INDIAN STATISTICAL INSTITUTE, KOLKATA
End Semester Examination, First Semester 2025-2026
Analysis III, B.Stat 2nd year
Total Marks -60, Time : 3 hours , Date : 17/11/2025 .

1. Let B be the open unit ball around $(0,0)$ in $\mathbb{R}^2$, and $f : B \to \mathbb{R}$ be a bounded function and for $0 < \epsilon < 1$ let
$$S(\epsilon) := \{(x,y) \in B || (x,y)| = \epsilon\}.$$
Suppose
   (a) $f|_{S(\epsilon)}$ is continuous for all $0 < \epsilon < 1$ such that $max_{\alpha \in S(\epsilon)}(f(\alpha)) - min_{\beta \in S(\epsilon)}(f(\beta))$ tends to 0 as ϵ tends to 0.
   (b) $f|_{y=0}$ is continuous.

   Then show that f is continuous at $(0,0)$. Find out an example of a bounded function $f : B \to \mathbb{R}$ satisfying (a) but not satisfying (b) such that f is not continuous at $(0,0)$. [6 +4]

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable function such that there exists $\alpha \in \mathbb{R}^2$ with $f(\alpha) = g(\alpha)$ and $Df|_\alpha \neq Dg_\alpha$. Show that there exists infinitely many solutions of the equation $f(x,y) = g(x,y)$.[5]

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that $Df|_{(0,0,\ldots,0)} \neq (0,0,\ldots,0)$. Show that there exists a unique direction w (i.e. $w \in \mathbb{R}^n$ with $|w| = 1$) such that w is the direction of greatest decrease of f compared to $f(0,0,\ldots,0)$ (i.e $Df_w|_{(0,0,\ldots,0)}$ is minimum).[10]

4. Let $\Omega \subset \mathbb{R}^n$ be a region and f be a bounded integrable function on Ω. Let g be a bounded function on Ω such that $g = f$ on $\Omega \setminus S$ with $content(S) = 0$. Then
   (a) g is integrable on Ω.[2]
   (b) $\int_\Omega f = \int_\Omega g$.[3]

5. Let P be the parallelogram in $\mathbb{R}^2$ with vertices $(1,1)$, $(3,4)$, $(5,8)$ $(3,5)$. Find suitable map $T : I^2 \to P$ to convert the integral $\int_P e^{x-y} dxdy$ to an integral over the unit square I^2 and evaluate the integral. [10]

6. Let S be a compact region in $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ be a non-negative continuous function. Then the set
$$\Omega := \{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} | (x_1,\ldots,x_n) \in S, 0 \le x_{n+1} \le f(x_1,x_2,\ldots,x_n)\}$$
is a region in $\mathbb{R}^{n+1}$ and
$$V(\Omega) = \int_S f.$$
[5]

7. Let $f : [0,1] \to \mathbb{R}$ be a continuous positive function such that f is continuously differentiable on $(0,1)$. Let

$$\Omega := \left\{ (x,y) \in \mathbb{R}^2 | x \in [0,1], 0 \leq y \leq f(x) \right\}.$$

(a) Compute $\int_\Omega (x^3 + y) dx \wedge dy$. [5]

(b) Find a 1 from ω on $\mathbb{R}^2$, such that $d\omega = dx \wedge dy$.[5]

(c) Verify Green's theorem on Ω for the chosen 1-form ω in the previous step. [5]