WEAK CONVERGENCE AND EMPIRICAL PROCESSES
M. STAT. IIND YEAR SEMESTER 2
INDIAN STATISTICAL INSTITUTE

Semestral Examination
Date: May 5, 2025 Time: 3 Hours
This exam is of 70 marks.
Answer as many as you can.
Maximum you can score is 60.

1. Let $\mathrm{P}_n \Rightarrow \mathrm{P}$ on a separable metric space (X, d). Let $\mathbb{F}$ be a collection of uniformly bounded equicontinuous family of real valued functions on X. Show that $\int f d\,\mathrm{P}_n \to \int f d\,\mathrm{P}$ uniformly in $f \in \mathbb{F}$. [8]

2. Let $\{X_n\}$ and $\{Y_n\}$ are random variables on a common probability space and taking values in Polish spaces S and T respectively. If X_n converges weakly to a random variable X and Y_n converges in probability to a constant $c \in T$, show that (X_n, Y_n) converges weakly to (X, c). [6]
If we modify the hypothesis so that X_n converges weakly to X, while Y_n converges in probability to a nondegenerate random variable Y taking values in T, will the same conclusion hold? [4]

3. Show that for each $t \in [0, 1]$ the evaluation map e_t defined on $C[0, 1]$ by $e_t(x) = x(t)$ is continuous. Let $\mathcal{B}$ be the Borel σ-field on $C[0, 1]$, which means, it is the smallest σ-field on the space containing its open sets. Show that this is also the smallest σ-field making all the functions e_t measurable. Show that this is also the smallest σ-field making the maps $\{e_t : 0 \leq t \leq 1, t \in \mathbb{Q}\}$ measurable.
Is the Borel σ-field of $C[0, 1]$ countably generated? [4+4+4+2=14]

4. (a) Show that the projection map $\pi : C \times [0, 1] \to \mathbb{R}$ given by $\pi(x, t) = x(t)$ is jointly continuous. [4]
(b) If $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a sequence of Poisson variables with $\mathrm{E}[\Lambda_n] = n$, show $\Lambda_n/n \to 1$ in probability. [4]
(c) Let $\{X_n\}_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables with mean 0 and variance 1, and $\{\Lambda_n\}_{n\in\mathbb{N}}$ be another sequence of Poisson random variables with $\mathrm{E}[\Lambda_n] = n$ independent of the sequence $\{X_n\}$, all defined on same probability space. Show that
$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\Lambda_n} X_i \Rightarrow N(0, 1).$$
[8]

5. Let $\{F_n\}$ be a sequence of distribution functions on $[0, 1]$. If they converge weakly, will they converge in Skorohod topology as elements of $D[0, 1]$? [6]

6. Let $\{f_n\}$ be a sequence of functions in $D[0, 1]$ converging to another function f in Skorohod topology, where f is continuous. If g is another $D[0, 1]$ function, show that $gf_n \to gf$ in Skorohod topolgy, where two functions are multiplied pointwise. [7]

7. For a class of real valued functions $\mathcal{F}$ defined on a set X, define the bracketing number $N_{[\,]}(\epsilon, \mathcal{F}, \|\cdot\|)$, where $\|\cdot\|$ is a norm for the fucntions in $\mathcal{F}$. [2]
For two classes of measurable functions $\mathcal{F}$ and $\mathcal{G}$, define the class
$$\mathcal{F} + \mathcal{G} = \{f + g : f \in \mathcal{F}, g \in \mathcal{G}\}.$$
Then, for any probability measure Q and $1 \leq r \leq \infty$, show that
$$N_{[\,]}(2\epsilon, \mathcal{F} + \mathcal{G}, \|\cdot\|) \leq N_{[\,]}(\epsilon, \mathcal{F}, \|\cdot\|)N_{[\,]}(\epsilon, \mathcal{G}, \|\cdot\|).$$
[7]