Essays in
Individual and Collective Choice

A dissertation submitted by

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Abstracts

Chapter 1

We consider a social choice model where voters have single-peaked preferences over a finite and ordered set of alternatives that are aggregated to produce contiguous sets or *intervals* of a fixed cardinality. This is applicable in situations where the alternatives can be arranged in a line (e.g. plots of land) and a contiguous subset of these is required (e.g. a hospital or a school). We define interval-social choice correspondences (I-SCCs) on profiles of single-peaked preferences which select intervals. We extend single-peaked preferences to *intervals* using *responsiveness*. We show that *generalized median-interval (GMI)* rules are the only *strategy-proof, anonymous* and *interval efficient* I-SCCs. These rules are interval versions of generalized median voter rules which consist of the median, min and max rules. We show that responsiveness over intervals is necessary for the strategy-proofness of the GMI rule if preferences over alternatives are single-peaked.

Chapter 2

We consider the problem of allocating a perfectly divisible heterogeneous good (e.g. land or advertisement slots) where agents have a preference for location and quantity. We introduce a new preference domain in which preferences are single-peaked in quantity, i.e., *semi-single-peaked* which can be represented by continuous indifference curves (ICs). We first provide results for two agents. We prove the existence of envy-free allocations using a fixed point argument. We show that an allocation rule is *envy-free* and Pareto efficient allocations if and only if it is the one produced by the Balanced-Curve Allocation (BCA) rule. We show that there is no *strategy-proof, envy-free* and Pareto efficient allocation rule. We provide some insights to obtain an *envy-free* and *Pareto efficient* allocation for any n number of agents when preferences are *q-constant* and *monotonic*.

Chapter 3

We consider a decision maker (DM) who has to choose a pair of 'complementary' goods, one from each of two sets, A and B. We propose a heuristic in which the DM chooses as if for each alternative in A she has a (complementarity) order over alternatives of B and vice-versa. The choices made by the DM are represented by joint choice functions. The proposed heuristic encapsulates a minimal notion of complementarity. We call a joint choice function satisfying this heuristic a weak-complements choice function (weak-CCF). We characterize the family of weak-CCFs using three testable axioms. Further, we look at a stronger notion of complementarity, represented by a sub-class of weak-CCFs called strong-complements choice functions (strong-CCFs). A strong-CCF may not exist in the universal domain of (complementarity) orders. We show that the 'single-crossing' property on the (complementarity) orders is sufficient for the existence of strong-CCFs.

To,
Mummy, Papa and Osheen

22/07/2025

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Introduction

This thesis consists of three chapters that address problems in social choice theory, fair division of a heterogeneous good and choosing a pair of complementary goods, respectively.

The first chapter deals with designing a voting mechanism. There are a set of finite alternatives arranged according to an exogenous order. The output of the mechanism must be a (fixed-cardinality) set of contiguous alternatives, that are referred to as intervals. We find a mechanism that is Pareto efficient and strategy-proof. In the second chapter we study the division of a heterogeneous resource. Each agent must be allocated a continuous (interval) subset of the interval [0,1]. The agents have preferences that are single-peaked in one dimension (quantity) but not in another (location). We characterize the full set of Pareto efficient and envy-free (i.e. fair) allocations. We also show that there is no rule that is strategy-proof, envy-free and Pareto efficient. In the third chapter we propose a heuristic on how a decision-maker (DM) might choose a pair of complementary goods, one alternative each from two sets. The modeling of complementarity is done without relying on prices or utility functions. The choices of the DM are represented by joint choice functions. We first define the concepts of weak and strong-complements. Further we characterize what we call weak-complements choice functions and then provide necessary and sufficient conditions for the existence of strong-complements choice function.

Chapter 1

In chapter 1 we consider a social choice setting in which the alternatives are arranged according to an exogenous order. The objective is to select a contiguous set of alternatives of a fixed cardinality, L. We consider aggregators or social choice correspondences which only pick intervals of cardinality L, henceforth referred to as interval-social choice correspondences (I-SCCs). The voters have single-peaked preferences over the alternatives. Preferences are said to be single-peaked when there is a 'best' alternative and alternatives nearer to the best alternative (w.r.t. the exogenous order) are at least as good as ones that are further away. Single-peakedness is a standard domain restriction when the policy space is ordered or one-dimensional as shown in Hotelling (1929) and Downs (1957).

There are many scenarios where such a model could be useful for analysis, for example (i) selecting L adjacent plots of land for a public project or (ii) choosing members for a committee. Since the outputs of I-SCCs are 'L-intervals', we extend the preferences of voters on alternatives to intervals using *responsiveness over intervals*. Responsiveness over intervals implies that if two intervals differ by just one alternative, the one with the better alternative is preferred over the other. Two preliminary results in the chapter are that (i) the top L alternatives of any single-peaked preferences form an interval and (ii) under single-peaked preferences over alternatives, single-peakedness over intervals and responsiveness over intervals are equivalent.[1]

There are three demands from the outcomes of the I-SCCs that we hope to discover. First that the outcomes are efficient (interval-efficient, in our setting) which is the translation of standard Pareto efficiency in our setting. And second, that the outcomes are non-manipulable, meaning that no voter should be able to misreport their true preferences to achieve a better outcome for himself. Finally I-SCCs must be anonymous.

The resulting class of rules, called the generalized median interval (GMI) rules are inspired by the rules characterized by Moulin (1980a). We show using an example that arranging preference orderings according to the top alternatives, as in the median-voter rule, does not work. The rule requires that voters be arranged not according to their top alternatives but according to the left end-point of the top-L interval.[2] The definition of the general rule requires having $n-1$ fixed intervals for n voters. The rule picks the median of the so arranged top-L intervals. The axioms are defined on

[1] Top-L intervals refers to the set of top L alternatives according to the preferences on alternatives arranged according to an exogenous order.

[2] or any other fixed k-th ranked alternative in the top interval.

preferences over alternatives that are the primitives and not on induced preferences over intervals and hence, voters have a larger number of possible manipulations. Therefore, the result does not directly follow from Moulin (1980a). The GMI rules also then coincide with Moulin's median rule when L = 1, i.e. when only one alternative is chosen. The GMI rules also include the min and max rules, which select the first or last top interval (when arranged as described above). We show that any strategy-proof and interval efficient rule requires only the information of top-L interval of the voters. This result is used to characterize the GMI rules.

Finally we show the necessity of responsiveness for strategy-proofness of GMI rules. We show that any rule, that admits preferences that result in preferences over intervals that are not single-peaked, is manipulable. Additionally, we provide an example of single-peaked preference over intervals that is induced by preferences over alternatives that are not single-peaked, highlighting the argument that responsiveness over intervals is a fairly weak assumption over intervals of length L. This work has been published in the International Journal of Game Theory (Bhattacharya and Khare (2024)).

Chapter 2

In chapter 2, we consider the fair division of a heterogeneous, perfectly divisible resource distributed along the interval $[0, 1]$. Each agent must receive a (connected) sub-interval of this resource. As the good is heterogeneous, the agents not only care about the quantity (length of the interval) received but also 'where' the interval is located. Each interval is described by the starting point $x \in [0, 1]$ (location) and the length, $q \in (0, 1 - x]$ (quantity).[3] This then becomes a two-dimensional setup. Any 'feasible' bundle (x, q) is located within the region of a right-angled isosceles triangle with edges, $x = 0, q = 0$ and $x + q = 1$. There are n agents, and a feasible allocation is such that for each $i \in N$, $0 < q_i$, $x_i + q_i \leq 1$ and $\sum_i q_i \leq 1$. We allow for free disposal of the resource. The fairness criteria in our setup requires that

[3] We assume that each agent receives a strictly positive quantity.

the allocation is Pareto efficient and envy-free (Varian (1973)). Most of the work in this literature either assumes homogeneous resource (notably, Sprumont (1991)) or a heterogeneous resource (cake-cutting literature) where the preference for quantity is monotonic.[4]

The model captures scenarios where an agent's optimal amount of resource depends on where that resource is situated. For instance, while looking for shops the agent may prefer a smaller plot in a mall over a larger one in a remote location. Similarly, for advertising time slots, an advertiser might prefer a shorter slot during prime hours over a larger one say during the night. Single-peakedness here may capture the inadequacy of smaller intervals than ideal interval and higher taxes or production cost for a larger than ideal interval.

We introduce a new domain of preferences that we call 'semi-single-peaked' where the preferences of agents for quantity at any location is single-peaked. This means that given a location there is a 'peak' (ideal) quantity and as the quantity decreases or increases around the peak, the agents prefer the bundles less. We include standard assumptions on preferences being complete, transitive, and continuous, but the crucial one is the existence of indifference curves functions (ICFs) so that each ICF intersects the $x = 0$ and $x + q = 1$ line exactly once. Finally the ICFs are arranged in a single-peaked manner i.e. there is a top indifference curve, and as we move away from it, the bundles are preferred lesser and lesser. This includes the case where the agent's preferences are monotonic i.e. the best point is $(0, 1)$. We consider 2 agents for most of the analysis.

The most important implication of the semi-single-peaked preferences, Proposition 3, is that for each preference in the domain there exists a *partition* of the interval in two 'indifferent parts' i.e. two (sub)intervals that lie on the same IC. The ICF on which they lie is called the balanced IC. If both agents have identical preferences and

[4] Thomson (2011) and Thomson (2016) provide excellent surveys of envy-free allocations, while the cake cutting literature include works like Procaccia (2016), Brams et al. (2013), Chen et al. (2013), Aumann and Dombb (2015) etc.

prefer the whole resource the most, then the only allocation that is Pareto efficient and envy-free is on the balanced IC. If the agents' preferences are not identical then there will be a balanced IC corresponding to each agent's preference and they will enclose what we call the 'balanced region'.

In Theorem 1, we use the balanced region to characterize the full set of envy-free and Pareto efficient allocations. If both agents can be allocated on their respective top curves then we allocate them bundles on their respective top IC. If they both demand 'high-enough' quantities then only the allocations within the balanced region are envy-free and Pareto efficient. Further, we provide the exact descriptions of the full set of fair allocations. The set of fair allocations is not single-valued, so we look for allocations rules that are also strategy-proof. But we find that there are no allocation rules that are strategy-proof, Pareto efficient and envy-free.

Finally we provide some results for $n > 2$ agents where the agents have constant slope ICFs. In summary, Chapter 2 expands the fair allocation literature to a two-dimensional setting by proposing a new preference domain (semi-single-peaked preferences) and fully characterizing the fair outcomes in that domain for two agents, along with an existence proof (via a fixed-point argument on balanced indifference curves) for the general n-agent case.

Chapter 3

In chapter 3 we study the choices of a decision-maker (DM) who chooses one alternative each from two (finite) sets, say X and Y. We represent the choices of the DM using joint choice functions as in Chambers et al. (2021) but in a discrete setting. The menus are $(A, B) \subseteq (X, Y)$ and $c(A, B) \in (A, B)$.

Standard consumer theory defines complementarity via cross-price effects – two goods are complements if the demand for one increases when the price of the other falls (negative cross-elasticity). Here the setting requires the alternatives to have prices and a well defined utility function. Manzini et al. (2019) considers the complementarity of (free-entry) public goods such as parks and museums and when it can be said that these goods are complementary. Chambers et al. (2021), Kashaev et al. (2024) etc. look at separable and entangled choice functions in a stochastic setting.[5]

The setup we construct can be used to describe many situations such as (i) choosing a public project from a list of public projects- park, parking, community center etc. and a list of locations (ii) a manager choosing a two person team consisting of a hardware and a software engineer that best complement each other from a pool of engineers etc.

Consider the pairs $(a, b), (a', b)$ in a menu (A, B). We believe complementarity can be plausible modeled by asking the questions, "does good a complement good b more than a', i.e. $a \succ_b a'$ or $a' \succ_b a$?" The heuristic we propose, each good in a set has a (complementarity) order over alternatives in the other set. Both the sets A and B are dealt with symmetrically in the analysis.

For any menu (A, B), we define the concept of 'demand', where every good a demands what complements it the most from B according to $\succ_a$ and similarly each good b demands an $a' \in A$. We say (a, b) are strong-complements when a demands b in B and b demands a in A. We show that if the domain of (complementarity) orders is all strict linear orders then strong-complements may not exist in every menu. Hence, we define weak-complements. The pair (a, b) are said to be weak-complements if either (i) a demands b in B and b demands a from the set of alternatives that demand b in B as opposed to the entire set A or (ii) b demands a in A and a demands b from the set of alternatives that demand a in A as opposed to the entire set B. Note that all strong-complements are also weak-complements but not the other way round.

We say that a joint choice function is weak-complements choice function if there exists a profile of (complementarity) orders $\succ$ such that choices are weak-complements in each menu.[6] We characterize the family of weak-complements choice functions

[5] Further references include Gentzkow (2007), Csaba (2019), Brady and Rehbeck (2016), Aguiar (2015) etc.

[6] $\succ = \{\succ_a\}_{a \in A} \cup \{\succ_b\}_{b \in B}$

using three testable axioms.

After this we impose a 'monotonicity' condition on the domain of (complementarity) orders that is sufficient to guarantee existence of strong-complements in each menu. This part is inspired by the strategic complementarities literature (Vives (1990), Topkis (1978), Topkis (1979)). This part may be used to model situations where alternatives are arranged according to external exogenous orders. We then weaken the monotonicity condition to obtain a necessary and sufficient condition that ensure existence of strong-complements in each menu.

In summary, Chapter 3 develops a novel framework for analyzing complementary choices using joint choice functions and (complementarity) orders.

Chapter 1

Strategy-proof interval-social choice correspondences over extended single-peaked domains†

1.1 Introduction

There are many voting situations over a one-dimensional and discrete policy space where a contiguous set of alternatives of a fixed cardinality (say, L) needs to be chosen. We will call these contiguous sets of cardinality L, *intervals* or more specifically, L-*intervals.* Consider the following examples:

- Choosing L plots of land: Individuals have preferences over single plots of land or 'alternatives' which are ordered on a line. A public good like a hospital or school needs to be constructed which requires L number of contiguous *plots* or an L-interval. The fact that exactly L plots of land are required may be imposed by a central planner.
- Choosing a committee with L members: A committee needs to be selected from the set of 'candidates' which are located on a line, and a connected set of L candidates need to be chosen. The connectedness of candidates location-wise may be imposed to reduce costs of traveling, while the cardinality requirement may be predetermined by a mandate.

In this chapter, we assume that the cardinality of the interval to be chosen is given exogenously. We will be assume it to be fixed and equal to L throughout the chapter. We consider aggregators or social choice correspondences which only pick

†This chapter is joint work with Mihir Bhattacharya, Ashoka University, Sonipat. This work has been published in the International Journal of Game Theory, Bhattacharya and Khare (2024). We would like to thank Debasis Mishra for providing feedback at various stages of the project. We would like to thank the Associate Editor and two anonymous referees of IJGT for their detailed comments and suggestions, which have greatly helped improve the paper. We would like to thank Bhaskar Dutta, Nicolas Gravel and Arunava Sen for very helpful comments.

intervals of cardinality L (*interval-social choice correspondences (I-SCCs)*). In many cases, voters may have an incentive to lie about their preferences if they can obtain a better outcome. Therefore, it is imperative to design an aggregation rule which is immune to such unilateral manipulations or is *strategy-proof*. In this chapter, we study *strategy-proof* aggregation of 'extended' single-peaked preferences which pick L-intervals.

In our model, voters have single-peaked preferences over the set of alternatives (as defined in Black (1948) or in Arrow (2012)) which will be extended to L-intervals. It is natural to assume that preferences are single-peaked when the policy space is ordered or one-dimensional as shown in Hotelling (1929) and Downs (1957).[1] Since L-intervals are chosen, we will extend preferences of voters over alternatives to obtain preferences over L-intervals using *responsiveness over intervals*. Responsiveness over intervals requires that if an alternative a has been removed from an L-interval A and another alternative b has been added to create a new L-interval B, then interval A is preferred to interval B if and only if alternative a is preferred to alternative b. We provide an example.

Example 1. Suppose there are five alternatives $a_1 < a_2 < a_3 < a_4 < a_5$ and let $L = 3$. There are three intervals of cardinality 3 from left to right: $[a_1] = \{a_1, a_2, a_3\}$, $[a_2] = \{a_2, a_3, a_4\}$ and $[a_3] = \{a_3, a_4, a_5\}$ where we denote $[a_l]$ as the L-interval where a_l is the left-end point of the interval, for any $l \in \{1, 2, 3\}$.

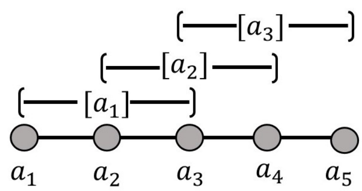


Figure 1: Intervals of cardinality 3 with 5 alternatives

We now illustrate responsiveness of preferences. Suppose a voter i has the following

[1]See Thomson (1997) and Amorós (2002) for applications of single-peaked preferences to public goods model.

single-peaked preference over alternatives, P_i: $a_2 P_i a_3 P_i a_1 P_i a_4 P_i a_5$. Therefore, peak of voter i is a_2, and preference is decreasing on either side of the peak. Responsiveness over intervals requires that interval $[a_1] = \{a_1, a_2, a_3\}$ is preferred to interval $[a_2] = \{a_2, a_3, a_4\}$ because a_1 is better than a_4 ($a_1 = [a_1] \setminus [a_2]$, $a_4 = [a_2] \setminus [a_1]$). Note that responsiveness does not put any restrictions on intervals which differ by more than two alternatives, for example, $[a_1]$ and $[a_3]$. The *lexicographic max* preference extension over intervals works as follows: the alternatives in each interval are listed in descending order of preference in $[a_1] : a_2 P_i a_3 P_i a_1$, and in $[a_3] : a_3 P_i a_4 P_i a_5$ and each corresponding ranked alternative is compared one by one till the lowest k-th ranked alternative in one sequence that is strictly better than its k-th ranked counterpart alternative in the other sequence (if all the alternatives ranked higher than the k-th ranked alternative in the first sequence are the same as the corresponding ranked alternative in the other sequence). In the above example, since the best alternative in $[a_1]$, i.e. a_2, is strictly better than the best ranked alternative in $[a_3]$, i.e. a_3, the lexicographic max preference extension would rank $[a_1]$ higher than $[a_3]$. Therefore, responsiveness over intervals is weaker than assuming *lexicographic max* preference extension over intervals.[2] We only use this property to compare adjacent intervals: two intervals are adjacent if their left end-points are adjacent (e.g. $[a_1]$ and $[a_2]$ above are adjacent). Various preference extensions used in the literature on social choice theory (Pattanaik and Peleg (1984), Bossert (1995), Bossert et al. (2000) and Sato (2008)) and matching theory (Roth (1985) and Alcalde and Barberà (1994)) satisfy responsiveness.[3]

This framework is applicable to settings where the voter preferences over alternatives can be used to make decisions on public goods over intervals whose cardinality is fixed throughout. Another advantage of this framework is that the cardinality of the interval may not be known *a priori*. Once the cardinality of the interval is known,

[2] One can check that all lexicographic max extensions are responsive over intervals but the converse is not true.

[3] See Barberà et al. (2004) for a survey on various preference extensions. All the extensions mentioned in Sato (2008) are *responsive* when restricted to intervals.

the aggregation of preferences over intervals can be done using the preferences over alternatives. Therefore, voters only need to report their preferences over alternatives. An I-SCC is *strategy-proof* if no voter can benefit by misreporting her preference over the set of alternatives for *any* responsive preference extension.

The classic works on *strategy-proof social choice functions*, Gibbard (1973) and Satterthwaite (1975), show that the only rules which are *strategy-proof* on the unrestricted domain with more than three alternatives are dictatorial rules. Similar results have been shown for SCCs in many works (see Gärdenfors (1976), Barberà (1977), Kelly (1977), Sato (2008), and Özyurt and Sanver (2009)). Barberà et al. (2001) and Ching and Zhou (2002) provide similar results for *strategy-proof* mechanisms in a cardinal setting, Schummer and Vohra (2002) for trees, and Border and Jordan (1983) for the n-dimensional Euclidean space. When preferences are single-peaked, Moulin (1980a) showed that *generalized median voter rules* are the only *strategy-proof*, *anonymous* and *Pareto optimal* social choice functions.[4] Our results generalize the results of the latter to the setting with L-intervals.

There are works which study similar social choice problems on a set of alternatives that is a subset of Euclidean space. Klaus and Protopapas (2020a) considers a model where the set of alternatives is $[0, 1]$. In their model, the preferences are single-peaked and based on absolute distance from peak. The extension to sets uses comparisons of the best and worst elements. A similar extension is used in Klaus and Protopapas (2020b) to characterize target-set correspondences. Klaus and Storcken (2002) studies a multidimensional model where the preferences over alternatives are single-peaked with a best point and *separable-quadratic* with respect to distance from the best point. However, all these papers study social choice over a connected subset of a Euclidean space. In our chapter, we consider the set of discrete and finite set of alternatives.

Another paper which studies interval social choice in a discrete domain is Cara-

[4]Barberà et al. (1993) provides a generalization of this result to the n-dimensional ‘box’ space.

muta (2010) where they consider two types of preferences: separable and additive. The result is an extension of Barberà et al. (1991)'s. They obtain a negative result (dictatorial SCF) with additivity, and a positive result with separability, where they characterize the interval variant of *voting by committees* rule. Our result adds to the literature on strategy-proof SCCs in a restricted domain- extended single-peaked domains.

Our first result (Proposition 1) states that when voters have single-peaked preferences over alternatives then their extended preferences to intervals will be *single-peaked over intervals* if and only if they are *responsive on intervals*. This implies that there is a peak interval and other intervals which are further away from this interval are strictly worse. This is an important insight in this domain which allows us to list the intervals from left to right. This result is proved using the fact that the 'peak-interval' or top-ranked interval is the set of top-L ranked alternatives in a voter's preference. Responsiveness then implies that the intervals adjacent to it on the left and right must be less preferred to the peak interval since the alternatives further away are strictly worse.

We characterize *generalized median interval (GMI)* rules which assign $n-1$ fixed intervals of cardinality L (where n is the number of voters) and outputs an L-interval. The top-L intervals of n voters and the $n-1$ fixed intervals are listed from left to right with respect to their lower-end points. These rules then pick the median interval which may not be the top-L interval of the median voter. These rules are the interval-versions of rules characterized in Moulin (1980a) and coincide with it when $L = 1$.

We show that GMI rules are the only *strategy-proof*, *anonymous* and *interval efficient* I-SCCs (Theorem 1). The first two axioms are standard in the literature, however, the version of *strategy-proofness* we use is not a direct extension of the condition to intervals. This axiom is defined on I-SCCs which choose intervals but the manipulations made by voters are over alternatives. Therefore, the voters have more deviations than they would if they could only manipulate 'intervals'. Due to this, the proof of the main theorem does not follow directly from the result in Moulin (1980a). Additional properties of the domain need to be proved in order to rule out these deviations. In fact, these additional arguments arise due to the fact that voters report preferences over alternatives and not intervals. The last axiom is a weaker, interval-variant of Pareto efficiency and can be stated as follows. *Interval efficiency* of the I-SCC requires that there should not exist any L-interval that makes all the voters strictly better-off compared to the outcome of the I-SCC for any responsive preference extension.

The proof of the first theorem proceeds in two steps. We first use Proposition 1 which provides preferences of voters over intervals. We then show that a *strategy-proof* I-SCC must be *top-L only* (Proposition 2). This implies that such I-SCCs are invariant to changes in the preference profile made outside the top-L intervals of voters. We first identify the fixed intervals used by the GMI rules using profiles where voters have preferences at the end points of the policy interval. We use induction on the number of voters who do not have such preferences to show that the rule must pick the median of the top-intervals and the fixed intervals across all profiles. The final arguments proceed by contradiction: if the I-SCC is not the GMI rule defined in the earlier steps, we construct preference profiles where a voter can deviate beneficially.

Finally, we show the necessity of responsiveness over intervals for the strategy-proofness of the GMI rule. This holds under the condition that voter preferences over alternatives is single-peaked and the top ranked L-interval is the set of top L ranked alternatives. We show that if there exists any preference which is not consistent with single-peaked preferences over intervals, then we can find a GMI rule with specified fixed intervals and a profile over which it is not strategy-proof. This validates the argument that responsiveness of intervals is a fairly weak assumption over preference extensions over L-intervals. The chapter is organized as follows. Section 2 will describe the model and definitions. Section 3 and 4 presents the set of axioms and results respectively. We conclude in Section 5.

1.2 The Model

The set of voters is $N = \{1, 2, \ldots, n\}$, and the set of alternatives is $X = \{a_1, a_2, \ldots, a_m\}$. The alternatives are arranged according to an ordering $<$ on X such that $a_1 < a_2 < \cdots < a_m$. We will denote by a_j and a_{j+1} as two consecutive alternatives according to $<$.

Voter preferences over alternatives: Each voter i's preference, P_i, is *single-peaked* on X, i.e., there exists a 'peak', $\tau(P_i)$, such that for any $x, y \in X$,

$$\left[y < x \leq \tau(P_i) \text{ or } \tau(P_i) \leq x < y\right] \Rightarrow xP_iy,$$

where the peak, $\tau(P_i)$, is the top-ranked alternative in X for voter $i \in N$. Moreover, we require that P_i is a linear order on X.[5] Let $\mathcal{S}$ be the set of all single-peaked preferences over X according to $<$ and let $P = (P_1, \ldots, P_n)$ denote a profile of single-peaked preferences where each $P_i \in \mathcal{S}$. Let $\mathcal{S}^n$ be the set of all single-peaked profiles on X. We only consider aggregation rules which pick contiguous sets of cardinality L or L-intervals which we define below.

Interval of cardinality L: For any $L \in \{1, 2, 3, \ldots, m\}$ we define an *interval of cardinality* L or L*-interval* as $[a_l] = \{a \in X \mid a_l \leq a \leq a_{l+L-1}\}$ where $l \in \{1, \ldots, m - L + 1\}$. For example, if $X = \{a_1, a_2, a_3, a_4, a_5\}$, then the interval $[a_2]$ of cardinality $L = 3$ and $L + l - 1 = 3 + 2 - 1 = 4$ is the set, $[a_2] = \{a_2, a_3, a_4\}$. Therefore $[a_l]$ includes all l alternatives from a_l to a_{l+L-1} according to the order $<$. We denote the set of all intervals of cardinality L over X as $\mathcal{I}_L$ for any $L \in \{1, \ldots, m\}$. We fix the cardinality of intervals to be L throughout the rest of this chapter. For simplicity, we

[5] A binary relation P defined on X is a *linear order* if it is (i) complete: either xPy or yPx $\forall x \neq y$, (ii) transitive: $[xPy \text{ and } yPz] \Rightarrow [xPz]$ $\forall x, y, z \in X$ and (iii) asymmetric: $[xPy] \Rightarrow \neg[yPx]$, $\forall x, y \in X$.

will refer to L-intervals as just intervals.

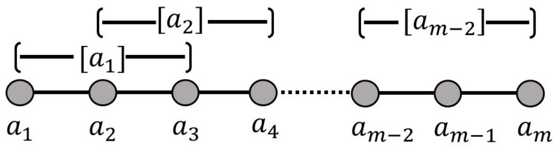


Figure 2: Intervals of cardinality 3

Ordering, $<_L$, over L-intervals: For any two intervals $[a_l]$ and $[a_r]$, define an ordering $<_L$ as follows: $[a_l] <_L [a_r]$ if and only if $l < r$. Therefore, in the Figure 2 with m alternatives and $L = 3$, we have $[a_1] < [a_2] < [a_3]$. Any two intervals $[a_l]$ and $[a_{l+1}]$ are *adjacent* for any $l \in \{1, 2, ..., m - L + 1\}$, i.e., the starting points of the two L-intervals are at the l-th and $(l + 1)$-th position respectively.

Three intervals are shown from left to right in Figure 2: $[a_1], [a_2]$ and so on till $[a_{m-2}]$ for a given length of $L = 3$. Interval $[a_1]$ is adjacent to $[a_2]$, $[a_2]$ is adjacent to $[a_3]$ and so on. For any preference $P_i \in \mathcal{S}$, the set of top L ranked alternatives is denoted by $P_i^L = \{x | \#\{y : yP_ix\} \leq L - 1\}$. Therefore, if P_i is $a_2P_ia_1P_ia_3P_ia_4P_ia_5$, then $P_i^3 = \{a_1, a_2, a_3\}$ for $L = 3$ since for each $x \in P_i^3$ the number of alternatives which are strictly preferred to x is less than or equal to 2. Given a single-peaked preference P_i for any $i \in N$ we show that the top-L ranked alternatives in P_i is an L-interval. We prove this claim formally.

Claim 1. *For any $L \in \{1, ..., m\}$ and any $i \in N$, the set of top-L ranked alternatives, P_i^L, of any single-peaked preference $P_i \in \mathcal{S}$ is an interval of cardinality L.*

Proof. We prove Claim 1 by contradiction. It is trivially satisfied for $L \in \{1, m\}$. Suppose the set of top-L ranked alternatives is not an interval, for some $L \in \{2, ..., m-1\}$. Then, there exist distinct alternatives $x, y \in X \setminus \{\tau(P_i)\}$ such that $x \in P_i^L$ and $y \notin P_i^L$, and either: (i) $x < y < \tau(P_i)$ or (ii) $x > y > \tau(P_i)$. Since $y \notin P_i^L$ and $x \in P_i^L$, by definition of the top-L ranked set, xP_iy. This along with the fact that either (i) or (ii) holds, implies that yP_ix. This is a contradiction since P_i is asymmetric. Therefore, P_i^L is an interval for all $i \in N$. ■

An *interval-social choice correspondence (I-SCC)*, $f : \mathcal{S}^n \to \mathcal{I}_L$ produces an L-interval $f(P) \in \mathcal{I}_L$ for every profile $P \in \mathcal{S}^n$. In order to compare the outcomes of I-SCCs with other outcomes which are sets of cardinality L, we need to extend voters' preferences over alternatives to subsets of fixed cardinality L. We define preference extensions below.

Extension of preferences to $\mathcal{I}_L$

Extension of P_i: A weak order $\succsim_i$ for $i \in N$ defined over $\mathcal{I}_L$ is an *extension* of $P_i \in \mathcal{S}$.[6] Note that if $[a_l]$ and $[a_{l+1}]$ are two intervals then $[a_l] = ([a_{l+1}] \setminus \{a_{l+L}\}) \cup \{a_l\}$ and $[a_{l+1}] = ([a_l] \setminus \{a_l\}) \cup \{a_{l+L}\}$. In Example 1, $[a_1]$ and $[a_2]$ are adjacent, where $[a_1] = ([a_2] \setminus \{a_4\}) \cup \{a_1\}$ and $[a_2] = ([a_1] \setminus \{a_1\}) \cup \{a_4\}$. In other words, two intervals are adjacent if the left end point alternative (a_l) of an interval $[a_l]$ is removed from it and is replaced by another alternative, a_{l+L}, to create a new interval $[a_{l+1}]$. We impose a property of *responsiveness* on intervals which is only applicable to the adjacent intervals in $\mathcal{I}_L$.

Responsiveness on intervals: Consider any two *adjacent* intervals $[a_l], [a_{l+1}] \in \mathcal{I}_L$. Any extension $\succsim_i$ of P_i is *responsive on intervals* if,

$$\text{(i)} \quad a_l P_i a_{l+L} \iff [a_l] \succ_i [a_{l+1}], \textit{ and}$$
$$\text{(ii)} \quad a_{l+L} P_i a_l \iff [a_{l+1}] \succ_i [a_l].$$

Responsiveness over intervals can also be interpreted as follows: if an alternative a_l is removed from an interval $[a_l]$ and it is replaced by another alternative a_{l+L} to create a new interval $[a_{l+1}]$ (thus making the two intervals adjacent) then the new interval is preferred over the old one if and only if a_{l+L} is strictly preferred over a_l. This version of responsiveness is similar to the one used in Bossert (1995) and is also used widely in the matching theory literature (for instance in Roth (1985) and

[6] A binary relation $\succsim$ defined on $\mathcal{I}_L$ is a *weak order* if it is (i) complete: either $[x] \succsim [y]$ or $[y] \succsim [x]$ $\forall [x], [y] \in \mathcal{I}_L$ and (ii) transitive: $[[x] \succsim [y]$ and $[y] \succsim [z]] \Rightarrow [[x] \succsim [z]]$ $\forall [x], [y], [z] \in \mathcal{I}_L$.

Alcalde and Barberà (1994)). Responsiveness over intervals is a fairly weak condition since it only imposes restrictions on adjacent intervals. For example, for a preference extension $\succsim_i$ of a single-peaked preference P_i with $L = 3$, we can have $[a_1] \succ_i [a_2]$ if and only if $a_1 P_i a_4$. Responsiveness does not impose anything on two intervals $[a_1]$ and $[a_3]$. However, our next proposition will show that preference extensions $\succsim_i$ are single-peaked over $\mathcal{I}_L$ according to $<_L$ if and only if they are *responsive on intervals.* We define single-peakedness over intervals first. We denote the top L-interval according to $\succsim_i$ as $\tau(\succsim_i)$, i.e. $\tau(\succsim_i) \succsim_i [a_k]$ for all $[a_k] \in \mathcal{I}_L$ for any $i \in N$.

Single-peakedness over intervals in $\mathcal{I}_L$: A preference extension $\succsim_i$ of P_i for voter $i \in N$ is *single-peaked over $\mathcal{I}_L$ according to $<_L$* if there exists a unique 'peak interval' $\tau(\succsim_i)$ such that for any two intervals $[a_k]$ and $[a_{k'}]$,

$$\left[\tau(\succsim_i) \leq_L [a_k] <_L [a_{k'}] \text{ or } [a_{k'}] <_L [a_k] \leq_L \tau(\succsim_i)\right] \implies [a_k] \succ_i [a_{k'}].$$

Single-peakedness over intervals is similar to single-peakedness over alternatives where a peak interval is at the top of the preference and intervals further away according to $<_L$ are strictly worse. Moreover, two intervals on different sides of the peak interval can be ranked in either way. Let $\mathcal{S}(\mathcal{I}_L)$ denote the set of single-peaked preference (extensions), $\succsim_i$, over intervals for any $i \in N$. Our next proposition establishes a connection between single-peakedness over alternatives and single-peakedness over intervals.

Proposition 1. *An extension, $\succsim_i$, of a single-peaked preference $P_i \in \mathcal{S}$ is single-peaked over $\mathcal{I}_L$ according to $<_L$ with $\tau(\succsim_i) = P_i^L$ if and only if it is responsive on intervals.*

Proof: We first show that *responsiveness* over intervals of $\succsim_i$ implies that it is single-peaked over $\mathcal{I}_L$. Consider an extension $\succsim_i$ of P_i that is responsive on intervals. We first show that the top-L interval of P_i is $\tau(\succsim_i) = P_i^L$ i.e. the set of top L ranked alternatives in P_i is also the top-ranked interval in $\succsim_i$. Let $P_i^L = [a_t]$ for some $t \in \{1, 2, ..., m - L + 1\}$. We will first show that (i) $[[a_{l-1}] <_L [a_l] \leq_L \tau(\succsim_i) =$

$[a_t]] \implies [[a_l] \succ_i [a_{l-1}]]$ for all $l \in \{1, 2, ..., t\}$ and (ii) $[\tau(\succsim_i) = [a_t] \leq_L [a_{l-1}] <_L [a_l]] \implies [[a_{l-1}] \succ_i [a_l]]$ for all $l \in \{t, ..., m - L + 1\}$. Transitivity of $\succsim_i$ will imply that $\tau(\succsim_i)$ is preferred to all the intervals on the 'left' according to $<_L$ and similar arguments for the intervals on the 'right' will then imply that $\tau(\succsim_i) = P_i^L = [a_t]$. We provide arguments for part (i) above. Part (ii) can be proved similarly.

Case 1: For any $l \in \{1, .., t\}$, let $[a_{l-1}]$ and $[a_l]$ be two intervals such that $\tau(P_i) \in [a_l]$. Let $P_i^L = \{a_t, a_{t+1} \ldots, \tau(P_i), a_{r1}, a_{r2}, \ \ldots, a_{rk}\}$ where $a_{r1}, \ldots, a_{rk}$ are elements to the right of $\tau(P_i)$ and are listed in order according to $<$. Note that $[a_{t-1}] = ([a_t] \setminus \{a_{rk}\}) \cup \{a_{t-1}\}$. Also note that $a_{rk} \in P_i^L = [a_t]$ and $a_{t-1} \notin P_i^L = [a_t]$. Therefore, $a_{rk} P_i a_{t-1}$ and by responsiveness on $[a_t]$ and $[a_{t-1}]$, $P_i^L = [a_t] \succ_i [a_{t-1}]$. Similarly by responsiveness on intervals, $[a_{t-1}]$ and $[a_{t-2}]$, we have $a_{rk-1} P_i a_{t-2} \Rightarrow [a_{t-1}] \succ_i [a_{t-2}]$.

Case 2: For any $l \leq t$, let $[a_{l-1}]$ and $[a_l]$ such that $\tau(P_i) \notin [a_l]$. By single-peakedness of P_i, $a_{l-1} < a_{l+L-1} < \tau(P_i)$ implies that $a_{l+L-1} P_i a_{l-1}$. By responsiveness $[a_l] \succ_i [a_{l-1}]$ since $[a_{l-1}] = ([a_l] \setminus \{a_{l+L-1}\}) \cup \{a_{l-1}\}$. By transitivity of $\succsim_i$, for all $[a_l] \leq_L P_i^L = [a_t]$, we have $[[a_l] \succ_i [a_{l-1}]$ and $[a_{l-1}] \succ_i [a_{l-2}]] \Rightarrow [[a_l] \succ_i [a_{l-2}]]$. Repeated application of transitivity of $\succsim_i$ implies that $[a_l] \succ_i [a_{l-k}]$ for all $[a_l] \leq_L P_i^L$ for all $k \in \{1, ..., l-1\}$. Similar arguments can be made for intervals to the 'right' of P_i^L according to $<_L$. Therefore $\succsim_i$ is single-peaked on $\mathcal{I}_L$ with respect to $<_L$ and its peak interval is given by $\tau(\succsim_i) = P_i^L$.

We now show the converse. Consider an extension of P_i, $\succsim_i$ that is single-peaked on $\mathcal{I}_L$ with respect to $<_L$ and $\tau(\succsim_i) = P_i^L = [a_t]$ for some $t \in \{1, 2, ..., m-L+1\}$. To show responsiveness on intervals, we need to show that for any two adjacent intervals $[a_l]$, $[a_{l-1}]$, (i) $[a_l] \succ_i [a_{l-1}]$ if and only if $a_{l+L-1} P_i a_{l-1}$ and (ii) $[a_{l-1}] \succ_i [a_l]$ if and only if $a_{l-1} P_i a_{l+L-1}$ for any $l \in \{1, ..., m-L+1\}$. Note that due to single peakedness on $\mathcal{I}_L$ (i) is true when both intervals are on the 'left' of $\tau(\succsim_i)$ while (ii) holds when both intervals are on the 'right' of $\tau(\succsim_i)$. We prove part (i). Consider intervals $[a_{l-1}]$ and $[a_l]$ on the 'left' of $\tau(\succsim_i)$ according to $<_L$, i.e. $[a_{l-1}] <_L [a_l] \leq_L \tau(\succsim_i)$. We have two further sub-cases: Case 1: Suppose $\tau(P_i) \notin [a_l]$. By single peakedness of P_i, $a_{l-1} < a_l < a_{l+L-1} < \tau(P_i)$ implies $a_{l+L-1} P_i a_{l-1}$. Case 2: Suppose $\tau(P_i) \in [a_l]$. Here, $a_{l+L-1} \in P_i^L$ while $a_{l-1} \notin P_i^L$. Therefore, $a_{l+L-1} P_i a_{l-1}$. Similar arguments can be made for part (ii), where $\tau(\succsim_i) \leq_L [a_{l-1}] <_L [a_l]$. ■

Proposition 1 provides an important insight into the nature of preference extensions $\succsim_i$ of single-peaked preferences P_i for any $i \in N$. It states that if voters have single-peaked preferences over alternatives and if $\tau(\succsim_i)$ is equal to P_i^L, then *any* preference extension of P_i over $\mathcal{I}_L$ is *single-peaked over intervals* if and only if it is *responsive on intervals*. The requirement that $\tau(\succsim_i)$ be equal to P_i^L is only used for the "only if" part of the proof. In the "if" part, we show that responsiveness of $\succsim_i$ implies that there exists a unique L-sized interval or 'peak interval' which is top-ranked in $\succsim_i$ and is also the set of top L ranked alternatives. Henceforth, we will denote the 'top' interval $\tau(\succsim_i) = P_i^L$ as the set of top L ranked alternatives for any $i \in N$. Other intervals which are further away from the peak-interval on the same side of the peak interval are strictly worse. We provide an example below.

Example 2. Suppose the set of voters is $N = \{1, 2, 3\}$, there are five alternatives which are arranged as follows: $a_1 < \cdots < a_5$ and $L = 3$. Consider the following preferences:

P_1	P_2	P_3
a_2	a_3	a_4
a_3	a_4	a_3
a_4	a_5	a_2
a_5	a_2	a_1
a_1	a_1	a_5

By Proposition 1, the preference extensions $(\succsim_1, \succsim_2, \succsim_3)$ on $\mathcal{I}_L$ will be single-peaked if and only if they are responsive. We elaborate below.

Voter 1: $\tau(\succsim_1) = [a_2]$ since $[a_2] = \{a_2, a_3, a_4\}$ is the set of top 3 ranked alternatives of voter 1. Hence, by Proposition 1, $[a_2] \succ_1 [a_3]$ and $[a_2] \succ_1 [a_1]$, since preferences over $\mathcal{I}_L$ are single-peaked. However, we impose no restriction on how $[a_1]$ and $[a_3]$ are to be compared as long as $\succsim_1$ is complete. Similarly, voter 2's peak is $\tau(\succsim_2) = [a_3]$ and her preference extension is such that $[a_3] \succ_2 [a_2] \succ_2 [a_1]$. Preference extension, $\succsim_3$, of voter 3 is such that $\tau(\succsim_3) = [a_2]$, $[a_2] \succ_3 [a_3]$ and $[a_2] \succ_3 [a_1]$.

We introduce some definitions to define *generalized median interval rules*:

Median of a sequence of alternatives: Consider any integer $p > 0$ and a sequence of alternatives $B = (x_1, x_2, \ldots, x_{2p-1})$ where repetitions are allowed and alternatives are arranged according to $<$. An alternative $x \in B$ is the *median* of this sequence, denoted by $med(x_1, x_2, ..., x_{2p-1})$, if

$$|\{x' \in B : x' \leq x\}| \geq p \text{ and } |\{x' \in B : x \leq x'\}| \geq p.$$

Note that the median of a sequence with $2p-1$ alternatives is the p-th alternative, and the order of the sequence does not matter. For example, $med(a_1, a_5, a_2, a_3, a_2, a_4, a_5) = med(a_1, a_2, a_2, a_3, a_4, a_5, a_5) = a_3$ where $p = 4$, and $2p-1 = 2(4)-1 = 7$ since there are four alternatives (weakly) above and below a_3 (including itself) when the alternatives are arranged in ascending order (with repetitions). The median can be found by enumerating the alternatives from left to right and picking the p-th alternative out of a total of $2p - 1$ alternatives.

Median of a sequence of intervals: Similarly, we can define the *median of a sequence of intervals*, $([x_1], ..., [x_{2k-1}])$, as $med([x_1], ..., [x_{2k-1}]) = [x_k]$ for any integer $k > 0$. For example,
$med([a_1], [a_2], [a_2], [a_3], [a_4]) = [a_2]$ since there are three intervals on either side of $[a_2]$ (counting itself twice).

Fixed Intervals: A sequence of $n-1$ fixed intervals will be denoted by $[\alpha_1], \ldots, [\alpha_{n-1}]$. These will be added to the vector of top-L intervals of the voters to compute the outcome of our main rule, which we define now.

Generalized median interval (GMI) rules: An I-SCC, $f^{\alpha} : \mathcal{S}^n \to \mathcal{I}_L$, is a *GMI* rule if there exist $n-1$ fixed intervals $\alpha = ([\alpha_1], \ldots, [\alpha_{n-1}])$ such that for any $P \in \mathcal{S}^n$,

$$f^{\alpha}(P) = med(\tau(\succsim_1), \ldots, \tau(\succsim_n), [\alpha_1], \ldots, [\alpha_{n-1}]).$$

GMI rules pick the median interval from the sequence consisting of the top-L interval of voters $\{\tau(\succsim_i)\}_{i=1}^{n}$ and the given fixed intervals $\{[\alpha_i]\}_{i=1}^{n-1}$. Note that for a given GMI rule the fixed intervals are defined independently of the profiles and remain fixed for all $P \in \mathcal{S}^n$. Therefore, different sets of fixed intervals define different GMI rules.

Consider Example 2 again. Let f^{α} be the following GMI rule with two fixed intervals: $[\alpha_1] = [a_1]$ and $[\alpha_2] = [a_3]$. For the preference extensions derived in Example 2, we have $f^{\alpha}(P) = med(\tau(\succsim_1), \tau(\succsim_2), \tau(\succsim_3), [\alpha_1], [\alpha_2]) = med([a_2], [a_3], [a_2], [a_1], [a_3]) = [a_2]$ since it is the third interval while enumerating from left (or right). Note that GMI rules only take into account the top-L intervals of voters. Also note that the median interval may not be the top-L interval of the median voter (except when $L = 1$ as in Moulin (1980a)). We provide an example to illustrate.

Example 3. Suppose $N = \{1, 2, 3\}$ and $X = \{a_1, a_2, ..., a_5\}$ where $a_1 < \cdots < a_5$ and $L = 3$. Consider the following preferences over alternatives:

P_1	P_2	P_3
a_2	a_3	a_4
a_3	a_4	a_3
a_4	a_5	a_2
a_5	a_2	a_1
a_1	a_1	a_5

By Proposition 1, we get the following profile of single-peaked preference extensions $\{\succsim_1, \succsim_2, \succsim_3\}$:

Voter 1: $\tau(\succsim_1) = [a_2]$ with $[a_2] \succ_1 [a_3]$, $[a_2] \succ_1 [a_1]$, and for completeness we can have either $[a_1] \succ_1 [a_3]$ or $[a_3] \succ_1 [a_1]$. Voter 2: $\tau(\succsim_2) = [a_3]$ with $[a_3] \succ_2 [a_2] \succ_2 [a_1]$. Voter 3: $\tau(\succsim_3) = [a_2]$ with $[a_2] \succ_3 [a_3]$, $[a_2] \succ_3 [a_1]$, and for completeness we can have either $[a_1] \succ_3 [a_3]$ or $[a_3] \succ_3 [a_1]$.

Note that even though voters 1 and 3 have different preferences over alternatives, they may have similar preferences over intervals. Suppose f^α is the *GMI rule* with fixed intervals $[\alpha_1] = [a_1]$ and $[\alpha_2] = [a_3]$. By definition of *GMI rule*, $f^\alpha(P) = med(\succsim_1, \succsim_2, \succsim_3, [\alpha_1], [\alpha_2]) = med([a_2], [a_3], [a_2], [a_1], [a_3]) = [a_2]$. The median voter according to the peaks is voter 2, however the outcome of the GMI rule is *not* her peak interval $[a_3]$, rather the outcome is the peak interval of voters 1 and 3, i.e., $[a_2]$.

We show in Section 4 that *strategy-proofness* implies the top-L only property which means that these I-SCCs only take as input the top-L intervals of voters. However, GMI rules are not the only I-SCCs which are top-L only. We define some top-L only I-SCCs over the single-peaked domain.

Dicatorial I-SCC: A GMI rule, f^i is dictatorial if for all $P \in \mathcal{S}^n$, $f^i(P) = \tau(\succsim_i)$. Dictatorial rules pick the dictator's (voter i's) peak interval for all profiles.

Min (max) I-SCCs: A GMI rule, f^{min} (f^{max}) is a min (max) rule if for all $P \in \mathcal{S}^n$, $f^{min}(P) = min\{\tau(\succsim_i)\}_{i\in N}$ ($f^{max}(P) = max\{\tau(\succsim_i)\}_{i\in N}$), where $min\{.\}_{i\in N}$ ($max\{.\}_{i\in N}$) picks the interval with the smallest (largest) interval according to $<_L$. Min and max I-SCCs are a sub-class of GMI rules if $\alpha = ([a_1], ..., [a_1])$ for min and $\alpha = ([a_{m-L+1}], ..., [a_{m-L+1}])$ for max. A median I-SCC can also be defined as a GMI rule with $\alpha = (\underbrace{[a_1], ..., [a_1]}_{\frac{n-1}{2}}, \underbrace{[a_m], ..., [a_m]}_{\frac{n-1}{2}})$ when n is odd.[7] We now present the axioms.

1.3 Axioms

In this section we define the axioms which will characterize GMI rules.

[7] A similar rule can be defined for the case when n is even, using the left or right median.

Anonymity: An I-SCC, f, satisfies *anonymity* if for every preference profile $P \in \mathcal{S}^n$, and for each permutation σ of N, $f(P) = f(P^\sigma)$, where $P^\sigma = (P_{\sigma(1)}, ..., P_{\sigma(n)})$.

Anonymity implies that the outcome of an I-SCC is independent of the identities of voters. All the rules mentioned above except the dictatorial rules are anonymous.

Strategy-proofness: An I-SCC, f, is said to be *strategy-proof* if for every profile $(P_i, P_{-i}) \in \mathcal{S}^n$,

$$f(P_i, P_{-i}) \succsim_i f(P'_i, P_{-i}) \quad \forall P'_i \in \mathcal{S}, \text{ for all responsive } \succsim_i .$$

In other words, *strategy-proofness* states that unilateral deviations do not make a voter strictly better-off for any responsive preference extension over intervals. Note that the deviations of voters are in terms of preferences over alternatives, and the outcomes are intervals. Since the outcome of I-SCCs are intervals, a natural extension of efficiency would be to compare intervals in $\mathcal{I}_L$ which we define below. All the rules mentioned above are strategy-proof. The proof for GMI rules (including min and max) rules is provided in the proof of our main result.

Interval efficiency: An I-SCC, f, is said to be *interval efficient* if for any $P \in \mathcal{S}^n$ and any $[a_l] \in \mathcal{I}_L$,

$$[\exists\, j \in N \text{ s.t. } [a_l] \succ_j f(P)] \Rightarrow [\exists\, k \in N \text{ s.t. } f(P) \succ_k [a_l]] \quad \text{ for all responsive } \succsim_i .$$

An I-SCC satisfies *interval efficiency* if for any voter who can be made strictly better-off by any interval $[a_l]$ there will be another voter who is made strictly worse-off by that interval. Interval efficiency can be interpreted as the interval version of Pareto efficiency. The dictatorial rule is interval efficient since it always picks the top-interval of the dictator. GMI rules are interval efficient as will be proved in the proof of Theorem 1.

1.4 Results

We first show that all strategy-proof I-SCC must be *top-L only.* The latter property is the interval version of the *tops-only property* commonly used in the social choice literature. This implies that only changes in the top L intervals of voters can affect the outcome of a strategy-proof I-SCC. We define top-L only.

Top-L only: An I-SCC, f, is said to be *top-L only* if for all $P, P' \in \mathcal{S}^n$ such that $\tau(\succsim_i) = \tau(\succsim'_i)$ for all $i \in N$, $f(P) = f(P')$.

It states that if every voter i reports the same set of top L ranked alternatives (which is always an interval, by Claim 1) in two preference profiles P and P', then the outcomes under P and P' are the same.

Proposition 2. *Suppose $f : \mathcal{S}^n \to \mathcal{I}_L$ is strategy-proof. Then it is top-L only.*

Proof: Suppose f is strategy-proof. We argue that f must be top-L only. Take any two profiles P and P' in $\mathcal{S}^n$, with the same top-L intervals, i.e., $P_i^L = P_i'^L$ for all $i \in N$. We show that $f(P) = f(P')$. We construct a sequence of profiles, starting from P and finishing at P', where each voter's preference changes from P_i to P'_i sequentially but the outcome of the rule does not change between any two consecutive profiles. Consider the following sequence of profiles:

$P^0 = P = (P_1, P_2, P_3, \ldots, P_n)$, $P^1 = (P'_1, P_2, P_3, \ldots, P_n)$, $P^2 = (P'_1, P'_2, P_3, \ldots, P_n)$,..., and $P^n = P' = (P'_1, P'_2, P'_3, \ldots, P'_n)$.

In the above sequence, the profile P is transformed one step at a time to the profile P'. We show that $f(P^q) = f(P^{q+1})$ for all $q \in \{0, ..., n-1\}$. We first provide the argument for $q = 0$. Similar arguments can be made for other values of q. Assume for contradiction that $f(P) = f(P^0) = [a_l] \neq f(P'_1, P_{-1}) = f(P^1) = [a_r]$. Assume w.l.o.g that $P_1^L = P_1'^L \leq_L [a_l]$. There are three cases:

Case 1: Suppose $P_1^L = P_1'^L \leq_L [a_l] <_L [a_r]$. Voter 1 can deviate at profile P^1 from P'_1 to P_1 and be better-off at the profile P^0 by single-peakedness of $\succsim_i$. Since f is strategy-proof this is a contradiction. Case 2: Suppose $[a_r] <_L [a_l]$. Similar contradiction arises in the following two sub-cases: Case 2.1: $P_1^L \leq_L [a_r] <_L [a_l]$. Voter 1 can deviate at profile P^0 from P_1 to P_1' and be better-off by single-peakedness of $\succsim_i$. Case 2.2: $[a_r] \leq_L P_1^L = P_1'^L \leq_L [a_l]$ with at least one inequality holding strictly. Suppose that $[a_r] = P_1^L = P_1'^L <_L [a_l]$. Consider the extension $\succsim_1$ of P_1 where $[a_r] \succ_1 [a_l]$. Note that such a preference extension, $\succsim_1 \in \mathcal{S}(\mathcal{I}_L)$, exists for any given $P_1 \in \mathcal{S}$ as described above. Moreover, it is consistent with responsiveness over intervals. The deviation by voter 1 from P_1 to P_1' to obtain $[a_r]$ will make her strictly better-off since $[a_r] \succ_1 [a_l]$. This is a contradiction to the fact that f is strategy-proof. A similar contradiction is reached if $[a_r] <_L P_1^L = P_1'^L = [a_l]$ with the preference extension $\succsim_1'$ of P_1' where $[a_l] \succ_1' [a_r]$. In this case, voter 1 will deviate from P_1' to P_1. ■

Therefore, the outcome of a strategy-proof I-SCC only depends on the top-L intervals of voters irrespective of the ordering of alternatives within that interval. We now show that the outcome any I-SCC lies 'between' the leftmost and the rightmost top L intervals of voters. Denote a closed interval (of any cardinality) as $[a_j, a_k] = \{x \in X | a_j \leq x \leq a_k\}$ and an open interval as $(a_j, a_k) = \{x \in X \mid a_j < x < a_k\}$. We define $[\underline{a}, \overline{a}]$ as the smallest interval which contains the top-L intervals of all voters, i.e. (i) $\bigcup_{i \in N}\{P_i^L\} \subseteq [\underline{a}, \overline{a}]$ for some $\underline{a}, \overline{a} \in X$ and (ii) $\nexists\ \underline{a}', \overline{a}' \in (\underline{a}, \overline{a})$ such that $\bigcup_{i \in N}\{P_i^L\} \subseteq [\underline{a}', \overline{a}]$ or $\bigcup_{i \in N}\{P_i^L\} \subseteq [\underline{a}, \overline{a}']$. We have constructed the set $[\underline{a}, \overline{a}]$ using only the top-L alternatives of the voters. We use interval efficiency to show that for all $P \in \mathcal{S}^n$, $f(P) \subseteq [\underline{a}, \overline{a}]$.

Claim 2. *An interval $[a_l]$ is interval efficient if and only if $[a_l] \subseteq [\underline{a}, \overline{a}]$.*

Proof We prove necessity first. All voters prefer $[\underline{a}, \underline{a}+L-1]$ to any other interval $[a_l] <_L [\underline{a}, \underline{a}+L-1]$, since their top-$L$ alternatives are on the right of $[a_l]$. Similarly, all voters prefer $[\overline{a}-L+1, \overline{a}]$ to any other interval $[a_l] >_L [\overline{a}-L+1, \overline{a}]$. We prove sufficiency. Suppose an L-interval $[a_l] \subseteq [\underline{a}, \overline{a}]$ is the outcome. Any distinct interval on the right of $[a_l]$ makes all voters i such that $[P_i^L] \leq_L [a_l]$ strictly worse-off and any distinct interval on the left makes all voters i such that $[a_l] \leq_L [P_i^L]$ strictly worse-off.

Therefore, $f(P) \subseteq [\underline{a}, \overline{a}]$ for all $P \in \mathcal{S}^n$. ■

Next, our main theorem provides a characterization of all strategy-proof, interval efficient and anonymous I-SCCs.

Theorem 1. *Suppose the extension $\succsim_i$ of preferences P_i for each voter $i \in N$ is responsive on intervals. An I-SCC, $f : \mathcal{S}^n \to \mathcal{I}_L$, is anonymous, strategy-proof and interval efficient if and only if it is a GMI rule.*

Proof: We prove necessity first. Anonymity follows from the definition of GMI rules. To show $f$is strategy-proof, we consider any voter $i \in N$ and a given profile $P \in \mathcal{S}^n$. There are three cases: (i) If $f(P) = P_i^L$ then there is no profitable deviation that leads to a strictly better outcome for i. (ii) If $f(P) <_L P_i^L$ then the only way to change the outcome is to change the median of the reported top-L intervals by all the voters $\{P_i^L\}_{i=1}^n$ and the fixed intervals $\{[\alpha_i]\}_{i=1}^{n-1}$. This can be done by reporting P_i' such that $P_i'^L <_L f(P)$. By single-peakedness over intervals and by the definition of GMI rule, $[f(P_i', P_{-i}) \leq_L f(P_i, P_{-i}) <_L P_i^L] \implies [f(P_i, P_{-i}) \succsim_i f(P_i', P_{-i})]$. Therefore, no unilateral deviation can be strictly beneficial. (iii) If $P_i^L <_L f(P)$, then the only way to change the outcome is to report a top-L interval on the right of $f(P)$, i.e., $f(P) <_L P_i'^L$. By single-peakedness over intervals and by the definition of GMI rule, $[P_i^L <_L f(P_i, P_{-i}) <_L f(P_i', P_{-i})] \implies [f(P_i, P_{-i}) \succsim_i f(P_i', P_{-i})]$. Therefore, no unilateral deviation can be strictly beneficial.

To show that f is interval efficient, take any profile $P \in \mathcal{S}^n$ and suppose $[a_l] <_L f(P)$ for some interval $[a_l] \in \mathcal{I}_L$. Since $f(P) = med(P_1^L, ..., P_n^L, [\alpha_1], ..., [\alpha_{n-1}])$ all voters i with $P_i^L \geq_L f(P)$ strictly prefer $f(P)$ over $[a_l]$ by single-peakedness over intervals. Therefore, any such interval $[a_l]$ cannot make all the voters strictly better-off compared to $f(P)$. Similar arguments can be made for the intervals on the right of $f(P)$. Therefore, GMI rules are interval efficient.

We now prove sufficiency. Suppose the I-SCC, $f : \mathcal{S}^n \to \mathcal{I}_L$ is anonymous, strategy-proof and interval efficiency. We will show that it is a GMI rule. Note that interval efficiency of f implies *unanimity* i.e. if $P_i^L = P_j^L$ for all $i, j \in N$, then $f(P) = P_i^L$

since any other outcome would not be interval efficient (it can be improved upon by selecting P_i^L). By Proposition 2 the rule is top-L only. Our proof proceeds in two steps. We first consider voter profiles where varying number of voters have peaks either at a_1 or a_m. We identify the fixed intervals, $[\alpha_1], ..., [\alpha_{n-1}]$, as the outcomes at those profiles. Finally, we show that the outcome for every profile is a GMI rule according to the given α's. We elaborate on the first step below.

Let $\underline{P}$: $a_1 \underline{P} a_2 \ldots \underline{P} a_m$ with the top-L interval as $\underline{P}^L = [a_1]$ and let $\overline{P}$: $a_m \overline{P} a_{m-1} \ldots \overline{P} a_1$ with the top-L interval as $\overline{P}^L = [a_{m-L+1}]$. Let $(\underline{P}^{n-k}, \overline{P}^k) \in \mathcal{S}^n$ be a profile where $n-k$ voters have the preference $\underline{P}$ and k voters have the preference $\overline{P}$ for any $k \in \{1, \ldots, n-1\}$. We identify the fixed intervals for the GMI rule f^α as follows. Let $[\alpha_k] = f(\underline{P}^{n-k}, \overline{P}^k)$ for all $k \in \{1, \ldots, n-1\}$. Therefore, $[\alpha_k]$ is the outcome of the I-SCC f where k voters have the preference $\overline{P}$ and the remaining voters have the preference $\underline{P}$. These α's define a GMI rule f^α with $\{[\alpha_i]\}_{i=1}^{n-1}$. To show that the two rules coincide i.e. $f = f^\alpha$ we use the following Lemma.

Lemma 1. *For all $i \in \{1, \ldots, n-2\}$ we have $[\alpha_i] \leq_L [\alpha_{i+1}]$.*

Proof: Note $f(\underline{P}^{n-k}, \overline{P}^k) = [\alpha_k]$ and $f(\underline{P}^{n-k-1}, \overline{P}^{k+1}) = [\alpha_{k+1}]$ for all $k \in \{1, \ldots, n-2\}$. Pick a voter $l \in N$, such that $P_l = \underline{P}$. Strategy-proofness of f requires that the deviation from $\underline{P}$ to $\overline{P}$ by voter l should not be beneficial. This implies that $[\alpha_k] \succsim_l [\alpha_{k+1}]$. Since $\tau(\succsim_l) = [a_1]$, by single-peakedness of $\succsim_l$, $[\alpha_k] \leq_l [\alpha_{k+1}]$. ■

We now show that f is the GMI rule f^α with fixed intervals as $\alpha = \{[\alpha_i]\}_{i=1}^{n-1}$, i.e. for any $P \in \mathcal{S}^n$, $f(P) = f^\alpha(P)$. We apply induction on the number of voters (say, κ) who do not have their top as a_1 or a_m. Let κ be the induction variable such that $\kappa = |\{i : \tau(P_i) \notin \{a_1, a_m\}\}|$. We first argue that the statement holds for the base case where $\kappa = 0$, i.e. $P_i \in \{\underline{P}, \overline{P}\}$ for all $i \in N$. There are two cases here. Let $|\{i|\tau(P_i) = a_1\}| = n-k$ and $|\{i|\tau(P_i) = a_m\}| = k$. Case 1: Suppose $k \in \{0, n\}$. For all $i \in N$, either $\tau(P_i) = a_1$ ($P_i = \underline{P}$) or $\tau(P_i) = a_m$ ($P_i = \overline{P}$). By interval efficiency, $f(\underline{P}^n) = f^\alpha(\underline{P}^n) = \underline{P}^L = [a_1]$ and $f(\overline{P}^n) = f^\alpha(\overline{P}^n) = \overline{P}^L = [a_{m-L+1}]$ respectively. Case 2: Suppose $0 < k < n$. By Lemma 1, $[\alpha_i] \leq_L$

$[\alpha_{i+1}]$ for all $i \in \{1, \ldots, n-2\}$. Therefore, by construction, $f(P) = f(\underline{P}^{n-k}, \overline{P}^k) = [\alpha_k] = med\{P_1^L, \ldots, P_n^L, [\alpha_1], \ldots, [\alpha_{n-1}]\} = f^\alpha(P)$. We have shown that when the top-ranked alternatives of voters is either a_1 or a_m, the outputs of f and f^α coincide, which is the base case ($\kappa = 0$). Using induction we will show that for any arbitrary profile the outputs of f and f^α coincide.

Induction Hypothesis: Suppose that for any $\kappa \in \{0, \ldots, n-1\}$ the outputs of the two functions f and f^α coincide. We prove that for $\kappa + 1$, the outputs of f and f^α will also coincide. Consider any $P \in \mathcal{S}^n$ where $\kappa + 1$ voters do not have their top alternatives as a_1 or a_m. Pick any such voter j with top-L interval P_j^L. Assume for contradiction $[a_l] = f(P_j, P_{-j}) \neq f^\alpha(P_j, P_{-j}) = [b_l]$. We show that there is a unilateral deviation which will benefit voter j.

W.l.o.g. assume that $[a_l] <_L [b_l]$. There are three cases.

Case 1: $P_j^L = [a_t] \leq_L [a_l] <_L [b_l]$. Consider the preference profile $P'' = (P_j'', P_{-j})$, where $P_j''^L = \underline{P}$. This falls under the case where κ voters do not have their top alternative as a_1 or a_m. Therefore, by the induction hypothesis, $f^\alpha(P'') = f(P'')$. By definition of GMI rule f^α, since j has moved her top-L interval on the same side of the previous outcome (which is a median), the outcome remains unchanged, i.e., $f^\alpha(P) = f^\alpha(P'')$. This implies that $f(P) = [a_l] <_L f(P'') = f^\alpha(P) = f^\alpha(P'') = [b_l]$. Consider the deviation by voter j at profile P'' from P_j'' to P_j. By single-peakedness over intervals, $\left[P_j''^L = \underline{P}^L <_L [a_l] <_L [b_l]\right] \implies \left[[a_l] \succ_j'' [b_l]\right] \implies \left[f(P) \succ_j'' f(P'')\right]$. Therefore, voter j is strictly better-off.

Case 2: $[a_l] <_L P_j^L = [a_t] <_L [b_l]$. Let $P_j'' = \underline{P}$ as above. By single-peakedness over intervals, $[a_l] \succ_j'' [b_l]$ as $\tau(\succ_j'') = \underline{P}$. Then the deviation from P_j'' to P_j at profile P'' would be beneficial for voter j. This violates strategy-proofness of f.

Case 3: Suppose $[a_l] <_L [b_l] \leq_L P_j^L = [a_t]$. Let, $P_j''^L = \overline{P}$. The outcomes of f^α must coincide at P and $P'' = (P_j'', P_{-j})$ since the median cannot change. By induction hypothesis, $f(P'') = f^\alpha(P'') = [b_l]$. By single-peakedness over intervals, $\left[[a_l] <_L [b_l] <_L P_j''^L = \overline{P}^L\right] \implies \left[[b_l] \succ_j [a_l]\right] \implies [f(P'') \succ_j f(P)]$. Hence the deviation from P_j to P''_j by j is beneficial.

Similar arguments can be made when $[b_l] <_L [a_l]$. Therefore, in both the cases, when $[a_l] \neq [b_l]$, there exists a profitable deviation for a given preference profile. This contradicts the assumption that f is strategy-proof. Therefore, $f(P) = [a_l] = f^{\alpha}(P) = [b_l]$ and the two rules coincide. This completes the induction argument, and the claim is true for all $\kappa \in \{1, 2, ..., n\}$. Therefore for all $P \in \mathcal{S}^n$, $f(P) = f^{\alpha}(P)$ with the fixed intervals $\alpha = ([\alpha_1], ..., [\alpha_{n-1}])$ as defined above. ■

We provide an intuitive sketch of the proof of Theorem 1. Necessity is straightforward. GMI rules are anonymous since the rule is invariant to permutation of voters' preferences. GMI rules are interval efficient since they always pick an L-interval which lies between the left-most and right-most top interval of voters. This implies that any other interval which makes a voter strictly better-off will also make a voter strictly worse-off. We show that GMI rules are strategy-proof. Since GMI rules only take into account the top interval of voters, a voter i has to change her own top interval to change the outcome. Since the GMI rules picks the median of the top intervals and the fixed intervals, the only way to change the outcome is to 'report' the top interval on the other side of the interval outcome, $f(P)$ (the outcome under truthful reporting). As a result of the deviation, the outcome moves further away from the 'true' top-L interval of voter i. Since the extension $\succsim_i$ of P_i is single-peaked over intervals (Proposition 1), any such deviation will make voter i worse-off.

Sufficiency of the axioms is proved in multiple steps and adapts the proof by Moulin (1980a) for alternatives. By Proposition 1, a preference $\succsim_i$ over *intervals* is single-peaked according to $<_L$. This implies that intervals can be arranged from left to right and can be seen as 'alternatives' in the relevant interval-based single-peaked domain. This along with Proposition 2 implies that only the top intervals determine the outcome of such I-SCCs. The next part of the proof involves identifying the fixed intervals, $[\alpha_1], \ldots, [\alpha_{n-1}]$. This is done by starting with a profile where all voters have peak intervals starting at either $[a_1]$ or $[a_{m-L+1}]$, i.e., the extreme left and right intervals respectively. Once the fixed intervals for the GMI rule have been identified, an induction argument is applied on the number of voters who do not have the extreme left or right peak intervals.

1.4.1 Necessity of responsiveness for the strategy-proofness of the GMI rule

In this section we show that responsiveness over intervals is *necessary* for the GMI rule to be strategy-proof given the ordering over the set of alternatives in the following sense. Let $\mathbb{D}$ be the set of all preference extensions $\succsim_i$ over $\mathcal{I}_L$ which are weak orders such that $\tau(\succsim_i) = P_i^L$. We show that if the set of preferences in a given domain $\mathbb{D}$ contains *any* preference which is not consistent with responsiveness over intervals, then there exists a GMI rule $f^\alpha : \mathbb{D}^n \rightarrow \mathcal{I}_L$ which is not strategy-proof on $\mathbb{D}^n$. In light of Proposition 1, we only need to show the following: if $\mathbb{D} \nsubseteq \mathcal{S}(\mathcal{I}_L)$ then there exists a profile $\pi \in \mathbb{D}^n$ and $\alpha \in \mathcal{I}_L^{n-1}$ such that f^α is not strategy-proof, i.e. an individual can unilaterally deviate in a strictly beneficial manner.

An I-SCC $f : \mathbb{D}^n \rightarrow \mathcal{I}_L$ is defined over n-tuples, $\pi = (\succsim_1, \succsim_2, ..., \succsim_n) \in \mathbb{D}^n$, and produces an alternative $f(\pi) \in \mathcal{I}_L$. The GMI rule is defined as before since it only takes into account the top-L intervals of voters: an I-SCC, $f^\alpha : \mathbb{D}^n \rightarrow \mathcal{I}_L$, is a *GMI* rule if there exist $n-1$ fixed intervals, $[\alpha_1], \ldots, [\alpha_{n-1}]$, such that for any $\pi \in \mathbb{D}^n$,

$$f^\alpha(\pi) = med(\tau(\succsim_1), \ldots, \tau(\succsim_n), [\alpha_1], \ldots, [\alpha_{n-1}]).$$

Theorem 2. *GMI rules are strategy-proof on* $\mathbb{D}$ *if and only if* $\mathbb{D} \subseteq \mathcal{S}(\mathcal{I}_L)$.

Proof. If $D \subseteq \mathcal{S}(\mathcal{I}_L)$, then by Theorem 1, GMI rules are strategy-proof. Suppose $\mathbb{D} \nsubseteq \mathcal{S}(\mathcal{I}_L)$. We show that there exists a GMI rule, $f^\alpha : \mathbb{D}^n \rightarrow \mathcal{I}_L$ with fixed intervals, $[\alpha_1], \ldots, [\alpha_{n-1}]$ and a profile $\pi \in \mathbb{D}^n$ on which f^α is not strategy-proof, i.e., an individual can deviate beneficially at π. Since $a_1 < a_2 < ... < a_m$ and $\mathbb{D} \nsubseteq \mathcal{S}(\mathcal{I}_L)$, this implies that there exists a preference $\succsim^* \in \mathbb{D} \setminus \mathcal{S}(\mathcal{I}_L)$. Therefore, due to violation of single-peakedness on intervals, there exist intervals $A, B, C \in \mathcal{I}_L$ such that either (i) $A < B < C$ or $C < B < A$, and (ii) $\tau(\succsim^*) = A$, while $C \succ^* B$.

Suppose w.l.o.g that $A < B < C$ and $\tau(\succsim^*) = A$. We take the following GMI rule, f^α which has the following fixed intervals: $[\alpha_1] = [\alpha_2] = ... = [\alpha_{n-1}] = C$. Consider the following profile, $\pi \in \mathbb{D}^3$ with the following preferences: $\succsim_1 = \succsim^*$ and $\succsim_i \in \mathcal{S}(\mathcal{I}_L)$ such that $\tau(\succsim_i) = B$ for all $i \in \{2, 3, ..., n\}$. Note that $\pi \notin \mathcal{S}(\mathcal{I}_L)^3$ since $\tau(\succsim_1) = A \succ_1 C \succ_1 B$ even though $A < B < C$. By definition of the GMI rule,

$$\begin{aligned}
f^\alpha(\pi) &= med(\tau(\succsim_1), ..., \tau(\succsim_n), [\alpha_1], ..., [\alpha_{n-1}]) \\
&= med(\underbrace{A}_{\text{Voter 1}}, \underbrace{B, ..., B}_{\text{Voters } 2,..,n}, \underbrace{C, ..., C}_{n-1}) \\
&= med(\underbrace{A}_{1}, \underbrace{B, ..., B}_{n-1}, \underbrace{C, ..., C}_{n-1}) = B
\end{aligned}$$

However, if voter 1 deviates to $\succsim_1'$ with $\tau(\succsim_1') = C$, then the outcome at $\pi' = (\succsim_1', \succsim_2, ..., \succsim_n)$ is,

$$f^\alpha(\pi') = med(\underbrace{C}_{1}, \underbrace{B, ..., B}_{n-1}, \underbrace{C, ..., C}_{n-1}) = C$$

Since $C \succ_1 B$, this move is beneficial for voter 1. Therefore, f^α is not strategy-proof. This implies that $\mathbb{D} \subseteq \mathcal{S}(\mathcal{I}_L)$. By Proposition 1, since preferences P_i are single-peaked and $\succsim_i \in \mathbb{D} \subseteq \mathcal{S}(\mathcal{I}_L)$, $\succsim_i$ is responsive over intervals. ■

Therefore, not only is responsiveness over intervals weaker than the max or min preference extension, it is 'somewhat' necessary for the GMI rule to be strategy-proof. The condition under which the latter holds is that the preferences over alternatives be single-peaked. If the latter condition is violated, there is no guarantee that a single-peaked preference extension over intervals will imply single-peakedness over alternatives, thereby violating responsiveness over intervals (since Proposition 1 would no longer hold).[8]

Theorem 2 cannot be made stronger in the following sense: *If* $\mathbb{D} \nsubseteq \mathcal{S}(\mathcal{I}_L)$*, then* ***for any*** $([\alpha_1], \ldots, [\alpha_{n-1}]) \in \mathcal{I}_L^{n-1}$*, the GMI rule* f^α *is not strategy-proof on* $\mathbb{D}^n$. In order to show that this statement is not true, we show that there exists a preference $\succsim \in \mathbb{D} \setminus \mathcal{S}(\mathcal{I}_L)$ for which f^α will continue to be strategy-proof for some $([\alpha], \ldots, [\alpha_{n-1}]) \in \mathcal{I}_L^{n-1}$. We provide an example.

Example 4. Suppose there are four alternatives, $a_1 < a_2 < a_3 < a_4$, $L = 2$ and there are three voters, $N = \{1, 2, 3\}$. Consider the following preference $\succsim$ such that $\tau(\succsim) = [a_1]$ is the peak interval, but $[a_3] \succ [a_2]$. Therefore, $\succsim \in \mathbb{D} \setminus \mathcal{S}(\mathcal{I}_L)$. Consider the GMI rule with $\alpha = (\alpha_1, \alpha_2) = ([a_1], [a_1])$, i.e., this GMI rule always picks the min of the top intervals of voters. Suppose $\pi = (\succsim_1, \succsim_2, \succsim_3) \in \mathbb{D}^3$ such that $\succsim_1 = \succsim$. Then, for any given preferences of the other voters, $\succsim_2, \succsim_3 \in \mathbb{D}$, it is always beneficial for voter 1 to truthfully report $[a_1]$ as the top-L interval of length 2. To see why, if no other voter is reporting $[a_1]$, then by reporting her top interval truthfully voter 1 gets her top-ranked interval since $med([a_1], [a_1], [a_1], \tau(\succsim_2), \tau(\succsim_3)) = [a_1]$. In any other situation, voter 1 is not strictly worse-off by truthfully reporting her peak interval $[a_1]$. It is easy to verify that voters 2 and 3 have no incentive to misreport their preferences either.

However, the same is not true for the GMI rule with $\alpha = ([a_3], [a_3])$. Voter 1 may get $[a_2]$ if no other voter reports $[a_3]$ and if she reports $[a_1]$, i.e., $med([a_1], [a_2], [a_2], [a_3], [a_3]) = [a_2]$. However, by reporting $[a_3]$, voter 1 gets $med([a_3], [a_2], [a_2], [a_3], [a_3]) = [a_3]$. Therefore, voter 1 gets better-off by misreporting her top interval. In this case, this GMI rule is not strategy-proof.

[8] We show by an example that assuming single-peaked preferences over intervals is weaker than assuming single-peaked preferences over alternatives. Preferences over alternatives need not be single-peaked for preferences over intervals to be single-peaked. e.g.: consider an ordered set of alternatives $a_1 < a_2 < a_3 < a_4 < a_5$ and $L = 3$. The preference P_i where $a_2 P_i a_4 P_i a_3 P_i a_1 P_i a_5$ is not single-peaked but the following responsive over intervals preference extension $\succsim_i$ is: $[a_2] \succsim_i [a_1]$ and $[a_2] \succsim_i [a_3]$.

1.5 Conclusion

We characterize *generalized median interval rules* on an extended single-peaked domain which satisfy responsiveness on intervals. It remains to be seen what the class of *strategy-proof* and *interval efficient* SCCs would be without responsiveness over intervals or if the aggregation rules picked non-intervals. This will depend on the nature of assumptions made on preference extensions to intervals or non-intervals.

Chapter 2

Fair allocation with semi-single-peaked preferences over location and quantity†

2.1 Introduction

There are many settings where a heterogeneous and perfectly divisible resource (e.g. land or advertisement slots) is to be allocated among multiple interested agents who have preferences over different intervals. The preferences may depend on the length of the interval (or quantity) and on the location of the interval. Some examples include allocating a piece of land for the construction of a facility or allocating advertisement slots during the day. We propose a new preference domain where the preference for quantity is single-peaked. In this chapter, we explore *envy-free* and Pareto efficient (i.e. fair) allocations in this setting for agents.[1]

We consider an allocation model where agents have preferences over intervals represented by the tuple (x, q) where x denotes the starting point or the 'location' of the interval and q denotes the length or the quantity of the interval. Therefore, each point (x, q) corresponds to an interval, $[x, x + q]$, which is a closed connected subset of the unit interval $[0, 1]$. We assume that agents have complete, transitive and continuous preference orderings over the set of all intervals.[2] However, we do not assume that

†This chapter is joint work with Mihir Bhattacharya, Ashoka University, Sonipat. We thank Debasis Mishra for providing valuable feedback at various stages of the project. We are also thankful to Bhaskar Dutta, Ariel Rubinstein and Arunava Sen for comments and suggestions. Comments and suggestions received at the Internal Seminars at the Department of Economics, Ashoka University, the Economics and Planning Unit, ISI Delhi and the Department of Economics, Shiv Nadar University have also been helpful.

[1] Varian (1973) defines an allocation to be *fair* if it is Pareto efficient and equitable (i.e. envy-free), this is inspired by the idea that fairness of allocation may be contributed by agents' judgment of their allocation, others' allocation and the comparison thereof.

[2] A binary relation R is called a *preference ordering* if it is (i) complete: xRy or yRx for all x, y and (ii) transitive: xRy and yRz implies xRz for all x, y, z. A preference, $\succ$ on X is continuous on X if and only if for any $x \in X$, the lower and upper contour sets, $\{y : x \succsim y\}$ and $\{y : y \succsim x\}$

preferences over location and quantity are separable. The set of all feasible bundles (x, q) can be represented as points in the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ as shown in Figure 3.

Consider the following motivating examples:

- Plot of land: an interval of land has to be allocated to each agent. An agent may prefer to have more (or less) quantity depending on the location due to additional benefits (or cost) at those locations. The marginal gain from a bigger piece of land may be less than the additional cost.
- Advertisement slots: agents advertise their product in different slots over the unit time interval. An agent may want slots at different times when their 'target' audience is attentive. Smaller slots may not be sufficient and larger slots may not be worth the additional cost.

Our model can be seen as a generalization of one-dimensional allocation models where the preferences are single-peaked. We allow agents to have preferences over location and quantity. We assume that agents have single-peaked preferences over quantity for every given location in the unit interval. This implies that for any given location there is a 'peak' quantity which makes that interval the best interval for that location and intervals with quantities closer to the best quantity are strictly preferred over the ones which are further away at that location.[3] However, a significant difference between our model and the standard notion of single-peakedness is that preferences over location are not single-peaked. Therefore, we use the term *semi-single-peaked* to denote these preferences.[4]

Envy-freeness requires that, in any allocation, each agent should weakly prefer their own bundle to that of any other agent. The idea of envy-freeness was introduced in Foley (1966) and further studied in Varian (1973) and Thomson and Varian (1984).

respectively, of x are closed.

[3]For formal treatment of single-peaked preferences, see Sprumont (1991) and Moulin (1980b). If preferences are monotonic, then $q = 1 - x$ is the peak quantity for every location $x \in [0, 1]$.

[4]Some additional assumptions are imposed which are discussed in detail in Section 2.2.

Envy-freeness is one of the central axioms of fairness and economic equity in the fair allocation literature (Thomson and Varian (1984), Moulin (2004), Fleurbaey and Maniquet (2011)). Most of the work in the literature only considers a uni-dimensional resource. In this chapter we consider the problem of dividing a resource with two dimensions (location and quantity) when the preference for quantity is single-peaked. We characterize the set of envy-free and Pareto efficient allocations for two agents and we provide some insights to obtain a fair allocation for any n number of agents.

Some crucial assumptions are required on the domain of semi-single-peaked preferences in our model. The first one requires that the 'peak' intervals at every location are connected by a top indifference curve (top IC) with end-points at the two axes of the domain (one such IC is shown in Figure 3a).[5] Another IC which plays an important role in our model is the *balanced IC* (Figure 5). For each preference ordering there is a balanced IC that intersects each of the lines $x = 0$ and $x + q = 1$ exactly once; and the two (sub)intervals represented by the end points of the balanced IC partition the good $[0, 1]$. The agent is indifferent between the two intervals since they are on the same IC. We prove the existence of the balanced IC for any given set of preferences using a fixed point argument. Moreover, the balanced IC is unique (Proposition 1).[6]

We first show that when both agents have *monotonic* preferences (more quantity is always strictly better irrespective of the location), both agents receive an allocation on or above their respective balanced ICs. This is similar to the Divide-and-Choose method used in the cake-cutting literature where one of the agents divides the cake or resource into two equal pieces according to her evaluation and the other agent picks the more valuable piece.[7] This is known to produce an envy-free allocation because the *divider*, being indifferent between the two pieces, has no envy, and the

[5]Each indifference curve (IC) represents a level of utility. In our model two ICs may have the same utility level only if they are identical or on different 'sides' of the top IC.

[6]If there is more than one balanced IC for a given set of preferences, then those ICs would have to intersect. However, this would contradict transitivity of preferences.

[7]See Brams and Taylor (1996) for a description of the Divide-and-Choose algorithm.

chooser selects the more preferred piece, ensuring she does not envy the divider. In our model, the balanced IC divides resource in a similar manner and allows us to identify an envy-free allocation in addition to guaranteeing that such an allocation exists.

We describe the *balanced curve allocation (BCA)* rule to characterize the full set of envy-free and Pareto efficient allocations. When agents have 'high enough' top ICs, in most cases, both agents receive allocations in the balanced IC region: the region of allocations between the two balanced ICs where one agent receives a bundle $(0, q)$ on the $x = 0$ axis and the other agent receives the remaining allocation $(q, 1 - q)$.[8] Therefore, in such cases, the two agents receive allocations on the two lines, $x = 0$ and $x + q = 1$ respectively. Observation 1 shows that when agents have identical and monotonic preferences, any envy-free and Pareto efficient allocation must assign them both a bundle on their balanced IC (here both have the same balanced IC since their preferences are same). The BCA ensures that an agent receives a bundle that is 'closer' to her top IC than the bundle received by the other agent.

Our main insight from the first result is that the set of envy-free and Pareto efficient allocations is large. When both agents prefer to have more quantity at any location, each agent gets a bundle on or above her balanced IC up to the point where the other agent does not get a bundle closer to her top IC. Whenever an agent does not prefer to have a large quantity, i.e., when her top IC is located closer to the $q = 0$ axis, the BCA gives her a bundle on the top IC, if possible, also ensuring Pareto efficiency.

Given that the set of envy-free and Pareto efficient set of rules is large, a natural question arises: are some of these rules strategy-proof, i.e., do they ensure truthful revelation of preferences? We find that for two agents, there are no strategy-proof, envy-free and Pareto efficient rules. However, characterizing allocation rules that are strategy-proof and Pareto efficient is still an open question due to the fact that there

[8] Note that when one agent receives the interval $(0, q)$ the remaining interval is $(q, 1-q)$ since the starting point of the left-over interval is q. We will use this notation throughout the chapter.

exists atleast one such rule (serial dictator rule defined in Section 2.4).

For more than two agents, we provide some arguments for the existence of a k-balanced IC which ensures that any k agents can be given allocations on the same IC without wastage (excess or deficit). We provide a necessary condition for an envy-free and Pareto efficient allocation. We indicate the possibility of generalizing the BCA allocation from 2 to any k number of agents. However, further restrictions will have to be imposed on the preferences to guarantee the existence of fair allocations in such cases.

There are some works studying envy-free and group envy-free allocations in the Walrasian equilibrium setting (see Varian (1973), Foley (1966), Cato (2010) and Donnini and Pesce (2021)). However, all these models assume canonical preferences which are convex and monotonic. The preferences in our model need not be monotonic with respect to quantity and hence, their results do not apply to our model.

Thomson (1994) studies a model of resource allocation in the single-peaked setting where the only dimension is quantity. They show that the uniform rule is the only rule that is Pareto efficient, envy-free and peaks-only. Sprumont (1991) considers a similar setting to characterize the uniform rule as the only strategy-proof, anonymous and efficient allocation rule. The BCA can be seen as the two-dimensional variant of the uniform rule. However, our model does not reduce to the one-dimensional single-peaked model when location 'does not matter' (when the ICs have zero slope everywhere).[9]

Allocation of multiple perfectly divisible goods is also well-studied in the literature. Richter and Rubinstein (2020) define a general model of equilibrium, without institutions, with envy-freeness as one of the properties of the equilibrium. However, they assume convexity of preferences and the convexity of the feasible set of allocations.[10] Therefore, our model of fair allocation cannot be represented as a special case

[9] Thomson (2011) and Thomson (2016) provide excellent surveys of the literature on envy-free allocations.

[10] The feasible set is the set which consists of all the feasible allocations. For example, in our

of their model.[11] Morimoto et al. (2013) characterize the uniform rule with several commodities and Euclidean single-peaked preferences using strategy-proofness, symmetry, unanimity and non-bossiness. Anno and Sasaki (2013) characterize generalized uniform rule in the multidimensional single-peaked setting.

The problem of 'cake-cutting' is widely studied in the literature as well, where the cake is assumed to be heterogeneous and agents have valuation functions, which may be piece-wise uniform over an interval representing the cake. Brams et al. (2013) and Brams et al. (1995) highlight the difficulties in finding a fair allocation of the cake when the number of agents is large. The latter work provides moving-knife algorithms to obtain envy-free allocations. Lindner and Rothe (2015) notes that "*...despite intense efforts over decades, up to this date no one has succeeded in finding a finite-bounded cake cutting protocol that guarantees envy-freeness for any number of players...*". Similarly, Stromquist (2008) provides an impossibility result for envy-free cake divisions by finite protocols. Many papers in the cake-cutting literature consider normalized valuation functions to get positive results (see Chen et al. (2013), Aumann and Dombb (2015) and Caragiannis et al. (2012)). The approach taken in our chapter is different, due to the ordinal nature of preferences. The restriction on the domain of preferences, i.e., single-peakedness in quantity, allows us to obtain positive results in this setting.

Bogomolnaia and Moulin (2023) studies the divide-and-choose and moving knife rules and provides conditions for minimum guarantees when preferences are represented by continuous utility functions (but may not be monotone or convex). In our model, when agents have monotonic preference over quantity, the BCA guarantees at least the worst allocation in the balanced IC region. This property may also hold for more than 3 agents. We hope that future work will provide further insights into this. Aziz and Mackenzie (2016) provides envy-free and bounded algorithms for cake-cutting for any number of agents. However, their procedure requires

model, for two agents, the allocation, $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ is feasible but $\{[0, \frac{1}{2}], [\frac{1}{3}, 1]\}$ is not.

[11] I am thankful to Ariel Rubinstein for this comment.

an exponentially high number of queries. Our approach in this chapter is closer to the literature on allocation models with single-peaked preferences since we do not assume that more quantity is always better. Due to this and the additional dimension, i.e., *location*, characterizing the set of fair allocations in our model is notably more challenging.

The model in this chapter is fairly general and the preferences in our model cannot always be represented by a single-variable valuation function over a one-dimensional space as in the cake-cutting literature (Procaccia (2016)). This is due to the fact that agents' preference over quantities depend on the location of the interval. Therefore, for continuous valuation functions, a strict subset of the interval that has the highest valuation will always have a strictly smaller valuation if the valuation at *every* location of the bigger interval is positive. Since we do not assume this property, a single valuation function would not be able to represent semi-single-peaked preferences.[12] The analogous cardinal version of our model would require a valuation function $f_x(q)$ for every given location x in the unit interval. However, computing envy-free allocations would be even more challenging.

The contribution of our result is more in line with the literature on axiomatic characterization of fair allocations. We provide some insights on how to obtain an envy-free and Pareto efficient allocation for more than two agents when preferences are representable by linear ICs and when more quantity is better. For this we prove the existence of a general k-pieced balanced IC using an extended fixed-point argument.

The chapter is organized as follows. Section 2 discusses the model, preferences and the assumptions that characterize semi-single-peaked preferences. Section 3 describes the axioms and Section 4 provides the results for two agents. Section 5 provides some observations for more than two agents and Section 6 provides the conclusion. Ap-

[12] More specifically if $I = [a, b] \subset [0, 1]$ is the interval with the highest valuation, which is given by $V(I) = \int_a^b f(x)dx$ for a continuous valuation function $f(x)$, then for any interval $I' \subset I$, $V(I') \leq V(I)$. However, the preferences in our model allow for the set $I' = [a', b'] \subset I$ to be higher valued if $a' \neq a$.

pendix A.1 provides the proofs that were omitted in the main text and Appendix A.2 provides some additional observations. The references are provided at the end.

2.2 The Model

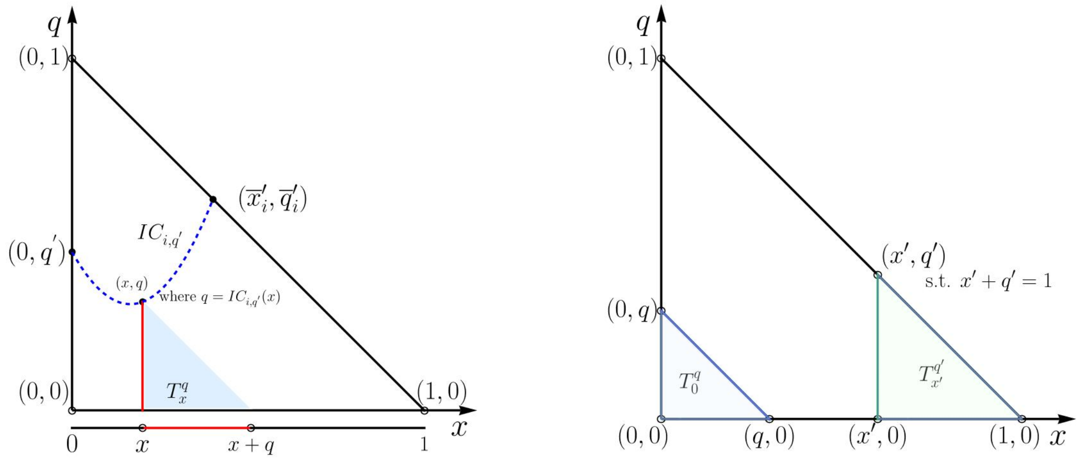
A heterogeneous and perfectly divisible resource is distributed on the interval $[0, 1]$ and has to be divided into disjoint intervals (except at end-points) among n agents. The set of agents is $N = \{1, 2, \ldots, n\}$. The interval $[0, 1]$ has to be divided such that each agent receives a closed interval $[x, x + q]$ where x is the starting point of the interval and q is the quantity of the interval. We will refer to the tuple (x, q) as a **bundle** which represents the interval $[x, x + q] \subseteq [0, 1]$. Formally, an **allocation** is a collection of bundles $\{a_i\}_{i=1}^n \equiv \{(x_i, q_i)\}_{i=1}^n$ where x_i and q_i are the starting points and length of the interval respectively for the agent i.

However, not all allocations are feasible. For example, the allocation $\{a_1, a_2\} = \{(0, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3})\}$, where allocating the interval $[0, \frac{1}{2}]$ to one agent and $[\frac{1}{3}, 1]$ to another agent is not feasible since the resource cannot be shared (e.g. if these are advertisement slots). An allocation $a = \{(x_i, q_i)\}_{i \in N}$, is said to be **feasible** if (i) for any $i \in N$, $q_i > 0$, $\sum_i q_i \leq 1$ and (ii) for any pair of distinct agents i and j, $x_i + q_i \leq x_j$ or $x_j + q_j \leq x_i$. The first condition ensures that the resource is allocated among the n agents with each receiving a positive quantity which may sum to less than 1. This also allows for *free disposal*, i.e., we do not need to allocate the full resource to the agents.[13] The second condition ensures that the intervals do not intersect except at a single point, which are sets of measure zero.[14]

Another way to visualize the allocation bundles is to arrange the agents who have received allocations from left to right, i.e., let $i^* \in \{1, ..., n\}$ be a re-ordering of agents such that $x_{1^*} < x_{2^*} < ... < x_{n^*}$. Then it must be the case that $x_{2^*} \geq x_{1^*} + q_{1^*}$,

[13]This is a fair assumption to make if the resource is advertisement time slots or a piece of land. However, if the resource is a set of contiguous tasks or work-shift slots which need to be fully allocated, this assumption would have to be relaxed. In this chapter, we focus on the former.

[14]Sets of measure zero have no consequence on the valuation or utility of the bundles. This is a standard assumption in the fair division literature (Procaccia (2016), Thomson (2011)).



(a) Allocation corresponding to (x, q) (b) Illustration for T_0^q and $T_{x'}^{q'}$

Figure 3: The set of alternatives X is the set of all the points in the triangle T_0^1 with vertices $\{(0,0),(0,1),(1,0)\}$

$x_{3^*} \geq x_{2^*} + q_{2^*}, ..., x_{n^*} \geq x_{n^*-1} + q_{n^*-1}$. For example, for three agents, the feasible allocation: $a = (a_1, a_2, a_3)$ such that $a_1 = (0, \frac{1}{2})$, $a_2 = (\frac{1}{2}, \frac{3}{10})$ and $a_3 = (\frac{8}{10}, \frac{2}{10})$ indicates that agent 1 receives the interval $[0, \frac{1}{2}]$, agent 2 has receives the interval $[\frac{1}{2}, \frac{8}{10}]$ and agent 3 receives the interval $[\frac{8}{10}, 1]$. The set of all feasible allocations for n agents shall be denoted by $\mathcal{A}$.[15]

The problem defined above can be simplified by visualizing any bundle (x, q) (corresponding to the interval $[x, x+q]$) where $x, q \in [0,1]$ and $x + q \leq 1$ as a point in the right-angled isosceles triangle with vertices at $(0,0)$, $(0,1)$ and $(1,0)$ (Figure 3). Every point in this triangle is a bundle (x, q) which corresponds to an interval $[x, x+q]$ where $x, q \in [0,1]$ and $x + q \leq 1$. This makes the problem easier to study since we can directly define preferences on the set of alternatives or bundles denoted as $X = \{(x,q) \in [0,1]^2 : x + q \leq 1\}$.

For any $x, q \in [0,1]$ such that $x + q \leq 1$, we denote a right-angled isosceles triangle with vertices at $(x,0)$, $(x+q,0)$ and (x,q) by T_x^q. Therefore, the set of alternatives can also be denoted as $X = T_0^1$. We are now ready to define preferences on X through

[15] Note that this set cannot be represented as a cross-product of any set since one's allocation depends on the other.

a sequence of assumptions.

Semi-single-peaked preferences. We define a new preference domain, the semi-single-peaked preference domain over $X = T_0^1$ which is a subset of the set of complete, transitive and continuous preferences over X. These are motivated by the interpretation of single-peaked preferences defined over a line (see Sprumont (1991) for example). In this chapter we will consider preferences which are single-peaked with respect to quantity for *any given* location $x \in [0, 1]$. Our first assumption requires functions to be complete, transitive and continuous, which is standard in the literature.

Assumption 0: *We assume that preference $\succsim_i$ of each agent $i \in N$ is complete, transitive and continuous.*[16]

We will denote the preferences by $\succsim$ where $\succ$ denotes the asymmetric part of the relation and $\sim$ denotes indifference. Let $\mathcal{C}$ be the set of all complete, transitive and continuous functions. We note that by Debreu et al. (1954) and Debreu (1959) these preferences can be represented by continuous utility functions, $u : T_0^1 \to \mathbb{R}$. These are further representable by continuous indifference curves (ICs). However, we will impose further restrictions on the structure of preferences and the corresponding ICs which guarantee the existence of fair allocations.

The first assumption is that preferences can be represented by ICs which are Indifference Curve Functions (ICFs) which map every location in its domain to a quantity in X. These ICs connect the two axes: $x = 0$ and $x + q = 1$ at the locations $(0, q)$ and $(\overline{x}_i, \overline{q}_i)$ respectively. Therefore, $\overline{x}_i$ (resp. $\overline{q}_i$) is the location (quantity) of the bundle on the given IC on the $x + q = 1$ axis.[17]

Assumption 1: **Existence of Indifference Curve Functions (ICFs).** *There exist K-Lipschitz continuous indifference curve functions (ICFs) with $K \in (0, 1]$,*

[16] Completeness: For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ or both. Transitivity: If $x \succeq y$ and $y \succeq z$, then $x \succeq z$. Continuity (of preferences): For all $x \in X$, the sets $\{y \in X \mid y \succeq x\}$ and $\{y \in X \mid x \succeq y\}$ are closed.

[17] The first bundle $(0, q)$ does not have the subscript i since for any other $j \in N$, $IC_{j,q}$ would have the same bundle on that IC on the axis $x = 0$. However, the bundle on the right axis will be denoted by $(\overline{x}_j, \overline{q}_j)$.

$IC_{i,q} : [0, \overline{x}] \to [0, 1]$ *for any* $q \in [0, 1]$ *where* $(0, q) \sim_i (\overline{x}_i, \overline{q}_i)$ *s.t.* $IC_{i,q}(\overline{x}_i) = \overline{q}_i$ *and* $\overline{x}_i + \overline{q}_i = 1$.[18]

Assumption 1 requires that for every $q \in [0, 1]$ and for every $i \in N$ there exists an indifference curve function (ICF), denoted $IC_{i,q}$, such that for each $x \in [0, \overline{x}_i]$, the bundle $(x, IC_{i,q}(x))$ is indifferent to the bundle $(0, q)$; that is, $(0, q) \sim_i (x, IC_{i,q}(x))$.[19] The endpoint $\overline{x}_i$ is determined such that $\overline{q}_i = IC_{i,q}(\overline{x}_i)$ and $\overline{x}_i + \overline{q}_i = 1$. This implies that the indifference curve intersects the line $x + q = 1$ at the point $(\overline{x}_i, \overline{q}_i)$. We use subscripts i and j to distinguish between agents, particularly when referring to their respective IC endpoints. Thus, $IC_{i,q}$ refers to all bundles $(x, IC_{i,q}(x))$ that are indifferent to $(0, q)$, for all $x \in [0, \overline{x}_i]$.

The above assumption requires that ICs are continuous mappings from locations to quantities within T_0^1. The fact that IC functions are K-Lipschitz continuous with $K \in (0, 1]$ guarantees that for every per unit change in the location of the interval, the change in quantity required to stay on the same indifference curve must be less than or equal to K. In other words, this ensures that sudden changes in preferences are precluded. This assumption also plays an important role for the existence of fair allocations in the semi-single-peaked preference domain. The importance of this assumption is made clear by Observation 1.[20]

We provide some examples of preferences in terms of utility functions which are representable by ICFs and one which cannot be represented by ICFs.

Example 5. Consider the following utility functions representing different preferences:

(i) $u(x, q) = q$ represents monotonic preferences, i.e., more quantity is always strictly preferred irrespective of the location. These preferences are representable by ICFs which are constant functions (horizontal lines parallel to the

[18] A function $f(x) : \mathbb{R} \to \mathbb{R}$ is K-Lipschitz continuous if for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq K|x - y|$.

[19] However, an IC may not represent all the bundles which are indifferent to these end-points. For non-monotonic preferences and every IC above the top IC there is a corresponding IC below the top IC which represents the same level of utility.

[20] Specifically, Observation 3 states that there may be no fair allocations if $K > 1$.

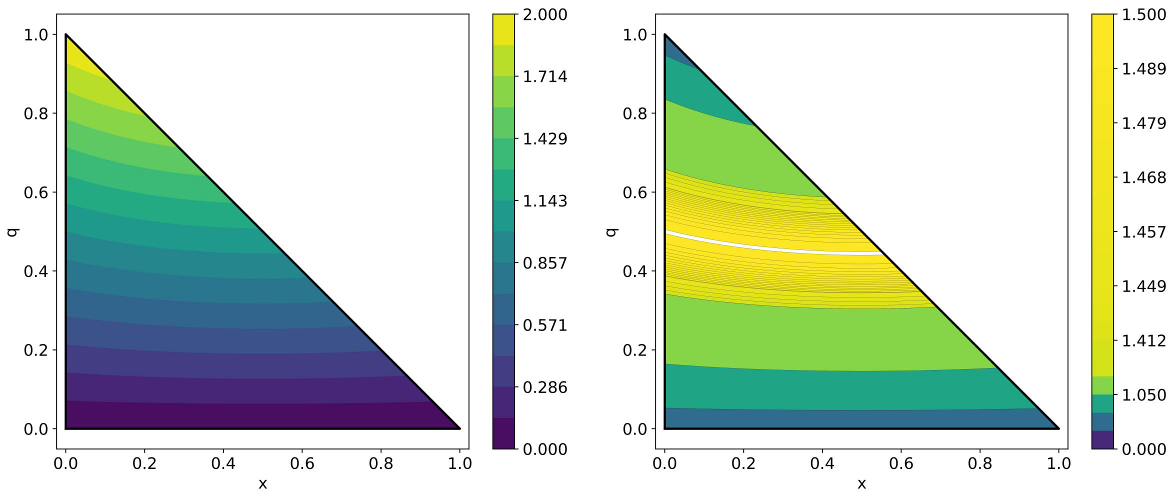
$q = 0$ line).

(ii) $u(x, q) = -(0.6 - q)^2$ represents a preference where the agent prefers to have a quantity of $q = 0.6$ and location does not matter. These preferences are also representable by straight line ICs (ICFs are constant functions) but the top IC is at $q = 0.6$.

(iii) The following are some examples of preferences where location matters and the ICFs are convex functions:

(iii.a) $\forall x \in [0, 1]$, $q = 1 - x$, the utility function $u_1(x, q) = q(2 + x - x^2)$ gives convex ICs and is monotonic i.e. top IC is the point $(0, 1)$.

(iii.b) $\forall x \in [0, 1]$, $q = 1 - x$, the utility function $u_2(x, q) = 1.5 - (u_1(x, y) - 1)^2$, the top IC passes through (x_1, q_1) such that $u_1(x_1, q_1) = 1$.



(a) ICs corresponding to $u_1(x, q)$

(b) ICs corresponding to $u_2(x, q)$

Figure 4: Plotting ICs as contour plots of utility functions in Example 5 (iii)

(iv) The utility function, $u(x, q) = x + 2q$ and the corresponding preferences are representable by ICFs which are sloping downwards in X. However, these preferences will be ruled out by our next assumption which does not allow ICs consisting of bundles with positive quantity to intersect the $q = 0$ line (bundles with zero quantity must be on the lowest IC). In the given preference,

$u(1,0) = u(0, \frac{1}{q})$ which is ruled out by our next Assumption.

Assumption 1 also guarantees the existence of a '**balanced IC**' IC_{i,q_i^f} (or IC_i^f for simplicity) or a balanced IC which cuts the two axes at $(0, q_i^f)$ and $(\overline{x}_i^f, \overline{q}_i^f)$ respectively where $q_i^f = \overline{x}_i^f$ (Figure 5). This IC allows us to find two allocations on the two axes such that the agent i is indifferent between the two bundles. We show later that this property is necessary for the characterization of envy-free and Pareto efficient allocations (Proposition 3).

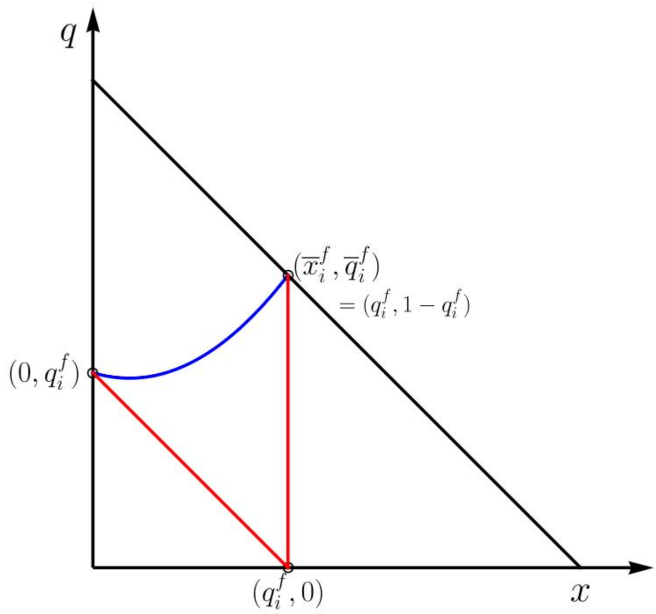


Figure 5: Balanced IC for $\succsim_i$ where $(0, q_i^f) \sim_i (q_i^f, 1 - q_i^f)$

The next assumption states that bundles with zero quantity are least preferred.

Assumption 2: **Lowest at zero quantity.** *For any* $\succsim_i \in \mathcal{C}$, $i \in N$ *for all* $x \in (0,1]$, $(0,0) \sim_i (x,0)$ *for all* $q > 0$, $(0,q) \succ_i (0,0)$.

The preferences in Part (iv) of Example 5 are ruled out by Assumption 2 above. All the other three preferences in the example satisfy assumptions 0, 1 and 2. All the assumptions till now do not rule out "thick" indifference curves since we have not imposed non-satiation. Since we want to capture single-peakedness in quantity, our preference domain will allow for satiation at the top IC (which is assumed to be unique in the next assumption). Our next assumption states that bundles on the $x = 0$ (or $x + q = 1$) axis which are 'closer' to the top IC are strictly preferred to the ones further away if they are on the same side of the top IC. This is a natural extension of single-peakedness on a line to single-peakedness with respect to quantity in our model. An important implication of this assumption is that it allows us to rank bundles on two different ICs on the same side of the top IC.

Assumption 3: **Single-peakedness in quantity.** *For any* $\succsim_i \in \mathcal{C}$*,* $i \in N$ *there exists a unique top ICF,* $IC_i^{\tau} : [0, \overline{x}_i^{\tau}] \to [0,1]$ *such that for any* $q', q'' \in (0,1)$*,*

$$(i)\ [q_i^{\tau} \leq q' < q'' \text{ or } q'' < q' \leq q_i^{\tau}] \implies [(0,q') \succ_i (0,q'')],\ or$$
$$(ii)\ [\overline{x}_i^{\tau} \leq \overline{x}' < \overline{x}'' \text{ or } \overline{x}'' < \overline{x}' \leq \overline{x}_i^{\tau}] \implies [(\overline{x}',\overline{q}') \succ_i (\overline{x}'',\overline{q}'')].$$

Assumption 3 requires the existence of a unique top ICF, $IC_{i,q_i^{\tau}} \equiv IC_i^{\tau}$ which represents the best set of $\succsim_i$ with end-points $(0, q_i^{\tau})$ and $(\overline{x}_i^{\tau}, \overline{q}_i^{\tau})$. Part (i) of Assumption 3 requires that bundles at $x = 0$ which are closer in quantity to the bundle $(0, q_i^{\tau})$ on either side of the top IC are strictly better than those that are further away. This is the standard extension of single-peakedness from the one-dimensional settings applied to the axis $x = 0$. Part (ii) of Assumption 3 requires that bundles along the $x + q = 1$ which are closer to the top IC are strictly preferred to the ones further away.

We note that Assumption 3 requires either part (i) or part (ii) to hold. Either part allows us to define 'closeness' of ICs to the top IC on the same side of the latter. Transitivity of the preferences implies that the ICs do not intersect (Mas-Colell et al. (1995)). Therefore, we can say that an $IC_{i,q}$ is 'closer' to the top IC, IC_i^{τ} than another $IC_{i,q'}$ if $q_i^{\tau} \leq q < q'$ or $q' < q \leq q_i^{\tau}$. This also implies that for any vertical line $x = k$ for some $k \in (0,1)$ intersecting the three ICs, the bundle lying on $IC_{i,q}$ will lie between the bundles on the other two ICs. In other words, for any two bundles: one on IC_i^{τ} and the other on $IC_{i,q'}$, there is a bundle on $IC_{i,q}$ which lies on a line joining the two bundles.

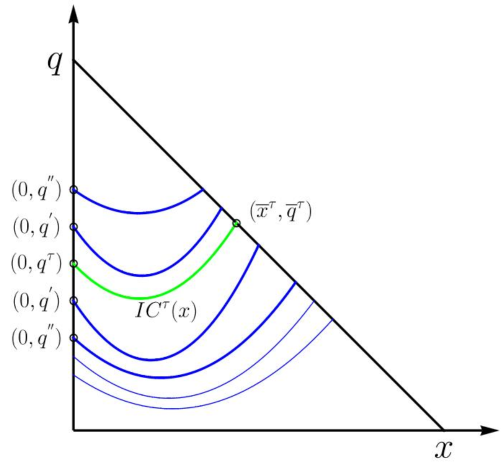


Figure 6: An example of semi-single-peaked preferences

The set of all preferences $\succsim \in \mathcal{C}$ satisfying Assumptions 0 to 3 over $X = T_0^1$ is the set of all semi-single-peaked preferences denoted by $\mathcal{D} \subset \mathcal{C}$. A preference profile $\mathbf{P} = (\succsim_1, \succsim_2, ..., \succsim_n) \in \mathcal{D}^n$ is a tuple of n preferences each belonging to $\mathcal{D}$.

A *allocation rule* is a function, $f : \mathcal{D}^n \to \mathcal{A}$, that takes in a preference profile $\mathbf{P} \in \mathcal{D}^n$ and produces a feasible allocation $f(P) = (f_1(\mathbf{P}), f_2(\mathbf{P}), \ldots, f_n(\mathbf{P})) \in \mathcal{A}$, where $f_i(\mathbf{P}) \in T_0^1$ is the allocation of agent $i \in N$.

Remark 1. *We do not require ICFs to be either convex or concave. However, Assumption 1 requires ICFs to be K-Lipschitz functions with $K \in (0, 1]$. This ensures that an ICF, $IC_{i,q}$ which starts at $(0, q)$ does not lie in the interior of the triangle T_0^q. For our results, this is sufficient to guarantee existence of fair allocations as highlighted in Observation 3. We do not impose any restrictions on differentiability or concavity/convexity of the ICFs.*

2.3 Axioms

The first axiom is the standard notion of efficiency: an allocation is Pareto efficient if no agent can be made strictly better-off without making another agent strictly worse-off.

Pareto Efficiency. *An allocation rule,* $f : \mathcal{D}^n \to \mathcal{A}$*, is Pareto efficient if for any preference profile* $\mathbf{P} \in \mathcal{D}^n$ *there does not exist another feasible allocation* $\{(x_i, q_i)\}_{i \in N} \in \mathcal{A}$ *s.t.* $(x_i, q_i) \succsim_i f_i(\mathbf{P})$ *for all* $i \in N$ *and* $(x_j, q_j) \succ_j f_j(\mathbf{P})$ *for some* $j \in N$*.*

In other words, an allocation is Pareto efficient if for any other allocation whenever an agent is strictly better-off, there is another agent who is strictly worse-off. The next axiom is an important one in the theory of equity.

Envy-free. *An allocation rule,* $f : \mathcal{D}^n \to \mathcal{A}$*, is said to be envy-free if for all* $\mathbf{P} \in \mathcal{D}^n$*, for all* $i, j \in N$*,* $f_i(\mathbf{P}) \succsim_i f_j(\mathbf{P})$*.*

An allocation rule is envy-free if every agent prefers her own bundle to any other agent's bundle. Our final axiom, strategy-proofness, is an inter-profile condition.

Strategy-proof. An allocation rule, $f : \mathcal{D}^n \to \mathcal{A}$, is said to be *strategy-proof* if for any $i \in N$ and for any $(\succsim_i, \succsim_{-i}) \in \mathcal{D}^n$,

$$f_i(\succsim_i, \succsim_{-i}) \succsim_i f_i(\succsim_i', \succsim_{-i}) \quad \text{for all } \succsim_i' \in \mathcal{D}.$$

An allocation is strategy-proof if it does not provide any incentive for individual agents to benefit strictly by individually misreporting one's preference.

2.4 Results

In this section we study envy-free and Pareto efficient allocation functions. It is easy to verify that when agents have identical preferences, envy-freeness requires agents to be indifferent between any bundle allocated to any of the agents, i.e., for any $\mathbf{P} = (\succsim_1, \succsim_2) \in \mathcal{D}^n$ such that $\succsim_1 = \succsim_2 = \succsim$ if $f(\mathbf{P}) \equiv (f_1(\mathbf{P}), f_2(\mathbf{P}))$ is an envy-free allocation, then $f_i(\mathbf{P}) \succsim_i f_j(\mathbf{P})$ for $i, j \in \{1, 2\}$. This implies that $f_1(\mathbf{P}) \sim_i f_2(\mathbf{P})$ for $i \in \{1, 2\}$. However, this does not necessarily imply that they are on the same IC unless preferences are monotonic. We define monotonic preferences which is a subset of the set of semi-single-peaked preferences. These preferences will be motivated to illustrate the importance of a balanced IC.

Monotonic preferences. *A semi-single-peaked preference* $\succsim \in \mathcal{D}$ *is monotonic if* $\tau(\succ_i) = \{(0,1)\}$. An agent with monotonic preferences prefers a bundle with higher quantity at any given location.

Additionally, we show that when both agents have identical and monotonic preferences both the bundles $f_1(\mathbf{P}), f_2(\mathbf{P}) \in X$ must lie on the same IC of both agents. We will call such ICs *balanced ICs*. We define them formally below and provide a result on its existence.

Balanced IC: For any $i \in N$ the *balanced IC* of agent i is an IC of the form IC_i^f[21] if there exists a $q_i^f \in (0,1)$ such that $(0, q_i^f) \sim_i (\overline{x}_i^f, \overline{q}_i^f)$ with $\overline{x}_i^f + \overline{q}_i^f = 1$. A balanced IC of agent i consists of two allocations $(0, q_i^f)$ and $(\overline{x}_i^f, \overline{q}_i^f)$ which lie on the same IC. Figure 5 provides an illustrative example of a balanced IC. We show in Observation 1 that the existence of balanced ICs is necessary for the existence of envy-free and Pareto efficient allocation. However, the existence of a balanced IC for every preference in $\mathcal{D}$ is not obvious.

Claim 1. *Suppose* $\succsim \in \mathcal{D}$ *is monotonic. For all* $(x,q), (x,q'), (x',q') \in X$*,*

$$(i)\ [q > q'] \implies [(x,q) \succ (x,q')],$$

$$(ii)\ [\overline{x} + \overline{q} = \overline{x}' + \overline{q}' = 1 \text{ and } \overline{x} < \overline{x}'] \implies [(\overline{x}, \overline{q}) \succ (\overline{x}', \overline{q}')].$$

The proof of the Claim is omitted since it follows from transitivity of preferences and single-peakedness in quantity.

Therefore, if $\succ \in \mathcal{D}$ is monotonic, ICs 'closer' to the point $(0,1)$ are strictly better than those further away since every bundle on the $IC_{i,q}$ is preferred to every bundle on $IC_{i,q'}$ for $q > q'$. We provide the set of fair allocations for such preferences below which plays an important role in the design of the more general fair allocation rule.

Observation 1. *Suppose* $\mathbf{P} = (\succsim_1, \succsim_2)$ *such that* $\succsim_1 = \succsim_2 = \succsim$*, and* $\tau(\succsim_i) = \{0,1\}$

[21] For ease of exposition, henceforth, let $IC_{iq_i^f}(x) \equiv IC_i^f(x)$.

for each $i \in \{1,2\}$. Any envy-free and Pareto efficient allocation rule f such that $f(\mathbf{P}) = (f_1(\mathbf{P}), f_2(\mathbf{P})) = ((x_1, q_1), (x_2, q_2))$, allocates the bundles, $f_1(\mathbf{P})$ and $f_2(\mathbf{P})$ which are both on the common balanced IC, $IC_i^f(x)$ which implies that $q_1 + q_2 = 1$. In other words, $f_1(\mathbf{P}) \sim_1 f_2(\mathbf{P})$ and $f_1(\mathbf{P}) \sim_2 f_2(\mathbf{P})$.

To see why the above is true, first note that both agents have the same balanced IC (since preferences are identical), and they must receive bundles on the same IC in order to not envy each other: if the two allocation bundles are not on the same IC, then the agent on the lower IC will envy the other agent. Suppose the allocation is not on the balanced IC, $IC_1^f = IC_2^f$. Note that any allocation where both bundles are on the same IC cannot be on an IC above the balanced IC, since that would not be feasible. To see this, suppose $f_1(\mathbf{P}), f_2(\mathbf{P}) \in \{(0, q), (\overline{x}, \overline{q})\}$ with $f_1(\mathbf{P}) \neq f_2(\mathbf{P})$ and $q > q_1^f$. Note that $(0, q)$ and $(\overline{x}, \overline{q})$ are on $IC_{i,q}$ while the two bundles $(0, q_1^f)$ and $(\overline{x}_1^f, \overline{q}_1^f)$ are on the balanced IC, IC_1^f. Therefore, $\overline{q} > \overline{q}_1^f$. But this implies that $q + \overline{q} > \overline{x}_1^f + \overline{q}_1^f = q_1^f + \overline{q}_1^f = 1$. This implies that the allocation $f_1(\mathbf{P}), f_2(\mathbf{P}) \in \{(0, q), (\overline{x}, \overline{q})\}$ for any $f_1(\mathbf{P}) \neq f_2(\mathbf{P})$ (with both bundles of the allocation on the same IC) is not feasible since the sum of quantities of the two bundles $f_1(\mathbf{P})$ and $f_2(\mathbf{P})$ exceed the total available quantity.

Suppose the allocation of the agents lie on an IC below the balanced IC. Suppose that $(f_1(\mathbf{P}), f_2(\mathbf{P})) = (a_1, a_2)$ such that a_1 and a_2 are on an IC below the balanced IC, $IC_1^f = IC_2^f$. We argue that this is not Pareto efficient, since both would strictly prefer receiving either of the two allocations $(0, q_i^f)$ or $(\overline{x}_i^f, \overline{q}_i^f)$ for $i \in \{1,2\}$ on the balanced IC since it is on an IC closer to their top IC. Recall that IC_i^f is closer to IC_i^τ (here, $IC_i^\tau = (0,1)$) than any other $IC_{i,q}$ which lies completely below IC_i^f. Therefore, both agents would be strictly better-off since their top allocation is $(0,1) \in X$. Therefore, any fair allocation has to be on the balanced IC when the top IC is at $(0,1)$ and both agents have the same preference. We will use a more general version of this observation in the proof of our main theorem. Note that most preferred bundle of both agents is $[0,1]$. Since both agents prefer to have as much quantity as possible at any given location, Pareto efficiency requires that $f(\mathbf{P}) = (a_1, a_2) = ((x_1, q_1), (x_2, q_2))$ such that $q_1 + q_2 = 1$.

Our next proposition proves that there always exists a balanced IC for every $\succsim \in \mathcal{D}$ and that it is unique.

Proposition 1. *For any $i \in N$ with preference $\succsim_i \in \mathcal{D}$, there exists an IC, $IC_{i,q_i^f} : [0, q_i^f] \to [0,1]$ for some $q_i^f \in (0,1)$ such that $(0, q_i^f) \sim_i (\overline{x}_i^f, \overline{q}_i^f)$. Moreover, q_i^f is unique for every preference $\succsim_i \in \mathcal{D}$. We will denote IC_{i,q_i^f} by IC_i^f.*

Proposition 1 proves the existence of a unique balanced IC for any given preference $\succsim_i \in \mathcal{D}$. This will be used frequently to describe the allocation rule to find an envy-free and Pareto efficient allocation. When both agents have monotonic preferences (as in part (ii) in Example 5), the existence of a balanced IC ensures that the whole resource can be allocated by allocating each end-point bundle of the balanced IC to the two agents. Therefore, the formulation of preferences as ICFs defined on X guarantees existence of fair allocations.[22] The proof is provided in the Appendix A.1.

The following Claims provide bounds on the location and quantity of the bundles which are Pareto efficient and envy-free. The proofs of these Claims are provided in the appendix.

Claim 2. *Suppose $f : \mathcal{D}^2 \to \mathcal{A}$ is Pareto efficient. Then, for any $\mathbf{P} \in \mathcal{D}^2$,*

$$[f_i(\mathbf{P}) = (0, a) \text{ and } f_j(\mathbf{P}) = (a, 1-a)] \implies [\, a \leq q_i^\tau \text{ and } \overline{q}_j^\tau \geq 1 - a].$$

The above Claim states that for any Pareto efficient allocation where both agents are allocated bundles on the two axis such that no resource is left-over, then it must be the case that agent on the left axis at $(0, a)$ must be receiving a quantity, a, which is less than or equal to the quantity required to allocate him on his top IC given the location $x = 0$. Similarly, for the agent j who is allocated on the right axis at $(a, 1-a)$, must be receiving less quantity, $1-a$, than the quantity, $\overline{q}_j^\tau$, required to be

[22] Similar properties are observed in the one-dimensional single-peaked allocation models (Thomson (1994)).

at his top IC on the axis $x + q = 1$. The proof is provided in the Appendix A.1. The The next claim identifies a case where envy is bound to occur.

Claim 3. *Suppose $f : \mathcal{D}^2 \rightarrow \mathcal{A}$. Let $P \in \mathcal{D}^2$ and $f_i(\mathbf{P}) = (0, a)$ and $f_j(\mathbf{P}) = (a, 1-a)$. Then,*

$$(i)\ [a < q_i^f < q_i^\tau \text{ and } 1 - q_i^f < 1 - a \leq \overline{q}_i^\tau] \Rightarrow (a, 1-a) \succ_i (0, a)$$
$$(ii)\ [q_j^f < a \leq q_j^\tau \text{ and } 1 - a < 1 - q_j^f < \overline{q}_j^\tau] \Rightarrow (0, a) \succ_j (a, 1-a).$$

Part (i) of the above Claim states that if agent i receives a bundle on the left axis and agent j receives a bundle on the right such that the latter's bundle $(a, 1-a)$ is on an IC closer to agent i's top IC (implied by $1 - q_i^f < 1 - a \leq \overline{q}_i^\tau$), then agent i will envy agent j. Similarly, part (ii) of the Claim states that if agent j's allocation on the right axis is on an IC further away from his top IC than the allocation of agent i on the other axis (implied by $1 - a < 1 - q_j^f < \overline{q}_j^\tau$), then agent j will envy i. The proof is provided in the Appendix A.1.

Definition 1 (Balanced-Curve Allocation (BCA))**.** *An allocation rule $f : \mathcal{D}^2 \rightarrow \mathcal{A}$ is the BCA rule if for every $\mathbf{P} \in \mathcal{D}^2$ it produces an allocation according to the rule provided below.*

The allocations which have property that $x_1 \leq x_2$, i.e., agent 1 is given an allocation on the left of that of agent 2 will be denoted as LR allocations, where L denotes 'left' for agent 1, and R denotes 'right' for agent 2. Similarly, allocations where $x_2 \leq x_1$ will be denoted as RL allocations.

Case 1: *$\overline{q}_2^\tau \leq 1 - q_1^\tau$ or $\overline{q}_1^\tau \leq 1 - q_2^\tau$ or both. Here, both agents get an allocation on their top IC. There are three sub-cases.*

Case 1(i): $\overline{q}_2^\tau \leq 1 - q_1^\tau$ and $\overline{q}_1^\tau \leq 1 - q_2^\tau$. In this case, both LR and RL type allocations are possible.

LR: If $\overline{q}_2^\tau = 1 - q_1^\tau$ then $f_1(\mathbf{P}) = (0, q_1^\tau)$, and $f_2(\mathbf{P}) = (\overline{x}_2^\tau, \overline{q}_2^\tau)$. If $\overline{q}_2^\tau < 1 - q_1^\tau$ define $\tilde{x}$ such that $\tilde{x} + IC_1^\tau(\tilde{x}) = 1 - \overline{q}_2^\tau$. BCA allocates the following:

$f_1(\mathbf{P}) = (x_1, IC_1^\tau(x_1))$ *where* $x_1 \in [0, \tilde{x}]$ *and* $f_2(\mathbf{P}) = (x_2, IC_2^\tau(x_2))$ *where* $x_2 \in [x_1 + IC_1^\tau(x_1), 1 - \overline{q}_2^\tau]$.

RL: If $\overline{q}_1^\tau = 1 - q_2^\tau$ *then* $f_2(P) = (0, q_2^\tau)$ *and* $f_1(P) = (\overline{x}_1^\tau, \overline{q}_1^\tau)$*, otherwise, define* $\tilde{x}$ *such that* $\tilde{x} + IC_2^\tau(\tilde{x}) = 1 - \overline{q}_1^\tau$*. Let* $f_2(\mathbf{P}) = (x_2, IC_2^\tau(x_2))$ *where* $x_2 \in [0, \tilde{x}]$ *and* $f_1(\mathbf{P}) = (x_1, IC_1^\tau(x_1))$ *where* $x_1 \in [x_2 + IC_2^\tau(x_2), 1 - \overline{q}_1^\tau]$.

Case 1(ii): Suppose $\overline{q}_2^\tau \leq 1 - q_1^\tau$ *and* $1 - q_2^\tau < \overline{q}_1^\tau$*. Here only LR allocations are possible and these allocations are the same as in LR for case 1(i) above. RL allocations are not Pareto efficient since both agents cannot be provided on the top IC in the RL allocation. Moreover, any such envy-free RL allocation can be Pareto improved by giving the agents an LR allocation as described above.*

Case 1(iii): Suppose $1 - q_1^\tau < \overline{q}_2^\tau$ *and* $\overline{q}_1^\tau \leq 1 - q_2^\tau$*. Here, there are only RL allocations, which are the same as in RL allocations for case 1(i) above. Any envy-free LR allocations will not be Pareto efficient since neither agent will get an allocation on her top IC. Claim 2 and claim 3 will be used to narrow down the set of allocations that are Pareto efficient and envy-free in the remaining cases.*

Cases 2 and 3 consider the case where $1 - q_1^\tau < \overline{q}_2^\tau$ *and* $1 - q_2^\tau < \overline{q}_1^\tau$*. In this case, there are no fair allocations where both agents get an allocation on the top IC simultaneously. Pareto efficiency implies that the whole resource is allocated when both the agents prefer to have 'sufficiently' high quantities, i.e., when* $1 - q_1^\tau < \overline{q}_2^\tau$ *and* $1 - q_2^\tau < \overline{q}_1^\tau$*. For allocations of the form* $f_i(\mathbf{P}) = (0, a)$ *and* $f_j(\mathbf{P}) = (a, 1 - a)$*, claim 2 implies that* $a \in [1 - \overline{q}_j^\tau, q_i^\tau]$.

Case 2: *Suppose* $IC_i^\tau \leq IC_i^f$ *for some* $i \in \{1, 2\}$ *and* $IC_j^f < IC_j^\tau$*, for* $j \neq i$*, and so,* $q_i^\tau \leq q_i^f \leq 1 - \overline{q}_i^\tau$ *and* $1 - \overline{q}_j^\tau < q_j^f < q_j^\tau$*. This is illustrated in Figure 7. W.l.o.g. let* $i = 1$ *and* $j = 2$*. The inequalities imply that* $1 - \overline{q}_2^\tau < q_1^\tau \leq 1 - \overline{q}_2^\tau < q_2^\tau$.

The following allocations are allocated under the BCA:

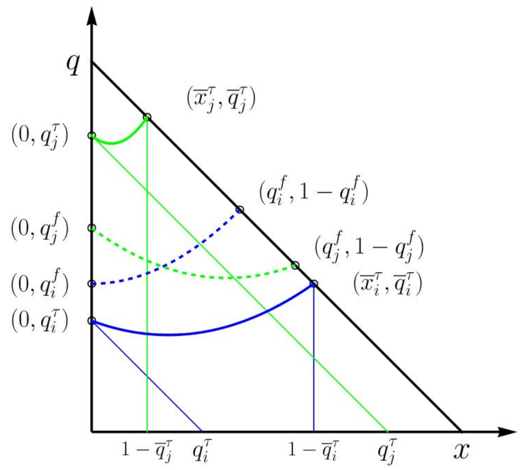


Figure 7: Case 2: $1 - \overline{q}_j^\tau < q_i^\tau \leq 1 - \overline{q}_i^\tau < q_j^\tau$

LR allocation: $f_1(\mathbf{P}) = (0, a)$*and* $f_1(\mathbf{P}) = (a, 1-a)$ *where* $a \in [\alpha, \beta]$ *such that* $\beta = \min\{q_2^f, q_1^\tau\}$ *and* $\alpha = \min\{b \in [1 - \overline{q}_2^\tau, \beta) : (0, a) \succsim_1 (a, 1-a)\}$.

RL allocation: $f_1(\mathbf{P}) = (a, 1-a)$ *and* $f_2(\mathbf{P}) = (0, a)$ *where* $a \in [\alpha, \beta]$ *such that* $\alpha = \max\{q_2^f, 1 - \overline{q}_1^\tau\}$ *and* $\beta = \max\{b \in (\alpha, q_j^\tau] : (a, 1-a) \succsim_1 (0, a)\}$.

Case 3: *The top IC of both agents is above the balanced IC of the corresponding agent, i.e.,* $IC_i^f < IC_i^\tau$ *for both agents* $i \in \{1, 2\}$, *and* $1 - \overline{q}_1^\tau < q_1^f < q_1^\tau$ *and* $1 - \overline{q}_2^\tau < q_2^f < q_2^\tau$.

Case 3(i): *If* $q_i^f = q_j^f = q^f$, *then there are two possible allocations (i)* $f_i(\mathbf{P}) = (0, q^f)$ *and* $f_j(\mathbf{P}) = (q^f, 1-q^f)$ *and (ii)* $f_j(\mathbf{P}) = (0, q^f)$ *and* $f_i(\mathbf{P}) = (q^f, 1-q^f)$.

a) If $q_1^f = q_2^f = q^f$, *then* $f_1(\mathbf{P}) = (0, q^f)$ *and* $f_2(\mathbf{P}) = (q^f, 1-q^f)$ *or* $f_2(\mathbf{P}) = (0, q^f)$ *and* $f_1(\mathbf{P}) = (q^f, 1-q^f)$.

b) If $q_i^f < q_j^f$ *then allocate,* $f_i(\mathbf{P}) = (0, a)$ *and* $f_j(\mathbf{P}) = (a, 1-a)$ *where* $a \in [\max\{1 - \overline{q}_j^\tau, q_i^f\}, \min\{q_i^\tau, q_j^f\}]$.

Apart from the allocations in the balanced region as defined earlier, there can be allocations outside it under the following cases.

Case 3(ii): *Both agents' top ICs cross,* $i \neq j$, $q_i^\tau \leq q_j^\tau$ *and* $\overline{q}_j^\tau \leq \overline{q}_i^\tau$ *and* ($\max\{q_i^f, q_j^f\} < q_i^\tau$ *or* $1 - \overline{q}_j^\tau < \min\{q_i^f, q_j^f\}$). *Above inequalities imply* $1 - \overline{q}_i^\tau \leq 1 - \overline{q}_j^\tau <$

$q_i^\tau \leq q_j^\tau$. *This case is illustrated in Figure 8.*

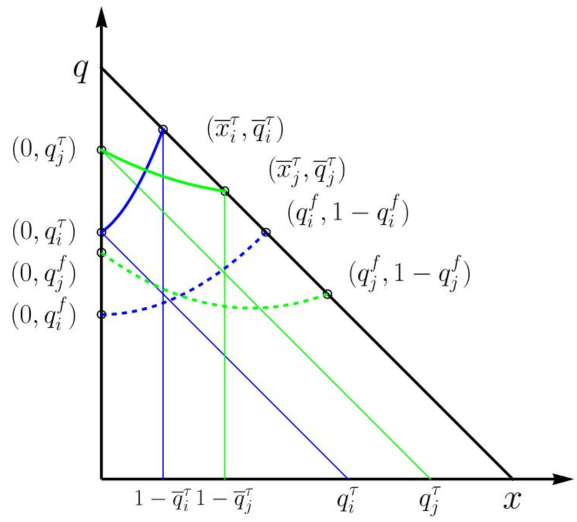


Figure 8: Case 3(ii) $1 - \overline{q}_i^\tau \leq 1 - \overline{q}_j^\tau < q_i^\tau \leq q_j^\tau$

Apart from the above allocations in 3(i) we have the following additional allocations of the form $f_i(\mathbf{P}) = (a, 1 - a)$ *and* $f_j(\mathbf{P}) = (0, a)$ *with the following conditions:*

(a) If $\max\{q_i^f, q_j^f\} < q_i^\tau$ *then* $a \in (q_i^\tau, q_j^\tau]$ *s.t.* $(a, 1 - a) \succsim_i (0, a)$.

(b) If $1 - \overline{q}_j^\tau < \min\{q_i^f, q_j^f\}$ *where* $a \in [1 - \overline{q}_i^\tau, 1 - \overline{q}_j^\tau)$ *s.t.* $(0, a) \succsim_j (a, 1 - a)$.

Case 3(iii): Both agents top ICs do not cross, $q_j^\tau < q_i^\tau$ *and* $\overline{q}_j^\tau \leq \overline{q}_i^\tau$ *and (*$\max\{q_i^f, q_j^f\} < q_j^\tau$ *or* $1 - \overline{q}_j^\tau < \min\{q_i^f, q_j^f\}$*). Above inequalities imply* $1 - \overline{q}_i^\tau \leq 1 - \overline{q}_j^\tau < q_j^\tau < q_i^\tau$. *This is illustrated in Figure 9.*

Apart from the above allocations in 3(i) we have the following additional allocations,

(a) If $\max\{q_i^f, q_j^f\} < q_j^\tau$*, BCA provides the following allocations:* $f_i(\mathbf{P}) = (0, a)$ *and* $f_j(\mathbf{P}) = (a, 1 - a)$ *where* $a \in (q_j^\tau, q_i^\tau]$ *s.t.* $(a, 1 - a) \succsim_j (0, a)$.

(b) If $1 - \overline{q}_j^\tau < \min\{q_i^f, q_j^f\}$*, then the BCA gives the following allocations:* $f_i(\mathbf{P}) = (a, 1 - a)$ *and* $f_j(\mathbf{P}) = (0, a)$ *where* $a \in [1 - \overline{q}_i^\tau, 1 - \overline{q}_j^\tau)$ *such that* $(0, a) \succsim_j (a, 1 - a)$.

Remark 2. *Note that the BCA rule is not single-valued and for a given profile, the*

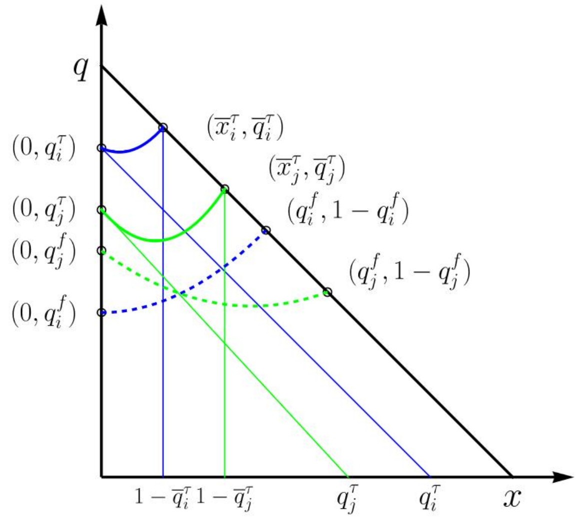


Figure 9: Case 3(iii) $1 - \overline{q}_i^\tau \leq 1 - \overline{q}_j^\tau < q_j^\tau < q_i^\tau$

BCA provides a (non-empty) multi-valued set of allocations. The following theorem states that BCA allocations described above are the only Pareto efficient and envy-free allocations.

The balanced-cure allocation (BCA) as described below gives a set of possible allocations for each preference profile $\mathbf{P} \in \mathcal{D}^2$. Consider a preference profile $\mathbf{P} = (\succsim_1, \succsim_2) \in \mathcal{D}^2$. By Proposition 1, there is a unique balanced IC for each agent's preference $\succsim_i$ denoted by IC_i^f for $i \in \{1, 2\}$.

The allocations $\{(x_1, q_1), (x_2, q_2)\}$ which have the property that $x_1 \leq x_2$, i.e., agent 1 is given a bundle on the left of that of agent 2 will be denoted as LR allocations, where L denotes 'left' for agent 1, and R denotes 'right' for agent 2. Similarly, allocations where $x_2 \leq x_1$ will be denoted as RL allocations.

We give a brief description of the allocations selected by the BCA in different cases. Case 1 considers the cases where both agents can be allocated bundles on their respective top ICs. There are two considerations: (i) We allocate LR and RL allocations if they are both Pareto efficient and if not, then the Pareto dominating one is allocated, (i) We also consider interior allocations (which are neither RL nor LR allocations) on top ICs if some subset of the resource remains unallocated. The remaining two cases (Case 2 and Case 3) consider the cases where at most one agent can be allocated a bundle on their top IC and the other agent receives a bundle on the other axis (we use 2 here). Case 2 is when one agent top IC is lower than their balanced IC (agent 1) and for the other it is higher (agent 2). As we know by Lemma 1 for each IC above the top IC their is a corresponding one below the top IC such that bundles on each are indifferent to each other. This are used to ensure that agent 1 never envies agent 2, by construction agent 2 does not envy agent 1. Case 3 is when the top IC of both agents is above their respective balanced IC. Here we use both claim 2 and claim 3 to ascertain allocations that are both envy-free and Pareto efficient within the balanced region. Moreover we get some additional allocations due to Lemma 1.

We note that the class of rules under BCA are not single-valued. This is observable from the fact that for a given preference profiles, the set of allocations may contain multiple allocations prescribed by the BCA. The following Theorem will prove that all allocations obtained by the BCA are Pareto efficient and envy-free.

Theorem 1. *An allocation rule $f : \mathcal{D}^2 \rightarrow \mathcal{A}$ is Pareto efficient and envy-free if and only if it is in the set of allocations specified by the BCA.*

Theorem 1 provides a characterization of fair (envy-free and Pareto efficient) allocations. The formal proof is provided in Appendix A.1. We provide a sketch of the proof here.

We have shown in Proposition 3 that if preferences are monotonic i.e. if both agents prefer the whole interval to any other bundle, then Pareto efficient and envy-free allocations must be in the region between the respective balanced ICs of the two agents where the agent whose balanced IC cuts the other's balanced IC from below gets an allocation on the left axis while the other agent gets the full remaining bundle on the right axis. In such cases, due to Assumption 4, an agent is either provided a bundle on the right, i.e., $(a, 1-a)$, or on the left, i.e., $(0, a)$ when both prefer to have the maximum quantity (by Proposition 3).

In all the other cases, if an agent (on the right axis) needs less than the quantity provided in the balanced IC region, she is given up to her top IC and quantity is decreased from her top IC to the infimum (or supremum) of the location of the bundle which that agent prefers to the bundles given to the other agent. This prevents envy while maintaining Pareto efficiency. Our next result shows that within the class of fair allocation rules, the set of strategy-proof rules is empty.

Theorem 2. *There is no allocation rule $f : \mathcal{D}^2 \to \mathcal{A}$ that is strategy-proof, envy-free and Pareto efficient.*

Theorem 2 states that adding strategy-proofness to the set of axioms leads to an impossibility. The proof is provided in the Appendix A.1. It is natural to assume that strategy-proofness is the main axiom driving the result of Theorem 2. This could be due to the richness of the domain vis-à-vis semi-single-peakedness of the domain with respect to quantity. However, this is not the case. We provide an example of a rule that is strategy-proof and Pareto efficient but not envy-free. Consider the following rule.

Serial dictator: One of the two agents is selected to be the 'first' dictator, i, and the other agent will be j. Agent i always receives allocation on IC_i^τ i.e. either $(0, q_i^\tau)$ or $(\overline{x}_i^\tau, \overline{q}_i^\tau)$ such that agent j gets the best possible allocation from either of the two 'left-over' regions, $T_{q_i^\tau}^{1-q_i^\tau}$ or $T_0^{\overline{x}_i^\tau}$ (Lemma 2).

Let (x^1, q^1) be an allocation such that $(x^1, q^1) \succsim_j (x, q)$ for all $(x, q) \in T_{q_i^\tau}^{1-q_i^\tau}$ and (x^2, q^2) be such that $(x^2, q^2) \succsim_j (x, q)$ for all $(x, q) \in T_0^{\overline{x}_i^\tau}$. If $(x^1, q^1) \succsim_j (x^2, q^2)$ then allocate $(f_1(\mathbf{P}), f_2(\mathbf{P})) = ((0, q_i^\tau), (x^1, q^1))$ else allocate $(f_1(\mathbf{P}), f_2(\mathbf{P})) = ((\overline{x}_i^\tau, \overline{q}_i^\tau), (x^2, q^2))$.

The above allocations are Pareto efficient as agent i always gets a bundle on the top IC and agent j gets the best bundle from the remaining interval. Any improvement for agent j will result in a worse bundle for agent i. It is easy to see that this rule is strategy-proof. Agent i has no incentive to deviate since she always gets her top bundle on the left or right. The only different bundle agent j can obtain by misreporting her preference is to get the other left-over bundle, which by construction cannot be strictly better. Note that this rule is not envy-free: if both agents prefer to have a very high quantity, only agent i will get her a top bundle. Agent j will then envy i. Therefore, the non-existence result of Theorem 2 is due to the fact that envy-free and Pareto efficient allocations are disjoint from the set of strategy-proof and Pareto efficient allocations. It is still an open problem to characterize the set of all strategy-proof and Pareto efficient allocation rules in this setting.

2.5 More than two agents

We first provide an allocation rule to obtain envy-free and Pareto efficient allocations for 3 agents when all the agents have a preference for greater quantity i.e. $\tau(\succsim_i) = \{(0, 1)\}$ for each $i \in N \equiv \{1, 2, 3\}$ and for *q-constant* preferences. A preference $\succsim_i \in \mathcal{D}$ is *q-constant* if there exits $a_q \in \mathbb{R}$ such that for any $q \in (0, 1)$ for all $x_1, x_2 \in (0, \overline{x}_i)$, $\frac{|IC_{i,q}(x_2)-IC_{i,q}(x_1)|}{|x_2-x_1|} = a_q$. These are K-Lipschitz continuous with $K \in \mathbb{R}$. We show that in our setting $K > 1$. Consider any $IC_{i,q}$ with $q > 0$. It cannot intersect the $x = 0$ axis which implies that $a_q \geq -1$ for all $q > 0$. We provide the following argument to show this. Suppose for some $q' > 0$, $a_{q'} < -1$, then it is easy to check that $IC_{i,q'}$ will intersect the $x = 0$ axis (say, at $(x', 0)$) which is a contradiction to Assumption 2 of Lowest at zero quantity. This is due to the fact that for any $q'' \in (0, q')$, $IC_{i,q''}$ would also intersect $x = 0$ (say, at some point $(x'', 0)$ with $x'' < x'$). However, this implies that the two bundles $(x', 0)$ and $(x'', 0)$ are on two different ICs, which is a contradiction to Assumption 2.

A *k-balanced IC* for agent $i \in N = \{1, 2, ..., k\}$ is an IC, IC_i^f such that there exist allocations $(0, q_{i1})$, $(x_{i2}, q_{i2}), ..., (x_{ik}, q_{ik})$ where $0 = x_{i1} < x_{i2} < ... < x_{ik}$, $x_{i2} = x_{i1} + q_{i1}, ..., x_{ik} = x_{ik-1} + q_{ik-1}$ and $\sum_{j=1}^{k} q_{ij} = 1$. The existence of a k-balanced IC can be proved using similar arguments as the ones used in Proposition 1. For any preference $\succsim_i \in \mathcal{D}$ and any given IC, $IC_{i,q}$ which represents it, we can construct a left-over or wastage function $\delta(x(q))$ as done for two agents earlier where x is the length of the portion left-over (surplus or deficit) after the first $k-1$ allocations $a_1, ..., a_{k-1}$ all lie on the $IC_{i,q}$. More specifically, if $q_1, .., q_{k-1}$ are the quantities in the allocations $a_i = (x_i, q_i)$ for $i \in \{1, ..., k-1\}$ then the wastage is given by $\delta(x(q)) = 1 - \sum_{i=1}^{k-1} q_i$.[23]

An implication of our Assumptions 1-4 in Section 2 is that as q approaches zero, we can find $q \in (0,1)$ and an $IC_{i,q}$ such that $\delta(x(q)) > 0$ (since the slope of ICs approach zero). Similarly, we can always find another $q' \in (0,1)$ such that $\delta(x(q')) < 0$. By continuity of preferences, $\delta(x(q))$ is a continuous function of $x(.)$ and $x(q)$ is a continuous function of q. Therefore, by the intermediate value theorem, there exists a $\hat{q} \in (0,1)$ such that $\delta(x(\hat{q})) = 0$ (Figure 10).

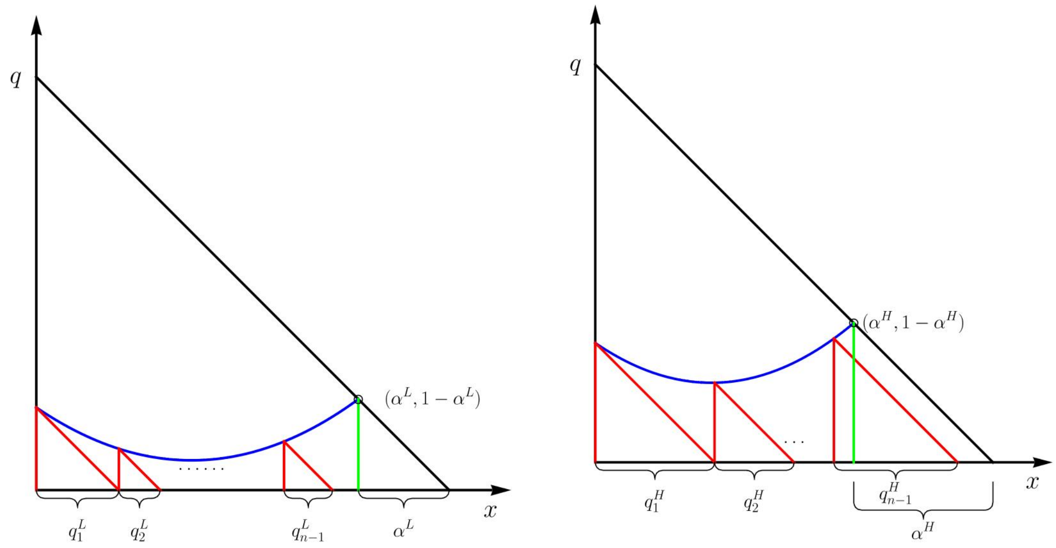


Figure 10: Balanced IC for k pieces

We denote the domain of preferences which are constant and monotone by $\mathcal{D}_1 \subset \mathcal{D}$. An allocation rule for $k \geq 3$ agents is a mapping $f : \mathcal{D}_1^k \to \mathcal{A}$.

Balanced curve allocation rule for 3 agents (BCA-3) Suppose the 3-piece-

[23]For very high ICs it may be the case that strictly less than $k-1$ allocations can be allocated on the given IC. In such cases, we can define $\delta(x(q)) = 1 - \sum_{i=1}^{k(q)}$ where $k(q) \in \{1, ..., k-1\}$ is the maximum number of allocations that can be given on $IC_{i,q}$ which are less than or equal to $k-1$.

balanced ICs for the three agents are $IC_{1,q_1^f}, IC_{2,q_2^f}$ and IC_{3,q_3^f} respectively. Further assume that the slopes of the balanced ICs of the three agents are in the following order: $a_{q_1^f} > a_{q_2^f} > a_{q_3^f}$ respectively. Let $a_1 = (x_{21}, q_{21})$, $a_2 = (x_{22}, q_{22})$ and $a_3 = (x_{23}, q_{23})$ be the three allocations that lie on agent 2's balanced IC, IC_{2,q_2^f}. Note that $q_{21} + q_{22} + q_{23} = 1$ i.e. there is no resource left over. Take any profile $\mathbf{P} \in \mathcal{D}_1^3$ i.e. when preferences are constant and monotone. We show that the following allocation $f_1(\mathbf{P}) = a_1, f_2(\mathbf{P}) = a_2$ and $f_3(\mathbf{P}) = a_3$ is envy-free and Pareto efficient. Clearly, the allocation is Pareto efficient since preferences are monotone. Moreover, $a_1 \sim_2 a_2 \sim_2 a_3$ so agent 2 does not envy any other agent. Since $a_{q_1^f} > a_{q_2^f}$, a_1 and a_2 lie on IC_1^f and $IC_{1,q'}$ such that $q' > q_1^f$. Therefore, $a_1 \succ_1 a_2$. By similar arguments we can show that $a_1 \succ_1 a_3$. For agent 3, note that $a_{q_2^f} > a_{q_3^f}$, so a_2 and a_3 lies on $IC_{3,q''}$ and IC_3^f respectively such that $q_{23} > \overline{q}_3$. Therefore, q_{23} lies on a higher IC than IC_3^f which implies that $a_3 \succ_3 a_2$. By similar arguments we can show that $a_3 \succ_3 a_1$. Therefore, the given allocation is envy-free as well. A similar allocation for more than 3 agents will be notably more challenging. We do not explore it in this chapter. We provide a property that must be satisfied for any fair allocation when preferences are constant and monotonic for any number of agents.

Proposition 2. *For any profile $\mathbf{P} \in \mathcal{D}_1^n$ with $n \geq 2$, suppose the allocation $\{a_i\}_{i=1}^n = \{x_i, q_i\}_{i=1}^n$ is fair (envy-free and Pareto efficient). For simplicity assume that $x_1 < x_2 < ... < x_n$ and let the slope of an agent i's IC through her own allocation bundle a_i be a_{q_i}. Then it must be the case that $a_{q_1} \geq a_{q_2} \geq .... \geq a_{q_n}$.*

Proof. Suppose for contradiction that agent 1's IC's slope through her own bundle is less than that of another agent through their bundle, i.e., there exists another agent $k \neq 1$ such that $a_{q_1} < a_{q_k}$. By envy-freeness, all the other bundles must lie below agent 1's IC through (x_1, q_1), i.e., $(x_1, q_1) \succsim_1 (x_j, q_j)$ for all j. But if agent $k \neq 1$ has a greater slope at the allocation (x_k, q_k) which lies below the IC of agent 1 through her own bundle, then agent k will prefer to have the bundle (x_1, q_1) since $x_1 < x_k$ and k's IC through the (x_1, q_1) will be higher than the IC through her own bundle,

(x_k, q_k). Therefore, by monotonicity of preferences and Claim 2, $(x_1, q_1) \succ_k (x_k, q_k)$. This is a contradiction to the fact the allocation is envy-free. Similar arguments show that any agent j's IC through her own bundle must have a greater slope than all the succeeding agents' ICs through their respective bundle. ■

2.6 Conclusion

We consider an allocation model where agents have a preference for location and quantity and the preference for quantity is single-peaked. We characterize the set of envy-free and Pareto efficient allocations for two agents. We show that there do not exist any strategy-proof, envy-free and Pareto efficient allocation rules. We provide observations for more than two agents which can be used to extend the BCA. The existence of the balanced IC for k portions can be utilized for obtaining fair allocations for more than two agents. However, additional assumptions are required for the existence of fair allocation for more than 3 agents.

For future research, other preference restrictions on the unit interval can be explored. However, the existence of a balanced IC might be crucial in such cases as well since the agents must be allocated on the same IC when they have identical preferences. Other extensions include a model where agents are allowed to trade between themselves with or without a given a set of endowments.

A.1 Appendix - Proofs

Proof of Proposition 1 We prove by contradiction. Suppose this is not the case. Then for each $q \in (0, 1)$, either (i) $IC_{i,q}$ cuts the $x + q = 1$ line above the point $(x, 1 - x)$ or (ii) $IC_{i,q}$ cuts the $x + q = 1$ line below the point $(x, 1 - x)$. Consider the IC at $q = \frac{1}{2}$, $IC_{i,\frac{1}{2}}$ which is the IC passing through the points $(0, \frac{1}{2})$ and $(\gamma, 1 - \gamma)$ for some $0 < \gamma < 1$. We provide arguments for different cases. Note that if $\gamma = \frac{1}{2}$, then $q_i^f = \frac{1}{2}$ and our claim is true.

Case (i): $\gamma < \frac{1}{2}$, then for some $\alpha \in (0, \frac{1}{2})$, we will have $IC_{i,\frac{1}{2}-\alpha}(\frac{1}{2}) = \frac{1}{2}$. Now consider ICs between $IC_{i,\frac{1}{2}}(x)$ and $IC_{i,\frac{1}{2}-\alpha}(x)$ s.t. $IC_{i,\frac{1}{2}-\varepsilon}(\frac{1}{2} - \delta_1(\varepsilon)) = \frac{1}{2} + \delta_1(\varepsilon)$ for some function $\delta_1 : [0, \alpha] \to [0, \frac{1}{2} - \gamma]$ such that, $\left(0, \frac{1}{2} - \varepsilon\right) \sim_i \left(\frac{1}{2} - \delta_1(\varepsilon), \frac{1}{2} + \delta_1(\varepsilon)\right)$ for any $\varepsilon \in [0, \alpha]$. Note the following properties of $\delta(\varepsilon)$: (i) $\delta_1(0) = \frac{1}{2} - \gamma$ and (ii) $\delta_1(\alpha) = 0$. The ICs are illustrated in Figure 11.

Since preferences are continuous on $X = T_0^1$, $\delta_1(\varepsilon)$ is a continuous and monotonic function of ε. Define $g_1(\varepsilon) = \delta_1(\varepsilon) - \varepsilon$. Note that $g_1(0) = \frac{1}{2} - \gamma > 0$ and $g_1(\alpha) = -\alpha < 0$. Since $g_1(.)$ is a continuous function, we can apply the *intermediate value theorem* which implies that there exist ε_1^* such that $g_1(\varepsilon_1^*) = \delta_1(\varepsilon_1^*) - \varepsilon_1^* = 0$. The function is illustrated in Figure 12.

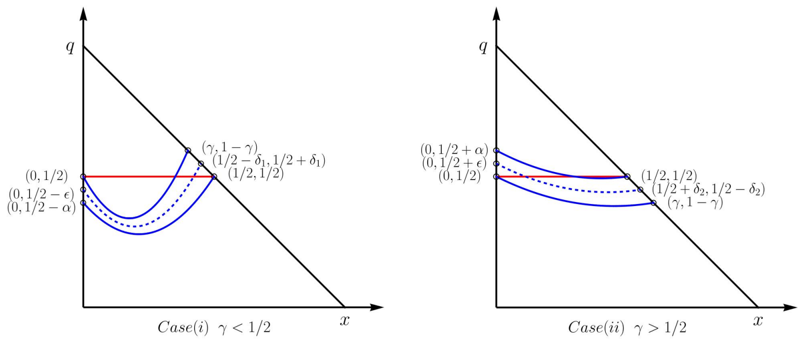


Figure 11: (a) and (b): Proving existence of a balanced IC

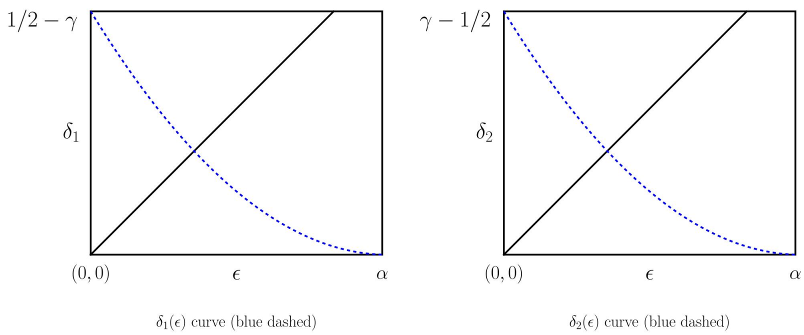


Figure 12: (a) and (b) illustrate $\delta_1(.)$ and $\delta_2(.)$ resp.

Therefore, $\delta_1(\varepsilon_1^*) = \varepsilon_1^*$ and this results in a contradiction since we assumed that such points do not exist. Therefore, there exist q_i^f with $IC_{i,q_i^f}(q_i^f) = 1 - q_i^f$ for each $i \in \{1, 2\}$.

Case (ii): $\frac{1}{2} < \gamma$, then for some $\alpha \in (0, \frac{1}{2})$, we will have $IC_{i,\frac{1}{2}+\alpha}(\frac{1}{2}) = \frac{1}{2}$. Now consider ICs between $IC_{i,\frac{1}{2}}(x)$ and $IC_{i,\frac{1}{2}+\alpha}(x)$ such that $IC_{i,\frac{1}{2}+\varepsilon}(\frac{1}{2}+\delta_2(\varepsilon)) = \frac{1}{2}-\delta_2(\varepsilon)$ for some function $\delta_2 : [0, \alpha] \to [0, \gamma - \frac{1}{2}]$ and $\big(0, \frac{1}{2} + \varepsilon\big) \sim_i \big(\frac{1}{2} + \delta_2(\varepsilon), \frac{1}{2} - \delta_2(\varepsilon)\big)$ for any $\varepsilon \in [0, \alpha]$. Note the following properties of δ_2, (i) $\delta_2(0) = \gamma - \frac{1}{2}$ and (ii) $\delta_2(\alpha) = 0$.

Define $g_2(\varepsilon) = \delta_2(\varepsilon) - \varepsilon$. Note that $g_2(0) = \gamma - \frac{1}{2} > 0$ and $g_2(\alpha) = -\alpha < 0$. Since δ_2 is a continuous function, we can apply the *intermediate value theorem* which implies that there exist ε_2^* such that $g_2(\varepsilon_2^*) = \delta_2(\varepsilon_2^*) - \varepsilon_2^* = 0$. Therefore, $\delta_2(\varepsilon_2^*) = \varepsilon_2^*$ and this results in a contradiction since we assumed that such points do not exist. Therefore, there exists $q_i^f \in (0, 1)$ with $IC_{i,q_i^f}(q_i^f) = 1 - q_i^f$ for each $i \in \{1, 2\}$.

We now show that for each individual $i \in N$, q_i^f is unique. Suppose for contradiction that there are two balanced ICs: IC_i^f and $IC_i^{'f}$ and that w.l.o.g. $q_i^f < q_i^{'f}$ which implies $1 - q_i^{'f} < 1 - q_i^f$. However, this further implies that $IC_i^f(x)$ and $IC_i^{'f}(x)$ intersect. This is a contradiction to transitivity of the preferences.

■

Proposition 3. *Suppose* $\mathbf{P} = (\succsim_1, \succsim_2) \in \mathcal{D}^2$ *where* $\tau(\succsim_i) = \{(0, 1)\}$ *for* $i \in \{1, 2\}$. *Then the following allocation rule* $f : \mathcal{D}^2 \to \mathcal{A}$ *is Pareto efficient and envy-free:*
(i) If $q_1^f = q_2^f$, *then* $f_i(\mathbf{P}) = (0, q_1^f)$ *and* $f_j(\mathbf{P}) = (q_1^f, 1 - q_1^f)$, *where* $i \neq j$, $i, j \in \{1, 2\}$.
(ii) If $q_i^f < q_j^f$, *then* $f_i(\mathbf{P}) = (0, \alpha)$ *and* $f_j(\mathbf{P}) = (\alpha, 1 - \alpha)$ *for all* $\alpha \in [q_i^f, q_j^f]$.

Proof. (i) If $q_1^f = q_2^f = q^f$ then either $f(\mathbf{P}) = ((0, q^f), (q^f, 1 - q^f))$ or $f(\mathbf{P}) = ((q^f, 1 - q^f), (0, q^f))$ is envy-free as both get allocations on the same IC and Pareto efficient as whole resource is allocated. The balanced region is illustrated in black in Figure 13.

(ii) We provide arguments for the case when $q_1^f < q_2^f$. Similar arguments can be used to prove the other case. Note that if $q_i^f \neq q_j^f$ then IC_i^f and IC_j^f can only intersect once i.e. $q_i^f < q_j^f \Rightarrow 1 - q_i^f > 1 - q_j^f$.

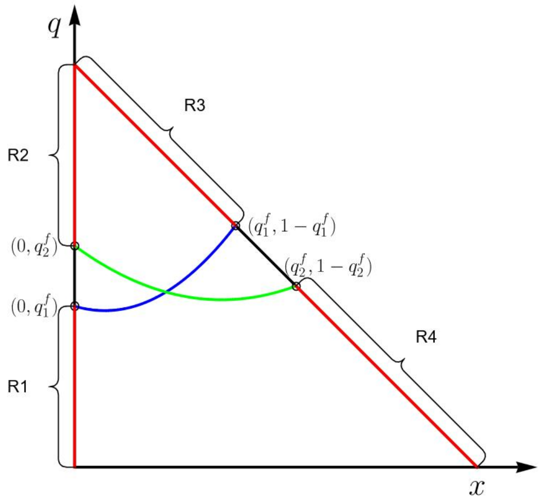


Figure 13: Illustrative balanced IC region

We define the following sets to argue that the provided allocations are fair: $R_1 = \{(0,\alpha)|\alpha \in [0, q_1^f)\}$, $R_2 = \{(0,\alpha)|\alpha \in (q_2^f, 1]\}$, $R_3 = \{(\alpha, 1-\alpha)|\alpha \in [0, q_1^f)]\}$ and $R_4 = \{(\alpha, 1-\alpha)|\alpha \in (q_2^f, 1]\}$. We will show that if f is Pareto efficient and envy-free, then $f(\mathbf{P}) = ((0,a),(a,1-a))$ where $(0,a) \notin R_1 \cup R_2$, and $(a, 1-a) \notin R_3 \cup R_4$. In other words, it must be the case that $a \in [q_1^f, q_2^f]$.

Consider any $(0,\alpha) \in R_1$, then the corresponding $(\alpha, 1-\alpha)$ will be in R_3. Here, both the agents will prefer $(\alpha, 1-\alpha)$ over $(0,\alpha)$ since their peak interval is $(0,1)$. Therefore, giving $(0,\alpha)$ to any agent will not be envy-free. Similarly, for any $(0,\alpha) \in R_2$, the corresponding $(\alpha, 1-\alpha)$ will be in R_4. By definition of monotonic preferences and Claim 1, both the agents will prefer $(0,\alpha)$ over $(\alpha, 1-\alpha)$ so giving $(\alpha, 1-\alpha)$ to any agent will again lead to envy.

For any $(0,\alpha)$ s.t. $\alpha \in [q_1^f, q_2^f]$, corresponding $(\alpha, 1-\alpha)$ will be on the $x + q = 1$ axis between $(q_1^f, 1-q_1^f)$ and $(q_2^f, 1-q_2^f)$. By definition of monotonic preferences and Claim 1, $(0,\alpha) \succ_1 (\alpha, 1-\alpha)$ and $(\alpha, 1-\alpha) \succ_2 (0,\alpha)$. Therefore, for any such allocation, neither of the two agents will envy the other's allocation. Moreover, this is Pareto efficient since there full interval is given away. ■

Proof of Claim 2 The above observation states that any Pareto efficient allocation where the whole resource is allocated is only possible when neither of the agents does not get more quantity than what is required for them to get a bundle on their top IC. If $a > q_i^\tau$ then agent i gets an allocation on an IC above IC_i^τ. This cannot be Pareto efficient since a lower quantity allocation at the same location on the top IC is feasible and also strictly preferred to her current allocation. Similarly if $a < \overline{x}_j^\tau$ then this is not Pareto efficient since agent j has received the bundle $(a, 1-a)$ which is strictly worse than the available bundle $(\overline{x}_j^\tau, \overline{q}_j^\tau)$ on her top IC, IC_j^τ.

■

Proof of Claim 3 i) If $a < q_i^f \leq \overline{q}_i^\tau$ and agent j's allocation is on an $IC_{i,q}$ between IC_i^f and IC_i^τ while than agent i's allocation is on an $IC_{i,q'}$ below IC_i^f with $q' < q < q_i^\tau$. Hence by semi-single peakedness of preferences, $(a, 1-a) \succ_i (0, a)$.

ii) If $q_j^f < a \leq q_j^\tau$ and agent i's allocation is on an $IC_{j,q}$ between IC_j^f and IC_j^τ while than agent j's allocation is on an $IC_{j,q'}$ below IC_j^f with $q' < q < q_j^\tau$. Hence by semi-single peakedness of preferences agent $(0, a) \succ_j (a, 1-a)$.

■

Proof of Theorem 1 We use the following Claims 2 and 3 with the following lemmas to show that BLA allocations are the only ones that are PE and envy-free.

Lemma 1. *If for any $i \in N$ with $q_i^\tau \in (0, 1)$, for all $q \in (q_i^\tau, 1]$ there exists $q' \in (0, q_i^\tau)$ such that $(0, q) \sim_i (0, q')$*

Proof. By assumption 2 we know that $(0, 1) \succ_i (0, 0)$. By single-peakedness in quantity (Assumption 3) and existence of top IC (Assumption 2): for any $q \in (q_i^\tau, 1)$,

$(0, q_i^\tau) \succ_i (0, q) \succ_i (0, 1) \succ_i (0, 0)$. This along with continuity of preferences (Assumption 0) implies that there exists $q_1, q' \in (0, q_i^\tau)$ s.t. $0 < q_1 < q' < q_i^\tau < q < 1$ such that $(0, q_1) \sim_i (0, 1)$ and $(0, q') \sim_i (0, q)$. ■

The lemma states that for any top IC which connects $(0, q_i^\tau)$, there exists a q' such that the interval on the right axis, $(0, q')$ is indifferent to $(0, q)$.

Lemma 2. *Consider $0 < x_i < x_j < 1 - \overline{q}'$. If allocation of the form $(f_i(\mathbf{P}), f_j(\mathbf{P})) = ((x_i, IC_{i,q}(x_i)), (x_j, IC_{j,q'}(x_j))$ is feasible then the allocation $(f_i(\mathbf{P}), f_j(\mathbf{P})) = ((0, q), (\overline{x}', \overline{q}'))$ is also feasible.*

Proof. Consider agent i. The interval $[0, x_i]$ is unallocated and agent i is allocated the interval $[x_i, x_i + IC_{i,q}(x_i)]$. By K-Lipschitz continuity of $IC_{i,q}(x)$ we have

$$
\begin{aligned}
& |IC_{i,q}(x_i) - q| \leq Kx_i \\
\iff & -Kx_i \leq IC_{i,q}(x_i) - q \leq Kx_i \\
\iff & q - Kx_i \leq IC_{i,q}(x_i) \leq q + Kx_i && \text{(since } K \leq 1 \text{)} \\
\iff & q < x_i + IC_{i,q}(x_i) && \text{(since } K > 0\text{).}
\end{aligned}
$$

Hence $(0, q)$ is feasible. Now consider agent j. Agent j was allocated from $[x_j, x_j + IC_{j,q'}(x)]$ and interval $[x_j + IC_{j,q'}(x), 1]$ was unallocated. By assumption $x_j < 1 - \overline{q}' \Rightarrow x_j < \overline{x}'$ and by the property that $IC_{j,q'}$ is $K-$Lipschitz continuous with $K \in (0, 1]$, $(\overline{x}', \overline{q}')$ is available after allocating $(0, q)$ to agent i. ■

One can easily check that the converse is not true. Consider $0 < q, q'$ such that $q + (1 - \overline{q}') = 1$. If the following allocation is feasible, $(f_i(\mathbf{P}), f_j(\mathbf{P})) = ((0, q), (\overline{q}', 1 - \overline{q}'))$ then any allocation $((x_i, IC_{i,q}(x_i)), (x_j, IC_{j,q'}(x_j))$ where $0 < x_i$ and $x_j < \overline{q}'$ is not feasible since $x_j < x_i + IC_{i,q}(x_i)$

Cases. We prove in separate cases, that each of the allocations prescribed by the BCA in the definition of the rule are the only Pareto efficient and envy-free allocations. For the sake of simplicity, we do not reiterate the conditions under different cases and the allocations, we simply provide the proofs of why such allocations are the only Pareto efficient and envy-free allocations.

Case 1. Case 1(i): LR allocation: These allocations are Pareto efficient and envy-free as both agents get allocation on their top ICs. Giving $f_1(\mathbf{P}) = (0, q_1^\tau)$, leaves more than enough for agent 2 on its top IC, since $\overline{q}_2^\tau \leq 1 - q_1^\tau$.

RL allocation: These allocations are Pareto efficient and envy-free as both agents get allocation on their top ICs. Giving $f_2(\mathbf{P}) = (0, q_2^\tau)$, leaves more than enough for agent 1 on its top IC, since $\overline{q}_1^\tau \leq 1 - q_2^\tau$.

Case 1(ii): Here, only LR allocations are Pareto efficient and envy-free. Since $1 - q_2^\tau < \overline{q}_1^\tau$, in a RL allocation, both agents can not get an allocation on their top IC and hence it is Pareto dominated by the LR allocation.

Case 1(iii): Here, only RL allocations are Pareto efficient and envy-free. Because in LR $1 - q_1^\tau < \overline{q}_2^\tau$ both agents can not get allocation on their top IC and hence it is not PI to RL. No other set of allocations can be Pareto efficient and envy-free since in all the allocations above, both the agents get an allocation on their respective top IC.

Case 2. Left-allocation to agent i, Right-allocation to agent j for $i, j \in \{1, 2\}, i \neq j$: Since agent i gets $(0, a)$, when $a < q_i^\tau$, the quantity given to i must be at the minimum level where she does not envy agent j as allocations are on opposite side of IC_i^τ. This implies that $a \in [\alpha, \beta]$ such that $\beta = \min\{q_j^f, q_i^\tau\}$ and $\alpha = \min\{b \in [1 - \overline{q}_j^\tau, \beta) : (0, a) \succ_i (a, 1 - a)\}$. If $a < \alpha$, then agent i will envy agent j, and if $a > \beta$, then either (i) $q_i^\tau > q_j^f$: in this case, agent j will envy agent 1 since the latter's allocation will be closer to the top IC of agent j than her own allocation, or (ii) $q_i^\tau \leq q_j^f$: in this case, agent i gets more than her top quantity and this cannot be Pareto efficient.

Left-allocation to agent j, and right-allocation to agent i: Here agent i gets an allocation $(a, 1 - a)$, while agent j gets $(0, a)$, where $a \in [\alpha, \beta]$ such that $\alpha =$

$\max\{q_j^f, 1-\overline{q}_i^\tau\}$ and $\beta = \max\{b \in (\alpha, q_j^\tau] : (a, 1-a) \succ_i (0, a)\}$. Here, if $a < \alpha$, then if (i) $q_j^f > 1-\overline{q}_i^\tau$, then agent j will envy i and if (ii) $q_j^f \leq 1-\overline{q}_i^\tau$, then $a < 1-\overline{q}_i^\tau$ will not be Pareto optimal since the quantity $1-a$ to agent i is more than required to be at her top IC at the same location. Similarly, if $a > \beta$, then agent i will envy agent j since $(0, a) \succ_i (a, 1-a)$.

Case 3: Case 3(i): If $q_i^f = q_j^f = q^f$, then both $f(\mathbf{P}) = ((0, q^f), (q^f, 1-q^f))$ and $f(\mathbf{P}) = ((q^f, 1-q^f), (0, q^f))$ allocations are envy-free as $(0, q^f) \sim_k (q^f, 1-q^f)$ for $k \in \{i, j\}$; and since the whole resource is allocated, it is Pareto efficient. By similar arguments as in Proposition 3, any other allocation will result in envy.

Allocations in the balanced region when $q_i^f \neq q_j^f$: If $q_i^f < q_j^f$, in the LR allocation, $a < q_i^f$ will imply that agent i will envy agent j and $a < 1-\overline{q}_j^\tau$ will not be Pareto efficient as agent j will have more than required for an allocation on the top IC. Hence, $\max\{1-\overline{q}_j^\tau, q_i^f\} \leq a$. If $q_j^f < a$, then agent j will envy agent i and if $q_i^\tau < a$ the allocation will not be Pareto efficient as agent i will have more than required for allocation on top IC. Hence, $a \leq \min\{q_i^\tau, q_j^f\}$.

If $q_j^f < q_i^f$: in the RL allocation, if $a < q_j^f$, then agent j will envy i and if $a < 1-\overline{q}_i^\tau$, then the allocation is not Pareto efficient as agent i will have more than required for allocation on top IC. Hence, $\max\{1-\overline{q}_i^\tau, q_j^f\} \leq a$. If $q_i^f < a$, then agent i will envy agent j and if $q_j^\tau < a$ the allocation will not be Pareto efficient as agent i will have more than required for allocation on top IC. Hence, $a \leq \min\{q_j^\tau, q_i^f\}$. We present another Lemma to prove the final case.

Case 3(ii): By claim 3, $q_i^\tau < a$. Therefore, the specified allocation is Pareto efficient and envy-free.

Case 3(iii): Left-allocation to agent i and right-allocation to agent j: By claim 3, $a > q_j^\tau$. Left-allocation to agent j and right-allocation to agent i. When $f_i(\mathbf{P}) = (a, 1-a)$ and $f_j(\mathbf{P}) = (0, a)$: By claim 3, $a < 1-\overline{q}_j^\tau$. Therefore, the allocation specified for this case is fair.

∎

Proof of Theorem 2 Suppose $f : \mathcal{D}^2 \to \mathcal{A}$ is strategy-proof, envy-free and Pareto efficient. Consider the following three types of preferences where the peak allocation is $\{(0,1)\}$ and all the ICs are *linear*. Type 1: $\succsim_i$, characterized by ICs with slope 0 i.e. $\frac{d}{dx}(IC_{i,q}(x)) = 0$ for all $q \in [0,1)$ and $x \in (0,1)$, Type 2: $\succsim_i'$, characterized by ICs with positive slope (except for $IC_{i,0}'$), i.e., $\frac{d}{dx}(IC_{i,q}(x)) > 0$ for all $q \in (0,1)$ and $x \in (0,1)$ and Type 3: $\succsim_i''$, with all ICs of negative slope (except for $IC_{i,0}''$) i.e., $\frac{d}{dx}(IC_{i,q}(x)) < 0$ for all $q \in (0,1)$ and $x \in (0,1)$. The actual value of the slope does not matter as long as it is positive (or negative) in the corresponding regions. Let q^f, q'^f and q''^f be the respective quantities of the balanced ICs at $x = 0$. All types of preferences used in the proof are illustrated in Figure 14.

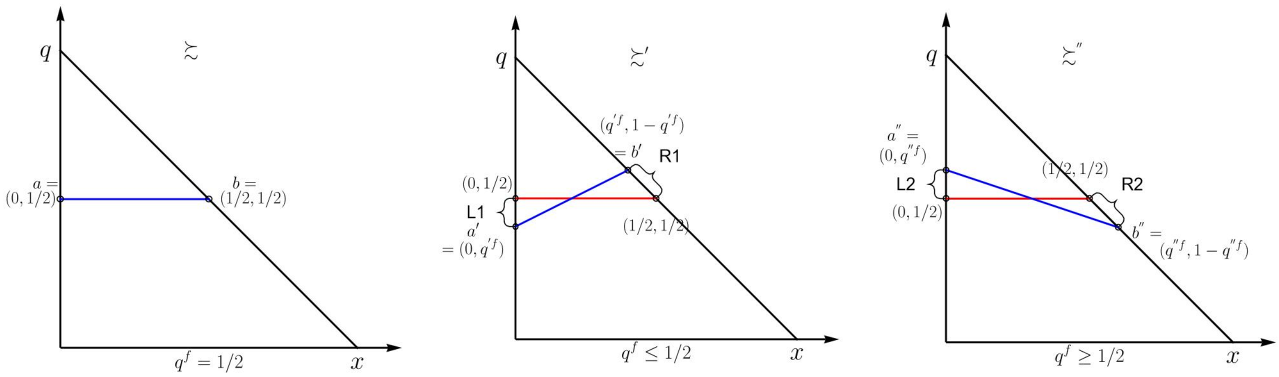


Figure 14: Construction of preferences for Theorem 2

We define the sets, $L_1 = \{(0,q) | q'^f \leq q \leq \frac{1}{2}\}$, $L_2 = \{(0,q) | \frac{1}{2} \leq q \leq q''^f\}$, $R_1 = \{(q, 1-q) | \frac{1}{2} \leq q \leq q'^f\}$ and $R_2 = \{(q, 1-q) | q''^f \leq q \leq \frac{1}{2}\}$. To prove our claim we construct profiles that consist of preferences from the set $\{\succsim_i, \succsim_i', \succsim_i''\}$.

Let $\mathbf{P}^1 = \{\succsim_1, \succsim_2\}$. Since f is envy-free and Pareto efficient, $f_1(\mathbf{P}^1) = a = (0, \frac{1}{2})$ and $f_2(\mathbf{P}^1) = b = (\frac{1}{2}, \frac{1}{2})$. Let $\mathbf{P}^2 = \{\succsim_1', \succsim_2\}$. According to Proposition 3, agent 1 has to be allocated in L_1 and agent 2 in R_1, to prevent envy. To prevent agent 1 from deviating from $\mathbf{P}^2$ to $\mathbf{P}^1$ via $\succsim_1$, by strategy-proofness, $f_1(\mathbf{P}^2) = a$ and $f_2(\mathbf{P}^2) = b$. Note that $q_1^{'f} = q_2^{'f} = q^{'f}$ and $q_1^{''f} = q_2^{''f} = q^{''f}$.

Suppose $\mathbf{P}^3 = \{\succsim_1', \succsim_2'\}$. Since both have identical preferences, envy-freeness requires that allocations be on either end of the corresponding balanced IC i.e. $(0, q'^f)$ and $(q'^f, 1 - q'^f)$. To prevent agent 2 from deviating from profile $\mathbf{P}^2$ to profile $\mathbf{P}^3$ via $\succsim_2'$, by strategy-proofness $f_2(\mathbf{P}^3) = a'$. Note, $a' \succ_2' b$ and $b' \succ_2 b \succ_2 a'$. By Pareto efficiency, $f_1(\mathbf{P}^3) = b'$.

Let $\mathbf{P}^4 = \{\succsim_1, \succsim_2'\}$. Here, according to Proposition 3, agent 2 has to be allocated in L_1 and agent 1 in R_1. To prevent agent 1 from deviating from $\mathbf{P}^4$ to $\mathbf{P}^3$ via $\succsim_1'$, $f_1(\mathbf{P}^4) = b'$. By Pareto efficiency, $f_2(\mathbf{P}^4) = a'$. Suppose profile $\mathbf{P}^5 = \{\succsim_1'', \succsim_2'\}$. By Proposition 3, agent 2 has to be allocated in $L_1 \cup L_2$ and agent 1 in $R_1 \cup R_2$. To prevent agent 1 from deviating from $\mathbf{P}^5$ to $\mathbf{P}^4$ via $\succsim_1'$, by strategy-proofness and Pareto efficiency, $f_1(\mathbf{P}^5) = b'$ and $f_2(\mathbf{P}^5) = a'$.

Finally, let $\mathbf{P}^6 = \{\succsim_1'', \succsim_2\}$. According to Proposition 3 and envy-freeness, agent 2 must be allocated in L_2 and agent 1 in R_2. But this will result in agent 2 deviating at $\mathbf{P}^5$ to $\mathbf{P}^6$ via $\succsim_2$ where her bundle will be strictly better since $f(\mathbf{P}^6) = (0, \alpha) \succ_2' a'$ for any $\alpha \in [\frac{1}{2}, q_f'']$. This is a contradiction to the fact that f is strategy-proof.

■

A.2 Appendix - Other observations

When both the agents have top ICs above region between the two balanced ICs and the domain of preferences is $\mathcal{D}$, any Pareto efficient allocation has to allocate the whole resource. In other words, any Pareto efficient allocation will assign allocations on the left and right axis respectively to the agents without leaving over any resource.

It is easy to verify the above observation. Once an allocation $(0, q)$ is given to an agent, the remaining feasible region is T_q^{1-q}. Since preferences are monotonic, for the other agent, the highest IC will intersect at $(q, 1 - q)$. Similarly, if an agent gets an allocation $(q, 1-q)$ on the right axis, then the remaining feasible region is given by T_0^q. Given assumption 4 above and by Claim 1, the highest IC of the other agent in T_0^q will intersect at $(0, q)$. We will say that two allocations $((x_1, q_1), (x_2, q_2))$ and $((x_1', q_1'), (x_2', q_2'))$ are *Pareto indifferent* to each other if both are Pareto efficient.

We say that $IC_{i,q} \leq IC_{i,q}$ when for all x, $IC_{i,q}(x) \leq IC_{i,q}(x)$. The strict and equal versions can be similarly defined. These are well-defined since ICs do not intersect and equality holds when the ICs coincide.

Observation 2. *For any $i \in \{1, 2\}$,*

(i) (a) $IC_i^\tau \leq IC_i^f \Leftrightarrow q_i^\tau \leq q_i^f \leq 1 - \overline{q}_i^\tau$.

(b) $IC_i^f < IC_i^\tau \Leftrightarrow 1 - \overline{q}_i^\tau < q_i^f < q_i^\tau$.

(ii) If $\overline{q}_j^\tau \leq 1 - q_i^\tau$, then the allocation $\{(0, q_i^\tau), (\overline{x}_j^\tau, \overline{q}_j^\tau)\}$ is feasible. Moreover the allocation is Pareto efficient and envy-free as both get allocated on their top ICFs.

Observation 2 lists some implications of the cases when the top IC is above, below or equal to the balanced IC. Another Observation is used in the description of the allocation rule.

We now describe the allocation rule that will characterize the full set of Pareto efficient and envy-free allocation for any given preference profile $\mathbf{P} \in \mathcal{D}^2$.

Observation 3. *Suppose IC functions are K-Lipschitz continuous with $K > 1$. Then, there exists a preference profile $\mathbf{P} \in \mathcal{D}^2$ for which there does not exist any Pareto efficient and envy-free allocations.*

We provide an example to prove the above observation. Consider a preference profile $\mathbf{P}$ where both agents have the same preference $\succsim_1 = \succsim_2 = \succsim$ and suppose for some $x > 0$, $|IC_{i,q}(x) - IC_{i,q}(0)| > |x - 0| = |x|$, i.e., $IC_{i,q}(.)$ is K-Lipschitz continuous with $K > 1$. An implication of this is that the allocation $(x, IC_{i,q}(x))$ belongs to the interior of the triangle T_0^q for every $x \in N_\varepsilon(0)$ (Figure 15). By the above observation, a feasible fair allocation must be on the same IC of the two agents since they have

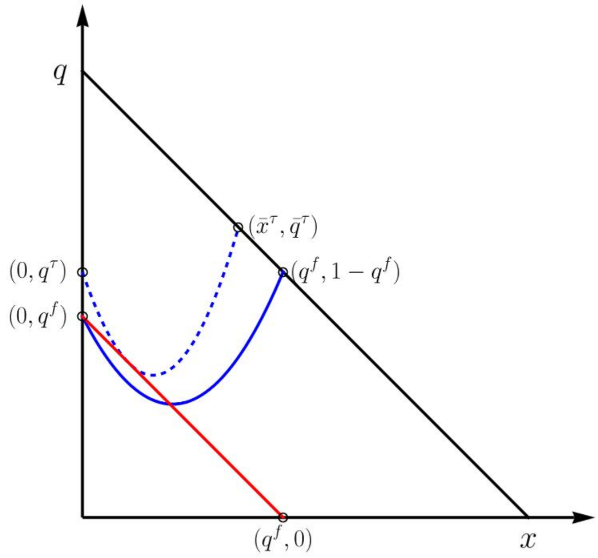


Figure 15: When preferences are “too steep”

identical preferences. Here if any agent is allocated $(0, q^f)$, there is a strict Pareto improvement possible if the agent is allocated another allocation at the point where the relevant IC is tangent (Figure 15). This does not require more quantity, so it does not affect the allocation of the other agent. Therefore, the only possible allocation is at the tangency point. Let this point be (x_1, q_1) with $x_1 > 0$ and $q_1 > q_f$ and $(x_1, q_1) \succ_i (0, q^f)$ for $i \in \{1, 2\}$.

The IC through which is passes must be balanced i.e., $(x_1, q_1) \sim_i (x_2, 1 - q_2)$ for some q_2 such that $1 - q_2 > 1 - q^f$ for $i \in \{1, 2\}$. However, this is not possible since $q_1 + 1 - q_2 > q^f + 1 - q^f = 1$ implies that the allocation $f_1(\mathbf{P}) = (x_1, q_1)$ and $f_2(\mathbf{P}) = (x_2, q_2)$ (or the allocation where the two agents get the permutation of this) is not feasible. Therefore, there is not fair allocation for the given set of preferences. However, these preference profiles are ruled out by the assumption that the IC functions are K-Lipschitz continuous.

Chapter 3

Choosing a pair of complementary goods[†]

3.1 Introduction

Complementarity is a well-studied phenomenon in economic theory. Consumers often buy and consume goods in bundles. In standard consumer demand theory, goods are considered complementary if the cross-elasticity of their demand functions is negative, i.e., when the price of one falls, the demand for the other increases. To work with this setup, one requires utility functions and, more importantly, the goods 'need' to have prices. Consider a problem with public goods as discussed in Manzini et al. (2019) of parks and museums where entry can be free. When can we say that such public goods are complementary?

Complementarity has also been studied in contexts other than just goods consumed together. In the transferable utility coalitional setup, convex games are studied extensively. A convex game represents complementarities in value creation. In a many-to-one matching setting involving hospitals and doctors, a hospital may prefer to hire a more diverse team instead of hiring only surgeons. In all the above contexts, the idea of objects being complementary is natural. However, it is unclear whether the choices arising out of preference for *better* complements can be inferred from the observed data of such choices. In this paper, we aim to formalize a choice model that does exactly that. For our purposes, we will not need knowledge of prices or utility functions. The model uses an ordinal setup through which we rationalize the observed choice data.

There has been a recent resurgence in modeling using joint choice functions. Cham-

[†]This chapter is joint work with Varun Bansal, Indian Statistical Institute, Delhi Center. We thank Debasis Mishra for providing valuable feedback at various stages of the project. We are also thankful to Arunava Sen and Dinko Dimitrov for helpful comments and suggestions.

bers et al. (2021) looked at stochastic joint choice functions and conditions that make the joint choice functions separable i.e. they characterize functions where choices over one set do not influence the other. Kashaev et al. (2024) further studied entangled choices that occur when the joint choice functions are not separable. We believe deterministic joint choice functions are suitable for analyzing choices of complementary goods, i.e., if an individual has to choose a pair of goods from a pair of sets that consist of complements. How each good in the pair is ranked according to complementarity affects the choices of the individual.

In our setting, the decision maker (DM) is presented with menus comprising of two sets and must choose one alternative from each set. We represent the choices made by a DM using joint choice functions like in Chambers et al. (2021) but in a discrete setup. We propose a heuristic where each alternative in a set, say A, has a (complementarity) order over alternatives of another set, say B, and similarly, each alternative in set B has a (complementarity) order over alternatives in set A.

We believe that joint choice functions naturally lend themselves for studying complementarity. Complementarity can be modeled, using linear orders, by asking the question "does alternative a_1 complement alternative b more than a_2 ?" i.e. $a_1 \succ_b a_2$ or $a_2 \succ_b a_1$.

We now illustrate with motivating examples:

Example 6. Public Goods: The government has a budget provisioned to construct one public project. Consider the set of possible projects {park, science center, sports complex } with a set of possible locations {Location 1, Location 2, Location 3}. While Location 1 may be ideal for constructing a park, its residents might instead prefer a sports complex. How would the government select a (project, location) pair?

Example 7. Choosing a team: A manager has to choose a two-person team for a task. There are two pools, one of software engineers and one of hardware engineers. Each software engineer may have skills that best complement those of a particular hardware engineer and vice versa. The manager has to select a team that can best accomplish the given task.

To formalize the heuristic, we begin by defining the concept of ‘demand’. We say b demands a in A if a is the maximal element according to $\succ_b$ when restricted to A. In this heuristic, the chosen pairs (a, b) are such that either (i) a demands b from the set of alternatives that demand a or (ii) b demands a from the set of alternatives that demand b. We will formalize the notion in the next section as well as provide a characterization of the family of choice functions, called weak-complements choice functions (weak-CCF). We are also able to retrieve *uniquely* the (complementarity) orders from the observed choice data.

We follow this up with a stronger notion of complementarity. The alternatives in a pair (a, b) are said to be strong-complements if a demands b in B *and* b demands a in A. A joint choice function is a strong-complements choice function (strong-CCF) if for all menus, the chosen pair (a, b) is such that the alternatives are strong-complements. For the universal domain of (complementarity) orders, an strong-CCF may not exist.[1] We then show that if the (complementarity) orders satisfy **single-crossing** property, then it is sufficient to guarantee the existence of strong-complements in all menus. We define single-crossing property in the same spirit as in Gans and Smart (1996) and Barberà and Moreno (2011).

The single-crossing property imposes restrictions on (complementarity) orders based on some underlying attribute of the alternatives in both sets.

Example 8. Weather: When deciding whether to go out (without an umbrella), stay at home or go out (with an umbrella) – the decision will depend on ”how likely” it is to rain. It is natural to presume that given a level of expected rain, if you prefer to go out with an umbrella, then for any level of expected rain more than this, you would still prefer to go out with an umbrella.

Example 9. Strategic Complementarity (Vives (1990)): Suppose two firms are choosing prices to sell their product from the set of feasible prices for a similar product. If

[1] Universal domain refers to the set of all strict linear orders over the relevant set of alternatives.

firm 1 increases its price then it is in the best interest of firm 2 to do the same.

Example 10. Food Choices: Suppose the choice for food depends on underlying nutrition considerations, for instance calories (there is an optimal calorie count which the DM wants to be as close to). A DM has to choose a main course and a dessert. Suppose an agent prefers to pair a higher-calorific dessert with a main to a lower-calorific dessert, then for any lower-calorific main, the preferred dessert's calorie count should not be lower.

In each of the above examples, there is an underlying order that restricts the (complementarity) orders. In Example 3, the expected rain is the underlying attribute; in Example 4, it is the price charged by the other firm and in Example 5, it is the calorie count of the main dish.

Another notion of complementarity that can be modeled in our setup is the notion of strategic complementarity. Since choices are made from two sets, the two sets can be interpreted as two action sets. Vives (1990) analyzed the existence and structure of Nash equilibria under strategic complementarity. These properties enabled them to apply the results by Tarski (1955) as in Topkis (1978) and Topkis (1979). The single-crossing property that we define on (complementarity) orders is an ordinal property in line with the increasing differences property that is exhibited by games with strategic complementarities. Single-crossing property is sufficient for the existence of a strong-complements choice function. Preferences in Examples 8 and 10 behave in a way that satisfies the single-crossing property.

Sprumont (2000) modeled collective choices using joint choice functions and analyses when these choices are Pareto rationalizable and Nash rationalizable. A joint choice function is Nash rationalizable if there exists an ordered set of preferences over all pairs such that the chosen pair from each choice problem is a Nash equilibrium. Example 13 shows how the strong-complement notion can also be interpreted as a Nash equilibrium if the primitives are motivated accordingly. However, the preferences in our model are not on pairs but on one side of the pairs, and so the weak-complements pairs can not be Pareto ranked as in Sprumont (2000).

3.1.1 Related Literature

Recent attempts have been made to analyze complementarity that is not dependent on price and income elasticities or based on parameters other than prices. Gentzkow (2007) and Manzini et al. (2019) analyze complementarity (or substitutability) by looking at online and print newspaper subscriptions as potentially complementary (or substitutable) goods. Gentzkow (2007) emphasized the need for models which allow for agents to choose multiple good simultaneously and constructs a discrete demand model that achieves the same. Manzini et al. (2019) analyze a choice model where the agents are making the choice to i) buy just the print version of the newspaper, ii) buy just the online version of the newspaper iii) buy both versions, and (iv) buy neither, to look at various measures of complementarity that satisfy different desirable properties. In our model, we compare different alternatives in terms of which alternative complements it more using (discrete) joint choice. We do not study measures of complementarity but how choices are likely to be made when complementarity is driving the choices.

Other models that study complementarity include Honda (2021), where they define perception complementarity i.e. the presence of one good may improve the perception of another good, which improves its chances of being chosen. Brady and Rehbeck (2016) look at complementarity in feasibility i.e. products that are more likely to be available together. For instance, dairy products supplied by the same supplier will face unavailability simultaneously if the truck supplying them is delayed.

Csaba (2019) and Aguiar (2015) look at complementarity in attention. Csaba (2019) argues that limited attention could imply that demand for traditional substitutes (for instance, buses and flights) could behave as if they were complements. For instance, making buses more efficient (extra bus lanes to ease traffic) frees up time to look at cheap flights, making it more likely that flights are chosen. Aguiar

(2015) explores a fuzzy attention model where alternatives can be complementary in attention, i.e., they are more likely to be considered together than individually. They call this the attraction effect.

In the literature discussed above, there is an attempt to study complementarity among alternatives, whether in utility, attention, or feasibility. The modeling was done through stochastic choice functions. The availability of joint choice functions as a tool allows us to construct a deterministic choice model that represents choices of complementary objects (a sales and a purchase head, a main and a dessert etc).

The paper is organized as follows. In Section 2, we present the model and describe our notions of complementarity and joint functions that arise from them. The axiomatic characterization of the weak-CCF is presented in Section 3 along with a discussion about the identification results. In section 4, we formally define the strong-CCF and discuss the single-crossing property that guarantees its existence. Section 5 concludes our paper.

3.2 The Model

There are two finite sets X and Y, with $|X| = m$ and $|Y| = n$. The set of all non-empty subsets of X and Y are denoted by $\mathcal{X}$ and $\mathcal{Y}$ respectively. The decision maker (DM) is presented with menus of the form (A, B) where $(A, B) \in (\mathcal{X}, \mathcal{Y})$. The choices of a DM are represented by a **joint choice function** $c(A, B)$.[2]

Definition 1. *A joint choice function is a map* $c : (\mathcal{X}, \mathcal{Y}) \to (X, Y)$ *such that* $c(A, B) \in (A, B)$ *for all* $(A, B) \in (\mathcal{X}, \mathcal{Y})$.

To formally introduce our heuristic, we will now define some concepts and corresponding notations. Each $a \in X$ is endowed with a strict linear order over alternatives in Y, which we denote by $\succ_a$ where $\succ_a \subseteq Y \times Y$.[3] Similarly, each $b \in Y$ is endowed with a strict linear order over alternatives in X, which we denote by $\succ_b \subseteq X \times X$.

[2] We will use the terms 'joint choice function' and 'choice function' interchangeably.

[3] A strict linear order is a complete, transitive and asymmetric binary relation.

The profile consisting of strict linear orders of all alternatives will be denoted by $\succ = \{\succ_a\}_{a\in X} \cup \{\succ_b\}_{b\in Y}$. For any given set, the universal domain of orders refers to the set of all strict linear orders on the set.

For any menu $(A, B) \in (\mathcal{X}, \mathcal{Y})$ and a profile of (complementarity) orders $\succ$, we say b **demands** a in the set A if $a \succ_b a'$ for all $a' \in A \setminus \{a\}$ and similarly a demands b in the set B if $b \succ_a b'$ for all $b' \in B \setminus \{b\}$. For any given menu (A, B), let the 'demanders' of $a \in A$ be defined as

$$D_a^{\succ}(A, B) := \{b \in B : a \succ_b a' \ \ \forall a' \in A \setminus \{a\}\}.$$

Similarly, we define demanders of $b \in B$ as

$$D_b^{\succ}(A, B) := \{a \in A : b \succ_a b' \ \ \forall b' \in B \setminus \{b\}\}.$$

Definition 2. *Given $(A, B) \in (\mathcal{X}, \mathcal{Y})$ and a profile of orders $\succ$, (a, b) are* ***strong-complements*** *in (A, B) if*

$$a \in D_b^{\succ}(A, B) \text{ and } b \in D_a^{\succ}(A, B)$$

The following example demonstrates that a pair of strong-complements may not exist in a menu i.e. there exists a profile of orders $\succ$ such that in a menu the set of strong-complements is empty.

Example 11. Consider the following menus: $A = \{a_1, a_2\} \subseteq X$ and $B = \{b_1, b_2\} \subseteq Y$ with the orders

$\succ_{a_1}$	$\succ_{a_2}$	$\succ_{b_1}$	$\succ_{b_2}$
b_1	b_2	a_2	a_1
b_2	b_1	a_1	a_2

Table 1: An example to show non-existence of strong-complements

One can observe from the set of demanders below that there is no pair of strong-complements.

$$D^{\succ}_{b_1}(A,B) = \{a_1\} \qquad D^{\succ}_{a_2}(A,B) = \{b_1\}$$
$$D^{\succ}_{b_2}(A,B) = \{a_2\} \qquad D^{\succ}_{a_1}(A,B) = \{b_2\}$$

Before we examine the question of the existence of strong-complements, we define the notion of weak-complements. Weak-complements are guaranteed to exist in each menu for the universal domain of orders. Further we will characterize the family of joint choice functions arising from them.

Definition 3. *Given $(A,B) \in (\mathcal{X},\mathcal{Y})$ and and a profile of orders $\succ$, (a,b) are* ***weak-complements*** *in (A,B) if*

$$a \in D^{\succ}_{b}(A,B) \text{ and } b \in D^{\succ}_{a}(D^{\succ}_{b}(A,B),B),$$
$$or$$
$$b \in D^{\succ}_{a}(A,B) \text{ and } a \in D^{\succ}_{b}(A,D^{\succ}_{a}(A,B))$$

The above notion is weaker than strong-complements in the sense that if (a,b) are weak-complements in (A,B) then either i) a demands b in B and b demands a only in the set consisting of alternatives that demand it i.e. $D_b(A,B)$ and not A or ii) b demands a in A and a demands b only in the set consisting of alternatives that demand it $D^{\succ}_{a}$ and not B. [4]

Lemma 1. *For any profile of orders $\succ$ and for every $(A,B) \in (\mathcal{X},\mathcal{Y})$, the set of weak-complements is non-empty.*

Proof. Note that for any $(A,B) \in (\mathcal{X},\mathcal{Y})$, each $a \in A$ demands some $b \in B$ because $\succ_a$ is a strict linear order. Hence there exists $b \in B$ such that $|D^{\succ}_{b}(A,B)| \geq 1$, similarly there exists an a' such that $|D^{\succ}_{a'}(A,B)| \geq 1$. And again due to $\succ_{a'}$ and $\succ_b$

[4]We may suppress the arguments from $D^{\succ}_{a}(A,B)$ when they are clear from the context.

being strict linear orders, a' and b will demand something in $D_{a'}^{\succ}(A,B)$ and $(D_b^{\succ}(A,B)$. This will result in non-empty set of weak-complements. ■

Example 12. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ and consider the following (complementarity) orders

a_1	a_2	a_3	b_1	b_2	b_3
b_1	b_2	b_2	a_3	a_1	a_3
b_3	b_1	b_3	a_2	a_3	a_2
b_2	b_3	b_1	a_1	a_2	a_1

Table 2: (Complementarity) orders used in Example 12

Consider (A,B). $D_{a_1}^{\succ} = \{b_2\}$ and $D_{b_1}^{\succ} = \{a_1\}$. Similarly $D_{a_3}^{\succ} = \{b_1, b_3\}$ and $D_{b_2}^{\succ} = \{a_2, a_3\}$. Since $b_3 \succ_{a_3} b_1$ and $a_3 \succ_{b_2} a_2$, we get our set of weak-complements as $\{(a_1, b_2), (a_1, b_1), (a_3, b_2), (a_3, b_3)\}$ in (A,B).

Definition 4. *A joint choice function is a **weak-complements choice function (weak-CCF)** if there exists a profile of orders $\succ$ such that for every $(A,B) \in (\mathcal{X}, \mathcal{Y})$ with $c(A,B) = (a,b)$ implies (a,b) are a pair of weak-complements with the profile of orders $\succ$.*

Some notational remarks: we refer to the menus of the form $(\{a, a'\}, \{b\})$ or $(\{a\}, \{b, b'\})$ as 'binary menus'.[5]

Also note that for a given $\succ$ there can be many weak-complements choice functions that are compatible with it as cardinality of the set of weak-complements may be greater than 1 for any non-binary menu.

[5]For notational convenience, we may write the menu $(\{a\}, \{b, b'\})$ as (a, bb') and $A \setminus \{a\}$ as $A \setminus a$.

3.3 Characterization of the weak-CCFs

In this section, we characterize the family of weak-complements choice functions by using the following three axioms.

Axiom 1. *Acyclicity: We say that a choice function, c, is acyclic if*

(i) $\forall a \in X, \forall b_1, b_2, b_3 \in Y$,

$$[c(a, b_1b_2) = (a, b_1) \text{ and } c(a, b_2b_3) = (a, b_2)] \Rightarrow [c(a, b_1b_3) = (a, b_1)]$$

(ii) $\forall a_1, a_2, a_3 \in A, \forall b \in Y$,

$$[c(a_1a_2, b) = (a_1, b) \text{ and } c(a_2a_3, b) = (a_2, b)] \Rightarrow [c(a_1a_3, b) = (a_1, b)]$$

The acyclicity axiom restricts how the choices behave in binary menus. This axiom is quite standard and helps us identify the underlying profile of (complementarity) orders.

For further axioms we define, **always chosen**. For any $(A, B) \in (\mathcal{X}, \mathcal{Y})$, we say a is always chosen with b in A if for all $a' \in A \setminus a$, $c(aa', b) = (a, b)$ similarly b is always chosen with a in B if for all $b' \in B \setminus b$, $C(a, bb') = (a, b)$.

We say a is *not always chosen* with b in A if there exists $a' \in A \setminus a$ such that $c(aa', b) = (a', b)$. We define b being not always chosen with a in B analogously.

The following axiom, consistent with binary choice, ensures that if an alternative, say a, is always chosen with two alternatives b, b' in A then if (a, b) was not chosen in binary menu (a, bb') then it will not be chosen in (A, B)

Axiom 2. *Consistent with binary choice (CBC): A joint choice function c is consistent with binary choice (CBC) if for every* $(A, B) \in (\mathcal{X}, \mathcal{Y})$ *and*

(i) $(a, b), (a, b') \in (A, B)$, *a is always chosen with b and b′ in A*

$$[c(a, bb') \neq (a, b)] \Rightarrow [c(A, B) \neq (a, b)]$$

(i) $(a, b), (a', b) \in (A, b)$, *b is always chosen with* a *and* a' *in* B

$$[c(aa', b) \neq (a, b)] \Rightarrow [c(A, B) \neq (a, b)]$$

The following axiom, no-upgrade, ensures that if for any pair $(a, b) \in (A, B) \in (\mathcal{X}, \mathcal{Y})$ there exists $a' \in A$ such that $c(aa', b) \neq (a, b)$ **and** $b' \in B$ such that $c(a, bb') \neq (a, b)$ then (a, b) can not be upgraded to be chosen in (A, B).

Axiom 3. *No-upgrade (NU): A joint choice function* c *satisfies no-upgrade (NU) if for every* $(A, B) \in (\mathcal{X}, \mathcal{Y})$ *and* $(a, b) \in (A, B)$

$$[(a \textit{ is not always chosen with } b \textit{ in } A) \textbf{ and } (b \textit{ is not always chosen with } a \textit{ in } B)] \Rightarrow [c(A, B) \neq (a, b)]$$

CBC axiom captures the 'demands' of underlying orders. It encapsulates the idea that if in underlying orders two alternatives a, a' demand the same alternative b then b will not be chosen with the one that it did not demand in the binary menu. Similarly, NU captures the idea that if neither alternative demands the other, then that pair can not be chosen.

The following theorem states that the above testable axioms characterize the weak-complements choice functions.

Theorem 1. *A choice function, c, is a weak-complements choice function if and only if it satisfies acyclicity,Consistency with Binary Choice (CBC), and No-Upgrade (NU).*

Moreover, the identified profile of orders that rationalize the choice function is unique.

Proof. **Necessity**

Acyclicity: Consider any $a, a' \in X$ and $b \in Y$ and binary menu (aa', b). Both a and a' will demand b in B and if $a \succ_b a'$ then $c(aa', b) = (a, b)$. The necessity of the acyclicity follows from the transitivity of $\succ_b$. Similarly, for the other side by considering binary menus of the form (a, bb') where $a \in X$ and $b \in Y$.

CBC: Consider any $(A, B) \in (\mathcal{X}, \mathcal{Y})$. If a is always chosen with both b and b' in A then both b and b' demand a in A and if $c(a, bb') \neq (a, b)$ then it implies that $b' \succ_a b$. Hence, a does not demand b in $D_a^{\succ}$. Similarly for the other side.

NU: If a is not always chosen with any b in A then that implies that there exists $a' \in A \setminus a$ such that $a' \succ_b a$, similarly there is a $b' \in B \setminus b$ s.t. $b' \succ_a b$. Hence, (a, b) can not be weak-complements as $a \notin D_b^{\succ}$ and $b \notin D_a^{\succ}$.

Sufficiency: Consider a choice function $c : (\mathcal{X}, \mathcal{Y}) \to (X, Y)$ that satisfies acyclicity, CBC and NU.

(Unique) Identification of $\{\succ_a\}_{a \in X}$ *and* $\{\succ_b\}_{b \in Y}$: Define revealed complementarity orders, $b \succ_a^R b'$ if $c(a, bb') = (a, b)$ and $a \succ_b^R a'$ if $c(aa', b) = (a, b)$. $\succ_a^R$ and $\succ_b^R$ defined such would be complete as $c(A, B)$ is non-empty for all $(A, B) \in (\mathcal{X}, \mathcal{Y})$. Moreover, c is a function; hence, the resulting orders are asymmetric. The transitivity of these orders is guaranteed by the acyclicity of the choice function, c.

Using $\succ_a^R$ and $\succ_b^R$ for each $(A, B) \in (\mathcal{X}, \mathcal{Y})$ construct the set of weak-complements. It remains to show that for all $(A, B) \in (\mathcal{X}, \mathcal{Y})$, the choice $c(A, B)$ are weak-complements with respect to the revealed $\succ^R$. For all binary menus, the set of weak-complements is a singleton $c(A, B)$. Further, this can only be true only if $\succ_a = \succ_a^R$ and $\succ_b = \succ_b^R$ as each of these orders is asymmetric.

Consider any non-binary menu $(A, B) \in (\mathcal{X}, \mathcal{Y})$, we need to show that if $c(A, B) = (a, b)$ then (a, b) are weak-complements wrt $\succ^R$. This is equivalent to saying that if (a, b) are not weak-complements wrt $\succ^R$ then they could not have been $c(A, B)$.

Assume for contradiction that (a, b) are not weak-complements then one of the following three cases must occur.

i $a \notin D_b^{\succ}$ and $b \notin D_a^{\succ}$. This implies that there exists $b' \in B$ and $a' \in A$ such that $b' \succ_a b$ and $a' \succ_b a$ and hence a is not always chosen with b in A and b is not always chosen with a in B. By NU, $c(A, B) \neq (a, b)$

ii $a \in D_b^{\succ}$ and there exists $a' \in D_b^{\succ}$ and $a' \succ_b a$. Hence, by CBC $c(A, B) \neq (a, b)$ as $c(aa', b) \neq (a, b)$

iii $b \in D_a^{\succ}$ and there exists $b' \in D_a^{\succ}$ and $b' \succ_a b$. Hence, by CBC $c(A, B) \neq (a, b)$ as $c(a, bb') \neq (a, b)$

■

We have characterized weak-complements choice functions. And we are able to retrieve the (complementarity) orders that give rise to these choice functions. Now, we consider the following.

Definition 5. *A joint choice function is a* ***strong-complements choice function (strong-CCF)*** *if there exists a profile of orders $\succ$ such that for every $(A, B) \in (\mathcal{X}, \mathcal{Y})$ with $c(A, B) = (a, b)$ implies (a, b) are a pair of strong-complements with respect to $\succ$.*

In the next section, we try to answer the question "What are the sufficient conditions on the (complementarity) orders so that strong-complements exist in all menus?" and hence, a strong-complements choice function exists.

3.4 Strong-complements

As discussed in the model section, the notion of strong-complements provides a natural formalization of complementarity based on mutual demand. We restate the definition of strong-complements. Given $(A, B) \in (\mathcal{X}, \mathcal{Y})$ and a profile of orders $\succ$, (a, b) are **strong-complements** in (A, B) if $a \in D_b^{\succ}(A, B)$ and $b \in D_a^{\succ}(A, B)$.

If a DM can pick a pair of alternatives from a menu that work best with each other, i.e. demand each other, then there is no clear improvement possible.

Consider any choice function, $c : (\mathcal{X}, \mathcal{Y}) \to (X, Y)$, that is weak-complements choice function i.e. there exists a profile of orders $\succ$ such that for all $(A, B) \subseteq (\mathcal{X}, \mathcal{Y})$, $c(A, B)$ are weak-complements. When can we say that for each (A, B), there exists $(a, b) \in (A, B)$ such that (a, b) are strong-complements?

Note also that any strong-CCF is also a weak-CCF, so it is a specific kind of weak-CCF. Just like the weak-CCF, the same profile of orders can potentially support multiple strong-CCF choice functions, each of which reflects that profile uniquely.

As shown in Example 11, the existence of strong-complements is not guaranteed in the universal domain of (complementarity) orders. We now provide sufficient conditions for the existence of strong-complements in all menus.

3.4.1 Single-Crossing

In this section, we show that the well-known single-crossing property is sufficient to guarantee the existence of strong-complements. The idea is that the underlying complementarity between alternatives follows some regularity.

Consider the following motivating example. While choosing a meal, the DM has to choose a curry and a dessert. Suppose the curries have different levels of spice, while the desserts have different levels of sweetness. If the DM pairs a dessert with a particular curry, it would be natural for him to prefer a dessert that is at least as sweet as the previous one with a spicier curry. We intend to restrict (complementarity) orders such that they respect such conditions. The single crossing property described below is motivated by the idea of increasing differences (Topkis (1978) and Vives (1990)) for the ordinal setting in particular as defined in Gans and Smart (1996) and Barberà and Moreno (2011). If a dessert works better for a curry than a less sweet dessert, then it must work better than the less sweet dessert for a spicier curry.

The literature on supermodular games (Vives (1990), Topkis (1978), Topkis (1979)) and monotone comparative statics (Milgrom and Shannon (1994), Milgrom and Roberts (1994), etc.) builds significantly on the notion of strategic complementarity. Since our notion of strong-complements can also capture game-theoretic environments — as illustrated in Example 13 — we draw on insights from this literature to establish a sufficient condition for the existence of a strong-complements choice function. When strong complements are interpreted as Nash equilibria, as in Example 13, our result provides a sufficient condition on the underlying primitives to ensure the existence of Nash equilibria in every sub-game of a two-player finite normal form game. This approach is aligned with the spirit of Sprumont (2000).

Example 13. Consider a game-theoretic setting; let A be the set of actions of Player 1 and B be the set of actions for Player 2. For each action of player 2, player 1 will rank his actions and vice versa. So we can interpret $a \succ_b^1 a'$ as if player 2 plays b, then player 1 prefers playing a over a'. Moreover, if the orders are strict linear, then $a \in D_b^{\succ}$ will mean a is the best response to b. Hence, any (a, b) being strong-complements can be interpreted as being the Nash equilibrium of a two-player game $\langle\{1, 2\}, \{A, B\}, \{\succ^1, \succ^2\}\rangle$.

The sufficient condition is that the (complementarity) orders satisfy a single crossing property (Milgrom and Shannon (1994)). Single crossing is an important condition observed in ordinal models of social choice (Gans and Smart (1996), Barberà and Moreno (2011)) and, in a similar spirit, lends itself to our model. We define single-crossing (complementarity) orders as in Barberà and Moreno (2011):

Single Crossing: A profile of orders $\succ$ satisfies **single-crossing** property if there exists strict linear orders $<_X$ and $<_Y$ on alternatives of X and Y respectively, and for any $a <_X a'$, $b <_Y b'$

(SP.i) $[b' \succ_a b] \Rightarrow [b' \succ_{a'} b]$ and

(SP.ii) $[a' \succ_b a] \Rightarrow [a' \succ_{b'} a]$

Lemma 2. *Consider $a, a' \in A \subseteq X$ and $b, b' \in B \subseteq Y$ and strict linear orders $<_X, <_Y$*

(i) If $a <_X a'$ and a demands b in B and a' demands b' in B then $b <_Y b'$ or $b = b'$

(ii) If $b <_X b'$ and b demands a in A and b' demands a' in A then $a <_Y a'$ or $a = a'$

Proof. Consider any $(A, B) \in (\mathcal{X}, \mathcal{Y})$, and any $a, a' \in A$. As $a <_X a'$ and a demands b in B and a' demands b' in B we have $b \succ_a b'$ and $b' \succ_{a'} b$. As $<_Y$ is a strict linear order, assume for contradiction $b' <_Y b$. But this violates (SP.i). Similarly, we can show (ii). ■

Single-crossing property and similar conditions has been used in literature (Athey (2001), Konishi et al. (1997), Bilancini and Boncinelli (2016), Lizzeri and Persico (2000) etc.) to show existence of pure-strategy Nash equilibrium in a game-theoretic setting and we do a similar exercise below to show existence of strong-complements.

Proposition 1. *If the profile of (complementarity) orders $\succ$ satisfies single-crossing property with respect to strict linear orders $<_X$ and $<_Y$, then for every menu $(A, B) \in (\mathcal{X}, \mathcal{Y})$ there exists a pair $(a, b) \in (A, B)$ such that (a, b) are strong-complements in (A, B).*

Proof. We will essentially start looking at the demands of alternatives starting from the leftmost alternative (according to the exogenous orders). Because of the single-crossing property, we will either reach a fixed point (strong-complement) or continue moving rightwards in both orders. Since the sets are finite, the process must eventually terminate at a pair of strong-complements.

Let the leftmost alternatives according to $<_X$ in A and according to $<_Y$ in B be a_s and b_s respectively; similarly, let the rightmost alternatives be a_l and b_l. Below, we will outline an algorithm that will terminate at a fixed point.

- let, a_s demands b_{q_1} in B. If b_{q_1} demands a_s in A, STOP
  - else let, b_{q_1} demands a_{p_1} in A. If a_{p_1} demands b_{q_1} in B, STOP
    - else let, a_{p_1} demands b_{q_2} in B. If b_{q_2} demands a_{p_1} in A, STOP
      - else let, b_{q_2} demands a_{p_2} in A. If a_{p_2} demands b_{q_2} in B, STOP

else keep moving forward i.e. towards right. Note that due to Lemma 2, $a_s <_X a_{p_1} <_X a_{p_2}$ and $b_s <_Y b_{q_1} <_Y b_{q_2}$.

If no fixed point is reached till a_{p_3} s.t. a_{p_3} demands b_l in B. We know by algorithm that a_{p_3} would have reached only if b_{q_3} demanded a_{p_3} in A, such that $b_{q_3} <_Y b_l$. So either b_l demands a_{p_3} in A or some a_{p_4} s.t. $a_{p_3} <_X a_{p_4}$. Due to Lemma 2, all a such that $a_{p_4} <_X a$ will demand b_l in B and we have a fixed point. Note that by symmetry the case where for some b_k where b_k demands a_l in A then for all alternatives $b_{k'}$ such that $b_k <_Y b_{k'}$ we will have $b_{k'}$ demand a_l in A. And a_l demands b_k in B or some $b_{k'}$ which will give us a fixed point.

The above can be seen in Figure 16 where the arrows represent demands.

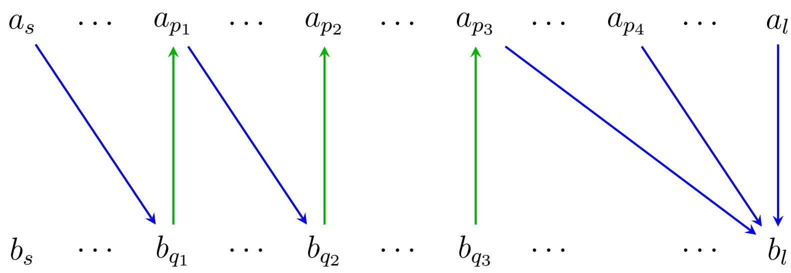


Figure 16: Illustration of steps to reach a pair of strong-complements

■

3.5 Conclusion

In this paper, we study the choices of a DM who is choosing a pair of complementary goods. The choices are driven by how well the goods complement each other. We have introduced a natural and novel heuristic to make such a choice by utilizing the tool of joint choice functions. We have defined when we say (a, b) are weak-complements and strong-complements. Our minimal notion of complementarity, weak-complements, gives rise to weak-CCFs. We provide an axiomatic characterization of this family of joint choice functions using three testable axioms. We then examine the existence of strong-complements, which may fail to exist under the universal domain of (complementarity) orders. We provide sufficient conditions to ensure it's existence.

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