S)$, $I - K$ has dense range or equivalently, trivial nullspace. In conjunction with Observations 1 and 2, we have $K_M \in \mathscr{M} \cap \mathcal{F}(\mathcal{H})$. By a similar argument as in part (i) above, it follows that $K_M = (K')_M$ for every pair of self-adjoint extensions, K, K', of $\mathfrak{K}(S)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$. ■

Remark 5.1. Let S be a densely-defined positive symmetric operator. Let K be the Krein transform of a positive self-adjoint extension of S (such an extension always exists as discussed above). Note that K is a positive self-adjoint extension of $\mathfrak{K}(S)$, and by Proposition 5.3, the extensions

$$\mathfrak{K}(S)_{\min} := K - \mathcal{D}(I + K; F), \quad \mathfrak{K}(S)_{\max} := K + \mathcal{D}(I - K; F), \tag{5.0.9}$$

are respectively the minimal and maximal self-adjoint extensions of $\mathfrak{K}(T)$ lying inside $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$. Although already noted in equations (5.0.3) and (5.0.4), we take a moment to emphasize that the expressions in (5.0.9) evaluate to the minimal and maximal extensions, respectively, *independent* of the choice of K.

Using the inverse Krein transform and Proposition 5.2, we conclude that S has a minimal and maximal self-adjoint extension, respectively given by,

$$\mathfrak{K}^{-1}\big(\mathfrak{K}(S)_{\min}\big), \quad \mathfrak{K}^{-1}\big(\mathfrak{K}(S)_{\max}\big).$$

In fact, as shown in [Ska79, §4], these extensions respectively recover the Krein-von Neumann and Friedrichs extensions of S, that is,

$$S^{\mathbf{Kr}} = \mathfrak{K}^{-1}(\mathfrak{K}(S)_{\min}), \quad S^{\mathbf{Fr}} = \mathfrak{K}^{-1}(\mathfrak{K}(S)_{\max}). \tag{5.0.10}$$

Let $(\mathscr{M}_{\mathrm{aff}}^{c})^{+}$ denote the set of closed positive operators affiliated with $\mathscr{M}$. It is now clear that if $S \in (\mathscr{M}_{\mathrm{aff}}^{c})^{+}$ and $B \in \mathscr{M}_{\mathrm{aff}}^{cb} \cap \mathcal{F}(\mathcal{H})$, then $\mathfrak{K}(S) \in \mathscr{M}_{\mathrm{aff}}^{cb} \cap \mathcal{F}(\mathcal{K})$ and $\mathfrak{K}^{-1}(B) \in (\mathscr{M}_{\mathrm{aff}}^{c})^{+}$.

Lemma 5.2. *Let $(\mathscr{M}; \mathcal{H}), (\mathscr{N}; \mathcal{K})$ be represented von Neumann algebras and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. Then $\Phi_{\mathrm{aff}} \circ \mathfrak{K} = \mathfrak{K} \circ \Phi_{\mathrm{aff}}$ on $(\mathscr{M}_{\mathrm{aff}}^{c})^{+}$, that is, the* *following diagram commutes:*

$$\begin{tikzcd}
(\mathscr{M}_{\text{aff}}^{c})^{+} \arrow[r, "\Phi_{\text{aff}}"] \arrow[d, "\mathfrak{K}"'] & (\mathscr{N}_{\text{aff}}^{c})^{+} \arrow[d, "\mathfrak{K}"] \\
\mathscr{M}_{\text{aff}}^{cb} \arrow[r, "\Phi_{\text{aff}}"'] & \mathscr{N}_{\text{aff}}^{cb}
\end{tikzcd}$$

Proof. For $S \in (\mathscr{M}_{\text{aff}}^{c})^{+}$, from equation (5.0.1) and Theorem 3.6 we have,

$$\begin{aligned}\Phi_{\text{aff}}\big(\mathfrak{K}(S)\big) &= \Phi_{\text{aff}}\left(I - 2(S+I)^{-1}\right) = I - 2\Phi_{\text{aff}}\left((S+I)^{-1}\right) \\ &= I - 2\left(\Phi_{\text{aff}}(S) + I\right)^{-1} = \mathfrak{K}\big(\Phi_{\text{aff}}(S)\big).\end{aligned}$$ ■

In the next theorem, we show that Φ_{aff} preserves $\mathfrak{K}(S)_{\min}$ and $\mathfrak{K}(S)_{\max}$ which yields our main result that Φ_{aff} maps $S^{\mathbf{Kr}}, S^{\mathbf{Fr}} \in \mathscr{M}_{\text{aff}}^{\text{MvN}}$ to their respective counterparts for $\Phi_{\text{aff}}(S)$.

Theorem 5.3. *Let $(\mathscr{M};\mathcal{H}), (\mathscr{N};\mathcal{K})$ be represented von Neumann algebras and $\Phi : \mathscr{M} \to \mathscr{N}$ be a unital normal $*$-homomorphism. Let S be a densely-defined closed positive operator in $\mathscr{M}_{\text{aff}}$. (Note that by Theorem 4.8-(i) and Theorem 4.14, $\Phi_{\text{aff}}(S)$ is a densely-defined closed positive operator in $\mathscr{N}_{\text{aff}}$.) Then Φ_{aff} maps the Krein-von Neumann extension (the Friedrichs extension, respectively) of S to the Krein-von Neumann extension (the Friedrichs extension, respectively) of $\Phi_{\text{aff}}(S)$, that is,*

$$\Phi_{\text{aff}}(S^{\mathbf{Kr}}) = \Phi_{\text{aff}}(S)^{\mathbf{Kr}} \text{ and } \Phi_{\text{aff}}(S^{\mathbf{Fr}}) = \Phi_{\text{aff}}(S)^{\mathbf{Fr}}.$$

Proof. Let $\mathcal{D}(\cdot\,;\cdot)$ be as in Lemma 5.1. For $A \in \mathscr{M}^{+}, E \in \mathscr{M}_{\text{proj}}$, it is clear from Lemma 1.10 and equation (5.0.2) that,

$$\Phi\big(\mathcal{D}(A;E)\big) = \mathcal{D}\big(\Phi(A);\Phi(E)\big). \tag{5.0.11}$$

Let K be a bounded self-adjoint extension of $\mathfrak{K}(S)$ lying in $\mathscr{M} \cap \mathcal{F}(\mathcal{H})$; (The Krein transform of the Friedrichs extension of S is one such extension.)

Observation 1: $\Phi(K)$ is a bounded self-adjoint extension of $\mathfrak{K}\big(\Phi_{\text{aff}}(S)\big)$ lying in $\mathscr{N} \cap \mathcal{F}(\mathcal{K})$.

Proof of Observation 1: From Theorem 3.8, we observe that $\Phi(K)$ is a bounded self-adjoint extension of $\Phi\big(\mathfrak{K}(S)\big) = \mathfrak{K}\big(\Phi_{\text{aff}}(S)\big)$. Since $K \in \mathcal{F}(\mathcal{H})$, note that $\|K\| \le 1$ and $\text{null}(I-K) = \{0\}_{\mathcal{H}}$. Thus $\|\Phi(K)\| \le 1$ and using Lemma 1.10, we get $\text{null}\big(I - \Phi(K)\big) = \{0\}_{\mathcal{K}}$, so that $\Phi(K) \in \mathcal{F}(\mathcal{K})$.

Observation 2: $\Phi\big(\mathfrak{K}(S)_{\min}\big) = \Big(\mathfrak{K}\big(\Phi_{\text{aff}}(S)\big)\Big)_{\min}$, $\Phi\big(\mathfrak{K}(S)_{\max}\big) = \Big(\mathfrak{K}\big(\Phi_{\text{aff}}(S)\big)\Big)_{\max}$.

Proof of Observation 2: Using Observation 1 in the context of Remark 5.1, we note that,

$$\begin{aligned}\Big(\mathfrak{K}\big(\Phi_{\text{aff}}(S)\big)\Big)_{\min} &\overset{(5.0.9)}{=} \Phi(K) - \mathcal{D}\big(I + \Phi(K);\Phi(F)\big) \\ &\overset{(5.0.11)}{=} \Phi\big(K - \mathcal{D}\left(I + K;F\right)\big) \\ &\overset{(5.0.9)}{=} \Phi\big(\mathfrak{K}(S)_{\min}\big).\end{aligned}$$

A similar approach gives us, $\Phi\big(\mathfrak{K}(S)_{\max}\big) = \Big(\mathfrak{K}\big(\Phi_{\text{aff}}(S)\big)\Big)_{\max}$.

Thus we have,

$$\begin{aligned}
\mathfrak{K}\left(\Phi_{\text{aff}}(S)^{\mathbf{Kr}}\right) &= \Big(\mathfrak{K}\big(\Phi_{\text{aff}}(S)\big)\Big)_{\text{min}}, \text{ by (5.0.10)} \\
&= \Phi\left(\mathfrak{K}(S)_{\text{min}}\right), \text{ by Observation 2} \\
&= \Phi\big(\mathfrak{K}(S^{\mathbf{Kr}})\big), \text{ by (5.0.10)} \\
&= \mathfrak{K}\big(\Phi_{\text{aff}}(S^{\mathbf{Kr}})\big), \text{ by Lemma 5.2.}
\end{aligned}$$

An application of the inverse Krein transform on both sides leads to the desired conclusion,

$$\Phi_{\text{aff}}(S)^{\mathbf{Kr}} = \Phi_{\text{aff}}(S)^{\mathbf{Kr}}.$$

The proof for $S^{\mathbf{Fr}}$ is similar. ■

Chapter 6

An Application of Functoriality

Experience shows that the domains of unbounded operators can behave in mysterious ways, which is a common source of frustration for the uninitiated. The literature abounds in examples and counterexamples discussing these peculiarities. The literature is equally sparse regarding analogous results in the context of affiliated operators. In this chapter, we hope to remedy this.

Let $\mathscr{M}$ be a properly infinite von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Consider the following three questions.

Question 1. Is there a positive self-adjoint operator T in $\mathscr{M}_{\text{aff}}^{c}$ and a unitary operator $U \in \mathscr{M}$ such that $\text{dom}(T) \cap \text{dom}(UTU^*) = \{0\}_{\mathcal{H}}$?

Question 2. Is there a densely-defined closed symmetric operator A in $\mathscr{M}_{\text{aff}}^{c}$ such that $\text{dom}(A^2) = \{0\}_{\mathcal{H}}$?

Question 3. Are there non-singular positive self-adjoint operators $A, B \in \mathscr{M}_{\text{aff}}^{c}$ satisfying $\text{dom}(A) \cap \text{dom}(B) = \text{dom}(A^{-1}) \cap \text{dom}(B^{-1}) = \{0\}_{\mathcal{H}}$?

Since the commutant of $\mathcal{B}(\mathcal{H})$ is trivial, recall that every closed operator on $\mathcal{H}$ is affiliated with $\mathcal{B}(\mathcal{H})$, that is, $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})_{\text{aff}}^{c}$. Specialists in the theory of unbounded operators will immediately recognize the inspiration behind the above-mentioned questions. We now elaborate on the classical results that motivate each of them.

In [Neu29], von Neumann proved that for any densely defined self-adjoint operator T on a Hilbert space $\mathcal{H}$, there exists a unitary operator U such that

$$\text{dom}(T) \cap \text{dom}(UTU^*) = \{0\}.$$

This result illustrates the extreme behavior possible with domains under unitary conjugation and motivates Question 1 in the setting of affiliated operators.

The second problem has been studied by Neumark (see [Neu40b]), with stronger versions explored by Schmüdgen (see [Sch83]), and an explicit example also given by Chernoff in [Che83]. Here we describe Chernoff's example.

Let $\mathcal{H} := L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Define the function Ω by

$$\Omega(t) := \begin{cases} \exp\left(ie^{-\frac{1}{t}}\right) & \text{if } 0 < t < \pi, \\ -1 & \text{if } \pi \le t \le 2\pi. \end{cases}$$

Then $\Omega \in L^\infty(\mathbb{T})$ has unit modulus, so the multiplication operator M_Ω on $L^2(\mathbb{T})$ is an isometry. Let $\mathcal{V} := H^2(\mathbb{T})$ denote the Hardy space, which is a norm-closed subspace of $L^2(\mathbb{T})$ (see [Mur90, §3.5]). Define the operator S on $L^2(\mathbb{T})$ by $\text{dom}(S) := \mathcal{V}$ and $Sx := M_\Omega x$ for

$x \in \mathcal{V}$. Chernoff showed that $\operatorname{ran}(S - I_{\mathcal{H}}) = (\Omega - 1)H^2(\mathbb{T})$ is dense in $L^2(\mathbb{T})$. Define the Cayley transform

$$A := i(S + I_{\mathcal{H}})(S - I_{\mathcal{H}})^{-1}.$$

Then A is a densely-defined closed symmetric operator on $L^2(\mathbb{T})$, and Chernoff showed that

$$\operatorname{dom}(A^2) = \{0\},$$

thus providing an affirmative answer to Question 2 in the classical setting.

The third question is motivated by a construction due to Kosaki in [Kos06], which builds on an idea originally from Chernoff. Consider the strongly continuous one-parameter unitary group $\{T_t\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{R})$ given by

$$(T_t\psi)(x) := \psi(x+t), \quad \psi \in L^2(\mathbb{R}),\ x \in \mathbb{R}.$$

By Stone's theorem (see [KR97, Theorem 5.6.36]), there exists a self-adjoint operator H such that $T_t = \exp(itH)$ for all $t \in \mathbb{R}$. Define the positive self-adjoint operator

$$A := \exp(-H),$$

which is also invertible. Next, define the multiplication function ϕ by

$$\phi(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

and let M_ϕ be the corresponding multiplication operator on $L^2(\mathbb{R})$. Define

$$B := M_\phi A M_\phi.$$

Kosaki showed that A and B are positive self-adjoint and invertible, with

$$\operatorname{dom}(A) \cap \operatorname{dom}(B) = \operatorname{dom}(A^{-1}) \cap \operatorname{dom}(B^{-1}) = \{0\}_{L^2(\mathbb{R})},$$

thereby providing an affirmative answer to Question 3 for $\mathscr{M} = \mathcal{B}\left(L^2(\mathbb{R})\right)$.

Using the functoriality of the near-semiring of affiliated operators, we provide a strategy for answering questions of a similar nature that may occur during the day-to-day musings of the working operator algebraist. The main ingredient is the following well-known proposition.

Proposition 6.1 (see [Tak02, Proposition V.1.22])**.** *Let $\mathscr{M}$ be a properly infinite von Neumann algebra, and $\mathcal{H}$ be a separable Hilbert space. Then there is a unital normal embedding of $\mathcal{B}(\mathcal{H})$ into $\mathscr{M}$.*

Proof. Without loss of generality, we may assume that $\mathcal{H} = \ell^2(\mathbb{N})$. Using the so-called halving lemma for properly infinite projections ([KR86, Lemma 6.3.3]), it can be shown that that there exists a sequence of mutually orthogonal projections $\{E_n\}_{n=1}^\infty$ such that $E_n \sim E_m$ for all n, m and $\sum_n E_n = I$; the details may be found in [KR86, Theorem 6.3.4]. This yields a countable system of matrix units in $\mathscr{M}$, and the assertion follows. ∎

Proposition 6.1 helps us in viewing properly infinite von Neumann algebras as a larger universe containing $\mathcal{B}(\mathcal{H})$. Since a unital normal embedding of $\mathcal{B}(\mathcal{H})$ into $\mathscr{M}$ lifts to a canonical embedding of $\mathcal{C}(\mathcal{H})$ into $\mathscr{M}_{\text{aff}}$, note that $\mathscr{M}_{\text{aff}}$ provides a larger mathematical universe where one can do physics. For example, pseudo-differential operators in $\mathcal{C}\big(H^k(\mathbb{R})\big)$ (with $H^k(\mathbb{R})$ a Sobolev space) and differential equations may be transported to the setting of $\mathscr{M}_{\text{aff}}$; the Heisenberg commutation relation can find a home in $\mathscr{M}_{\text{aff}}$, that is, there are closed operators

$P, Q \in \mathscr{M}_{\rm aff}^{\rm MvN}$ such that $PQ - QP$ is pre-closed and densely-defined with closure I. Although what is to be gained from such an exercise is a matter of speculation at this moment, some evidence in its favour may be gleaned from an application of the continuous dimension theory of type II_∞ von Neumann algebras to finite difference equations given by Schaeffer in [Sch72].

As a proof of concept, we provide an affirmative answer to Question 3 and leave the first two questions to the reader.

Theorem 6.1. *Let $\mathscr{M}$ be a properly infinite von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Then there is a pair of non-singular positive self-adjoint operators A, B affiliated with $\mathscr{M}$ with their domains satisfying,*

$$\mathrm{dom}(A) \ \cap \ \mathrm{dom}(B) = \mathrm{dom}(A^{-1}) \ \cap \ \mathrm{dom}(B^{-1}) = \{0\}_{\mathcal{H}}.$$

Proof. Noting that $L^2(\mathbb{R}) \cong \ell^2(\mathbb{N})$ as Hilbert spaces, by [Kos06, Proposition 13], there is a pair of non-singular positive self-adjoint operators A', B' on $\ell^2(\mathbb{N})$ satisfying the above-mentioned domain conditions, that is,

$$\mathrm{dom}(A') \ \cap \ \mathrm{dom}(B') = \mathrm{dom}(A'^{-1}) \ \cap \ \mathrm{dom}(B'^{-1}) = \{0\}_{\ell^2(\mathbb{N})}.$$

Let $\Phi : \mathcal{B}(\ell^2(\mathbb{N})) \to \mathscr{M}$ be a unital normal embedding (say, as in Proposition 6.1), and $\Phi_{\rm aff} : \mathcal{B}(\ell^2(\mathbb{N}))_{\rm aff}^c \to \mathscr{M}_{\rm aff}^c$ be the canonical extension. From Theorem 3.6-(iii), note that $\Phi_{\rm aff}(A'), \Phi_{\rm aff}(B')$ are non-singular, and

$$\Phi_{\rm aff}(A'^{-1}) = \Phi_{\rm aff}(A')^{-1}, \ \Phi_{\rm aff}(B'^{-1}) = \Phi_{\rm aff}(B')^{-1}.$$

By Theorem 4.14-(ii)-(v) and Theorem 4.8-(iii), we observe that $\Phi_{\rm aff}(A), \Phi_{\rm aff}(B)$ are positive self-adjoint operators in $\mathscr{M}_{\rm aff}^c$. From Theorem 4.8-(i) and Theorem 2.7-(i), we note that,

$$\begin{aligned} \mathrm{dom}\big(\Phi_{\rm aff}(A')\big) \cap \mathrm{dom}\big(\Phi_{\rm aff}(B')\big) &= \Phi_{\rm aff}^s\big(\mathrm{dom}(A') \cap \mathrm{dom}(B')\big) \\ &= \Phi_{\rm aff}^s(\{0\}_{\ell^2(\mathbb{N})}) = \{0\}_{\mathcal{H}}, \\ \mathrm{dom}\big(\Phi_{\rm aff}(A')^{-1}\big) \cap \mathrm{dom}\big(\Phi_{\rm aff}(B')^{-1}\big) &= \Phi_{\rm aff}^s\big(\mathrm{dom}(A'^{-1}) \cap \mathrm{dom}\big(B'^{-1}\big)\big) \\ &= \Phi_{\rm aff}^s(\{0\}_{\ell^2(\mathbb{N})}) = \{0\}_{\mathcal{H}}. \end{aligned}$$

Thus, $A = \Phi_{\rm aff}(A'), B = \Phi_{\rm aff}(B')$ satisfy the conditions stipulated in the assertion. ■

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