Indian Statistical Institute

Semsetral examination : (2025-26)

M. Math. II year

Differential Geometry I

Date : 14/11/25 Maximum marks : 50 Duration : 3 hours.

Answer ALL questions. Marks are indicated in bracket which add up to 55. However, the maximum you can score is 50.

(1) Let M be a smooth manifold of dimension n and let TM be the set $\{(x, v) : x \in M, v \in T_xM\}$. Prove the following:

(i) TM is a smooth manifold of dimension $2n$.

(ii) TM is orientable. [5+10=15]

2. Let M be the level set of some regular smooth map $\Phi : R^n \to R^m$ $(m < n)$, say $M = \Phi^{-1}(\{c\})$ for some $c \in \Phi(R^n) \subset R^m$. Let $\mathcal{N}(M)$ be the set of all pairs (p, v) where $p \in M$ and $v \in (T_pM)^{\perp}$. Prove that this set has the structure of a smooth manifold. [10]

3. Let x, y, z be the usual Euclidean coordinate variables of R^3 and $\partial_x, \partial_y, \partial_z$ denote the corresponding globally defined vector fields, given by partial derivative w.r.t. x, y, z respectively. Let R be the vector field defined by, $R(x, y, z) = x\partial_x + y\partial_y + z\partial_z$ and let $\mathcal{D}$ be the family of subspaces of tangent spaces defined by $\mathcal{D}_p = \{v \in TR^3_p : \ < v, R(p) >= 0\}$, for any **nonzero** $p \in R^3$ and where $< \cdot, \cdot >$ denotes the Euclidean inner product. Prove that $\mathcal{D}$ is a smooth 2-dimensional distribution on $R^3 \setminus \{0\}$ and it is integrable. What are the maximal integral submanifolds? [10]

4. Find out (with proof) all the geodesics on the conical surface $S \subset R^3$ given by

$$S = \{(x, y, z) : \ x^2 + y^2 = z^2, \ z > 0\}.$$

[10]

5. Prove that there cannot be a globally defined, smooth vector field X on the 2-sphere $S^2 = \{(x, y, z) : \ x^2 + y^2 + z^2 = 1\}$ which is nonzero at every point. [Hint: You can assume the fact proved in class about the Gaussian curvature of the sphere.] [10]