

Note: This is a **closed notes/closed book** examination. Notations, if not explicitly explained, are to be interpreted as defined in class. For the branching process problems, you may use without proof the fact that the extinction probability is the smallest solution of the equation $\varphi(z) = z$ in the interval $[0, 1]$, where φ is the probability generating function of the offspring distribution.

Unless stated otherwise, assume that all the random variables mentioned in the questions below are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. Let X_1, X_2, \dots be independent random variables with $\mathbb{E}[\exp(X_i)] < \infty$ for each i . For $n \geq 1$, let $M_n := \exp\left(\sum_{i=1}^n (X_i - \log \mathbb{E} \exp(X_i))\right)$.

(a) Prove that M_n converges a.s. to some random variable M .

(b) Prove that M_n converges to M in L^1 if and only if $\mathbb{E}[M] = 1$. 2 + 3

2. Let Z_n denote the size of the n -th generation in a branching process with offspring distribution $(p_k)_{k \geq 0}$. Suppose that $Z_0 = 1$, $p_1 \neq 1$ and $\sum_{k \geq 0} k p_k \leq 1$. Prove that Z_n is a supermartingale, and from this deduce that $\mathbb{P}(Z_n = 0 \text{ eventually}) = 1$. 5

3. In a shop there are two items for sale, call them item A and item B. Item A is more popular than item B in the sense that a typical customer chooses to buy item A over item B with probability $3/4$. Suppose at each round a customer arrives and buys one of the items. You may assume that the customers behave independently of one another. By setting up an appropriate martingale, compute the probability that item B outsells item A at some point. 15

4. Let $\Gamma = \{0, 1\}^{\mathbb{N}}$ denote the space of all binary sequences. For $j \geq 1$, let $\pi_j : \Gamma \rightarrow \{0, 1\}$ denote the j -th coordinate projection map, i.e. $\pi_j(x_1, x_2, \dots, x_j, \dots) = x_j$. Let $\mathcal{G}_n = \sigma(\pi_1, \dots, \pi_n)$ and set $\mathcal{G}_\infty = \sigma(\cup_{n \geq 1} \mathcal{G}_n)$ (these are defined with respect to the discrete σ -algebra on $\{0, 1\}$). Let P denote the product Bernoulli measure on $(\Gamma, \mathcal{G}_\infty)$ with probability $1/2$ (i.e. the distribution of an infinite sequence of i.i.d. fair coin tosses). Also, let Q denote the product Bernoulli measure on $(\Gamma, \mathcal{G}_\infty)$ with probability $1/3$.

(a) Show that Q is absolute continuous with respect to P in the measurable space (Γ, \mathcal{G}_n) and compute the Radon-Nikodym derivative $\left(\frac{dQ}{dP}\right)_{\mathcal{G}_n}$ explicitly.

(b) Show that $\left(\frac{dQ}{dP}\right)_{\mathcal{G}_n}$ is a martingale in the filtered probability space $(\Gamma, (\mathcal{G}_n)_{n \geq 1}, P)$ that converges almost surely. Does it converge in L^1 ? Is Q absolutely continuous with respect to P on $(\Gamma, \mathcal{G}_\infty)$? 4 + (3 + 1 + 2)

5. Let Z_n, Z'_n denote the population size processes corresponding to two independent branching processes with identical offspring distribution $(p_k)_{k \geq 0}$ and arbitrary, possibly random, Z_0, Z'_0 with $Z_0 + Z'_0 \geq 1$ a.s. Let

$$Y_n := \begin{cases} \frac{Z_n}{Z_n + Z'_n} & \text{if } Z_n + Z'_n \neq 0; \\ Y_{n-1} & \text{if } Z_n + Z'_n = 0. \end{cases}$$

Prove that Y_n is a martingale with respect to its natural filtration. Does Y_n converge almost surely? Does it converge in L^1 ? Suppose that $p_0 > 0, p_1 < 1$, and $\sum_{k \geq 0} k p_k > 1$.

Prove that there is a non-zero chance that $Y_n \rightarrow 1$. 10 + 1 + 2 + 2