INDIAN STATISTICAL INSTITUTE

Mid Semester Examination

M.Tech CS, 2025 – 2026 (Semester – I)

Discrete Mathematics

Date: 9 September 2025 | Maximum Marks: 60 | Duration: 2 hours

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General comment. Answer as much as you can, but the maximum you can score from both Group-A and Group-B is 30.

Notations and definitions. $\mathbb{N}$ denotes the set of natural numbers. Given any set X, 2^X denotes the collection of all subsets of X.

Let P be a subset of $\mathbb{R}^d$, convex hull $\text{Conv}(P)$ denotes the smallest convex set in $\mathbb{R}^d$ containing P.

For all $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

A permutation $\pi : [n] \to [n]$ is a derangement if for all $i \in [n]$ we have $\pi(i) \neq i$.

Group-A

(AQ1) Let P be a finite subset of $\mathbb{R}^d$ with $|P| \geq d + 1$. Show that every point in the convex hull $\text{Conv}(P)$ can be written as a convex combination of at most $d + 1$ points of P. [10]

(AQ2) Prove that every finite poset is the intersection of finitely many of its linear extensions. [10]

(AQ3) Determine the number of ways to distribute n distinct balls into m identical bins. [10]

(AQ4) Let $\mathcal{S} \subseteq 2^{[n]}$. Suppose that for all distinct sets $A, B \in \mathcal{S}$ we have neither $A \subseteq B$ nor $B \subseteq A$. Show that

$$|\mathcal{S}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

[10]

Group-B

(BQ1) (a) A set C of propositions is called logically closed if it satisfies both of the following conditions:

(i) C contains all tautologies (propositions whose truth tables have only T in the output column).

(ii) For any propositions p and q, if both p and $p \implies q$ belong to C, then so does q.

Prove that if C is logically closed and both p and $\neg p$ are in C, then C contains every proposition.

(b) Negate the following mathematical statement:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \in \mathbb{R}, \; \big(|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \epsilon\big).$$

[6+4=10]

(BQ2) Let $P_1, \ldots, P_n \subset \mathbb{R}^2$ be rectangles with sides parallel to the coordinate axes, such that every two rectangles in the collection intersect. Show that all n rectangles have a common intersection, i.e.,

$$\bigcap_{i=1}^{n} P_i \neq \emptyset.$$

[10]

(BQ3) Prove the following identity using a combinatorial argument:

$$\sum_{s=k}^{\lfloor n/2 \rfloor} \binom{n+1}{2s+1} \binom{s}{k} = \binom{n-k}{k} 2^{n-2k}.$$

[10]

(BQ4) (a) Show that if $k \log k = \Theta(n)$, then $k = \Theta\left(\frac{n}{\log n}\right)$.

(b) Using the substitution method, solve the recurrence

$$f(n) = 4f(n/2) + Cn.$$

[4+6=10]