Some Combinatorial Designs in VLSI Architectures and Statistics

by

Soumen Maity

A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirement for the degree of Doctor of Philosophy

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Soumen Maity

Chapter 1

Introduction

In this dissertation, we consider the following combinatorial problems: some characterization, enumeration, construction and optimization problems in both VLSI linear and VLSI two-dimensional arrays; and construction of two combinatorial designs as used by statisticians: nearly strongly balanced uniform repeated measurements designs (NSBURMDs) and balanced near uniform repeated measurements designs (BNURMDs). We give below, chapter-wise, the problems considered and a brief outline of the solutions.

1.1 Enumerating Catastrophic Fault Patterns in VLSI Linear Arrays with Bidirectional or Unidirectional Links

Systolic systems consist of a large number of identical and elementary processing elements locally connected in a regular fashion. Each element receives data from its neighbors, computes and then sends the results again to its neighbors. Few particular elements located at the extremes of the systems (these extremes depend on the particular system) are allowed to communicate with the external world.

The simplest systolic model is the VLSI linear array. In such a system the processing elements (PEs) are connected in a linear fashion: processing elements are arranged in linear order and each element is connected with the previous and the following element. Figure 1.1 shows a linear array of processing elements.

Figure 1.1: VLSI linear array.

Despite their simplicity, VLSI linear arrays have been used to solve several problems. It is well-known how to use a VLSI linear array for the matrix-vector multiplication; several other numerical problems (e.g. convolutions, triangular linear systems) have been solved using VLSI linear arrays (see, for example [85]). The use of VLSI linear arrays is not limited to numerical problems. For example, various algorithms that solve the longest common subsequence problem on a VLSI array have been devised [57].

Fault tolerant techniques are very important to systolic systems. Here we assume that only processors can fail. Indeed, since the number of processing elements is very large, the probability that a set of processing elements becomes faulty is not small. In a linear array of N processing elements, one faulty element is sufficient to stop the flow of information from one side to the other. Without the provision of fault-tolerance capabilities, the yield of VLSI chips for such an architecture would be so poor that the chip would be unacceptable. Thus, fault-tolerant mechanisms must be provided in order to avoid faulty processing elements taking part in the computation. A widely used technique to achieve reconfigurability consists of providing redundancy to the desired architecture [6, 14, 52].

In VLSI linear arrays the redundancy consists of additional processing elements, called spares, and additional connections, called bypass links. Bypass links are links that connect each processor with another processor at a fixed distance greater than 1. The redundant processing elements are used to replace any faulty processing element; the redundant links are used to bypass the faulty processing elements and reach others.

The effectiveness of using redundancy to increase fault tolerance clearly depends on both the amount of redundancy and the reconfiguration capability of the system. It does however depend also on the distribution of faults in the system. There are sets of faulty processing elements for which no reconfiguration strategy is possible. Such sets are called catastrophic fault patterns. From a network perspective, such fault patterns can cause network disconnection.

We now recall the relevant definitions and concepts from the literature. The basic components of a linear array are the *processing elements*, or simply processors, and the *links*. There are two kind of links: *regular and bypass*. Regular links connect neighboring processors, while the bypass links connect non-neighbors. The bypass links are used only for reconfiguration purposes when faulty processors are detected, otherwise they are considered to be the redundant links.

More precisely, let $A = \{p_1,\ p_2, \ldots,\ p_N\}$ denote a linear array of identical processing elements connected by regular links $(p_i,\ p_{i+1})$, $1 \leq i < N$.

Definition 1.1.1 Let $G = \{g_1,\ g_2,\ \ldots,\ g_k\}$ be an ordered set of integers such that $2 \leq g_1 < g_2 < \ldots < g_k$. We say that A has *link redundancy* or *link configuration* G if, the bypass links are (p_i, p_{i+g_t}) for $1 \leq i \leq N - g_t$ and $1 \leq t \leq k$.

Note that the set G does not contain the regular links even though they exist. We denote by g the length of the longest bypass link, i.e., $g = g_k$.

At the extremities of the array two special processors, called I (for input) and O (for output), are responsible for the I/O function of the system. We assume that I is connected to $p_1, p_2, \ldots, p_g$ while O is connected to $p_{N-g+1}, p_{N-g+2}, \ldots, p_N$ so that bottlenecks at the borders of the array are avoided.

Example 1.1.1 Figure 1.2 shows a linear array of 15 processors with redundancy $G = \{4\}$.

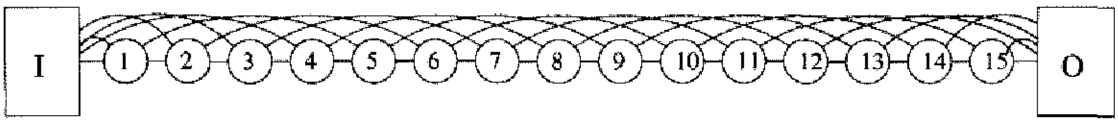


Figure 1.2: A linear array of processors

We refer to this structure as a *redundant linear array* or as a *redundant array*. A redundant array is called *bidirectional* or *unidirectional* according to the nature of its links. We sometimes refer to a processor p_i as processor i.

Definition 1.1.2 A *fault pattern* $F = \{f_1, f_2, \ldots, f_m\}$ for A is the set of faulty processors which can be any non-empty subset of A.

Definition 1.1.3 The *width* ω_F of a fault pattern F is defined to be the number of processors between and including the first and the last fault in F. That is, if $F = \{f_1, f_2, \ldots, f_m\}$ then $\omega_F = f_m - f_1 + 1$.

Definition 1.1.4 A fault pattern F is *catastrophic* for an array A with link redundancy G if I and O are not connected in the presence of such an assignment of faults.

In other words, given a redundant linear array A, a fault pattern F is catastrophic for A if and only if no path exists between I and O, once the faulty processors, and their incident links are removed. For example, in a linear array of processing elements with no link redundancy, a single PE fault in any location is sufficient to stop the flow of information from one side to the other.

Example 1.1.2 Consider the following two fault patterns $F_1 = \{4,\ 5,\ 7\}$ and $F_2 = \{3,\ 5,\ 7\}$ for a linear array with link redundancy $G = \{3\}$. We see from Figure 1.3, that the input processor I and the output processor O are connected by a path $[I,\ 1,\ 2,\ 3,\ 6,\ 9,\ O]$. Hence F_1 is not a catastrophic fault pattern by Definition 1.1.4. It is easy to check that, F_2 is catastrophic.

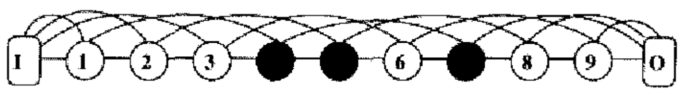


Figure 1.3: Fault Pattern $F_1 = \{4,\ 5,\ 7\}$.

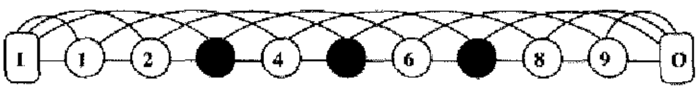


Figure 1.4: Fault Pattern $F_2 = \{3,\ 5,\ 7\}$.

We denote a fault pattern by FP and a catastrophic fault pattern by CFP. If we have to reconfigure a system when a fault pattern occurs, it is necessary to know if the fault pattern is catastrophic or not. Therefore it is important to study the properties of catastrophic fault patterns. A Characterization of catastrophic fault patterns was given in [70, 71, 74]. Nayak, Santoro, and Tan [71] proved that the number of faulty processing elements in any catastrophic fault pattern is greater than or equal to the length of the longest bypass link. That is, F is catastrophic with respect to $G = \{g_1, g_2, \ldots, g_k\}$ implies that the cardinality of F, $||F|| \geq g_k$. Suppose to the contrary that $||F|| < g_k$. Then partition the linear array of processing elements into blocks of g_k elements and list the blocks as consecutive rows of a "matrix". In Figure 1.5, we show the matrix for the array given in Figure 1.2.

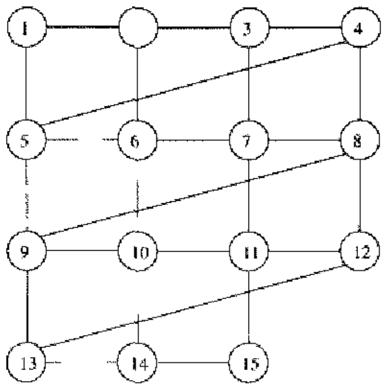


Figure 1.5: Matrix for the array given in Figure 1.2.

Observe that in this matrix, going down a column corresponds to using the bypass link g_k. Since the number of faulty elements is less than the size of the block, there must be a column with no faulty element, regardless of the distribution of the fault pattern. Thus F cannot be catastrophic since we can repeatedly use the bypass links of length g_k to avoid any of the faulty PEs.

As done in [74, 76], we consider only fault patterns of cardinality g_k, so, in general, $F = \{f_1, f_2, \ldots, f_{g_k}\}$. Also, the width of a fault pattern must fall within precise bounds for the pattern to be catastrophic; these bounds were established on the width ω_F of the fault pattern for different link configurations. Let $F = \{f_1, f_2, \ldots, f_{g_k}\}$ be a fault pattern for a linear array A with link redundancy $G = \{g_1, g_2, \ldots, g_k\}$. Necessary condition for F

to be catastrophic is

$$g_k \leq \omega_F \leq (\lceil \frac{g_k}{2} \rceil - 1)g_k + \lfloor \frac{g_k}{2} \rfloor + 1,$$

in the case of bidirectional links and

$$g_k \leq \omega_F \leq (g_k - 1)^2 + 1,$$

in the case of unidirectional links [74].

Nayak, Santoro and Tan [74] give an algorithm for constructing a catastrophic fault pattern with maximum width. Nayak, Pagli and Santoro [73] and De Prisco, Monti and Pagli [21] give algorithms for testing whether a fault pattern is catastrophic or not.

From now on, by catastrophic fault pattern we mean a catastrophic fault pattern having minimum number of faulty processing elements, i.e., with size equal to the length of the longest bypass link. Given a linear array with a set of bypass links, an important problem is to count the number of catastrophic fault patterns. The knowledge of this number enables us to estimate the probability that the system operates correctly.

Example 1.1.3 Figure 1.6 shows all catastrophic fault patterns for a linear array with bidirectional link redundancy $G = \{4\}$. Links are not drawn in Figure 1.6. Figure 1.7 shows the only fault pattern $F = \{4,\ 7,\ 10,\ 13\}$ which is not catastrophic for bidirectional links but is catastrophic for unidirectional link redundancy $G = \{4\}$. Hence, the number of CFPs for a linear array with unidirectional link redundancy $G = \{4\}$ is 5.

Enumeration of catastrophic fault patterns for link redundancy $G = \{g\}$ has been done in [22] for unidirectional case. In Chapter 2, we extend this to the case of link redundancy $G = \{2,\ 3, \ldots,\ k,\ g\}$, $2 \leq k < g-1$. See also [63, 64]. We characterize catastrophic fault patterns for both unidirectional and bidirectional cases and, using random walk as a tool, enumerate them. It is easy to check that, the run (or cluster) of g faulty processors is the only catastrophic fault pattern for $G = \{2, 3, 4, \ldots g-1, g\}$.

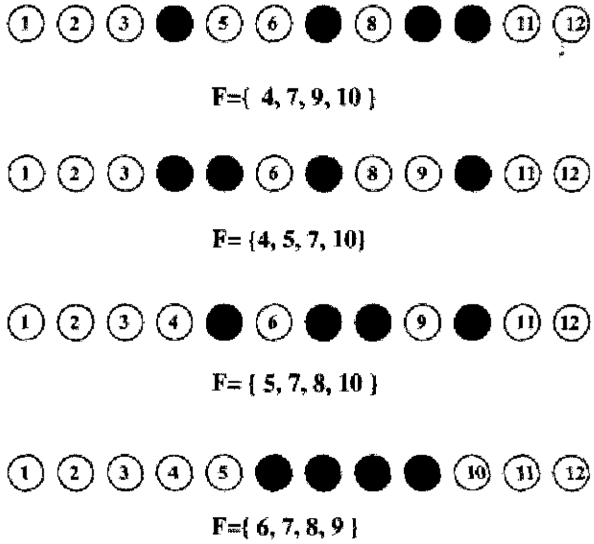


Figure 1.6: Catastrophic Fault Patterns for bidirectional link redundancy $G = \{4\}$.

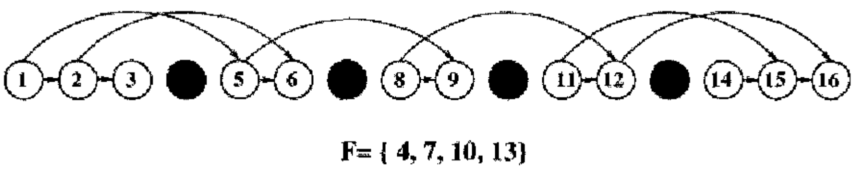


Figure 1.7: Catastrophic Fault Pattern F for unidirectional link redundancy $G = \{4\}$.

We use the following *matrix representation* [72] for fault patterns based on Boolean matrices. Consider an arbitrary fault pattern $F = \{f_1, f_2, \ldots, f_{g_k}\}$, consisting of g_k faults for an arbitrary link configuration $G = \{g_1, g_2, \ldots, g_k\}$. Without loss of generality, assume that $f_1 = 1$. The links can be either unidirectional or bidirectional. We represent F by a $\omega_F^+ \times g_k$ Boolean matrix W defined as follows:

$$W(i,j) = \begin{cases} 1 & \text{if } (ig_k + j + 1) \in F \\ 0 & \text{Otherwise} \end{cases}$$

Here $\omega_F^+ = \lceil \frac{\omega_F}{g_k} \rceil$. Notice that $W(0,0) = 1$ which indicates the location of the first fault.

Example 1.1.4 Consider the fault pattern $F = \{1, 5, 8, 12, 14, 15, 18, 19\}$ with 8 faults and with $\omega_F = 19$ for an array of PEs with link configuration $G = \{8\}$ as shown in Figure 1.8. The Boolean matrix representation of F is shown in Figure 1.9.

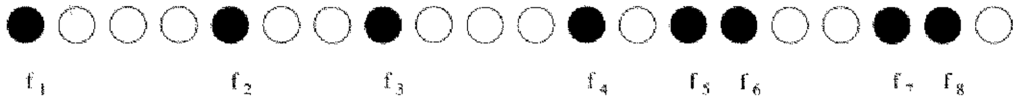


Figure 1.8: A fault pattern F for $G = \{8\}$

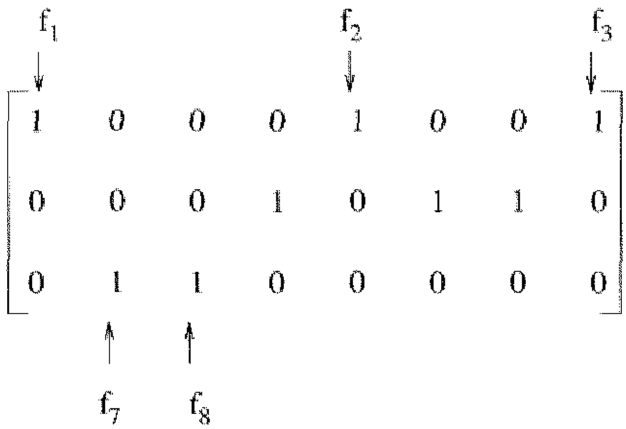


Figure 1.9: The matrix representation for F in Figure 1.8.

Observe that in the matrix W each regular link corresponds either to two consecutive elements in the same row or to the last element in a row and the first element in the following row, whereas the highest bypass link (g_k) corresponds to two consecutive elements in the same column. Let W be the matrix representation of a minimal fault pattern F. Notice that any minimal catastrophic fault pattern satisfies the necessary condition that for each j, there is only one i for which $W(i,j) = 1$. Indeed, if there is a column of W with two 1's, then there would be a column of W with only zero entries, as F has cardinality g_k. Using the longest bypass links (g_k) of that column we can pass over the fault zone, contradicting the hypothesis that F is catastrophic. Therefore, we are only interested in fault patterns whose corresponding matrix W has exactly one non-zero entry in every column. A CFP can be represented by the set of row indices corresponding to the entry 1 in columns. Formally, the *row representation* of a CFP F is the g_k-tuple $(r_0, r_1, \ldots, r_{g_k-1})$, where each r_i is the unique integer such that $W(r_i, i) = 1$. Another convenient way to represent a CFP is the *catastrophic sequence* [76]. Here a catastrophic fault pattern is represented as a sequence of $g_k - 1$ integer *moves* $(m_1, m_2, \ldots, m_{g_k-1})$, where $m_i = r_{i-1} - r_i$.

Example 1.1.5 Let $G = \{8\}$ and $F = \{1, 5, 8, 12, 14, 15, 18, 19\}$. Its row representation is (0, 2, 2, 1, 0, 1, 1, 0) and its catastrophic sequence is $(-2,\ 0,\ 1,\ 1,\ -1,\ 0,\ 1)$.

For given link configuration $G = \{2,\ 3, \ldots\ k,\ g\}$, $2 \le k < g-1$, we characterize minimal CFPs in terms of the catastrophic sequence. We prove that, catastrophic sequence of CFP with respect to bidirectional (resp., unidirectional) link configuration is same as symmetric (resp., asymmetric) random walk. Finally using random walk as a tool, we enumerate catastrophic fault patterns for both unidirectional and bidirectional cases and obtain the following results.

Theorem 1.1.1 *The number of CFPs for a linear array with bidirectional bypass links of length* 2, 3, ..., k *and* g *(i.e., with link redundancy* $G = \{2,\ 3 \ldots,\ k,\ g\}$, $k < g-1$*) is*

$$F^B(2,3\ldots,k,g) = 1 + \sum_{n=1}^{\lfloor \frac{g-k}{2} \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r} \right] \binom{g-k-2(n-r)(k-1)}{2n}$$

Theorem 1.1.2 *The number of CFPs for a linear array with unidirectional bypass links of length* 2, 3, ..., k *and* g *(i.e., with link redundancy* $G = \{2,\ ,3\ \ldots,\ k,\ g\}$, $k < g-1$*) is*

$$F^U(2,3,\ldots,k,g) = 1 + \sum_{n=1}^{\lfloor \frac{g-k}{2} \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2} \right] \binom{g-k-(n-r)(k-2)}{2n}.$$

1.2 Identification of Maximal Link Redundancy for which a Given Fault Pattern is Catastrophic in VLSI linear Arrays

Consider a linear array A of processing elements. For a given link configuration G, there exist many fault patterns which are catastrophic for the linear array. Similarly, a given fault pattern can be catastrophic for different link configurations. In Chapter 3, we consider the problem of finding maximal link configuration for which a given fault pattern is catastrophic. See also [65]. We consider maximality with respect to two parameters: the length g of the longest bypass link in G and the number $|G|$ of bypass links in G. The problem of minimization of the parameters is trivial since any F is catastrophic when $G = \emptyset$.

This is important because, for example, if a maximal (with respect to g) link configuration for a given fault pattern F has g-value 5, this shows that using bypass links of length 6, one can bypass all the faulty PEs of F. Similarly, if a maximal (with respect to $|G|$) link configuration for a given fault pattern F has $|G|$-value 4, this shows that using any five distinct bypass links, one can bypass all the faulty PEs of F.

A *fault pattern* for A is the set of faulty processors which can be any non-empty subset of A. However, we define it below in terms of runs of faulty processors.

Definition 1.2.1 For a redundant linear array A, a *fault pattern* F is an ordered set of pairs of positive integers $F = \{(f_1, \ell_1),\ (f_2, \ell_2),\ \ldots,\ (f_n, \ell_n)\}$, where $f_i + \ell_i < f_{i+1}$ for $1 \leq i \leq n-1$ and $f_n + \ell_n - 1 \leq N$

The pair (f_i, ℓ_i) identifies the i-th run of faulty processors $p_{f_i},\ p_{f_i+1},\ \ldots,\ p_{f_i+\ell_i-1}$. Hence, a processor p_z is faulty if and only if $f_i \leq z < f_i + \ell_i$ for some i, $1 \leq i \leq n$.

Non-faulty processors are working processors. A *path* from a working processor i to a processor j is a sequence of distinct processors $i = i_0,\ i_1,\ \ldots,\ i_s,\ i_{s+1} = j$ such that, for each $r = 0, 1, \ldots, s$, processor i_r is a working processor connected by a link to processor i_{r+1}. The *length* of the path is $s + 1$. An *escape path* with respect to F is a path from I to O.

Definition 1.2.2 Given a redundant array A, a fault pattern F is *catastrophic* for A if and only if no escape path exists with respect to F.

Given a fault pattern $F = \{(f_1, \ell_1),\ (f_2, \ell_2),\ \ldots,\ (f_n, \ell_n)\}$ for a redundant array A, we focus our attention on the part of A beginning at processor p_{f_1-g+1} and ending at processor $p_{f_n+\ell_n+g-2}$. We call this part of the array the *fault zone*. Since all the processors are indistinguishable, we will assume without loss of generality that the fault zone begins at processor p_1, i.e., $f_1 = g$. We denote by g the length of the longest bypass link, i.e., $g = g_k$.

By a run of working (respectively faulty) processors we mean a set of consecutive working (respectively faulty) processors which is not contained in any larger set of consecutive working (respectively faulty) processors. A run of working processors in the fault zone will be called a *chunk*. More formally, we give the following definition:

Definition 1.2.3 Let $F = \{(f_1, \ell_1),\ (f_2, \ell_2),\ \ldots,\ (f_n, \ell_n)\}$ be a fault pattern for A. Then for $1 \leq i \leq n-1$, $chunk_i$ is the run of working processors $p_{f_i+\ell_i}$, $p_{f_i+\ell_i+1}$, , $p_{f_{i+1}-1}$ between processor $f_i + \ell_i - 1$ and processor f_{i+1}. We also define $chunk_0$ to be the run of working processors $p_{f_1-g+1},\ p_{f_1-g+2},\ \ldots,\ p_{f_1-1}$ and $chunk_n$ to be the run of working processors $p_{f_n+\ell_n}, p_{f_n+\ell_n+1}, \ldots, p_{f_n+\ell_n+g-2}$.

Example 1.2.1 Consider the fault pattern $F = \{(4,\ 1),\ (7,\ 3)\}$ for a bidirectional linear array of 14 processors with link redundancy $G = \{4\}$. Then the fault zone begins at processor p_1 and ends at processor p_{12}. There are three chunks: $chunk_0$ begins at processor p_1 and ends at processor p_3; $chunk_1$ begins at processor p_5 and ends at processor p_6; $chunk_2$ begins at processor p_{10} and ends at processor p_{12}. See Figure 1.10.

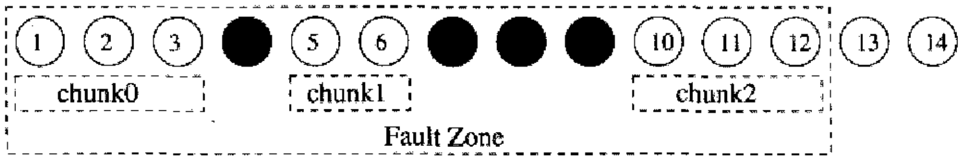


Figure 1.10: A fault pattern and the chunks

Let A be an array of N processors with bidirectional link redundancy $G = \{g_1, g_2, \dots, g_k\}$, and let F be a set of m faults grouped into $n \leq m$ runs of faulty processors. Then a graph $H = (V, E)$ was defined in [21] as follows:

Let $C_0, C_1, \dots, C_n$ be the chunks of F. Then $V = \{C_0, C_1, \dots, C_n\}$ and $(C_i, C_j) \in E$ if and only if there are two processors, $p_x \in C_i$ and $p_y \in C_j$ such that $|y-x| \in \{g_1, g_2, \dots, g_k\}$, that is, if and only if some processor in C_i and some processor in C_j are connected in A by a bypass link.

We call the graph H the *derived graph* of the fault pattern F. By definition of derived graph it follows that a fault pattern F is not catastrophic for an array A, if and only if C_0 is connected with C_n in the derived graph. We use the concept of derived graph to solve our problems and get the following:

Suppose we are given a fault pattern of m faulty processors grouped into $n \leq m$ runs of faulty processors. Then we prove that the maximum value of g can be found in $O(mn)$ time. We also show that the problem of finding max $|G|$ can be reduced to a graph problem, which looks somewhat similar to a min-cut problem, as follows:

Let H be the derived graph for the given fault pattern F and link redundancy $G = \{u\}$, $u \leq |F|$. Suppose C_0 and C_n are not connected in H. Let $S_0, S_1, \dots, S_h$ $(h \geq 1)$ be the components of H where $C_0 \in S_0$, $C_n \in S_h$. We then form a graph $H^* = (V^*,\ E^*)$ thus: $V^* = \{S_0,\ S_1,\ \ldots,\ S_h\}$. For $0 \le i \ne j \le h$, let $L_{ij} = \{\alpha \quad 1 \le \alpha \le u-1$ and $\alpha = |x-y|$ for some PE $x \in S_i$ and some PE $y \in S_j\}$. Then $(S_i,\ S_j) \in E^*$ if and only if $L_{ij} \ne \phi$. We call L_{ij} the label set of edge (S_i, S_j). Then we show that the problem of finding max $|G|$ is equivalent to finding a partition of V^* into V_1^* and V_2^* such that $S_0 \in V_1^*$, $S_h \in V_2^*$ and $|\bigcup_{i \in V_1^*, j \in V_2^*} L_{ij}|$ is minimum.

1.3 Catastrophic Faults in Reconfigurable VLSI Two-dimensional Array

In Chapter 4, we will focus on *two-dimensional networks*. See also [61]. The basic components of such a network are the processing elements(PEs) indicated by circles in Figure 1.11. The links are bidirectional. There are two kinds of links : *regular* and *bypass*. Regular links connect neighboring (either horizontal or vertical) PEs while bypass links connect non-neighbors. We have used broken lines to denote vertical links in the figure. The bypass links are used strictly for reconfiguration purposes when a fault is detected, otherwise they are considered to be the redundant links.

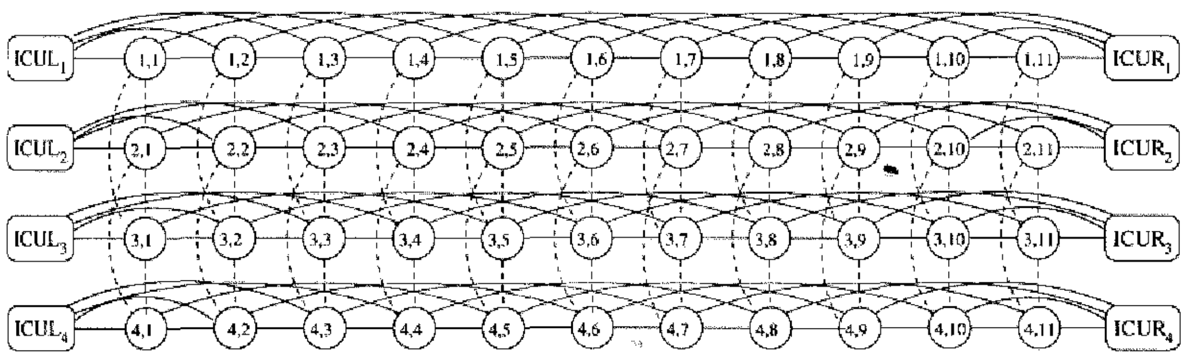


Figure 1.11: Two-dimensional network of PEs

We now introduce the following definitions:

Definition 1.3.1 A two-dimensional network $\mathcal{N}$ consists of a set V of PEs and a set E of links (where a link joins a pair of distinct PEs) satisfying the conditions listed below.

V is the union of three disjoint sets: the set $ICUL = \{ICUL_1, ICUL_2, \ldots, ICUL_{N_1}\}$ of left interface control units, the set $ICUR = \{ICUR_1, ICUR_2, \ldots, ICUR_{N_1}\}$ of right interface control units and a two-dimensional array $A = \{p_{ij} : 1 \leq i \leq N_1, 1 \leq j \leq N_2\}$ of PEs. We sometimes refer to the processing element p_{ij} as (i, j).

E consists of the links obtained as follows. Fix integers $1 = g_1 < g_2 < \ldots < g_k \leq N_2 - 1$ and $1 = v_1 < v_2 < \ldots < v_l \leq N_1 - 1$. Join p_{ij} to $p_{i'j'}$ by a link if and only if (i) $i = i'$ and $|j - j'|$ is one of $g_1, g_2, \ldots, g_k$ or (ii) $j = j'$ and $|i - i'|$ is one of $v_1, v_2, \ldots, v_l$. Also join $ICUL_i$ to $p_{i1}, p_{i2}, \ldots, p_{ig_k}$ and join $p_{i,N_2-g_k+1}, p_{i,N_2-g_k+2}, \ldots, p_{iN_2}$ to $ICUR_i$ by links, for $i = 1, 2, \ldots, N_1$.

We assume that $N_2 >> g_k$ and $N_1 > v_l$.

Definition 1.3.2 We call $g_1, g_2, \ldots, g_k$ the *horizontal link redundancies* of $\mathcal{N}$ and $v_1, v_2, \ldots, v_l$ the *vertical link redundancies* of $\mathcal{N}$. We refer to $G = (g_1, g_2, \ldots, g_k \mid v_1, v_2, \ldots, v_l)$ as the *link redundancy* of $\mathcal{N}$.

Figure 1.11 shows a two-dimensional network with $N_1 = 4$, $N_2 = 11$ and $G = (1, 4 \mid 1, 2)$. A link joining two PEs of the type p_{ij} and $p_{i,j+1}$ is called a *horizontal direct link* and a link joining two PEs of the type p_{ij} and $p_{i+1,j}$ is called a *vertical direct link*. Direct links are also called *regular links*. Links joining p_{ij} and $p_{i,j+g}$ with $g > 1$ are called *horizontal bypass links* and links joining p_{ij} and $p_{i+v,j}$ with $v > 1$ are called *vertical bypass links*. The horizontal and vertical bypass links are shown in red and red broken lines respectively in Figure 1.11.

The *length* of the horizontal bypass link joining p_{ij} to $p_{i,j+g}$ is g and the *length* of the vertical bypass link joining p_{ij} to $p_{i+v,j}$ is v.

Note that no links exist in the network $\mathcal{N}$ except the ones specified by G as in Definition 1.3.1.

Definition 1.3.3 Given a two-dimensional array A, a *fault pattern* (FP) for A is simply a non-empty subset F of the set of processing elements in A. An assignment of a fault pattern F to A means that every processing element belonging to F is faulty (and the others operate correctly).

Given a fault pattern F, define $m = \min\{j \ : \ (i,\ j) \in F\}$ and $M = \max\{j \ : \ (i,\ j) \in F\}$. We will focus our attention on the part of A beginning at the m-th column and ending at the M-th column, assuming that there are more than g_k columns before the m-th column and after the M-th column. It is assumed that $ICUL$ and $ICUR$ always operate correctly and we are considering information flow from $ICUL$ to $ICUR$.

Definition 1.3.4 The *window* W_F of a fault pattern F is the sub-array of A consisting of $\{p_{ij} \ : \ 1 \le i \le N_1,\ m \le j \le M\}$. By the *width* ω_F of F we mean $M - m + 1$.

Definition 1.3.5 The part of A beginning at $(m - g_k + 1)$-th column and ending at $(M + g_k - 1)$-th column is called *fault zone* of the array A.

Example 1.3.1 Consider the fault pattern $F = \{(1,5), (1,6), (1,8), (1,11), (2,5), (2,8), (2,10), (2,11), (3,6), (3,8), (3,9), (3,11), (4,7), (4,8), (4,10), (4,13)\}$ with link redundancy $G = (1,\ 4 \mid 1)$ as shown in Figure 1.12. Then, the fault zone begins at 2-nd column and ends at 16-th column. Links are not drawn in the figure. F has width $\omega_F = 13 - 5 + 1 = 9$.

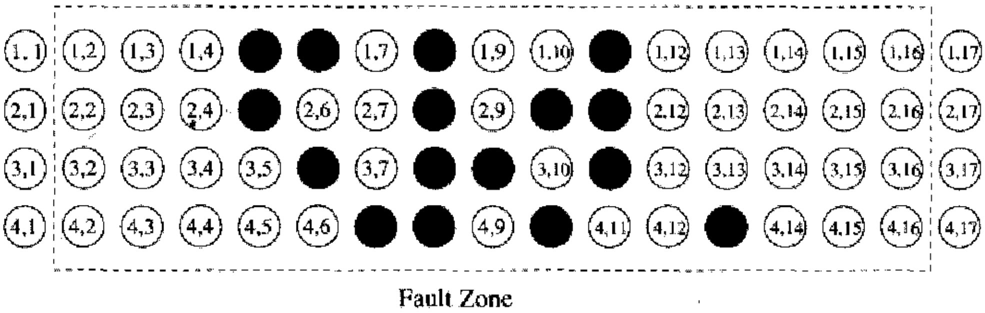


Figure 1.12: Fault Pattern F with link redundancy $G = (1,\ 4 \mid 1)$.

Definition 1.3.6 A fault pattern is *catastrophic* for the network $\mathcal{N}$ if $ICUL$ and $ICUR$ are not connected (i.e. there is no path connecting any $ICUL_i$ to any $ICUR_{i'}$ which does not involve a faulty PE) when the fault pattern F is assigned to A.

Example 1.3.2 Consider the fault pattern F of Example 1.3.1. We see from Figure 1.13, that the removal of the processing elements belonging to F along with their incident links disconnects ICUL and ICUR. Hence F is catastrophic. It is easy to check that F is not catastrophic with respect to link redundancy $G = (1, 4 \mid 1, 2)$.

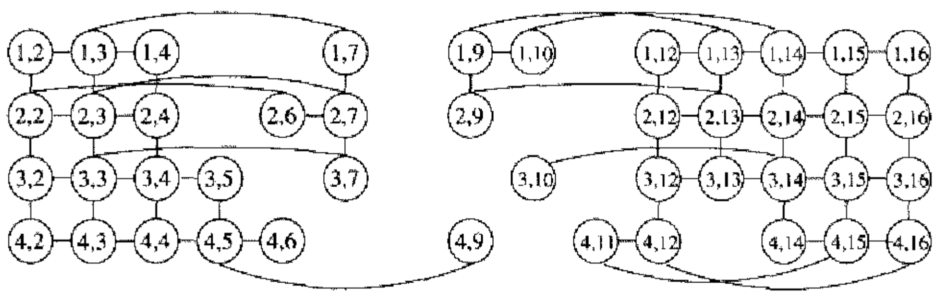


Figure 1.13: Network $\mathcal{N}$ after the removal of F and their incident links.

Definition 1.3.7 Let F be a fault pattern in a two-dimensional network $\mathcal{N}$ with link redundancy $G = (1,\ g_2,\ \ldots,\ g_k \mid 1,\ v_2,\ \ldots,\ v_l)$. If we remove all faulty PEs, their adjacent links and all bypass links from $\mathcal{N}$ then a component in the fault zone of $\mathcal{N}$

will be called a *chunk*. Let C_0, C_1, , C_n be the chunks of F where C_0 starts at $(m - g_k + 1)$-th column, C_n ends at$(M + g_k - 1)$-th column and other C_i's are labeled arbitrarily.

Example 1.3.3 Consider the fault pattern F of Example 1.3.1. There are six chunks: $C_0 = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (3,5), (4,2), (4,3), (4,4), (4,5), (4,6)\}$, $C_1 = \{(1,7), (2,6), (2,7), (3,7)\}$, $C_2 = \{(1,9), (1,10), (2,9)\}$, $C_3 = \{(4,9)\}$, $C_4 = \{(3,10)\}$ and $C_5 = \{(1,12), (1,13), (1,14), (1,15), (1,16), (2,12), (2,13), (2,14), (2,15), (2,16), (3,12), (3,13), (3,14), (3,15), (3,16), (4,11), (4,12), (4,14), (4,15), (4,16)\}$. Chunks are shown in Figure 1.14.

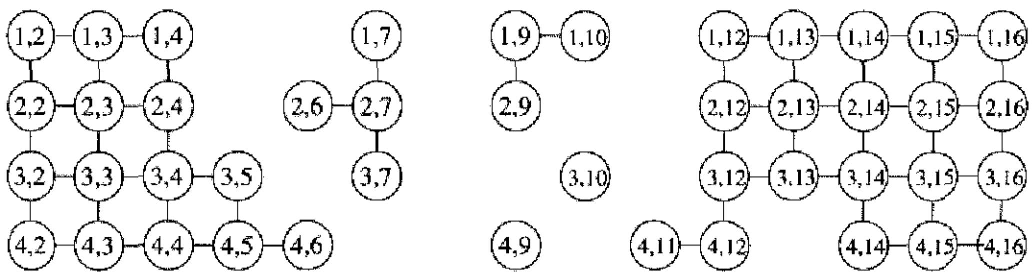


Figure 1.14: Chunks of the fault pattern F

Our main contribution here is a complete characterization of catastrophic fault patterns for two-dimensional arrays. Let $\mathcal{N}$ be a two-dimensional network with link redundancy $G = (g_1, g_2, \ldots, g_k \mid v_1, v_2, \ldots, v_l)$, and let F be a fault pattern. Then

1. We prove that, F is catastrophic with respect to $\mathcal{N}$ implies that the cardinality of F, $|F| \geq N_1 g_k$.

2. We outline an algorithm for the construction of a CFP with the maximum width for a given link redundancy G.

3. We give necessary and sufficient conditions for a fault pattern F to be catastrophic with respect to link redundancy $G = (g_1, g_2, \ldots, g_k \mid v_1, v_2, \ldots, v_l)$.

4. We provide an algorithm to test whether a given F is catastrophic with respect to link redundancy $G = (g_1, g_2, \ldots, g_k \mid v_1, v_2, \ldots, v_l)$.

For the above, we introduced a cuboid representation of A and the height matrix of F. We now describe these briefly.

Suppose we are given a fault pattern F in a two-dimensional array with link redundancy $G = (g_1, g_2, \ldots, g_k \mid v_1, v_2, \ldots, v_l)$. Without loss of generality we will assume that the first column of A contains a fault. We partition the two-dimensional array A of PEs into blocks of g_k columns as $A = (A_1 \vdots A_2 \vdots \cdots \vdots A_c)$ where $c = \lceil \frac{N_2}{g_k} \rceil$ and place the blocks as consecutive floors to form a cuboid. In Figure 1.15, we show the cuboid for the array given in Figure 1.11 (vertical bypass links are not drawn in the cuboid). Observe that, in this cuboid representation, each horizontal regular link joins two consecutive elements in the same row of a floor or the last element of a row of a floor with the first element of the corresponding row of the floor just above it whereas each vertical regular link joins two consecutive elements in the same column of a floor. On the other hand each horizontal bypass link of the maximum length joins two consecutive elements in the same *pillar*. So, in this cuboid, going down a pillar corresponds to using the longest horizontal bypass links. Suppose the number of faulty elements $|F|$ is less than the size of a block which is also the number of pillars $(N_1 g_k)$. Then there must be a pillar with no faulty element, regardless of the distribution of the fault pattern. Since the bottom and top of each pillar are linked to ICUL and ICUR respectively, F cannot be catastrophic since we can use the horizontal bypass links of length g_k to avoid the faulty PEs. This shows that,

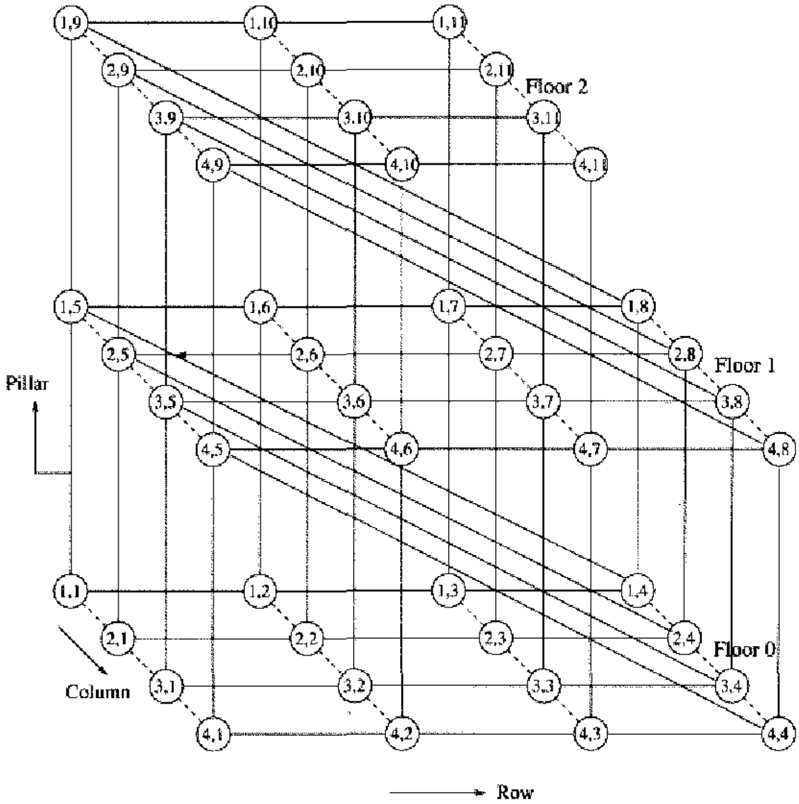


Figure 1.15: Cuboid representation of a 4×11 array with link redundancy $G = (1,\ 4 \mid 1)$.

F is catastrophic with respect to $\mathcal{N}$ implies that the cardinality of F, $|F| \geq N_1 g_k$.

We label the N_1 rows in any floor of the cuboid with $0,\ 1,\ \ldots,\ N_1 - 1$ and g_k columns in any floor with $0,\ 1,\ \ldots,\ g_k - 1$. The floors are labeled using $0,\ 1,\ 2,\ \ldots$. With every PE $(i,\ j)$ we can uniquely associate the triple $(x,\ y,\ z)$ where x, y and z are the row label, column label and floor label of the position $(i,\ j)$ occupies in the cuboid. (Note that $x = i - 1$, y is the remainder obtained when $j - 1$ is divided by g_k and z is $\lfloor \frac{j-1}{g_k} \rfloor$). We then write

$$W(x,\ y,\ z) = \begin{cases} 1 & \text{if } (i,\ j) \in F \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now F is a fault pattern such that for any $(x,\ y)$, there is exactly one z for which $W(x,\ y,\ z) = 1$, i.e., there is exactly one faulty PE in each pillar (note that every minimal CFP has this property). We then denote this z by h_{xy} and call the matrix

$$H = \begin{pmatrix} h_{00} & h_{01} & \cdots & h_{0,g_k-1} \\ h_{10} & h_{11} & \cdots & h_{1,g_k-1} \\ \vdots & \vdots & & \vdots \\ h_{N_1-1,0} & h_{N_1-1,1} & \cdots & h_{N_1-1,g_k-1} \end{pmatrix}$$

the height matrix of F.

Example 1.3.4 Consider the CFP $F = \{(1,5),\ (1,6),\ (1,8),\ (1,11),\ (2,4),\ (2,5),\ (2,6),\ (2,7),\ (3,5),\ (3,7),\ (3,8),\ (3,10),\ (4,1),\ (4,4),\ (4,6),\ (4,7)\}$ with 16 faults for a two-dimensional array A with link redundancy $G = (1,\ 4 \mid 1)$ which has $\omega_F = 11$ as shown in Figure 1.16.

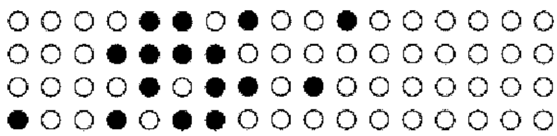


Figure 1.16: A fault Pattern

The height matrix for this CFP is

$$H = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

We characterize minimal CFPs in term of the height matrix.

1.4 Construction of Nearly Strongly Balanced Uniform Repeated Measurements Designs

An experiment in which an experimental unit is exposed repeatedly to a sequence of treatments is called a repeated measurements design(RMD). If there are p periods, v treatments and n experimental units then a repeated measurements design is an $n \times p$ array, say $D = ((d_{ij}))$, where d_{ij} denotes the treatment assigned to i-th unit in the j-th period. Generally in statistical literature, the transpose of the matrix D, D^T is defined as RMD(v, n, p); however, we will use D and not D^T in Chapter 5 and Chapter 6. For convenience, a repeated measurements design with v treatments, n units and p periods is abbreviated as RMD(v, n, p).

Example 1.4.1 Figure 1.17 shows an arrangement D of the treatments 1, 2 and 3 in a 12×3 array. If we take rows (resp., columns) of D as experimental units (resp., periods) then D is an RMD$(3, 12, 3)$.

The applications of these designs are not limited to any single field of study but are gaining importance over such diverse fields as agriculture, medicine, pharmacology, industry, social sciences, animal husbandry and psychology. The designs have proved to be attractive because of their economic use of experimental units and because of the more sensitive treatment comparisons that result from elimination of inter-unit variation. The use of RMDs rather than the classical designs can be justified in many setting such as when:

(i) One of the objectives of the experiment is to determine the effect of different sequences of treatment applications as in drug, nutrition or learning experiments.

Periods

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{Units}$$

Figure 1.17: RMD $(3, 12, 3)$

(ii) The experimenters might be interested in discovering whether or not a trend can be traced among the responses obtained by successive applications of several treatments on a single experimental unit. For example, if one wants to measure the degree of adaptation to darkness over time, the most efficient use of subjects requires that each subject be tested at all times of interest.

(iii) Experimental units are scarce and have to be used repeatedly. This is often the case in small clinics or in the development of large military systems, such as aerospace vehicles, airplanes, radar, computers, etc.

(iv) The nature of the experiments is such that it calls for special training over a long period of time. Therefore, to minimize cost and time, the experimenter should take advantage of the trained experimental units for repeated measurements.

For details of models, practical applicability and examples, one may refer to Hedayat and Afsarinejad [36], Afsarinejad [3], Patterson [79], Patterson and Lucas [82], Davis and Hall [20].

The application of a sequence of treatments to the same unit in RMDs, however, has the potential of producing residual or carryover treatment effects in the periods following the application of the treatment. A residual effect which persists in the i-th period after its application is called a residual effect of the i-th order. In most of the work done, till date, it has been assumed that second- and higher-order residual effects are negligible. Consequently, most of the designs developed so far, permit only the estimation of first order residual effects along with the treatment effects. In this discourse also, we restrict our attention to the first order residual effects.

Before proceeding to a survey of the literature on the subject, we recall the relevant definitions and concepts from the literature.

Definition 1.4.1 An RMD is said to be *uniform on periods* if in each period the same number of units is assigned to each treatment.

Definition 1.4.2 An RMD is said to be *uniform on units* if on each unit each treatment appears in the same number of periods.

Definition 1.4.3 An RMD is said to be *uniform* if it is uniform on periods and uniform on units simultaneously.

We see that in Figure 1.17, in each unit, each treatment occurs once and in each period, each treatment occurs four times. Hence D is both uniform on periods and uniform on units, i.e., the design D is uniform.

Definition 1.4.4 The underlying statistical model is called *circular* if in each unit the residual in the initial period is incurred from the last period.

Definition 1.4.5 The underlying statistical model is said to be *linear or non-circular or without pre-periods* if in each unit there is no residual effect in the initial period.

Only the linear case is being considered here.

Definition 1.4.6 Under linear model an RMD(v, n, p) is called *balanced* if the collection of ordered pairs $(d_{ij}, d_{i,j+1})$, $1 \leq i \leq n$, $1 \leq j \leq p-1$ contains each ordered pair of distinct treatments the same number of times, say λ, and does not contain the pairs of identical treatments at all.

Definition 1.4.7 Under linear model an RMD(v, n, p) is called *strongly balanced* if the collection of ordered pairs $(d_{ij}, d_{i,j+1})$, $1 \leq i \leq n$, $1 \leq j \leq p-1$ contains each ordered pair of treatments, distinct or identical, the same number of times.

We see from Figure 1.17, that each ordered pair of distinct treatments occurs four times. For example, the four occurrences of pair (1, 3) are shown in Figure 1.17 by underlining. Hence, D is a balanced uniform RMD. But D is not strongly balanced as it does not contain ordered pairs of identical treatments.

Cheng and Wu [15] showed that in the class of RMD (v, n, p), the strongly balanced designs are universally optimal for the estimation of direct as well as residual effects. They also showed that the necessary conditions for the existence of such a design are that $v^2|n$ and $p \geq 2v$. So even a minimal design in this class of designs needs $2v^3$ observations to be collected. As a result, when the number of treatments is large, the design becomes impractical. So, attempts were made to cut down the size of the experiment by relaxing some of the requirements of such designs. Kunert [49] has considered relaxation of the number of units keeping number of periods fixed. He considered the situations where $p = sv$, $s \geq 2$ and $v \mid n$ but $v^2 \nmid n$. He assumed $n = Av^2 + Bv$, $A \geq 0$, $1 \leq B \leq v-1$. Usually for practical purposes A is taken to be zero.

Definition 1.4.8 A square matrix C is said to be *completely symmetric* if all the diagonal elements of C are equal and all the off-diagonal elements of C are equal.

For any RMD, let m_{ij} be the number of appearances of treatment i immediately preceded by treatment j on the same unit and $M = ((m_{ij}))$, $1 \leq i, j \leq v$. For example, in D, $m_{31} = 4$ as seen above.

Definition 1.4.9 (Kunert [49]) For given (v, n, p), an RMD(v, n, p) is called *nearly strongly balanced* (NSB) if MM^T is completely symmetric and m_{ij}'s assume values

$$\left[\frac{n(p-1)}{v^2}\right] \text{ or } \left[\frac{n(p-1)}{v^2}\right] + 1, \quad 1 \leq i, j \leq v$$

where $[x]$ denotes the largest integer $\leq x$ and M^T is transpose of the matrix M.

Example 1.4.2 If $v = 3$, $n = p = 6$, then $A = 0$ and $B = 2$ and a nearly strongly balanced uniform RMD(3, 6, 6) is shown in Figure 1.18.

Periods

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 2 & 3 & 1 & 1 & 3 & 2 \\ 3 & 1 & 2 & 2 & 1 & 3 \\ 1 & 1 & 2 & 3 & 3 & 2 \\ 2 & 2 & 3 & 1 & 1 & 3 \\ 3 & 3 & 1 & 2 & 2 & 1 \end{bmatrix} \text{ Units}$$

Figure 1.18: NSBURMD(3, 6, 6)

Here the matrix M is $\begin{bmatrix} 3 & 3 & 4 \\ 4 & 3 & 3 \\ 3 & 4 & 3 \end{bmatrix}$. Note that, $MM^T = \begin{bmatrix} 34 & 33 & 33 \\ 33 & 34 & 33 \\ 33 & 33 & 34 \end{bmatrix}$

Kunert [49] showed that a nearly strongly balanced uniform RMD is universally optimal under both fixed- and mixed-effects models for the estimation of direct treatment effects among all competing designs which are uniform on units and on the last period. But he did not consider the construction of such designs. In Chapter 5, we provide a method of construction for a class of nearly strongly balanced uniform RMDs using suitable SBIBD's (symmetric balanced incomplete block designs) constructed through difference technique. See also [60]. For construction of nearly strongly balanced uniform RMDs, an adaptation of R. C. Bose's method of "symmetrically repeated differences" [11] has been used. Consider a group G with v elements and operation "$+$". If there is a p-tuple $B = (a_0,\ a_1,\ a_2, \ldots, a_{p-1})$ with elements belonging to G then the $p-1$ elements $a_i - a_{i+1}$ for $i = 0,\ 1,\ \ \ \ , p-2$ are said to be the *backward differences* arising from the p-tuple B. B will be referred to as a difference vector and $C = \{a_i - a_{i+1} : i = 0, 1, \ldots p-2\}$ as the set of backward differences in B.

Given any p-tuple $B = (a_0, a_1, ..., a_{p-1})$ with elements belonging to G, the set of p-tuples $B + \theta = (a_0 + \theta,\ a_1 + \theta,\ \ \ \ ,\ a_{p-1} + \theta)$ obtained as θ runs over the elements of G, is said to be the set of p-tuples obtained by *developing* B.

Example 1.4.3 Consider the group $Z_7 = \{0,\ 1,\ 2, \ldots,\ 6\}$ with operation "$+$" (i.e. addition modulo 7). Consider the triple $B = (1,\ 2,\ 4)$. Then the backward differences arising from the triple are $1 - 2 = 6,\ \ 2 - 4 = 5$. The set of triples obtained by developing $B = (1,\ 2,\ 4)$ are

$$(1,\ 2,\ 4),\ \ (2,\ 3,\ 5),\ \ (3,\ 4,\ 6),\ \ (4,\ 5,\ 0),\ \ (5,\ 6,\ 1),\ \ (6,\ 0,\ 2),\ \ (0,\ 1,\ 3).$$

We now introduce the following definition:

Definition 1.4.10 Consider the group $Z_v = \{0,\ 1,\ 2,\ \ \ldots,\ v-1\}$ with operation "$+$" (i.e. addition modulo v). Then for $i = 0,\ 1,\ 2\ \ \ldots,\ v-1$ and $s \geq 2$, $D_i^{s,v}$ denotes an sv-tuple of elements from Z_v such that each element of Z_v appears s times in it and among the *backward differences* in it each element of Z_v occurs s times except i which occurs $(s-1)$ times.

For example, $D_0^{2,7} = (0,\ 6,\ 1,\ 5,\ 2,\ 4,\ 3,\ 3,\ 4,\ 2,\ 5,\ 1,\ 6,\ 0)$.

In Chapter 5, we prove that, for odd v, $D_i^{s,v}$ can be constructed for all $s \geq 2$ and for all $i \in Z_v$. We then prove the existence of a nearly strongly balanced uniform RMD for any n of the type $Av^2 + (v-k)v$, $A \geq 0$ and $p = sv$, $s \geq 2$, assuming the existence of SBIBD(v, k, λ) constructed through difference technique as indicated below.
Given an SBIBD(v, k, λ) constructed through difference technique, let us consider any block, say, the j-th block. We will construct an sv-tuple, $D_t^{s,v}$, for each treatment t which is not present in the j-th block. Since any block of SBIBD(v, k, λ) contains k treatments, we have to construct $(v-k)$ sv-tuples. Let $t_1, t_2, \dots t_{v-k}$ be the treatments not occurring in the j-th block of SBIBD(v, k, λ). We write the $(v-k)$ sv-tuples, i.e., $D_{t_1}^{s,v}$, $D_{t_2}^{s,v}$, $\dots$, $D_{t_{v-k}}^{s,v}$ as the rows of an array and write the developed sv-tuples under

them as shown below

$$D = \begin{bmatrix} D_{t_1}^{s,v} \\ D_{t_1}^{s,v} + 1 \\ \vdots \\ D_{t_1}^{s,v} + v - 1 \\ D_{t_2}^{s,v} \\ D_{t_2}^{s,v} + 1 \\ \vdots \\ D_{t_2}^{s,v} + v - 1 \\ \vdots \\ D_{t_{v-k}}^{s,v} \\ D_{t_{v-k}}^{s,v} + 1 \\ \vdots \\ D_{t_{v-k}}^{s,v} + v - 1 \end{bmatrix}$$

where the elements of $D_t^{s,v} + \theta$ are obtained by adding θ to the elements of $D_t^{s,v}$. Then the $v(v-k) \times sv$ array D, is a nearly strongly balanced uniform RMD with parameters v, $n = v(v-k)$ and $p = sv$.

1.5 Construction of Balanced Near Uniform Repeated Measurements Designs

Cheng and Wu [15] showed that in the class of RMD (v, n, p), the strongly balanced designs are universally optimal for the estimation of direct as well as residual effects. They also showed that the necessary conditions for the existence of such a design are that $v^2 | n$ and $p \geq 2v$. So even a minimal design in this class of designs needs $2v^3$ observations to be collected. As a result, when the number of treatments is large, the design becomes impractical. So, attempts were made to cut down the size of the experiment by relaxing some of the requirements of such designs.

It is observed in the literature that Kunert [49] has proposed designs, which cut down the size of experimental units. Firstly, Kunert relaxed the condition of strong balance for RMDs and used the concept of nearly strongly balanced, viz., a design where the frequencies of ordered pairs of treatments, distinct or identical, in the design instead of being equal, differ by at most one. He considered designs where $v|n$ and $p \geq 2v$, studied optimality and characterized these designs in terms of optimality. It may be noted that n is small but p is large for such designs. Dey, Gupta and Singh [24] considered $p < v$ to be a realistic situation. They studied the universal optimality in a restricted class of designs and found that universally optimal designs are necessarily balanced. It is easy to check that, p is small but n still continues to be large for such designs. In an effort to make RMDs realistic and cost effective, we propose to reduce both n and p in the class of balanced designs. We now introduce the following definitions:

Definition 1.5.1 An RMD which is uniform on periods and for which frequencies of administration of different treatments on experimental units do not differ by more than one is called a *near uniform* RMD.

Definition 1.5.2 In case v does not divide p, we call an RMD *balanced near uniform* if it is uniform on periods, frequencies of administration of different treatments on an experimental unit do not differ by more than one, and balanced for first order residual under a linear model, i.e., the collection of ordered pairs $(d_{ij},\ d_{i,j+1})$, $1 \leq i \leq n$, $1 \leq j \leq p-1$, contains each ordered pair of distinct treatments the same number of times, say λ times. A balanced near uniform RMD$(v,\ n,\ p)$ will be abbreviated as BNURMD$(v,\ n,\ p)$.

Example 1.5.1 Figure 1.19 shows a balanced near uniform RMD with parameters $v = 5$, $n = 10$ and $p = 3$. We see in Figure 1.19 that,

Periods

$$D=\begin{bmatrix} 0 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \\ 4 & 3 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \\ 4 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix} \quad \textit{Units}$$

Figure 1.19: BNURMD(5,10,3)

- in each period each treatment appears the same number of times, i.e., the design is uniform on periods.

- each ordered pair of distinct treatments occurs once whereas no ordered pair of identical treatments occurs at all. That is,

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

- in each unit, each treatment occurs at most once.

In Chapter 6, we prove that, $p-1$ divides $\lambda(v-1)$ is a necessary condition for the existence of a BNURMD with the parameters v, p and λ. When this happens, the number of units n is an integer multiple of v, the number of treatments. See also [62]. We then give a procedure for construction of balanced near uniform RMDs when every ordered pair of distinct treatments appears exactly once or twice (i.e., $\lambda = 1$ or 2), except for the case v an odd integer and $p = v$. This exceptional case is equivalent to a row complete Latin square of odd order.

For construction of these designs, an adaptation of R. C. Bose's method of "symmetrically repeated differences" [11] has been used. Let the treatments in an RMD correspond to the elements of a group G with size v and with operation "$+$". Suppose there exist m p-tuples (called initial units) $B_1, B_2, \ldots, B_m$ such that (i) for any B_i, the frequencies of the different elements of G in B_i differ by at most 1 and (ii) among the $m(p-1)$ backward differences obtained from $B_1, B_2, \ldots, B_m$, each of the $v-1$ non-zero elements of G occurs exactly λ times. Then by developing the initial units $B_1, B_2, \ldots, B_m$, we obtain a BNURMD with parameters $v,\ n = mv,\ p$.

1.6 Literature Survey

Fault tolerance is the survival attribute of computer systems. A common approach for achieving fault tolerance in VLSI-based systolic architectures is through the incorporation of redundancy. The popularity of this approach rests on the fact that, with modern technology, it is now possible to incorporate a large degree of redundant processing elements (PEs) and additional circuitry into a single chip. The redundant PEs are used to replace any faulty PE(s); the redundant links are used to bypass the faulty PEs and reach others.

Most of the decisions made at design time with regard to fault tolerance are therefore focused on two particular aspects: amount of redundancy and reconfiguration technique. In the following, the VLSI design will be used as the leading example.

Intuitively, a system incorporating a large number of redundant PEs and "long" redundant links should be able to tolerate a large number of PE failures. A long redundant link can bypass a large block of consecutive faulty PEs. However, long wires are not always possible in such systems due to layout constraints. A long wire will introduce larger propagation delays which in turn might create synchronization problems and become the limiting factor in the performance of the system. Furthermore, note that to increase the number of redundant PEs in a chip requires an extra overhead of interconnections and switching circuitry which implies a higher likelihood of failure. Provided that these technical difficulties can be successfully overcome, it still seems natural to assume that a very high degree of tolerance can be achieved by simply providing a sufficiently large number of spare PEs with a large number of long connections.

The effectiveness of the approach of using structural redundancy to increase fault-tolerance clearly does not depend solely on the amount of redundancy. In fact, the availability of a large number of redundant PEs is useless if these PEs cannot be successfully employed to replace the faulty ones. Thus, a main measure of fault-tolerance in such redundant arrays is the *reconfiguration capability* (or reconfiguration effectiveness); that is, the ability to map faulty elements to spares (using bypass links) while preserving the high degree of regularity and locality of reference required by the system to perform correctly.

It is therefore not surprising that a large amount of research has been devoted to the design of reconfiguration algorithms for redundant arrays as well as proposing redundant architectures which facilitate the reconfiguration process [8, 33, 38, 46, 45, 53, 56, 58, 75, 87, 91, 93, 94]. Related work has focused on the study of low yield problem [105], i.e., on improving spare utilization.

A computer architecture (or network) can be characterized by its topological properties. It is called regular if the underlying graph is regular; that is, it is connected and every node has the same degree. The focus of this thesis will be on the regular architectures.

The effectiveness of using redundancy to increase fault tolerance in a regular architecture clearly depends on both the amount of redundancy and the reconfiguration capability of the system. It does however depend also on the distribution of the faults in the system. In fact, faults occurring at strategic locations in a regular architecture may have catastrophic effect on the entire structure and this cannot be overcome by any amount of clever design. If we have to reconfigure a system when a fault pattern occurs, it is necessary to know if the fault pattern is catastrophic or not. Therefore it is important to study the properties of catastrophic fault patterns. A characterization of catastrophic fault patterns was given in [70, 71, 74] for VLSI linear array. Nayak, Santoro and Tan [74] proved that a catastrophic fault pattern must contain a number of faulty processing elements which is greater than or equal to the length of the longest bypass link. They give an algorithm for constructing a catastrophic fault pattern with maximum width. Nayak, Pagli and Santoro [73] and De Prisco, Monti and Pagli [21] give algorithms for testing whether a fault pattern is catastrophic or not.

Given a linear array with a set of bypass links, an important problem is to count the number of catastrophic fault patterns. The knowledge of this number enables us to estimate the probability that the system operates correctly. Pagli and Pucci [76] obtained tight upper and lower bounds for the number $F^B(g)$ of catastrophic fault patterns of size g for a linear array with one bidirectional bypass link of length g. Enumeration of catastrophic fault patterns for link redundancy $G = \{g\}$ has been done in De Prisco and

Santis [22] for unidirectional case. Maity, Roy and Nayak [63, 64] extend this to the case of $G = \{2,\ 3, \ldots,\ k,\ g\}$, $2 \leq k < g-1$. They characterize catastrophic fault patterns for both unidirectional and bidirectional cases and, using random walk as a tool, enumerate them.

For a given link configuration G, there exist many fault patterns which are catastrophic for the linear array. Similarly, a given fault pattern can be catastrophic for different link configurations. Maity, Roy and Nayak [65] consider the problem of finding optimal link configuration for which a given fault pattern is catastrophic. Optimality is considered with respect to two parameters: the length g of the longest bypass link in G and the number $|G|$ of bypass links in G. Optimization here means maximizing the parameters; the problem of minimization of the parameters is trivial since any F is catastrophic when $G = \emptyset$.

Maity and Roy [61] completely characterize catastrophic fault patterns for two-dimensional arrays. They prove that the minimum number of faults in a catastrophic fault pattern is a function of the length of the longest horizontal bypass link and the number of rows in the two-dimensional array. From a practical viewpoint, this result provides some answers to the question about the guaranteed level of fault tolerance of a design. Guaranteed fault tolerance refers to an affirmative answer to the question: will the system withstand upto s faults always regardless of how and where they occur? They analyze catastrophic sets having the minimal number of faults and give an algorithm for constructing a catastrophic fault pattern with maximum width. They also give an algorithm for testing whether a set of faults is catastrophic or not.

The results for arrays apply to a large variety of commercially available array processors such as Geometric Arithmetic Parallel Processor (GAPP) [16] of NCR, Distributed Array Processor (DAP) [88] of ICL, England, NASA's Massively Parallel Processor (MPP)

[7], and Connection Machines [109] of Thinking Machines Corporation. The results presented in this paper also apply to a large number of WSI-based and VLSI-based processor arrays which include the Systolic Arrays [50], Reconfigurable Array of Processors ELSA (European Large SIMD Array) [95], and a variety of special-purpose VLSI and experimental WSI devices for applications such as signal processing, image processing, and numerical computations. Furthermore, the results are equally applicable to the memory chips. Memory chips are the most obvious candidates since the underlying architecture is highly regular and has a large number of identical cells.

We now turn to a survey of the literature on the construction of the combinatorial designs as used by statisticians which are considered in this thesis. A number of papers in the statistical literature in recent years have considered the structure of designs with certain desirable statistical properties. As we consider only construction of RMDs in this thesis, we present a survey of literature on construction of RMDs followed by a summary of important work done.

Different authors have used various methods of construction, like, cyclic arrangements of the treatments when the number of periods is less than that of treatments, construction based on finite fields and sequenceable non-abelian groups for generating uniform and balanced designs. Cochran, Autrey and Cannon [17] were probably the first to point out that the classical designs are not suitable for estimation of direct and residual effects in their dairy cattle feeding experiment. Williams [113] introduced and constructed balanced minimal repeated measurements designs when the number of periods is equal to the number of treatments whenever the number of treatments is even. Houston [39] showed that it is impossible to construct a balanced minimal repeated measurements design based on a cyclic group when the number of treatments is odd. It is known that no balanced minimal repeated measurements design exists if the number of treatments is three, five or seven. Mendelsohn [68] constructed a balanced repeated measurements design for 21 treatments based on a noncyclic group. Balanced minimal repeated measurements designs for 9 and 15 treatments were given by Hedayat and Afsarinejad [37], who attributed them to K. B. Mertz and E. Sonnemann respectively. Denes and Keedwell [23] reported that Wang has constructed balanced minimal repeated measurements design for 21 treatments based on a noncyclic group.

A major shortcoming with the above-mentioned designs is that each experimental unit is used for v tests. That is, each experimental unit must receive all the treatments. This may not be possible in many experiments, such as drug testing or other medical experiments. In many other experiments this limitation is undesirable.

Patterson [79] considered the case in which the number of periods is less than the number of treatments and constructed a series of balanced minimal repeated measurements designs. Patterson and Lucas [82] gave a catalogue of repeated measurements designs in which the number of periods is equal to or less than the number of treatments. Atkinson [5], Davis and Hall [20], Hedayat and Afsarinejad [36] and Constantine and Hedayat [19] have also constructed families of balanced minimal repeated measurements designs. Afsarinejad [2] has constructed, by an easily remembered method, balanced minimal repeated measurements designs when p, the number of periods, is less than v, the number of treatments.

Another useful subset of RMDs, viz., strongly balanced uniform repeated measurements designs, were first discussed in Berenblut [9]. Afsarinejad [2] has constructed strongly balanced RMDs when $p < v$. If v^2 divides n and $\frac{p}{v}$ is an even integer then Cheng and Wu [15] showed that the necessary conditions for the existence of a strongly balanced uniform RMD are that $v^2|n$ and $p \geq 2v$ and gave a method to construct such designs. Berenblut [9], Patterson [80, 81] and Kok and Patterson [44] have constructed similar designs. Sen and Mukherjee [96] have given a method of construction of such designs when v^2 divides n and $\frac{p}{v}$ is an odd integer, using MOLS of order v. Clearly the method fails for $v = 6$. However, they listed a different method of construction for the same design which works for $v = 6$. But these latter designs lack some properties compared to the ones based on MOLS. It may be noted that even a minimal design in this class of strongly balanced uniform RMDs needs $2v^3$ observations to be collected. As a result, when the number of treatments is large, the design becomes impractical. So, attempts were made to cut down the size of the experiment by relaxing some of the requirements of such designs. It is observed in the literature that Kunert [49] has proposed designs, which cut down the size of experimental units. He relaxed the condition of strong balance for RMDs and used the concept of nearly strongly balanced, viz., a design where *the frequencies of ordered pairs of treatments, distinct or identical, in the design, instead* of being equal, differ by at most one. Maity and Roy [60] have given a method of construction for a class of nearly strongly balanced uniform RMDs using suitable SBIBD's (symmetric balanced incomplete block designs) constructed through difference technique. In an effort to make RMDs realistic and cost effective, Maity, Dutta and Roy [62] have proposed to reduce both n and p in the class of balanced designs. They introduced and characterized balanced near uniform RMDs and gave a method to construct such designs.

Sonnemann, quoted in Kunert [48], gives a method of construction for circular balanced uniform RMDs with a minimum number of experimental units whenever $v > 2$ is an even integer. Afsarinejad [3], using disjoint directed Hamiltonian cycles, constructs circular balanced uniform RMDs with minimum number of experimental units whenever v is an odd number. Roy [92] and Dutta and Roy [27] have constructed circular balanced uniform RMDs, using a different method, when v divides n, $\frac{p}{v}$ is an odd integer and $v = 0,\ 1,\ 3\ (mod\ 4)$. Sharma [102] constructs circular strongly balanced uniform RMDs whenever $n = v$ and $\frac{p}{v}$ is an even integer.

In situations with correlated errors, the optimal designs usually prove to be variants of the designs constructed by Williams [113] (called Williams designs by Kunert [48]). For $v = 2 \pmod 4$, Street [106] gives a method of construction of Williams design with a circular structure. Matthews [67] obtains optimal designs under a linear fixed effects model with auto-correlated errors for three- and four-period designs.

An extreme form of an RMD is the one in which the entire experiment is planned on a single experimental unit. Details on this can be found in Williams [115]. For a general survey of RMDs one can refer to Hedayat and Afsarinejad [36], Bishop and Jones [10], Hedayat [36], Street [106].

Chapter 2

Enumerating Catastrophic Fault Patterns in VLSI Linear Arrays with Bidirectional or Unidirectional Links

2.1 Introduction

Given a linear array A with a set of bypass links, an important problem is to count the number of catastrophic fault patterns. The knowledge of this number enables us to estimate the probability that the system operates correctly. Pagli and Pucci [76] obtained tight upper and lower bounds for the number $F^B(g)$ of catastrophic fault patterns of size g for a linear array with one bidirectional bypass link of length g. Enumeration of catastrophic fault patterns for link redundancy $G = \{g\}$ has been done in [22] for unidirectional case. In this chapter, we extend this to the case of link redundancy $G = \{2, 3, \ldots, k, g\}$, $2 \leq k < g - 1$. See also [63, 64]. We characterize catastrophic fault patterns for both unidirectional and bidirectional cases and, using random walk as a tool, enumerate them.

2.2 Characterization of Catastrophic Fault Patterns

We use the following *matrix representation* [72] for fault patterns based on Boolean matrices. Consider an arbitrary fault pattern $F = \{f_1,\ f_2,\ \ \ ,\ f_{g_k}\}$, consisting of g_k faults for an arbitrary link configuration $G = \{g_1,\ g_2,\ \ \ ,\ g_k\}$. Without loss of generality, assume that $f_1 = 1$. The links can be either unidirectional or bidirectional. We represent F by a $\omega_F^+ \times g_k$ Boolean matrix W defined as follows:

$$W(i,j) = \begin{cases} 1 & \text{if } (ig_k + j + 1) \in F \\ 0 & \text{otherwise} \end{cases}$$

Here $\omega_F^+ = \lceil \frac{\omega_F}{g_k} \rceil$. We will sometimes refer to (i, j) as the location of the PE $(ig_k + j + 1)$. We now recall the following definitions from the literature.

Definition 2.2.1 Let F be a minimal CFP. Let (x, y) be the location of faulty PE f. Then the PE of A corresponding to the location (i, y) is said to be *interior*, *border* or *exterior* with respect to f according as $i < x$, $i = x$ or $i > x$.

The definition of interior, border, and exterior can now be extended from element to regions as follows:

Definition 2.2.2 For a given fault pattern F, the *interior* $I(F)$ of F is the set of all interior elements, the *border* $B(F)$ of F is the set of all border elements, and the *exterior* $E(F)$ of F is the set of all exterior elements.

Example 2.2.1 Consider the fault pattern $F = \{1, 5, 8, 12, 14, 15, 18, 19\}$ with 8 faults in an array with link redundancy $G = \{3, 8\}$. The interior, border, and exterior elements are shown in Figure 2.1.

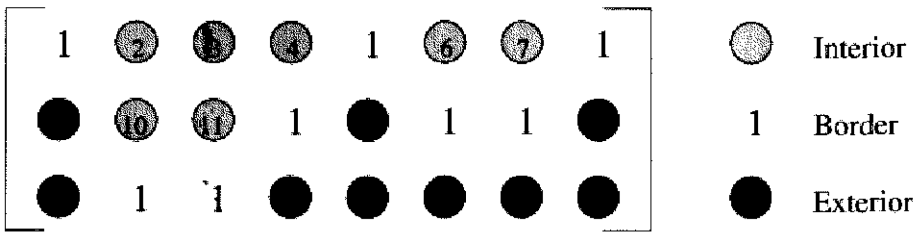


Figure 2.1: Interior, exterior and border of the fault pattern F.

Lemma 2.2.1 (Nayak et al. [73]) A fault pattern is catastrophic for an array A with bidirectional link redundancy G iff it is not possible to reach any exterior element (resp. interior element) from any interior element (resp. exterior element) using the links in G.

Proof. It is easy to see that all interior elements are reachable from I and all exterior elements are reachable from O. The lemma follows from Definition 1.1.4

Lemma 2.2.2 (Nayak et al. [73]) A fault pattern is catastrophic for an array A with unidirectional link redundancy G iff it is not possible to reach any exterior element from any interior element using the links in G.

Example 2.2.2 Figure 2.2 shows the matrix representation of the fault pattern $F = \{1, 2, 5, 6, 10, 15\}$ with 6 faults for $G = \{6\}$. Its row representation is (0, 0, 2, 1, 0, 0) and its catastrophic sequence is (0, −2, 1, 1, 0). Note that, the exterior PE 8 is connected to interior PE 9 by a regular link. An escape path [O, 14, 8, 9, 3, I] is shown in the figure. Hence F is not catastrophic for bidirectional link redundancy $G = \{6\}$ by Lemma 2.2.1. It is easy to check that, F is catastrophic for unidirectional link redundancy $G = \{6\}$ by Lemma 2.2.2.

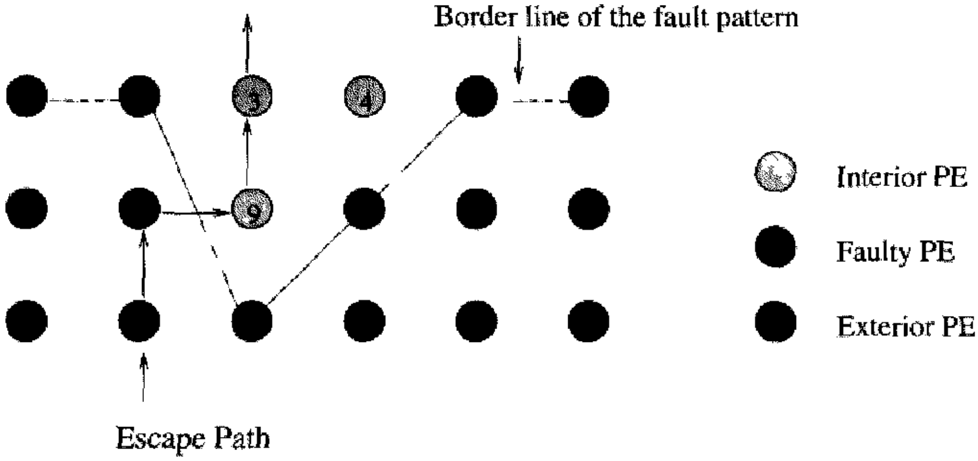


Figure 2.2: A fault Pattern F which is not catastrophic for bidirectional $G = \{6\}$.

2.3 Some Results on Random Walk

Definition 2.3.1 (Feller [28]). A *random walk* is a sequence $\omega = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \varepsilon_n)$ where each $\varepsilon_i = +1$ or -1. Such a sequence can be represented by the polygonal line $[(0,0),\ (1,\varepsilon_1),\ (2,\varepsilon_1+\varepsilon_2),\ \ldots,\ (n,\sum_{i=1}^{n}\varepsilon_i)]$, usually called a *path*. We will denote $\sum_{i=1}^{k}\varepsilon_i$ by S_k.

For example, the random walk $(1,-1,-1,1,-1,-1)$ is represented by the path joining $(0,0)$ to $(6,-2)$ with $(1,1)$, $(2,0)$, $(3,-1)$, $(4,0)$, $(5,-1)$ as the intermediate vertices. See Figure 2.3.

Definition 2.3.2 A subsequence $(\varepsilon_{s+1}, \varepsilon_{s+2}, \ldots, \varepsilon_{s+r})$ of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $r \geq 1$ is called a *run* of length r if $\varepsilon_s \neq \varepsilon_{s+1} = \varepsilon_{s+2} = \ldots = \varepsilon_{s+r} \neq \varepsilon_{s+r+1}$.

Let ρ_1 and ρ_{-1} denote the numbers of runs whose elements are 1 and -1 respectively and $R = \rho_1 + \rho_{-1}$. Note that R is the number of runs.

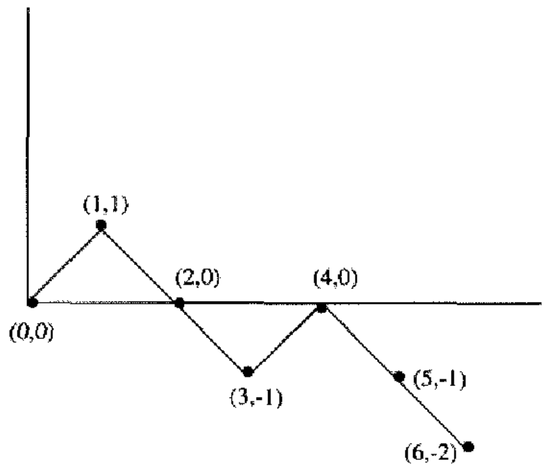


Figure 2.3: Random walk $\omega = (1, -1, -1, 1, -1, -1)$

We use the following notations from Vellore [110].

Notations: We assume $m \geq 0$.

$E_{n,m}$: A path from $(0, 0)$ to (n, m).

$E^{R}_{n,m}$: An $E_{n,m}$ path with R runs.

$E^{R+}_{n,m}$: An $E^{R}_{n,m}$ path starting with a positive step.

$E^{R-}_{n,m}$: An $E^{R}_{n,m}$ path starting with a negative step.

$E^{R+,t}_{n,m}$: An $E^{R+}_{n,m}$ path crossing the line $y = t,\ t > 0$ at least once.

$E^{R-,t}_{n,m}$: An $E^{R-}_{n,m}$ path crossing the line $y = t,\ t > 0$ at least once.

$N(A)$: The number of all A paths. e.g., $N(E_{n,m}) = \binom{n}{\frac{n-m}{2}}$.

Theorem 2.3.1 *(Feller [28]). Among the $\binom{2n}{n}$ paths joining the origin to the point $(2n, 0)$ there are exactly $\frac{1}{n+1}\binom{2n}{n}$ paths such that $S_1 \leq 0,\ S_2 \leq 0, \quad , S_{2n-1} \leq 0,\ S_{2n} = 0$*

Theorem 2.3.2 *(Vellore [110]). Let $m \geq 0$. Then for $max(1,m) \leq t < \frac{m+n}{2}$,*

$$N(E_{n,m}^{(2r-1)+,t}) = \binom{\frac{n-m}{2}+t-1}{r-2}\binom{\frac{n+m}{2}-t-1}{r-1}$$

$$N(E_{n,m}^{2r-,t}) = \binom{\frac{n-m}{2}+t-1}{r-2}\binom{\frac{n+m}{2}-t-1}{r}.$$

Theorem 2.3.3 *The number of paths from the origin to the point $(2n,0)$ such that $S_1 \leq 0,\ S_2 \leq 0,\ \ldots, S_{2n-1} \leq 0,\ S_{2n} = 0$ and there are exactly $2r$ runs, is*

$$\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}.$$

Proof: The required number of paths equals

$$N(E_{2n,0}^{2r-}) - N(E_{2n,0}^{2r-,0\ *}) \tag{2.1}$$

Where $E_{2n,0}^{2r-,0\ *}$ is an $E_{2n,0}^{2r-}$ path crossing the line $y=0$ at least once. It is known that

$$N(E_{2n,0}^{2r-}) \quad = \quad \binom{n-1}{r-1}^2 \tag{2.2}$$

(see Wald and Wolfowitz [111]). To find $N(E_{2n,0}^{2r-,0\ *})$, let P be an $E_{2n,0}^{2r-,0\ *}$ path, i.e., a path from $(0,0)$ to $(2n,0)$ with the first step negative, with $2r$ runs and crossing $y=0$ at least once. Then dropping the first step and taking $(1,-1)$ as the new origin we have an $E_{2n-1,1}$ path Q. Note that Q crosses the line $y=1$ (w.r.t. the new origin) and has $2r-1$ runs if it starts with a positive step and $2r$ runs if it starts with a negative step. Moreover, any path Q with these properties arises from a unique $E_{2n,0}^{2r-,0\ *}$ path P as above. Thus

$$\begin{aligned} N(E_{2n,0}^{2r-,0\ *}) &= N(E_{2n-1,1}^{2r-,1}) + N(E_{2n-1,1}^{(2r-1)+,1}) \\ &= \binom{n-1}{r-2}\binom{n-2}{r} - \binom{n-1}{r-2}\binom{n-2}{r-1} \qquad \text{[Using Theorem 2.3.2]} \\ &= \binom{n-1}{r-2}\binom{n-1}{r}. \end{aligned} \tag{2.3}$$

The theorem follows from 2.1, 2.2 and 2.3.

2.4 The Case of Bidirectional Links

We start with a characterization of CFPs in terms of the catastrophic sequence.

Proposition 2.4.1 (Pagli et al. [76]). *Necessary and sufficient conditions for* $(m_1, m_2, \ldots, m_{g-1})$ *to be the catastrophic sequence of a minimal CFP for a bidirectional linear array with link redundancy* $G = \{g\}$ *are:*

1. $m_i = -1, 0$ or 1, for $1 \leq i \leq g-1$,
2. $S_k = \sum_{i=1}^{k} m_i \leq 0$ for $k = 1, 2, \ldots, g-2$,
3. $S_{g-1} = \sum_{i=1}^{g-1} m_i = 0$.

We shall illustrate Proposition 2.4.1 by an example.

Example 2.4.1 Consider the fault pattern $F = \{1,\ 2,\ 5,\ 9,\ 13\}$ in a linear array A with link redundancy $G = \{5\}$. Note that in the matrix representation for F there is exactly one faulty PE in each column. See Figure 2.4.

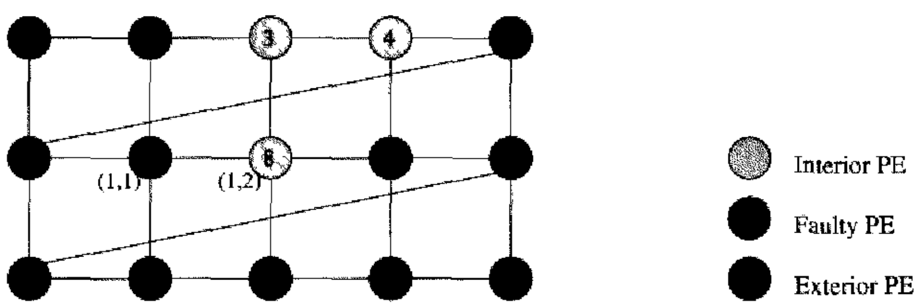


Figure 2.4: Interior, exterior and border of the fault pattern F.

The row representation for this fault pattern is (0, 0, 2, 1, 0) and the catastrophic sequence is $(m_1,\ m_2,\ m_3,\ m_4) = (0,\ -2,\ +1,\ +1)$. Note that, $m_2 = -2$ which violates condition (1) of Proposition 2.4.1. We see from Figure 2.4, that the interior processor at location $(1,2)$ and the exterior processor at location $(1,1)$ are connected by a regular link. Hence F is not a catastrophic fault pattern by Lemma 2.2.1. However, it can easily be verified that F satisfies conditions (2) and (3) of Proposition 2.4.1 even though F is not catastrophic.

Theorem 2.4.1 *For $G = \{g\}$, the number of CFPs for bidirectional links is given by*

$$F^B(g) = \sum_{n=0}^{\lfloor \frac{g-1}{2} \rfloor} \frac{1}{n+1}\binom{2n}{n}\binom{g-1}{2n}$$

Proof. The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-1})$ satisfying the conditions of Proposition 2.4.1. Any such sequence with n -1's has n $+1$'s and $g-1-2n$ 0's and can be obtained by starting with a path from $(0,0)$ to $(2n, 0)$ such that $S_1 \le 0,\ S_2 \le 0,\ \ldots,\ S_{2n-1} \le 0,\ S_{2n} = 0$ and plugging in $g-1-2n$ 0's in the $(2n+1)$ distinct places for each such path (i.e., $2n-1$ intermediate places and two more places before and after the sequence). Clearly the number of such paths is $\frac{1}{n+1}\binom{2n}{n}$ and for each path, the number of ways of plugging in the 0's is $\binom{(g-1-2n)+(2n+1)-1}{g-1-2n} = \binom{g-1}{2n}$. Hence the theorem follows.

Proposition 2.4.2 Necessary and sufficient conditions for $(m_1, m_2, \ldots, m_{g-1})$ to be the catastrophic sequence of a minimal CFP for a bidirectional linear array with link redundancy $G = \{2,\ g\}$ are:

1. $m_{g-1} = 0$

2. $m_j = -1,\ 0,\ +1$ for $j = 1,\ 2,\ 3, \ldots,\ g-2$

3. $\sum_{j=1}^{k} m_j \leq 0$ for $k = 1, 2, 3, \ldots, g-3$

4. $\sum_{j=1}^{g-2} m_j = 0$,

5. $m_i + m_{i+1} = -1, 0 \ +1$ for $i = 1, 2, 3, \quad , g-3$.

   That is, two or more consecutive $+1$ or -1 is not allowed.

We shall illustrate Proposition 2.4.2 by an example.

Example 2.4.2 Consider the fault pattern $F = \{1, 6, 8, 10, 11, 15\}$ in a linear array A with link redundancy $G = \{2, 6\}$. Note that in the matrix representation for F there is exactly one faulty PE in each column.

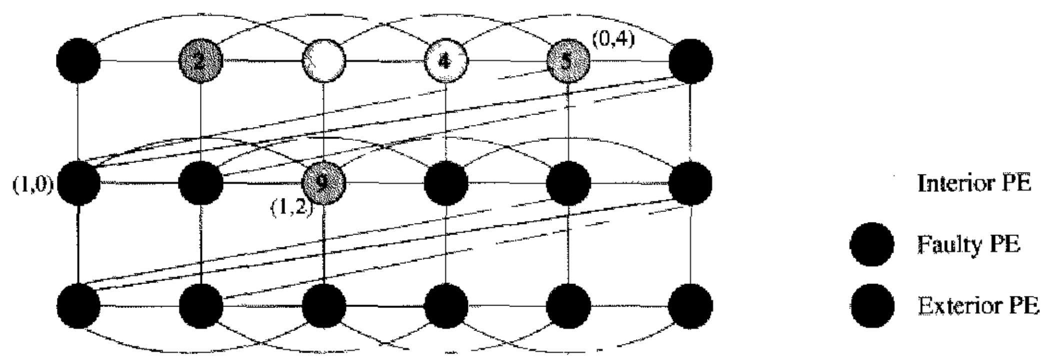


Figure 2.5: Interior, exterior and border of the fault pattern F.

The row representation for this fault pattern is (0, 1, 2, 1, 1, 0) and the catastrophic sequence is $(m_1, m_2, m_3, m_4, m_5) = (-1, -1, +1, 0, +1)$. Note that, $m_5 = +1$ which violates condition (1) of Proposition 2.4.2. We see from Figure 2.5, that the interior processor at location $(0, 4)$ and the exterior processor at location $(1, 0)$ are connected by a bypass link of length 2. Hence F is not a catastrophic fault pattern by Lemma 2.2.1. Similarly, condition (5) of Proposition 2.4.2 is violated since $m_1 + m_2 = -2$. Note that, locations $(1, 0)$ and $(1, 2)$ contain an interior processor and an exterior processor respectively which are connected by a bypass link of length 2. However, it can easily be verified that F satisfies conditions (2), (3) and (4) of Proposition 2.4.2 even though F is not catastrophic.

In general we have the following characterization.

Proposition 2.4.3 Necessary and sufficient conditions for $(m_1,\ m_2,\ \quad,\ m_{g-1})$ to be the catastrophic sequence of a minimal CFP for a bidirectional linear array with link redundancy $G = \{2,\ 3,\ \quad,\ k,\ g\}$, $k < g-1$ are:

1. $m_{g-1} = m_{g-2} = \ldots = m_{g-k+1} = 0$
2. $m_j = -1, 0, +1$ for $j = 1,\ 2,\ 3,\ \quad,\ g-k$
3. $\sum_{j=1}^{k} m_j \leq 0$ for $k = 1,\ 2\ \ 3\ \ \ldots,\ g-k-1$
4. $\sum_{j=1}^{g-k} m_j = 0,$
5. $m_i + m_{i+1} + \ldots + m_{i+s} = -1, 0, +1$ for $s = 1, 2, 3, \ldots, k-1,$ for $i = 1, 2, \ldots, g-k-s.$

The characterizations described in Proposition 2.4.2 and 2.4.3 are easy to visualize and hence the proofs are omitted.

Theorem 2.4.2 *The number of CFPs for a linear array with bidirectional bypass links of lengths* 2 *and* g *(i.e., with link redundancy* $G = \{2,\ g\}$*) is*

$$F^B(2,g) = 1 + \sum_{n=1}^{\lfloor \frac{g-2}{2} \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r} \right] \binom{g-2(n-r)-2}{2n}.$$

Proof. The number of CFPs is equal to the number of catastrophic sequences $(m_1,\ m_2,\ \ldots,\ m_{g-2})$ satisfying the conditions (2)-(5) of Proposition 2.4.2. Any such sequence with n -1's has n $+1$'s and $g-2-2n$ 0's and can be obtained by starting with a path from $(0,0)$ to $(2n,0)$ such that $S_1 \leq 0,\ S_2 \leq 0,\ \ldots,\ S_{2n-1} \leq 0,\ S_{2n} = 0$ and having exactly $2r$ runs. The number of such paths is $\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2}$ by Theorem 2.3.3. R(run)= 1+ number of changes of the type $(-1,+1)$ or $(+1,-1)$. All these paths have $(n-r)$ identical pairs of the type $(+1,+1)$ and $(n-r)$ identical pairs of the type $(-1,-1)$. Now to satisfy condition (5) of Proposition 2.4.2 we have to plug in a 0 between every two consecutive $+1$'s and every two consecutive -1's. So the number of 0's plugged in is $2(n-r)$. The remaining $g-2-2n-2(n-r) = g-4n+2r-2$ places are also to be filled up with 0's. There are $(2n+1)$ distinct places for each such path in which $(g-4n+2r-2)$ 0's can be plugged in $\binom{(g-4n+2r-2)+(2n+1)-1}{g-4n+2r-2} = \binom{g-2(n-r)-2}{2n}$ ways. Since n can vary from 1 to $\lfloor\frac{g-2}{2}\rfloor$, the total number of such paths is

$$\sum_{n=1}^{\lfloor\frac{g-2}{2}\rfloor}\sum_{r=1}^{n}\left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}\right]\binom{g-2(n-r)-2}{2n}.$$

Note that these paths do not include the trivial path corresponding to the sequence $(0,0,0,\ldots,0)$. Hence the theorem follows.

Theorem 2.4.3 *The number of CFPs for a linear array with bidirectional bypass links of length* 2, 3, $\ldots$, k *and* g *(i.e., with link redundancy* $G = \{2,\ 3\ \ \ldots,\ k,\ g\}$, $k < g-1$*) is*

$$F^B(2,3\ldots,k,g) = 1+\sum_{n=1}^{\lfloor\frac{g-k}{2}\rfloor}\sum_{r=1}^{n}\left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}\right]\binom{g-k-2(n-r)(k-1)}{2n}$$

Proof. The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-k})$ satisfying conditions $(2)-(5)$ of Proposition 2.4.3. The proof of the present theorem is similar to that of Theorem 2.4.2. Here to satisfy condition (5) of Proposition 2.4.3, we have to plug in $(k-1)$ 0's between every two consecutive +1's and between every two consecutive −1's.

2.5 The Case of Unidirectional Links

Proposition 2.5.1 (Pagli *et al.* [76]). Necessary and sufficient conditions for $(m_1, m_2, \ldots, m_{g-1})$ to be the catastrophic sequence of a minimal CFP for a unidirectional linear array with link redundancy $G = \{g\}$ are:

1. $m_j \leq 1$ for $j = 1, 2, \ldots, g-1$
2. $\sum_{j=1}^{k} m_j \leq 0$ for $k = 1, 2 \quad , g-2$
3. $\sum_{j=1}^{g-1} m_j = 0$

We shall illustrate Proposition 2.5.1 by an example.

Example 2.5.1 Consider the fault pattern $F = \{1, 4, 11, 12, 14, 15\}$ in a linear array A with link redundancy $G = \{6\}$. Note that in the matrix representation for F there is exactly one faulty PE in each column. See Figure 2.6.

Its row representation is (0, 2, 2, 0, 1, 1) and its catastrophic sequence is $(m_1, m_2, m_3, m_4, m_5) = (-2, 0, +2, -1, 0)$. Note that, $m_3 = +2$ which violates condition 1 of Proposition 2.5.1. We see from Figure 2.6, that the interior processor at location $(1, 2)$ and the exterior processor at location $(1, 3)$ are connected by a regular link. Hence F is not a catastrophic fault pattern by Lemma 2.2.1. Similarly condition 3 of Proposition 2.5.1 is violated since $\sum_{j=1}^{g-1} m_j = -1$. Note that, locations $(0, 5)$ and $(1, 0)$ contain an interior and an exterior processor respectively which are connected by a regular link. However, it

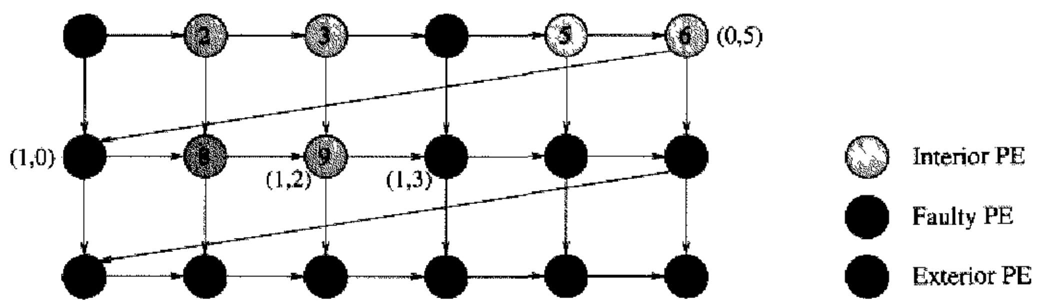


Figure 2.6: Interior, exterior and border of the fault pattern F.

can easily be verified that F satisfies condition (3) of Proposition 2.5.1 even though F is not catastrophic.

In general, we have the following characterization for $k > 1$.

Proposition 2.5.2 Necessary and sufficient conditions for $(m_1, m_2, \ldots, m_{g-1})$ to be the catastrophic sequence of a minimal CFP for a unidirectional linear array with link redundancy $G = \{2,\ 3,\ \ldots,\ k,\ g\}$, $k < g-1$ are:

1. $m_{g-1} = m_{g-2} = \ldots = m_{g-k+1} = 0$
2. $m_j \leq 1$ for $j = 1,\ 2,\ \ldots,\ g-k$
3. $\sum_{j=1}^{p} m_j \leq 0$ for $p = 1,\ 2,\ 3,\ \ldots,\ g-k-1$
4. $\sum_{j=1}^{g-k} m_j = 0$
5. $m_i + m_{i+1} + \ldots + m_j \leq 1$ if $1 \leq j - i \leq k-1$.

Theorem 2.5.1 *(De Prisco [22]). The number of CFPs for a linear array with unidirectional bypass links of length g (i.e., with link redundancy $G = \{g\}$) is*

$$F^U(g) = \frac{1}{g}\binom{2g-2}{g-1}.$$

Proof: The proof given here is little different from that of De Prisco [22] and is relevant to the proof of theorems to follow. The theorem is proved by establishing a bijective mapping between the set of all sequences $(m_1, m_2, \ldots m_{g-1})$ satisfying the conditions of Proposition 2.5.1 and the set of all paths from $(0,0)$ to $(2g-2, 0)$ such that $S_1 \leq 0,\ S_2 \leq 0,\ \quad,\ S_{2g-3} \leq 0,\ S_{2g-2} = 0$ since the number of such paths is $\frac{1}{g}\binom{2g-2}{g-1}$. Let F be a CFP and $(m_1, m_2, \ldots m_{g-1})$ its catastrophic sequence. Let $s(m_i)$ be the string $-1, -1, \ldots, -1, +1$ of length $2 - m_i$. To the CFP F associate the string $s(F)$ obtained by concatenating $s(m_1), s(m_2), \ldots, s(m_{g-1})$. From Proposition 2.5.1, it is clear that $s(F)$ corresponds to a path from $(0,0)$ to $(2g-2, 0)$ with the properties stated above.

Theorem 2.5.2 *The number of CFPs for a linear array with unidirectional bypass links of lengths 2 and g (i.e., with link redundancy $G = \{2,\ g\}$) is*

$$F^U(2,g) = \sum_{n=0}^{\lfloor \frac{g-2}{2} \rfloor} \frac{1}{n+1}\binom{2n}{n}\binom{g-2}{2n}.$$

Proof: The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-2})$ satisfying conditions (2)-(5) of Proposition 2.5.2 with $k = 2$. Given such a catastrophic sequence, by using the above mapping we get $s(F) = (x_1, x_2, \ldots, x_{2(g-2)})$ with the following properties:

$$\begin{array}{ll} (1)\ x_i = -1 \text{ or } +1 \text{ for } i = 1,\ 2,\ \ldots,\ 2(g-2) & \\ (2)\ \sum_{i=1}^{k} x_i \leq 0 \text{ for } k = 1,\ 2,\ \ldots,\ 2(g-2)-1 & \\ (3)\ \sum_{i=1}^{2(g-2)} x_i = 0 & \qquad (A) \\ (4)\ x_i + x_{i+1} + x_{i+2} \leq 1 \text{ for } i = 1,\ 2,\ \ldots,\ 2(g-2)-2 & \end{array}$$

Given any sequence $(x_1, x_2, \ldots, x_{2(g-2)})$ satisfying (A), let y_i be the number of $+1$'s between the i-th -1 and the $(i+1)$-th -1 and $z_i = y_i - 1$. Then (4), (2) and (3) of (A) give

$$\begin{aligned} &(a)\ z_i = -1, 0 \text{ or } +1 \text{ for } i = 1,\ 2\ \ \ldots,\ g-3 \\ &(b)\ z_1 + z_2 + \ldots + z_k \leq 0 \text{ for } k = 1,\ 2,\ \ldots,\ g-4 \\ &(c)\ z_1 + z_2 + \ldots + z_{g-3} = 0 \text{ or } -1 \end{aligned} \qquad (B)$$

respectively. It is also clear that any sequence $(z_1,\ z_2,\ \ \ ,\ z_{g-3})$ satisfying (B) arises from a unique sequence $(x_1, x_2, \ldots, x_{2(g-2)})$ satisfying (A). So the number of CFPs is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-3})$ satisfying (B). Now the number of such sequences is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-2})$ satisfying the following conditions (where z_{g-2} is taken to be 0 or 1 according as $z_1 + z_2 \ldots + z_{g-3}$ is 0 or -1):

$$\begin{aligned} &(a)\ z_i = -1,\ 0 \text{ or } +1 \text{ for } i = 1,\ 2\ \ \ ,\ g-2 \\ &(b)\ z_1 + z_2 + \ldots + z_k \leq 0 \text{ for } k = 1,\ 2,\ \ldots,\ g-3 \\ &(c)\ \textstyle\sum_{i=1}^{g-2} z_i = 0. \end{aligned}$$

Any such sequence with n -1's has n $+1$'s and $g-2-2n$ 0's and can be obtained by starting with a path from $(0,0)$ to $(2n, 0)$ such that $S_1 \leq 0,\ S_2 \leq 0,\ \ldots,\ S_{2n-1} \leq 0,\ S_{2n} = 0$ and plugging in $g-2-2n$ 0's in the $(2n+1)$ distinct places ($2n-1$ intermediate places and two more places before and after the sequence). Clearly the number of such paths is $\frac{1}{n+1}\binom{2n}{n}$ and for each path, the number of ways of plugging in the 0's is $\binom{(g-2-2n)+(2n+1)-1}{2n} = \binom{g-2}{2n}$. Hence the theorem follows.

Theorem 2.5.3 *The number of CFPs for a linear array with unidirectional bypass links of lengths 2, 3 and g (i.e., with link redundancy $G = \{2,\ 3,\ g\}$) is*

$$F^U(2,3,g) = 1 + \sum_{n=1}^{\lfloor \frac{g-3}{2} \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2} \right] \binom{g-3-(n-r)}{2n}$$

Proof: The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-3})$ satisfying conditions (2)-(5) of Proposition 2.5.2 with $k = 3$. Given a catastrophic sequence, by using the mapping given in the proof of Theorem 2.5.1, we get $s(F) = (x_1, x_2, \ldots, x_{2(g-3)})$ with the following properties:

$$\begin{aligned}
&(1)\ x_i = -1 \text{ or } +1 \text{ for } i = 1, 2, 3, \ldots, 2(g-3) \\
&(2)\ \textstyle\sum_{i=1}^{k} x_i \leq 0 \text{ for } k = 1, 2, \ldots, 2(g-3) - 1 \\
&(3)\ \textstyle\sum_{i=1}^{2(g-3)} x_i = 0 \qquad\qquad (C) \\
&(4.1)\ x_i + x_{i+1} + x_{i+2} \leq 1 \text{ for } i = 1, 2, \ldots, 2(g-3) - 2 \text{ and} \\
&(4.2)\ x_i + x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4} \leq 1 \text{ for } i = 1, 2, \ldots, 2(g-3) - 4.
\end{aligned}$$

Given any sequence $(x_1, x_2, \ldots, x_{2(g-3)})$ satisfying (C), let y_i be the number of $+1$'s between the i-th -1 and the $(i+1)$-th -1 and $z_i = y_i - 1$. Then (4.1), (4.2), (2) and (3) of (C) give

$$\begin{aligned}
&(a.1)\ z_i = -1, 0 \text{ or } +1 \text{ for } i = 1, 2, \ldots, g-4 \\
&(a.2)\ -2 \leq z_i + z_{i+1} \leq 1 \text{ for } i = 1, 2, \ldots, g-5 \qquad (D) \\
&(b)\ z_1 + z_2 + \ldots + z_k \leq 0 \text{ for } k = 1, 2, \ldots, g-5 \\
&(c)\ z_1 + z_2 + \ldots + z_{g-4} = -1 \text{ or } 0.
\end{aligned}$$

respectively. It is also clear that any sequence $(z_1, z_2, \ldots, z_{g-4})$ satisfying (D) arises from a unique sequence $(x_1, x_2, \ldots, x_{2(g-3)})$ satisfying (C). So the number of CFPs is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-4})$ satisfying (D). Now the number of such sequences is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-3})$ satisfying the following conditions (where z_{g-3} is taken to be 0 or 1 according as $z_1 + z_2 + \ldots + z_{g-4}$ is 0 or -1):

$$\begin{aligned}
&(a.1)\ z_i = -1,\ 0 \text{ or } +1 \text{ for } i = 1, 2, \ldots, g-3 \\
&(a.2)\ -2 \leq z_i + z_{i+1} \leq 1 \text{ for } i = 1, 2, \ldots, g-4 \qquad (E) \\
&(b)\ z_1 + z_2 + \ldots + z_k \leq 0 \text{ for } k = 1, 2, \ldots, g-4 \\
&(c)\ \textstyle\sum_{i=1}^{g-3} z_i = 0.
\end{aligned}$$

Any such sequence with n -1's has n $+1$'s and $g-3-2n$ 0's and can be obtained by starting with a path from $(0,0)$ to $(2n,0)$ such that $S_1 \leq 0$, $S_2 \leq 0$, , $S_{2n-1} \leq 0$, $S_{2n}=0$ and having exactly $2r$ runs. The number of such paths is $\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2}$ by Theorem 2.3.3. R(run)= 1+ *number of changes of the type* $(-1,+1)$ or $(+1,-1)$. All these paths have $(n-r)$ identical pairs of the type $(+1,+1)$ and $(n-r)$ identical pairs of the type $(-1,-1)$. Now to satisfy condition (a.2) of (E) we have to plug in a 0 between every two consecutive $+1$'s. So the number of 0's plugged in is $(n-r)$. The remaining $g-3-2n-(n-r) = g-3n+r-3$ places are also to be filled up with 0's. There are $(2n+1)$ distinct places for each such path in which $g-3n+r-3$ 0's can be plugged in $\binom{g-3-(n-r)}{2n}$ ways. Since n can vary from 1 to $\lfloor \frac{g-3}{2} \rfloor$, the total number of such paths is

$$\sum_{n=1}^{\lfloor \frac{g-3}{2} \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2} \right] \binom{g-3-(n-r)}{2n}$$

Note that these paths do not include the trivial path corresponding the sequence $(0,0,\ldots,0)$. Hence the theorem follows.

Theorem 2.5.4 *The number of CFPs for a linear array with unidirectional bypass links of length* 2, 3 , k *and* g *(i.e., with link redundancy* $G = \{1,2,\ldots,k,g\}$, $k < g-1$*) is*

$$F^U(2,3,\ldots,k,g) = 1 + \sum_{n=1}^{\lfloor \frac{g-k}{2} \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2} \right] \binom{g-k-(n-r)(k-2)}{2n}.$$

Proof: The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-k})$ satisfying conditions $(2)-(5)$ of Proposition 2.5.2. By using the same argument as in the proof of Theorem 2.5.3, the number of such sequences is equal to the number of the sequences $(z_1, z_2, \ldots, z_{g-k})$ satisfying the following conditions

$$
\left.\begin{array}{l}
(a.1)\ z_i = -1,\ 0 \text{ or } +1 \text{ for } i = 1,\ 2,\ \ldots,\ g-k \\
(a.2)\ -2 \le z_i + z_{i+1} \le 1 \text{ for } i = 1,\ 2,\ \ldots,\ g-k-1 \\
\\
(a.k-1)\ \ -(k-1) \le z_i + z_{i+1} \ldots + z_{i+k-2} \le 1 \text{ for } i = 1, 2, \ldots, g-2k+2 \\
(b)\ z_1 + z_2 + \ldots + z_p \le 0 \text{ for } p = 1, 2, \ldots, g-k-1 \\
(c)\ \sum_{i=1}^{g-k} z_i = 0.
\end{array}\right\} \quad (F)
$$

Counting the number of such sequences is done as in Theorem 2.5.3 except that instead of plugging in one 0 we have to plug in $(k-2)$ 0's between any two consecutive +1's to satisfy conditions $(a.2) - (a.k-1)$ of (F). Hence the theorem follows.

Chapter 3

Identification of Maximal Link Redundancy for which a Given Fault Pattern is Catastrophic in VLSI Linear Arrays

3.1 Introduction

In this chapter, we consider the problem of finding the maximal link configuration for which a given fault pattern F is catastrophic. See also [65]. We consider maximality with respect to two parameters: the length g of the longest bypass link in G and the number $|G|$ of bypass links in G. The problem of minimization of the parameters is trivial since any F is catastrophic when $G = \emptyset$.

In reality, the problem of finding a minimal link configuration G for which a given fault pattern F is not catastrophic is more important. Since the designer can adopt G to ensure that F cannot disrupt the flow of information from I to O. Depending on designer choice, minimality can be with respect to various parameters like: the length g of the longest bypass link, the number $|G|$ of bypass links in G or the sum $\sum_{i=1}^{k} g_i$ of the lengths of bypass links in G. The problem of finding the minimum value of the length g of the longest bypass link in G for which a given fault pattern F is not catastrophic is easy. Since we will now show that if $F = \{(f_1, \ell_1), (f_2, \ell_2), \ldots, (f_n, \ell_n)\}$ then the minimum value of g is $\ell+1$ where $\ell = \max(\ell_1, \ell_2, \ldots, \ell_n)$. We first show that F is not catastrophic with respect to link configuration $G_0 = \{2, 3, \ldots, \ell+1\}$. Let $H = (V, E)$ be the derived graph for link redundancy G_0. Let $C_0, C_1, \ldots, C_n$ be the chunks of F. Then $V = \{C_0, C_1, \ldots, C_n\}$ and $(C_i, C_{i+1}) \in E$ for all $i = 0, 1, \ldots, n-1$. Hence C_0 and C_n are connected in H and so F is not catastrophic with respect to G_0. If possible, suppose F is also not catastrophic with respect to some G with the length g of the longest bypass link $\leq \ell$. W.l.g. we assume $= \ell_1$. Then in the derived graph $H = (V, E)$ for link redundancy G, $(C_0, C_i) \notin E$ for all $i = 1, 2, \ldots, n$. So C_0 and C_n are not connected in H and F is catastrophic with respect to G, a contradiction. This proves that min $g = \ell + 1$. However, studying minimality with respect to the other two parameters, i.e., $|G|$ and $\sum_{i=1}^{k} g_i$, seems to be difficult.

The maximization problem we consider gives a partial solution to the minimization roblem thus: If max g and max $|G|$ for the maximization problem are g_0 and s then F is not catastrophic with respect to any G with the length of the longest bypass link $\geq g_0 + 1$, as well as with respect to any G with $|G| \geq s + 1$. For example, for the fault pattern F given in Example 3.3.1, max $g = 8$, the corresponding maximal G being $\{8\}$ and max $|G| = 3$, the corresponding maximal G being $\{4, 5, 8\}$. Thus if $g \geq 9$ or $|G| \geq 4$ then the fault pattern is not catastrophic with respect to G. Hence one can use $\{9\}$ or $\{2, 3, 4, 5\}$ as a link redundancy. However, these are not optimal (i.e. minimal) since the given fault pattern is not catastrophic with respect to each of $\{6\}$ and $\{2, 4\}$. Thus the solution to the maximization problem we consider gives a reasonably good feasible solution, which may not be optimal, for the minimization problem.

Given a fault pattern of m faults grouped into $n \leq m$ runs of faulty processors, we show that the maximum value of g can be found in $O(mn)$ time and that the problem of finding max $|G|$ is equivalent to a graph problem, which looks somewhat similar to a min-cut problem.

Since the concept of derived graph is the main tool used in this chapter, we recall its definition. Let A be a bidirectional array of N processors with link redundancy $G = \{g_1, g_2, .., g_k\}$, and let F be a set of m faults grouped into $n \leq m$ runs of faulty processors. Then a graph $H = (V, E)$ was defined in [21] as follows:
Let $C_0, C_1,$, C_n be the chunks of F. Then $V = \{C_0, C_1, \ldots, C_n\}$ and $(C_i, C_j) \in E$ if *and only if there are two processors,* $p_x \in C_i$ and $p_y \in C_j$ such that $|y-x| \in \{g_1, g_2, \ldots, g_k\}$, that is, if and only if some processor in C_i and some processor in C_j are connected in A by a bypass link.

We call the graph H the *derived graph* of the fault pattern F. By definition of derived graph it follows that a fault pattern F is not catastrophic for an array A, if and only if C_0 is connected with C_n in the derived graph.

Figure 3.1 shows an algorithm, called GRAPH, which constructs the derived graph. Inputs to GRAPH are the fault pattern F and the link redundancy G. The output is he derived graph represented by its adjacency lists. The following may be noted: $L(C_i)$ denotes the adjacency list of node C_i at any stage, for $i = 0, 1,$, n. The first and last processors of chunk C_i are denoted by x_i and y_i respectively.

Lemma 3.1.1 [21] Algorithm GRAPH constructs the derived graph.

Lemma 3.1.2 [21] Algorithm GRAPH requires $O(kn)$ time.

```
GRAPH(F, {g_1, g_2, ..., g_k}, H)
    for i = 0 to n - 1 do
        L(C_i) = φ (null set)
    endfor
    for t = 1 to k do
        j = 1
        for i = 0 to n - 1 do
            while x_i + g_t > y_j do
                j = j + 1
            endwhile
            while j ≤ n and x_j ≤ y_i + g_t do
                L(C_i) = L(C_i) ∪ C_j
                L(C_j) = L(C_j) ∪ C_i
                j = j + 1
            endwhile
            if y_{j-1} ≥ x_{i+1} + g_t then j = j - 1
        endfor
    endfor
    H is given by L(C_i) for i = 0, 1, ... n
return (H)
```

Figure 3.1: Algorithm GRAPH

3.2 Algorithm to find a G, with maximum length of the longest bypass link, for which a given fault pattern is catastrophic

We now give an algorithm to find a link configuration G, with maximum length of the longest bypass link, with respect to which a given fault pattern is catastrophic.

Algorithm 1 Input: A fault pattern F of m faults grouped into $n \leq m$ runs of faulty processors.

Output: A link configuration G, with the maximum length of the longest bypass link, with respect to which F is catastrophic.

Step 1: Set $u = m$.

Step 2: Construct GRAPH($F, \{u\}, H$)

if C_0 and C_n are connected in H, decrease u by 1 and go to step 2.

else return ($\{u\}$).

Lemma 3.2.1 Algorithm 1 generates a link configuration G, with maximum length of the longest bypass link, for which a given fault pattern F is catastrophic.

Proof. Let l be the maximum length of a longest bypass link in a link configuration with respect to which F is catastrophic. Let $G = \{g_1,\ g_2,\ \ldots,\ g_k = l\}$ be a link configuration with respect to which F is catastrophic. It is known that $l \leq m$. By the definition of l, it is clear that for any u with $l + 1 \leq u \leq m$, F is not catastrophic with respect to the link redundancy $\{u\}$. Also F is catastrophic with respect to the link redundancy $\{l\}$ since $\{l\} \subseteq G$. Hence it follows that the algorithm starts with $u = m$ and goes on reducing u by 1 until u attains the value l and then stops. Thus the algorithm returns the value l. ∎

Lemma 3.2.2 *Algorithm 1 requires $O(mn)$ time.*

Proof. Note that, GRAPH(F, $\{u\}$, H) requires $O(n)$ time [Lemma 3.1.2]. Since, Step 2 can repeat atmost m times hence Algorithm 1 requires $O(mn)$ time. ∎

3.3 Algorithm to find a G with maximum number of bypass links, for which a given fault pattern is catastrophic

We next give an algorithm to find a link configuration G, with maximum number of bypass links, for which a given fault pattern is catastrophic.

Algorithm 2:

Input: A fault pattern F of m faults grouped into $n \leq m$ runs of faulty processors.

Output: A link configuration G, with maximum number of bypass links, with respect to which F is catastrophic.

Step 1: Set $u = m$, and $G = \phi$.

Step 2: If $u \leq |G| + 1$, go to Step 5. Otherwise go to Step 3.

Step 3: Construct GRAPH(F, $\{u\}$, H). If C_0 and C_n are connected in H then decrease u by 1 and go to step 2. If C_0 and C_n are not connected in H, go to Step 4.

Step 4: Let $S_0, S_1, \ldots, S_h$ $(h \geq 1)$ be the components of H where $C_0 \in S_0$, $C_n \in S_h$. Now construct a new graph $H^* = (V^*, E^*)$ thus: $V^* = \{S_0, S_1, \ldots, S_h\}$. For $0 \leq i \neq j \leq h$, let $L_{ij} = \{\alpha : 1 \leq \alpha \leq u-1 \text{ and } \alpha = |x-y| \text{ for some PE } x \in S_i \text{ and some PE } y \in S_j\}$. Then $(S_i, S_j) \in E^*$ if and only if $L_{ij} \neq \phi$. We call L_{ij} the label set of edge (S_i, S_j). If $X \subseteq V^*$ is such that $S_0 \in X$ and $S_h \in \bar{X} := V^* - X$, we say that $(X, \bar{X})$ is a cut and define $L_X = \bigcup\{L_{ij} : S_i \in X \text{ and } S_j \in \bar{X}\}$. Find a cut $(X_0, \bar{X}_0)$ for which $|L_X|$ is minimum and let $G^* = \{2, 3, \ldots, u\} - L_{X_0}$. If $|G^*| > |G|$ then set $G = G^*$. Decrease u

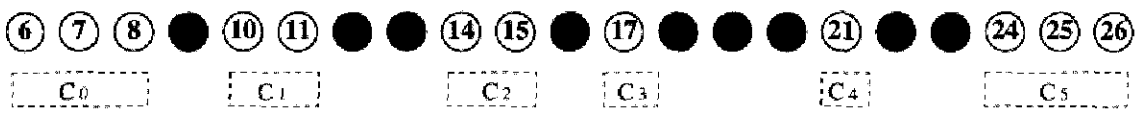
by 1 and go to Step 2.

Step 5: Stop. G is a link redundancy with maximum number of bypass links with respect to which F is catastrophic.

Example 3.3.1 Consider the fault pattern

$$F = \{(9,\ 1),\ (12,\ 2),\ (16,\ 1),\ (18,\ 3), (22, 2)\}$$

of 9 faulty PEs grouped into 5 runs of faulty processors.



We set $u = 9$ and $G = \phi$. Since $u > |G| + 1$, we construct the derived graph H for link redundancy $\{9\}$. Note that there are six chunks: $C_0 = \{1,\ 2,\ \ldots,\ 8\}$, $C_1 = \{10,\ 11\}$, $C_2 = \{14,\ 15\}$, $C_3 = \{17\}$, $C_4 = \{21\}$, $C_5 = \{24,\ 25,\ \ldots,\ 30\}$. The graph H is shown in Figure 3.2.

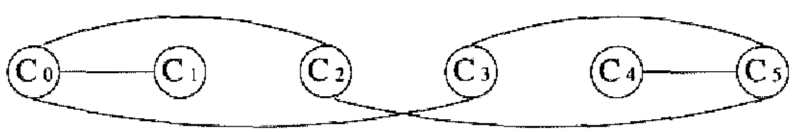


Figure 3.2: Graph H.

Clearly, C_0 and C_5 are connected, so we decrease u to 8. Since $u = 8 > |G| + 1 = 1$, we construct the derived graph H for link redundancy $\{8\}$ which is shown in Figure 3.3.

Here C_0 and C_5 are not connected, so construct the graph $H^* = (V^*,\ E^*)$ which is shown in Figure 3.4.

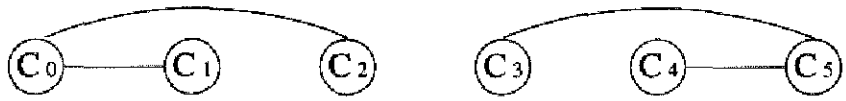


Figure 3.3: Graph H.

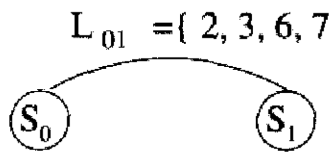


Figure 3.4: Graph H^*.

Note that $V^* = (S_0,\ S_1)$ where $S_0 = \{C_0,\ C_1,\ C_2\}$ and $S_1 = \{C_3,\ C_4,\ C_5\}$. The PEs $15 \in S_0$ and $17 \in S_1$ give $2 \in L_{01}$. Note that 11 and 17 as well as 15 and 21 give $6 \in L_{01}$. It can be verified that $L_{01} = \{2,\ 3\ \ 6,\ 7\}$. Here there is only one cut, viz. $X = \{S_0\}$ and $L_{X_0} = L_X = L_{01} = \{2,\ 3\ \ 6,\ 7\}$. Therefore we take $G^* = \{2,\ 3,\ 4,\ \ \ ,\ 8\} - \{2,\ 3,\ 6,\ 7\} = \{4,\ 5,\ 8\}$. It may be noted that F is catastrophic with respect to link redundancy $G^* = \{4,\ 5,\ 8\}$. Since $3 = |G^*| > |G| = 0$, we set $G = \{4,\ 5,\ 8\}$. Then we decrease u to 7. Since $u = 7 > |G| + 1 = 4$, we construct the derived graph H for link redundancy $\{7\}$ which is shown in Figure 3.5.

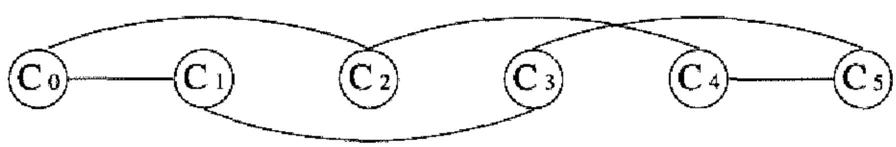


Figure 3.5: Graph H.

Since C_0 and C_5 are connected in H, we reduce u to 6. Since $u = 6 > |G| + 1 = 4$, we construct the derived graph H for link redundancy $\{6\}$ which is shown in Figure 3.6.

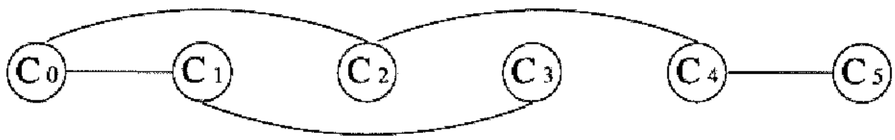


Figure 3.6: Graph H.

Since C_0 and C_5 are connected in H, we reduce u to 5. Since $u = 5 > |G| + 1 = 4$, we construct the derived graph H for link redundancy $\{5\}$ which is shown in Figure 3.7.

Figure 3.7: Graph H.

Here C_0 and C_5 are not connected. So we construct the graph $H^* = (V^*,\ E^*)$ which is hown in Figure 3.8.

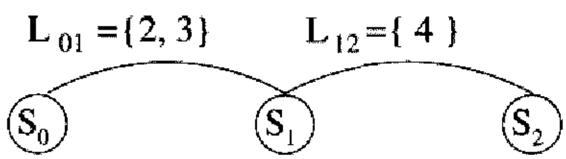


Figure 3.8: Graph H^*.

Note that $V^* = (S_0,\ S_1,\ S_2)$ where $S_0 = \{C_0,\ C_1,\ C_2\}$, $S_1 = \{C_3\}$ and $S_3 = \{C_4,\ C_5\}$. Here with $X = \{S_0\}$, we have $|L_X| = |\{2,\ 3\}| = 2$ and with $X = \{S_0,\ S_1\}$, we have $|L_X| = |\{4\}| = 1$. Hence we take $X_0 = \{S_0,\ S_1\}$ and $G^* = \{2,\ 3,\ 4,\ 5\} - \{4\} = \{2,\ 3,\ 5\}$. Since $|G^*| \not> |G| = 3$, we do not alter G. We next decrease u to 4. Now $u \leq |G| + 1$, so the algorithm stops and $G = \{4,\ 5,\ 8\}$ is a link redundancy with maximum number of bypass links with respect to which the given F is catastrophic. ■

In Lemma 3.3.1, Lemma 3.3.2 and Theorem 3.3.1, a link redundancy G is called catastrophic link redundancy with respect to a given fault pattern F if the given fault pattern F is catastrophic with respect to the link redundancy G.

Lemma 3.3.1 In Algorithm 2, G^* obtained in Step 4 is catastrophic link redundancy and the largest element of G^* is current u.

Proof: Suppose there is a path from I to O using bypass links of lengths belonging to G^*. Then it is easy to see that there exist PEs $x \in X_0$ and $y \in X_0$ joined by a bypass link of length ω (say) belonging to G^*. But then $\omega \in L_{X_0}$, a contradiction to the definition of G^*. This proves that F is catastrophic with respect to G^*. ■

Lemma 3.3.2 In Algorithm 2, G^* has the maximum size among all catastrophic link redundancies with the (current) u as the length of the longest bypass link.

proof: Suppose G' is a catastrophic link redundancy with u as the length of the longest bypass link and $|G'| > |G^*|$. Let H be the derived graph with respect to link redundancy $\{u\}$ and let $S_0, S_1, \ldots, S_h$ be the components of H, where $C_0 \in S_0$ and $C_n \in S_h$. Since G' is catastrophic link redundancy, there is a partition of $\{S_0, S_1, \quad, S_h\}$ into X' and $\bar{X}'$ with $S_0 \in X'$ and $S_h \in \bar{X}'$ such that if a PE $x \in X'$ and a PE $y \in \bar{X}'$ then $|x-y| \notin G'$. So if $\alpha \in L_{X'}$ then $\alpha \notin G'$. Hence $L_{X'} \subseteq \{2, 3 \quad \ldots, u\} - G'$. Since $|G'| > |G^*|$ and $G^* = \{2, 3 \quad , u\} - L_{X_0}$, we get $|L_{X'}| \leq (u-1) - |G'| < (u-1) - |G^*| = |L_{X_0}|$, a contradiction to the minimality of $|L_{X_0}|$. ■

Theorem 3.3.1 *Algorithm 2 provides a link configuration G, with maximum number of bypass links, for which a given fault pattern F is catastrophic.*

Proof: By Lemma 3.3.1, the G_0 returned by the Algorithm 2 is catastrophic link redundancy. Moreover, when the algorithm terminates, the final value of u, say u_0, is $|G_0| + 1$.

Suppose now there exists a catastrophic link redundancy G' with $|G'| > |G_0|$. Let v be the length of the longest bypass link in G'. Then $v \geq |G'| + 1 > |G_0| + 1 = u_0$. It is known that $v \leq m$. Hence when u equals v, the algorithm would have found a catastrophic link redundancy G^* with $|G^*| \geq |G'|$ and since the size of G increases as the algorithm processes, it follows that $|G_0| \geq |G'|$, a contradiction. This proves that G_0 is a link configuration, with maximum number of bypass links, with respect to which F is catastrophic. ∎

3.4 Conclusion

Note that, given a fault pattern of m faults grouped into $n \leq m$ runs of faulty processors we have reduced the problem of finding a link configuration G, with maximum number of bypass links, to the following graph problem:

Given a graph $H^* = (V, E)$ with two specified vertices $s, t \in V$, called the "source" and "terminus" respectively and a set $L_{ij} \subseteq \{1, 2, \ldots, u\}$ for each edge (i, j). The problem is to find a partition of V into V_1 and V_2 such that $s \in V_1$, $t \in V_2$ and $|\bigcup_{i \in V_1, j \in V_2} L_{ij}|$ is minimum. However we mention that we do not have any good algorithm to solve this graph problem which looks somewhat similar to the min-cut problem but seems to be more difficult.

Example 3.4.1 Figure 3.9 shows a graph $H^* = (V, E)$ with $L_{sa} = \{2,\ 4\}$, $L_{sb} = \{1,\ 3,\ 6\}$, $L_{ab} = \{5\}$, $L_{at} = \{4,\ 6\}$ and $L_{bt} = \{2,\ 4,\ 5\}$. Here a partition which minimizes $|\bigcup_{i \in V_1,\ j \in V_2} L_{ij}|$ is $V_1 = \{s,\ b\}$, $V_2 = \{a,\ t\}$.

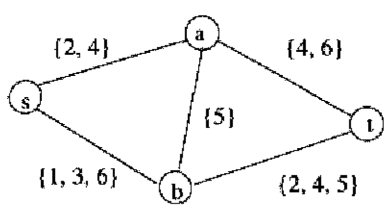


Figure 3.9: A graph

Chapter 4

Catastrophic Faults in Reconfigurable VLSI Two-dimensional Array

4.1 Introduction

The main contribution of this chapter is complete characterization of catastrophic fault patterns for two-dimensional arrays. See also [61]. We prove that the minimum number of faults in a catastrophic fault pattern is a function of the length of the longest horizontal bypass link and the number of rows in the two-dimensional array. From a practical viewpoint, this result provides some answers to the question about the guaranteed level of fault tolerance of a design. Guaranteed fault tolerance refers to an affirmative answer to the question: will the system withstand upto s faults always regardless of how and where they occur? We analyze catastrophic sets having the minimal number of faults and give an algorithm for constructing a catastrophic fault pattern with maximum width. We also give an algorithm for testing whether a set of faults is catastrophic or not.

4.2 Characterization of Catastrophic Fault Patterns

In this section, we will characterize the catastrophic fault patterns for two-dimensional networks and prove that the minimum number of faults in a catastrophic fault pattern is a function of N_1 and the length of the longest horizontal bypass link.

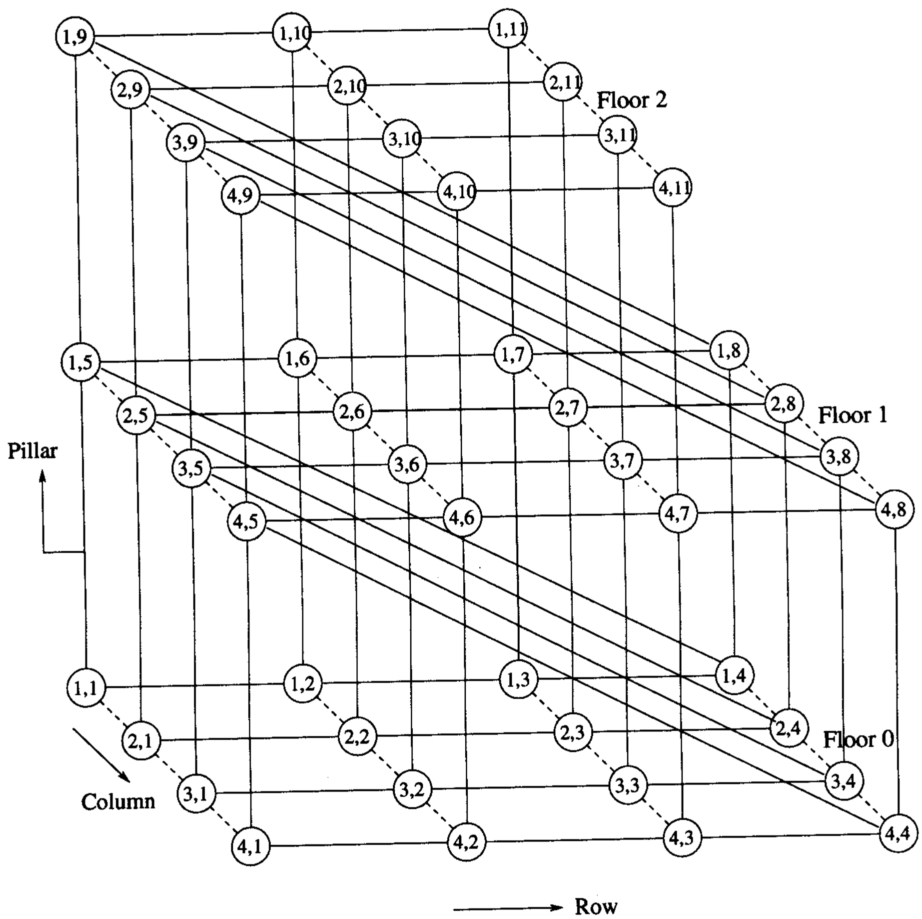


Figure 4.1: Cuboid representation of a 4×11 array with link redundancy $G = (1,\ 4 \mid 1)$.

Theorem 4.2.1 *F is catastrophic with respect to $\mathcal{N}$ implies that the cardinality of F, $|F| \geq N_1 g_k$.*

Proof: Suppose to the contrary that $|F| < N_1 g_k$. Then partition the two-dimensional array A of PEs into blocks of g_k columns as $A = (A_1 \vdots A_2 \vdots \ \ldots \ \vdots A_c)$ where $c = \lceil \frac{N_2}{g_k} \rceil$ and place the blocks as consecutive floors to form a cuboid. In Figure 4.1, we show the cuboid for the array given in Figure 1.11 (vertical bypass links are not drawn in the cuboid). Observe that, in this cuboid representation, each horizontal regular link joins two consecutive elements in the same row of a floor or the last element of a row of a floor with the first element of the corresponding row of the floor just above it whereas each vertical regular link joins two consecutive elements in the same column of a floor. On the other hand each horizontal bypass link of the maximum length joins two consecutive elements in the same pillar. So, in this cuboid, going down a pillar corresponds to using the longest bypass links. Since the number of faulty elements $|F|$ is less than the size of a block which is also the number of pillars, there must be a pillar with no faulty element, regardless of the distribution of the fault pattern. Since the bottom and top of each pillar are linked to ICUL and ICUR respectively, F cannot be catastrophic since we can use the bypass links of length g_k to avoid the faulty PEs, a contradiction which proves the theorem. ∎

This theorem gives us a necessary condition on the minimum number of faults required for blocking a two-dimensional array. This also tells us that fewer than $N_1 g_k$ faults occurring in A will not be catastrophic. *In the following we will restrict ourselves to the case where there are at least $N_1 g_k$ faults, and we will characterize the blocking fault patterns containing exactly $N_1 g_k$ faults.*

Not all fault patterns consisting of $N_1 g_k$ faults are catastrophic. Some additional properties must be satisfied. Before we describe further characteristics of a catastrophic fault pattern (CFP), we outline an algorithm for the construction of a CFP with the maximum width for a given link redundancy G when links are bidirectional, and backtracking is allowed (i.e. one may go from $p_{i,j+g}$ to p_{ij} where g is a horizontal bypass link) in attempts to bypass faulty PEs.

Algorithm 1: Construction of a Catastrophic Fault Pattern for Bidirectional Horizontal and Bidirectional Vertical Links

Input: G.

Output: A catastrophic fault pattern F with the maximum width.

Step 1. Partition the two-dimensional array of PEs into blocks of g_k columns and list the blocks as the floors of a cuboid. Mark the first element of the N_1-th row in floor 0 by an X and set $f = 1$.

Step 2 If there exists an unmarked element $u = (i, j)$ in floor f such that the element $v = (i, j - g_k)$ below u in floor $f - 1$ is marked, choose one such u and go to Step 4. Otherwise go to Step 3.

Step 3 If there is an unmarked element in floor f, then increase f by 1 and go to Step 2. Otherwise, go to Step 5.

Step 4. If v is marked Y, then mark u by Y and go to Step 2.

If v is marked X, then mark u by Y and mark every unmarked element w which is of the form $(i, j \pm g)$ where $g \in \{g_1, g_2, \ldots, g_{k-1}\}$ or $(i \pm v, j)$ where $v \in \{v_1, v_2, \ldots, v_l\}$. Mark w by Y if the pillar of w contains another marked element; otherwise mark w by X. Go to Step 2.

Step 5 Stop. Note that all elements in floor f are marked. The elements marked X form a catastrophic fault pattern F with maximum width for link redundancy G.

Note that the algorithm assigns exactly $N_1 g_k$ number of X's.

Example 4.2.1 Figure 4.2 shows a CFP, obtained by the above algorithm, consisting of 16 faults corresponding to $G = (1,\ 4 \mid 1,\ 2)$ in a 4×18 array A when the links are bidirectional.

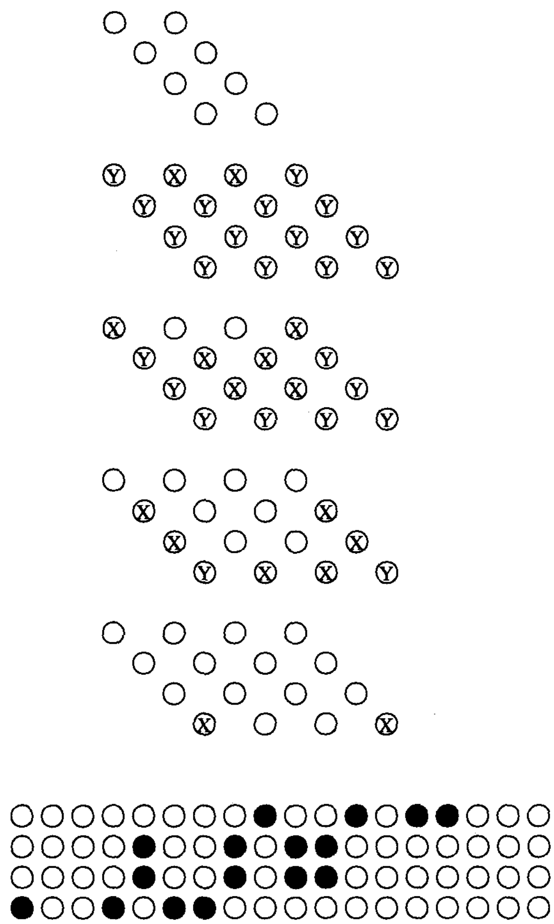


Figure 4.2: Fault Pattern for $G = (1,\ 4 \mid 1,\ 2)$

Theorem 4.2.2 *Algorithm 1 generates a catastrophic fault pattern.*

Proof: We make the following simple observations on the algorithm:

- Marking takes place only in Step 4. When some PE is marked there are two cases: (a) if the pillar has no marked PE then the current PE is marked by X and (b) if the pillar has at least one marked PE then all marked PEs are below the current PE and the current PE is marked by Y.

When the algorithm terminates, we also have the following:

- There is exactly one X in each pillar, hence $|F| = N_1 g_k$.
- For each pillar the X occurs below the Y's.
- Let the final value of f be f_0. If a PE p is marked Y and is adjacent to a PE q then q is marked (with X or Y) unless p is in floor f_0 and q is in a floor $f_0 + 1$.

We next prove that any PE α_1 marked with a Y is inaccessible, i.e., there is no way to reach this PE from $ICUL$ without using any faulty PEs (those marked with X). Suppose α_1 is accessible, i.e., α_1 is connected to $ICUL$ by a path $\mu = [\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n,\ ICUL]$ not containing any faulty PEs (those marked X). We consider two cases.

Case 1. α_i lies in a floor $\leq f_0$ for $i = 1,\ 2,\ \ldots, n$. Then it follows that $\alpha_1,\ \alpha_2,\ \ldots, \alpha_n$ are all marked Y. Now, the PE's adjacent to $ICUL$ all lie in the 0-th floor. But no PE in the 0-th floor can be marked Y by the algorithm, a contradiction which proves that α_1 is not accessible from $ICUL$.

Case 2. Atleast one α_i lies in a floor $\geq f_0 + 1$. Let α_{j-1} be the last such PE. Then α_j lies on floor f_0. Since all PEs in floor f_0 are marked and μ does not contain any PE marked X, it follows that α_j is marked Y. Now considering the path $[\alpha_j,\ \alpha_{j+1}, \ldots,\ \alpha_n,\ ICUL]$, we get a contradiction as in Case 1. This completes the proof.

Clearly when the algorithm terminates, all PEs in floor f_0 are either faulty or inaccessible from $ICUL$, so no PE in any floor $\geq f_0 + 1$ is accessible from $ICUL$; in particular, $ICUR$ is not accessible from $ICUL$ and F is a CFP.

We prove that the algorithm terminates by showing that the number of marked elements in a floor increases strictly until it reaches $N_1 g_k$. Note that all elements of the floor $f + 1$, which are above marked elements in floor f, are marked. If there is an unmarked element in the f-th floor, note that some marked element v is adjacent to an unmarked element z in the f-th floor. Then, by the algorithm, the element w in floor $f+1$ above z will be marked. ∎

Conjecture 4.2.1 *The CFP F, generated by Algorithm 1 for bidirectional horizontal and bidirectional vertical links, has the maximum width of the window, W_F.*

We now indicate our basis for the above conjecture. Let F^* be any catastrophic fault pattern with $N_1 g_k$ faulty PEs. We assume that PE $(N_1,\ 1)$ belongs to F^*. We now consider the cuboid representation of A used in the proof of Theorem 4.2.1. Note that, since there are only $N_1 g_k$ faulty PEs, each pillar can contain only one faulty PE. Now mark the faulty PEs by X's and in each pillar mark the PEs above the faulty PE by Y's. A PE $(i,\ j)$ marked by X and $\neq (N_1,\ 1)$ is called *fair PE* if no PE among the PEs $(i,\ j \pm g)$ where $g \in \{g_1,\ g_2,\ \ldots,\ g_{k-1}\}$ or $(i \pm v,\ j)$ where $v \in \{v_1,\ v_2, \ldots, v_l\}$ is marked by Y. In other words, a PE (marked by X) is called *fair PE* if it is not adjacent to any PE marked by Y (except the PE immediately above it) using the links in G. In the given fault pattern F^*, if there exists a fair PE$(i,\ j)$, we obtain a new fault pattern called *derived fault pattern* by making the PE $(i,\ j+g_k)$ faulty instead of PE $(i,\ j)$. Then PE $(i,\ j+g_k)$ becomes faulty whereas PE $(i,\ j)$ becomes accessible from ICUL. Note that, the derived fault pattern is still catastrophic and its width is greater than or equal to the width of F^*. If there is any fair PE (i', j') in the derived fault pattern then make PE $(i', j'+g_k)$ faulty instead of PE $(i',\ j')$. Continuing this process we get a derived fault pattern F^0 with no fair PE. It is easy to check that this process terminates in finite number of steps. Clearly $\omega_{F^0} \geq \omega_{F^*}$. Let F be the fault pattern obtained from Algorithm 1. Then we think F will be the only catastrophic fault pattern which does not have any fair PE, so $F^0 = F$ and $\omega_F \geq \omega_{F^*}$. Since F^* is arbitrary, F has the maximum width.

This conjecture gives us the framework for achieving specific upper bounds and exact bounds on the size of the largest window for a given link configuration. Given a link configuration G, we can obtain, by applying the algorithm, a catastrophic fault pattern F which is contained in the largest window; that is, ω_F is the maximum value possible.

We study the effect of G on the maximum width of window of a CFP. We start by showing that the window size decreases as the size of G increases.

Theorem 4.2.3 *Let $G = (G_1 \mid G_2)$ and $G' = (G'_1 \mid G'_2)$ be two link redundancies with the same largest horizontal bypass link. If $G_1 \subseteq G'_1$ and $G_2 \subseteq G'_2$ and, W_F and $W_{F'}$ are the corresponding widest fault windows, then $\omega_F \geq \omega_{F'}$.*

In Algorithm 1, we note that there will be more X's for G' than for G in each floor, so the algorithm terminates sooner.

We now present some results which give the maximum width of a window of a CFP when there are at most two horizontal and at most two vertical link redundancies and an upper bound for the width of a window of a CFP in the general case. These results follow directly from Conjecture 4.2.1 and application of Algorithm 1.

Result 4.2.1 *Let $G = (1,\ g \mid 1)$. Then the maximum width of the window of a CFP with $N_1 g$ faults is $(\lceil \frac{g}{2} \rceil + N_1 - 2)g + \lfloor \frac{g}{2} \rfloor + 1$.*

Proof: Let F be the fault pattern obtained from Algorithm 1. Let $F_i \subseteq F$ be the set of faulty PEs occur only in the i-th row of A. Note that all F_i's are identical and of width $(\lceil \frac{g}{2} \rceil - 1)g + \lfloor \frac{g}{2} \rfloor + 1$. See Nayak, Santoro and Tan [74]. Also if F_i starts at j-th column then F_{i-1} can start atmost from PE$(i-1, j+g)$. Suppose to the contrary that

F_{i-1} starts after PE $(i-1,\ j+g)$. Then PE $(i-1,\ j+g)$ will be a accessible processor whereas PE$(i,\ j+g)$ is inaccessible and these are connected by a vertical regular link. Which contradict that F is catastrophic. Since A has N_1 rows, clearly the width of F is $(\lceil \frac{g}{2} \rceil + N_1 - 2)g + \lfloor \frac{g}{2} \rfloor + 1$.

Result 4.2.2 *Let $G = (1,\ g \mid 1,\ v)$. Then the maximum width of the window of a CFP with $N_1 g$ faults is given by given by*

$$\begin{cases} (\lceil \frac{g}{2} \rceil + \lceil \frac{v}{2} \rceil + \lfloor \frac{N_1-1}{v} \rfloor - 2)g + (\lfloor \frac{g}{2} \rfloor + 1) & \text{if } v \mid (N_1 - 1) \\ (\lceil \frac{g}{2} \rceil + \lceil \frac{v}{2} \rceil + \lfloor \frac{N_1-1}{v} \rfloor - 2)g + (\lfloor \frac{g}{2} \rfloor + 1) & \text{if } N_1 - 1 = v\lfloor \frac{N_1-1}{v} \rfloor + r,\ r \le \lceil \frac{v}{2} \rceil - 1 \\ (\lceil \frac{g}{2} \rceil + \lceil \frac{v}{2} \rceil + \lfloor \frac{N_1-1}{v} \rfloor - 1)g + (\lfloor \frac{g}{2} \rfloor + 1) & \text{if } N_1 - 1 = v\lfloor \frac{N_1-1}{v} \rfloor + r,\ r > \lceil \frac{v}{2} \rceil - 1 \end{cases}$$

In view of Theorem 4.2.3, when $G = (1,\ g_2, \ldots,\ g_k \mid 1,\ v_2,\ \ldots,\ v_l)$, we get an upper bound for the width of the window of a CFP with $N_1 g_k$ faults by replacing g and v by g_k and v_l respectively in the expression given in the preceding result. A similar statement holds for Result 4.2.1.

4.3 Cuboid Representation for Fault Pattern

Suppose we are given a fault pattern F with $N_1 g_k$ faults in a two-dimensional array A with link redundancy $G = (g_1,\ g_2, \ldots,\ g_k \mid v_1,\ v_2,\ \ldots,\ v_l)$. W.l.g. we will assume that the first column of A contains a fault. We now consider the cuboid representation of A used in the proof of Theorem 4.2.1. However, we label the N_1 rows in any floor of the cuboid with $0,\ 1,\ \ldots,\ N_1 - 1$ instead of $1,\ ,2\ \ldots,\ N_1$ and the g_k columns in any floor with $0,\ 1,\ \ldots,\ g_k - 1$ instead of $1,\ 2,\ \ldots,\ g_k$. The floors are labeled using $0,\ 1,\ 2,\ \ldots$ as before. With every PE $(i,\ j)$ we can uniquely associate the triple $(x,\ y,\ z)$ where x, y and z are the row label, column label and floor label of the position $(i,\ j)$ occupies in the cuboid. (Note that $x = \imath - 1$, y is the remainder obtained when $\jmath - 1$ is divided by g_k and z is $\lfloor \frac{\jmath-1}{g_k} \rfloor$). We will write

$$W(x,\ y,\ z) = \begin{cases} 1 & \text{if } (\imath,\ \jmath) \in F \\ 0 & \text{otherwise.} \end{cases}$$

We will sometimes refer to $(x,\ y,\ z)$ as the *location of the PE* $(\imath,\ \jmath)$

Suppose now F is a fault pattern such that for any $(x,\ y)$, there is exactly one z for which $W(x,\ y,\ z) = 1$ (i.e., there is exactly one faulty PE in each pillar). We then denote this z by h_{xy} and call the matrix

$$H = \begin{pmatrix} h_{00} & h_{01} & \cdots & h_{0,g_k-1} \\ h_{10} & h_{11} & \cdots & h_{1,g_k-1} \\ \vdots & \vdots & & \vdots \\ h_{N_1-1,0} & h_{N_1-1,1} & \cdots & h_{N_1-1,g_k-1} \end{pmatrix}$$

the height matrix of F.

Example 4.3.1 Consider the CFP $F = \{(1,5),(1,6),(1,8),(1,11),(2,4),(2,5),(2,6),(2,7),(3,5),(3,7),(3,8),(3,10),(4,1)(4,4),(4,6),(4,7)\}$ with 16 faults for a two-dimensional array A with link redundancy $G = (1,\ 4 \mid 1)$ which has $\omega_F = 11$ as shown in Figure 4.3.

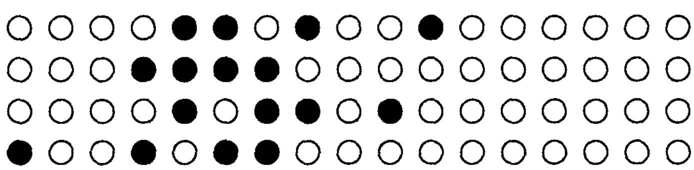


Figure 4.3: A fault Pattern

The height matrix for this CFP is

$$H = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Note that every minimal CFP (i.e., CFP with $N_1 g_k$ faulty PEs) satisfies the conditions stated at the beginning of the preceding paragraph. We now define the interior, exterior and border elements in the cuboid representation of a minimal CFP.

Definition 4.3.1 Let F be a minimal CFP. Then the PE of A corresponding to the location $(x,\ y,\ z)$ is said to be *interior*, *border* or *exterior* with respect to F according as $z < h_{xy}$, $z = h_{xy}$ or $z > h_{xy}$. The interior $I(F)$ of F, the border $B(F)$ of F and the exterior $E(F)$ of F are defined to be the set of all interior elements, the set of all border elements and the set of all exterior elements of F, respectively.

Example 4.3.2 Consider the fault pattern F of Example 4.3.1. The interior, border, and exterior elements are shown in Figure 4.4.

Lemma 4.3.1 *A fault pattern F is catastrophic for a two-dimensional network $\mathcal{N}$ with bidirectional link redundancy G iff it is not possible to reach any exterior element from any interior element and also any interior element from any exterior element using the links in $\mathcal{N}$.*

Proof: It is easy to see that all interior elements are reachable from $ICUL$ and all exterior elements are reachable from $ICUR$. The lemma follows from Definition 1.3.6.

4.4 Counting Minimal Catastrophic Fault Patterns

We start with a characterization of minimal CFPs in terms of the height matrix

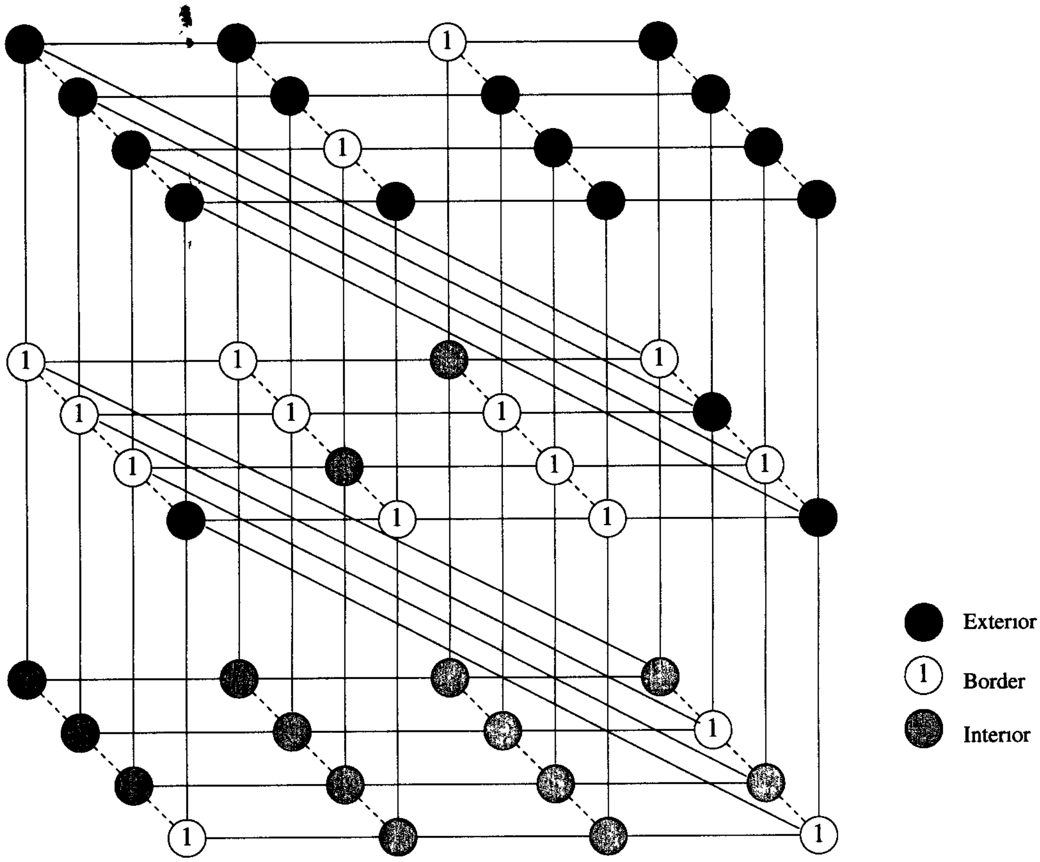


Figure 4.4: Interior, Exterior and Border of the fault pattern given in Example 4.3.1

Proposition 4.4.1 An $N_1 \times g_k$ matrix $H = ((h_{xy}))$ with non-negative integer entries is the height matrix of a minimal CFP for $\mathcal{N}$ with bidirectional link redundancy $G = (1,\ g \mid 1)$ iff the following conditions are satisfied:

(i) $h_{x0} - h_{x,g-1} = 0$ or $+1$ for all x such that $0 \leq x \leq N_1 - 1$; $h_{x0} = h_{x,g-1} = 0$ for at least one x.

(ii) $h_{xy} - h_{x,y+1} = -1,\ 0$ or $+1$ whenever $0 \leq x \leq N_1 - 1$ and $0 \leq y \leq g - 2$ and

(iii) $h_{xy} - h_{x+1,y} = -1,\ 0$ or $+1$ whenever $0 \leq x \leq N_1 - 2$ and $0 \leq y \leq g - 1$.

Clearly the number of minimal CFPs for $\mathcal{N}$ with bidirectional link redundancy $G = (1,\ g \mid 1)$ is equal to the number of height matrices H which satisfy the conditions of Proposition 4.4.1. We shall illustrate Proposition 4.4.1 by an example.

Example 4.4.1 Consider the fault pattern $F = \{(1,\ 4),\ (1,\ 5),\ (1,\ 10),\ (1,\ 11),\ (2,\ 1),\ (2,\ 2),\ (2,\ 7),\ (2,\ 8),\ (3,\ 5),\ (3,\ 10),\ (3,\ 11),\ (3,\ 8)\}$ in a 3×12 array A with link redundancy $G = (1,\ 4 \mid 1)$. Note that in the cuboid representation for F there is exactly one faulty PE in each pillar.

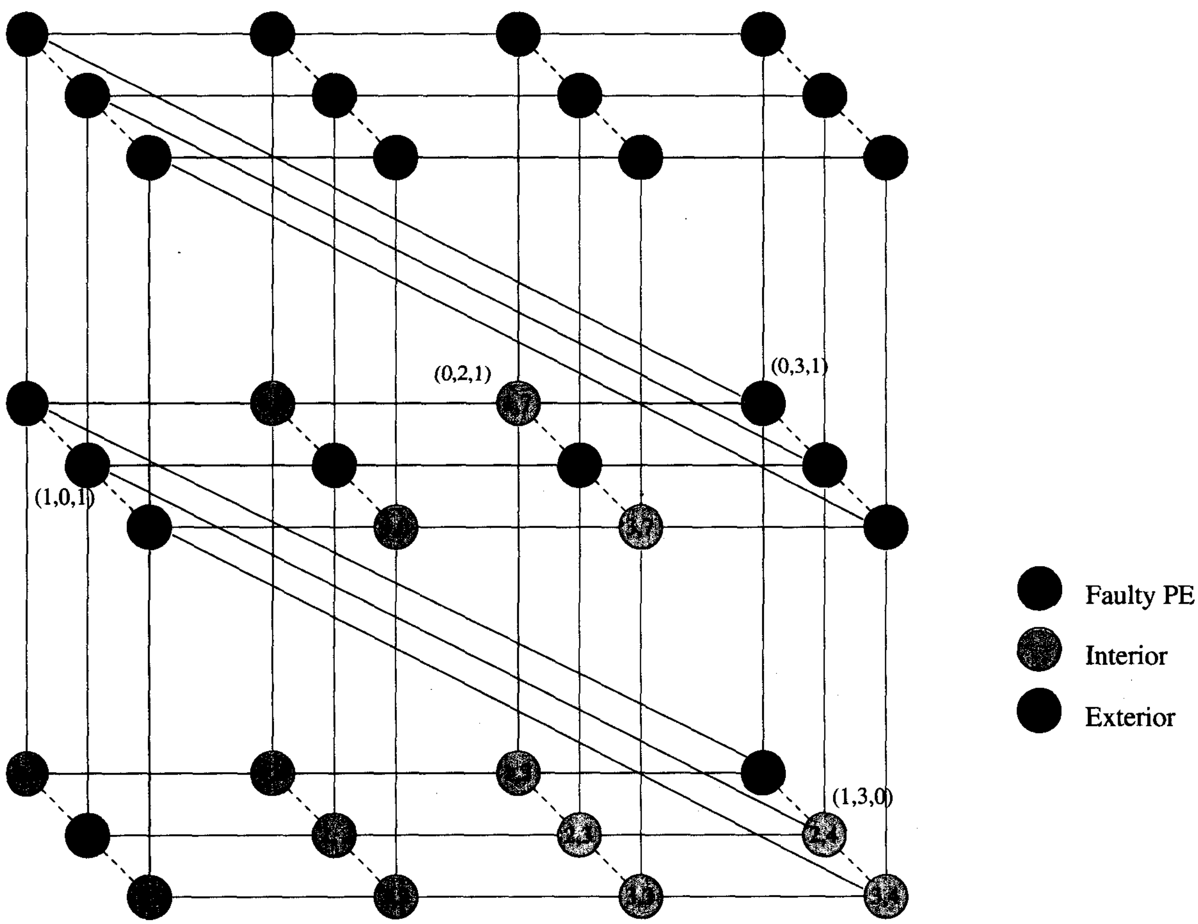


Figure 4.5: Cuboid representation of F

The height matrix for this fault pattern is

$$H = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix}$$

Note that, $h_{10} - h_{13} = -1$ which violates condition (i) of Proposition 4.4.1. We see from Figure 4.5, that the exterior processor at location (1, 0, 1) and the interior processor at location (1, 3, 0) are connected by a horizontal regular link. Hence F is not a catastrophic fault pattern by Lemma 4.3.1. Similarly condition (ii) of Proposition 4.4.1 is violated since $h_{02} - h_{03} = 2$. Note that, locations (0, 2, 1) and (0, 3, 1) contain an interior processor and an exterior processor respectively which are connected by a horizontal regular link. However, it can easily be verified that F satisfies conditions (iii) of Proposition 4.4.1 even though F is not catastrophic.

In the general case we have the following proposition:

Proposition 4.4.2 An $N_1 \times g_k$ matrix $H = ((h_{xy}))$ with non-negative integer entries is the height matrix of a minimal CFP for $\mathcal{N}$ with bidirectional link redundancy $G = (g_1,\ g_2 \ldots,\ g_k \mid v_1,\ v_2, \ldots,\ v_l)$ iff the following conditions are satisfied:

(i) $h_{x0} - h_{x,g_k-1} = 0$ or $+1$ for all x such that $0 \leq x \leq N_1 - 1$; $h_{x0} = h_{x,g_k-1} = 0$ for at least one x.

(ii) $h_{xy} - h_{x,y+g_i} = -1,\ 0$ or $+1$ for all g_i, $1 \leq i \leq k-1$ whenever $0 \leq x \leq N_1 - 1$ and $0 \leq y \leq g_k - g_i - 1$ and

(iii) $h_{xy} - h_{x+v_i,y} = -1, 0, +1$ for all v_i, $1 \leq v_i \leq l$ whenever $0 \leq x \leq N_1 - v_i - 1$ and $0 \leq y \leq g_k - 1$.

As before, the number of minimal CFPs for $\mathcal{N}$ with bidirectional link redundancy $G = (g_1,\ g_2 \ldots,\ g_k \mid v_1,\ v_2, \ldots,\ v_l)$ is equal to the number of height matrices H which satisfy the conditions of Proposition 4.4.2.

4.5 A testing algorithm for two-dimensional array with bidirectional links

Let $\mathcal{N}$ be a bidirectional array of N_1N_2 processors with link redundancy $G = (g_1,\ g_2, \ldots,\ g_k \mid v_1,\ v_2, \ldots,\ v_l)$, and let F be a fault pattern with m faults. A simple way to test if F is catastrophic for $\mathcal{N}$ is to consider a graph whose set of vertices is given by the chunks of working processors. More formally, we construct a graph $H = (V,\ E)$ as follows: The set V of vertices is $\{C_0,\ C_1,\ \ldots,\ C_n\}$, where C_i's represent chunks of F and $(C_i,\ C_j) \in E$ if and only if there are two processors, $p_{xy} \in C_i$ and $p_{x'y'} \in C_j$ such that $y = y'$ and $|x - x'| \in \{v_1,\ v_2,\ \ldots,\ v_l\}$ or $x = x'$ and $|y - y'| \in \{g_1,\ g_2, \ldots,\ g_k\}$, that is, such that these two processors are connected in $\mathcal{N}$ by a bypass link.

Fact 1. A fault pattern F is not catastrophic for a network $\mathcal{N}$, if and only if C_0 and C_n are connected in the graph H.

Example 4.5.1 Consider the fault pattern F of Example 1.3.3. Figure 4.6 shows the graph H for the fault pattern F and link redundancy $G = (1,\ 4 \mid 1,\ 2)$. To see that $(C_0,\ C_1) \in E$ one may use, for example, p_{23} and p_{27} or p_{46} and p_{26}. Note that, C_0 and C_5 are connected in H. The path $[C_0,\ C_3,\ C_2,\ C_5]$ in H gives a path $[ICUL,\ (4,\ 5),\ (4,\ 9),\ (2,\ 9),\ (2,\ 13),\ RICU]$ in $\mathcal{N}$ and this shows that the fault pattern F is not catastrophic with respect to link redundancy $G = (1,\ 4 \mid 1,\ 2)$.

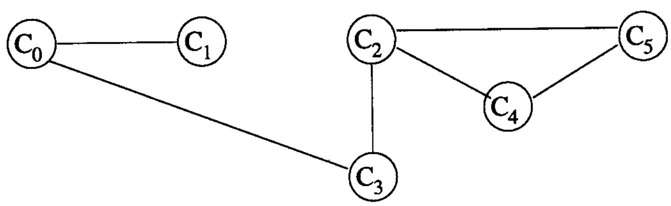


Figure 4.6: Graph H.

Chapter 5

Construction of Nearly Strongly Balanced Uniform Repeated Measurements Designs

5.1 Introduction

In this chapter, we provide a method of construction for a class of nearly strongly balanced uniform RMDs using suitable symmetric balanced incomplete block designs constructed through difference technique. See also [60]. For construction of nearly strongly balanced uniform RMDs, an adaptation of R. C. Bose's method of "symmetrically repeated differences" [11] has been used under the heading of "Method of differences".

5.2 Method of Differences

Consider a group G with v elements and operation "+". If there is a p-tuple $B = (a_0,\ a_1,\ a_2, \ldots, a_{p-1})$ with elements belonging to G then the $p-1$ elements $a_i - a_{i+1}$ for $i = 0,\ 1,\ \ldots,\ p-2$ are said to be the *backward differences* arising from the p-tuple B. B will be referred to as a difference vector and $C = \{a_i - a_{i+1} :\ i = 0, 1, \ldots p-2\}$ as the set of backward differences in B.

Given any p-tuple $B = (a_0, a_1, ..., a_{p-1})$ with elements belonging to G, the set of p-tuples $B + \theta = (a_0 + \theta,\ a_1 + \theta,\ \ldots,\ a_{p-1} + \theta)$ obtained as θ runs over the elements of G, is said to be the set of p-tuples obtained by *developing* B.

Example 5.2.1 Consider the group $Z_7 = \{0,\ 1,\ 2, \ldots,\ 6\}$ with operation " + " (i.e. addition modulo 7). Consider the triple $B = (1,\ 2,\ 4)$. Then the backward differences arising from the triple are $1-2=6,\ \ 2-4=5$. The set of triples obtained by developing $B = (1,\ 2,\ 4)$ are

$$(1,\ 2,\ 4),\ \ (2,\ 3,\ 5),\ \ (3,\ 4,\ 6),\ \ (4,\ 5,\ 0),\ \ (5,\ 6,\ 1),\ \ (6,\ 0,\ 2),\ \ (0,\ 1,\ 3).$$

Notation: If A is any n-tuple $(a_1,\ a_2,\ ...,\ a_n)$ then A' is the n-tuple $(a_n,\ a_{n-1},\ ...,\ a_1)$. If $A = (a_1,\ a_2,\ ...,\ a_n)$ and $B = (b_1,\ b_2,\ ...,\ b_m)$ then by AB we mean the $(n+m)$-tuple $(a_1,\ a_2,\ ...,\ a_n,\ b_1,\ b_2,\ ...,\ b_m)$.

Definition 5.2.1 A *shift starting at* a_i of $(a_0,\ a_1,\ a_2, ...,\ a_{v-1})$ is $(a_i,\ a_{i+1},\ ...,\ a_{v-1},\ a_0,\ a_1,\ a_2, ...,\ a_{i-1})$.

We now introduce the following definition:

Definition 5.2.2 Consider the group $Z_v = \{0,\ 1,\ 2,\ \ldots,\ v-1\}$ with operation " + " (i.e. addition modulo v). Then for $i = 0,\ 1,\ 2,\ \ldots,\ v-1$ and $s \geq 2$, $D_i^{s,v}$ denotes an sv-tuple of elements from Z_v such that each element of Z_v appears s times in it and among the *backward differences* in it each element of Z_v occurs s times except i which occurs $(s-1)$ times.

Theorem 5.2.1 *If in an RMD the treatments correspond to the elements of the group $Z_v = \{0,\ 1,\ 2,\ \ldots,\ v-1\}$ with operation "+" (i.e. addition modulo v), then developing $D_i^{s,v}$ we get a uniform RMD with parameters $v,\ n = v,\ p = sv$ and*

$$
M = \begin{array}{c} \\ 0 \\ 1 \\ \\ v\text{-}i\text{-}1 \\ v\text{-}i \\ v\text{-}i\text{+}1 \\ \\ v\text{-}1 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & 2 & & i\text{-}1 & i & i\text{+}1 & i\text{+}2 & & v\text{-}1 \end{array} \\
\left[\begin{array}{cccccccccc}
s & s & s & \cdots & s & s\text{-}1 & s & s & \cdots & s \\
s & s & s & \cdots & s & s & s\text{-}1 & s & \cdots & s \\
 & & & & \vdots & & & & & \\
s & s & s & \cdots & s & s & s & s & \cdots & s\text{-}1 \\
s\text{-}1 & s & s & \cdots & s & s & s & s & \cdots & s \\
s & s\text{-}1 & s & \cdots & s & s & s & s & \cdots & s \\
 & & & & \vdots & & & & & \\
s & s & s & \cdots & s\text{-}1 & s & s & s & \cdots & s
\end{array}\right]
\end{array}
$$

Before proving the theorem we shall illustrate it by an example.

Example 5.2.2 Consider the group $Z_7 = \{0,\ 1,\ 2,\ \ldots,\ 6\}$ with operation + (i.e., addition modulo v). We write the seven 14-tuples obtained by developing the 14-tuple $D_0^{2,7} = (0,\ 6,\ 1,\ 5,\ 2,\ 4,\ 3,\ 3,\ 4,\ 2,\ 5,\ 1,\ 6,\ 0)$ as the rows of an array as shown in Figure 5.1.

$$\begin{bmatrix} 0&6&1&5&2&4&3&3&4&2&5&1&6&0\\ 1&0&2&6&3&5&4&4&5&3&6&2&0&1\\ 2&1&3&0&4&6&5&5&6&4&0&3&1&2\\ 3&2&4&1&5&0&6&6&0&5&1&4&2&3\\ 4&3&5&2&6&1&0&0&1&6&2&5&3&4\\ 5&4&6&3&0&2&1&1&2&0&3&6&4&5\\ 6&5&0&4&1&3&2&2&3&1&4&0&5&6 \end{bmatrix}$$

Figure 5.1: An RMD(7,7,14)

This array is a uniform RMD with parameters $v = 7,\ n = 7,\ p = 14$, and with

$$M = \begin{bmatrix} 1&2&2&2&2&2&2\\ 2&1&2&2&2&2&2\\ 2&2&1&2&2&2&2\\ 2&2&2&1&2&2&2\\ 2&2&2&2&1&2&2\\ 2&2&2&2&2&1&2\\ 2&2&2&2&2&2&1 \end{bmatrix}$$

Proof of Theorem 5.2.1: Write the v sv-tuples obtained by developing the sv-tuple $D_i^{s,v}$ as the rows of an array as shown below

$$\begin{bmatrix} D_i^{s,v} \\ D_i^{s,v}+1 \\ \cdots \\ D_i^{s,v}+v-1 \end{bmatrix}$$

where the elements of $D_i^{s,v} + \theta$ are obtained by adding θ to the elements of $D_i^{s,v}$. In the design so obtained it is clear that the first three parameters are $v,\ n = v,\ p = sv$. In each unit, each treatment occurs s times and in each period, each treatment occurs once. Consider the pair of treatments $(c,\ d)$. The difference arising from this pair is $c-d$. Let $c - d = i$. Then there are exactly $s - 1$ pairs in $D_i^{s,v}$ which give rise to the same difference i. Let $(a_x,\ a_y)$ be one such pair occurring in $D_i^{s,v}$. Then $a_x - a_y = c - d$. If $a_x + \theta = c$ then $a_y + \theta = d$. Hence the pair $(c,\ d)$ occurs in the tuple $D_i^{s,v} + \theta$, where $\theta = c - a_x = d - a_y$. Thus corresponding to each of the pairs in the initial tuple, $D_i^{s,v}$, which give rise to the difference $c-d$, there will be one tuple of the design where the pair $(c,\ d)$ occurs. Thus $m_{dc} = s - 1$. On the other hand, if $c - d = j\ (\neq i)$ then there are exactly s pairs in $D_i^{s,v}$ which give rise to the same difference j. Then $m_{dc} = s$. Since the pairs $(i,\ 0),\ (i+1,\ 1),\ \ldots,(v-1,\ v-i-1),\ (0,\ v-i),\ (1,\ v-i+1),\ldots,(i-1,\ v-1)$ give rise to the difference i, $m_{0i} = m_{1,i+1} = \ldots = m_{v-i-1,v-1} = m_{v-i,0} = m_{v-i+1,1} = \ldots = m_{v-1,i-1} = s - 1$. ∎

Lemma 5.2.1 *For odd v, $D_i^{s,v}$ can be constructed for all $s \geq 2$ and for all $i \in Z_v$.*

Proof : Case 1: $s = 2,\ i = 0$.

For odd number of treatments v consider the v-tuple

$$A = (0,\ v-1,\ 1,\ v-2,\ 2,\ v-3,\ \cdots,\ \frac{v-1}{2})$$

as constructed by Sen and Mukerjee [96].

Note that, AA' is a $2v$-tuple such that frequency of each element of Z_v in AA' is two and among the backward differences in AA' each element of Z_v occurs exactly twice except 0 which occurs once. Hence $D_0^{2,v} = AA'$.

Case 2: $s = 3,\ i = 0$.

For odd number of treatments $v(= 2k+1)$ consider the difference vector

$$D_0^{2,v} = (0,\ 2k,\ 1,\ 2k-1, \ldots,\ k-1,\ k+1,\ k,\ k,\ k+1,\ k-1, \ldots,\ 2k-1,\ 1,\ 2k,\ 0).$$

Now replace one occurrence of i by the triplet $(i,\ 2k-i,\ i)$ for $i = 0, 1, \ldots, k-1$ and replace one occurrence of k by the ordered pair$(k,\ k)$. Consider this modification on $D_0^{2,v}$.

Then the modified $D_0^{2,v}$ contains each element of Z_{2k+1} exactly thrice and the collection of backward differences in modified $D_0^{2,v}$ also contains each element of Z_{2k+1} exactly thrice except 0 which occurs exactly twice. Therefore, modified $D_0^{2,v}$ is our required $D_0^{3,v}$. For details see Dutta and Roy [27]. Now $D_0^{s,v}$ can be constructed for all $s \geq 2$ juxtaposing $D_0^{2,v}$'s and $D_0^{3,v}$'s suitably.

Let $D_0^{s,v} = (d_1, d_2, ..., d_{sv})$. If $d_r - d_{r+1} = \imath$ then $D_\imath^{s,v}$ is the shift starting at d_{r+1} of $D_0^{s,v}$. It is clear that the collection of backward differences in $D_\imath^{s,v}$ contains each element of Z_v exactly s times except $\imath$ which occurs $(s-1)$ times. So for odd v, $D_\imath^{s,v}$ can be constructed for all $s \geq 2$ and for all $\imath \in Z_v$. ■

Illustration: Let $v = 7$, $k = 3$ and $D_0^{2,7} = (0, 6, 1, 5, 2, 4, 3, 3, 4, 2, 5, 1, 6, 0)$. We then replace one 0 by $(0, 6, 0)$, one 1 by $(1, 5, 1)$, one 2 by $(2, 4, 2)$ and one 3 by $(3, 3)$. Thus we get $D_0^{3,7} = (0, 6, 0, 6, 1, 5, 1, 5, 2, 4, 2, 4, 3, 3, 3, 4, 2, 5, 1, 6, 0)$.

5.3 Symmetric Balanced Incomplete Block Designs

We recall the definition and basic properties of SBIBD for later use. Suppose there are v objects or *treatments*, which are to be arranged in b sets or *blocks* satisfying the following conditions:

1. Each block contains k treatments.

2. Each treatment occurs in r blocks.

3. Every pair of treatments occurs together in λ blocks.

Such an arrangement, if it exists, is called a *balanced incomplete block (BIB) design*, with parameters (v, b, r, k, λ).

Consider a BIB design with parameters $(v,\ b,\ r,\ k,\ \lambda)$. Then the $v \times b$ matrix

$$N = ((n_{ij}))$$

is called the *incidence matrix* of the design if the element n_{ij} in the i-th row and the j-th column is 1 or 0, according as the i-th treatment occurs or does not occur in the j-th block.

Example 5.3.1 The following is a BIB design with parameters $(7,\ 7,\ 3,\ 3,\ 1)$:

Block	Treatments in the Block
1	(1, 2, 4)
2	(2, 3, 5)
3	(3, 4, 6)
4	(4, 5, 7)
5	(5, 6, 1)
6	(6, 7, 2)
7	(7, 1, 3)

The incidence matrix of the design is

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Theorem 5.3.1 *(Bose and Manvel [12]) If $N = (n_{ij})$ is the incidence matrix of a BIB design with parameters $(v,\ b,\ r,\ k,\ \lambda)$, then*

$$NN^T = (r - \lambda)I_v + \lambda J_{vv}$$

where I_v is the identity matrix of order v and J_{vv} is the $v \times v$ matrix with all elements 1.

Definition 5.3.1 A BIB design is said to be *symmetric* if the number of blocks is equal to the number of treatments, that is, $v = b, \ r = k$.

Theorem 5.3.2 *Existence of an SBIBD (v, k, λ) implies existence of a $v \times v$ matrix M with elements a and $a + 1$, $a \geq 0$ such that MM^T is completely symmetric.*

Proof. Let N be the incidence matrix of an SBIB design with parameters (v, k, λ). Note that the elements of the matrix $N + aJ_{vv}$, $a \geq 0$, are a and $a + 1$. Using Theorem 5.3.1, it is easy to check that $(N + aJ_{vv})(N + aJ_{vv})^T$ is completely symmetric. Hence $N + aJ_{vv}$ can be taken to be M.

5.4 Construction of Nearly Strongly Balanced Uniform RMDs

We now show how to construct a family of nearly strongly balanced uniform RMDs using SBIBDs.

Suppose there exists a uniform RMD(v, n, p) whose M matrix is of the form $M = N + \left[\frac{n(p-1)}{v^2}\right] J_{vv}$, where N is the incidence matrix of an SBIBD(v, k, λ). It is clear that, elements of the matrix M are $\left[\frac{n(p-1)}{v^2}\right]$ and $\left[\frac{n(p-1)}{v^2}\right]+1$, and MM^T is completely symmetric (Theorem 5.3.2). Since in each row of N, 1 occurs k times and 0 occurs $(v - k)$ times, in each row of M, $\left[\frac{n(p-1)}{v^2}\right] + 1$ occurs k times and $\left[\frac{n(p-1)}{v^2}\right]$ occurs $(v - k)$ times. That is, for a fixed treatment i there exist k treatments j such that $m_{ij} = \left[\frac{n(p-1)}{v^2}\right] + 1$ and $(v - k)$ treatments j' such that $m_{ij'} = \left[\frac{n(p-1)}{v^2}\right]$. So among the v^2 ordered treatment pairs, vk pairs occur $\left[\frac{n(p-1)}{v^2}\right] + 1$ times each and the remaining $(v^2 - vk)$ treatment pairs occur $\left[\frac{n(p-1)}{v^2}\right]$ times each. Note that, in an RMD (v, n, p), the number of ordered pairs $(d_{ij}, d_{i,j+1})$, $1 \leq i \leq n$, $1 \leq j \leq p-1$ is $n(p-1)$. Hence we get the following equation

$$vk\left(\left[\frac{n(p-1)}{v^2}\right]+1\right)+(v^2-vk)\left[\frac{n(p-1)}{v^2}\right]=n(p-1).$$

Using $n = Av^2 + Bv$ and $p = sv$ we get

$vk[A(sv-1)+sB]+(v^2-vk)[A(sv-1)+sB-1]=(Av^2+Bv)(sv-1)$

and so $B = v - k$ and $n = Av^2 + (v-k)v$. In Theorem 5.4.1 we will prove the existence of such a uniform RMD for any n of the type $Av^2 + (v-k)v$, $A \geq 0$ and $p = sv$, $s \geq 2$, assuming the existence of SBIBD(v, k, λ).

On the other hand, if there exists a uniform RMD for which $M = aJ_{vv} + (J_{vv} - N)$ where N is the incidence matrix of an SBIBD(v, k, λ) then from similar considerations we can obtain $B = k$ and so $n = Av^2 + kv$, $A \geq 0$ and $p = sv$, $s \geq 2$.

Theorem 5.4.1 *Existence of an SBIBD(v, k, λ) (constructed through difference technique) implies existence of a family of nearly strongly balanced uniform RMDs with parameters $n = Av^2 + (v-k)v$, $A \geq 0$ and $p = sv$, $s \geq 2$.*

Proof : Given an SBIBD(v, k, λ) constructed through difference technique, let us consider any block, say, the j-th block. We will construct an sv-tuple, $D_t^{s,v}$, for each treatment t which is not present in the j-th block. Since any block of SBIBD(v, k, λ) contains k treatments, we have to construct $(v-k)$ sv-tuples. Let $t_1, t_2, \ldots t_{v-k}$ be the treatments not occurring in the j-th block of SBIBD(v, k, λ). We write the $(v-k)$ sv-tuples, i.e., $D_{t_1}^{s,v}$, $D_{t_2}^{s,v}$, $\ldots$, $D_{t_{v-k}}^{s,v}$ as the rows of an array and write the developed sv-tuples under them as shown below

$$D = \begin{bmatrix} D_{t_1}^{s,v} \\ D_{t_1}^{s,v} + 1 \\ \vdots \\ D_{t_1}^{s,v} + v - 1 \\ D_{t_2}^{s,v} \\ D_{t_2}^{s,v} + 1 \\ \vdots \\ D_{t_2}^{s,v} + v - 1 \\ \vdots \\ D_{t_{v-k}}^{s,v} \\ D_{t_{v-k}}^{s,v} + 1 \\ \vdots \\ D_{t_{v-k}}^{s,v} + v - 1 \end{bmatrix}$$

where the elements of $D_t^{s,v} + \theta$ are obtained by adding θ to the elements of $D_t^{s,v}$. In the design so obtained it is clear that the first three parameters are $v,\ n = v(v-k),\ p = sv$. From the definition of $D_t^{s,v}$, it is clear that D is uniform on units. Note that, in each column each treatment occurs exactly $v - k$ times. Hence D is uniform on periods.

Consider the pair of treatments $(c,\ d)$. The backward difference arising from this pair is $c - d$. If $(c - d) \notin \{t_1,\ t_2,\ ...,\ t_{v-k}\}$ then there are exactly s pairs in each $D_{t_l}^{s,v}$ for $l = 1,\ 2,\ \ldots,\ v - k$ which give rise to the same difference $(c - d)$. Let $a_i,\ a_j$ be one such pair occurring, say, in $D_{t_1}^{s,v}$. Then $a_i - a_j = c - d$. If $a_i + \theta = c$, then $a_j + \theta = d$. Hence the pair $(c,\ d)$ occurs in the block $D_{t_1}^{s,v} + \theta$, where $\theta = c - a_i = d - a_j$. Thus corresponding to each of the pairs in $D_{t_1}^{s,v}$ which give rise to the difference $c - d$, there will be one tuple, obtained by developing $D_{t_1}^{s,v}$ where the pair $(c,\ d)$ occurs. Since there are exactly s pairs in each $D_{t_l}^{s,v}$ for $l = 1, 2, ..v - k$ which give rise to the same difference $c - d$, $m_{dc} = s(v - k)$.

On the other hand, suppose $c - d \in \{t_1, t_2, \ldots, t_{v-k}\}$ and without loss of generality let $(c - d) = t_1$. So there are exactly $s - 1$ pairs in $D_{t_1}^{s,v}$ and exactly s pairs in each $D_{t_l}^{s,v}$ for $l = 2, ..v - k$ which give rise to the same difference $c - d$. Then from the similar considerations we get $m_{dc} = s(v-r)-1$. Hence m_{ij}'s assume values $s(v-r)$ or $s(v-r)-1$. Now by using Theorem 5.2.1 we get $M = s(v-r)J_{vv}-(J_{vv}-N)^T$, where N is the incidence matrix of the given SBIBD(v, k, λ). Note that MM^T is completely symmetric. ■

Example 5.4.1 Let us take the SBIBD(7, 3, 1) as given below:

$$\{(1\ 2\ 4),\ (2\ 3\ 5),\ (3\ 4\ 6),\ (4\ 5\ 0),\ (5\ 6\ 1),\ (6\ 0\ 2),\ (0\ 1\ 3)\}.$$

Without loss of generality let us consider the initial block. Since it contains the treatments 1, 2, 4, we construct $D_0^{2,7}$, $D_3^{2,7}$, $D_5^{2,7}$, $D_6^{2,7}$ where

$D_0^{2,7} = (0,\ 6,\ 1,\ 5,\ 2,\ 4,\ 3,\ 3,\ 4,\ 2,\ 5,\ 1,\ 6,\ 0)$

$D_3^{2,7} = (5,\ 2,\ 4,\ 3,\ 3,\ 4,\ 2,\ 5,\ 1,\ 6,\ 0,\ 0,\ 6,\ 1)$

$D_5^{2,7} = (4,\ 3,\ 3,\ 4,\ 2,\ 5,\ 1,\ 6,\ 0,\ 0,\ 6,\ 1,\ 5,\ 2)$

$D_6^{2,7} = (4,\ 2,\ 5,\ 1,\ 6,\ 0,\ 0,\ 6,\ 1,\ 5,\ 2,\ 4,\ 3,\ 3)$.

It is easy to verify that, for $i = 0,\ 3,\ 5,\ 6$, among the backward differences in $D_i^{2,7}$ each element of Z_7 occurs twice except i which occurs once. Now developing $D_0^{2,7}$, $D_3^{2,7}$, $D_5^{2,7}$ and $D_6^{2,7}$ over Z_7 respectively we obtain our required design as shown in Figure 5.2.

Here the matrix M is
$$\begin{bmatrix} 7 & 8 & 8 & 7 & 8 & 7 & 7 \\ 7 & 7 & 8 & 8 & 7 & 8 & 7 \\ 7 & 7 & 7 & 8 & 8 & 7 & 8 \\ 8 & 7 & 7 & 7 & 8 & 8 & 7 \\ 7 & 8 & 7 & 7 & 7 & 8 & 8 \\ 8 & 7 & 8 & 7 & 7 & 7 & 8 \\ 8 & 8 & 7 & 8 & 7 & 7 & 7 \end{bmatrix} = 8J_{77} + (J_{77} - N)^T.$$

Periods

$$
D = \left[\begin{array}{cccccccccccccc}
0 & 6 & 1 & 5 & 2 & 4 & 3 & 3 & 4 & 2 & 5 & 1 & 6 & 0 \\
1 & 0 & 2 & 6 & 3 & 5 & 4 & 4 & 5 & 3 & 6 & 2 & 0 & 1 \\
2 & 1 & 3 & 0 & 4 & 6 & 5 & 5 & 6 & 4 & 0 & 3 & 1 & 2 \\
3 & 2 & 4 & 1 & 5 & 0 & 6 & 6 & 0 & 5 & 1 & 4 & 2 & 3 \\
4 & 3 & 5 & 2 & 6 & 1 & 0 & 0 & 1 & 6 & 2 & 5 & 3 & 4 \\
5 & 4 & 6 & 3 & 0 & 2 & 1 & 1 & 2 & 0 & 3 & 6 & 4 & 5 \\
6 & 5 & 0 & 4 & 1 & 3 & 2 & 2 & 3 & 1 & 4 & 0 & 5 & 6 \\
 & & & & & & & & & & & & & \\
5 & 2 & 4 & 3 & 3 & 4 & 2 & 5 & 1 & 6 & 0 & 0 & 6 & 1 \\
6 & 3 & 5 & 4 & 4 & 5 & 3 & 6 & 2 & 0 & 1 & 1 & 0 & 2 \\
0 & 4 & 6 & 5 & 5 & 6 & 4 & 0 & 3 & 1 & 2 & 2 & 1 & 3 \\
1 & 5 & 0 & 6 & 6 & 0 & 5 & 1 & 4 & 2 & 3 & 3 & 2 & 4 \\
2 & 6 & 1 & 0 & 0 & 1 & 6 & 2 & 5 & 3 & 4 & 4 & 3 & 5 \\
3 & 0 & 2 & 1 & 1 & 2 & 0 & 3 & 6 & 4 & 5 & 5 & 4 & 6 \\
4 & 1 & 3 & 2 & 2 & 3 & 1 & 4 & 0 & 5 & 6 & 6 & 5 & 0 \\
 & & & & & & & & & & & & & \\
4 & 3 & 3 & 4 & 2 & 5 & 1 & 6 & 0 & 0 & 6 & 1 & 5 & 2 \\
5 & 4 & 4 & 5 & 3 & 6 & 2 & 0 & 1 & 1 & 0 & 2 & 6 & 3 \\
6 & 5 & 5 & 6 & 4 & 0 & 3 & 1 & 2 & 2 & 1 & 3 & 0 & 4 \\
0 & 6 & 6 & 0 & 5 & 1 & 4 & 2 & 3 & 3 & 2 & 4 & 1 & 5 \\
1 & 0 & 0 & 1 & 6 & 2 & 5 & 3 & 4 & 4 & 3 & 5 & 2 & 6 \\
2 & 1 & 0 & 2 & 0 & 3 & 6 & 4 & 5 & 5 & 4 & 6 & 3 & 0 \\
3 & 2 & 2 & 3 & 1 & 4 & 0 & 5 & 6 & 6 & 5 & 0 & 4 & 1 \\
 & & & & & & & & & & & & & \\
4 & 2 & 5 & 1 & 6 & 0 & 0 & 6 & 1 & 5 & 2 & 4 & 3 & 3 \\
5 & 3 & 6 & 2 & 0 & 1 & 1 & 0 & 2 & 6 & 3 & 5 & 4 & 4 \\
6 & 4 & 0 & 3 & 1 & 2 & 2 & 1 & 3 & 0 & 4 & 6 & 5 & 5 \\
0 & 5 & 1 & 4 & 2 & 3 & 3 & 2 & 4 & 1 & 5 & 0 & 6 & 6 \\
1 & 6 & 2 & 5 & 3 & 4 & 4 & 3 & 5 & 2 & 6 & 1 & 0 & 0 \\
2 & 0 & 3 & 6 & 4 & 5 & 5 & 4 & 6 & 3 & 0 & 2 & 1 & 1 \\
3 & 1 & 4 & 0 & 5 & 6 & 6 & 5 & 0 & 4 & 1 & 3 & 2 & 2
\end{array}\right] \text{Units}
$$

Figure 5.2: Nearly strongly balanced uniform RMD (7, 28, 14)

Note that MM^T is completely symmetric. The above design is a nearly strongly balanced uniform RMD(7, 28, 14).

5.5 Notes

1. *In Theorem 5.4.1, we construct an* sv*-tuple,* $D_t^{s,v}$*, for each treatment* t *which is present in the* j*-th* block (instead of those which are not present *in the* j*-th block)*

and develop such $D_t^{s,v}$'s over Z_v to get nearly strongly balanced uniform RMDs with $n = kv$, $p = sv$, $s \geq 2$.

2. *To construct nearly strongly balanced uniform RMDs with* $n = Av^2 + (v-k)v$ *and* $p = sv$, $s \geq 2$, *append the following A times below the design obtained in Theorem 5.4.1:*

$$\begin{bmatrix} D_0^{s,v} \\ D_0^{s,v} + 1 \\ \vdots \\ D_0^{s,v} + v - 1 \\ D_1^{s,v} \\ D_1^{s,v} + 1 \\ \vdots \\ D_1^{s,v} + v - 1 \\ \vdots \\ D_{v-1}^{s,v} \\ D_{v-1}^{s,v} + 1 \\ \vdots \\ D_{v-1}^{s,v} + v - 1 \end{bmatrix}.$$

Chapter 6

Construction of Balanced Near Uniform Repeated Measurements Design

6.1 Introduction

In this chapter, we give a procedure for construction of balanced near uniform RMDs when every ordered pair of distinct treatments appears exactly once or twice, except for the case v an odd integer and $p = v$. See also [62]. For construction of BNURMDs, we use an adaptation of R. C. Bose's method of "symmetrically repeated differences" [11] under the heading of "method of differences". Throughout this chapter, we assume that the underlying statistical model is linear and the second order and higher-order residual effects are negligible.

6.2 Some Observations

It is easy to show that the five parameters v, n, p, r, λ of a BNURMD satisfy

$$v \text{ divides } n \tag{6.1}$$

$$\lambda v(v-1) = (p-1)n \tag{6.2}$$

$$vr = np \tag{6.3}$$

where r is the total number of times each treatment appears in the design. Note that given v, p and λ, n and r are given by:

$$n = \frac{\lambda v(v-1)}{p-1} \tag{6.4}$$

$$r = \frac{p\lambda(v-1)}{p-1} \tag{6.5}$$

Conversely, given v, n and p, r and λ can be computed from (6.2) and (6.3). Thus we may take either v, n and p or v, p and λ as the parameters of the design. Since p and $(p-1)$ are relatively prime, for r to be an integer we must have: $(p-1)$ divides $\lambda(v-1)$. When this happens, the number of units n is automatically an integer multiple of v, the number of treatments. Thus

$$p-1 \text{ divides } \lambda(v-1) \tag{6.6}$$

is a necessary condition for the existence of a BNURMD with the parameters v, p and λ.

6.3 Method of Differences

Consider a group G with v elements and operation "+". If there is a p-tuple $B = (a_0,\ a_1,\ a_2,\ \ldots,\ a_{p-1})$ with elements belonging to G then the $p-1$ elements $a_i - a_{i+1}$ for $i = 0,\ 1,\ \ldots,\ p-2$ are said to be the *backward differences* arising from the p-tuple B. B will be referred to as a difference vector and $C = \{a_i - a_{i+1} :\ i = 0,\ 1,\ \ldots,\ p-2\}$ as the set of backward differences in B.

Given any p-tuple $B = (a_0,\ a_1,\ ...,\ a_{p-1})$ with elements belonging to G, the set of p-tuples $B+\theta = (a_0+\theta,\ a_1+\theta,\ \ldots,\ a_{p-1}+\theta)$ obtained as θ runs over the elements of G, is said to be the set of p-tuples obtained by *developing* B. We present a simple theorem which allows an easy and direct construction of BNURMD $(v,\ n,\ p)$ for a given λ.

Theorem 6.3.1 *Let the treatments in an RMD correspond to the elements of a group G with size v and with operation "+". Suppose there exist m p-tuples (called initial units) $B_1,\ B_2,\ \ldots,\ B_m$ such that (i) for any B_i, the frequencies of the different elements of G in B_i differ by at most 1 and (ii) among the $m(p-1)$ backward differences obtained from $B_1,\ B_2,\ \ldots,\ B_m$, each of the $v-1$ non-zero elements of G occurs exactly λ times. Then by developing the initial units $B_1,\ B_2,\ \ldots,\ B_m$, we obtain a BNURMD with parameters $v,\ n = mv,\ p$.*

Proof: We write the m p-tuples $B_1,\ B_2,\ \ldots,\ B_m$ as the rows of an array and write the developed p-tuples (or units) under them as shown below

$$\begin{bmatrix} B_1 \\ B_1+1 \\ \vdots \\ B_1+v-1 \\ B_2 \\ B_2+1 \\ \vdots \\ B_2+v-1 \\ \vdots \\ B_m \\ B_m+1 \\ \vdots \\ B_m+v-1 \end{bmatrix}$$

where the elements of $B_i+\theta$ are obtained by adding θ to the elements of B_i. In the design so obtained it is clear that the parameters are v, $n = mv$, p. Since the frequencies of the elements of G in each B_i differ by at most one, it follows that the design is nearly uniform on units. It only remains to check that each ordered pair of distinct treatments is present in exactly λ units of the design. Consider the pair of distinct treatments $(c,\ d)$. The backward difference arising from this pair is $c-d$. Note that, there are exactly λ pairs in the initial units $B_1,\ B_2,\ \ldots,\ B_m$ which gives rise to the same backward difference $c-d$. Let $(a_i,\ a_j)$ be one such pair occurring, say, in the initial unit B_u. Then $a_i - a_j = c - d$. If $a_i + \theta = c$, then $a_j + \theta = d$. Hence the pair $(c,\ d)$ occurs in the unit $B_u + \theta$, where $\theta = c - a_i = d - a_j$. Thus corresponding to each of the pairs in the initial units which gives rise to the backward difference $c - d$, there will be one unit of the design where the pair $(c,\ d)$ occurs. Hence the pair $(c,\ d)$ occurs in exactly λ units of the design. By developing the initial units $B_1,\ B_2, \ldots, B_m$ we shall therefore get a BNURMD with the parameters $v,\ n = mv,\ p$. ■

Lemma 6.3.1 For the BNURMD constructed in the preceding theorem,

$$m = \frac{\lambda(v-1)}{(p-1)}. \tag{6.7}$$

Remark 6.3.1 We have seen that $(p-1)$ divides $\lambda(v-1)$ is a necessary condition for the existence of BNURMD with parameters v, p and λ. So if we can obtain a k-tuple $(a_1,\ a_2,\ \ldots,\ a_k)$ where $k = \lambda(v-1)+1$ such that in any p consecutive elements of this tuple the frequencies of the elements of G differ by at most 1 then the required initial units can easily be generated as

$$\begin{aligned}
B_1 &= (a_1,\ a_2,\ \ldots,\ a_p) \\
B_2 &= (a_p,\ a_{p+1},\ \ldots,\ a_{2p-1}) \\
&\ \ \vdots \\
B_m &= (a_{(m-1)p-(m-2)},\ \ a_{(m-1)p-(m-3)},\ \ldots,\ a_k)
\end{aligned}$$

where $m = \frac{\lambda(v-1)}{(p-1)}$.

For all the constructions in this paper we use Theorem 6.3.1 and exploit the observation made in Remark 6.3.1 to obtain the initial units, except when $p = v$, v an odd integer. Before we proceed with the constructions, we define the following notations:

If A is any n-tuple $(a_1,\ a_2,\ \ldots,\ a_n)$ then A' is the n-tuple $(a_n,\ a_{n-1},\ \ldots,\ a_1)$.
If $A = (a_1,\ a_2,\ \ldots,\ a_n)$ and $B = (b_1,\ b_2,\ \ldots,\ b_m)$ then by AB we mean the $(n+m)$-tuple $(a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2,\ \ldots,\ b_m)$

6.4 Construction when $\lambda = 1$

Now the necessary condition (6.6) implies that $(p-1)$ divides $(v-1)$ and from (6.7) we know that the number of initial units, m, is $\frac{(v-1)}{(p-1)}$.

6.4.1 Case: v odd and $p < v$

Let G be the group $Z_v = \{0,\ 1,\ 2,\ \ldots,\ v-1\}$ with operation "+" (i.e. addition modulo v) and k_1 and k_2 be two odd integers such that $k_1 + k_2 = v - 1$ and $0 \leq k_1 - k_2 \leq 2$. Now consider the following two tuples:

$$\begin{aligned} A &= (a_1,\ a_2,\ \ldots,\ a_{k_1}) \\ &= \Big(0,\ v-1,\ 1,\ v-2,\ \ldots,\ \frac{k_1-3}{2},\ v-\frac{k_1-1}{2},\ \frac{k_1-1}{2}\Big) \text{ and} \end{aligned}$$

$$\begin{aligned} B &= (a_{k_1+1},\ a_{k_1+2},\ \ldots,\ a_{k_1+k_2},\ a_v) \\ &= \Big(\frac{k_1+1}{2},\ v-\frac{k_1+1}{2},\ \frac{k_1+3}{2},\ v-\frac{k_1+3}{2},\ \ldots,\ v-\frac{k_1+k_2}{2}+1,\ \frac{k_1+k_2}{2},\ 0\Big). \end{aligned}$$

Let $D = AB$. Note that the set of backward differences over D contains all non-zero elements of Z_v exactly once and all elements of D are distinct except a_1 and a_v. So from Theorem 6.3.1 and Remark 6.3.1 it follows that there exist m p-tuples $B_1,\ B_2,\ \ldots,\ B_m$ which on developing over Z_v produce a BNURMD$(v,\ n,\ p)$ with $\lambda = 1$.

Example 6.4.1 Let $v = 11$, $p = 6$ and $\lambda = 1$. Then (6.7) gives $m = 2$. Note that $n = 22$ and $k_1 = k_2 = 5$. So $D = (0,\ 10,\ 1,\ 9,\ 2,\ 3,\ 8,\ 4,\ 7,\ 5,\ 0)$. Then $B_1 = (0,\ 10,\ 1,\ 9,\ 2,\ 3)$ and $B_2 = (3,\ 8,\ 4,\ 7,\ 5,\ 0)$. Figure 6.1 shows a BNURMD $(11,\ 22,\ 6)$ with $\lambda = 1$.

Note: This procedure does not work for $p = v$, since $a_1 = a_v$ in v-tuple D, which affects the near uniform condition of BNURMD.

Periods

Units

0	10	1	9	2	3
1	0	2	10	3	4
2	1	3	0	4	5
3	2	4	1	5	6
4	3	5	2	6	7
5	4	6	3	7	8
6	5	7	4	8	9
7	6	8	5	9	10
8	7	9	6	10	0
9	8	10	7	0	1
10	9	0	8	1	2
3	8	4	7	5	0
4	9	5	8	6	1
5	10	6	9	7	2
6	0	7	10	8	3
7	1	8	0	9	4
8	2	9	1	10	5
9	3	10	2	0	6
10	4	0	3	1	7
0	5	1	4	2	8
1	6	2	5	3	9
2	7	3	6	4	10

Figure 6.1: BNURMD(11, 22, 6)

6.4.2 Case: v odd and $p = v$

A latin square with elements 1, 2, ..., v is called *row complete* if, for any ordered pair of elements α, β $(1 \leq \alpha, \beta \leq v,\ \alpha \neq \beta)$, there exists a row of the latin square in which α and β appear as adjacent elements. It is easy to see that a BNURMD(v, v, v) is basically a row complete latin square of order v since $\lambda = 1$. But there is no general result on the existence of such squares for odd values of v. Gordon [32] has shown that the existence of a *sequencible group* of order v is a sufficient condition for the existence of a row complete Latin square of order v. Sequencible groups have been found for orders 21, 27, 39, 55 and 57 (see Denes and Keedwell [23], Mendelsohn [68] and Wang [112]), while there is no sequencible group of order 9 or 15. However, Archdeacon, Dinitz, Stinson and Tillson [4] give row complete Latin squares of orders 9, 15, 21 and 27. The square of order 9 is the smallest possible odd order row-complete latin square. So BNURMDs $(v,\ v,\ v)$ do not exist for $v = 3,\ 5$ or 7. Figure 6.2 shows a BNURMD $(v,\ v,\ v)$ with $v = 9$ (and $\lambda = 1$).

Periods

Units									
	1	4	7	2	6	8	5	9	3
	2	5	8	3	4	9	6	7	1
	3	6	9	1	5	7	4	8	2
	4	2	1	8	7	3	9	5	6
	5	3	2	9	8	1	7	6	4
	6	1	3	7	9	2	8	4	5
	7	8	6	5	2	4	3	1	9
	8	9	4	6	3	5	1	2	7
	9	7	5	4	1	6	2	3	8

Figure 6.2: BNURMD(9, 9, 9)

6.4.3 Case: v even

Let G be the group Z_v as before. Consider the v-tuple

$$D = (a_1,\ a_2,\ \ldots,\ a_v) = \left(0,\ v-1,\ 1,\ v-2,\ 2,\ v-3,\ \ldots,\ \frac{v}{2}-1,\ \frac{v}{2}\right).$$

D is a standard difference set for constructing row complete latin square of even order. Note that all the elements of D are distinct and the set of $v-1$ backward differences in D contains all elements of Z_v except 0. So it follows from Theorem 6.3.1 and Remark 6.3.1 that D can be decomposed in m p-tuples $B_1,\ B_2,\ \ldots,\ B_m$ which when developed over Z_v give the necessary BNURMD $(v,\ n,\ p)$ for $\lambda = 1$. This is true for all p such that $(p-1)$ divides $(v-1)$. This construction is basically the same as that of Afsarinejad [2] for $\lambda = 1$. In fact the balanced uniform RMD $(v,\ n,\ p)$ for even v as constructed by him is the same as a BNURMD $(v,\ n,\ p)$.

6.5 Construction when $\lambda = 2$

Now by (6.6), $(p-1)$ divides $2(v-1)$ and from (6.7) we have $m = \frac{2(v-1)}{p-1}$. In this section we consider three cases: first when $p = v$, next when v is odd and $p \neq v$ and finally when v is even and $p \neq v$.

6.5.1 Case: $p = v$

Consider Z_v and define two p-tuples B_1 and B_2 as

$$B_1 = \left(0,\ v-1,\ 1,\ v-2,\ 2,\ v-3,\ \ldots,\ \left[\frac{v}{2}\right]\right)$$

and

$$B_2 = B_1' = \left(\left[\frac{v}{2}\right],\ \ldots,\ v-3,\ 2,\ v-2,\ 1,\ v-1,\ 0\right),$$

where $[\frac{v}{2}]$ is the integral part of $\frac{v}{2}$. Note that all the elements of Z_v occur exactly once in B_1 and exactly once in B_2. Also among the totality of $2(v-1)$ backward differences in the two v-tuples B_1 and B_2, each of the $(v-1)$ non-zero elements of Z_v appears twice. So the design can be obtained as in Theorem 6.3.1.

Example 6.5.1 Let $v = p = 5$ and $\lambda = 2$. Then (6.7) gives $m = 2$. So $B_1 = (0,\ 4,\ 1,\ 3,\ 2)$ and $B_2 = (2,\ 3,\ 1,\ 4,\ 0)$ and we get the BNURMD $(5,\ 10,\ 5)$ with $\lambda = 2$ exhibited in Figure 6.3.

Periods

Units

0	4	1	3	2
1	0	2	4	3
2	1	3	0	4
3	2	4	1	0
4	3	0	2	1
2	3	1	4	0
3	4	2	0	1
4	0	3	1	2
0	1	4	2	3
1	2	0	3	4

Figure 6.3: BNURMD(5, 10, 5)

6.5.2 Case: v odd and $p \neq v$

Suppose first $p > v$. Since $p - 1$ divides $2(v - 1)$ and $p - 1 > v - 1$, it follows that $p = 2(v - 1) + 1$. Now consider B_1 and B_2 defined in Section 6.5.1. Note that the last element of B_1 and the first element of B_2 are the same. So we redefine B_2 as a $(v - 1)$-tuple by deleting the first element of B_2 and call it B_3. $D = B_1B_3$ is an initial unit for generating BNURMD $(v,\ n,\ p)$ for $\lambda = 2$ when $p = 2v - 1$.

Next let $p < v$. Then define tuples A and B as in Section 6.4.1. From the k_1-tuple A define a $(k_1 - 1)$-tuple C as $C = (a_2,\ a_3,\ \ldots,\ a_{k_1})$. Let $D = ABCB$. Note that D is a $(2v - 1)$-tuple, the set of backward differences in D contains each non-zero element of Z_v twice and any p consecutive elements of D are distinct. So the required m p-tuples for a BNURMD $(v,\ n,\ p)$ for $\lambda = 2$ can be obtained from D as in Theorem 6.3.1 and Remark 6.3.1. These when developed over Z_v give the required design.

6.5.3 Case: v even and $p \neq v$

The construction procedure for BNURMD (v, n, p) for $\lambda = 2$ when $p > v$ and v is an even integer is the same as that for an odd integer v.

Example 6.5.2 Let $v = 4$, $p = 7$ and $\lambda = 2$. Then $B_1 = (0,\ 3,\ 1,\ 2)$ and $B_2 = (2,\ 1,\ 3,\ 0)$. So $B_3 = (1,\ 3,\ 0)$ is obtained from B_2 by deleting 2. Now $D = (0,\ 3,\ 1,\ 2,\ 1,\ 3,\ 0)$ is an initial unit for BNURDM (4, 4, 7). On developing D over Z_4 we get a BNURMD(4, 4, 7) as shown in Figure 6.4.

Periods

Units	0	3	1	2	1	3	0
	1	0	2	3	2	0	1
	2	1	3	0	3	1	2
	3	2	0	1	0	2	3

Figure 6.4: BNURMD(4, 4, 7)

Now let $p < v$. Consider Z_v and let k_1 and k_2 be two odd integers such that $k_1 + k_2 = v - 2$ and $0 \leq k_1 - k_2 \leq 2$. Construct two tuples A and B as

$$\begin{aligned} A &= (a_1,\ a_2,\ \ldots,\ a_{k_1}) \\ &= \left(0,\ v-1,\ 1,\ \ v-2,\ \ldots,\ \frac{k_1-3}{2},\ v-\frac{k_1-1}{2},\ \frac{k_1-1}{2}\right) \end{aligned}$$

$$\begin{aligned} B &= \left(a_{k_1+1},\ a_{k_1+2},\ \ldots,\ a_{k_1+k_2},\ a_{k_1+k_2+1}\right) \\ &= \left(\frac{k_1+1}{2},\ \ v-\frac{k_1+3}{2},\ \ \frac{k_1+3}{2},\ \ v-\frac{k_1+5}{2},\ \ldots,\ v-\frac{k_1+k_2}{2},\ \ \frac{k_1+k_2}{2},\ \ 0\right) \end{aligned}$$

Let $C = AB$. It can be easily seen that the set of backward differences arising from the $(v-1)$-tuple C contains all the elements of Z_v exactly once except for the elements 0 and

Now consider the v-tuple D defined in Section 6.4.3. The last element of D is $\frac{v}{2}$ while the first element of C is 0. For the ordered pair $(\frac{v}{2}, 0)$, the backward difference is $\frac{v}{2}$ over Z_v. Define $E = DC$.

Suppose first that $p \leq v-2$. Then in E, the set of backward differences contains every non-zero element of Z_v twice and any p consecutive elements of E are distinct. So E can be decomposed into the required m blocks $B_1, B_2, \ldots, B_m$ following the steps outlined in Remark 6.3.1 which when developed over Z_v produce a BNURMD (v, n, p) for $\lambda = 2$.

Next let $p = v - 1$. Since v is an even integer and $(p-1)$ divides $2(v-1)$, v must be 4. In this case, we may take $B_1 = (0, 3, 1)$, $B_2 = (1, 0, 2)$ and $B_3 = (0, 1, 2)$ as the required three initial units for generating a BNURMD (4, 12, 3) with $\lambda = 2$.

Example 6.5.3 Let $v = 6$, $p = 3$ and $\lambda = 2$. Then we get $m = 5$ from (6.7). Here $D = (0, 5, 1, 4, 2, 3)$, $A = (0, 5, 1)$ and $B = (2, 0)$. So, $E = DAB = (0, 5, 1, 4, 2, 3, 0, 5, 1, 2, 0)$ and hence $B_1 = (0, 5, 1)$, $B_2 = (1, 4, 2)$, $B_3 = (2, 3, 0)$, $B_4 = (0, 5, 1)$ and $B_5 = (1, 2, 0)$.

Now $B_1 + g$, $B_2 + g$, $B_3 + g$, $B_4 + g$, $B_5 + g$; $g \in Z_6$ give a BNURMD (6, 30, 3) with $\lambda = 2$.

Chapter 7

Concluding Remarks and Open Problems

Enumeration of catastrophic fault patterns for link redundancy $G = \{g\}$ has been done in [22] for unidirectional case. In Chapter 2, we extend this to the case of $G = \{2, 3, \ldots, k, g\}$, $2 \leq k < g - 1$. We characterize catastrophic fault patterns for both unidirectional and bidirectional cases and, using random walk as a tool, enumerate them. A method of enumeration of CFPs for an arbitrary link configuration G was discussed in Sipala [104], but no closed form solution was obtained. The number of catastrophic fault patterns for an arbitrary link configuration $G = \{g_1, g_2, \ldots, g_k\}$ is still unknown. The fault model of the problems mentioned in the present thesis assumes only PE failures. In reality, both PEs and links can fail and can do so simultaneously. Not much is known in the case of both PE and link failures.

In Chapter 3, we consider the problem of finding the maximal link configuration for which a given fault pattern F is catastrophic. We consider maximality with respect to two parameters: the length g of the longest bypass link in G and the number $|G|$ of bypass links in G. The problem of minimization of the parameters is trivial since any F is catastrophic when $G = \emptyset$. In reality, the problem of finding a minimal link configuration G for which a given fault pattern F is not catastrophic is more important. Since the designer can adopt G to ensure that F cannot disrupt the flow of information from I to O. Depending on designer choice, minimality can be with respect to various parameters like: the length g of the longest bypass link, the number $|G|$ of bypass links in G or the sum $\sum_{i=1}^{k} g_i$ of the lengths of bypass links in G. The problem of finding the minimum value of the length g of the longest bypass link in G for which a given fault pattern F is not catastrophic is easy as was shown in Section 3.1. However, studying minimality with respect to the other two parameters, i.e., $|G|$ and $\sum_{i=1}^{k} g_i$, seems to be difficult.

In Chapter 4, our main contribution is a complete characterization of catastrophic fault patterns for two-dimensional arrays. Let $\mathcal{N}$ be a two-dimensional network with link redundancy $G = (g_1,\ g_2, \ldots,\ g_k \mid v_1,\ v_2, \ldots,\ v_l)$, and let F be a fault pattern. Then we prove that, F is catastrophic with respect to $\mathcal{N}$ implies that the cardinality of F, $|F| \geq N_1 g_k$. We outline an algorithm for the construction of a CFP with the maximum width for a given link redundancy G. We give necessary and sufficient conditions for a fault pattern F to be catastrophic with respect to link redundancy $G = (g_1,\ g_2, \ldots,\ g_k \mid v_1,\ v_2, \ldots,\ v_l)$. We provide an algorithm to test whether a given F is catastrophic with respect to link redundancy $G = (g_1,\ g_2, \ldots,\ g_k \mid v_1,\ v_2, \ldots,\ v_l)$. The number of catastrophic fault patterns is not known even for the link redundancy $G = (1,\ g \mid 1,\ v)$.

In Chapter 5, we provide a method of construction for a class of nearly strongly balanced uniform RMDs using suitable symmetric balanced incomplete block designs constructed through difference technique. We do not know whether one can give a method of construction for nearly strongly balanced uniform RMDs using any SBIBD not necessarily constructed through difference technique or without using SBIBD.

In Chapter 6, we give a procedure for construction of balanced near uniform RMDs when every ordered pair of distinct treatments appears exactly once or twice (i.e., $\lambda = 1$ or 2), except for the case v an odd integer and $p = v$. This exceptional case is equivalent to finding a row complete Latin square of odd order. We could not obtain any result for general λ and the problem remains open for $\lambda \geq 3$.

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