INDIAN STATISTICAL INSTITUTE

End-Semester Examination: 2024-25

B. Stat. I Year

Vectors and Matrices II

Date: 09/05/2025 Maximum Marks: 60 Duration: 3 Hours

Note: Give proper justification to all your answers.
State clearly all the results you are using.

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(1) Let A be an $n \times n$ matrix over $\mathbb{C}$ and $a_{i,j}$ be its (i,j)th entry for $1 \leq i, j \leq n$. Let $j_1, j_2, \ldots, j_n$ and $k_1, k_2, \ldots, k_n$ be integers between 1 to n. Define

$$D(A) = a_{j_1,k_1} a_{j_2,k_2} \ldots a_{j_n,k_n}.$$

Prove that D is n-linear if and only if the integers $j_1, j_2, \ldots, j_n$ are distinct. [8]

(2) Reduce the matrix

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 4 & 1 & -1 & -1 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

to a matrix of the form $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ and then compute its determinant. [6]

(3) Let $M_n(\mathbb{C})$ denote the vector space of $n \times n$ matrices over $\mathbb{C}$. Fix $A \in M_n(\mathbb{C})$ and define the linear map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $T(B) = AB$ for $B \in M_n(\mathbb{C})$. Prove the following.

(a) λ is an eigenvalue of T if and only if λ is an eigenvalue of A.

(b) p is the minimal polynomial of T if and only if p is the minimal polynomial of A. [10]

(4) Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & 5 \\ -1 & 0 & -1 & -1 \\ 0 & -3 & 3 & -3 \\ 2 & 2 & 4 & 0 \end{bmatrix}.$$

Find a spectral decomposition of A of the form $A = P \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$, where P is a non-singular matrix of order 4 and Δ is a non-singular diagonal matrix of order $\rho(A)$. [10]

[Hint. The first two columns of A form a column basis. Find a rank factorization $A = PQ$ and obtain the eigenvalues and eigenvectors of A from those of QP.]

P.T.O.

(5) Let $A \in M_n(\mathbb{C})$.

(a) Show that A is idempotent (i.e., $A^2 = A$) if and only if A is diagonalizable and each eigenvalue of A is 0 or 1.

(b) If A is diagonalizable and $A^2 = A^3$, then show that A is idempotent. 8

(6) Consider the quadratic form

$$q(x) = x_1x_2 + x_2x_3 + x_1x_3, \quad x \in \mathbb{R}^3.$$

Reduce the quadratic form to a diagonal form of the type

$$Q(u) = d_1u_1^2 + d_2u_2^2 + d_3u_3^2, \quad u \in \mathbb{R}^3,$$

where $d_i = 1, -1$ or 0 for $i = 1, 2, 3$. Also, obtain the transforming matrix P such that $x = Pu$ reduces $q(x)$ to $Q(u)$. 10

(7) Let A be an $n \times n$ positive definite matrix, $d \in \mathbb{R}$, and $b \in \mathbb{R}^n$. Define

$$M = \begin{bmatrix} A & b \\ b^t & d \end{bmatrix}.$$

Let $c = d - b^tA^{-1}b$. Prove the following.

(a) If $c > 0$, then M is positive definite.

(b) If $c = 0$, then M is non-negative definite.

(c) If $c < 0$, then M is indefinite. 10

Hint. (a) $\det M = c \det A$.

(b) Let $A = B^tB$ and $C = [B \mid (B^{-1})^tb]$. Observe that $M = C^tC$.

(c) Take $y^t = [-b^tA^{-1} \mid 1]$ and evaluate y^tMy. Find $x \in \mathbb{R}^n$ such that $x^tMx = a_{11}$.