Coxeter Groups and Positive Matrices

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To my parents:
L. N. Lall
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Arvind K. Lal

Errata

Page	Line	Replace	By
2	8	$l(w)$	$\ell(w)$
33	2	page 8	page 9
51	-5	C_2	C_2 and C_3
53	11	$1 \leq m_i \leq N$	$1 \leq \ell \leq N$
54	7	list C_2	list C_2 or C_3
54	-3	sugraphs	subgraphs
56	-10	C_2	C_2 or C_3

On page 26, the graph D_{n+2} represents the graph $B_{n,2}$ which is

Errata

Page	Line	error	replace by
2	5	have	has
2	8	$l(w)$	$\ell(w)$
4	1	advoc	adhoc
4	6	undirected graphs	directed graphs
4	8	undirected graphs	combinatorially symmetric directed graphs
8	6	graph	connected graph
9	-2	$\tilde{D}_n$	$\tilde{B}_n$
11	13	directed	combinatorially symmetric directed
12	8	have	has
13	-3	$P_1(\lambda) = \lambda$.	$P_1(\lambda) = \lambda$ and $P_0(\lambda) = 1$.
26	4		D_{n+2} is same as as the graph $B_{n,2}$.
33	1	$\tilde{B}_n$	$\tilde{D}_n$
33	-6	David and John	Ford and Mckay
53	11	m_i	ℓ
63	-9	$(-1)^{\epsilon(\sigma)}$	$\epsilon(\sigma)$
69	9	ia	is
75	14	page 61	page 68
77	12	page 67	page 75
88	-5	e	v
88	-4	where e	where v
88	-3	of $\Pi_q(A)$.	of $\Pi_q(A)$ where A is a 3×3 positive definite matrix.
115	-8	Laslo Lovasz	László Lovász

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Chapter 0

Introduction

In this thesis, we study positive matrices (matrices whose entries are nonnegative as well as matrices which are positive semidefinite) with Coxeter groups as the underlying theme. For an exposition on Coxeter groups see Humphreys (1990).

A *Coxeter system* consists of a pair (W, S); where W is a group and S is a set which consists of the generators of the group W. The elements of the set S have only the relations of the form $(ss')^{m(s,s')} = 1$; where $m(s,s) = 1$, $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S. In case no relation occurs for a pair s, s', we make the convention that $m(s,s') = \infty$.

To represent a *Coxeter system* (W, S) we need a finite set S of generators and a symmetric matrix M whose rows and columns are indexed by S with entries in $\mathbb{Z} \cup \{\infty\}$ subject only to the conditions : $m(s,s) = 1$, $m(s,s') \geq 2$ if $s \neq s'$. Equivalently, one can draw a graph G with S as the vertex set, joining vertices s and s' by an edge labelled $m(s,s')$ whenever this number (∞ allowed) is at least 3. If distinct vertices s and s' are not joined, we understand that $m(s,s') = 2$. G is called the *Coxeter graph* corresponding to the Coxeter system (W, S). A Coxeter system is said to be *irreducible* if the corresponding Coxeter graph G is connected.

The *adjacency matrix* of a Coxeter graph G, usually denoted by $A(G) = ((a_{ij}))$

is defined to be a square matrix of order $|V|$ ($|V|$ denotes the cardinality of V); where $a_{ij} = 2cos(\pi/p)$ if the edge (i,j) is labelled with the integer p, and 0 if there is no edge joining the vertex i with vertex j. Note that $p = \infty$ corresponds to $a_{ij} = 2$ for $i \neq j$ and $a_{ii} = 0$ for $i = 1, 2, \ldots, n$.

Since each of the generators $s \in S$ have order 2 in W, each $w \neq 1$ (*identity*) in W can be written in the form $w = s_1 s_2 \ldots s_r$ for some s_i (not necessarily distinct) in S. If r is the smallest integer for which the above expression is possible, then r is called the *length* of w, written $l(w)$.

The *root system* Φ of W is the collection of all vectors $w(\alpha_s)$; where $w \in W$ and $s \in S$. To understand the root system Φ, we look at the geometrical representation of W. Let (W, S) be the given Coxeter system. We consider a vector space V over $I\!R$, having a basis $\{\alpha_s | s \in S\}$ which has one-to-one correspondence with S. We give a geometry on V in such a way that the 'angle' between α_s and $\alpha_{s'}$ is $\frac{\pi}{m(s,s')}$. Based on this geometry, we define a symmetric bilinear form B on V by taking:

$$B(\alpha_s, \alpha_{s'}) = -\cos\frac{\pi}{m(s,s')}.$$

(This expression is interpreted to be -1 in case $m(s,s') = \infty$.) We then get a relation

$$A(G) = 2(I - B).$$

There are several criteria for the finiteness of a Coxeter group (see Proposition 4.1 of Deodhar (1982)). We just state four of them which are based on the definitions given above.

Assume (W, S) to be irreducible. Then the following statements are equivalent:

1. W is finite,
2. Φ is finite,
3. The set $\{\ell(w) | w \in W\}$ is bounded above,

4. The bilinear form $B(\cdot\ ,\ \cdot)$ is positive definite.

In the first chapter, we give a new criterion for the finiteness of a Coxeter group. The matrices which appear in the first chapter have nonnegative entries. We have denoted by A_n the adjacency matrix of the path on n vertices. For any positive integer k, we denote by P_k the characteristic polynomial of A_k. We also make the convention that $P_0 \equiv 1$. We have

$$P_k(\lambda) = \det(\lambda I - A_k) \quad \text{for } k = 1, 2, \ldots$$

It can be easily seen that $P_k(\lambda)$ is a polynomial in λ of degree k and corresponds to the Chebyshev polynomial of the second kind.

For any Coxeter graph G, we evaluate the matrix $P_k(A(G))$. For a fixed integer k, $P_k(A(G))$ is defined to be the *kth path-matrix* of the Coxeter graph G. A (Coxeter) graph G is said to be *path-positive of order m,* if for all $k = 1, 2, \ldots, m$, the kth path-matrix of G is nonnegative. Furthermore, the (Coxeter) graph is said to be *path-positive,* if for all positive integers k the kth path-matrix is nonnegative .

We first show that the graphs which are not path-positive exhibit periodicity in the sequence $P_k(A(G))$. We then show that if $\tilde{G}$ is a supergraph of the graph G and if G is path-positive then so is $\tilde{G}$. This Lemma makes our task easy in the sense that for characterising all the Coxeter graphs, we need to show the path-positivity of fewer graphs. In the fourth section of the first chapter, we show path-positivity of some graphs using the following technique: for showing the path-positivity of a Coxeter graph G, we use the spectral decomposition of its adjacency matrix. The adjacency matrix $A(G)$ of any undirected graph G is a real symmetric matrix. So, there exists an orthogonal matrix Q such that $A(G) = QDQ^t$; where the matrix D is a diagonal matrix whose entries are the eigenvalues of $A(G)$. Since Q is orthogonal,

$$P_k(A(G)) = P_k(QDQ^t) = QP_k(D)Q^t$$

This technique enables us to give the exact form of the matrices $P_k(A(G))$ for any Coxeter graph G and for any $k \in \mathbb{Z}^+$, and this in turn proves the conjectures of Bapat and Lal (1991). In Appendix 4, using an advoc method, we show that the graphs Z_4 and Z_5 are path-positive.

The main result in this chapter is the following : *A connected Coxeter graph is path-positive if and only if it is not in $\mathcal{C}_1$ of Appendix 1.* The result can be interpreted as a new criterion for the infiniteness of a Coxeter group.

In the last section, we discuss undirected graphs. A *directed graph* $D = (V, \vec{E})$ consists of a finite nonempty set V of vertices together with a collection $\vec{E}$ of *ordered* pairs of distinct vertices. *Using the main result, we go on to characterise undirected graphs as well on the basis of path-positivity.*

In the second and third chapter, we consider matrices which are positive semidefinite. It is well known that $\mathcal{S}_n$, the symmetric group on n elements, is a Coxeter group. Using the Coxeter group $\mathcal{S}_n$, we define the *q-permanent* of a matrix A, denoted by $\mathrm{per}_q(A)$ as

$$\mathrm{per}_q(A) = \sum_{\sigma \in \mathcal{S}_n} q^{i(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)} \ ;$$

where for $\sigma \in \mathcal{S}_n$, $i(\sigma)$ denotes the number of inversions of σ. Observe that $\mathrm{per}_1(A) = \mathrm{per}\, A$, $\mathrm{per}_{-1}(A) = \det A$ and $\mathrm{per}_0(A) = \prod_{i=1}^{n} a_{ii}$; where "per" and "det" denote the permanent and determinant respectively.

It follows from a result of Bożejko and Speicher (1990) that if $A \geq 0$ then

$$\mathrm{per}_q(A) \geq 0, \qquad -1 \leq q \leq 1.$$

In the second chapter, we construct a similar function based on the sign change group $\mathcal{B}_n$, which like $\mathcal{S}_n$, is a Coxeter group. If A is an $n \times n$ matrix, then for a complex q, the *q- permanent of A with respect to the sign change group $\mathcal{B}_n$*, denoted by $Q_q(A)$, is defined as

$$Q_q(A) = \sum_{\tilde{\sigma} \in \mathcal{B}_n} (-1)^{|J_{\tilde{\sigma}}|} q^{n(\tilde{\sigma})} \prod_{j=1}^{n} a_{j|\tilde{\sigma}(j)|} \ ;$$

where

$$|J_{\tilde{\sigma}}| = \mathrm{card}(\{i \in \{1, 2, \ldots, n\} : \tilde{\sigma}(i) \in \{-1, -2, \ldots, -n\}\}) \ ,$$

and for $\tilde{\sigma} \in \mathcal{B}_n$, $n(\tilde{\sigma})$ is the number of inversions of $\tilde{\sigma}$. In Lemma 2.1.1, we explicitly calculate the coefficient of the term $\prod_{j=1}^{n} a_{j\sigma(j)}$ and hence show that

$$Q_q(A) = \sum_{\sigma\in\mathcal{S}_n} q^{i(\sigma)} \prod_{l=1}^{n} \{1 - q^{2(l-\beta_l^\sigma)-1}\} \prod_{j=1}^{n} a_{j\sigma(j)};$$

where for $1 \leq \ell \leq n$, β_ℓ^σ, denotes the number of inversions in

$$(1,\ell), (2,\ell), \ldots, (\ell-1,\ell)$$

for a fixed $\sigma \in \mathcal{S}_n$. For the sake of convenience, we take $\beta_1^\sigma = 0$.

We also show that if A is positive semidefinite then $Q_q(A)$ is nonnegative for $-1 \leq q \leq 1$. It can be easily seen that the above function generalises the determinant function.

In the third chapter, we study the behaviour of the function $\mathrm{per}_q(A)$. Let $A = ((a_{ij}))$ be any $n \times n$ positive semidefinite matrix. In 1893, Hadamard proved that

$$\det A \leq \prod_{i=1}^{n} a_{ii}.$$

Fifteen years later, E. Fischer showed that if A is partitioned into blocks

$$A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} \tag{0.0.1}$$

where B and D are square matrices then

$$\det A \leq \det B . \det D \leq \prod_{i=1}^{n} a_{ii}. \tag{0.0.2}$$

In 1963, M. Marcus proved the Hadamard theorem for permanents, namely

$$\mathrm{per} A \geq \prod_{i=1}^{n} a_{ii}.$$

Shortly thereafter, E. H. Lieb (1966) proved the corresponding dual of (0.0.2), which shows that

$$\mathrm{per} A \geq \mathrm{per} B . \, \mathrm{per} D \geq \prod_{i=1}^{n} a_{ii};$$

where A is partitioned as in (0.0.1). The main aim of section two in the third chapter is to prove a generalised version of the Lieb's inequality and then obtain a generalised Hadamard inequality.

It is conjectured in Bapat (1991) that if A is positive definite and is not a diagonal matrix then $\text{per}_q(A)$ is strictly increasing in [-1,1]. The conjecture has been verified for $n \leq 3$ in Bapat (1991). In the third section of this chapter, we settle the conjecture for a special case, namely, when the matrix A is tridiagonal. We also show that $\text{per}_q(A)$ is strictly increasing in [-1, 1], for a 3×3 positive definite matrix A, using a different method than the one used in Bapat (1991).

We now define the Schur $q-$power matrix : We know that for any $n \times n$ matrix $A = ((a_{ij}))$, the Schur power matrix of A is the matrix $\Pi(A)$ of order $n! \times n!$. The rows and columns of this matrix are indexed by the elements of the group $\mathcal{S}_n$. For $\sigma,\ \tau \in \mathcal{S}_n$,

$$(\Pi(A))_{\sigma,\tau} = \prod_{i=1}^{n} a_{i\ \tau\sigma^{-1}(i)}.$$

We define, the *Schur $q-$power matrix* of A to be the matrix

$$\Pi_q(A) = \Pi(A) \circ \mathcal{M} = ((\prod_{j=1}^{n} a_{j\ \tau\sigma^{-1}(j)} q^{i(\tau\sigma^{-1})}))_{\sigma,\tau\in\mathcal{S}_n}.$$

In the fourth section, we show, following the technique in Bapat and Sunder (1986), that for a 3×3 positive semidefinite matrix A, $\text{per}_q(A)$ is the largest eigenvalue of $\Pi_q(A)$, $0 \leq q \leq 1$. Here, we also conjecture that $\text{per}_q(A)$ is the largest eigenvalue of $\Pi_q(A)$ for $0 \leq q \leq 1$.

In the last section of this chapter, we show the relation between the two conjectures and then go on to show that the conjecture that " $\text{per}_{-q}(A)$ is the smallest eigenvalue of $\Pi_q(A)$ for $0 \leq q \leq 1$" is false.

Chapter 1

Path-Positivity and infinite Coxeter Groups

1.1 Introduction

We start with a few definitions which will be needed throughout this chapter.

Definition 1.1.1 : A *graph* $G = (V, E)$ consists of a finite nonempty set V (the *vertex set*) together with a set E (the edge set) of ***unordered*** pairs of distinct elements of V.

Definition 1.1.2 : A *Coxeter graph* is a connected graph, whose edges are labelled with integers ≥ 3 or with the symbol ∞. Since the label 3 occurs frequently, we omit it when representing a graph by a picture.

Definition 1.1.3 : A *directed graph* $D = (V, \vec{E})$ consists of a finite nonempty set V of vertices together with a collection $\vec{E}$ of ***ordered*** pairs of distinct elements of V.

Definition 1.1.4 : The *adjacency matrix* of a Coxeter graph G, usually denoted by $A(G) = ((a_{ij}))$ is defined to be a square matrix of order $|V|$ ($|V|$ denotes the cardinality of V); where $a_{ij} = 2cos(\pi/p)$ if the edge (i, j) is labelled with the integer p, and 0 if there is no edge joining the vertex i with vertex j. Note that $p = \infty$ corresponds to $a_{ij} = 2$ for $i \neq j$ and $a_{ii} = 0$ for $i = 1, 2, \ldots, n$.

In the first five sections of this chapter, we consider only Coxeter graphs and devote the last section for directed graphs. For basic concepts in Graph Theory which are not explicitly defined here, we refer to Harary (1969).

By a *supergraph* of a graph $G = (V, E)$, we mean a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ such that $V \subseteq \tilde{V}$, $E = \tilde{E}$, or $E \subsetneq \tilde{E}$, and $m \leq \tilde{m}$ if $e \in E \subset \tilde{E}$ has label m in G and $\tilde{m}$ in $\tilde{G}$.

A *path on n vertices* (or *a path of length* $n - 1$) is the graph with n vertices, say $1, 2, ..., n$, with $n - 1$ edges such that vertex i and $i + 1$ are adjacent, $i = 1, 2, ..., n - 1$. The adjacency matrix of the path is clearly the tridiagonal matrix with 1 on the super- and sub- diagonals and zeros elsewhere. We will denote the adjacency matrix of the path on n vertices by A_n.

For any positive integer k, we denote by P_k the characteristic polynomial of A_k. We also make the convention that $P_0 \equiv 1$. We have

$$P_k(\lambda) = \det(\lambda I - A_k) \quad \text{for } k = 1, 2, ... \tag{1.1.1}$$

Throughout this chapter, for any matrix A, by $A \geq 0$ we mean the matrix A to be a nonnegative matrix , that is, the entries of the matrix A are nonnegative. Also, $A \geq B$ will mean $a_{ij} \geq b_{ij}$ for all i, j. We introduce the following definitions :

Definition 1.1.5 : The *kth path-matrix* of the graph G is defined to be the matrix $P_k(A(G))$.

Definition 1.1.6 : A (Coxeter / directed) graph G is said to be *path-positive of order* m, if for all $k = 1, 2, ..., m$, the kth path-matrix of G is nonnegative. Furthermore, the (Coxeter / directed) graph is said to be *path-positive,* if for all positive integers k the kth path-matrix is nonnegative .

It was conjectured in Bapat and Lal (1991) that any connected graph (that is, a connected Coxeter graph, each of whose edges is labelled three (also known as a simply-laced Coxeter graph)) is path-positive with the exception of the cases A, D and E (see list $\mathcal{C}_1$ in Appendix 1). It was brought to our notice by Professor J. J. Seidel that the proof of this conjecture is contained in de la Harpe and Wenzl (1987). We briefly describe the main result in that paper: (See Goodman et al (1989) for the definitions and basic prerequisites in von Neumann algebras.)

Let $\mathcal{E}$ be the set of all algebraic integers which appear as spectral radii of symmetric matrices with entries in $\mathcal{N} = \{0, 1, 2, \cdots\}$.

Then for any $\lambda \in \mathcal{E}$ with $\lambda \geq 2$, there exists a pair $M_0 \subset M_1$ (M_0 and M_1 are semisimple complex algebras of finite dimension, where M_0 is a subalgebra of M_1 which contains identity) with the inclusion matrix Λ which is symmetric, irreducible, and has spectral radius λ.

It is also true that any symmetric matrix Λ with entries in $\mathcal{N}$ can be viewed (see [11], §2.3) as an inclusion matrix describing $M_0 \subset M_1$; where M_0 and M_1 are both semisimple complex algebras of finite dimension.

Let $M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$ be the basic tower construction of finite dimensional C^*- algebras (see [16]) with every inclusion matrix being Λ. If the spectral radius λ of Λ is at least 2, then by [14], the matrix $P_k(\Lambda)$ is the inclusion matrix of $pM_0 \subset pM_kp$, for any k, and for a certain projection p in $M_0' \cap M_k$. Hence $P_k(\Lambda)$ has nonnegative entries, and Λ is path-positive.

Since it is well-known that $A_\ell, D_\ell, E_n($ for $n = 6, 7, 8.)$ are the only graphs with spectral radius less than 2, the conjecture mentioned earlier is seen to be true.

It appears that the technique that was used in de la Harpe and Wenzl (1987) will not give the path-positivity of the additional cases, namely $\tilde{D}_n$, $\tilde{C}_n$, Z_4, and Z_5, due to the following reasons. In de la Harpe and Wenzl (1987), it is shown that if Λ is a symmetric matrix over the nonnegative integers, and if

$$M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$$

is the basic tower of finite- dimensional C^*- algebras with every inclusion matrix being Λ, then $P_k(\Lambda)$ is the inclusion matrix of $pM_0 \subset pM_kp$, for a certain projection p in $M_0' \cap M_k$.

The proof depends on the fact that if Γ is the inclusion matrix of $M_0 \subset M_1$, and if $\{p_1, p_2, \cdots, p_N\}$ is a partition of unity into projections in $M_0' \cap M_k$, then

$$\Gamma = \sum_{\ell=1}^{N} \Gamma_\ell \quad ;$$

where Γ_ℓ is essentially the inclusion matrix of $p_\ell M_0 \subset p_\ell M_k p_\ell$ (augmented by suitable zero rows and columns).

If Λ is a symmetric matrix with entries from

$$[2, \infty) \cup \{2\cos(\pi/n) : n \geq 2\}$$

(the adjacency matrix of a Coxeter graph as we consider in the present paper is one such matrix) then Λ may be viewed (see [11], §3.5) as an inclusion matrix describing $M_0 \subset M_1$ where M_0 and M_1 are both finite direct sums of finite factors. If

$$M_0 \subset M_1 \subset M_2 \subset \cdots$$

is the result of applying the basic construction to $M_0 \subset M_1$, then it is true that Λ still describes the inclusion matrix of $M_n \subset M_{n+1}$. However, it is not clear to the author that the inclusion matrix of $M_0 \subset M_k$ will necessarily have nice decompositions corresponding to the partitions of unity in $M_0' \cap M_k$. This is because there is no reason why $\mathrm{tr}_{M_0'}$ and tr_{M_k} should agree on $M_0' \cap M_k$ – which does happen when M_0 and M_k are finite- dimensional, the case discussed in de la Harpe and Wenzl (1987). In particular, we do not know if the Theorem in de la Harpe and Wenzl (1987) about $P_k(\Lambda)$ describing suitable *compressions* of $M_0 \subset M_k$ remains valid in the more general setting.

Our technique is as follows: for showing the path-positivity of a Coxeter graph G (marked/unmarked), we use the spectral decomposition of its adjacency matrix. The adjacency matrix $A(G)$ of any undirected graph G is a real symmetric matrix. So, there exists an orthogonal matrix Q such that $A(G) = QDQ^t$; where the matrix D is a diagonal matrix whose entries are the eigenvalues of $A(G)$. Since Q is orthogonal,

$$P_k(A(G)) = P_k(QDQ^t) = QP_k(D)Q^t.$$

This technique enables us to give the exact form of the matrices $P_k(A(G))$ for any Coxeter graph G and for any $k \in \mathbb{Z}^+$, and this in turn proves the conjectures of Bapat and Lal (1991).

In this chapter, we prove the conjectures. Thus the main result in this chapter is the following : *A connected Coxeter graph is path-positive if and only if it is not in C_1 of Appendix 1.* The result can be interpreted as a new criterion for the infiniteness of a Coxeter group. *Using the main result, we go on to characterise directed graphs as well on the basis of path-positivity.*

Several equivalent criteria for the finiteness of a Coxeter group can be found in Proposition 4.1 of Deodhar (1982). We will just state four of them. For this we state a few definitions which have been taken from Chapter 5 of Humphreys (1990).

Definition 1.1.7 : A *Coxeter system* is a pair (W, S); where W is a group and $S \subset W$ is the set of generators, subject only to the relations of the form

$$(ss')^{m(s,s')} = 1;$$

where $m(s,s) = 1$, $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S. In case no relation occurs for a pair s, s', we make the convention that $m(s,s') = \infty$.

To represent a Coxeter system (W, S), we need a finite set S of generators and a symmetric matrix M whose rows and columns are indexed by S, with entries in $\mathbb{Z} \cup \{\infty\}$ subject only to the conditions : $m(s,s) = 1$, $m(s,s') \geq 2$ if $s \neq s'$. Equivalently, one can draw a graph G with S as the vertex set, joining vertices s and s' by an edge labelled $m(s,s')$ whenever this number (∞ allowed) is at least 3. If distinct vertices s and s' are not joined, we understand that $m(s,s') = 2$.

The label $m(s,s') = 3$ is omitted as it occurs frequently and we call G a *Coxeter graph.*

Definition 1.1.8 : A Coxeter system is said to be *irreducible* if the corresponding Coxeter Graph G is connected.

Since each of the generators $s \in S$ have order 2 in W, each $w \neq 1$ (*identity*) in W can be written in the form $w = s_1 s_2 \ldots s_r$ for some s_i (not necessarily distinct) in S. If r is the smallest integer for which the above expression is possible, then r is called the *length* of w, written $\ell(w)$.

Before, we define the root system Φ of W, let us first see the geometric representation of W, with the help of which the concept of the root system Φ can be understood : suppose, we are given a Coxeter system (W,S). We begin with a vector space V over $\mathbb{R}$, having a basis $\{\alpha_s | s \in S\}$ which has one-to-one correspondence with S. Now, we impose a geometry on V in such a way that the 'angle' between α_s and $\alpha_{s'}$ is compatible with the given number $m(s,s')$, that is, the angle between α_s and $\alpha_{s'}$ is $\frac{\pi}{m(s,s')}$.

The *root system* Φ of W consists of a set of unit vectors in V permuted by W, that is, Φ is the collection of all vectors $w(\alpha_s)$, where $w \in W$ and $s \in S$. Accordingly, we define a symmetric bilinear form B on V by requiring :

$$B(\alpha_s, \alpha_{s'}) = -\cos\frac{\pi}{m(s,s')}.$$

(This expression is interpreted to be -1 in case $m(s,s') = \infty$.) Evidently $B(\alpha_s, \alpha_s) = 1$, while $B(\alpha_s, \alpha_{s'}) \leq 0$ if $s \neq s'$. It is easily seen that by Definition 1.1.4,

$$A(G) = 2(I - B). \tag{1.1.2}$$

The first four criteria in the following list are from Proposition 4.1 of Deodhar

(1982) and the fifth is new. For the equivalence between the fourth and the fifth criterion see Lemma 1.5.1, Corollary 1.5.3 and the Observation 1.1.2.

Assume (W, S) to be irreducible. Then the following statements are equivalent:

1. W is finite
2. Φ is finite
3. The set $\{\ell(w) | w \in W\}$ is bounded above
4. The bilinear form $B(\cdot\ ,\ \cdot)$ is positive definite
5. The Coxeter graph corresponding to (W, S) is not path-positive.

1.2 Preliminaries

We begin with the observation that the matrix A_n is given by

$$(A_n)_{ij} = \begin{cases} 1, & \text{if } |i-j| = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2.1}$$

We see that A_n is a symmetric matrix and hence any power of A_n or any polynomial in A_n will give rise to a symmetric matrix. On expanding the determinant in (1.1.1) along the first row, we get the recurrence relation given by

$$P_k(\lambda) = \lambda P_{k-1}(\lambda) - P_{k-2}(\lambda) \qquad \text{for } k = 2, 3, \ldots \tag{1.2.2}$$

with $P_1(\lambda) = \lambda$. The recurrence relation can be used to get a closed expression for P_k, see Lovasz (1979), page 72, which is

$$P_k(\lambda) = \sum_{u=0}^{[k/2]} \binom{k-u}{u} \lambda^{k-2u} (-1)^u \ ; \tag{1.2.3}$$

where $[x]$ denotes the greatest integer less than or equal to x. It will be used in the proof of Lemma 1.4.4.

Lemma 1.2.1: For any positive integer k and square matrices A and B of the same order, we have the following identity:

$$P_k(A+B) = P_k(A) + \sum_{s=0}^{k-1} P_s(A)BP_{k-1-s}(A+B). \tag{1.2.4}$$

Proof: We use induction on k. For $k = 1$, the L.H.S. of (1.2.4) is equal to $P_1(A+B) = A + B = P_1(A) + B$, which is same as the R.H.S. of (1.2.4). Let (1.2.4) be true for $1 \le k \le m$. Using (1.2.2), we get

$$\begin{aligned}
P_{m+1}(A+B) &= (A+B)P_m(A+B) - P_{m-1}(A+B) \\
&= AP_m(A+B) - P_{m-1}(A+B) + BP_m(A+B) \\
&= A\{P_m(A) + \sum_{s=0}^{m-1} P_s(A)BP_{m-1-s}(A+B)\} + BP_m(A+B) \\
&\quad -\{P_{m-1}(A) + \sum_{s=0}^{m-2} P_s(A)BP_{m-2-s}(A+B)\} \\
&= AP_m(A) - P_{m-1}(A) + ABP_{m-1}(A+B) + BP_m(A+B) + \\
&\quad \sum_{s=1}^{m-1}\{AP_s(A) - P_{s-1}(A)\}BP_{m-1-s}(A+B) \\
&= P_{m+1}(A) + BP_m(A+B) + ABP_{m-1}(A+B) \\
&\quad + \sum_{s=1}^{m-1} P_{s+1}(A)BP_{m-(s+1)}(A+B) \\
&= P_{m+1}(A) + \sum_{s=0}^{m} P_s(A)BP_{m-s}(A+B).
\end{aligned}$$

Hence the result is proved. ■

Lemma 1.2.2: *Let $G = (V, E)$ be a graph with $|V| = n$. Suppose $\tilde{G}$ is a supergraph of G. If G is path-positive, then $\tilde{G}$ is also path-positive.*

Proof: We consider two cases,

Case (i): $V \subset \tilde{V}$ and $E \subsetneqq \tilde{E}$. Suppose $\tilde{G} = (\tilde{V}, \tilde{E})$; where $\tilde{V} = V \cup i_1$ for $i_1 \notin V$ and $\tilde{E} = E \cup (i_1, l)$ for some $l \in V$. Number the vertices of the graph $\tilde{G}$ in such a way that the vertex i_1 is given number 1 and the vertex $l \in V$ is given number 2. The renumbering of the vertices does not affect the path-positivity of the graph G.

We have to show that, the kth path matrix of $\tilde{G}$ is nonnegative for all positive integers k.

The matrix $A(\tilde{G})$ is a square matrix of order $n + 1$ and

$$A(\tilde{G}) = \begin{pmatrix} 0 & e_1^t \\ e_1 & A(G) \end{pmatrix} = A + B \ \text{(say)};$$

where $e_1^t = (1, 0, ..., 0) \in \mathbb{R}^n$ (t denotes transpose),

$$A = \begin{pmatrix} 0 & 0^t \\ 0 & A(G) \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & e_1^t \\ e_1 & 0 \end{pmatrix}.$$

We shall use induction on k to show that the kth path-matrix of $\tilde{G}$ is nonnegative for all positive integers k.

For $k = 1$, the claim is trivial. Let the claim hold true for all k, $1 \leq k \leq m$. We shall prove it for $k = m + 1$. By Lemma 1.2.1, we get

$$\begin{aligned} P_{m+1}(A(\tilde{G})) &= P_{m+1}(A + B) \\ &= P_{m+1}(A) + \sum_{s=0}^{m} P_s(A) B P_{m-s}(A + B). \end{aligned} \tag{1.2.5}$$

We observe the following about the expression on the RHS of (1.2.5) :

1. $P_s(A) \geq 0$ for all $s = 0, 1, 2, ..., m + 1$, except possibly for the entry $(1, 1)$ which may be -1.
2. $B \geq 0$;
3. By induction hypothesis, $P_\ell(A + B) \geq 0$ for all $\ell, 0 \leq \ell \leq m$; and

4. The matrix $P_k(A(\tilde{G}))$ is a symmetric matrix as $A(\tilde{G})$ is a symmetric matrix.

Hence, we see that by (1.2.5), the $(m+1)$st path-matrix of $A(\tilde{G})$ is nonnegative except possibly for the entry $(1,1)$. We shall show that

$(P_{m+1}(A(\tilde{G})))_{11} \geq 0$.

We have

$$\begin{aligned} P_{m+1}(A(\tilde{G})) &= P_{m+1}(A+B) \\ &= (A+B)P_m(A+B) - P_{m-1}(A+B). \qquad \text{using (1.2.2)} \end{aligned}$$

Since the first row of A is zero, we get,

$$\begin{aligned} (P_{m+1}(A(\tilde{G})))_{11} &= (P_m(A+B))_{21} - (P_{m-1}(A+B))_{11} \\ &= [P_m(A) + \sum_{s=0}^{m-1} P_s(A)BP_{m-1-s}(A+B)]_{21} \\ &\quad -(P_{m-1}(A+B))_{11} \qquad \text{using Lemma 1.2.1} \\ &= (P_m(A))_{21} + \sum_{s=1}^{m-1}(P_s(A)BP_{m-1-s}(A+B))_{21} \\ &\quad +(P_0(A)BP_{m-1}(A+B))_{21} - (P_{m-1}(A+B))_{11} \\ &= \sum_{s=1}^{m-1}(P_s(A)BP_{m-1-s}(A+B))_{21} \\ &\quad +(P_{m-1}(A+B))_{11} - (P_{m-1}(A+B))_{11} \\ &= \sum_{s=1}^{m-1}(P_s(A)BP_{m-1-s}(A+B))_{21} \\ &\geq 0; \end{aligned}$$

the absence of $(P_m(A))_{21}$ in the fourth equality is due to the fact that $(P_m(A))_{21} = 0$. We get the last inequality due to the following: the second row of $P_s(A)$ is nonnegative for all s, $1 \leq s \leq m-1$; $B \geq 0$, and by induction hypothesis the first column of $P_\ell(A+B)$ is nonnegative for $1 \leq \ell \leq m-1$. Hence the result.

Case(ii): $V = \tilde{V}$, $E = \tilde{E}$, and $m \leq \tilde{m}$ for some $e \in E = \tilde{E}$; where e has label m in G and $\tilde{m}$ in $\tilde{G}$. In this case, we write $A(\tilde{G}) = A(G) + B = A + B$ (say); where B is of order n with each entry nonnegative.

Let us prove this case also by induction on k. Clearly $P_1(A+B) \geq 0$. Let $P_k(A+B) \geq 0 \ \forall \ k, \ 1 \leq k \leq m$.

Since the graph G is path-positive, $P_s(A) \geq 0$ for all positive integer s. Also the matrix $B \geq 0$. Therefore, using Lemma 1.2.1, and the induction hypothesis, we get

$$\begin{aligned} P_{m+1}(A+B) &= P_{m+1}(A) + \sum_{s=0}^{m} P_s(A) B P_{m-s}(A+B) \\ &\geq 0. \end{aligned}$$

Hence the result. ∎

1.3 Periodicity of the graphs A_n, B_n, and D_n

In this section, we show the periodic nature of the graphs A_n, B_n, and D_n. It can easily be seen that all the graphs which appear in the list $\mathcal{C}_1$ (see Appendix 1) behave periodically. We do not discuss the remaining as those cases can be easily verified. Let us denote by $A_{i.}$, $A_{.j}$ the ith row and the jth column of the matrix A respectively. The following result is essentially contained in Beezer (1984). We include a proof for completeness.

Lemma 1.3.1: *For* $1 \leq k \leq n$

$$(P_k(A_n))_{ij} = \begin{cases} 1, & \text{if } i+j = k+2r \text{ with } i,j \geq r \text{ and } r = 1,2,\cdots,n-k; \\ 0, & \text{otherwise.} \end{cases} \qquad (1.3.1)$$

Proof: We will prove the lemma by induction on k.

Step 1. For $k = 1$, $P_1(A_n) = A_n$. Here we have $i + j = 1 + 2r$ with $i, j \geq r$ and $r = 1, 2, \cdots, n-1$. That is,

(i) if $i = r$ then $j = r+1$, and (ii) if $i = r+1$ then $j = r$ with $r = 1, 2, \cdots, n-1$. Hence, we see that in either case $|i - j| = 1$. Therefore, (1.3.1) gives us

$$(P_1(A_n))_{ij} = \begin{cases} 1, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

which is same as (1.2.1).

Step 2. Let the lemma be true for $1 \leq k \leq m \leq n-1$. We have

$$P_{m+1}(A_n) = A_n P_m(A_n) - P_{m-1}(A_n)$$

and hence

$$(P_{m+1}(A_n))_{i.} = (A_n P_m(A_n))_{i.} - (P_{m-1}(A_n))_{i.} \tag{1.3.2}$$

Case 1: $i = 1$. The RHS of (1.3.2) is given by

$$\begin{aligned}
& (A_n P_m(A_n))_{1.} - (P_{m-1}(A_n))_{1.} \\
= \ & (P_m(A_n))_{2.} - (P_{m-1}(A_n))_{1.} \\
= \ & (0,\cdots,0,\underbrace{1}_{mth},0,1,0,\cdots,0) - (0,\cdots,0,\underbrace{1}_{mth},0,\cdots,0) \quad \text{by (1.3.1)} \\
= \ & (0,\cdots,0,\underbrace{1}_{(m+2)th},0,\cdots,0) = (P_{m+1}(A_n))_{1.} \quad \text{by (1.3.1)}
\end{aligned}$$

Case 2: $2 \leq i \leq n-1$. We will give a proof for $i = 2$. The cases $i = 3, 4, \cdots, n-1$ can be handled is a similar way. The RHS of (1.3.2) is given by

$$\begin{aligned}
& (A_n P_m(A_n))_{2.} - (P_{m-1}(A_n))_{2.} \\
= \ & (P_m(A_n))_{1.} + (P_m(A_n))_{3.} - (P_{m-1}(A_n))_{2.} \\
= \ & (0,\cdots,0,\underbrace{1}_{(m+1)st},0,\cdots,0) + (0,\cdots,0,\underbrace{1}_{(m-1)st},0,1,0,1,0,\cdots,0) \\
& -(0,\cdots,0,\underbrace{1}_{(m-1)st},0,1,0,\cdots,0) \quad \text{by (1.3.1)} \\
= \ & (0,\cdots,0,\underbrace{1}_{(m+1)st},0,1,0,\cdots,0) = (P_{m+1}(A_n))_{2.} \quad \text{by (1.3.1)}
\end{aligned}$$

Case 3: $i = n$. The RHS of (1.3.2) is given by

$$\begin{aligned}
& (A_n P_m(A_n))_{n.} - (P_{m-1}(A_n))_{n.} \\
= \ & (P_m(A_n))_{n-1.} - (P_{m-1}(A_n))_{n.} \\
= \ & (0,\cdots,0,\underbrace{1}_{(n-m-1)st},0,1,0,\cdots,0) - (0,\cdots,0,\underbrace{1}_{(n-m+1)st},0,\cdots,0) \quad \text{by (1.3.1)} \\
= \ & (0,\cdots,0,\underbrace{1}_{(n-m-1)st},0,\cdots,0) = (P_{m+1}(A_n))_{n.} \quad \text{by (1.3.1)}
\end{aligned}$$

Hence the result. ■

Corollary 1.3.2: *For $k = 1, 2, \cdots, n$, $(P_k(A_n))_{ij} \geq 0$ for $1 \leq i, j \leq n$.*

Proof: By Lemma 1.3.1 each entry of $P_k(A_n)$ either takes the value 1 or 0. ■

Note that according to Corollary 1.3.2, the path on n vertices is path- positive of order n. The Cayley- Hamilton theorem for A_n may also be derived from Lemma 1.3.1 as follows.

Corollary 1.3.3: *For $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n, (P_n(A_n))_{ij} = 0$.*

Proof: Since $k = n$, the result follows trivially from (1.3.1) . ■

Lemma 1.3.4: *For any integer $k \in \mathbb{Z}^+$, $1 \leq \ell \leq k$, and for any matrix A,*

$$P_k(A) = P_\ell(A)P_{k-\ell}(A) - P_{\ell-1}(A)P_{k-\ell-1}(A). \tag{1.3.3}$$

Proof: For $\ell = 1$, the RHS of (1.3.3) is given by

$$\begin{aligned} P_1(A)P_{k-1}(A) - P_0(A)P_{k-2}(A) &= AP_{k-1}(A) - P_{k-2}(A) \\ &= P_k(A) \qquad \text{using (1.2.2).} \end{aligned}$$

Now let (1.3.3) be true for $1 \leq \ell \leq m \leq k - 1$. Using induction hypothesis and (1.2.2), we get

$$\begin{aligned} P_k(A) &= P_m(A)P_{k-m}(A) - P_{m-1}(A)P_{k-m-1}(A) \\ &= P_m(A)\{AP_{k-m-1}(A) - P_{k-m-2}(A)\} - P_{m-1}(A)P_{k-m-1}(A) \\ &= \{P_m(A)A - P_{m-1}(A)\}P_{k-m-1}(A) - P_m(A)P_{k-m-2}(A) \\ &= P_{m+1}(A)P_{k-m-1}(A) - P_m(A)P_{k-m-2}(A). \end{aligned}$$

Hence the result follows. ■

Theorem 1.3.5: *Let $k = t(n+1) + i$ where $i = 0, 1, \cdots, n$. Then*

$$P_k(A_n) = \begin{cases} P_i(A_n), & \text{if } t \text{ is even;} \\ -P_{n-(i+1)}(A_n), & \text{if } t \text{ is odd;} \end{cases} \tag{1.3.4}$$

where $P_{-1}(A_n)$ is the zero matrix.

Proof: We will prove the result by induction on t. Clearly the expression (1.3.4) holds for $t = 0$.

Case 1: Let us assume that (1.3.4) holds for $0 \leq t \leq 2m$. Now consider the case $t = 2m+1$, that is, $k = (2m+1)(n+1) + i$ with $i = 0, 1, \cdots, n$. Since $P_n(A_n) = 0$, we have $P_\ell(A_n) = 0$ for $\ell = 2m(n+1) + n$. We also have, $k - \ell = i + 1$. Now, using Lemma 1.3.4, we get

$$\begin{aligned} P_k(A_n) &= P_\ell(A_n)P_{k-\ell}(A_n) - P_{\ell-1}(A_n)P_{k-\ell-1}(A_n) \\ &= P_{k-\ell}(A_n) \times 0 - P_i(A_n)P_{n-1}(A_n) \quad (\text{ by the induction hypothesis}) \\ &= -P_i(A_n)P_{n-1}(A_n) \\ &= P_{n+i+1}(A_n) \\ &= -P_{n-(i+1)}(A_n) \qquad [n+i+1 = 1(n+1)+i]. \end{aligned}$$

Note that

$$\begin{aligned} P_{n+i+1}(A_n) &= P_n(A_n)P_{i+1}(A_n) - P_{n-1}(A_n)P_i(A_n) \\ &= -P_{n-1}(A_n)P_i(A_n) \quad \text{using Lemma 1.3.4.} \end{aligned}$$

Case 2: Let us assume that (1.3.4) holds for $0 \leq t \leq 2m+1$. Now, consider the case $t = 2m+2$, that is, $k = (2m+2)(n+1) + i$ with $i = 0, 1, \cdots, n$. Since $P_{-1}(A_n) = 0$, we have $P_\ell(A_n) = 0$ for $\ell = (2m+1)(n+1) + n$. We also have, $k - \ell = i + 1$. Now, using Lemma 1.3.4, we get

$$P_k(A_n) = P_\ell(A_n)P_{k-\ell}(A_n) - P_{\ell-1}(A_n)P_{k-\ell-1}(A_n)$$

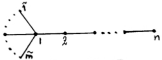


Figure 1.1: $B_{n,m}$

$$\begin{aligned} &= 0 - \{-P_0(A_n)\}P_i(A_n) \quad (\text{ by the induction hypothesis}) \\ &= P_i(A_n)P_0(A_n) \\ &= P_i(A_n). \end{aligned}$$

Hence (1.3.4) holds for all k. ■

Remark. We see that $P_k(A_n)$ is periodic with period $2n+2$, as $P_{-1}(A_n) = P_{2n+1}(A_n)$, $P_0(A_n) = P_{2n+2}(A_n)$, $P_1(A_n) = P_{2n+3}(A_n), \cdots$, $P_{2n}(A_n) = P_{4n+2}(A_n)$, $P_{2n+1}(A_n) = P_{4n+3}(A_n)$, and so on.

Evaluating P_k at a broom

In this subsection, first of all we describe the pattern of $P_k(B_{n,m})$ for any integer m and $k \leq n$. The graph of $B_{n,m}$ can be seen in figure 1.1 above. The order of the matrix $B_{n,m}$ is $(n+m) \times (n+m)$. Let

$$B_{n,m} = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \end{array}$$

with $B_{11} = 0$, $B_{12} = (e_m, 0) = B_{21}^t$, and $B_{22} = A_n$. Here the zero matrix in B_{12} is of order $m \times (n-1)$ and $e_m = (1, 1, \cdots, 1)^t \in I\!R^m$.

Theorem 1.3.6: *For $k \leq n$, if we partition the matrix $P_k(B_{n,m}) = P_k^m$ (say) according to the partition of the matrix $B_{n,m}$, then we have :*

(1)
$$(P_k^m)_{11} = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ D_\ell, & \text{if } k \text{ is even with } k = 2\ell, \end{cases} \tag{1.3.5}$$

where

$$(D_\ell)_{ij} = \begin{cases} \frac{m-1}{m}\{(m-1)^{\ell-1} + (-1)^{\ell}\}, & \text{if } i = j \\ (D_\ell)_{11} + (-1)^{\ell+1}, & \text{if } i \neq j. \end{cases} \tag{1.3.6}$$

(2)

$$((P_k^m)_{12})_{ij} = \begin{cases} (m-1)^r, & \text{if } j = k - 2r \text{ with } r = 0, 1, 2, \cdots, [\frac{(k-1)}{2}], \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.7}$$

Note that the RHS of (1.3.7) does not depend on i; *hence* $((P_k^m)_{12})_{ij} = ((P_k^m)_{12})_{1j}$.

(3)

$$(P_k^m)_{22} = P_k(A_n) + C_m^k; \tag{1.3.8}$$

where

$$(C_k^m)_{ij} = \begin{cases} m(m-1)^{r-1}, & \text{if } i + j = k + 2 - 2r \text{ with } r = 1, 2, \cdots, [\frac{k}{2}], \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.9}$$

Proof: Let us use induction on k. For $k = 1$, $P_1^m = B_{n,m}$. Now let the theorem be true for $1 \leq k \leq t \leq n - 1$. Consider

$$P_{t+1}(B_{n,m}) = B_{n,m} P_t(B_{n,m}) - P_{t-1}(B_{n,m}). \tag{1.3.10}$$

We will prove the result for t odd, the proof for t even can be given on similar lines.

(1) Using (1.3.10), we get $(P_{t+1}^m)_{11} = B_{12}(P_t^m)_{21} - (P_{t-1}^m)_{11}$, that is,

$$\begin{aligned} ((P_{t+1}^m)_{11})_{ij} &= ((P_t^m)_{21})_{1j} - ((P_{t-1}^m)_{11})_{ij} \\ &= ((P_t^m)_{12})_{j1} - ((P_{t-1}^m)_{11})_{ij}. \end{aligned}$$

since t is odd, $\frac{t-1}{2} \in \mathbb{Z}^+$. Therefore

$$((P_{t+1}^m)_{11})_{ij}$$

$$\begin{aligned} &= \begin{cases} (m-1)^r - \frac{m-1}{m}\{(m-1)^{r-1} + (-1)^r\} - (-1)^{r+1}, & \text{if } i \neq j, \\ (m-1)^r - \frac{m-1}{m}\{(m-1)^{r-1} + (-1)^r\} & \text{if } i = j, \end{cases} \\ &= \begin{cases} \frac{m-1}{m}\{(m-1)^r + (-1)^{r+1}\} + (-1)^{r+2}, & \text{if } i \neq j, \\ \frac{m-1}{m}\{(m-1)^r + (-1)^{r+1}\} & \text{if } i = j, \end{cases} \end{aligned}$$

where $r = \frac{t-1}{2}$, which is equal to the expression on the RHS of (1.3.6) after replacing ℓ by $r+1$.

(2) From (1.3.10), we get

$$\begin{aligned} ((P_{t+1}^m)_{12})_{ij} &= (B_{12}(P_t^m)_{22})_{ij} - ((P_{t-1}^m)_{12})_{ij} \\ &= ((P_t^m)_{22})_{1j} - ((P_{t-1}^m)_{12})_{ij} \\ &= (P_t(A_n))_{1j} + (C_t^m)_{1j} - ((P_{t-1}^m)_{12})_{ij} \end{aligned}$$

that is,

$$\begin{aligned} & ((P_{t+1}^m)_{12})_{i.} \\ = \ & (P_t(A_n))_{1.} + (C_t^m)_{1.} - ((P_{t-1}^m)_{12})_{i.} \\ = \ & (0, \cdots, 0, \underbrace{1}_{(t+1)\ st}, 0, \cdots, 0) + (0, \alpha_a, 0, \alpha_{a-1}, 0, \cdots, \underbrace{\alpha_0}_{(t-1)\ st}, 0, \cdots, 0) \\ & -(0, \beta_b, 0, \cdots, \underbrace{\beta_0}_{(t-1)\ st}, 0, \cdots, 0), \end{aligned}$$

where $a = [\frac{t}{2}] - 1 = [\frac{t-2}{2}] = b$, $\alpha_j = m(m-1)^{j-1}$, and $\beta_j = (m-1)^j$. So

$$((P_{t+1}^m)_{12})_{i.} = (0, \nu_{a+1}, 0, \nu_a, 0, \cdots, 0, \underbrace{\nu_0}_{(t+1)\ st}, 0, \cdots, 0),$$

where $\nu_j = (m-1)^j$ which is same as the row obtained by using (1.3.7) .

(3) Here we have $((P_{t+1}^m)_{22})_{ij} = (B_{21}(P_t^m)_{12})_{ij} + (B_{22}(P_t^m)_{22})_{ij} - ((P_{t-1}^m)_{22})_{ij}$.

Case (i): For $i = 1$, we get

$$((P_{t+1}^m)_{22})_{1.}$$

$$\begin{aligned}
&= m((P_t^m)_{12})_{1.} + ((P_t^m)_{22})_{2.} - ((P_{t-1}^m)_{22})_{1.} \\
&= m\{(0, \alpha_a, 0, \alpha_{a-1}, 0, \cdots, \underbrace{\alpha_0}_{t\ th}, 0, \cdots, 0)\} + (0, \cdots, 0, \underbrace{1}_{t\ th}, 0, 1, 0, \cdots, 0) \\
&\quad + (\nu_b, 0, \nu_{b-1}, 0, \cdots, \underbrace{\nu_0}_{(t-2)\ nd}, 0, \cdots, 0) \\
&\quad - (0, \cdots, 0, \underbrace{1}_{t\ th}, 0, \cdots, 0) - (\nu_c, 0, \cdots, \underbrace{\nu_0}_{(t-2)\ nd}, 0, \cdots, 0),
\end{aligned}$$

where $a = [\frac{t-1}{2}]$, $b = [\frac{t}{2}] - 1 = [\frac{t-1}{2}] - 1 = c$, $\alpha_j = (m-1)^j$, and $\nu_j = m(m-1)^{j-1}$. So

$$((P_{t+1}^m)_{22})_{1.} = (\nu_d, 0, \nu_{d-1}, 0, \cdots, \nu_1, 0, 1, 0, \cdots, 0),$$

where $d = [\frac{t+1}{2}]$, which is same as the row obtained by using (1.3.8) and (1.3.9) .

Case (ii): For $i \neq 1,\ n$, we have

$$\begin{aligned}
&((P_{t+1}^m)_{22})_{i.} \\
&= (B_{21}(P_t^m)_{12})_{i.} + (B_{22}(P_t^m)_{22})_{i.} - ((P_{t-1}^m)_{22})_{i.} \\
&= ((P_t^m)_{22})_{i-1.} + ((P_t^m)_{22})_{i+1.} - ((P_{t-1}^m)_{22})_{i.} \\
&= (P_t(A_n) + C_t^m)_{i-1.} + (P_t(A_n) + C_t^m)_{i+1.} - (P_{t-1}(A_n) + C_{t-1}^m)_{i.} \\
&= (P_{t+1}(A_n))_{i.} + (C_t^m)_{i-1.} + (C_t^m)_{i+1.} - (C_{t-1}^m)_{i.}.
\end{aligned}$$

Since $i \geq 1$, $i + j \geq 2$. Now let $i = 2$, so we get

$$((P_{t+1}^m)_{22})_{2.} = (P_{t+1}(A_n))_{2.} + (C_t^m)_{1.} + (C_t^m)_{3.} - (C_{t-1}^m)_{2.}.$$

But

$$\begin{aligned}
&(C_t^m)_{1.} + (C_t^m)_{3.} - (C_{t-1}^m)_{2.}. \\
&= (0, \nu_a, 0, \cdots, \underbrace{\nu_0}_{(t-1)\ st}, 0, \cdots, 0) + (0, \nu_b, 0, \cdots, 0, \underbrace{\nu_0}_{(t-3)\ rd}, 0, \cdots, 0) \\
&\quad - (0, \nu_c, 0, \cdots, 0, \underbrace{\nu_0}_{(t-3)\ rd}, 0, \cdots, 0),
\end{aligned}$$

where $a = [\frac{t}{2}] - 1$, $b = [\frac{t-2}{2}] - 1 = [\frac{t-3}{2}] - 1 = c$, $\nu_j = m(m-1)^{j-1}$.

Therefore,

$$(C_t^m)_{1.} + (C_t^m)_{3.} - (C_{t-1}^m)_{2.} = (C_t^m)_{1.} = (C_{t+1}^m)_{2.},$$

as t is odd and hence $[\frac{t}{2}] = [\frac{t+1}{2}]$. In a similar way, we can handle $i = 3, 4, \cdots, n-1$.

Case (iii): For $i = n$, consider

$$\begin{aligned}
& ((P_{t+1}^m)_{22})_{n.} \\
= \quad & ((P_t^m)_{22})_{n-1.} - ((P_{t-1})_{22})_{n.} \\
= \quad & (P_t(A_n))_{n-1.} + (C_t^m)_{n-1.} - (P_{t-1}(A_n))_{n.} - (C_{t-1}^m)_{n.} \\
= \quad & (P_{t+1}(A_n))_{n.} + (C_t^m)_{n-1.} - (C_{t-1}^m)_{n.}.
\end{aligned}$$

For $t \leq n-1$, we have the condition as $n + j \leq n$. This holds true only for $j = 0$. Hence $(C_t^m)_{n-1.} = (0, \cdots, 0) = (C_{t-1}^m)_{n.}$. Also, $(C_{t+1}^m)_{n.} = (0, \cdots, 0)$, using (1.3.9) . Hence

$$((P_{t+1}^m)_{22})_{n.} = (P_{t+1}(A_n))_{n.} + (C_{t+1}^m)_{n.}.$$

Now combining the three cases, we see that the theorem is true for t odd. Hence the result. ■

Corollary 1.3.7 : *For $m = 2$, we have*

(1)

$$(P_k^2)_{11} = \begin{cases} 0, & \textit{if } k \textit{ is odd,} \\ D_\ell & \textit{if } k \textit{ is even with } k = 2\ell, \end{cases}$$

where

$$(D_\ell)_{ij} = \begin{cases} \frac{1}{2}\{1 + (-1)^\ell\}, & \textit{if } i = j, \\ (D_\ell)_{11} + (-1)^{\ell+1} & \textit{if } i \neq j. \end{cases}$$

(2)

$$((P_k^2)_{12})_{ij} = \begin{cases} 1 & \textit{if } j = k - 2r \textit{ with } r = 0, 1, 2, \cdots, [\frac{(k-1)}{2}], \\ 0 & \textit{otherwise,} \end{cases}$$

and (3) $(P_k^2)_{22} = P_k(A_n) + C_k^2$; *where*

$$(C_k^2)_{ij} = \begin{cases} 2 & \text{if } i+j = k+2-2r \text{ with } r = 1,2,\cdots,[\frac{k}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

Proof: In Theorem 1.3.6, put $m = 2$. ∎

Consider $q_{n+2}(\lambda) = \det(\lambda I - D_{n+2})$. Expanding the determinant along the first row, we get

$$\begin{aligned} q_{n+2}(\lambda) &= \lambda\{P_{n+1}(\lambda) - P_{n-1}(\lambda)\} \\ &= \lambda P_{n+1}(\lambda) - P_n(\lambda) + P_n(\lambda) - \lambda P_{n-1}(\lambda) \\ &= P_{n+2}(\lambda) - P_{n-2}(\lambda). \end{aligned}$$

By Cayley- Hamilton Theorem, $q_{n+2}(D_{n+2}) = 0$. Hence

$$\left. \begin{aligned} &P_{n+2}(D_{n+2}) = P_{n-2}(D_{n+2}), \\ &D_{n+2}\{P_{n+1}(D_{n+2}) - P_{n-1}(D_{n+2})\} = 0. \end{aligned} \right\} \tag{1.3.11}$$

Lemma 1.3.8 : *For n even, $P_{n+1}(D_{n+2}) = P_{n-1}(D_{n+2})$.*

Proof: Let n be an even positive integer, so that $n+1$ and $n-1$ are odd. Therefore, Corollary 1.3.7 gives us $(P_{n+1}^2)_{11} = (P_{n-1}^2)_{11} = 0$. Let us partition the matrix $P_{n+1}(D_{n+2}) - P_{n-1}(D_{n+2})$ into four blocks according to the partition of the matrix D_{n+2}, and let them be P_{11}, P_{12}, P_{21}, and P_{22}. But $P_{11} = 0 = (P_{n+1}^2)_{11} - (P_{n-1}^2)_{11}$. Therefore, using (1.3.11), we get $B_{22}P_{21} = 0$, with $B_{22} = A_n$.
But the matrix A_n is nonsingular for n even, and thus $P_{21} = 0 = P_{12}^t$. Again $B_{22}P_{22} = 0$ with $B_{22} = A_n$, and hence $P_{22} = 0$. Therefore, for n even,

$$P_{n+1}(D_{n+2}) = P_{n-1}(D_{n+2}). \qquad ∎$$

Corollary 1.3.9 : *For n even, $P_{n+k}(D_{n+2}) = P_{n-k}(D_{n+2})$ for $k = 1, 2, \cdots, n$.*

Proof: It was proved in Lemma 1.3.8 that for n even $P_{n+1}(D_{n+2}) = P_{n-1}(D_{n+2})$, that is, the result is true for $k = 1$. Now let the result be true for $1 \le k \le n-1$. Using the induction hypothesis and (1.2.2), we get

$$\begin{aligned} P_{n+k+1}(D_{n+2}) &= D_{n+2}P_{n+k}(D_{n+2}) - P_{n+k-1}(D_{n+2}) \\ &= D_{n+2}P_{n-k}(D_{n+2}) - P_{n-k+1}(D_{n+2}) \\ &= P_{n-k-1}(D_{n+2}). \end{aligned}$$

Hence the result. ■

We will just state the analogous result for n odd, which can be proved on similar lines.

Fact: *Let k be an odd nonnegative integer. Then,*
(i) if $n + k = 4m$ for $m \in \mathbb{Z}^+$, we have

$$P_{n+k}(D_{n+2}) - P_{n-k}(D_{n+2}) = \begin{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix},$$

and (ii) if $n + k = 4m + 2$ for $m \in \mathbb{Z}^+$, we have

$$P_{n+k}(D_{n+2}) - P_{n-k}(D_{n+2}) = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 1.3.10 : *For all $n \in \mathbb{Z}^+$, the matrix $P_{2n+1}(D_{n+2}) = 0$.*

Proof: We will consider the cases n odd and n even separately. Using Lemma 1.3.4, note the following:

$$P_{2n+1}(\lambda)$$

$$
\begin{aligned}
&= P_{n+2}(\lambda)P_{n-1}(\lambda) - P_{n+1}(\lambda)P_{n-2}(\lambda) \\
&= P_{n+2}(\lambda)P_{n-1}(\lambda) - P_{n-1}(\lambda)P_{n-2}(\lambda) + P_{n-1}(\lambda)P_{n-2}(\lambda) - P_{n+1}(\lambda)P_{n-2}(\lambda) \\
&= P_{n-1}(\lambda)\{P_{n+2}(\lambda) - P_{n-2}(\lambda)\} + P_{n-2}(\lambda)\{P_{n-1}(\lambda) - P_{n+1}(\lambda)\}.
\end{aligned}
$$

Case 1: Let n be even. Then using the above, we have

$$
\begin{aligned}
P_{2n+1}(D_{n+2}) &= P_{n-1}(D_{n+2})\{P_{n+2}(D_{n+2}) - P_{n-2}(D_{n+2})\} \\
&\quad + P_{n-2}(D_{n+2})\{P_{n-1}(D_{n+2}) - P_{n+1}(D_{n+2})\} \\
&= P_{n-1}(D_{n+2}) \times 0 + P_{n-2}(D_{n+2}) \times 0 \\
&= 0, \qquad \text{using Lemma 1.3.8}
\end{aligned}
$$

Case 2: Let n be odd. Since n is odd $n-2$ is odd. Therefore $P_{n-2}(\lambda) = \lambda R(n, \lambda)$, (say). Thus

$$
\begin{aligned}
P_{2n+1}(D_{n+2}) &= P_{n-1}(D_{n+2})\{P_{n+2}(D_{n+2}) - P_{n-2}(D_{n+2})\} \\
&\quad + R(n, D_{n+2})\, D_{n+2}\{P_{n-1}(D_{n+2}) - P_{n+1}(D_{n+2})\}.
\end{aligned}
$$

Now using (1.3.11), we get $P_{2n+1}(D_{n+2}) = 0$. ∎

The following result is analogous to Theorem 1.3.5:

$$
P_k(D_{n+2}) = \begin{cases} P_i(D_{n+2}), & \text{if } t \text{ is even;} \\ -P_{2n-i}(D_{n+2}) & \text{if } t \text{ is odd;} \end{cases} \tag{1.3.12}
$$

where $k = t(2n+2) + i$ with $i = 0, 1, 2, \cdots, 2n+1$, and $P_{-1}(D_{n+2}) = 0$. To prove this, use induction, observing that $P_{2n+1}(D_{n+2}) = 0 = P_{-1}(D_{n+2})$ implies that $P_s(D_{n+2}) = 0$ for $s = m(2n+2) + 2n + 1$ whenever $m < t$. Now proceed along the lines of the proof of Theorem 1.3.5 to get (1.3.12) .

Now, let us describe $P_k(B_n)$, for any integer $k \in \mathbb{Z}^+$. We will write A in place of the adjacency matrix $A(B_{n+1})$. Let us partition the matrix A in four blocks as:

$$A = \begin{array}{c} \\ 1 \\ n \end{array} \begin{array}{c} \begin{array}{cc} 1 & n \end{array} \\ \begin{pmatrix} 0 & b^t \\ b & A_n \end{pmatrix} \end{array}$$

where $b^t = \sqrt{2}\, e_1^t$.

Lemma 1.3.11 : *For $k \leq n$, if we partition the matrix $P_k(A)$ according to the partition of the matrix A, then:*

(1) *we have*

$$(P_k(A))_{11} = \begin{cases} 1, & \text{if } k \text{ is even} \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.13}$$

(2)

$$((P_k(A))_{12})_{1j} = \begin{cases} \sqrt{2}, & \text{if } j = k - 2r \text{ with } r = 0, 1, 2, \cdots, [\frac{k}{2}] \\ 0, & \text{otherwise,} \end{cases} \tag{1.3.14}$$

and (3) $(P_k(A))_{22} = P_k(A_n) + B_k$; *where*

$$(B_k)_{ij} = \begin{cases} 2, & \text{if } i + j = k - 2r \text{ with } r = 0, 1, 2, \cdots, [\frac{k}{2}] - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.15}$$

Proof: Let us use induction on k. For $k = 1$, $P_1(A) = A$. Now, let the lemma be true for $1 \leq k \leq t \leq n - 1$. Consider

$$P_{t+1}(A) = AP_t(A) - P_{t-1}(A). \tag{1.3.16}$$

We will prove the result for t odd and the proof for t even can be given in a similar way.

1. Observe that t odd and (1.3.16) gives us

$$(P_{t+1}(A))_{11} \quad = \quad b^t(P_t(A))_{21} - (P_{t-1}(A))_{11}$$

$$\begin{aligned}
&= \sqrt{2}\, e_1^t (P_t(A))_{21} - 1 \qquad \text{using (1.3.13) and (1.3.14)}\\
&= \sqrt{2}\, ((P_t(A))_{21})_{11} - 1\\
&= 2 - 1 = 1 \qquad \text{using (1.3.14).}
\end{aligned}$$

that is, (1) is true with $k = t+1$ as t is odd.

2. Using (1.3.16) , we have

$$\begin{aligned}
& ((P_{t+1}(A))_{12})_{1.}\\
= \;& (b^t(P_t(A))_{22})_{1.} - ((P_{t-1}(A))_{12})_{1.}\\
= \;& ((P_t(A))_{22})_{1.} - ((P_{t-1}(A))_{12})_{1.}\\
= \;& (P_t(A_n))_{1.} + (B_t)_{1.} - ((P_{t-1}(A))_{12})_{1.}\\
= \;& \sqrt{2}\{(0,\cdots,0,\underbrace{1}_{(t+1)\,st},0,\cdots,0) + (0,2,0,\cdots,\underbrace{2}_{(t-1)\,st},0,\cdots,0)\}\\
& -(0,\sqrt{2},\cdots,0,\underbrace{\sqrt{2}}_{(t-1)\,st},0,\cdots,0)\\
= \;& (0,\sqrt{2},\cdots,0,\underbrace{\sqrt{2}}_{(t+1)\,st},0,\cdots,0)
\end{aligned}$$

which is same as the row obtained by using (1.3.14) .

3. Here, we have

$$((P_{t+1}(A))_{22})_{i.} = (b(P_t(A))_{12})_{i.} + (B(P_t(A))_{22})_{i.} - ((P_{t-1}(A))_{22})_{i.}$$

and we need to consider three subcases.

case (i): For $i = 1$, we get

$$\begin{aligned}
& ((P_{t+1}(A))_{22})_{1.}\\
= \;& \sqrt{2}\, ((P_t(A))_{12})_{1.} + ((P_t(A))_{22})_{2.} - ((P_{t-1}(A))_{22})_{1.}\\
= \;& \sqrt{2}(\sqrt{2},0,\cdots,\underbrace{\sqrt{2}}_{t\,th},0,\cdots,0) + (0,\cdots,0,1,0,\underbrace{1}_{(t+2)\,nd},0,\cdots,0)\\
& +(2,0,\cdots,\underbrace{2}_{(t-2)\,nd},0,\cdots,0) - (0,\cdots,0,\underbrace{1}_{t\,th},0,\cdots,0)
\end{aligned}$$

$$\begin{aligned}
& -(2,0,\cdots,\underbrace{2}_{(t-2)\ nd},0,\cdots,0) \\
= \; & (2,0,2,0,\cdots,2,0,\underbrace{1}_{(t+2)\ nd},0,\cdots,0)
\end{aligned}$$

which is again seen to be the same as the row obtained by using (3) of Lemma 1.3.11 .

case (ii): For $i \neq 1, n$, we have

$$\begin{aligned}
& ((P_{t+1}(A))_{22})_{i.} \\
= \; & ((P_t(A))_{22})_{i-1,.} + ((P_t(A))_{22})_{i+1,.} - ((P_{t-1}(A))_{22})_{i.} \\
= \; & (P_{t+1}(A_n))_{i.} + (B_t)_{i-1,.} + (B_t)_{i+1,.} - (B_{t-1})_{i.}
\end{aligned}$$

Since $i \geq 1,\ i + j \geq 2$. Now let $i = 2$, so that we have

$$\begin{aligned}
& (B_t)_{1,.} + (B_t)_{3,.} - (B_{t-1})_{2,.} \\
= \; & (0,2,0,\cdots,\underbrace{2}_{(t-1)\ st},0,\cdots,0) + (0,2,0,\cdots,\underbrace{2}_{(t-3)\ rd},0,\cdots,0) \\
& -(0,2,0,\cdots,\underbrace{2}_{(t-3)\ rd},0,\cdots,0) \\
= \; & (0,2,0,\cdots,\underbrace{2}_{(t-1)\ st},0,\cdots,0) \\
= \; & (B_{t+1})_{2,.}
\end{aligned}$$

case (iii): For $i = n$, we have

$$\begin{aligned}
& ((P_{t+1}(A))_{22})_{n.} \\
= \; & ((P_t(A))_{22})_{n-1.} - ((P_{t-1}(A))_{22})_{n.} \\
= \; & (P_t(A_n))_{n-1.} + (B_t)_{n-1.} - (P_{t-1}(A_n))_{n.} - (B_{t-1})_{n.} \\
= \; & (P_{t+1}(A_n))_{n.} + (B_t)_{n-1.} - (B_{t-1})_{n.}\ .
\end{aligned}$$

For $t \leq n - 1$, the condition becomes $n + j \leq n$ which holds true only for $j = 0$. Hence $(B_t)_{n-1.} = (0,\cdots,0) = (B_{t-1})_{n.}$. Also $(B_{t+1})_{n.} = (0,\cdots,0)$ using (1.3.15). Thus, we have

$$((P_{t+1}(A))_{22})_{n.} = (P_{t+1}(A_n))_{n.} + (B_{t+1})_{n.}\ .$$

Now combining all the above cases, we see that the result holds true for t odd. Hence the result. ■

Consider $r_{n+1}(\lambda) = \det(\lambda I - A)$. Expanding the determinant along the first row, we get

$$r_{n+1}(\lambda) = \lambda P_n(\lambda) - 2P_{n-1}(\lambda). \tag{1.3.17}$$

But $P_{n+1}(\lambda) = \lambda P_n(\lambda) - P_{n-1}(\lambda)$. Substituting for $\lambda P_n(\lambda)$ in (1.3.17), we get

$$r_{n+1}(\lambda) = P_{n+1}(\lambda) - P_{n-1}(\lambda).$$

By Cayley- Hamilton Theorem, $r_{n+1}(A) = 0$, and hence

$$P_{n+1}(A) = P_{n-1}(A).$$

Therefore, it can be easily shown by induction that

$$P_{n+k}(A) = P_{n-k}(A) \quad \text{for } k = 1, 2, \cdots, n,$$

and which gives us

$$P_{2n+1}(A) = 0.$$

The following result is analogous to Theorem 1.3.5 :

$$P_k(B_{n+1}) = \begin{cases} P_i(B_{n+1}), & \text{if } t \text{ is even,} \\ -P_{2n-i}(B_{n+1}), & \text{if } t \text{ is odd,} \end{cases} \tag{1.3.18}$$

where $k = t(2n+2) + i$ with $i = 0, 1, 2, \cdots, 2n+1$, and $P_{-1}(B_{n+1}) = 0$. To prove this, use induction, observing that $P_{2n+1}(B_{n+1}) = 0 = P_{-1}(B_{n+1})$ implies that $P_s(B_{n+1}) = 0$ for $s = m(2n+2) + 2n + 1$ whenever $m < t$. Now proceed along the lines of the proof of Theorem 1.3.5 to get (1.3.18) .

1.4 Path-positivity of graphs in C_2

As mentioned on page 8, the proof of the path-positivity of $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ is contained in de la Harpe and Wenzl (1987). There we have also made a remark that we are not able to use the same technique that was used in de la Harpe and Wenzl (1987), for proving the path-positivity of additional graphs $\tilde{B}_n$, $\tilde{C}_n$, Z_4 and Z_5. Hence, we use another technique, already mentioned. In the process, we give a complete description of all path-matrices of the graphs $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$. See Lemmas 1.4.3, 1.4.4, 1.4.5, 1.4.6, and Appendix 3. For showing the path-positivity of Z_4 and Z_5 we had to use a different ad hoc method.

The adjacency matrix $A(G)$ of any graph G is a real symmetric matrix. So, there exists an orthogonal matrix Q such that $A(G) = QDQ^t$; where the matrix D is a diagonal matrix whose entries are the eigenvalues of $A(G)$. Since Q is orthogonal,

$$P_k(A(G)) = P_k(QDQ^t) = QP_k(D)Q^t. \tag{1.4.1}$$

Hence we look at the behaviour of $P_k(\lambda)$ for $\lambda \in \mathbb{R}$.

The spectral decomposition of

$$\tilde{A}_n,\ \tilde{D}_n,\ \tilde{E}_6,\ \tilde{E}_7,\ \text{and } \tilde{E}_8$$

have been taken from David and John (1981) with slight modification while those of $\tilde{B}_n$ and $\tilde{C}_n$ were not found in the literature and are given here for the first time.

Lemma 1.4.1: *For any $\lambda \in \mathbb{R}$, the following difference equation*

$$P_k(\lambda) = \lambda P_{k-1}(\lambda) - P_{k-2}(\lambda) \qquad \textit{for } k = 2, 3, \ldots \tag{1.4.2}$$

with

$$P_0(\lambda) = 1 \qquad \textit{and} \qquad P_1(\lambda) = \lambda \tag{1.4.3}$$

has the solution

$$P_k(\lambda) = \begin{cases} \frac{\sin(k+1)x}{\sin x} & \text{if } \lambda = 2\cos x, \ 0 < x \le \frac{\pi}{2}; \\ k+1 & \text{if } \lambda = 2; \\ \frac{\sinh(k+1)x}{\sinh x} & \text{if } \lambda = 2\cosh x, \ 0 < x \ . \end{cases} \tag{1.4.4}$$

Proof: An easy computation gives the result. ■

Remark: 1. It can be checked that in general, we also have

$$P_k(\lambda) = \frac{1}{\sqrt{\lambda^2-4}}[(\frac{\lambda+\sqrt{\lambda^2-4}}{2})^{k+1} - (\frac{\lambda-\sqrt{\lambda^2-4}}{2})^{k+1}], \tag{1.4.5}$$

for $\lambda \neq 2$.

2. It is easily seen that $P_k(-\lambda) = (-1)^k P_k(\lambda)$.

Corollary 1.4.2: *We have the following:*

(i) $P_k(0) = 0$ *if* k *is odd and* $(-1)^{k/2}$ *if* k *is even.*

(ii) For $\lambda = \theta^\ell + \theta^{-\ell}, \quad \ell = 0, 1, 2, ..., r-1, \quad$ *where* $\theta = \exp(\frac{2\pi i}{r})$

$$P_k(\lambda) = \sum_{u=0}^{k} \theta^{\ell(k-2u)} \tag{1.4.6}$$

In particular, for positive integers m,

(a) $P_{6m}(1) = P_{6m+1}(1) = 1 = -P_{6m+3}(1) = -P_{6m+4}(1) \quad ,$

$P_{6m+2}(1) = P_{6m+5}(1) = 0.$

(b) $P_{4m}(\sqrt{2}) = (-1)^m = P_{4m+2}(\sqrt{2}), \quad P_{4m+3}(\sqrt{2}) = 0, \ P_{4m+1}(\sqrt{2}) = (-1)^m\sqrt{2}$

(c) For $\alpha_1 = \frac{-1+\sqrt{5}}{2}$ *and* $\alpha_2 = \frac{-1-\sqrt{5}}{2}$ *, we have*

$P_{5m}(\alpha_i) = 1 = -P_{5m+3}(\alpha_i), \quad P_{5m+4}(\alpha_i) = 0 \quad ,$

$P_{5m+1}(\alpha_i) = \alpha_i = -P_{5m+2}(\alpha_i).$

Proof: Assertion (i) is easy to prove.

(ii) Let $\lambda = \theta^\ell + \theta^{-\ell}$. Then using Lemma 1.4.1, we get

$$P_k(\lambda) = \frac{\sin(\frac{\ell(k+1)}{r} \cdot 2\pi)}{\sin(\frac{\ell}{r} \cdot 2\pi)}$$

which gives us

$$P_k(\lambda) = \frac{\theta^{\ell(k+1)} - \theta^{-\ell(k+1)}}{\theta^\ell + \theta^{-\ell}},$$

which in turn gives us the required result.

From now on, we use expression (1.4.5) to get the exact forms of $P_k(\lambda)$ for particular values of λ.

(a) For $\lambda = 1$, $\sqrt{\lambda^2 - 4} = \sqrt{3}i$ and

$$(\frac{1 \pm \sqrt{3}i}{2}) = \cos(\pi/3) \pm i\sin(\pi/3).$$

Therefore,

$$\begin{aligned}(\frac{1 \pm \sqrt{3}i}{2})^{6m} &= ((\frac{1 \pm \sqrt{3}i}{2})^3)^{2m} \\ &= ((\cos(\pi/3) \pm i\sin(\pi/3))^3)^{2m} = (-1)^{2m} = 1\end{aligned}$$

for any positive integer m. The above equation shows that we need to consider k of the form $6m + \ell$ *for* $\ell = 0, 1, 2, 3, 4, 5$, that is,

$$k \equiv \ell \pmod 6 \quad \textit{for} \ \ell = 0, 1, 2, \ldots, 5.$$

Now substituting for λ gives us the required result.

(b) If $\lambda = \sqrt{2}$, $\sqrt{\lambda^2 - 4} = \sqrt{2}i$

and

$$\frac{\sqrt{2} \pm \sqrt{2}i}{2} = \cos(\pi/4) \pm i\sin(\pi/4).$$

Therefore

$$\begin{aligned}(\frac{\sqrt{2} \pm \sqrt{2}i}{2})^{4m} &= (\cos(\pi) \pm i\sin(\pi))^m \\ &= (-1)^m\end{aligned}$$

Figure 1.2: $\tilde{A}_n$

for any positive integer m. The above equation shows that in this case we need to have

$$k \equiv \ell \pmod 4 \quad \textit{for} \quad \ell = 0, 1, 2, 3.$$

Substituting for λ gives us the desired result.

(c) If $\lambda = \alpha_i$ *for* $i = 1, 2,$ it can be seen that

$$\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} = \cos(2\pi/5) \pm i \sin(2\pi/5).$$

Therefore

$$\begin{aligned} \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}\right)^{5m} &= (\cos(2\pi/5) \pm i \sin(2\pi/5))^{5m} \\ &= (1)^m = 1 \end{aligned}$$

for any positive integer m. The above equation shows that here we need

$$k \equiv \ell \pmod 5 \quad \textit{for} \quad \ell = 0, 1, 2, 3, 4.$$

Again substituting for λ, gives the desired result. ∎

From now on we will be using the following notations:

(i) 1_X denotes the indicator function of the set X, and X^c denotes the complement of the set X.

(ii) We will write G in place of $A(G)$, for the adjacency matrix of the graph G.

Lemma 1.4.3: *The cycle $\tilde{A}_n$ for $n \geq 2$ is path-positive.*

Proof: The graph of the cycle $\tilde{A}_n$ on n vertices can be seen in Figure 1.2. The spectral decomposition of $\tilde{A}_n$ is $\tilde{A}_n = QDQ^t$; where

$$D = \text{diag}\,(w^j + w^{-j}) \ \text{ for } \ j = 0, 1, \cdots, n$$

and

$$Q = (q_{jk})_{(n+1)\times(n+1)} \ \text{ for } \ j,k = 0,1,...,n$$

with $q_{jk} = \frac{w_{jk}}{\sqrt{n+1}}$ and $w = \exp(\frac{2\pi i}{n+1})$.

For $1 \leq i \leq j \leq n+1$, consider

$$\begin{aligned}
(P_k(\tilde{A}_n))_{ij} &= \frac{1}{n+1}\sum_{s=0}^{n} w^{(i-j)s} P_k(w^s + w^{-s}) \\
&= \frac{1}{n+1}\sum_{s=0}^{n} w^{(i-j)s} \sum_{u=0}^{k} w^{(k-2u)s} \qquad \text{using (1.4.6)} \\
&= \frac{1}{n+1}\sum_{u=0}^{k}\sum_{s=0}^{n} w^{(i-j+k-2u)s}.
\end{aligned}$$

Let $X = X(i,j,k) = \{u : i-j+k-2u \equiv 0 \pmod{n+1}\}$. Then

$$\begin{aligned}
&(P_k(\tilde{A}_n))_{ij} \\
=\ & \frac{n+1}{n+1}\sum_{u=0}^{k} 1_X(u) + \frac{1}{n+1}\sum_{u=0}^{k} \frac{1-(w^{i-j+k-2u})^{n+1}}{1-w^{i-j+k-2u}} 1_{X^c}(u) \\
=\ & \sum_{u=0}^{k} 1_X(u) + \frac{1}{n+1}\sum_{u=0}^{k} \frac{1-(w^{n+1})^{i-j+k-2u}}{1-w^{i-j+k-2u}} 1_{X^c}(u) \\
=\ & \sum_{u=0}^{k} 1_X(u) \\
\geq\ & 0.
\end{aligned}$$

Hence the result. ■

Lemma 1.4.4: *The graph $\tilde{D}_n$ for $n \geq 4$ is path-positive.*

Proof: The graph of $\tilde{D}_n$ can be seen in Figure 1.3. The spectral decomposition of $\tilde{D}_n$ is

$$\tilde{D}_n = QDQ^t \ ;$$

where

$$D = \text{ diag}(\theta^\ell + \theta^{-\ell} \text{ for } \ell = 1,2,\cdots,n-3,\ 2,\ -2,0,0)$$

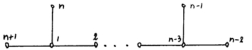


Figure 1.3: $\tilde{D}_n$

with $\theta = \exp(\frac{2\pi i}{2n-4})$ and

$$Q = \frac{1}{\sqrt{4n-8}} \left(\begin{array}{cc} \begin{pmatrix} \sqrt{2}(\theta^{jk} + \theta^{-jk}) \\ j,k = 1,2,\ldots,n-3. \end{pmatrix} & \begin{pmatrix} 2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 2 & (-1)^{n-3} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} (-1)^k\sqrt{2} \\ (-1)^k\sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix} & \begin{pmatrix} 1 & (-1)^n & -\sqrt{n-2} & \sqrt{n-2} \\ 1 & (-1)^n & \sqrt{n-2} & -\sqrt{n-2} \\ 1 & 1 & -\sqrt{n-2} & -\sqrt{n-2} \\ 1 & 1 & \sqrt{n-2} & \sqrt{n-2} \end{pmatrix} \end{array} \right).$$

We have to show $P_k(\tilde{D}_n) \geq 0$ for all positive integers k. We shall only consider those i, j for which there exists a path of length k, for if there does not exist a path of length k from the vertex i to vertex j, then (1.2.3) gives us $(P_k(\tilde{D}_n))_{ij} = 0$.

The entries of the matrix Q clearly tell us that we shall have to consider different cases. We will be repeatedly using a similar analysis as was used in Lemma 1.4.3.

Case(i): It is clear from the graph of $\tilde{D}_n$ that we need k to be even for a path from i to j to exist for $n \leq i \leq j \leq n+1$. Consider

$$\begin{aligned} & (P_k(\tilde{D}_n))_{ij} \\ = \ & \frac{1}{2}(-1)^{k/2+j-i} + \frac{k+1}{4n-8} + (-1)^k \frac{k+1}{4n-8} + \frac{1}{2n-4} \sum_{\ell=1}^{n-3} P_k(\theta^\ell + \theta^{-\ell}) \\ = \ & \frac{1}{2}(-1)^{k/2+j-i} + \frac{k+1}{2n-4} + \frac{1}{2n-4} \sum_{\ell=1}^{n-3} \sum_{u=0}^{k} \theta^{(k-2u)\ell} \qquad \text{using (1.4.6)} \\ = \ & \frac{1}{2}(-1)^{k/2+j-i} + \frac{k+1}{2n-4} + \frac{n-3}{2n-4} \sum_{u=0}^{k} 1\chi(u) \end{aligned}$$

$$+\frac{1}{2n-4}\sum_{u=0}^{k}\frac{1-\theta^{(k-2u)(n-3)}}{1-\theta^{k-2u}}\theta^{k-2u}1_{X^c}(u);$$

where

$$X = \{\, u : k-2u \equiv 0 \pmod{2n-4}\}.$$

Clearly $X \neq \emptyset$ as $k/2 \in X$. Since $k-2u$ is even, we have

$$\begin{aligned}
&(P_k(\tilde{D}_n))_{ij} \\
= \; & (-1)^{k/2+j-i}\frac{1}{2} + \frac{n-3}{2n-4}\sum_{u=0}^{k} 1_X(u) + \frac{1}{2n-4}\sum_{u=0}^{k}(1_X(u) + 1_{X^c}(u)) \\
& +\frac{1}{2n-4}\sum_{u=0}^{k}\frac{\theta^{k-2u} - (\theta^{n-2})^{2(k/2-u)}}{1-\theta^{k-2u}}1_{X^c}(u) \\
= \; & (-1)^{k/2+j-i}\frac{1}{2} + \frac{1}{2}\sum_{u=0}^{k} 1_X(u) + \frac{1}{2n-4}\sum_{u=0}^{k}\{1 + \frac{\theta^{k-2u} - 1^{k/2-u}}{1-\theta^{k-2u}}\}1_{X^c}(u) \\
= \; & (-1)^{k/2+j-i}\frac{1}{2} + \frac{1}{2}\sum_{u=0}^{k} 1_X(u) \\
\geq \; & 0.
\end{aligned}$$

A similar reasoning goes to show $(P_k(\tilde{D}_n))_{ij} \geq 0$, for $n-2 \leq i \leq j \leq n-1$.

Case(ii): It is again clear from the graph of $\tilde{D}_n$ that we need $k \pm j$ to be even for the path of length k from i to j to exist, for $n \leq i \leq n+1$ and $1 \leq j \leq n-3$. Consider

$$\begin{aligned}
(P_k(\tilde{D}_n))_{ij} &= \frac{2(k+1)}{4n-8} + \frac{2(-1)^{j-2+k}(k+1)}{4n-8} \\
&\quad +\frac{1}{2n-4}\sum_{\ell=1}^{n-3}(\theta^{(j-2)\ell} + \theta^{-(j-2)\ell})P_k(\theta^{\ell} + \theta^{-\ell}) \\
&= \frac{2(k+1)}{2n-4} + \frac{1}{2n-4}\sum_{\ell=1}^{n-3}\sum_{u=0}^{k}(\theta^{(j-2)\ell} + \theta^{-(j-2)\ell})(\theta^{k-2u})^{\ell}.
\end{aligned}$$

Let

$$X_1 = \{\, u : k-2u+j-2 \equiv 0 \pmod{2n-4}\}$$

and

$$X_2 = \{\, u : k-2u-j+2 \equiv 0 \pmod{2n-4}\}.$$

Since $k - 2u \pm j \pm 2$ is even, we get

$$\begin{aligned}
(P_k(\tilde{D}_n))_{ij} &= \frac{2(k+1)}{2n-4} + \frac{n-3}{2n-4}\sum_{u=0}^{k}(1_{X_1}(u) + 1_{X_2}(u)) \\
&\quad + \frac{1}{2n-4}\sum_{u=0}^{k}\frac{\theta^{k-2u+j-2} - (\theta^{n-2})^{2(\frac{k+j}{2}-u-1)}}{1-\theta^{k-2u+j-2}} 1_{X_1^c}(u) \\
&\quad + \frac{1}{2n-4}\sum_{u=0}^{k}\frac{\theta^{k-2u-j+2} - (\theta^{n-2})^{2(\frac{k-j}{2}-u-1)}}{1-\theta^{k-2u-j+2}} 1_{X_2^c}(u) \\
&= \frac{1}{2}\sum_{u=0}^{k}(1_{X_1}(u) + 1_{X_2}(u)) \\
&= \frac{1}{2}\sum_{u=0}^{k} 1_X(u) \\
&\geq 0;
\end{aligned}$$

where $X = X_1 \cup X_2$.

Case(iii): Let $n \leq i \leq n+1$ and $n-2 \leq j \leq n-1$. In this case we must have $k+n$ even for the path of length k to exist from i to j. Consider

$$\begin{aligned}
(P_k(\tilde{D}_n))_{ij} &= \frac{k+1}{4n-8} + \frac{(k+1)(-1)^{k+n}}{4n-8} \\
&\quad + \frac{1}{2n-4}\sum_{\ell=1}^{n-3}(-1)^{\ell}P_k(\theta^{\ell} + \theta^{-\ell}) \\
&= \frac{k+1}{2n-4} + \frac{1}{2n-4}\sum_{\ell=1}^{n-3}\sum_{u=0}^{k}(-\theta^{k-2u})^{\ell}.
\end{aligned}$$

Let

$$X = \{\, u : k - 2u \equiv 0 \pmod{n-2}\}$$

and

$$Y = \{\, u : k - 2u \equiv 0 \pmod{2n-4}\}.$$

Clearly $Y \subseteq X$, and when k and n are both odd $Y = \emptyset$.

Subcase (a): k and n are both odd. Then

$$(P_k(\tilde{D}_n))_{ij}$$

$$\begin{aligned}
&= \frac{k+1}{2n-4} + \frac{n-3}{2n-4}\sum_{u=0}^{k} 1_X(u) + \frac{1}{2n-4}\sum_{u=0}^{k} \frac{-\theta^{k-2u} - (-\theta^{k-2u})^{n-2}}{1+\theta^{k-2u}} 1_{X^c}(u) \\
&= \frac{1}{2}\sum_{u=0}^{k} 1_X(u) + \frac{1}{2n-4}\sum_{u=0}^{k} 1_{X^c}(u) + \frac{1}{2n-4}\sum_{u=0}^{k} \frac{-\theta^{k-2u} + (\theta^{n-2})^{k-2u}}{1+\theta^{k-2u}} 1_{X^c}(u) \\
&= \frac{1}{2}\sum_{u=0}^{k} 1_X(u) + \frac{1}{2n-4}\sum_{u=0}^{k} 1_{X^c}(u) + \frac{1}{2n-4}\sum_{u=0}^{k} \frac{-\theta^{k-2u} + (-1)^{k-2u}}{1+\theta^{k-2u}} 1_{X^c}(u) \\
&= \frac{1}{2}\sum_{u=0}^{k} 1_X(u) \qquad \text{as } k-2u \text{ is odd} \\
&\geq 0.
\end{aligned}$$

Subcase (b): k and n are both even. Then $n-3$ is odd and hence we have

$$\begin{aligned}
(P_k(\tilde{D}_n))_{ij} &= \frac{k+1}{2n-4} + \frac{n-3}{2n-4}\sum_{u=0}^{k} 1_{X\setminus Y}(u) \\
&\quad + \frac{1}{2n-4}\sum_{u=0}^{k} \frac{-\theta^{k-2u} - (-\theta^{k-2u})^{n-2}}{1+\theta^{k-2u}} 1_{(X\setminus Y)^c}(u) \\
&= \frac{1}{2}\sum_{u=0}^{k} 1_{X\setminus Y}(u) + \frac{1}{2n-4}\sum_{u=0}^{k} 1_{(X\setminus Y)^c}(u) \\
&\quad + \frac{1}{2n-4}\sum_{u=0}^{k} \frac{-\theta^{k-2u} - (\theta^{n-2})^{2(k/2-u)}}{1+\theta^{k-2u}} 1_{(X\setminus Y)^c}(u) \\
&= \frac{1}{2}\sum_{u=0}^{k} 1_{X\setminus Y}(u) + \frac{1}{2n-4}\sum_{u=0}^{k} 1_{(X\setminus Y)^c}(u) \\
&\quad + \frac{1}{2n-4}\sum_{u=0}^{k} \frac{-\theta^{k-2u} - 1}{1+\theta^{k-2u}} 1_{(X\setminus Y)^c}(u) \\
&= \frac{1}{2}\sum_{u=0}^{k} 1_{X\setminus Y}(u) \\
&\geq 0.
\end{aligned}$$

Case(iv): Let $1 \leq i \leq j \leq n-3$. Now again using the graph of $\tilde{D}_n$, we get $k \pm i \pm j$ to be even for the path of length k to exist from i to j. Consider

$$\begin{aligned}
& (P_k(\tilde{D}_n))_{ij} \\
= \; & \frac{4(k+1)}{4n-8} + \frac{4(k+1)(-1)^{k+i+j-4}}{4n-8} + \\
& \frac{1}{2n-4}\sum_{\ell=1}^{n-3} (\theta^{(i-2)\ell} + \theta^{-(i-2)\ell})(\theta^{(j-2)\ell} + \theta^{-(j-2)\ell}) P_k(\theta^{\ell} + \theta^{-\ell})
\end{aligned}$$

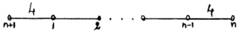


Figure 1.4: $\tilde{C_n}$

$$= \frac{1}{2n-4}\sum_{u=0}^{k}\sum_{\ell=1}^{n-3}(\theta^{(i+j-4)\ell}+\theta^{(i-j)\ell}+\theta^{(-i+j)\ell}+\theta^{(-i-j+4)\ell})(\theta^{k-2u})^{\ell}+\frac{4(k+1)}{2n-4}.$$

Here we have $X=\cup_{i=1}^{4}X_i$; where

$$X_1=\{\,u : k-2u+i+j-4\equiv 0 \pmod{2n-4}\},$$

$$X_2=\{\,u : k-2u+i-j\equiv 0 \pmod{2n-4}\},$$

$$X_3=\{\,u : k-2u-i+j\equiv 0 \pmod{2n-4}\},$$

and

$$X_4=\{\,u : k-2u-i-j+4\equiv 0 \pmod{2n-4}\}.$$

Proceeding as in case (ii), we get

$$\begin{aligned}(P_k(\tilde{D_n}))_{ij} &= \frac{1}{2}\sum_{u=0}^{k}1_X(u)\\ &\geq 0.\end{aligned}$$

Case(v): Let $1\leq i\leq n-3$ and $n-2\leq j\leq n-1$. A similar argument as in case(iii) gives us

$$(P_k(\tilde{D_n}))_{ij}\geq 0.$$

The above five cases give us the desired result. ■

Lemma 1.4.5: *The graph $\tilde{C_n}$ for $n\geq 2$ is path-positive.*

Proof: The graph of $\tilde{C_n}$ can be seen in Figure 1.4. The proof goes on the lines of the proof of Lemma 1.4.4; where

$$D = \operatorname{diag}(\theta^{\ell}+\theta^{-\ell} \text{ for } l=1,2,\cdots,n-1,\ 2,\ -2)$$

and

$$Q = \frac{1}{\sqrt{2n}} \begin{pmatrix} \begin{pmatrix} (\theta^{jk} + \theta^{-jk}) \\ j,k = 1,2,\ldots,n-1. \end{pmatrix} & \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \\ \vdots & \vdots \\ \sqrt{2} & (-1)^{n-1}\sqrt{2} \end{pmatrix} \\ \begin{pmatrix} (-1)^k\sqrt{2} \\ \sqrt{2} \end{pmatrix} & \begin{pmatrix} 1 & (-1)^n \\ 1 & 1 \end{pmatrix} \end{pmatrix},$$

with $\theta = \exp(\frac{2\pi i}{2n})$.

The following can be easily obtained:

(i) $(P_k(\tilde{C}_n))_{n+1,n+1} = \sum_{u=0}^{k} 1_X(u)$; where

$$X = \{\, u : k - 2u \equiv 0, \quad (\text{mod } 2n)\}.$$

The same is true for the entry (n,n) .

(ii) For $1 \leq j \leq n-1$, we have

$$(P_k(\tilde{C}_n))_{n+1,j} = \frac{1}{\sqrt{2}} \sum_{u=0}^{k} (1_{X_1}(u) + 1_{X_2}(u));$$

where

$$X_1 = \{\, u : k - 2u + j - 1 \equiv 0, \quad (\text{mod } 2n)\},$$

and

$$X_2 = \{\, u : k - 2u - j + 1 \equiv 0, \quad (\text{mod } 2n)\}.$$

(iii)

$$(P_k(\tilde{C}_n))_{n+1,n} = \begin{cases} \sum_{u=0}^{k} 1_X(u), & \text{if } n \text{ is odd;} \\ \sum_{u=0}^{k} 1_{X \setminus Y}(u), & \text{if } n \text{ is even.} \end{cases}$$

where

$$X = \{\, u : k - 2u \equiv 0, \quad (\text{mod } n)\},$$

and

$$Y = \{\, u : k - 2u \equiv 0, \quad (\text{mod } 2n)\}.$$

(iv) For $1 \leq i \leq j \leq n-1$, we have

$$(P_k(\tilde{C}_n))_{ij} = \frac{1}{2}\sum_{u=0}^{k} 1_X(u);$$

where

$$X = \cup_{i=1}^{4} X_i$$

with

$$X_1 = \{\, u : k - 2u + i + j - 2 \equiv 0 \pmod{2n}\},$$

$$X_2 = \{\, u : k - 2u + i - j \equiv 0 \pmod{2n}\},$$

$$X_3 = \{\, u : k - 2u - i + j \equiv 0 \pmod{2n}\},$$

and

$$X_4 = \{\, u : k - 2u - i - j + 2 \equiv 0 \pmod{2n}\}.$$

(v) For $1 \leq i \leq n-1$, we have

$$(P_k(\tilde{C}_n))_{i,n} = \begin{cases} \frac{1}{\sqrt{2}}\sum_{u=0}^{k}(1_{X_1}(u) + 1_{X_2}(u)), & \text{if } n \text{ is odd;} \\ \frac{1}{\sqrt{2}}\sum_{u=0}^{k}(1_{X_1\setminus Y_1}(u) + 1_{X_2\setminus Y_2}(u)), & \text{if } n \text{ is even.} \end{cases}$$

where

$$X_1 = \{\, u : k - 2u + i - 1 \equiv 0 \pmod{n}\},$$

$$Y_1 = \{\, u : k - 2u + i - 1 \equiv 0 \pmod{2n}\},$$

$$X_2 = \{\, u : k - 2u - i + 1 \equiv 0 \pmod{n}\},$$

and

$$Y_2 = \{\, u : k - 2u - i + 1 \equiv 0 \pmod{2n}\}.$$

Hence we get the desired result. ∎

Lemma 1.4.6: *The graph $\tilde{B}_n$ for $n \geq 3$ is path-positive.*

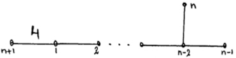


Figure 1.5: $\tilde{B}_n$

Proof: The graph of $\tilde{B}_n$ can be seen in Figure 1.5. Here we have separate spectral decomposition for n odd and n even.

Case(i): n even. The spectral decomposition of $\tilde{B}_n$ is $\tilde{B}_n = QDQ^t$;

where

$$D = \operatorname{diag}(\theta^{\ell} + \theta^{-\ell} \text{ for } l = 1, 2, \cdots, n-2,\ 2,\ -2,\ 0)$$

and

$$Q = \frac{1}{2\sqrt{n-1}} \begin{pmatrix} \begin{pmatrix} \sqrt{2}(\theta^{jk} + \theta^{-jk}) \\ j, k = 1, 2, \ldots, n-2. \end{pmatrix} & \begin{pmatrix} 2 & (-1)2 & 0 \\ 2 & (-1)^2 2 & 0 \\ \vdots & \vdots & \vdots \\ 2 & 2(-1)^{n-2} & 0 \end{pmatrix} \\ \begin{pmatrix} (-1)^k\sqrt{2} \\ (-1)^k\sqrt{2} \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & -1 & -\sqrt{2(n-1)} \\ 1 & -1 & \sqrt{2(n-1)} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \end{pmatrix},$$

with $\theta = \exp(\frac{2\pi i}{2n-2})$.

Subcase (i): $(P_k(\tilde{B}_n))_{n+1,n+1} = \sum_{u=0}^{k} 1_X(u)$; where

$$X = \{\, u : k - 2u \equiv 0 \pmod{2n-2}\}.$$

Subcase (ii): For $1 \leq j \leq n-2$, we have

$$(P_k(\tilde{B}_n))_{n+1,j} = \frac{1}{\sqrt{2}} \sum_{u=0}^{k} (1_{X_1}(u) + 1_{X_2}(u)) \,;$$

where

$$X_1 = \{\, u : k - 2u + j - 1 \equiv 0 \pmod{2n-2}\},$$

and

$$X_2 = \{\, u : k - 2u - j + 1 \equiv 0 \pmod{2n-2}\}.$$

Subcase (iii): For $n-1 \leq j \leq n$, we have

$$(P_k(\tilde{B_n}))_{n+1,j} = \frac{1}{\sqrt{2}} \sum_{u=0}^{k} 1_X(u) \; ;$$

where

$$X = \{\, u : k - 2u \equiv 0 \pmod{n-1}\}$$

as n is even.

Subcase (iv): For $1 \leq i \leq j \leq n-2$, we have

$$(P_k(\tilde{B_n}))_{ij} = \frac{1}{2} \sum_{u=0}^{k} 1_X(u) \; ;$$

where

$$X = \cup_{i=1}^{4} X_i$$

with

$$X_1 = \{\, u : k - 2u + i + j - 2 \equiv 0 \pmod{2n-2}\},$$

$$X_2 = \{\, u : k - 2u + i - j \equiv 0 \pmod{2n-2}\},$$

$$X_3 = \{\, u : k - 2u - i + j \equiv 0 \pmod{2n-2}\},$$

and

$$X_4 = \{\, u : k - 2u - i - j + 2 \equiv 0 \pmod{2n-2}\}.$$

Subcase (v): For $1 \leq i \leq n-2$ and $n-1 \leq j \leq n$, we have

$$(P_k(\tilde{B_n}))_{ij} = \frac{1}{2} \sum_{u=0}^{k} (1_{X_1}(u) + 1_{X_2}(u)) \; ;$$

where

$$X_1 = \{\, u : k - 2u + j - 1 \equiv 0 \pmod{n-1}\},$$

and

$$X_2 = \{\, u : k - 2u - j + 1 \equiv 0 \pmod{n-1}\}.$$

Subcase (vi): Let $n-1 \leq i,j \leq n$. Using the graph of $\tilde{B_n}$ we get k to be even for the path of length k to exist from i to j. For $i \neq j$, we have

$$(P_k(\tilde{B_n}))_{ij} = \frac{1}{2}\sum_{u=0}^{k} 1_X(u) - \frac{(-1)^{k/2}}{2} \ ;$$

where

$$X = \{\, u : k - 2u \equiv 0 \pmod{2n-2}\}$$

which is non-empty as $u = \frac{k}{2} \in X$. For $i = j$, we have

$$(P_k(\tilde{B_n}))_{ii} = \frac{1}{2}\sum_{u=0}^{k} 1_X(u) + \frac{(-1)^{k/2}}{2}$$

which is also non-empty as $u = \frac{k}{2} \in X$.

Case(ii): n odd. The spectral decomposition of $\tilde{B_n}$ is $\tilde{B_n} = QDQ^t$;

where

$$D = \text{diag}\ (\theta^\ell + \theta^{-\ell} \ \text{ for } \ l = 1,2,\cdots,n-2 \text{ with } \ k \neq \frac{n-1}{2},\ 2,\ -2,\ 0,\ 0)$$

and

$$Q = \left(\begin{array}{cc} \begin{pmatrix} \frac{\theta^{jk}+\theta^{-jk}}{\sqrt{2(n-1)}} \\ j,k = 1,2,\ldots,n-2, \\ \text{with } k \neq \frac{n-1}{2} \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{n-1}} & -\frac{1}{\sqrt{n-1}} & 0 & 0 \\ \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & -\sqrt{\frac{2}{n}} & -\sqrt{\frac{2}{n(n-1)}} \\ \frac{1}{\sqrt{n-1}} & -\frac{1}{\sqrt{n-1}} & 0 & 0 \\ \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \sqrt{\frac{2}{n}} & \sqrt{\frac{2}{n(n-1)}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n-1}} & -\frac{1}{\sqrt{n-1}} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \frac{(-1)^k}{\sqrt{2(n-1)}} \\ \frac{(-1)^k}{\sqrt{2(n-1)}} \\ \frac{1}{n-1} \end{pmatrix} & \begin{pmatrix} \frac{1}{2\sqrt{n-1}} & \frac{1}{2\sqrt{n-1}} & \frac{(-1)^{\frac{n+1}{2}}\sqrt{2}}{\sqrt{n}} & \frac{(n-2)(-1)^{\frac{n-1}{2}}}{\sqrt{2n(n-1)}} \\ \frac{1}{2\sqrt{n-1}} & \frac{1}{2\sqrt{n-1}} & 0 & \frac{(n)(-1)^{\frac{n+1}{2}}}{\sqrt{2n(n-1)}} \\ \frac{1}{\sqrt{2(n-1)}} & \frac{1}{\sqrt{2(n-1)}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} \end{pmatrix} \end{array}\right)$$

with $\theta = \exp\{\frac{2\pi i}{2n-2}\}$.

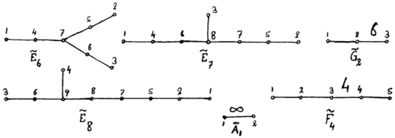


Figure 1.6:

For this case, we note that $n-1$ is even. Therefore,

$$\theta^{\frac{n-1}{2}} = \exp\{\frac{2\pi i}{2(n-1)} \cdot \frac{n-1}{2}\}. = \exp\{\frac{\pi i}{2}\} = i$$

and hence

$$\theta^{\frac{n-1}{2}} + \theta^{\frac{-(n-1)}{2}} = 0.$$

Thus, for any odd integer k, we get

$$\theta^{\frac{k(n-1)}{2}} + \theta^{\frac{-k(n-1)}{2}} = 0.$$

Therefore, in this case, we will get a similar expression for $(P_k(\tilde{B}_n))_{ij}$ for $1 \leq i \leq j \leq n+1$ as in the Case (i). Hence the result. ■

Lemma 1.4.7: *The graphs* $\tilde{E_6}$, $\tilde{E_7}$, $\tilde{E_8}$, $\tilde{F_4}$, $\tilde{G_2}$ *, and* $\tilde{A_1}$ *are all path-positive.*

Proof: The graphs of $\tilde{E_6}$, $\tilde{E_7}$, $\tilde{E_8}$, $\tilde{F_4}$, $\tilde{G_2}$, and $\tilde{A_1}$ can be seen in Figure 1.6. Let $\tilde{G}$ be any graph from the six graphs given above. Then the spectral decomposition of $\tilde{G}$ is $\tilde{G} = QDQ^t$; where D is a diagonal matrix and Q is an orthogonal matrix. For each case we give the matrices D and Q.

Case (i): For $\tilde{E_6}$, we have

$$D = \text{diag}\,(2, -2, 0, -1, -1, 1, 1)$$

and

$$Q = \frac{1}{\sqrt{48}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{12} & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ \sqrt{2} & \sqrt{2} & \sqrt{12} & 2w\sqrt{2} & 2w^2\sqrt{2} & 2w\sqrt{2} & 2w^2\sqrt{2} \\ \sqrt{2} & \sqrt{2} & \sqrt{12} & 2w^2\sqrt{2} & 2w\sqrt{2} & 2w^2\sqrt{2} & 2w\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} & 0 & -2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} & 0 & -2w\sqrt{2} & -2w^2\sqrt{2} & 2w\sqrt{2} & 2w^2\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} & 0 & -2w^2\sqrt{2} & -2w\sqrt{2} & 2w^2\sqrt{2} & 2w\sqrt{2} \\ 3\sqrt{2} & 3\sqrt{2} & -\sqrt{12} & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $w = \exp(\frac{2\pi i}{3})$.

Case (ii): For $\tilde{E}_7$, we have

$$D = \text{ diag } (2, -2, 0, -1, 0, 1, \sqrt{2}, -\sqrt{2})$$

and

$$Q = \frac{1}{\sqrt{48}} \begin{pmatrix} 1 & 1 & 2\sqrt{3} & 2\sqrt{2} & \sqrt{6} & 2\sqrt{2} & \sqrt{6} & \sqrt{6} \\ 1 & 1 & -2\sqrt{3} & 2\sqrt{2} & \sqrt{6} & 2\sqrt{2} & -\sqrt{6} & -\sqrt{6} \\ 2 & 2 & 0 & -2\sqrt{2} & 2\sqrt{6} & -2\sqrt{2} & 0 & 0 \\ 2 & -2 & 0 & -2\sqrt{2} & 0 & 2\sqrt{2} & 2\sqrt{3} & -2\sqrt{3} \\ 2 & -2 & 0 & -2\sqrt{2} & 0 & 2\sqrt{2} & -2\sqrt{3} & 2\sqrt{3} \\ 3 & 3 & -2\sqrt{3} & 0 & -\sqrt{6} & 0 & \sqrt{6} & \sqrt{6} \\ 3 & 3 & 2\sqrt{3} & 0 & -\sqrt{6} & 0 & -\sqrt{6} & -\sqrt{6} \\ 4 & -4 & 0 & 2\sqrt{2} & 0 & -2\sqrt{2} & 0 & 0 \end{pmatrix}.$$

Case (iii): For $\tilde{E}_8$, we have

$$D = \text{ diag } (2, -2, 0, -1, 1, \alpha_1, \alpha_2, -\alpha_2, -\alpha_1)$$

with

$$\alpha_1 = \frac{1}{2}(-1+\sqrt{5}) \text{ and } \alpha_2 = \frac{1}{2}(-1-\sqrt{5})$$

and

$$Q = \frac{1}{\sqrt{120}} \begin{pmatrix} 1 & 1 & \sqrt{30} & 2\sqrt{5} & 2\sqrt{5} & 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \\ 2 & -2 & 0 & -2\sqrt{5} & 2\sqrt{5} & 2\alpha_1\sqrt{3} & 2\alpha_2\sqrt{3} & -2\alpha_2\sqrt{3} & -2\alpha_1\sqrt{3} \\ 2 & -2 & 0 & -2\sqrt{5} & 2\sqrt{5} & 2\alpha_2\sqrt{3} & 2\alpha_1\sqrt{3} & -2\alpha_1\sqrt{3} & -2\alpha_2\sqrt{3} \\ 3 & 3 & -\sqrt{30} & 0 & 0 & -2\alpha_2\sqrt{3} & -2\alpha_1\sqrt{3} & -2\alpha_1\sqrt{3} & -2\alpha_2\sqrt{3} \\ 3 & 3 & -\sqrt{30} & 0 & 0 & -2\alpha_1\sqrt{3} & -2\alpha_2\sqrt{3} & -2\alpha_2\sqrt{3} & -2\alpha_1\sqrt{3} \\ 4 & 4 & 0 & 2\sqrt{5} & 2\sqrt{5} & -2\sqrt{3} & -2\sqrt{3} & -2\sqrt{3} & -2\sqrt{3} \\ 4 & -4 & 0 & 2\sqrt{5} & -2\sqrt{5} & -2\sqrt{3} & -2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \\ 5 & 5 & \sqrt{30} & -2\sqrt{5} & -2\sqrt{5} & 0 & 0 & 0 & 0 \\ 6 & -6 & 0 & 0 & 0 & 2\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} & -2\sqrt{3} \end{pmatrix}.$$

In this case we also have the following relations:

$$\alpha_1 + \alpha_2 = -1 = \alpha_1 \cdot \alpha_2,$$

$$\alpha_1{}^2 + \alpha_2{}^2 = 3,$$

and

$$\alpha_1{}^3 + \alpha_2{}^3 = -4.$$

Case (iv): For $\tilde{F}_4$, we have

$$D = \text{diag}\,(2, 1, 0, -1, -2)$$

and

$$Q = \frac{1}{\sqrt{24}} \begin{pmatrix} 1 & 2\sqrt{2} & \sqrt{6} & 2\sqrt{2} & 1 \\ 2 & 2\sqrt{2} & 0 & -2\sqrt{2} & -2 \\ 3 & 0 & -\sqrt{6} & 0 & 3 \\ 2\sqrt{2} & -2 & 0 & 2 & -2\sqrt{2} \\ \sqrt{2} & -2 & 2\sqrt{3} & -2 & \sqrt{2} \end{pmatrix}.$$

Case (v): For $\tilde{G}_2$, we have

$$D = \text{diag}\,(2, 0, -2)$$

and

$$Q = \frac{1}{\sqrt{24}} \begin{pmatrix} \sqrt{3} & 3\sqrt{2} & \sqrt{3} \\ 2\sqrt{3} & 0 & -2\sqrt{3} \\ 3 & -\sqrt{6} & 3 \end{pmatrix}.$$

Case (vi): For $\tilde{A}_1$, we have

$$D = \text{ diag } (2, -2)$$

and

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Except for the cases (v) and (vi), the real number 1 is an eigenvalue of each graph. Hence, from (*ii*) of the Corollary 3.2, we see that we shall have to consider at least six subcases.

Therefore for each case, we have made a table to show that $(P_k(\tilde{G}))_{ij} \geq 0$ whenever there exists a path of length k from vertex i to vertex j.

From Appendix 3, where we have given the table for each case, we get the required result. ■

Lemma 1.4.8: *Z_4 and Z_5 are path-positive.*

Proof: See Appendix 4. ■

We have shown that all the graphs listed in $\mathcal{C}_2$ of Appendix 1 are path-positive.

1.5 The Main Theorem

We first obtain a preliminary result.

Lemma 1.5.1: *If G is any graph from the list $\mathcal{C}_1$ of Appendix 1, then G is not path-positive.*

Proof: By definition

$P_k(\lambda) = \det[\lambda I - A_k]$ for $k = 1, 2, \cdots$. with $P_0(\lambda) = 1$. Also

$$P_k(\lambda) = \prod_{j=1}^{k} \{\lambda - 2\cos(\frac{\pi j}{k+1})\} \tag{1.5.1}$$

as the eigenvalues of A_k are $2\cos(\frac{\pi j}{k+1})$ for $j = 1, 2, \cdots, k$.

We claim that for each G in $\mathcal{C}_1$ there exists a positive integer N such that

$$P_N(G) = 0.$$

This N depends on G. From page 125 of Mehta (1989) (see also Grove and Benson (1985) and Coxeter and Moser (1980)) the following can be seen:

1. Eigenvalues of A_n, $n \geq 1 : 2\cos(\frac{\pi j}{n+1})$, $j = 1, 2, \cdots, n$.
2. Eigenvalues of B_n, $n \geq 2 : 2\cos(\frac{\pi j}{2n})$, $j = 1, 3, 5, \cdots, 2n - 1$.
3. Eigenvalues of D_n, $n \geq 4 : 2\cos(\frac{\pi j}{2n-2})$, $j = 1, 3, 5, \cdots, 2n - 3, n - 1$.
4. Eigenvalues of $E_6 : 2\cos(\frac{\pi j}{12})$, $j = 1, 4, 5, 7, 8, 11$.
5. Eigenvalues of $E_7 : 2\cos(\frac{\pi j}{18})$, $j = 1, 5, 7, 9, 11, 13, 17$.
6. Eigenvalues of $E_8 : 2\cos(\frac{\pi j}{30})$, $j = 1, 7, 11, 13, 17, 19, 23, 29$.
7. Eigenvalues of $F_4 : 2\cos(\frac{\pi j}{12})$, $j = 1, 5, 7, 11$.
8. Eigenvalues of $I_2(m) : 2\cos(\frac{\pi j}{m})$, $j = 1, m - 1$.
9. Eigenvalues of $H_3 : 2\cos(\frac{\pi j}{10})$, $j = 1, 5, 9$; and
10. Eigenvalues of $H_4 : 2\cos(\frac{\pi j}{30})$, $j = 1, 11, 19, 29$.

Now, using the spectral decomposition of any graph G, we get $P_k(G) = QP_k(D)Q^t$; where Q is an orthogonal matrix and D is the diagonal matrix with entries the eigenvalues of G.

To show $P_N(G) = 0$ for some N, we need to show $P_N(D) = 0$ for some N, that is, to show $P_N(\lambda) = 0 \ \forall \ \lambda \in D$.

But from (1.5.1), we get $P_N(\lambda) = \prod_{i=1}^{N}\{\lambda - 2\cos(\frac{\pi i}{N+1})\}$. Hence, for each graph G, take N to be the maximum value of j appearing in the eigenvalue list. Thus, for $\lambda = 2\cos(\frac{\pi \ell}{N+1})$, we have

$$\begin{aligned} P_N(\lambda) &= P_N(2\cos(\frac{\pi \ell}{N+1})) \\ &= \prod_{j=1}^{N}\{2\cos(\frac{\pi \ell}{N+1}) - 2\cos(\frac{\pi j}{N+1})\} \\ &= 0, \quad \text{as } 1 \leq m_i \leq N. \end{aligned}$$

Hence $P_N(G) = 0$ for some N. The choice of N clearly depends on the graph G. We write N_G in place of N.

For another proof of the fact that there exists a nonnegative integer N such that $P_N(G) = 0$ for each of the above graphs G, see Seidel (1991).

We will show that the graphs in $\mathcal{C}_1$ are not path-positive. For each G in $\mathcal{C}_1$, consider $P_k(G)$. If $P_k(G)$ for some $k \leq N_G$ has at least one negative entry, then by definition G is not path-positive. On the contrary, let $P_k(G)$ be entrywise nonnegative for all integers $k \leq N_G$. Now, let T be the smallest integer for which $P_T(G) = 0$ and $P_{T-1}(G)$ has at least one entry positive (the integer T exists as we already have $P_{N_G}(G) = 0$, for each graph G in $\mathcal{C}_1$). Consider

$$\begin{aligned} P_{T+1}(G) &= GP_T(G) - P_{T-1}(G) \\ &= -P_{T-1}(G) \\ &\leq 0. \end{aligned}$$

Since $P_{T-1}(G)$ is entrywise nonnegative with at least one entry positive, we get, at least one entry of $P_{T+1}(G)$ to be negative .

Hence we see that G is not path-positive. ■

Theorem 1.5.2: *A graph G is not path-positive if and only if G is from the list $\mathcal{C}_1$ of Appendix* 1.

Proof: Suppose a graph G is not path-positive. Then by Lemma 1.2.2 , G cannot be a supergraph of any graph which is in list $\mathcal{C}_2$ of Appendix 1. That is, we have

(a) G does not have $\tilde{A}_n$ as a subgraph, hence G is a tree.

(b) G does not have $\tilde{D}_4$ as a subgraph, hence no more than three edges can originate at a given vertex of G.

(c) G does not have $\tilde{D}_n$ for $n \geq 5$ as a subgraph, hence atmost one vertex of G can have degree 3.

(d) G cannot have $\tilde{C}_n$ for some $n \geq 2$ as a subgraph, hence G cannot have two or more edges marked (that is, having label other than 3).

(e) In view of the path-positivity of $\tilde{G}_2$ and $\tilde{A}_1$, if the label of any edge of G exceeds 5, then the number of vertices n in G cannot be more than 2, and the label must be finite.

(f) G does not have $\tilde{F}_4$ as a subgraph, hence G can only be F_4.

(g) G does not have Z_4 and Z_5 as sugraphs, hence the graph G is either H_3 or H_4.

(h) G does not have $\tilde{B}_n$ as a subgraph, hence G can have a branch point, but no edge of G can be marked.

(i) G does not have $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ as subgraphs, hence G will have branches of lengths p, q, r with

$$(p,q,r) = (1,2,2),(1,2,3),(1,2,4),(1,1,m).$$

Therefore, we see that whenever G is not path-positive, we get the list $\mathcal{C}_2$ of Appendix 1.

Conversely, suppose G is from the list $\mathcal{C}_1$ of Appendix 1. Then by Lemma 1.5.1 G is not path-positive. ∎

Corollary 1.5.3: *The graphs in list $\mathcal{C}_1$ of Appendix 1 are the only connected graphs with Perron-eigenvalue less than 2.*

Proof: We will denote the Perron-eigenvalue of a graph G by $\rho(G)$.

Suppose on the contrary that there exists a connected graph G which does not appear in list $\mathcal{C}_1$ but has $\rho(G) < 2$.

Since $\rho(G) < 2$, we can write $\rho(G) = 2\cos(\frac{\pi}{q})$ for some positive real number $q > 2$. Hence $\frac{\pi}{q}$ lies in the first quadrant. Since $A(G)$ is a nonnegative matrix, by Perron-Frobenius theorem there exists a positive vector $\underline{x}$ (G is connected implies that $A(G)$ is irreducible) such that

$$A(G)\underline{x} = \rho(G)\underline{x}.$$

It can be easily seen that

$$P_k(A(G))\underline{x} = P_k(\rho(G))\underline{x}.$$

That is, $P_k(\rho(G))$ is the Perron-eigenvalue of $P_k(A(G))$ with Perron-eigenvector $\underline{x}$. This is same as saying,

$$P_k(\rho(G)) = \rho(P_k(A(G))).$$

Since G is not from the list $\mathcal{C}_1$, Theorem 1.5.2 tells us that G is path-positive. That is the matrix $P_k(A(G))$ is entrywise nonnegative. Hence we must have

$$\rho(P_k(A(G))) \geq 0 \tag{1.5.2}$$

But from (1.4.4), we have

$$\begin{aligned} P_k(\rho(G)) &= P_k(2\cos(\frac{\pi}{q})) \\ &= \frac{\sin(\frac{(k+1)\pi}{q})}{\sin(\frac{\pi}{q})}. \end{aligned}$$

Choose a positive integer k such that

$\pi < \frac{(k+1)\pi}{q} < \frac{3\pi}{2}$

that is, $1 < \frac{k+1}{q} < \frac{3}{2}$

that is, $q - 1 < k < \frac{3q}{2} - 1$.

Therefore the range for k is

$$\frac{3q}{2} - 1 - \{q - 1\} = \frac{q}{2} > 1$$

as $q > 2$. Hence we can always find a positive integer k such that $\pi < \frac{(k+1)\pi}{q} < \frac{3\pi}{2}$. That is, $\frac{(k+1)\pi}{q}$ lies in the third quadrant. Therefore $P_k(\rho(G)) < 0$, a contradiction to (1.5.2).

Conversely, if $G \in \mathcal{C}_1$, then as already seen $\rho(G) < 2$. Hence the result. ■

Remarks (i) : If G is in $\mathcal{C}_2$ then we have in fact proved that $P_k(A(G))$ is nonnegative and nonzero for all k. It then follows from Lemma 1.2.2 that any Coxeter graph G is either in $\mathcal{C}_1$ or has the property that $P_k(A(G))$ is nonnegative, nonzero for all k.

(ii) : Any connected graph on n vertices is path-positive of order n. This can be seen as follows: for A_n, B_n, and D_n see Section 1.3 .For other graphs which are in list $\mathcal{C}_1$ (see Appendix 1), it can be easily verified. Other connected graphs are path-positive by Theorem 1.5.2.

1.6 Directed Graphs

Directed graphs have already been defined (see Definition 1.1.3). We now define the adjacency matrix of the directed graph and combinatorial symmetry. The definition of combinatorial symmetry and the list of the directed graphs shown in Appendix 2 has been taken from Berman, Moody and Wonenburger (1972).

Definition 1.6.1 : A matrix $B = (b_{ij})$ is said to be *combinatorially symmetric* if

$$b_{ij} = 0 \text{ whenever } \quad b_{ji} = 0$$

Definition 1.6.2 : By the *adjacency matrix* of a directed graph $D = (V, \vec{E})$ we mean the square matrix $A(D) = (a_{ij})$ of order $|V|$; where a_{ij} is equal to the number of edges from vertex i to vertex j.

If in the graph, there is no edge from vertex j to vertex i but at least one edge from vertex i to vertex j then $a_{ji} = 1$ and a_{ij} is equal to the number of edges from vertex i to vertex j.

Definition 1.6.3 : A *directed graph* D is said to be *combinatorially symmetric* if the adjacency matrix $A(D)$ of the directed graph D is combinatorially symmetric. Throughout this section, we consider combinatorially symmetric graphs.

In this section our main result is the following: *A directed graph $\vec{D}$ is not path-positive if and only if $\vec{D}$ is from the list $\mathcal{D}_1$ of Appendix 2.*

Before coming to the main result, let us note that for this section the adjacency matrix is not symmetric. Hence our aim is to symmetrize the adjacency matrices in such a way that we are able to use the results of Section 1.4 .

Definition 1.6.4 : A matrix B is said to be *symmetrizable* if there exists a nonsingular diagonal matrix D such that DBD^{-1} is symmetric.

Definition 1.6.5 : The *directed graph $\vec{D}$* is said to be *symmetrizable* if its adjacency matrix is symmetrizable.

Lemma 1.6.1 : *The directed graphs in list $\mathcal{D}_2$ of Appendix 2 are symmetrizable.*

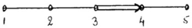


Figure 1.7: $\breve{F}_4$

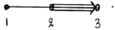


Figure 1.8: $\breve{G}_2$

Proof : For each of the five directed graphs in list $\mathcal{D}_2$ we give the nonsingular diagonal matrix D, such that the resultant symmetric matrix in each case is the adjacency matrix of the corresponding graph of list C_2 of Appendix 1.

$$\text{If } D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix},$$

then $D\breve{F}_4D^{-1} = \tilde{F}_4$.

If

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix},$$

then $D\breve{G}_2D^{-1} = \tilde{G}_2$.

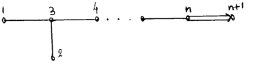


Figure 1.9: $\breve{B}_n$

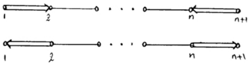


Figure 1.10: $\check{C}_n$

For $\check{B}_n$,

$$D = \begin{pmatrix} I_{n-1} & \underline{0} \\ \underline{0}^t & \sqrt{2} \end{pmatrix},$$

and hence $D\check{B}_n D^{-1} = \tilde{B}_n$.

For the first $\check{C}_n$,

$$D = \begin{pmatrix} 1 & \underline{0}^t & 0 \\ \underline{0} & \sqrt{2}I_{n-1} & \underline{0} \\ 0 & \underline{0}^t & 1 \end{pmatrix},$$

and for the second

$$D = \begin{pmatrix} 1 & \underline{0}^t & 0 \\ \underline{0} & \frac{1}{\sqrt{2}}I_{n-1} & \underline{0} \\ 0 & \underline{0}^t & 1 \end{pmatrix}.$$

In each case $D\check{C}_n D^{-1} = \tilde{C}_n$. Hence the result. ∎

The next Lemma is a generalization of Lemma 1.2.2 .

Lemma 1.6.2 : *Let $\vec{D} = (V, \vec{E})$ be a directed graph with $|V| = n$. Suppose $\check{D}$ is a supergraph of $\vec{D}$. If $\vec{D}$ is path-positive then $\check{D}$ is also path-positive.*

Proof : We consider two cases :

Case (i): Suppose $\check{D} = (\check{V}, \check{E})$; where $\check{V} = V \cup i_1$ for $i_1 \notin V$ and $\check{E} = \vec{E} \cup (i_1, \ell)$ for some $\ell \in V$. Since we are considering only combinatorially symmetric graphs, we get $\check{E} = \vec{E} \cup (i_1, \ell) \cup (\ell, i_1)$. Number the vertices of the graph $\check{D}$ in such a way that the vertex i_1 is given number 1 and the vertex $\ell \in V$ is given number 2. The renumbering of the vertices does not affect the path-positivity of the graph $\vec{D}$.

We have to show that, the kth path matrix of $\check{D}$ is nonnegative for all positive integers k.

The matrix $A(\check{D})$ is a square matrix of order $n+1$ and

$$A(\check{D}) = \begin{pmatrix} 0 & e_1^t \\ e_1 & A(\vec{D}) \end{pmatrix} = A + B \text{ (say);}$$

where $e_1^t = (1, 0, ..., 0) \in I\!R^n$ (t denotes transpose),

$$A = \begin{bmatrix} 0 & 0^t \\ 0 & A(\vec{D}) \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & e_1^t \\ e_1 & 0 \end{bmatrix}.$$

We shall use induction on k to show that the kth path-matrix of $\check{D}$ is nonnegative for all positive integers k.

For $k = 1$, the claim is trivial. Let the claim hold true for all k, $1 \le k \le m$. We shall prove it for $k = m+1$. By Lemma 1.2.1, we get

$$\begin{aligned} P_{m+1}(A(\check{D})) &= P_{m+1}(A+B) \\ &= P_{m+1}(A) + \sum_{s=0}^{m} P_s(A) B P_{m-s}(A+B). \end{aligned} \tag{1.6.1}$$

We observe the following:

1. $P_s(A) \ge 0$ for all $s = 0, 1, 2, ..., m+1$, except possibly for the entry $(1,1)$ which may be -1;
2. $B \ge 0$; and
3. by induction hypothesis, $P_\ell(A+B) \ge 0$ for all $\ell, 0 \le \ell \le m$.

Hence, we see that by (1.6.1), the $(m+1)$st path-matrix of $A(\check{D})$ is nonnegative except possibly for the first row. We shall show that $(P_{m+1}(A(\check{D})))_{1i} \ge 0$; for $1 \le i \le$

$n+1$. We have

$$\begin{aligned} P_{m+1}(A(\check{D})) &= P_{m+1}(A+B) \\ &= (A+B)P_m(A+B) - P_{m-1}(A+B), \qquad \text{using (1.2.2).} \end{aligned}$$

Since the first row of A is zero, we get,

$$\begin{aligned} (P_{m+1}(A(\check{D})))_{1i} &= (P_m(A+B))_{2i} - (P_{m-1}(A+B))_{1i} \\ &= [P_m(A) + \sum_{s=0}^{m-1} P_s(A)BP_{m-1-s}(A+B)]_{2i} \\ &\quad -(P_{m-1}(A+B))_{1i}, \qquad \text{using Lemma 1.2.1,} \\ &= (P_m(A))_{2i} + \sum_{s=1}^{m-1}(P_s(A)BP_{m-1-s}(A+B))_{2i} \\ &\quad +(P_0(A)BP_{m-1}(A+B))_{2i} - (P_{m-1}(A+B))_{1i} \\ &\geq \sum_{s=1}^{m-1}(P_s(A)BP_{m-1-s}(A+B))_{2i} \\ &\quad +(P_{m-1}(A+B))_{1i} - (P_{m-1}(A+B))_{1i} \\ &= \sum_{s=1}^{m-1}(P_s(A)BP_{m-1-s}(A+B))_{2i} \\ &\geq 0. \end{aligned}$$

The first inequality is due to the fact that $(P_m(A))_{2i} \geq 0$. We get the second inequality as the second row of $P_s(A) \geq 0$ for all s, $1 \leq s \leq m-1$, $B \geq 0$ and by induction hypothesis the i^{th} column $(1 \leq i \leq n+1)$ of $P_\ell(A+B) \geq 0$ for $1 \leq \ell \leq m-1$. Hence the result.

Case(ii): The proof is similar to the proof of Case (ii) of Lemma 1.2.2 . ■

Theorem 1.6.3 : *A graph $\vec{G}$ is not path-positive if and only if $\vec{G}$ is from the list $\mathcal{D}_1$ of Appendix 2.*

Proof : 'If' part: we want to show that if $\vec{G}$ is any graph from the list $\mathcal{D}_1$ of Appendix 2, then $\vec{G}$ is not path-positive. First note that all the directed graphs from the list $\mathcal{D}_1$ of Appendix 2 are symmetrizable and they correspond to the adjacency matrix of the

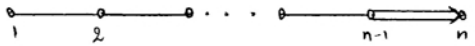


Figure 1.11: $\vec{B_n}$

corresponding graph of list $\mathcal{C}_1$ of Appendix 1. For example, consider $\vec{B_n}$. The graph of $\vec{B_n}$ can be seen in Figure 1.11. Since $\vec{B_n}$ is symmetrizable, we get a nonsingular diagonal matrix D such that $DA(\vec{B_n})D^{-1} = A(B_n)$. Hence by Lemma 1.5.1, there exists an $N = 2n-1$ such that $P_N(A(\vec{B_n})) = 0$. Since

$$A(\vec{B_n}) = \begin{pmatrix} A_{n-1} & 2e_{n-1} \\ e_{n-1}^t & 0 \end{pmatrix} \text{ we get } D = \begin{pmatrix} I_{n-1} & 0 \\ 0^t & \sqrt{2} \end{pmatrix}.$$

Therefore, we see that $DA(\vec{B_n})D^{-1} = A(B_n)$ and

$$P_N(A(\vec{B_n})) = P_N(D^{-1}A(B_n)D) = D^{-1}P_N(A(B_n))D = 0.$$

A similar argument will show that for each of the graphs $\vec{G}$ in list $\mathcal{D}_1$ of Appendix 2 there exists an N such that $P_N(A(\vec{G})) = 0$.

For the only if part, proceed as in Theorem 1.5.2 . Hence the result. ■

Corrolary 1.6.4 : *The graphs in list $\mathcal{D}_1$ of Appendix 2 are the only connected directed graphs with Perron-eigenvalue less than 2.*

Proof : A similar reasoning as in Corrolary 1.5.3 gives the required result. ■

Chapter 2

The Sign Change Group

2.1 The q- permanent with respect to $\mathcal{B}_n$

From this chapter, by $A > 0$ $(A \geq 0)$ we mean that the matrix A is hermitian positive definite (positive semidefinite). Also $A \geq B$ means that $A \geq 0$, $B \geq 0$, and $A - B \geq 0$. The determinant and the permanent of the matrix A will be denoted by $\det A$ and per A respectively. As usual, $\mathcal{S}_n$ denotes the symmetric group of degree n, and $\epsilon(\sigma) = 1$ or -1 according as $\sigma \in \mathcal{S}_n$ is even or odd. Thus

$$\det A = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)} \quad \text{and} \quad \text{per } A = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If $\sigma \in \mathcal{S}_n$, then $i(\sigma)$ will denote the number of inversions of σ. Recall that an inversion of σ is a pair $(i, j), 1 \leq i,\ j \leq n$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

Let A be an $n \times n$ matrix and q a complex number. Then the $q-$*permanent* of A, denoted by $\text{per}_q(A)$, is defined as

$$\text{per}_q(A) = \sum_{\sigma \in \mathcal{S}_n} q^{i(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)} \tag{2.1.1}$$

Observe that $\text{per}_1(A) = \text{per } A$, $\text{per}_{-1}(A) = \det A$ and $\text{per}_0(A) = \prod_{i=1}^{n} a_{ii}$. It follows from a result of Bożejko and Speicher (1990) that if $A \geq 0$ then

$$\text{per}_q(A) \geq 0, \qquad -1 \leq q \leq 1.$$

In this chapter, we construct a similar function $Q_q(A)$ using the sign change group $\mathcal{B}_n$, which like $\mathcal{S}_n$, is a Coxeter group. We also show that for $q \in [-1,1]$, $Q_q(A) \geq 0$.

Let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$ with $e_i^t = (0,\ldots,0,1,0,\ldots,0)$ where the 1 occurs at the i-th place, for $1 \leq i \leq n$. Recall (see Humphreys (1990), page 72 or Benson and Grove (1985), page 76.) that a root system corresponding to the Coxeter group $\mathcal{B}_n$ can be taken as

$$\Phi = \{\pm e_i, 1 \leq i \leq n; \;\; e_i \pm e_j, 1 \leq i,j \leq n\}.$$

The root system of any Coxeter group can be written as a disjoint union of two systems, namely, the *positive root system* Π and the *negative root system* $-\Pi$. For the Coxeter group $\mathcal{B}_n$ let us take

$$\Pi = \{e_i, 1 \leq i \leq n; \;\; e_i \pm e_j, 1 \leq j < i \leq n\}$$

and hence

$$-\Pi = \{-e_i, 1 \leq i \leq n; \;\; -(e_i \pm e_j), 1 \leq j < i \leq n\}.$$

Clearly

$$\Phi = \Pi \cup -\Pi.$$

For any element $\tilde{\sigma} \in \mathcal{B}_n$, the number of inversions of $\tilde{\sigma}$, denoted by $n(\tilde{\sigma})$, is defined as

$$\begin{aligned} n(\tilde{\sigma}) &= \text{Cardinality } (\Pi \cap \tilde{\sigma}^{-1}(-\Pi)) \\ &= \text{Cardinality } (\tilde{\sigma}(\Pi) \cap -\Pi) \\ &= \text{number of positive roots being sent to } -\Pi \text{ by } \;\tilde{\sigma}. \end{aligned}$$

For example, for $n = 3$, we have

$$\Pi = \{e_1, e_2, e_3, e_2 - e_1, e_2 + e_1, e_3 - e_2, e_3 + e_2, e_3 - e_1, e_3 + e_1\}$$

$$-\Pi = \{-e_1, -e_2, -e_3, -e_2 - e_1, e_1 - e_2, -e_3 - e_2, e_2 - e_3, -e_3 - e_1, e_1 - e_3\}.$$

(i) Let

$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \end{pmatrix}.$$

Then,

$$\tilde{\sigma}(\Pi) = \{-e_1, e_3, -e_2, e_3 + e_1, e_3 - e_1, -e_2 - e_3, -e_2 + e_3, -e_2 + e_1, -e_2 - e_1\}.$$

and hence

$$\begin{aligned} n(\tilde{\sigma}) &= \text{Card}\ \{-e_1, -e_2, -e_2 - e_3, -e_2 + e_1, -e_2 - e_1\} \\ &= 5. \end{aligned}$$

(ii) Let $\tilde{\sigma} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$. Then,

$$\tilde{\sigma}(\Pi) = \{-e_1, -e_2, -e_3, -e_2 - e_1, e_1 - e_2, -e_3 - e_2, e_2 - e_3, -e_3 - e_1, e_1 - e_3\}.$$

and hence $n(\tilde{\sigma}) = 9$.

It is an easy exercise (see, Humphreys (1990), page 14) to see that $n(\tilde{\sigma}) = i(\tilde{\sigma})$; where $i(\tilde{\sigma})$ is the length of the element $\tilde{\sigma} \in \mathcal{B}_n$.

Let us denote by β_ℓ^σ, $1 \leq \ell \leq n$, the number of inversions in

$$(1, \ell), (2, \ell), \ldots, (\ell - 1, \ell)$$

for a fixed $\sigma \in \mathcal{S}_n$. For the sake of convenience, let us take $\beta_1^\sigma = 0$.

Consider the following two maps:

$$(i) \qquad \epsilon : \{1, 2, \ldots, n\} \longrightarrow \{1, -1\}$$

and let

$$J_\epsilon = \{i \in \{1, 2, \ldots, n\} : \epsilon(i) = -1\},$$

and

$$(ii) \qquad \tau_{\sigma,\epsilon} : \{1, 2, \ldots, n\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm n\}$$

defined as

$$\tau_{\sigma,\epsilon}(i) = \epsilon(i)\sigma(i) \ ,$$

for each $i \in \{1, 2, \ldots, n\}$ and each fixed $\sigma \in \mathcal{S}_n$.

Lemma 2.1.1 : *For each fixed $\sigma \in \mathcal{S}_n$,*

$$\sum_{\tau_{\sigma,\epsilon}} (-1)^{|J_\epsilon|} q^{i(\tau_{\sigma,\epsilon})} = q^{i(\sigma)} \prod_{\ell=1}^{n} (1 - q^{2(\ell - \beta_\ell^\sigma) - 1}); \tag{2.1.2}$$

where $|J_\epsilon|$ denotes the cardinality of the set J_ϵ.

Proof: Since $i(\tilde{\sigma}) = n(\tilde{\sigma})$ for each $\tilde{\sigma} \in \mathcal{B}_n$, the equality (2.1.2) can be written as

$$\sum_{\tau_{\sigma,\epsilon}} (-1)^{|J_\epsilon|} q^{n(\tau_{\sigma,\epsilon})} = q^{n(\sigma)} \prod_{\ell=1}^{n} (1 - q^{2(\ell - \beta_\ell^\sigma) - 1}). \tag{2.1.3}$$

Note that the summation in the left hand side of (2.1.3) is over all the 2^n elements of $\mathcal{B}_n$ for a given fixed $\sigma \in \mathcal{S}_n$. We get the 2^n elements of $\mathcal{B}_n$ for a fixed $\sigma \in \mathcal{S}_n$ by taking $\epsilon(\ell)$ as 1 or -1 for each ℓ, $1 \leq \ell \leq n$. The 2^n elements generated by $\epsilon(\ell)$, for $1 \leq \ell \leq n$ are $\otimes_{\ell=1}^{n} \epsilon(\ell)$. That is, the 2^n elements of $\mathcal{B}_n$ corresponding to the fixed $\sigma \in \mathcal{S}_n$ are

$$\{\tau_{\sigma,\epsilon} : \epsilon \in \otimes_{\ell=1}^{n} \epsilon(\ell)\}.$$

Hence

$$\sum_{\tau_{\sigma,\epsilon}} (-1)^{|J_\epsilon|} q^{n(\tau_{\sigma,\epsilon})} = \prod_{\ell=1}^{n} \sum_{\epsilon(\ell)} q^{n(\tau_{\sigma,\epsilon})} \tag{2.1.4}$$

Let us now count the number of positive roots which are being sent to $-\Pi$ by $\tau_{\sigma,\epsilon(\ell)}$. For this, we partition the positive root system $\Pi = \cup_{\ell=1}^{n} \Pi_\ell$; where for each ℓ, $1 \leq \ell \leq n$,

$$\Pi_\ell = \{e_\ell, e_\ell \pm e_j; \ 1 \leq j < \ell\}.$$

We note the following about the element $\tau_{\sigma,\epsilon(\ell)} \in \mathcal{B}_n$:

Case (i) : if $\epsilon(\ell) > 0$ then $\tau_{\sigma,\epsilon(\ell)}(e_\ell - e_j) \in -\Pi$ whenever $\sigma(\ell) < \sigma(j)$. Therefore, we get

$$\ell > j \quad \text{and} \quad \sigma(\ell) < \sigma(j) \text{ for the pair } (\ell, j)$$

as $e_\ell - e_j \in \Pi$. Hence, whenever $\epsilon(\ell) > 0$, the number of roots of Π_ℓ being sent to $-\Pi$ is exactly β_l^σ.

Case (ii) : (a) if $\epsilon(\ell) < 0$ then $\tau_{\sigma,\epsilon(\ell)}(e_\ell) = -e_{\sigma(\ell)} \in -\Pi$.

(b) if $\epsilon(\ell) < 0$ and $\epsilon(j) > 0$ for $1 \leq j < \ell$ then

$$\tau_{\sigma,\epsilon(\ell)}(e_\ell - e_j) = -(e_{\sigma(\ell)} + e_{\sigma(j)}) \in -\Pi$$

and if $\epsilon(\ell) < 0$ and $\epsilon(j) < 0$ for $1 \leq j < \ell$ then

$$\tau_{\sigma,\epsilon(\ell)}(e_\ell + e_j) = -(e_{\sigma(\ell)} + e_{\sigma(j)}) \in -\Pi.$$

Hence, if $\epsilon(\ell) < 0$ then the number of roots of Π_ℓ being sent to $-\Pi$ is at least $\ell - 1$ as $1 \leq j < \ell$.

(c) if $\epsilon(\ell) < 0$ and $\epsilon(j) > 0$ for $1 \leq j < \ell$ then

$$\begin{aligned} \tau_{\sigma,\epsilon(\ell)}(e_\ell + e_j) \in -\Pi &\iff -e_{\sigma(\ell)} + e_{\sigma(j)} \in -\Pi \\ &\iff \sigma(\ell) > \sigma(j), \end{aligned}$$

and if $\epsilon(\ell) < 0$ and $\epsilon(j) < 0$ for $1 \leq j < \ell$, then

$$\begin{aligned} \tau_{\sigma,\epsilon(\ell)}(e_\ell - e_j) \in -\Pi &\iff -e_{\sigma(\ell)} + e_{\sigma(j)} \in -\Pi \\ &\iff \sigma(\ell) > \sigma(j). \end{aligned}$$

Therefore, using (a) , (b) and (c), we see that whenever $\epsilon(\ell) < 0$, the number of roots in Π_ℓ being sent to $-\Pi$ is exactly

$1 + \ell - 1 + (\text{ number of non-inversions}) \;=\; 1 + \ell - 1 + (\ell - 1 - \beta_l^\sigma) = 2\ell - 1 - \beta_l^\sigma.$

Therefore, for a given ℓ, $1 \leq \ell \leq n$, using the two cases, we have

$$(-1)^{\epsilon(\ell)} q^{n(\tau_{\sigma,\epsilon(\ell)})} = \begin{cases} q^{\beta_l^\sigma} & \text{if } \epsilon(\ell) > 0 \\ -q^{2\ell-1-\beta_l^\sigma} & \text{if } \epsilon(\ell) < 0 \end{cases}$$

Therefore, using (2.1.4), we get

$$\begin{aligned}\sum_{\tau_{\sigma,\epsilon}}(-1)^{|J_\epsilon|}q^{n(\tau_{\sigma,\epsilon})} &= \prod_{\ell=1}^{n}\{q^{\beta_\ell^\sigma}-q^{2\ell-1-\beta_\ell^\sigma}\}\\ &= \prod_{\ell=1}^{n}q^{\beta_\ell^\sigma}\{1-q^{2(\ell-\beta_\ell^\sigma)-1}\}\\ &= q^{\sum_{\ell=1}^{n}\beta_\ell^\sigma}\prod_{\ell=1}^{n}\{1-q^{2(\ell-\beta_\ell^\sigma)-1}\}\\ &= q^{n(\sigma)}\prod_{\ell=1}^{n}\{1-q^{2(\ell-\beta_\ell^\sigma)-1}\}\end{aligned}$$

■

It will be convenient for us to assume throughout this section and the next chapter that the elements of $\mathcal{S}_n$ have been ordered in the following way: if $\sigma, \tau \in \mathcal{S}_n$, then σ precedes τ if σ precedes τ in the lexicographic ordering, or equivalently, if the first non-zero difference $\sigma(i)-\tau(i)$, $i = 1, 2, \ldots, n$, is negative. Thus the elements of $\mathcal{S}_3$ are ordered as follows:

$$123, 132, 213, 231, 312, 321.$$

For any permutation $\tilde{\sigma} \in \mathcal{B}_n$, by $[\tilde{\sigma}]$, we denote the element of $\mathcal{S}_n$ corresponding to $\tilde{\sigma}$. That is,

$$[\tilde{\sigma}] = \begin{pmatrix} 1 & 2 & \ldots & n \\ |\tilde{\sigma}(1)| & |\tilde{\sigma}(2)| & \ldots & |\tilde{\sigma}(n)| \end{pmatrix} ;$$

where $|\cdot|$ denotes the absolute value.

For each fixed $\sigma \in \mathcal{S}_n$, we order the 2^n elements of $\mathcal{B}_n$ in the following way :

(i) τ_{σ,ϵ_1} precedes τ_{σ,ϵ_2} if $|J_{\epsilon_1}| < |J_{\epsilon_2}|$, or

(ii) if $|J_{\epsilon_1}| = |J_{\epsilon_2}|$ then τ_{σ,ϵ_1} precedes τ_{σ,ϵ_2} if the first non-zero difference $\sigma(i)\epsilon_1(i) - \sigma(i)\epsilon_2(i)$, $i = 1, 2, \ldots, n$, is negative. Thus, for the element 123 of $\mathcal{S}_3$ we have the ordering as

$$1\;2\;3, -1\;2\;3, 1-2\;3, 1\;2-3, -1-2\;3, -1\;2-3, 1-2-3, -1-2-3.$$

Therefore, for $\mathcal{B}_n$ we have the ordering as

(i) $\tilde{\sigma}$ precedes $\tilde{\eta}$ if $[\tilde{\sigma}]$ precedes $[\tilde{\eta}]$ as an element of $\mathcal{S}_n$, or

(ii) if $[\tilde{\sigma}] = [\tilde{\eta}]$ then $\tilde{\sigma}$ precedes $\tilde{\eta}$ according to the ordering defined for a fixed $\sigma \in \mathcal{S}_n$.

Thus the elements of $\mathcal{B}_2$ are ordered as follows:

$$1\,2, -1\,2, 1\,-2, -1\,-2, 2\,1, -2\,1, 2\,-1, -2\,-1.$$

Let $M = ((m_{\tilde{\sigma}\tilde{\eta}}))$ be a $2^n n! \times 2^n n!$ matrix with

$$m_{\tilde{\sigma}\tilde{\eta}} = q^{n(\tilde{\eta}\tilde{\sigma}^{-1})} \qquad \text{for} \quad \tilde{\eta}, \tilde{\sigma} \in \mathcal{B}_n. \tag{2.1.5}$$

If A ia an $n \times n$ matrix, then the *Schur power matrix* of A, denoted by $\Pi(A)$, is defined to be the $n! \times n!$ matrix

$$((\prod_{i=1}^{n} a_{\sigma(i)\tau(i)}));$$

see, for example, Merris (1987).

We will use the following notation :

1. For two matrices A and B, by $A \times B$, we mean the Kronecker product of the matrices A and B. If A and B are hermitian positive semidefinite matrices then it is well known that the Kronecker product gives rise to a hermitian positive semidefinite matrix, and
2. The Hadamard product of the two matrices $A = ((a_{ij}))$ and $B = ((b_{ij}))$ with the same dimensions and entries in a given ring is the entrywise product $A \circ B = ((a_{ij}b_{ij}))$, which has the same dimension as A and B. It is well known that if $A \geq 0$ and $B \geq 0$ are matrices of the same dimension, then $A \circ B \geq 0$.

Lemma 2.1.2: *The matrix M defined in (2.1.5) is positive semidefinite.*

Proof: The Lemma follows from Bożejko, Januszkiewics and Spatzier (1988); where it has been done for an arbitrary Coxeter group. ∎

If A is an $n \times n$ matrix, then for a complex q, the *q- permanent of A with respect to the sign change group $\mathcal{B}_n$*, denoted by $Q_q(A)$, is defined as

$$Q_q(A) = \sum_{\tilde{\sigma} \in \mathcal{B}_n} (-1)^{|J_{\tilde{\sigma}}|} q^{n(\tilde{\sigma})} \prod_{j=1}^{n} a_{j|\tilde{\sigma}(j)|} \ ;$$

where

$$|J_{\tilde{\sigma}}| = \text{card}(\{i \in \{1,2,\ldots,n\} : \tilde{\sigma}(i) \in \{-1,-2,\ldots,-n\}\}).$$

By Lemma 2.1.1,

$$Q_q(A) = \sum_{\sigma \in \mathcal{S}_n} q^{i(\sigma)} \prod_{\ell=1}^{n} \{1 - q^{2(\ell - \beta_l^{\sigma}) - 1}\} \prod_{j=1}^{n} a_{j\sigma(j)}.$$

Theorem 2.1.3 : *If $A \geq 0$ then $Q_q(A) \geq 0$ for $-1 \leq q \leq 1$.*

Proof: Let N be a $n! \times n!$ matrix. We index the rows and columns of the matrix N with the elements of the group $\mathcal{S}_n$. Define

$$(N)_{\sigma\pi} = \sum_{\tau_{\sigma,\epsilon_1}} \sum_{\tau_{\pi,\epsilon_2}} (-1)^{|J_\xi|} q^{n(\xi)} \ ; \tag{2.1.6}$$

where $\xi = \tau_{\pi,\epsilon_2}(\tau_{\sigma,\epsilon_1})^{-1}$ and both the summations run over all the 2^n elements of $\mathcal{B}_n$ for fixed σ and π in $\mathcal{S}_n$. Note that

$$|J_\xi| = |J_{\epsilon_1}| + |J_{\epsilon_2}| \qquad (\text{mod } 2).$$

We first claim that the matrix $N \geq 0$. We will show that

$$\sum_{\sigma,\pi} (N)_{\sigma\pi} r(\sigma) \overline{r(\pi)} \geq 0$$

for arbitrary $r : \mathcal{S}_n \longrightarrow \mathbb{C}$. Using (2.1.6), that is same as showing

$$\sum_{\sigma,\pi} \sum_{\tau_{\sigma,\epsilon_1}} \sum_{\tau_{\pi,\epsilon_2}} (-1)^{|J_\xi|} q^{n(\xi)} r(\sigma) \overline{r(\pi)} \geq 0$$

which is same as showing

$$\sum_{\sigma,\tau_{\sigma,\epsilon_1}} \sum_{\pi,\tau_{\pi,\epsilon_2}} (-1)^{|J_{\epsilon_1}|} r(\sigma) q^{n(\xi)} (-1)^{|J_{\epsilon_2}|} \overline{r(\pi)} \geq 0. \tag{2.1.7}$$

Let

$$y = r \otimes \Big((-1)^{|J_{\epsilon_1^1}|},\ (-1)^{|J_{\epsilon_1^2}|},\ \ldots,\ (-1)^{|J_{\epsilon_1^{2^n}}|} \Big) \in I\!R^{2^n n!} .$$

$$|J_{\epsilon_1^i}| = \text{card}\,(\{\ell \in \{1, 2, \ldots, n\} : \epsilon_1^i(\ell) = -1\})$$

for $1 \leq i \leq 2^n$, and $\tau_{\sigma,\epsilon_1^i}$ for $1 \leq i \leq 2^n$ are the 2^n elements of $\mathcal{B}_n$ corresponding to the fixed $\sigma \in \mathcal{S}_n$.

Therefore, from (2.1.5) and (2.1.7) we see that we just need to show that

$$y^* M y \geq 0\ ;$$

which by Lemma 2.1.2 is true.

Since $\Pi(A)$ is a principal submatrix of $A \times A \times \ldots \times A$ taken n times, it follows that for $A \geq 0$, $\Pi(A) \geq 0$. Therefore, the matrix $\Pi(A) \circ N \geq 0$.

Let us observe the following: For each fixed $\pi, \sigma \in \mathcal{S}_n$, we get a fixed $\eta = \sigma^{-1}\pi$. Thus, in each row and column of the matrix $((q^{n(\xi)}))_{2^n \times 2^n}$; where $\xi = (\tau_{\sigma,\epsilon_1})^{-1}\tau_{\pi,\epsilon_2}$, we get all the 2^n elements of $\mathcal{B}_n$ corresponding to the fixed $\eta \in \mathcal{S}_n$ as $\mathcal{B}_n$ is a group. Therefore,

$$\sum_{\tau_{\eta,\epsilon}} (-1)^{|J_\epsilon|} q^{n(\tau_{\eta,\epsilon})} = \frac{1}{2^n}(N)_{\sigma\pi}$$

and hence

$$\begin{aligned} Q_q(A) &= \frac{1}{2^n} \sum_{\sigma,\tau} (\Pi(A) \circ N)_{\sigma,\tau} \\ &\geq 0 \end{aligned}$$

■

Remark: If A is a $n \times n$ matrix, then for a complex q, the *q- permanent of A with respect to the Coxeter group* $\mathcal{D}_n$ *(the subgroup of* $\mathcal{B}_n$ *with* $|J_\epsilon|$ *even)*, denoted by $R_q(A)$, is defined as

$$R_q(A) = \sum_{\bar{\sigma} \in \mathcal{D}_n} q^{n(\bar{\sigma})} \prod_{j=1}^{n} a_{j|\bar{\sigma}(j)|}.$$

(i) An argument similar to the one used to prove $Q_q(A) \geq 0$ for $A \geq 0$ and $-1 \leq q \leq 1$ will show that $R_q(A) \geq 0$ for $A \geq 0$ and $-1 \leq q \leq 1$.

(ii) For each fixed $\sigma \in \mathcal{S}_n$, the closed form expression for the coefficient of $\prod_{j=1}^{n} a_{j\sigma(j)}$ is given by

$$q^{i(\sigma)} \prod_{\ell=2}^{n} (1 + q^{2(\ell-1-\beta_\ell^\sigma)}).$$

The proof of this goes on the lines of the proof adopted to get the Lemma 2.1.1 proved.

(iii) It can be easily seen that $Q_{-1}(A) = 2^n \det A$, $R_{-1}(A) = 2^{n-1} \det A$, $R_1(A) = 2^{n-1}$ per$A$, and $Q_0(A) = R_0(A) = \prod_{j=1}^{n} a_{jj}$. That is, we have constructed matrix functions which take nonnegative values on positive semidefinite matrices and which generalise the determinant in the case of $\mathcal{B}_n$, and the determinant and the permanent in the case of $\mathcal{D}_n$. The functions make use of the Coxeter groups $\mathcal{B}_n$ and $\mathcal{D}_n$. A similar function based on $\mathcal{S}_n$, denoted by $\text{per}_q(A)$, will be considered in greater detail in the next chapter.

Chapter 3

Some Inequalities for the $q-$ permanent

3.1 Introduction

Let $\mathcal{S}_n$ denote the symmetric group on n symbols. If $\sigma \in \mathcal{S}_n$, then $i(\sigma)$ will denote the number of inversions of σ. Recall that an inversion of σ is a pair $(i,j), 1 \leq i,\ j \leq n$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

We write $A \geq 0$ $(A > 0)$ to indicate that A is a hermitian positive semidefinite (positive definite) matrix. Also $A \geq B$ means that $A \geq 0,\ \ B \geq 0,$ and $A - B \geq 0$. If A is an $n \times n$ matrix, then for a complex q, the *$q-$permanent* of A, denoted by $\mathrm{per}_q(A)$ is defined as

$$\mathrm{per}_q(A) = \sum_{\sigma \in \mathcal{S}_n} q^{i(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}. \tag{3.1.1}$$

Observe that $\mathrm{per}_1(A) = \mathrm{per}\ A$, $\mathrm{per}_{-1}(A) = \det A$ and $\mathrm{per}_0(A) = \prod_{i=1}^{n} a_{ii}$; where "per" and " det" denote the permanent and determinant respectively.

It follows from a result of Bożejko and Speicher (1990) that if $A \geq 0$ then

$$\mathrm{per}_q(A) \geq 0, \qquad -1 \leq q \leq 1. \tag{3.1.2}$$

We now define the Schur $q-$ power matrix : we know that for any $n \times n$ matrix $A = ((a_{ij}))$ the Schur power matrix of A is the matrix $\Pi(A)$ of order $n! \times n!$. The rows and columns of this matrix are indexed by the elements of the group $\mathcal{S}_n$. For $\sigma,\ \tau \in \mathcal{S}_n$,

$$(\Pi(A))_{\sigma,\tau} = \prod_{i=1}^{n} a_{i\,\tau\sigma^{-1}(i)}.$$

We define, the *Schur $q-$power matrix* of the matrix A to be the matrix

$$\Pi_q(A) = \Pi(A) \circ \mathcal{M} = ((\prod_{j=1}^{n} a_{j\,\tau\sigma^{-1}(j)} q^{i(\tau\sigma^{-1})}))_{\sigma,\tau\in\mathcal{S}_n}.$$

It is conjectured in Bapat (1991) that if $A > 0$ and is not a diagonal matrix then $\mathrm{per}_q(A)$ is strictly increasing in [-1,1]. The conjecture has been verified for $n \leq 3$ in Bapat (1991). In the fourth section, for a 3×3 positive semidefinite matrix A, we show that $\mathrm{per}_q(A)$ is the largest eigenvalue of $\Pi_q(A), 0 \leq q \leq 1$. In the third section of this chapter, we settle the conjecture for a special case, namely, when the matrix A is tridiagonal. We also show that $\mathrm{per}_q(A)$ is strictly increasing in [-1, 1], for a 3×3 positive definite matrix A using a different method than the one used in Bapat (1991). In the second section, we prove a weaker form of the conjecture which generalizes Lieb's inequality, the classical Hadamard inequality as well as the permanental Hadamard inequality of Marcus (1963).

3.2 A Generalization of Lieb's Inequality

The following results are known for any $n \times n$ positive semidefinite matrix $A = ((a_{ij}))$.

In 1893, Hadamard published his celebrated theorem on determinants which states that

$$\det A \leq \Pi_{i=1}^{n} a_{ii}. \tag{3.2.1}$$

Fifteen years later, E. Fischer showed that if A is partitioned into blocks

$$A = \begin{bmatrix} B & C \\ C^{*} & D \end{bmatrix} \tag{3.2.2}$$

where B and D are square matrices then

$$\det A \leq \det B. \det D \leq \Pi_{i=1}^{n} a_{ii}. \tag{3.2.3}$$

In 1963, M. Marcus proved the Hadamard theorem for permanents, namely

$$perA \geq \Pi_{i=1}^{n} a_{ii}. \tag{3.2.4}$$

Shortly thereafter, E. H. Lieb (1966) proved the corresponding dual of (3.2.3). which shows that

$$perA \geq perB.\ perD \geq \Pi_{i=1}^{n} a_{ii}; \tag{3.2.5}$$

where A is partitioned as in (3.2.2).

The main aim of this section is to prove a generalised version of the Lieb's inequality and then obtain a generalised Hadamard inequality.

We now introduce some notation: by J_n, we mean the $n \times n$ matrix which has all the entries equal to 1.

We assume throughout this section that the elements of $\mathcal{S}_n$ have been ordered lexicographically, as we did on page 61 of Chapter 2.

For each fixed t, $1 \leq t \leq n$, we also assume throughout this section that the subsets of the numbers $\{1, 2, \ldots, n\}$ of cardinality t have been ordered in the following way:

consider two subsets $I = \{i_1 < i_2 < \ldots < i_t\}$ and $J = \{j_1 < j_2 < \ldots < j_t\}$. Clearly $|I| = |J| = t$; where $|S|$ denotes the cardinality of the set S. The subset I precedes the subset J if the first non-zero difference $i_k - j_k$ for $k = 1, 2, \ldots, t$ with $i_k \in I$ and $j_k \in J$ is negative. That is, the ordering is lexicographic.

Let S_I be the set of all permutation of the numbers from the set I. Similarly, we define the set S_J. For fixed I, the permutations in S_I are ordered lexicographically. If I precedes J then all the permutations in S_I precede all the permutations in S_J. For example, for $n = 4$ and $t = 3$, we have the ordering as

123, 132, 213, 231, 312, 321, 124, 142, 214, 241, 412, 421.

134, 143, 314, 341, 413, 431, 234, 243, 324, 342, 423, 432.

As already defined $\Pi(A) = ((\prod_{i=1}^{n} a_{\sigma(i)\tau(i)}))$ is a $n! \times n!$ matrix indexed by the elements of $\mathcal{S}_n$. Since $\Pi(A)$ is a principal submatrix of $A \times A \times \ldots \times A$, taken n times, it follows that for $A \geq 0$, $\Pi(A) \geq 0$.

For a matrix A, we define the *t-th compound matrix* $C_t(A) = ((c_{IJ}))$ (whose rows and columns are indexed by the subsets I and J of the numbers $\{1, 2, \ldots, n\}$ of cardinality t, arranged lexicographically.) of order $\binom{n}{t} \times \binom{n}{t}$ by

$$c_{IJ} = \operatorname{per}_q(A[I; J]);$$

where $\operatorname{per}_q(A[I; J])$ denotes the q- permanent of the submatrix of A which consists of the rows and columns of A corresponding to the subsets I and J respectively.

Lemma 3.2.1 : *Let $A \geq 0$ be partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ A_{21} & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \ldots \\ A_{k1} & A_{k2} & \ldots & A_{kk} \end{pmatrix};$$

where A_{ii} are square matrices, $i = 1, 2, \ldots, k$. Let $B = ((b_{ij}))$; where b_{ij} is the sum of all the entries in A_{ij}. Then $B \geq 0$.

Proof: An easy exercise gives us the result. ■

Lemma 3.2.2: *The compound matrix $C_t(A) \geq 0$ for $1 \leq t \leq n$.*

Proof: Let D be the principal submatrix of $C = \underbrace{J_n \times J_n \times \ldots J_n}_{n-t} \times \underbrace{A \times A \times \ldots \times A}_{t}$ corresponding to the rows and columns indexed by $\{\sigma : \sigma \in \mathcal{S}_n\}$ with σ arranged in lexicographic ordering defined earlier. Since $C \geq 0$, $D \geq 0$ and D is a $n! \times n!$ matrix.

Now, we take out a $\frac{n!}{(n-t)!} \times \frac{n!}{(n-t)!}$ principal submatrix E of D in the following way: for a fixed subset I of $\{1, 2, \ldots, n\}$ of cardinality t, consider a permutation $\tilde{\sigma} \in \mathcal{S}_I$. Now from the matrix D, consider the rows and columns which have been indexed by the permutation $\sigma \in \mathcal{S}_n$ with

$$\sigma(i) = \tilde{\sigma}(i) \qquad \text{for}, i = 1, 2, \ldots, t; \quad \text{and}$$

$$\sigma(j) > \sigma(j+1) \qquad \text{for } j = t+1, t+2, \ldots, n-1.$$

Once we have fixed the numbers $\{i_1 < i_2 < \ldots < i_t\}$ of the set $\{1, 2, \ldots, n\}$; we are left out with $n - t$ numbers $\{1, 2, \ldots, n\} \backslash \{i_1 < i_2 < \ldots < i_t\}$ which can be arranged in $(n-t)!$ ways. Therefore the matrix E which we have got from the matrix D is of order $\frac{n!}{(n-t)!} \times \frac{n!}{(n-t)!}$ and since $D \geq 0$, $E \geq 0$.

Let us permute the rows and columns of E symmetrically so as to get the matrix F, which has the rows and columns indexed by the ordering defined in page 67 of Chapter 3. Therefore, $F \geq 0$ as $E \geq 0$.

Consider the matrix $L = ((l_{\sigma\tau}))$ of order $t! \times t!$ whose rows and columns are indexed by the lexicographic ordering defined for S_t with $l_{\sigma\tau} = q^{i(\tau\sigma^{-1})}$. From page 9 of Bożejko and Speicher (1990), we get $L \geq 0$.

Hence the matrix $M = J_N \times L \geq 0$; where $N = \binom{n}{t}$ and thus the order of the matrix M is $(\binom{n}{t} t!) \times (\binom{n}{t} t!)$ which is same as $\frac{n!}{(n-t)!} \times \frac{n!}{(n-t)!}$.

We now get a $\binom{n}{t} \times \binom{n}{t}$ matrix $G = ((g_{IJ}))$ from the $\frac{n!}{(n-t)!} \times \frac{n!}{(n-t)!}$ matrix $F \circ M$, by taking

$$g_{IJ} = \sum_{\xi, \eta} (F \circ M)_{\xi, \eta};$$

where $\xi \in S_I$ and $\eta \in S_J$. Using Lemma 3.2.1 and the fact that $F \circ M \geq 0$, we get $G \geq 0$.

Let us now observe the entry g_{IJ} of the matrix G for a fixed subsets I and J of $\{1, 2, \ldots, n\}$ with cardinality t. We have $I = \{i_1 < i_2 < \ldots < i_t\}$ and $J = \{j_1 < j_2 < \ldots < j_t\}$. The summation over ξ and η tells us that we are summing $t! \times t!$ entries in all to get just one entry of the matrix G. This summation is nothing but the sum of all the entries of the type $(F \circ M)_{\ell_1 k_1, \ell_2 k_2, \ldots, \ell_t k_t}$; where for each i, $1 \leq i \leq t$, $\ell_i \in I$ and $k_i \in J$. But the $t! \times t!$ submatrix of $F \circ M$ corresponding to the rows and columns I and J respectively, has the same entries in each row and column as S_t is a group. Therefore, we get

$$c_{IJ} = \frac{1}{t!}\, g_{IJ}.$$

But $G \geq 0$ and hence $C_t(A) \geq 0$. ■

Theorem 3.2.3: *Let $A \geq 0$ be an $n \times n$ matrix. Suppose A is partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix};$$

where the submatrix A_{11} is of order $r \times r$, then

$$per_q(A) \geq per_q(A_{11})\; per_q(A_{22}) \quad \textit{for } 0 \leq q \leq 1.$$

Proof: For $q = 0$, we get

$$\mathrm{per}_0(A) = \prod_{i=1}^{n} a_{ii} = \prod_{i=1}^{r} a_{ii} \prod_{i=r+1}^{n} a_{ii} = \mathrm{per}_0(A_{11})\; \mathrm{per}_0(A_{22}).$$

First suppose $-1 \leq q \leq 1$, $q \neq 0$. Let

$$\alpha_t = \sum q^{i(\sigma)} \prod_{j=1}^{n} a_{j\sigma(j)};$$

where the summation is over all the permutations σ such that

$$|\, \{\sigma(1), \sigma(2), \ldots, \sigma(r)\} \cap \{1, 2, \ldots, r\} \,| = t.$$

Then $\alpha_r = \mathrm{per}_q(A_{11})\; \mathrm{per}_q(A_{22})$. We will show that

$$\alpha_t \geq 0, \;\; 0 \leq t \leq r-1.$$

For the sake of convenience, let us introduce some more notation. For a fixed t, $0 \leq t \leq r-1$:

1. by R_1 and C_1, we will mean the subsets $\{i_1 < i_2 < \ldots < i_t\}$ and $\{j_1 < j_2 < \ldots < j_t\}$ of the set $\{1, 2, \ldots, r\}$, respectively;
2. by R_2 and C_2, we will mean the subsets $\{\ell_1 < \ell_2 < \ldots < \ell_{n-2r+t}\}$ and $\{m_1 < m_2 < \ldots < m_{n-2r+t}\}$ of the set $\{r+1, r+2, \ldots, n\}$ respectively;
3. by R_1^c, we mean the subset $\{1, 2, \ldots, r\} \backslash R_1$ and similarly, we have

   $C_1^c = \{1, 2, \ldots, r\} \backslash C_1$, $R_2^c = \{r+1, r+2, \ldots, n\} \backslash R_2$, and

   $C_2^c = \{r+1, r+2, \ldots, n\} \backslash C_2$;
4. by $A[R; C]$, we mean the submatrix of A which has been taken corresponding to the rows and columns, indexed by the subsets R and C respectively.

Using the above notation, we have

$$\begin{aligned}
\alpha_t &= \sum_{R_1, R_2, C_1, C_2} q^u \mathrm{per}_q(A[R_1, C_1]) \mathrm{per}_q(A[R_2, C_2]) \mathrm{per}_q(A[R_1^c, C_2^c]) \mathrm{per}_q(A[R_2^c, C_1^c]) \\
&= q\langle C_t(A_{11}) \times C_{n-2r+t}(A_{22}) x,\ x\rangle; \quad \text{where} \qquad (3.2.6)
\end{aligned}$$

$$u = (r-t)^2 + (n-2r+t)(n+2r-t+1) - t(t+1) + \sum_{k=1}^{t} (i_k + j_k) - \sum_{k=1}^{n-2r+t} (\ell_k + m_k)$$

and x is a vector of order $\binom{r}{t}\binom{n-r}{n-2r+t} \times 1$ which is same as $\binom{r}{t}\binom{n-r}{r-t} \times 1$. We index the rows of the vector x with the lexicographic ordering defined for any subsets of the set $\{1, 2, \ldots, r\}$ and the anti-lexicographic ordering for any subsets of the set $\{r+1, r+2, \ldots, n\}$. (The anti-lexicographic ordering is the ordering got after reversing the positions in the elements in the lexicographic ordering.) The entry of the vector x corresponding to R_i's and C_i's is

$$q^k \mathrm{per}_q(A[R_1^c, C_2^c]),$$

where

$$k = \frac{(r-t)^2 - 1}{2} + \frac{(n-2r+t)(n+2r-t+1)}{2} - \frac{t(t+1)}{2} + \sum_{\ell=1}^{t} i_\ell - \sum_{\ell=1}^{n-2r+t} m_\ell.$$

Note that $(n-2r+t)(n+2r-t+1)$ and $t(t+1)$ are even numbers and since $0 \leq t \leq r-1$, $(r-t)^2 \geq 1$ we can take one q outside in (3.2.6).

Now let us see how we get the number u using the permutation, say σ, which has given rise to sets R_1, R_2, C_1 and C_2. The number u is nothing but the total number of inversions due to this permutation σ which we count as follows:

Case (i) : if $i, j \in R$ then the number of inversions is taken care of with the help of $\mathrm{per}_q(A[R;S])$; where R is any one of the sets R_1, R_2, R_1^c and R_2^c and S is the corresponding set of columns.

Case (ii) : if $i \in R_1$ and $j \in R_1^c$ then we get $\sigma(i) < \sigma(j)$, and therefore no inversion.

Case (iii) : if $i \in R_1^c$ and $j \in R_1$, we have $\sigma(i) > \sigma(j)$, and hence the number of inversions is equal to

$$\begin{aligned} & |\{(i,j) : i \in R_1^c,\ j \in R_1\ i < j\}| \\ = \; & (i_1 - 1) + (i_2 - 2) + \ldots + (i_t - t) \\ = \; & \sum_{\ell=1}^{t} i_\ell - \frac{t(t+1)}{2} \end{aligned}$$

Case (iv) : if $i \in R_1$ and $j \in R_2$ then we have $i < j$ and $\sigma(i) < \sigma(j)$. That is, there is no inversion. Similarly, if $i \in R_2$ and $j \in R_1$ then there will not be any inversion as we have $i > j$.

Case (v) : if $i \in R_1$ and $j \in R_2^c$ then we have $i < j$ and therefore, the number of inversions is equal to

$$\begin{aligned} & |\{(\ell,k) : \ell \in C_1,\ k \in C_1^c,\ \ell > k\}| \\ = \; & (j_1 - 1) + (j_2 - 2) + \ldots + (j_t - t) \\ = \; & \sum_{\ell=1}^{t} j_\ell - \frac{t(t+1)}{2} \end{aligned}$$

Case (vi) : if $i \in R_2^c$ and $j \in R_1$ then $i > j$, and hence no inversion.

Case (vii) : if $i \in R_1^c$ and $j \in R_2$ then we have $i < j$, and hence the number of inversions is equal to

$$|\{(\ell,k) : i \in C_2^c,\ k \in C_2,\ \ell > k\}|$$

$$\begin{aligned}
&= (n-m_1)+(n-m_2-1)+\ldots+(n-m_{n-2r+t}-(n-2r+t-1))\\
&= \frac{(n-2r+t)(n+2r-t+1)}{2}-\sum_{k=1}^{n-2r+t} m_k
\end{aligned}$$

Case (viii) : if $i \in R_2$ and $j \in R_1^c$ then $i > j$, and thus we don't get any inversion.

Case (ix) : if $i \in R_1^c$ and $j \in R_2^c$ then we have $i < j$ and $\sigma(i) > \sigma(j)$. Thus the number of inversion is equal to $(r-t)^2$.

Case (x) : if $i \in R_2^c$ and $j \in R_1^c$ then $i > j$, and hence no inversion.

Case (xi) : if $i \in R_2$ and $j \in R_2^c$ then we have $\sigma(i) > \sigma(j)$, and hence the number of inversions is equal to

$$\begin{aligned}
&|\{(\ell,k) : i \in R_2,\ k \in R_2^c,\ \ell < k\}|\\
&= (n-\ell_1)+(n-\ell_2-1)+\ldots+(n-\ell_{n-2r+t}-(n-2r+t-1))\\
&= \frac{(n-2r+t)(n+2r-t+1)}{2}-\sum_{k=1}^{n-2r+t} \ell_k
\end{aligned}$$

Case (xii) : if $i \in R_2^c$ and $j \in R_2$ then we have $\sigma(i) < \sigma(j)$, and therefore no inversion.

Now adding the different cases, we get the number u.

Suppose $0 < q \leq 1$. Now using Lemma 3.2.2 and the fact that Kronecker product of two positive semidefinite matrices gives rise to a positive semidefinite matrix, we get $\alpha_t \geq 0$ for $0 \leq t \leq r-1$. But then

$$\begin{aligned}
\operatorname{per}_q(A) &= \sum_{t=0}^{r} \alpha_t\\
&= \alpha_r + \sum_{t=0}^{r-1} \alpha_t
\end{aligned}$$

that is

$$\begin{aligned}
\operatorname{per}_q(A) - \operatorname{per}_q(A_{11})\operatorname{per}_q(A_{22}) &= \sum_{t=0}^{r-1} \alpha_t\\
&\geq 0.
\end{aligned}$$

Hence the result. ■

As a corollary to this, we get a generalization of Hadamard inequality for permanent as well as determinant.

Corollary 3.2.4: *Let $A \geq 0$ be an $n \times n$ matrix. Then*

$$per_q(A) \leq \prod_{i=1}^{n} a_{ii} \qquad -1 \leq q \leq 0$$

and

$$per_q(A) \geq \prod_{i=1}^{n} a_{ii} \qquad 0 \leq q \leq 1.$$

Proof: The case $0 \leq q \leq 1$ follows by a straightforward application of Theorem 3.2.3 and induction.

For $-1 \leq q < 0$, we partition the matrix A as

$$A = \begin{pmatrix} a_{11} & x^* \\ x & A(1,1) \end{pmatrix};$$

where $x^* = (a_{12}, a_{13}, \ldots, a_{1n})$ and $A(1,1)$ is the principal submatrix of A got by deleting the first row and the first column of A. Therefore, we have in this case

$$\mathrm{per}_q(A) = \sum_{t=0}^{1} \alpha_t = \alpha_0 + \alpha_1.$$

Using (3.2.6), we get $\alpha_1 = a_{11}\mathrm{per}_q(A(1,1))$ and $\alpha_0 = q\langle C_{n-2}(A(1,1))y,\ y\rangle$ and it can be easily seen that $y^* = (a_{12}, qa_{13}, \ldots, q^{n-2}a_{1n})$. Hence

$$\begin{aligned} \frac{\mathrm{per}_q(A) - a_{11}\mathrm{per}_q(A(1,1))}{q} &= \alpha_0 \\ &\geq 0. \end{aligned}$$

Since $-1 \leq q < 0$, we get $\mathrm{per}_q(A) \leq a_{11}\mathrm{per}_q(A(1,1))$ for $-1 \leq q \leq 0$. Now a repeated application gives us the required result. ∎

3.3 Monotonicity of the $q-$Permanent

The following notations will be used:

1. If $S \subset \{1, 2, \ldots, n\}$, then by $A(S)$ we mean the principal submatrix of A corresponding to S.
2. If $S = \{i, i+1, \ldots, j\}$, $1 \leq i \leq j \leq n$, then we will write $A[i, j]$ in place of $A(S)$.

   We also make the convention that $A[i, j] = 1$ (a 1×1 matrix) for $i > j$.

Let us denote by $\mathcal{S}_{n-1}^{(k)}$, the set of all bijections from $\{2, 3, \ldots, n\}$ to $\{1, 2, \ldots, \hat{k}, \ldots, n\}$; where $\hat{k}$ indicates that the index k is not included in the set $\{1, 2, \ldots, n\}$. Then we can write each $\sigma \in \mathcal{S}_n$ as $\sigma = \begin{pmatrix} 1 \longrightarrow k \\ \pi \end{pmatrix}$; where $\sigma(1) = k$ and $\pi \in \mathcal{S}_{n-1}^{(k)}$ with $\pi(l) = \sigma(l)$ for $l = 2, 3, \ldots, n$. The number of inversions $i(\pi)$ for $\pi \in \mathcal{S}_{n-1}^{(k)}$ is defined in the same way as for $\sigma \in \mathcal{S}_n$. It is easy to see that

$$i(\sigma) = k - 1 + i(\pi) \quad \text{if} \quad \sigma = \begin{pmatrix} 1 \longrightarrow k \\ \pi \end{pmatrix}.$$

The above observation gives us the following lemma:

Lemma 3.3.1: *Let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by removing its i-th row and j-th column. Let $A_{ij,k\ell}$ be the $(n-2) \times (n-2)$ matrix obtained from A by removing its rows i and k as well as columns j and ℓ, $i \neq k$, $j \neq \ell$. Then*

$$per_q(A) = a_{11} per_q(A_{11}) + \sum_{i=2}^{n} \sum_{j=2}^{n} q^{i+j-3} a_{i1} a_{1j} per_q(A_{11,ij}) \tag{3.3.1}$$

Remark : (3.3.1) is an analogue of Cauchy expansion for the determinant. ■

Lemma 3.3.2 : *Let A be an $n \times n$ tridiagonal matrix. Then*

$$per_q(A) = a_{11} per_q(A[2, n]) + q|a_{12}|^2 per_q(A[3, n]).$$

Proof: Using Lemma 3.3.1, we get

$$\begin{aligned}
\mathrm{per}_q(A) &= a_{11}\mathrm{per}_q(A_{11}) + \sum_{i=2}^{n}\sum_{j=2}^{n} q^{i+j-3}\bar{a}_{1i}a_{1j}\mathrm{per}_q(A_{11,ij}) \\
&= a_{11}\mathrm{per}_q(A_{11}) + q|a_{12}|^2\mathrm{per}_q(A_{11,22}) \\
&= a_{11}\mathrm{per}_q(A[2,n]) + q|a_{12}|^2\mathrm{per}_q(A[3,n])
\end{aligned}$$

as $a_{1i} = 0$ for $i = 3, 4, \dots, n$. Hence the result. ∎

Note that, using Lemma 3.3.2, we see that if $A > 0$ is a tridiagonal matrix and A is not a diagonal matrix then $\mathrm{per}_q(A)$ is strictly increasing for $0 \leq q \leq 1$.

Theorem 3.3.3: *Let A be an $n \times n$ tridiagonal matrix. Then for $n \geq 3$*

$$\begin{aligned}
\frac{d}{dq}per_q(A) &= |a_{12}|^2 per_q(A[3,n]) + \\
&\quad \sum_{i=0}^{n-3} per_q(A[1,i+1])|a_{i+2,i+3}|^2 per_q(A[i+4,n])
\end{aligned}$$

Proof : We shall prove the theorem by induction on the order of the matrix A. Suppose A is a 3×3 tridiagonal matrix, then using Lemma 3.3.2

$$\begin{aligned}
\frac{d}{dq}\mathrm{per}_q(A) &= \frac{d}{dq}\{a_{11}\mathrm{per}_q(A[2,3]) + q|a_{12}|^2\mathrm{per}_q(A[3,3])\} \\
&= a_{11}\frac{d}{dq}\mathrm{per}_q(A[2,3]) + |a_{12}|^2\mathrm{per}_q(A[3,3]) + \\
&\quad q|a_{12}|^2\frac{d}{dq}\mathrm{per}_q(A[3,3]) \\
&= a_{11}|a_{23}|^2 + |a_{12}|^2\mathrm{per}_q(A[3,3]) \\
&= |a_{12}|^2\mathrm{per}_q(A[3,3]) + \mathrm{per}_q(A[1,1])|a_{23}|^2\mathrm{per}_q(A[4,3])
\end{aligned}$$

which is same as the RHS for $n = 3$, as by convention $\mathrm{per}_q(A[4,3]) = 0$.

Let the theorem be true for all tridiagonal positive semidefinite matrices A of order n, $1 \leq n \leq m$. We shall now prove it for the matrix A of order m+1. We have, using Lemma 3.3.2,

$$\begin{aligned}
& \frac{d}{dq}\mathrm{per}_q(A) \\
= \; & \frac{d}{dq}\{a_{11}\mathrm{per}_q(A[2,m+1]) + q|a_{12}|^2\mathrm{per}_q(A[3,m+1])\} \\
= \; & a_{11}\frac{d}{dq}\mathrm{per}_q(A[2,m+1]) + |a_{12}|^2\mathrm{per}_q(A[3,m+1])\} + \\
& q|a_{12}|^2\frac{d}{dq}\mathrm{per}_q(A[3,m+1]) \\
= \; & a_{11}\sum_{i=1}^{m+1-3}\mathrm{per}_q(A[2,i+1])|a_{i+2,i+3}|^2\mathrm{per}_q(A[i+4,m+1]) + \\
& |a_{12}|^2\mathrm{per}_q(A[3,m+1]) + a_{11}|a_{23}|^2\mathrm{per}_q(A[4,m+1]) + \\
& q|a_{12}|^2\sum_{i=2}^{m+1-3}\mathrm{per}_q(A[3,i+1])|a_{i+2,i+3}|^2\mathrm{per}_q(A[i+4,m+1]) + \\
& q|a_{12}|^2|a_{34}|^2\mathrm{per}_q(A[5,m+1]) \quad \text{using induction} \\
= \; & |a_{12}|^2\mathrm{per}_q(A[3,m+1]) + a_{11}|a_{23}|^2\mathrm{per}_q(A[4,m+1]) + \\
& q|a_{12}|^2\sum_{i=2}^{m-2}\mathrm{per}_q(A[3,i+1])|a_{i+2,i+3}|^2\mathrm{per}_q(A[i+4,m+1]) + \\
& a_{11}\sum_{i=1}^{m-2}\mathrm{per}_q(A[2,i+1])|a_{i+2,i+3}|^2\mathrm{per}_q(A[i+4,m+1]) + \\
& q|a_{12}|^2|a_{34}|^2\mathrm{per}_q(A[5,m+1]) \\
= \; & |a_{12}|^2\mathrm{per}_q(A[3,m+1]) + a_{11}|a_{23}|^2\mathrm{per}_q(A[4,m+1]) + \\
& [a_{11}\mathrm{per}_q(A[2,2]) + q|a_{12}|^2]|a_{34}|^2\mathrm{per}_q(A[5,m+1]) + \\
& \sum_{i=2}^{m-2}[a_{11}\mathrm{per}_q(A[2,i+1]) + q|a_{12}|^2\mathrm{per}_q(A[3,i+1])]|a_{i+2,i+3}|^2\mathrm{per}_q(A[i+4,m+1]) \\
= \; & |a_{12}|^2\mathrm{per}_q(A[3,m+1]) + \mathrm{per}_q(A[1,1])|a_{23}|^2\mathrm{per}_q(A[4,m+1]) + \\
& \sum_{i=2}^{m-2}\mathrm{per}_q(A[1,i+1])|a_{i+2,i+3}|^2\mathrm{per}_q(A[i+4,m+1]) + \\
& \mathrm{per}_q(A[1,2])|a_{34}|^2\mathrm{per}_q(A[5,m+1]), \quad \text{using Lemma 3.3.2,} \\
= \; & |a_{12}|^2\mathrm{per}_q(A[3,m+1]) + \sum_{i=0}^{m+1-3}\mathrm{per}_q(A[1,i+1])|a_{i+2,i+3}|^2\mathrm{per}_q(A[i+4,m+1])
\end{aligned}$$

and hence the theorem is proved. ■

Corollary 3.3.4 : *If A is a tridiagonal positive semidefinite matrix and A is not a diagonal matrix, then $\frac{d}{dq}per_q(A)$ is positive for $q \in (-1,0)$.*

Proof: We shall prove the result by induction on the order of the matrix A. Suppose A is a 3×3 tridiagonal matrix, then from Theorem 3.3.3, we get

$$\begin{aligned}\frac{d}{dq}\mathrm{per}_q(A) &= a_{11}|a_{23}|^2 + |a_{12}|^2 a_{33} \\ &> 0, \quad \text{as } A \text{ is not a diagonal matrix and } A > 0.\end{aligned}$$

Let the result be true for all matrices A of order n, $1 \leq n \leq m$. For $n = m+1$, we have from Theorem 3.3.3

$$\begin{aligned}\frac{d}{dq}\mathrm{per}_q(A) &= |a_{12}|^2 \mathrm{per}_q(A[3, m+1]) + \\ &\sum_{i=0}^{m-2} \mathrm{per}_q(A[1, i+1])|a_{i+2,i+3}|^2 \mathrm{per}_q(A[i+4, m+1])\end{aligned}$$

As observed in Bapat (1991), if $A > 0$ then $\mathrm{per}_q(A) \geq \det A$ and the induction hypothesis implies $\mathrm{per}_q(A) > 0$ for $q \in (-1,0)$.

Hence $\frac{d}{dq}\mathrm{per}_q(A)$ is positive for $q \in (-1,0)$. ■

Note that the Corollary 3.3.4 is valid for $q \in (0,1)$ as well.

It follows from the Corollary 3.3.4 that if $A > 0$ is a tridiagonal matrix and A is not a diagonal matrix, then $\mathrm{per}_q(A)$ is strictly increasing for $q \in [-1,1]$.

Let $D_1 = \mathrm{diag}\ (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $D_2 = \mathrm{diag}\ (\beta_1, \beta_2, \cdots, \beta_n)$ be two diagonal matrices of order n. We note that, for any matrix A

$$\mathrm{per}_q(D_1 A D_2) = \sum_{\sigma \in S_n} q^{i(\sigma)} \prod_{j=1}^{n} (D_1 A D_2)_{j\sigma(j)}$$

$$\begin{aligned}
&= \sum_{\sigma\in\mathcal{S}_n} q^{i(\sigma)} \prod_{j=1}^{n} \alpha_{jj}(A)_{j\sigma(j)}\beta_{\sigma(j)\sigma(j)} \\
&= \sum_{\sigma\in\mathcal{S}_n} q^{i(\sigma)} \prod_{j=1}^{n}(A)_{j\sigma(j)} \prod_{j=1}^{n} \alpha_{jj} \prod_{k=1}^{n} \beta_{kk} \\
&= \prod_{j=1}^{n} \alpha_{jj}\{\sum_{\sigma\in\mathcal{S}_n} q^{i(\sigma)} \prod_{j=1}^{n}(A)_{j\sigma(j)}\} \prod_{k=1}^{n} \beta_{kk} \\
&= \mathrm{per}_q(D_1)\mathrm{per}_q(A)\mathrm{per}_q(D_2) \qquad (3.3.2)
\end{aligned}$$

Therefore, without loss of generality, we can assume that the diagonal entries of any matrix to consist just of 1's.

We now prove the following lemma.

Lemma 3.3.5 : *Let A be a 3×3 positive semidefinite matrix. Then $per_q(A)$ is strictly increasing in q.*

Proof: Because of (3.3.2), we can take the matrix A to be

$$A = \begin{pmatrix} 1 & d & e \\ d & 1 & f \\ e & \overline{f} & 1 \end{pmatrix}.$$

Since A is positive semidefinite, we must have $d^2,\ e^2,\ |f|^2 \leq 1$. We have, by definition

$$\mathrm{per}_q(A) = 1 + q(d^2 + |f|^2) + 2q^2\mathrm{Re}(def) + q^3e^2. \qquad (3.3.3)$$

Therefore, differentiating (3.3.3) with respect to q, we get

$$\frac{d\mathrm{per}_q(A)}{dq} = d^2 + |f|^2 + 4q\mathrm{Re}(def) + 3q^2e^2. \qquad (3.3.4)$$

This is a quadratic equation in q with

$$\left.\frac{d\mathrm{per}_q(A)}{dq}\right|_{q=0} = d^2 + |f|^2 \geq 0.$$

Let us denote the discriminant of the quadratic equation (3.3.4) by D. We now show that the discrimanant D, of the quadratic equation is negative, which in turn. will imply that $\frac{d\mathrm{per}_q(A)}{dq}$ is nonnegative for all q; $-1 \leq q \leq 1$. Consider

$$\begin{aligned}
D &= 16\{\mathrm{Re}(def)\}^2 - 4 \times 3e^2 \times (d^2 + |f|^2) \\
&= 4e^2\{4d^2(\ \mathrm{Re}(f))^2 - 3d^2 - 3|f|^2\} \\
&= 4e^2\{3d^2(\mathrm{Re}(f))^2 + d^2(\mathrm{Re}(f))^2 - 3d^2 - 3|f|^2\} \\
&\leq 4e^2\{3d^2[(\mathrm{Re}(f))^2 - 1] + d^2|f|^2 - 3|f|^2\} \\
&= 4e^2\{3d^2[(\mathrm{Re}(f))^2 - 1] + |f|^2[d^2 - 3]\} \\
&< 0
\end{aligned}$$

as $(\mathrm{Re}(f))^2 - 1 \leq 0$ and $d^2 - 3 < 0$. Hence the result. ∎

3.4 $\mathrm{per}_q(A)$ as the largest eigenvalue

In this section, following the technique in [3], we show that $\mathrm{per}_q(A)$ is the largest eigenvalue of the Schur $q-$ power matrix $\Pi_q(A)$, for a 3×3 positive semidefinite matrix A, with $0 \leq q \leq 1$.

We have $\mathrm{per}_q(A) = \sum_{\sigma \in \mathcal{S}_n} q^{i(\sigma)} \ \prod_{j=1}^{n} a_{j\sigma(j)}$ and $\mathcal{M} = ((q^{i(\tau\sigma^{-1})}))_{\sigma,\tau \in \mathcal{S}_n}$. It can be easily checked that

$$\Pi_q(A)\ e = \mathrm{per}_q(A)\ e;$$

where $e \in I\!R^{n!}$ with all entries 1.

Theorem 3.4.1 : *For* $0 \leq q \leq 1$, *$\mathrm{per}_q(A)$ is the largest eigenvalue of* $\Pi_q(A)$.

Proof: Let us order the elements of $\mathcal{S}_n$, lexicographically. Let

$$A = \begin{pmatrix} 1 & d & e \\ d & 1 & f \\ e & \overline{f} & 1 \end{pmatrix}$$

be a 3×3 positive semidefinite matrix. So we have the condition on the entries as $d^2,\ e^2,\ |f|^2 \leq 1$. Now

$$\Pi(A) = \begin{bmatrix} 1 & |f|^2 & d^2 & def & de\overline{f} & e^2 \\ |f|^2 & 1 & def & d^2 & e^2 & de\overline{f} \\ d^2 & de\overline{f} & 1 & e^2 & |f|^2 & def \\ de\overline{f} & d^2 & e^2 & 1 & def & |f|^2 \\ def & e^2 & |f|^2 & de\overline{f} & 1 & d^2 \\ e^2 & def & de\overline{f} & |f|^2 & d^2 & 1 \end{bmatrix}.$$

It can be seen that

$$\Pi_q(A) = \begin{bmatrix} 1 & q|f|^2 & qd^2 & q^2def & q^2de\overline{f} & q^3e^2 \\ q|f|^2 & 1 & q^2def & qd^2 & q^3e^2 & q^2de\overline{f} \\ qd^2 & q^2de\overline{f} & 1 & q^3e^2 & q|f|^2 & q^2def \\ q^2de\overline{f} & qd^2 & q^3e^2 & 1 & q^2def & q|f|^2 \\ q^2def & q^3e^2 & q|f|^2 & q^2de\overline{f} & 1 & qd^2 \\ q^3e^2 & q^2def & q^2de\overline{f} & q|f|^2 & qd^2 & 1 \end{bmatrix}.$$

Let

$$\widehat{\Pi_q(A)} = \begin{bmatrix} 1 + q|f|^2 & qd^2 + q^2def & q^2de\overline{f} + q^3e^2 \\ qd^2 + q^2de\overline{f} & 1 + q^3e^2 & q|f|^2 + q^2def \\ q^2def + q^3e^2 & q|f|^2 + q^2de\overline{f} & 1 + qd^2 \end{bmatrix}$$

which is obtained from $\Pi_q(A)$ after adding the two by two blocks and then dividing by $2!$.

The distinct eigenvalues of $\Pi_q(A)$ are $\mathrm{per}_q(A)$, $-\mathrm{per}_q(A)$, and two others which appear in pairs. It can be seen that if λ is an eigenvalue of $\widehat{\Pi_q(A)}$, then it is an eigenvalue of $\Pi_q(A)$, as well. Except for $-\mathrm{per}_q(A)$ the other three different eigenvalues of $\Pi_q(A)$, clearly are the eigenvalues of $\widehat{\Pi_q(A)}$. We will show that

$$\mathrm{per}_q(A)I - \widehat{\Pi_q(A)} \geq 0.$$

Clearly $\mathrm{per}_q(A)$ is an eigenvalue of $\widehat{\Pi_q(A)}$. We have

$$\mathrm{per}_q(A)I - \widehat{\Pi_q(A)} = \begin{bmatrix} \alpha+\beta & -\alpha & -\beta \\ -\overline{\alpha} & \overline{\alpha}+\gamma & -\gamma \\ -\overline{\beta} & -\overline{\gamma} & \overline{\beta+\gamma} \end{bmatrix};$$

where $\alpha = qd^2 + q^2 def$, $\beta = q^2 de\overline{f} + q^3 e^2$, and $\gamma = q|f|^2 + q^2 def$.

We observe the following:

(i)

$$\begin{aligned} \alpha+\beta &= q\{d^2 + qdef\} + q^2\{de\overline{f} + qe^2\} \\ &= \mathrm{per}_q(A) - \mathrm{per}_q(A(1,1)) \\ &\geq 0, \quad \text{using Hadamard inequality with } a_{11} = 1, \end{aligned}$$

where $A(1,1)$ is the 2×2 matrix obtained from A by deleting the first row and the first column.

(ii)

$$\begin{vmatrix} \alpha+\beta & -\alpha \\ -\overline{\alpha} & \overline{\alpha}+\gamma \end{vmatrix} = \alpha\,\gamma + \beta\,\gamma + \overline{\alpha}\,\beta$$

$$\begin{aligned} &\alpha\,\gamma + \beta\,\gamma + \overline{\alpha}\,\beta \\ =\ & q^2\{d^2 + qdef\}\{|f|^2 + qdef\} + q^3\{de\overline{f} + qe^2\}\{|f|^2 + qdef\} \\ &+ q^3\{de\overline{f} + qe^2\}\{d^2 + qde\overline{f}\} \\ =\ & q^2\{(d^2 + q^2e^2)|f|^2 + q^2e^2d^2 + q^2d^2e^2|f|^2 + 2q^2(\mathrm{Re}(def))^2 \\ &+ 2(qd^2 + q|f|^2 + q^3e^2)\mathrm{Re}(def)\} \\ =\ & q^2\{(d^2 + q^2e^2)|f|^2 + q^2e^2d^2 + q^2d^2e^2|f|^2 + 2q^2(\mathrm{Re}(def))^2 \\ &+ 2q(d^2 + |f|^2 + q^2e^2)\mathrm{Re}(def)\} \qquad (3.4.1) \end{aligned}$$

Let $f = a + ib$. We have $|f|^2 = a^2 + b^2$, $\mathrm{Re}(def) = dea$, and $(\mathrm{Re}(def))^2 = d^2e^2(\mathrm{Re}(f))^2 = d^2e^2(a^2 - b^2)$. Therefore, (3.4.1) can be written as

$q^2\{(d^2 + q^2e^2)(a^2 + b^2) + q^2e^2d^2 + q^2d^2e^2(a^2 + b^2) + 2q^2(a^2 - b^2)d^2e^2 + 2q(d^2 + a^2 + b^2 -$

$q^2e^2)dea\}$ which is same as
$q^2\{3q^2d^2e^2a^2+(d^2+q^2e^2+2qdea)b^2+2q(d^2+a^2+q^2e^2)dea+q^2(1-b^2)d^2e^2+a^2(d^2+q^2e^2)\}$.
Here

1.

$$\begin{aligned} & d^2+q^2e^2+2qdea \\ \geq\ & d^2+q^2e^2-2q|dea| \\ \geq\ & d^2+q^2e^2-2qde \qquad (|a|\leq|f|\leq 1) \\ =\ & (d-qe)^2 \\ \geq\ & 0. \end{aligned}$$

2. $d^2e^2a^2 \geq 0$.

3. $2q(d^2+a^2+q^2e^2)dea+q^2(1-b^2)d^2e^2+a^2(d^2+q^2e^2) \geq 0$ if $dea \geq 0$. Let $dea \leq 0$, then we have

$2q(d^2+a^2+q^2e^2)dea+q^2(1-b^2)d^2e^2+a^2(d^2+q^2e^2)$
$= x^tTx+2q^2(q-1)e^2dea$ which is nonnegative; where $x=(qde, ad, qea)$ and

$$T=\begin{pmatrix} 1-b^2 & d & e \\ d & 1 & a \\ e & a & 1 \end{pmatrix}.$$

To complete the proof, it would suffice to prove that $T \geq 0$. Without loss of generality, we assume that the matrix A is positive definite (The general case can then be proved by approximation). In this case, every principal submatrix of A is positive definite, in particular, $1-|f|^2>0$. Hence,

$$\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1-a^2 \geq 1-|f|^2 > 0.$$

Also,

$$\begin{aligned} \det T &= (1-b^2)(1-a^2)+2dea-d^2-e^2 \\ &= a^2b^2+\det A \\ &\geq \det A \ > \ 0. \end{aligned}$$

These two inequalities prove that T is positive definite, and the proof is complete.

(iii) $\det(\mathrm{per}_q(A)I - \widehat{\Pi_q(A)}) = 0$ as the row sums are separately zero.

Thus $\mathrm{per}_q(A)I - \widehat{\Pi_q(A)}$ is positive semidefinite matrix. Hence the result. ∎

3.5 Concluding Remarks

In this chapter, we have made the following two conjectures:

(i) Let $A \geq 0$. Then $\mathrm{per}_q(A)$ is strictly increasing for $q \in (-1, 1)$; and

(ii) $\mathrm{per}_q(A)$ is the largest eigenvalue of $\Pi_q(A)$ for $q \in [0, 1]$.

Both the conjectures have been verified for a 3×3 matrix A. In this section, we show that the second conjecture immediately implies the first conjecture for $q \in (0, 1)$. Before, coming to the proof, note the following result which is already in the literature: Let $A \geq 0$ and $B \geq 0$ be two matrices of the same order. Let us denote by $\rho(A)$ the largest eigenvalue of the matrix A. Then

$$\rho(A \circ B) \leq \rho(A) \times \text{ the largest diagonal entry of } B. \tag{3.5.1}$$

Let $q_1, q_2 \in (0, 1)$ with $q_1 \leq q_2$. We claim that $\mathrm{per}_{q_1}(A) \leq \mathrm{per}_{q_2}(A)$.

By definition, $\Pi_{q_1}(A) = \Pi(A) \circ \mathcal{M}_1$; where $\mathcal{M}_1 = ((q_1{}^{l(\sigma\tau^{-1})}))_{\sigma,\tau \in S_n}$ and hence $\mathcal{M}_1 \geq 0$. But

$$\begin{aligned} q_1{}^{l(\sigma\tau^{-1})} &= (\frac{q_1}{q_2} \cdot q_2)^{l(\sigma\tau^{-1})} \\ &= (\frac{q_1}{q_2})^{l(\sigma\tau^{-1})} q_2{}^{l(\sigma\tau^{-1})}. \end{aligned}$$

Let us define,

$$\mathcal{M}_2 = ((q_2{}^{l(\sigma\tau^{-1})}))_{\sigma,\tau \in S_n}; \quad \text{and} \quad \mathcal{M}_3 = (((\frac{q_1}{q_2})^{l(\sigma\tau^{-1})}))_{\sigma,\tau \in S_n}.$$

Since $q_1 \leq q_2$, $\frac{q_1}{q_2} \leq 1$ and hence $\mathcal{M}_3 \geq 0$. We also have $\mathcal{M}_2 \geq 0$. Clearly,

$$\Pi_{q_1}(A) = \Pi(A) \circ \mathcal{M}_2 \circ \mathcal{M}_3.$$

Now using (3.5.1), we see that

$$\begin{aligned}
\rho(\Pi_{q_1}(A)) &= \rho(\Pi(A) \circ \mathcal{M}_2 \circ \mathcal{M}_3) \\
&\leq \rho(\Pi(A) \circ \mathcal{M}_2) \times \text{ the largest diagonal entry of } \mathcal{M}_3 \\
&= \rho(\Pi_{q_2}(A)) \times 1
\end{aligned}$$

$$\text{that is,} \qquad \mathrm{per}_{q_1}(A) \leq \mathrm{per}_{q_2}(A).$$

Hence the claim holds true.

In view of the above remark and the second conjecture it seems natural to conjecture the following: Let $A \geq 0$. Then, $\mathrm{per}_{-q}(A)$ is the smallest eigenvalue of $\Pi_q(A)$ for $0 < q < 1$.

But by an example, we show that this conjecture is false. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and } q \in (0,1].$$

Consider

$$\Pi_q(A) = \Pi(A) \circ \mathcal{M} = \begin{bmatrix} 1 & q & q & q^2 & q^2 & q^3 \\ q & 1 & q^2 & q & q^3 & q^2 \\ q & q^2 & 1 & q^3 & q & q^2 \\ q^2 & q & q^3 & 1 & q^2 & q \\ q^2 & q^3 & q & q^2 & 1 & q \\ q^3 & q^2 & q^2 & q & q & 1 \end{bmatrix}.$$

The eigenvalues of $\Pi_q(A)$ are $\mathrm{per}_q(A)$, $\mathrm{per}_{-q}(A)$ and two other eigenvalues which appear in pairs. We will show that $\mathrm{per}_{-q}(A)$ is not the smallest eigenvalue of $\Pi_q(A)$. To show this, we consider the matrix

$$\widehat{\Pi_q(A)} = \begin{bmatrix} 1-q & -q+q^2 & q^2-q^3 \\ -q+q^2 & 1-q^3 & -q+q^2 \\ q^2-q^3 & -q+q^2 & 1-q \end{bmatrix}.$$

Here again, it can be seen that if λ is an eigenvalue of $\widehat{\Pi_q(A)}$ then it is an eigenvalue of $\Pi_q(A)$ as well. Except for $\mathrm{per}_q(A)$, the other three different eigenvalue of $\Pi_q(A)$ are clearly the eigenvalues of $\widehat{\Pi_q(A)}$. We have

$$\mathrm{per}_{-q}(A) = 1 - 2q + 2q^2 - q^3.$$

Then $\mathrm{per}_{-q}(A)$ is an eigenvalue of $\widehat{\Pi_q(A)}$ corresponding to the eigenvector $(1,1,1)^t$; where t denotes the transpose. We now show that

$$\widehat{\Pi_q(A)} - \mathrm{per}_{-q}(A)I$$

is not a nonnegative definite matrix. We have

$$\widehat{\Pi_q(A)} - \mathrm{per}_{-q}(A)I = \begin{bmatrix} q - 2q^2 + q^3 & -q + q^2 & q^2 - q^3 \\ -q + q^2 & 2q - 2q^2 & -q + q^2 \\ q^2 - q^3 & -q + q^2 & q - 2q^2 + q^3 \end{bmatrix},$$

that is,

$$\widehat{\Pi_q(A)} - \mathrm{per}_{-q}(A)I = \begin{bmatrix} -\alpha - \beta & \alpha & \beta \\ \alpha & -\alpha - \gamma & \gamma \\ \beta & \gamma & -\beta - \gamma \end{bmatrix};$$

where $\alpha = -q + q^2$, $\beta = q^2 - q^3$, and $\gamma = -q + q^2$. It can be easily seen that $\det(\widehat{\Pi_q(A)} - \mathrm{per}_{-q}(A)I) = 0$. We now see the following:

(i) $-\alpha - \beta = q - 2q^2 + q^3 = q(1 - 2q + q^2) = q(1-q)^2 \geq 0$.

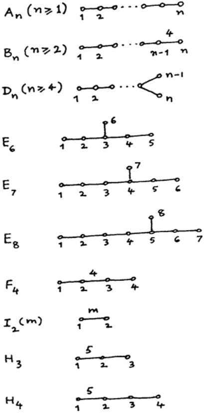
(ii)

$$\begin{aligned} & \begin{vmatrix} -\alpha - \beta & \alpha \\ \alpha & -\alpha - \gamma \end{vmatrix} \\ = \; & \alpha\gamma + \alpha\beta + \beta\gamma \\ = \; & \alpha\gamma + \beta(\alpha + \gamma) \\ = \; & (-q + q^2)(-q + q^2) + (q^2 - q^3)(-q + q^2 - q + q^2) \\ = \; & q^2\{(1-q)^2 + 2(1-q)(-1+q)q\} \\ = \; & q^2(1-q)^2(1-2q) \\ < \; & 0 \end{aligned}$$

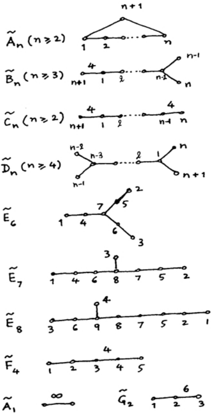
for $q \in (\frac{1}{2}, 1)$. Hence $\mathrm{per}_{-q}(A)$ is not the smallest eigenvalue of $\widehat{\Pi_q(A)}$. ■

Appendix 1

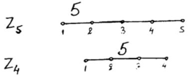
C_1



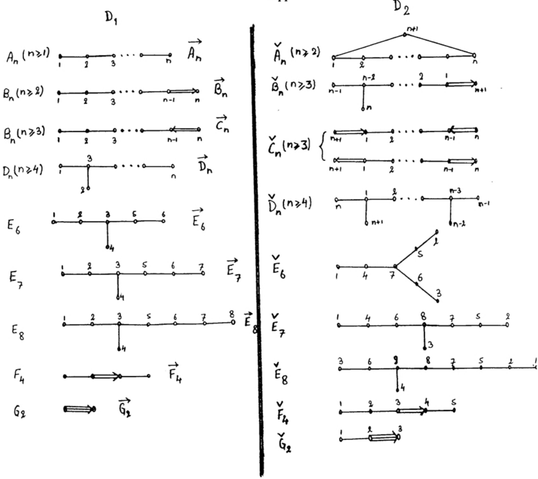
C_2



C_3



Appendix 2


D₁
A_n (n≥1) 1 2 3 n →A_n
B_n (n≥2) 1 2 3 n-1 n →B_n
B_n (n≥3) 1 2 3 n-1 n →C_n
D_n (n≥4) 1 2 3 n →D_n
E_6 1 2 3 4 5 6 →E_6
E_7 1 2 3 4 5 6 7 →E_7
E_8 1 2 3 4 5 6 7 8 →E_8
F_4 →F_4
G_2 →G_2
D₂
Ǎ_n (n≥2) 1 2 n n+1
B̌_n (n≥3) n-1 n-2 2 1 n+1 n
Č_n (n≥3) n+1 1 2 n-1 n
n+1 1 2 n-1 n
Ď_n (n≥4) n 1 2 n-3 n-1 n+1 n-2
Ě_6 1 4 7 5 2 6 3
Ě_7 1 4 6 8 7 5 2 3
Ě_8 3 6 9 8 7 5 2 1 4
F̌_4 1 2 3 4 5
Ǧ_2 1 2 3

Appendix 3: Path-matrices

Table 1.1 $\tilde{E}_6$

k even

$k =$	$6n$		$6n+2$		$6n+4$	
entry	n odd	n even	n odd	n even	n odd	n even
$(1,1)$	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$
$(1,2)$	$\frac{n-1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n+2}{2}$
$(1,7)$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$
$(4,4)$	$2n+1$	$2n+1$	$2n+1$	$2n+1$	$2n+1$	$2n+1$
$(4,5)$	$2n$	$2n$	$2n$	$2n$	$2n+2$	$2n+2$
$(7,7)$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$

k odd

$k =$	$6n+1$	$6n+3$	$6n+5$
(1,4)	$n+1$	n	n+1
(1,5)	n	n+1	n+1
(4,7)	3n+1	3n+2	3n+3

Table 1.2 $\tilde{F}_4$

k even

$k =$	$6n$		$6n+2$		$6n+4$	
entry	n odd	n even	n odd	n even	n odd	n even
(1,1)	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$
(1,3)	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$
(1,5)	$\frac{n-1}{\sqrt{2}}$	$\frac{n}{\sqrt{2}}$	$\frac{n+1}{\sqrt{2}}$	$\frac{n}{\sqrt{2}}$	$\frac{n+1}{\sqrt{2}}$	$\frac{n+2}{\sqrt{2}}$
(2,2)	2n+1	2n+1	2n+1	2n+1	2n+1	2n+1
(2,4)	$2n\sqrt{2}$	$2n\sqrt{2}$	$(2n+1)\sqrt{2}$	$(2n+1)\sqrt{2}$	$(2n+2)\sqrt{2}$	$(2n+2)\sqrt{2}$
(3,3)	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$
(3,5)	$\frac{3n+1}{\sqrt{2}}$	$\frac{3n}{\sqrt{2}}$	$\frac{3n+1}{\sqrt{2}}$	$\frac{3n+2}{\sqrt{2}}$	$\frac{3n+3}{\sqrt{2}}$	$\frac{3n+2}{\sqrt{2}}$
(4,4)	4n+1	4n+1	4n+2	4n+2	4n+3	4n+3
(5,5)	n	n+1	n+1	n	n	n+1

k odd

$k=$	$6n+1$	$6n+3$	$6n+5$
(1, 2)	$n+1$	n	n+1
(1,4)	$n\sqrt{2}$	$(n+1)\sqrt{2}$	$(n+1)\sqrt{2}$
(2,3)	3n+1	3n+2	3n+3
(2,5)	$n\sqrt{2}$	$(n+1)\sqrt{2}$	$(n+1)\sqrt{2}$
(3,4)	$(3n+1)\sqrt{2}$	$(3n+2)\sqrt{2}$	$(3n+3)\sqrt{2}$
(4,5)	2n+1	2n+1	2n+2

Table 1.3 $\tilde{G}_2$

$k=$	$2n$		$2n+1$
entry	n odd	n even	
(1,1)	$\frac{n-1}{2}$	$\frac{n+2}{2}$	0
(1,2)	0	0	n+1
(1,3)	$\frac{(n+1)\sqrt{3}}{2}$	$\frac{n\sqrt{3}}{2}$	0
(2,2)	2n+1	2n+1	0
(2,3)	0	0	$(n+1)\sqrt{3}$
(3,3)	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	0

Table 1.4 $\tilde{A}_1$

$k=$	2n	2n+1
entry		
(1,1)	n+1	0
(1,2)	0	n+1
(2,2)	n+1	0

Table 1.5 $\tilde{E}_7$

k even

$k =$	$12n$		$12n+2$		$12n+4$	
entry	n odd	n even	n odd	n even	n odd	n even
(1, 1)	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$
(1, 2)	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$
(1,3)	n	n	n	n	n+1	n+1
(1, 6)	$\frac{3n-1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n}{2}$
(1, 7)	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$
(3, 3)	$2n+1$	$2n+1$	$2n$	$2n$	$2n+1$	$2n+1$
(3, 6)	$3n$	$3n$	$3n+1$	$3n+1$	$3n+1$	$3n+1$
(4, 4)	$2n$	$2n+1$	$2n$	$2n+1$	$2n+1$	$2n$
(4, 5)	$2n+1$	$2n$	$2n+1$	$2n$	$2n$	$2n+1$
(4, 8)	$4n$	$4n$	$4n+1$	$4n+1$	$4n+2$	$4n+2$
(6, 6)	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$
(6, 7)	$\frac{9n+1}{2}$	$\frac{9n}{2}$	$\frac{9n+3}{2}$	$\frac{9n+2}{2}$	$\frac{9n+3}{2}$	$\frac{9n+4}{2}$
(8,8)	8n+1	8n+1	8n+2	8n+2	8n+3	8n+3

k even

$k =$	$12n+6$		$12n+8$		$12n+10$	
entry	n odd	n even	n odd	n even	n odd	n even
(1, 1)	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$
(1, 2)	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$
(1,3)	n	n	n+1	n+1	n+1	n+1
(1, 6)	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+4}{2}$
(1, 7)	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$
(3, 3)	$2n+1$	$2n+1$	$2n+2$	$2n+2$	$2n+1$	$2n+1$
(3, 6)	$3n+2$	$3n+2$	$3n+2$	$3n+2$	$3n+3$	$3n+3$
(4, 4)	$2n+2$	$2n+1$	$2n+1$	$2n+2$	$2n+1$	$2n+2$
(4, 5)	$2n+1$	$2n+2$	$2n+2$	$2n+1$	$2n+2$	$2n+1$
(4, 8)	$4n+2$	$4n+2$	$4n+3$	$4n+3$	$4n+4$	$4n+4$
(6, 6)	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$
(6, 7)	$\frac{9n+5}{2}$	$\frac{9n+6}{2}$	$\frac{9n+7}{2}$	$\frac{9n+6}{2}$	$\frac{9n+9}{2}$	$\frac{9n+8}{2}$
(8,8)	8n+5	8n+5	8n+6	8n+6	8n+7	8n+7

k odd

$k =$	$12n+1$		$12n+3$	$12n+5$	
entry	**n odd**	**n even**		**n odd**	**n even**
(1,4)	n	n+1	n	n+1	n
(1,5)	n+1	n	n	n	n+1
(1,8)	2n	2n	2n+1	2n+1	2n+1
(3,4)	2n	2n	2n+1	2n+1	2n+1
(3,8)	4n+1	4n+1	4n+1	4n+2	4n+2
(4,6)	3n	3n+1	3n+1	3n+2	3n+1
(4,7)	3n+1	3n	3n+1	3n+1	3n+2
(6,8)	6n+1	6n+1	6n+2	6n+3	6n+3

k odd

$k =$	$12n+7$	$12n+9$		$12n+11$
entry		n odd	n even	
(1,4)	n+1	n	n+1	n+1
(1,5)	n+1	n+1	n	n+1
(1,8)	2n+1	2n+2	2n+2	2n+2
(3,4)	2n+1	2n+2	2n+2	2n+2
(3,8)	4n+3	4n+3	4n+3	4n+4
(4,6)	3n+2	3n+3	3n+2	3n+3
(4,7)	3n+2	3n+2	3n+3	3n+3
(6,8)	6n+4	6n+5	6n+5	6n+6

Table 1.6 $\tilde{E}_8$

k even

$k =$	$30n$		$30n+2$		$30n+4$		$30n+6$	
entry	n odd	n even	n odd	n even	n odd	n even	n odd	n even
$(1,1)$	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$
$(1,4)$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n-1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$
$(1,5)$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n-1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n-1}{2}$	$\frac{3n}{2}$
$(1,6)$	$2n$	$2n$	$2n$	$2n$	$2n$	$2n$	$2n+1$	$2n+1$
$(1,8)$	$\frac{5n-1}{2}$	$\frac{5n}{2}$	$\frac{5n+1}{2}$	$\frac{5n}{2}$	$\frac{5n+1}{2}$	$\frac{5n+2}{2}$	$\frac{5n+1}{2}$	$\frac{5n}{2}$
$(2,2)$	$2n+1$	$2n+1$	$2n+1$	$2n+1$	$2n$	$2n$	$2n$	$2n$
$(2,3)$	$2n$	$2n$	$2n$	$2n$	$2n$	$2n$	$2n+1$	$2n+1$
$(2,7)$	$4n$	$4n$	$4n+1$	$4n+1$	$4n+1$	$4n+1$	$4n$	$4n$
$(2,9)$	$6n$	$6n$	$6n$	$6n$	$6n+1$	$6n+1$	$6n+2$	$6n+2$
$(3,3)$	$2n+1$	$2n+1$	$2n$	$2n$	$2n$	$2n$	$2n+1$	$2n+1$
$(3,7)$	$4n$	$4n$	$4n$	$4n$	$4n+1$	$4n+1$	$4n+1$	$4n+1$
$(3,9)$	$6n$	$6n$	$6n+1$	$6n+1$	$6n+1$	$6n+1$	$6n+1$	$6n+1$
$(4,4)$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+1}{2}$	$\frac{9n}{2}$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+3}{2}$	$\frac{9n+2}{2}$
$(4,5)$	$\frac{9n-1}{2}$	$\frac{9n}{2}$	$\frac{9n+1}{2}$	$\frac{9n}{2}$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+3}{2}$	$\frac{9n+2}{2}$
$(4,6)$	$6n$	$6n$	$6n+1$	$6n+1$	$6n+1$	$6n+1$	$6n+1$	$6n+1$
$(4,8)$	$\frac{15n+1}{2}$	$\frac{15n}{2}$	$\frac{15n+1}{2}$	$\frac{15n+2}{2}$	$\frac{15n+3}{2}$	$\frac{15n+2}{2}$	$\frac{15n+3}{2}$	$\frac{15n+4}{2}$
$(5,5)$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+3}{2}$	$\frac{9n+2}{2}$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+1}{2}$	$\frac{9n}{2}$
$(5,6)$	$6n$	$6n$	$6n$	$6n$	$6n+1$	$6n+1$	$6n+2$	$6n+2$
$(5,8)$	$\frac{15n+1}{2}$	$\frac{15n}{2}$	$\frac{15n+1}{2}$	$\frac{15n+2}{2}$	$\frac{15n+3}{2}$	$\frac{15n+2}{2}$	$\frac{15n+3}{2}$	$\frac{15n+4}{2}$
$(6,6)$	$8n+1$	$8n+1$	$8n+1$	$8n+1$	$8n+1$	$8n+1$	$8n+2$	$8n+2$
$(6,8)$	$10n$	$10n$	$10n+1$	$10n+1$	$10n+2$	$10n+2$	$10n+2$	$10n+2$
$(7,7)$	$8n+1$	$8n+1$	$8n+1$	$8n+1$	$8n+1$	$8n+1$	$8n+2$	$8n+2$
$(7,9)$	$12n$	$12n$	$12n+1$	$12n+1$	$12n+2$	$12n+2$	$12n+3$	$12n+3$
$(8,8)$	$\frac{25n+1}{2}$	$\frac{25n+2}{2}$	$\frac{25n+3}{2}$	$\frac{25n+2}{2}$	$\frac{25n+3}{2}$	$\frac{25n+4}{2}$	$\frac{25n+7}{2}$	$\frac{25n+6}{2}$
$(9,9)$	$18n+1$	$18n+1$	$18n+2$	$18n+2$	$18n+3$	$18n+3$	$18n+4$	$18n+4$

k even

$k =$	$30n+8$		$30n+10$		$30n+12$		$30n+14$	
entry	**n odd**	**n even**	**n odd**	**n even**	**n odd**	**n even**	**n odd**	**n even**
$(1,1)$	$\frac{n-1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$
$(1,4)$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$
$(1,5)$	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$
$(1,6)$	$2n+1$	$2n+1$	$2n$	$2n$	$2n+1$	$2n+1$	$2n+1$	$2n+1$
$(1,8)$	$\frac{5n+1}{2}$	$\frac{5n+2}{2}$	$\frac{5n+3}{2}$	$\frac{5n+2}{2}$	$\frac{5n+1}{2}$	$\frac{5n+2}{2}$	$\frac{5n+3}{2}$	$\frac{5n+2}{2}$
$(2,2)$	$2n$	$2n$	$2n+1$	$2n+1$	$2n+2$	$2n+2$	$2n+1$	$2n+1$
$(2,3)$	$2n+1$	$2n+1$	$2n$	$2n$	$2n+1$	$2n+1$	$2n+1$	$2n+1$
$(2,7)$	$4n+1$	$4n+1$	$4n+2$	$4n+2$	$4n+2$	$4n+2$	$4n+2$	$4n+2$
$(2,9)$	$6n+2$	$6n+2$	$6n+2$	$6n+2$	$6n+2$	$6n+2$	$6n+3$	$6n+3$
$(3,3)$	$2n$	$2n$	$2n+1$	$2n+1$	$2n+1$	$2n+1$	$2n+1$	$2n+1$
$(3,7)$	$4n+1$	$4n+1$	$4n+2$	$4n+2$	$4n+1$	$4n+1$	$4n+2$	$4n+2$
$(3,9)$	$6n+2$	$6n+2$	$6n+2$	$6n+2$	$6n+3$	$6n+3$	$6n+3$	$6n+3$
$(4,4)$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$	$\frac{9n+3}{2}$	$\frac{9n+4}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$
$(4,5)$	$\frac{9n+3}{2}$	$\frac{9n+4}{2}$	$\frac{9n+3}{2}$	$\frac{9n+2}{2}$	$\frac{9n+3}{2}$	$\frac{9n+4}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$
$(4,6)$	$6n+2$	$6n+2$	$6n+2$	$6n+2$	$6n+3$	$6n+3$	$6n+3$	$6n+3$
$(4,8)$	$\frac{15n+5}{2}$	$\frac{15n+4}{2}$	$\frac{15n+5}{2}$	$\frac{15n+6}{2}$	$\frac{15n+7}{2}$	$\frac{15n+6}{2}$	$\frac{15n+7}{2}$	$\frac{15n+8}{2}$
$(5,5)$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+3}{2}$	$\frac{9n+2}{2}$	$\frac{9n+1}{2}$	$\frac{9n+2}{2}$	$\frac{9n+1}{2}$	$\frac{9n}{2}$
$(5,6)$	$6n$	$6n$	$6n$	$6n$	$6n+1$	$6n+1$	$6n+2$	$6n+2$
$(5,8)$	$\frac{15n+5}{2}$	$\frac{15n+4}{2}$	$\frac{15n+5}{2}$	$\frac{15n+6}{2}$	$\frac{15n+7}{2}$	$\frac{15n+6}{2}$	$\frac{15n+7}{2}$	$\frac{15n+8}{2}$
$(6,6)$	$8n+2$	$8n+2$	$8n+3$	$8n+3$	$8n+4$	$8n+4$	$8n+4$	$8n+4$
$(6,8)$	$10n+3$	$10n+3$	$10n+4$	$10n+4$	$10n+4$	$10n+4$	$10n+5$	$10n+5$
$(7,7)$	$8n+2$	$8n+2$	$8n+3$	$8n+3$	$8n+4$	$8n+4$	$8n+4$	$8n+4$
$(7,9)$	$12n+4$	$12n+4$	$12n+4$	$12n+4$	$12n+5$	$12n+5$	$12n+6$	$12n+6$
$(8,8)$	$\frac{25n+7}{2}$	$\frac{25n+8}{2}$	$\frac{25n+9}{2}$	$\frac{25n+8}{2}$	$\frac{25n+11}{2}$	$\frac{25n+12}{2}$	$\frac{25n+13}{2}$	$\frac{25n+12}{2}$
$(9,9)$	$18n+5$	$18n+5$	$18n+7$	$18n+7$	$18n+8$	$18n+8$	$18n+9$	$18n+9$

k even

$k =$	$30n+16$		$30n+18$		$30n+20$		$30n+22$	
entry	n odd	n even	n odd	n even	n odd	n even	n odd	n even
(1,1)	$\frac{n-1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$
(1,4)	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$
(1,5)	$\frac{3n+1}{2}$	$\frac{3n}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+4}{2}$
(1,6)	2n+1	2n+1	2n+2	2n+2	2n+1	2n+1	2n+1	2n+1
(1,8)	$\frac{5n+3}{2}$	$\frac{5n+4}{2}$	$\frac{5n+3}{2}$	$\frac{5n+2}{2}$	$\frac{5n+3}{2}$	$\frac{5n+4}{2}$	$\frac{5n+5}{2}$	$\frac{5n+4}{2}$
(2,2)	2n	2n	2n+1	2n+1	2n+2	2n+2	2n+2	2n+2
(2,3)	2n+1	2n+1	2n+2	2n+2	2n+1	2n+1	2n+1	2n+1
(2,7)	4n+2	4n+2	4n+2	4n+2	4n+3	4n+3	4n+4	4n+4
(2,9)	6n+4	6n+4	6n+4	6n+4	6n+4	6n+4	6n+4	6n+4
(3, 3)	$2n+1$	$2n+1$	$2n+1$	$2n+1$	$2n+2$	$2n+2$	2n+1	2n+1
(3, 7)	$4n+3$	$4n+3$	$4n+2$	$4n+2$	$4n+3$	$4n+3$	4n+3	4n+3
(3,9)	6n+3	6n+3	6n+4	6n+4	6n+4	6n+4	6n+5	6n+5
(4, 4)	$\frac{9n+5}{2}$	$\frac{9n+6}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$	$\frac{9n+7}{2}$	$\frac{9n+6}{2}$
(4, 5)	$\frac{9n+5}{2}$	$\frac{9n+6}{2}$	$\frac{9n+7}{2}$	$\frac{9n+6}{2}$	$\frac{9n+5}{2}$	$\frac{9n+6}{2}$	$\frac{9n+7}{2}$	$\frac{9n+6}{2}$
(4,6)	6n+3	6n+3	6n+4	6n+4	6n+4	6n+4	6n+5	6n+5
(4,8)	$\frac{15n+9}{2}$	$\frac{15n+8}{2}$	$\frac{15n+9}{2}$	$\frac{15n+10}{2}$	$\frac{15n+11}{2}$	$\frac{15n+10}{2}$	$\frac{15n+11}{2}$	$\frac{15n+12}{2}$
(5,5)	$\frac{9n+3}{2}$	$\frac{9n+4}{2}$	$\frac{9n+5}{2}$	$\frac{9n+4}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$	$\frac{9n+9}{2}$	$\frac{9n+8}{2}$
(5,6)	6n+4	6n+4	6n+4	6n+4	6n+4	6n+4	6n+4	6n+4
(5,8)	$\frac{15n+9}{2}$	$\frac{15n+8}{2}$	$\frac{15n+9}{2}$	$\frac{15n+10}{2}$	$\frac{15n+11}{2}$	$\frac{15n+10}{2}$	$\frac{15n+11}{2}$	$\frac{15n+12}{2}$
(6,6)	8n+4	8n+4	8n+5	8n+5	8n+6	8n+6	8n+6	8n+6
(6,8)	10n+6	10n+6	10n+6	10n+6	10n+7	10n+7	10n+8	10n+8
(7,7)	8n+4	8n+4	8n+5	8n+5	8n+6	8n+6	8n+6	8n+6
(7,9)	12n+7	12n+7	12n+8	12n+8	12n+8	12n+8	12n+9	12n+9
(8,8)	$\frac{25n+13}{2}$	$\frac{25n+14}{2}$	$\frac{25n+17}{2}$	$\frac{25n+16}{2}$	$\frac{25n+17}{2}$	$\frac{25n+18}{2}$	$\frac{25n+19}{2}$	$\frac{25n+18}{2}$
(9,9)	18n+10	18n+10	18n+11	18n+11	18n+13	18n+13	18n+14	18n+14

k even

$k =$	$30n + 24$		$30n + 26$		$30n + 28$	
entry	n odd	n even	n odd	n even	n odd	n even
$(1,1)$	$\frac{n+1}{2}$	$\frac{n+2}{2}$	$\frac{n+1}{2}$	$\frac{n}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$
$(1,4)$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+4}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$
$(1,5)$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$	$\frac{3n+1}{2}$	$\frac{3n+2}{2}$	$\frac{3n+3}{2}$	$\frac{3n+2}{2}$
$(1,6)$	$2n+2$	$2n+2$	$2n+2$	$2n+2$	$2n+2$	$2n+2$
$(1,8)$	$\frac{5n+3}{2}$	$\frac{5n+4}{2}$	$\frac{5n+5}{2}$	$\frac{5n+4}{2}$	$\frac{5n+5}{2}$	$\frac{5n+6}{2}$
$(2,2)$	$2n+2$	$2n+2$	$2n+1$	$2n+1$	$2n+1$	$2n+1$
$(2,3)$	$2n+2$	$2n+2$	$2n+2$	$2n+2$	$2n+2$	$2n+2$
$(2,7)$	$4n+3$	$4n+3$	$4n+3$	$4n+3$	$4n+4$	$4n+4$
$(2,9)$	$6n+5$	$6n+5$	$6n+6$	$6n+6$	$6n+6$	$6n+6$
$(3,3)$	$2n + 2$	$2n + 2$	$2n + 2$	$2n + 2$	$2n + 1$	$2n + 1$
$(3,7)$	$4n + 3$	$4n + 3$	$4n + 4$	$4n + 4$	$4n + 4$	$4n + 4$
$(3,9)$	$6n+5$	$6n+5$	$6n+5$	$6n+5$	$6n+6$	$6n+6$
$(4,4)$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$	$\frac{9n+9}{2}$	$\frac{9n+8}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$
$(4,5)$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$	$\frac{9n+9}{2}$	$\frac{9n+8}{2}$	$\frac{9n+9}{2}$	$\frac{9n+10}{2}$
$(4,6)$	$6n+5$	$6n+5$	$6n+5$	$6n+5$	$6n+6$	$6n+6$
$(4,8)$	$\frac{15n+13}{2}$	$\frac{15n+12}{2}$	$\frac{15n+13}{2}$	$\frac{15n+14}{2}$	$\frac{15n+15}{2}$	$\frac{15n+14}{2}$
$(5,5)$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$	$\frac{9n+7}{2}$	$\frac{9n+6}{2}$	$\frac{9n+7}{2}$	$\frac{9n+8}{2}$
$(5,6)$	$6n+5$	$6n+5$	$6n+6$	$6n+6$	$6n+6$	$6n+6$
$(5,8)$	$\frac{15n+13}{2}$	$\frac{15n+12}{2}$	$\frac{15n+13}{2}$	$\frac{15n+14}{2}$	$\frac{15n+15}{2}$	$\frac{15n+14}{2}$
$(6,6)$	$8n+7$	$8n+7$	$8n+7$	$8n+7$	$8n+7$	$8n+7$
$(6,8)$	$10n+8$	$10n+8$	$10n+9$	$10n+9$	$10n+10$	$10n+10$
$(7,7)$	$8n+7$	$8n+7$	$8n+7$	$8n+7$	$8n+7$	$8n+7$
$(7,9)$	$12n+10$	$12n+10$	$12n+11$	$12n+11$	$12n+12$	$12n+12$
$(8,8)$	$\frac{25n+21}{2}$	$\frac{25n+22}{2}$	$\frac{25n+23}{2}$	$\frac{25n+22}{2}$	$\frac{25n+23}{2}$	$\frac{25n+24}{2}$
$(9,9)$	$18n+15$	$18n+15$	$18n+16$	$18n+16$	$18n+17$	$18n+17$

k odd

k = entry	30n+1	30n+3	30n+5	30n+7	30n+9	30n+11	30n+13
(1,2)	n+1	n	n	n	n	n+1	n+1
(1,3)	n	n	n	n+1	n	n	n+1
(1,7)	2n	2n+1	2n	2n	2n+1	2n+1	2n+1
(1,9)	3n	3n	3n+1	3n+1	3n+1	3n+1	3n+1
(2,4)	3n	3n	3n+1	3n+1	3n+1	3n+1	3n+1
(2,5)	3n+1	3n+1	3n	3n	3n+1	3n+2	3n+2
(2,6)	4n	4n	4n+1	4n+2	4n+2	4n+1	4n+2
(2,8)	5n	5n+1	5n+1	5n+1	5n+2	5n+2	5n+2
(3,4)	3n	3n+1	3n	3n+1	3n+1	3n+1	3n+2
(3,5)	3n	3n	3n+1	3n+1	3n+1	3n+1	3n+1
(3,6)	4n+1	4n	4n+1	4n+1	4n+2	4n+2	4n+2
(3,8)	5n	5n+1	5n+1	5n+1	5n+2	5n+2	5n+2
(4,7)	6n	6n+1	6n+1	6n+2	6n+2	6n+2	6n+3
(4,9)	9n+1	9n+1	9n+2	9n+2	9n+3	9n+4	9n+4
(5,7)	6n+1	6n+1	6n+1	6n+1	6n+2	6n+3	6n+3
(5,9)	9n	9n+1	9n+2	9n+3	9n+3	9n+3	9n+4
(6,7)	8n	8n+1	8n+2	8n+2	8n+3	8n+3	8n+3
(6,9)	12n+1	12n+2	12n+2	12n+3	12n+4	12n+5	12n+6
(7,8)	10n+1	10n+1	10n+2	10n+3	10n+3	10n+4	10n+5
(8,9)	15n+1	15n+2	15n+3	15n+4	15n+5	15n+6	15n+7

k odd

k = entry	30n+15	30n+17	30n+19	30n+21	30n+23	30n+25	30n+27	30n+29
(1,2)	n	n	n+1	n+1	n+1	n+1	n	n+1
(1,3)	n	n+1	n+1	n	n+1	n+1	n+1	n+1
(1,7)	2n+1	2n+1	2n+1	2n+2	2n+2	2n+1	2n+2	2n+2
(1,9)	3n+1	3n+2	3n+2	3n+2	3n+2	3n+2	3n+3	3n+3
(2,4)	3n+2	3n+2	3n+2	3n+2	3n+2	3n+3	3n+3	3n+3
(2,5)	3n+1	3n+1	3n+2	3n+3	3n+3	3n+2	3n+2	3n+3
(2,6)	4n+2	4n+3	4n+3	4n+2	4n+3	4n+4	4n+4	4n+4
(2,8)	5n+3	5n+3	5n+3	5n+4	5n+4	5n+4	5n+5	5n+5
(3,4)	3n+1	3n+2	3n+2	3n+2	3n+3	3n+2	3n+3	3n+3
(3,5)	3n+2	3n+2	3n+2	3n+2	3n+2	3n+3	3n+3	3n+3
(3,6)	4n+2	4n+2	4n+3	4n+3	4n+3	4n+4	4n+3	4n+4
(3,8)	5n+3	5n+3	5n+3	5n+4	5n+4	5n+4	5n+5	5n+5
(4,7)	6n+3	6n+4	6n+4	6n+4	6n+5	6n+5	6n+6	6n+6
(4,9)	9n+5	9n+5	9n+6	9n+7	9n+7	9n+8	9n+8	9n+9
(5,7)	6n+3	6n+3	6n+4	6n+5	6n+5	6n+5	6n+5	6n+6
(5,9)	9n+5	9n+6	9n+6	9n+6	9n+7	9n+8	9n+9	9n+9
(6,7)	8n+5	8n+5	8n+5	8n+6	8n+6	8n+7	8n+8	8n+8
(6,9)	12n+6	12n+7	12n+8	12n+9	12n+10	12n+10	12n+11	12n+12
(7,8)	10n+5	10n+6	10n+7	10n+7	10n+8	10n+9	10n+9	10n+10
(8,9)	15n+8	15n+9	15n+10	15n+11	15n+12	15n+13	15n+14	15n+15

Appendix 4

Let us write $\lambda = 2\cos(\pi/5) \simeq 1.618$. We will use the observations : (i) $\lambda^2 = \lambda + 1$, and (ii) $\lambda(\lambda - 1) = \lambda^2 - \lambda = 1$.

Lemma 1: *Z_4 is path-positive.*

Proof: For convenience, we will denote the adjacency matrix of Z_4 by Z. Thus

$$Z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The adjacency matrix is symmetric with respect to diagonal positions $(1,4), (2,3), (3,2)$ and $(4,1)$. The matrix is symmetric in the usual sense as well. Hence for showing Z is path-positive it is enough to show the following:

For each nonnegative integer k, $(P_k(Z))_{ij} \geq 0$ for

$$(i,j) \in S = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3)\}.$$

We shall prove this by induction on k. For $k = 1$, $P_k(Z) = Z$, which is a nonnegative matrix. Let the Lemma be true for all k, $1 \leq k \leq m$ and consider $k = m + 1$.

We claim that

$$(P_t(Z))_{3i} - (P_{t-1}(Z))_{2i} \geq 0 \text{ for } i = 1, 2, \tag{1}$$

and for any nonnegative integer t, $1 \leq t \leq m + 1$.

The claim is obviously true for $t = 1$. Let it be true for all t, $1 \leq t \leq u \leq m$ and consider $t = u + 1$. The L.H.S. of (1) for $t = u + 1$ is

$$(P_{u+1}(Z))_{3i} - (P_u(Z))_{2i}$$

$$
\begin{aligned}
&= (ZP_u(Z) - P_{u-1}(Z))_{3i} - (P_u(Z))_{2i} \\
&= \lambda(P_u(Z))_{2i} + (P_u(Z))_{4i} - (P_{u-1}(Z))_{3i} - (P_u(Z))_{2i} \\
&= (\lambda - 1)\{ZP_{u-1}(Z) - P_{u-2}(Z)\}_{2i} - (P_{u-1}(Z))_{3i} + \\
&\quad \{ZP_{u-1}(Z) - P_{u-2}(Z)\}_{4i} \\
&= (\lambda - 1)\{(P_{u-1}(Z))_{1i} + \lambda(P_{u-1}(Z))_{3i}\} - (P_{u-1}(Z))_{3i} - (P_{u-2}(Z))_{4i} - \\
&\quad (\lambda - 1)(P_{u-2}(Z))_{2i} + (P_{u-1}(Z))_{3i} \\
&= (P_{u-1}(Z))_{3i} + (\lambda - 1)[(P_{u-1}(Z))_{1i} - (P_{u-2}(Z))_{2i}] - (P_{u-2}(Z))_{4i} \\
&= \{ZP_{u-2}(Z) - P_{u-3}(Z)\}_{3i} - (P_{u-2}(Z))_{4i} - (\lambda - 1)(P_{u-2}(Z))_{2i} + \\
&\quad (\lambda - 1)\{ZP_{u-2}(Z) - P_{u-3}(Z)\}_{1i} \\
&= \lambda(P_{u-2}(Z))_{2i} + (P_{u-2}(Z))_{4i} - (P_{u-3}(Z))_{3i} - (P_{u-2}(Z))_{4i} + \\
&\quad (\lambda - 1)\{P_{u-2}(Z))_{2i} - (P_{u-3}(Z))_{1i} - (P_{u-2}(Z))_{2i}\} \\
&= \lambda(P_{u-2}(Z))_{2i} - (\lambda - 1)(P_{u-3}(Z))_{1i} - (P_{u-3}(Z))_{3i} \\
&= \lambda\{ZP_{u-3}(Z) - P_{u-4}(Z)\}_{2i} - (P_{u-3}(Z))_{3i} - (\lambda - 1)(P_{u-3}(Z))_{1i} \\
&= \lambda(P_{u-3}(Z))_{1i} + \lambda^2(P_{u-3}(Z))_{3i} - (P_{u-3}(Z))_{3i} - \\
&\quad \lambda(P_{u-4}(Z))_{2i} - (\lambda - 1)(P_{u-3}(Z))_{1i} \\
&= (P_{u-3}(Z))_{1i} + \lambda\{(P_{u-3}(Z))_{3i} - (P_{u-4}(Z))_{2i}\} \\
&\geq 0
\end{aligned}
$$

using the two induction hypotheses. Hence the claim is true. Consider the following cases:

Case (i): Using claim, we have

$$
\begin{aligned}
(P_{m+1}(Z))_{11} &= \{ZP_m(Z) - P_{m-1}(Z)\}_{11} \\
&= (P_m(Z))_{21} - (P_{m-1}(Z))_{11} \\
&= \{ZP_{m-1}(Z) - P_{m-2}(Z)\}_{21} - (P_{m-1}(Z))_{11} \\
&= \lambda(P_{m-1}(Z))_{31} - (P_{m-2}(Z))_{21}
\end{aligned}
$$

$$\geq\ (P_{m-1}(Z))_{31} - (P_{m-2}(Z))_{21}\ \geq\ 0$$

Case (ii): Using the claim again, we have

$$\begin{aligned}
(P_{m+1}(Z))_{14} &= (P_{m+1}(Z))_{41} \;=\; \{ZP_m(Z) - P_{m-1}(Z)\}_{41} \\
&= (P_m(Z))_{31} - (P_{m-1}(Z))_{41} \\
&= \{ZP_{m-1}(Z) - P_{m-2}(Z)\}_{31} - (P_{m-1}(Z))_{41} \\
&= \lambda(P_{m-1}(Z))_{21} - (P_{m-2}(Z))_{31} \\
&= \lambda\{ZP_{m-2}(Z) - P_{m-3}(Z)\}_{21} - (P_{m-2}(Z))_{31} \\
&= \lambda\{(P_{m-2}(Z))_{11} + \lambda(P_{m-2}(Z))_{31} - (P_{m-3}(Z))_{21}\} - \\
&\quad (P_{m-2}(Z))_{31} \\
&= \lambda(P_{m-2}(Z))_{11} + \lambda\{(P_{m-2}(Z))_{31} - (P_{m-2}(Z))_{21}\}\ \geq\ 0
\end{aligned}$$

Case (iii): For any nonnegative integer k, we have

$$(P_{k+1}(Z))_{11} = (P_k(Z))_{21} - (P_{k-1}(Z))_{11}$$

and hence

$$(P_{m+1}(Z))_{12} = (P_{m+1}(Z))_{21} = (P_{m+2}(Z))_{11} + (P_m(Z))_{11}\ \geq\ 0.$$

Case (iv):

$$\begin{aligned}
(P_{m+1}(Z))_{22} &= \{ZP_m(Z) - P_{m-1}(Z)\}_{22} \\
&= \lambda(P_m(Z))_{32} + (P_m(Z))_{12} - (P_{m-1}(Z))_{22} \\
&\geq\ (P_m(Z))_{12} + (P_m(Z))_{32} - (P_{m-1}(Z))_{22}\ \geq\ 0
\end{aligned}$$

by (1), we have

$$(P_{m+1}(Z))_{13} = (P_{m+1}(Z))_{31} \geq 0; \quad \text{and}$$

$$(P_{m+1}(Z))_{23} = (P_{m+1}(Z))_{32} \geq 0.$$

Hence considering the above four cases we get the required result. ∎

Lemma 2: *Z_5 is path-positive.*

Proof: The adjacency matrix is

$$Z_5 = \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 \\ \lambda & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For convenience, we will denote the matrix $P_k(Z_5)$ simply by P^k. Thus

$$P^k = Z_5 P^{k-1} - P^{k-2}\ , \quad k = 2, 3, \ldots \text{ and } P^0 = I.$$

We have to show P^k is nonnegative for all positive integer k. We will use induction on k.

For $k = 1$, we have

$$P^k = P^1 = Z_5 \geq 0.$$

Suppose P^k is nonnegative for all k, $1 \leq k \leq m$ and consider $k = m+1$. That is to show for all $1 \leq i < j \leq 5$, $P_{ij}^{m+1} \geq 0$.

We claim that, for $3 \leq u \leq m+1$

$$\lambda[P_{2i}^u - P_{1i}^{u-1}] - (\lambda - 1)P_{5i}^{u-3} \geq 0. \qquad \textbf{(2)}$$

It can be easily checked that the claim holds true for $u = 3, 4, \ldots, 8$.

Let the claim be true for $3 \leq u \leq t-1 \leq m$ and consider $u = t$. For $u = t$, the L.H.S. of **(2)**

$$= \lambda[P_{2i}^t - P_{1i}^{t-1}] - (\lambda - 1)P_{5i}^{t-3}$$

$$\begin{aligned}
&= \lambda\{\lambda P_{1i}^{t-1} + P_{3i}^{t-1} - P_{2i}^{t-2} - P_{1i}^{t-1}\} - (\lambda-1)P_{5i}^{t-3} \\
&= \lambda\{(\lambda-1)P_{1i}^{t-1} + P_{3i}^{t-1} - P_{2i}^{t-2}\} - (\lambda-1)P_{5i}^{t-3} \\
&= \lambda\{(\lambda-1)[\lambda P_{2i}^{t-2} - P_{1i}^{t-3}] + P_{4i}^{t-2} - P_{3i}^{t-3}\} - (\lambda-1)[P_{4i}^{t-4} - P_{5i}^{t-5}] \\
&= \lambda\{P_{2i}^{t-2} - (\lambda-1)P_{1i}^{t-3}]\} + \lambda\{P_{5i}^{t-3} - P_{4i}^{t-4}\} - (\lambda-1)[P_{4i}^{t-4} - P_{5i}^{t-5}] \\
&= \lambda\{\lambda P_{1i}^{t-3} + P_{3i}^{t-3} - P_{2i}^{t-4} - (\lambda-1)P_{1i}^{t-3}\} + \\
&\quad\ \lambda\{P_{5i}^{t-3} - P_{4i}^{t-4}\} - (\lambda-1)[P_{3i}^{t-5} - P_{4i}^{t-6}] \\
&= \lambda\{P_{1i}^{t-3} + P_{4i}^{t-4} - P_{3i}^{t-5}\} - \lambda P_{5i}^{t-5} - (\lambda-1)[P_{3i}^{t-5} - P_{4i}^{t-6}] \\
&= \lambda\{P_{1i}^{t-3} + P_{4i}^{t-4} - P_{3i}^{t-5} - P_{5i}^{t-5}\} - (\lambda-1)[P_{2i}^{t-6} - P_{3i}^{t-7}] \\
&= \lambda\{\lambda P_{2i}^{t-4} - P_{1i}^{t-5} - P_{4i}^{t-6}\} - (\lambda-1)[P_{2i}^{t-6} - P_{3i}^{t-7}] \\
&= \lambda\{\lambda[\lambda P_{1i}^{t-5} + P_{3i}^{t-5} - P_{2i}^{t-6}] - P_{1i}^{t-5} - P_{4i}^{t-6}\} - (\lambda-1)[P_{2i}^{t-6} - P_{3i}^{t-7}] \\
&= \lambda\{\lambda[P_{1i}^{t-5} + P_{3i}^{t-5} - P_{2i}^{t-6}] - P_{4i}^{t-6}\} - (\lambda-1)[P_{2i}^{t-6} - P_{3i}^{t-7}] \\
&= \lambda\{\lambda P_{1i}^{t-5} - P_{2i}^{t-6} + (\lambda-1)[P_{3i}^{t-5} - P_{2i}^{t-6}]\} + \\
&\quad\ \lambda\{P_{3i}^{t-5} - P_{4i}^{t-6}\} - (\lambda-1)[P_{2i}^{t-6} - P_{3i}^{t-7}] \\
&= \lambda\{\lambda[P_{2i}^{t-6} - P_{1i}^{t-7}] + (\lambda-1)[P_{4i}^{t-6} - P_{3i}^{t-7}]\} + \\
&\quad\ \lambda\{P_{2i}^{t-6} - P_{3i}^{t-7}\} - (\lambda-1)[P_{2i}^{t-6} - P_{3i}^{t-7}] \\
&= \lambda\{\lambda[P_{2i}^{t-6} - P_{1i}^{t-7}] + (\lambda-1)[P_{5i}^{t-7} - P_{4i}^{t-8}]\} + P_{2i}^{t-6} - P_{3i}^{t-7} \\
&= \lambda\{\lambda[P_{2i}^{t-6} - P_{1i}^{t-7}] - (\lambda-1)P_{5i}^{t-9}\} + \lambda P_{1i}^{t-7} - P_{2i}^{t-8} \\
&= \lambda\{\lambda[P_{2i}^{t-6} - P_{1i}^{t-7}] - (\lambda-1)P_{5i}^{t-9}\} + \lambda[P_{2i}^{t-8} - P_{1i}^{t-9}] \\
&\geq 0.
\end{aligned}$$

The first term is nonnegative using the induction assumption for (**2**). The second term is nonnegative as by using (**2**) for $u = t - 8$, we get

$$\lambda[P_{2i}^{t-8} - P_{1i}^{t-9}] - (\lambda-1)P_{5i}^{t-11} \geq 0,$$

that is,

$$\lambda[P_{2i}^{t-8} - P_{1i}^{t-9}] \geq (\lambda-1)P_{5i}^{t-11} \geq 0$$

using the first induction hypothesis.

Hence (2) is true for all i, $1 \leq i \leq 5$ and all positive integer u, $3 \leq u \leq m+1$.

Therefore, using the first induction hypothesis, we get

$$P_{2i}^{u} - P_{1i}^{u-1} \geq \frac{\lambda - 1}{\lambda} P_{5i}^{u-3} \geq 0. \tag{3}$$

We now show that for all $1 \leq i < j \leq 5$, $P_{ij}^{m+1} \geq 0$.

Case (i): For $1 \leq i \leq 5$, we have using (3)

$$P_{1i}^{m+1} = \lambda P_{2i}^{m} - P_{1i}^{m-1} \geq P_{2i}^{m} - P_{1i}^{m-1} \geq 0.$$

Case (ii): For $2 \leq i \leq 5$, using the first induction hypothesis, we get

$$P_{2i}^{m+1} \geq P_{1i}^{m} \geq 0.$$

Case (iii): For $3 \leq i \leq 5$, we have

$$\begin{aligned} P_{3i}^{m+1} &= P_{2i}^{m} + P_{4i}^{m} - P_{3i}^{m-1} = \lambda P_{1i}^{m-1} - P_{2i}^{m-2} + P_{4i}^{m} \\ &= \lambda\{P_{2i}^{m-2} - P_{1i}^{m-3}\} + P_{4i}^{m} \geq 0. \end{aligned}$$

Case (iv): For $4 \leq i \leq 5$, we have

$$\begin{aligned} P_{4i}^{m+1} &= P_{3i}^{m} + P_{5i}^{m} - P_{4i}^{m-1} = P_{2i}^{m-1} - P_{3i}^{m-2} + P_{5i}^{m} \\ &= \lambda P_{1i}^{m-2} - P_{2i}^{m-3} + P_{5i}^{m} = \lambda\{P_{2i}^{m-3} - P_{1i}^{m-4}\} + P_{5i}^{m} \\ &\geq 0. \end{aligned}$$

Case (v): For $i = 5$, we have

$$\begin{aligned} P_{55}^{m+1} &= P_{45}^{m} - P_{55}^{m-1} = P_{35}^{m-1} - P_{45}^{m-2} = P_{25}^{m-2} - P_{35}^{m-3} \\ &= \lambda P_{15}^{m-3} - P_{25}^{m-4} = \lambda\{P_{25}^{m-4} - P_{15}^{m-5}\} \geq 0. \end{aligned}$$

Hence the result. ∎