

Fourier Analysis: M. Math II: Final Examination

November 17, 2025.

Maximum Marks 60

Time 3 hrs.

State any result you use.

(1) Answer (e) and any three from (a), (b), (c) and (d). [7 + 3 × 6 = 25]

(a) Let f be a continuous function on \mathbb{R} which is periodic of period π , i.e. $f(x + \pi) = f(x)$ for all $x \in \mathbb{R}$. Consider f as a function on the circle \mathbb{T} and show that the convolution ideal generated by f in $L^1(\mathbb{T})$ is not dense in $L^1(\mathbb{T})$.

(b) Suppose for a function $f \in L^1(\mathbb{R})$, $\widehat{f}(\xi) = 0$ for almost every $\xi \in \mathbb{R}$. Show that $f(x) = 0$ almost everywhere.

(c) Take a nonzero $f \in C_c^\infty(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for some $1 < p < 2$. Suppose that $f * g = 0$. Prove or disprove that $g = 0$.

(d) For $\phi, \psi \in S(\mathbb{R})$, show that if $P(d/dx)\phi = P(d/dx)\psi$ then $\phi = \psi$, where P is a polynomial in 1-variable.

(e) Suppose $f \in L^p(\mathbb{R})$, $1 < p < 2$ is an even function. Take its double derivative f'' in the sense of distribution. Let $\widehat{f''}$ is the Fourier transform of f'' in the sense of distribution. Show that

$$\widehat{f''} = \Lambda \widehat{f}, \text{ where } \Lambda(\xi) = -4\pi^2 \xi^2.$$

(2) Answer the following questions:

[2+6=8]

(a) Define central and non-central Hardy-Littlewood maximal functions H_c and H_{nc} respectively.

(b) Let H_s be the central Hardy-Littlewood maximal function on \mathbb{R}^2 using solid squares, instead of balls (i.e. discs). Assuming H_c is strong type $p - p$ show that H_s is also strong type $p - p$.

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- (3) Define heat maximal function H_L on \mathbb{R} . Assuming that H_L is weak type $1 - 1$, prove that for $f \in L^1(\mathbb{R})$, $f * g_t(x) \rightarrow f(x)$ for almost every $x \in \mathbb{R}^n$, where g_t is the Gaussian:

$$g_t(x) = t^{-1/2} e^{-\frac{\pi x^2}{t}}. \quad [10]$$

- (4) Let I be a proper closed ideal of $L^1(\mathbb{R})$. Stating and using Wiener's Tauberian theorem on \mathbb{R} , show that I is inside a maximal ideal M of $L^1(\mathbb{R})$. You need to verify that the ideal M defined by you is a maximal ideal. [10]

- (5) Suppose a function $f \in S(\mathbb{R})$ is supported on $[-M, M]$. Show that \widehat{f} extends as an entire function (write only the steps) and

$$|\widehat{f}(x + iy)| \leq A e^{2\pi M|y|},$$

for some constants $A > 0$ and $M > 0$. Is the following stronger statement also true?

$$|(1 + z^2)\widehat{f}(z)| \leq A' e^{2\pi M'|\Im z|},$$

where $\Im z$ is the imaginary part of z . [4+6=10]

- (6) Consider the following initial value problem:

$$\begin{cases} \partial_t u(t, x) = \Delta u(t, x), & t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0, x) = f(x). \end{cases}$$

Prove that the solution $u(t, x)$ satisfies the following inequality:

$$\left\| \frac{d}{dx} u(t, \cdot) \right\|_{L^p(\mathbb{R})} \leq C |t|^{-\left(\frac{1}{2r} + \frac{1}{2}\right)} \|f\|_{L^q(\mathbb{R})},$$

where

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{r}.$$

[10]