INDIAN STATISTICAL INSTITUTE

Mid-semester Examination (2025–2026)

M. MATH. II

Advanced Functional Analysis

Date : 06.09.2025 Maximum Marks : 50 Time : 2 hrs.

This paper carries 60 marks. Maximum you can score is 50.
Precisely justify all your steps. Carefully state all the results you are using.
All vector topologies are assumed to be Hausdorff.

1. Let X be a locally convex topological vector space and X^* be the dual of X. Prove that $(X^*, \sigma(X^*, X))$ is normable if and only if X is finite dimensional. [15]

2. Let X, Y be topological vector spaces and $T : X \to Y$ be a linear map. T is said to be *bounded* if T takes bounded sets to bounded sets.

   (a) Show that if T is continuous, then T is bounded. [10]

   (b) Let X be bornological and Y locally convex. Then the converse is also true, that is, if T is bounded, then T is continuous.

   Recall that a *bornivore* in X is a convex balanced set that *absorbs* every bounded set in X. And that X is *bornological* if it is locally convex and every bornivore in X is a neighbourhood of 0. [15]

3. Let X be a real Banach space. A closed linear subspace $Y \subsetneq X$ is called proximinal if for every $x \in X$, there exists $y \in Y$ such that $\|x - y\| = dist(x, Y)$.

   Let $f \in S(X^*)$ and let $Y = \{x \in X : f(x) = 0\}$. Show that Y is proximinal if and only if f is norm attaining, that is, there exists $x_0 \in S(X)$ such that $f(x_0) = 1$. [20]

   [Hint: Show that $dist(x, Y) = |f(x)|$ for all $x \in X$.]