INDIAN STATISTICAL INSTITUTE

Semester Examination

M.Tech CS, 2025 – 2026 (Semester – I)

Discrete Mathematics

Date: 28 November 2025 Maximum Marks: 100 Duration: 3 hours

General comment. Answer as much as you can, but the maximum you can score from both Group-A and Group-B is 50.

Notations and definitions. $\mathbb{N}$ denotes the set of natural numbers. For all $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

A set $C \subseteq \mathbb{R}^d$ is called convex if, whenever two points are chosen from C, the whole line segment joining them lies entirely in C.

The random graph $G(n, 1/2)$ is defined on n vertices, where each pair of distinct vertices is joined by an edge independently with probability $1/2$.

The n-dimensional Hamming cube, denoted Q_n, is the graph defined as follows:

- The vertex set of Q_n is the set of all binary strings of length n, i.e., $V(Q_n) = \{0,1\}^n$.
- Two vertices $x, y \in \{0,1\}^n$ are connected by an edge if and only if x and y differ in exactly one coordinate.

A Hamiltonian cycle in a graph is a cycle that contains every vertex of the graph exactly once.

Recall the statements of the following two theorems from graph theory.

Theorem 1 (Kőnig's Theorem) In every bipartite graph, the size of a maximum matching is equal to the size of a minimum vertex cover.

Theorem 2 (Hall's Theorem) Let $G = (X \cup Y, E)$ be a bipartite graph with bipartition X and Y. Then G has a matching that saturates every vertex of X if and only if for every subset $S \subseteq X$, the neighbourhood $N(S)$ satisfies $|N(S)| \geq |S|$.

Group-A

(AQ1) Let L be a finite set of lines in $\mathbb{R}^2$ and P be a finite set of points in $\mathbb{R}^2$. Show that

$$I(P,L) = O\left(|L|^{2/3}|P|^{2/3} + |P| + |L|\right),$$

where $I(P,L) := |\{(p,\ell) \in P \times L \ : \ p \in \ell\}|$. [10]

(AQ2) Show that König's Theorem and Hall's Theorem are equivalent. [10]

(AQ3) Show that every planar graph is 5-colorable. The use of the Four Color Theorem is not allowed. [10]

(AQ4) Let $G(n,p)$ denotes the Erdős–Rényi random graph model. Show that

$$\lim_{n\to\infty} \Pr\left[G(n,1/2) \text{ is connected}\right] = 1$$

and

$$\lim_{n\to\infty} \Pr\left[G(n,1/2) \text{ is bipartite}\right] = 0$$

[5+5 = 10]

(AQ5) Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Prove that if the intersection of every $d+1$ sets in $\mathcal{F}$ is nonempty, then the intersection of all the sets in $\mathcal{F}$ is nonempty. [10]

(AQ6) Let $(P, \preceq)$ be a finite partially ordered set and let w be the size of a largest antichain in P. Then the minimum number of chains needed to partition P equals w. [10]

Group-B

(BQ1) Let $S = \{1, 2, \ldots, kn\}$, and suppose $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are both partitions of S into n sets of size k. Then there exists a set T of size n such that every intersection $T \cap A_i$ and $T \cap B_i$ has cardinality exactly 1. [10]

(BQ2) Let $\ell(m,n)$ denote the number of points inside a ball of radius m in $\mathbb{Z}^n$, and $d(m,n)$ denote the number of Delannoy paths from $(0,0)$ to (m,n). Using bijection establish that $\ell(m,n) = d(m,n)$. [10]

(BQ3) Suppose there are n lines in $\mathbb{R}^2$ such that every three of them can be intersected with a unit circle. Prove that all of them can be intersected with a unit circle. [10]

(BQ4) Let $\{A_1, \ldots, A_m\}$ be a collection of m distinct subsets of $[n]$ satisfying the following properties:

- For any A_i and A_j, we have $A_i \not\subseteq A_j$.
- For all A_i and A_j, $A_i \cap A_j \neq \emptyset$.
- For all A_i, $|A_i| \leq n/2$.

Show that

$$\sum_{i=1}^{m} \frac{1}{\binom{n-1}{|A_i|-1}} \leq 1.$$

[10]

(BQ5) Let T, T' be two spanning trees of a connected graph G. For an edge $e \in E(T) \setminus E(T')$, prove that there exists an edge $e' \in E(T') \setminus E(T)$ such that both $T' + e - e'$ and $T + e' - e$ are spanning trees of G. [10]

(BQ6) (a) For all $n \geq 2$, show that n-dimensional Hamming cube Q_n contain a Hamiltonian cycle for every $n \geq 2$?

(b) Construct a graph $G = (V, E)$ on n vertices satisfying the following two conditions:

- $|E| = \binom{n-1}{2} + 1$, and
- G is not a Hamiltonian graph.

[5+5 = 10]