Midsem
Discrete Mathematics
BStat 2025

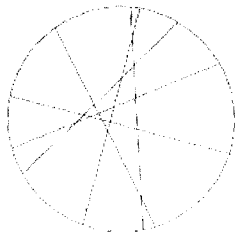
Time: 2.5 hours

26th February 2025

There are 8 problems with a total of 120 marks. You can attempt all the problems. The maximum score one can score is 100.

1. (3×5 marks) Prove or disprove the following set of asymptotic relations:
   (a) $(2.9)^{\log_2 n} = O(10 n^{\log_2 2.8} \log^2 n)$
   (b) $\log\log n = \Omega((\log\log\log n)^{\sqrt{\log\log\log n}})$
   (c) $n^4 = \Theta((1-1/n)^{(n+n^2)} n^4)$
   (d) $2^{(\log n)-(\log\log n)} \sim 2^{(1-1/n)\log n}$
   (e) $\log\binom{2n}{n} = o\left(\frac{\log(2n)!}{\log 2n}\right)$
2. (7 + 8 marks) Solve the following two recurrences: $T(0) = T(1) = T(2) = 5$ and for all integer $n \geq 2$ we have
   (a) $T(n) = T(\lfloor \frac{n}{4} \rfloor) + T(\lfloor \frac{n}{2} \rfloor) + n + O(\log n)$
   (b) $T(n) = T(\lceil \frac{n}{4} \rceil) + T(\lfloor \frac{4n}{5} \rfloor) + n$
3. (2 + 5 + 8 marks) Recall the definition of relations - transitive and equivalence relations.
   (a) How many relations can be there on a set of size n
   (b) Let $B(n)$ denote the number of equivalence relations on a given set of n elements. Prove that the following statement is false:
   $$\log_2 B(n) = \Theta(n^2).$$
   (c) Let $T(n)$ denote the number of transitive relations on a given set of n elements. Prove: $\log_2 T(n) = \Theta(n^2)$.

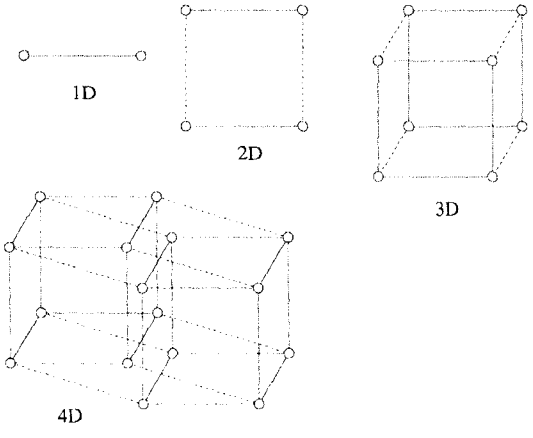
4. (15 marks) Suppose that we take a circular disk and cut it along straight lines (as though we were cutting cake, but very badly). Here is an example:



This divides the circle into several regions. Prove that no matter how you cut, those regions can always be colored black and white so that whenever two regions share a side, they are different colors.

5. (15 marks) Let Q_n be the hypercube graph. Its vertices are $\{0,1\}^n$: n-tuples $(x_1, x_2, \ldots, x_n)$, where each x_i is either 0 or 1. It has an edge between two vertices that differ in exactly one coordinate, and agree in the $n-1$ others. It is also called the n-dimensional or nD cube.

The $1D$, $2D$ $3D$ and $4D$ cube graphs are shown below.



The diameter of a graph is the longest distance between any two vertices in the graph.

Prove that for all $n \geq 1$, the diameter of Q_n is n.

6. (10 marks) Prove or disprove the following statement:

"Suppose every vertex in an n-vertex graph is the endpoint of at least two different edges. Then the graph must have at least $2n-3$ edges."

7. (15 marks) What is the number of ways the numbers $1, 2, \ldots, 2n$ can be arranged in a 2-by-n rectangle so that each row and each column is increasing? Give proof for your answer.

For example $\{1,2,3,4,5,6\}$ can be arranged in 5 ways as

123	124	125	134	135
456	356	346	256	246

8. (10 + 10 marks) For an undirected graph $G = (V, E)$ an independent vertex set is a subset $W \subseteq V$ such that W is an independent set, that is, for all $u, v \in W$, $(u, v) \notin E$. The number of independent vertex sets for the for cyclic graphs C_n $(n \geq 2)$ is denoted by $L(n)$. C_n is the cyclic graphs on n vertices.

   (a) Prove that for any $n \geq 4$, $L(n) = L(n-1) + L(n-2)$

   (b) Obtain a closed form expression for $L(n)$.