Study on Determination of Optimal Parameters for Shewhart Control Charts and Other Related Methodologies

A thesis submitted to the Indian Statistical Institute

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by

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under the guidance of

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Dedicated to My Family and

Dr. Arup Ranjan Mukhopadhyay

Certificate

Date:

This is to certify that the thesis entitled **“Determination of Optimal Parameters for Shewhart Control Charts and Other Related Methodologies"** being submitted by **Sandeep** to the Indian Statistical Institute, Kolkata, for the award of the degree of Doctor of Philosophy in **Quality, Reliability & Operations Research (QROR)** is a record of bonafide research work carried out by him for the last five years under my supervision. The results embodied in this thesis, to the best of my knowledge, have not been submitted for the award of any other degree/diploma anywhere. In my opinion, the thesis fulfills the requirement for the award of the degree.

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Chapter 1

Introduction

In this thesis, we try to find the optimal parameters for two different kinds of methodologies. These methodologies consist of finding the optimal diagnosis interval for online quality control methods and finding the optimal parameters of different kinds of control charts based on economic and statistical aspects.

1.1 Online Quality Control Methods

The use of online quality control methods is crucial in manufacturing processes to ensure that the products meet the desired quality standards. These methods aim to detect defects in the production process and prevent non-conforming products from reaching the end-user, thereby reducing the risk of product recalls, customer complaints, and damage to the company's reputation.

In the past, quality control methods relied mainly on manual inspection of products, which was time-consuming, labor-intensive, and prone to human error. With the advent of automation and computerization, these methods have become more sophisticated and prevalent, offering real-time monitoring and control of production processes.

On-line quality control methods essentially use the online data collected from the production process to identify sporadic shifts, sustained drifts, and other non-random process abnormalities that help to adopt appropriate remedial actions to conform to the desired quality standards. These methods or regulatory exercises enable simultaneously to improve the quality of products, enhance productivity, and reduce losses.

In this context, it may be worthwhile to note that the production systems of yesteryears gave due importance to incoming, in-process as well as final inventories with an apprehension of shortage of raw materials as well as the finished goods to be delivered to the customers or end users. However, in modern production systems, more emphasis has been laid on Just-in-Time (JIT) or lean management. JIT management or lean management has emerged in the production system to avoid the high cost of holding an inventory in a manufacturing system. Accordingly, an increased emphasis has been given to small-scale production and the associated systematic sampling through selecting items or parts or components of the appropriate population in a production system at a regular interval for exercising control or regulatory activities in the process. In this connection, the crucially important question is what should be the optimal sampling interval from economic consideration, while introducing and maintaining the procedure of systematic sampling in a production system - be it large or small.

It is worth noting that JIT or lean systems have many merits compared to conventional or traditional production systems. The essential feature of JIT or lean system is that the production cycles or runs are short. Consequently, the manufacturers can switch over conveniently from one product to another product. Additionally, JIT or lean system reduces costs on account of having a reduced need for warehouses for storing materials. The manufacturers spend less money on raw materials since they buy just enough quantum of resources to produce the ordered products. However, the major demerit of JIT or lean systems emerges if unwanted disruptions occur in the supply chain due to sudden machine breakdown or some such thing that can not deliver the goods promptly and can stall the entire production line. The delivery time to the consumers also can get adversely affected. Such demerits in JIT or lean systems reinforce the necessity of introducing and maintaining appropriate systematic sampling of products and processes for trouble-free operation keeping in view the determination of sampling interval of production processes in businesses from economic considerations.

The diagnosis interval is very important in production systems for many reasons, some of which are provided here. Regular diagnosis is an enabler for the early detection of potential issues or deviations from the desired contour of manufacturing parameters.

By monitoring at regular intervals, manufacturers can identify any anomalies or abnormalities that may affect the quality of the product at various stages of the production process. Regular inspections can identify wear and tear, equipment malfunctions, or any signs of potential breakdowns. By addressing these issues proactively, manufacturers can avoid unexpected equipment failures, reduce downtime, and ensure uninterrupted production schedules. Diagnosis intervals play a crucial role in maintaining consistent product quality. By monitoring essentially the critical and major process parameters and product characteristics at regular intervals, manufacturers can identify any deviations from the desired specifications. This allows timely intervention for necessary adjustments, calibrations, or corrective actions to bring the process back to its acceptable range. Through troubleshooting at the upstream stage of a process at the earliest opportunity, one can avoid the production of defective or substandard products. This results, in turn, substantial business benefits in terms of reduction of scrap, rework, and repair as internal failure costs, and customer complaints as well as returns from customers as external failure costs. Furthermore, the adoption of a systematic sampling through implementing the optimum diagnosis interval enables to minimize the unexpected breakdowns and associated costs. Delivery performance too can be adequately boosted due to this systematic process intervention through the implementation of optimum diagnosis interval. Thus, establishing and introducing regular diagnosis intervals helps optimize the overall manufacturing cost. Some of the works on online quality control methods for optimal diagnosis interval based on economic considerations were given by Taguchi [85, 86, 87, 88] and Taguchi et al. [89].

Parameters such as the rate of production(P_r), the loss due to false alarm(C_f), and the loss due to non-detection of process abnormality (C_a) are likely to significantly influence the determination of the optimal diagnosis interval ($h^\star$). While Taguchi et al. [89] provided methodologies for computing $h^\star$, their approaches did not consider these important parameters. However, we have considered them in our models. In addition, the present work introduces a formulation that is supposed to capture the practical reality in production engineering by incorporating a detailed and pragmatic breakup of the diagnosis cost ($\tilde{y}$). The novelty thus lies in the inclusion of these relevant additional parameters C_f, C_a, P_r, along with the pragmatic breakup of $\tilde{y}$ in the computation of $h^\star$.

Therefore, building upon the foundational ideas proposed by Taguchi et al. [89], modified approaches have been developed in Chapter 2 that delineate the revised methodologies for determining $h^\star$ in the context of online quality control methods.

1.2 Optimal Parameters for the Control Charts

A certain amount of variability is always present in every production process, regardless of its meticulous design or operation. This variability can arise either due to the presence of natural/chance causes or special/assignable causes. If a process possesses solely chance causes, it is stated to be in control. However, the process is declared out of control, if the presence of assignable causes is found in it.

Statistical process control procedures are required to monitor process variability and to detect whether the process is in control or out of control. Control charts are statistical process control tools that are employed to assess process stability. These charts monitor the pertinent characteristics of a process to detect any shifts or drifts present in the process.

Control charts were invented by Shewhart [80] essentially to differentiate between the assignable or the special causes of variation and the chance or the common causes of variation. The motive was that if there is an assignable cause in the process, one should be able to detect it and then eliminate it. The methodology behind the control charts consists of taking samples from a process periodically and resorting to process measurements like the mean of the quality characteristic of a process or the associated nonconforming percentage, as the case may be. By using these process measurements, control limits and a central line are established. No action for process stability is required, provided the measurements lie within the control limits. If the process measurements fall outside the control limits, a search for the assignable cause is undertaken followed by finding the assignable cause that needs to be eliminated from the process.

While we try to find these assignable causes, there are some costs associated with the endeavor like the cost of sampling, the cost of testing, the cost for removal of the assignable cause, the cost of generating a defective item, etc. Due to the association of these costs, it is imperative to consider the economic consequences of the design of a control chart.

There are three types of designs of the control charts which deal with the economic consequences. These three kinds are the economic design, economic statistical design, and multi-objective economic statistical design.

1.2.1 The Economic Design of Control Charts

Historically, there are seven important types of Shewhart control charts. These are $\bar{X}$-R chart, $\bar{X}$-S chart, I-MR chart, np-chart, p-chart, c-chart, and u-chart. While the first three charts deal with variable quality characteristics, the remaining four charts deal with attribute quality characteristics. Specifically, while np-chart and p-chart deal with defectives generated in a process, c-chart and u-chart deal with the number of defects in varied contexts. To use these kinds of charts, some important parameters should be specified like the sample size (n), the sampling interval (h), and the control limits' multiplier (k). The question arises about how one can find these parameters. Duncan [25] tried to answer this question during the exploration of the economic design of the $\bar{X}$-chart, in which he tried to minimize the sum of the pertinent costs involved. Ladany [47] came up with the economic design of the np-chart. The economic design of the np-chart was also given due consideration by Chiu [16], Heikes et al. [34], Montgomery et al. [60], Duncan [26], Gibra [30], Williams et al. [98], Kooli and Limam [45] and Kooli and Limam [46]. The general procedure to determine the economic design of control charts was given by Lorenzen and Vance [52]. The economic design of a moving average control chart for non-normal data using variable sampling intervals was studied by Patil and Shirke [69]. Huang [40] proposed an economic design of max charts using Taguchi's loss function.

In Chapter 3, taking inspiration from the study done by Duncan [25], a modified model has been proposed to minimize the loss function ($\mathcal{L}$).

In our model, we have incorporated parameters that were not given due consideration by Duncan [25], such as a pragmatic breakup of cost per unit of measuring an item alias diagnosis cost of a product into cost components like the sampling cost per unit (y_1), sample preparation cost per unit (y_2), cost per unit of testing an item of the product (y_3), the energy consumption cost per unit (y_4), the consumables cost, if any, per unit (y_5), and the reporting cost per unit (y_6) along with the rate of production (P_r). These parameters have the potential to play an important role in determining the optimal sampling interval (h) from the industrial engineering perspective.

To obtain the optimal values of the design parameters n, h, and k, two distinct approaches have been employed. The first method follows Duncan [25] by differentiating $\mathcal{L}$ with respect to n, despite n being an integer. To ensure mathematical soundness and industry-friendly implementation, a second method has been introduced, wherein $\mathcal{L}$ is differentiated with respect to h and k only. This second approach utilizes an iterative algorithmic approach to find the optimal parameters that fully incorporate the additional parameters overlooked by Duncan [25], thereby offering a more comprehensive and practically applicable solution.

1.2.2 The Economic Statistical Design of Control Charts

In assessing the effectiveness of the control charts, two key metrics are utilized: the in-control Average Run Length (ARL_0) and the out-of-control Average Run Length (ARL_δ). The Average Run Length (ARL) represents the average number of plotted points before indicating an out-of-control condition.

Based on the effectiveness of the control charts, some limitations over the use of the economic design of control charts were given by Woodall [99]. To overcome these limitations, Saniga [78] proposed an economic statistical design of $\bar{X}$-chart, where some conditions were imposed on ARL_0 and ARL_δ. Based on the approach introduced by Saniga [78], an economic statistical design for the S control chart using Taguchi's loss function was proposed by Yang [103].

The economic statistical design of control charts has already been studied by many researchers, some of which are mentioned here. An economic statistical design of np control charts using a full adaptive approach was given by Katebi and Moghadam [42].

Lee and Chou [51] proposed an economic-statistical design of synthetic Tukey's control chart with Taguchi's asymmetric loss functions under log-normal distribution. An economic-statistical design of $\bar{X}$ control charts with multiple assignable causes was put forth by Yu et al. [106]. A general model for the economic-statistical design of adaptive control charts for processes with multiple assignable causes was introduced by Nenes et al. [67]. An optimum variable-dimension EWMA chart for multivariate statistical process control was studied by Epprecht et al. [27]. An economic design of residuals MEWMA control chart with variable sampling intervals and sample size was proposed by Xue et al. [102]. The economic design approach for an SPC inspection procedure implementing the adaptive c-chart was given by Lupo [54]. Lupo [53] provided the economic design approach for implementing the adaptive c-chart. Inghilleri et al. [41] came up with the double sampling scheme for the c-chart. The economic design of cumulative sum control charts for monitoring a process with correlated samples was given by Lee [48].

To the best of our knowledge, there are no articles that deal with the economic statistical designs of u-chart and p-chart. Thus, for the first time, the economic statistical designs of u-chart and p-chart have respectively been proposed in Chapters 4 and 5.

1.2.3 The Multi-objective Economic Statistical Design of Control Charts

Along with the economic aspects associated with control charts, sometimes it is necessary to consider the statistical aspects like detecting the shift as early as possible. The multi-objective economic statistical design of control charts is one way to study the economic aspects along with the statistical aspects. So, in the multi-objective economic statistical design of control charts, two or more objectives are considered simultaneously. In the literature, there are several papers dealing with the multi-objective economic design of control charts, some of which are included here. Celano and Fichera [15] studied the multi-objective economic design of an $\bar{X}$ control chart. Yang et al. [104] used a multi-objective particle swarm optimization algorithm to develop a multi-objective model for the optimal design of $\bar{X}$ and S control charts. Safaei et al. [73] studied the multi-objective economic statistical design of $\bar{X}$ control chart considering Taguchi's loss function. Faraz and Saniga [29] examined a bi-objective optimization model for the economic-statistical design of control charts. The multi-objective design approach for the c-chart considering Taguchi's loss function was developed by Lupo [56]. Bashiri et al. [9] proposed a multi-objective economic statistical design for the cumulative count of conforming control chart. Morabi et al. [61] presented a multi-objective optimization model for designing an $\bar{X}$ control chart with fuzzy parameters to monitor the process mean.

As far as our knowledge goes, the studies given in the literature have not dealt with the multi-objective economic statistical design of u-chart, p-chart, and CUSUM control chart. Hence, the multi-objective economic statistical designs of u-chart, p-chart, and CUSUM control chart have respectively been proposed for the first time in Chapters 4, 5, and 6.

1.3 Methods Used for Solving the Proposed Methodologies

Two kinds of approaches have been used to solve the proposed methodologies. These approaches consist of the conventional partial derivative approach and meta-heuristic approaches like Genetic Algorithm (GA) and Non-dominated Sorting Genetic Algorithm II (NSGA II).

1.3.1 Conventional Partial Derivative Approach

The conventional partial derivative approach has been used to solve the proposed model for finding the optimal parameter of the online quality control methods and the optimal parameters of the economic design of $\bar{X}$-chart.

1.3.2 Genetic Algorithmic Approach

The concept of Genetic Algorithms (GAs) was initially introduced by John Holland Holland [38]. GAs are evolutionary algorithms inspired by Charles Darwin's theory of evolution. They aim to mimic natural selection, where the fittest individuals have a higher chance of reproducing and passing on their genes to the next generation. Despite being among the earliest evolutionary algorithms developed, GAs have been widely used for solving complex optimization problems.

The fundamental idea behind genetic algorithms is the survival of the fittest. In this context, individuals with higher fitness scores are more likely to be selected for reproduction. This selection process is typically facilitated through genetic operators, such as selection, crossover, and mutation. During each generation of the algorithm, the fittest individuals are chosen for mating for these operations. Offspring inherit genetic information from their parents through crossover operation, while new genetic diversity is introduced to the population via mutation operation.

To ensure the persistence of the most fit individuals across generations, genetic algorithms often employ elitism operators. These operators preserve the best-performing individuals from one generation to the next, preventing the loss of valuable genetic material. The pseudo-code for a genetic algorithm adhering to this description is presented in Algorithm 1, where p_c, p_m, and e_r denote the crossover probability, mutation probability, and elitism rate, respectively.

The improvised pseudocode for implementing the Genetic Algorithm is given through Algorithm 1. This algorithm originated from Scardua [79]. Following this algorithm, the code to implement the Genetic Algorithm has been appropriately modified for the models proposed in Chapters 4 and 5. A separate pseudocode for Elitism is also required to implement the Genetic Algorithm. The required pseudocode for implementing Elitism has been borrowed from Scardua [79].

Algorithm 1 Genetic Algorithm

```
1: p_c ← receive the crossover probability
2: p_m ← receive the mutation probability
3: e_r ← receive the elitism rate
4: f(.) ← receive the fitness function
5: f_s(.) ← receive the selection function
6: f_c(.) ← receive the crossover function
7: f_m(.) ← receive the mutation function
8: X ← randomly generate N individuals satisfying given constraints
```

$F = f(x) \leftarrow$ compute the fitness values of X
while termination criterion not satisfied **do**
  for k = 1, 2, ..., N **do**
    $\mathbf{p}_1, \mathbf{p}_2 = f_s(X)$ ▷ select two parents from X
    $\mathbf{c_k} \leftarrow f_c(\mathbf{p}_1, \mathbf{p}_2, p_c)$ ▷ Generate one child
    $\mathbf{c_k} \leftarrow f_m(\mathbf{c_k}, p_m)$ ▷ Mutate one child
  end for
  $F_c = f(C) \leftarrow$ compute the fitness values of $C = \{\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c_N}\}$
  $X, F = f_e(X, F, C, F_c, e_r)$ ▷ keep N individuals while preserving elite chromosomes
end while
Return the individual from X with the best fitness value

1.3.3 Non-dominated Sorting Genetic Algorithmic II (NSGA II) Approach

At iteration k, NSGA II initializes with a population P_k consisting of N candidate solutions. It then proceeds to a loop where N children are generated. Each child is created by selecting a pair of parents through binary tournament selection, with the criterion being the Crowded-comparison Operator (CCO) operator. The chosen parents undergo crossover to produce offspring, which subsequently undergoes mutation. All children are stored in a matrix Q_k.

A combined population $R_k = P_k \cup Q_k$ is formed. Utilizing fast non-dominated sorting, the Pareto fronts $F_0, F_1, \ldots$ of R_k are determined. Subsequently, the population for the next generation, P_{k+1}, is constructed.

P_{k+1} is initially an empty set. The algorithm then sets $P_{k+1} = P_{k+1} \cup F_0$. If the cardinality of P_{k+1}, denoted as $|P_{k+1}|$, equals N, the creation of P_{k+1} is concluded, and the algorithm proceeds to the next iteration. However, if $|P_{k+1}| < N$, the creation process continues. In this case, fronts are added to P_{k+1} in the order of their ranking until $|P_{k+1}| = N$.

While incorporating fronts into P_{k+1}, it's possible that one front, denoted as F_{last}, may not entirely fit into P_{k+1}. In such instances, the solutions of F_{last} are arranged in descending order of the CCO, and these ordered solutions are sequentially inserted into P_{k+1} until $|P_{k+1}| = N$.

The pseudocode for implementing NSGA II is given in Algorithm 2. This algorithm was given by Scardua [79]. Following this algorithm, the code to implement NSGA II has been appropriately modified for the models proposed in Chapters 4, 5, and 6. In order to write the code for implementing NSGA II certain pseudocodes are required. These pseudocodes pertaining to Pareto Ranking, Fast Non-dominated Sorting, and CCO are also given in Scardua [79].

Algorithm 2 Non-dominated Sorting Genetic Algorithm II (NSGA II)

$k \leftarrow 0$ ▷ index of the current iteration
$P_k \leftarrow$ initial population of N candidate solutions ▷ satisfying constraints
$Q_k \leftarrow \phi$ ▷ population of children
while termination criterion not satisfied **do**
  for $i \in \{1, ..., N-1\}$ **do**
    $(\mathbf{p}_1, \mathbf{p}_2) \leftarrow$ select two parents using binary tournament from P_k
    $\mathbf{r} \leftarrow$ recombine$(\mathbf{p}_1, \mathbf{p}_2)$ ▷ create a child by crossover
    $\mathbf{q} \leftarrow$ mutate$(\mathbf{r})$ ▷ mutate the child
    $Q_k \leftarrow Q_k \cup \{\mathbf{q}\}$ ▷ update the population of children
  end for
  $R_k \leftarrow P_k \cup Q_k$ ▷ create a combined population
  $\mathcal{F} \leftarrow$ all non-dominated fronts in R_k ▷ $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, ...\}$
  $P_{k+1} \leftarrow \{\}$ ▷ new population
  while P_{k+1} does not have N individuals **do**
    $i \leftarrow 0$
    if P_{k+1} has room for all elements of $\mathcal{F}_i$ **then**
      $P_{k+1} \leftarrow P_{k+1} \cup \mathcal{F}_i$ ▷ add i-th front to the parent population
    else
      $P_{k+1} \leftarrow P_{k+1} \cup$ the first $(N - |P_{k+1}|)$ points of $\mathcal{F}_i$
    end if
    $i \leftarrow i + 1$
  end while
  delete the fronts which could not be inserted in P_{k+1}
  $k \leftarrow k + 1$
end while

1.4 Notations and Abbreviations

Here are some notations and abbreviations that will be used throughout the thesis.

Notations

x	The loss caused by producing one unit of defective item.
x_d	The loss per unit incurred due to generating a defective item and the defective item is sent to the next stage of production or to the consumer.
$\tilde{y}$	The diagnosis cost.
y_1	It is the sampling cost. This cost includes the labor cost, shipping cost, and material cost while taking a sample.
y_2	It is the sample preparation cost. Sometimes a sample requires a prerequisite shape and size for certain tests. The cost incurred for reshaping the collected sample as per metrological requirements is the sample preparation cost.
y_3	It is the testing cost. The cost related to testing of a collected sample by individuals comes under testing cost.
y_4	It is the energy cost. It is essentially the cost of power consumption - electrical or otherwise.
y_5	It is the consumables' cost that may be incurred for certain tests. For example, to assess the chemical composition of hot liquid metal using Spectrographic analysis, Argon, an inert gas, is used.
y_6	The cost of reporting comes under this cost.
C_f	It is the loss due to the false alarms over the average trouble occurrence interval.
C_a	It is the loss due to non-detection of process abnormality over the diagnosis interval.
z	The adjustment cost for removing process abnormalities over the average trouble occurrence interval.
z_1	The cost of stopping the process for one unit of time.
z_2	The direct cost to recover the process including labor cost, material cost, and equipment cost.
t_0	The average recovery time.

$\bar{\eta}$	The average trouble occurrence interval.
l	The time lag.
h	The diagnosis interval or the sampling interval.
$h^{\star}$	The optimal diagnosis interval.
L	The total cost per item.
P_r	The rate of production.
μ	Mean when the process is in-control state.
σ	Standard deviation of the process.
$\bar{X}$	The sample mean.
δ	The magnitude of shift jolted by an assignable cause.
λ	The average rate at which an assignable cause occurs per unit of operating time.
I_0	The income per hour per piece of a product when the process is in-control state.
I_1	The income per hour per piece of a product whenever there is a shift in the mean under an out-of-control state.
V	The average income difference per piece of a product per hour between in-control and out-of-control state or $V = I_0 - I_1$. V can reasonably be assumed to be positive.
$\mathcal{L}$	The loss function.
p	The proportion defective.
p_0	The expected fraction defective produced when the process is in control.
p_1	The expected fraction defective produced when the process is out of control.
C_p	The penalty incurred per defective item.
n	The sample size.
k	The control limits' multiplier.
P or Power	The probability of detecting an assignable cause when an assignable cause has already occurred.

Q or Type II error (β)	The probability of an assignable cause remaining undetected.
Type I error (α)	The probability of wrongly saying a process is out-of-control.
ρ	The proportion of time for which a process is in in-control state.
ρ_1	The proportion of time for which a process is in out-of-control state.
C_0	Quality cost per hour when the process is in-control.
C_1	Quality cost per hour when the process is out-of-control.
C_2	The cost for searching an assignable cause when there is none.
C_3	The average cost of identifying and eliminating an assignable cause.
d	The cost per sample for maintaining a control chart in a process.
y	The variable cost of sampling an inspection unit.
τ	Expected time taken by an assignable cause to occur.
t	The expected time to take a sample and obtain the results.
T_0	The expected time taken to ascertain that an alarm is false or time associated with a false alarm.
T_1	The expected time required to discover an assignable cause.
T_2	The expected time required to eliminate an assignable cause.
γ_1	A binary variable that takes the value 1 if the production or process continues during the search for an assignable cause and 0 otherwise.
γ_2	A binary variable that takes the value 1 if the production or process continues during the elimination of an assignable cause through intervening in the process and 0 otherwise.
s	Average number of samples taken when the process is in-control.
T_δ	The expected production cycle time.
C_E	The expected cost per cycle.

Abbreviations

ESD	Economic Statistical Design
MoESD	Multi-objective Economic Statistical Design

JIT	Just-in-Time
CL	Central Line
LCL	Lower Control Limit
UCL	Uppar Control Limit
ARL	Average Run Length
ARL_0	in-control Average Run Length
ARL_δ	out-of-control Average Run Length
ATS	Average time to Signal.
GA	Genetic Algorithm.
NSGA II	Non-dominated Sorting Genetic Algorithm II
CUSUM	Cumulative Sum

1.5 Publications From the Thesis

Published Articles

1. Sandeep and Mukhopadhyay, A. R. (2024). Optimal diagnosis interval for online quality control methods. *Quality Engineering, 36(3), 594-608.*
   DOI: https://doi.org/10.1080/08982112.2023.2256372 [From Chapter 2]

2. Sandeep and Mukhopadhyay, A. R. (2025). Evolving Parameters of Shewhart's $\bar{X}$ Control Chart from Present-day Industrial Engineering Perspective. *Communications in Statistics - Theory and Methods, 54(16), 5257-5283.*
   DOI: https://doi.org/10.1080/03610926.2024.2435580 [From Chapter 3]

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2. Sandeep, Arup Ranjan Mukhopadhyay (2023). Optimal Diagnosis Interval for Online Quality Control Methods. *Technical Report No. - SQCOR-2023-02*

Publications Under Review

1. Economic Statistical Design of the u-chart for Poisson Distributed Quality Characteristic. [From Chapter 4]

2. Determining the Optimal Parameters of the Economic Statistical Design of the p-chart Using Genetic Algorithm. [From Chapter 5]

3. A Multi-objective Economic Statistical Design of the CUSUM Control Chart: NSGA II Approach. *Arxiv*

   DOI: https://doi.org/10.48550/arXiv.2409.04673 [From Chapter 6]

Chapter 2

Optimal Diagnosis Interval for On-line Quality Control Methods[1]

2.1 Introduction

Historically, on-line quality control methods for finding the optimal diagnosis interval, i.e., $h^\star$ were given by Taguchi [85, 86, 87, 88] and Taguchi et al. [89] based on economic considerations. Adams and Woodall [2] studied Taguchi's methods for variables and concluded that these methods can be decisive in some cases. In this Chapter, we have considered Taguchi's on-line quality control methods.

Two cases for finding $h^\star$ were studied by Taguchi [85, 86, 87, 88] and Taguchi et al. [89]. In Case I, the process shifts to produce all (100)% defective items from no defective item produced initially. In Case II, the process shifts to produce, on the contrary, $\pi(100)\%$ defective items from the initial no defective item. In both cases, however, it was recommended to inspect the units after every h units of production. The process is adjusted as soon as a defective item is found. The value of h is calculated in such a way that it minimizes the total cost per unit, i.e., L under the assumption of fixed cost for diagnosis, the cost incurred due to generating a defective item, and the pertinent adjustment cost, if any, to set the process right. Srivastava and Wu [83, 84], Nayebpour and Woodall [65], Wang and Yue [97], Borges et al. [13], Dasgupta [22], Chou and Wang [19], Trindade et al. [92], Ho et al. [37] and da Costa Quinino et al. [21] also proposed

[1] Optimal Diagnosis Interval for On-line Quality Control Methods. *Quality Engineering*, **36(3)**, 594-608, 2024.

on-line quality control methods for finding $h^\star$. On-line quality control method for a short-run production was given by Bessegato et al. [12]. The economical designs for on-line process control of attributes for a variable sampling interval were proposed by Bessegato et al. [11]. Ho and Da Costa Quinino [36] developed on-line quality control methods for a variable sampling interval.

The first economic design in the realm of control charts was proposed by Duncan [25]. The economic methods for finding $h^\star$ given by Taguchi et al. [89] and the economic design of np-control chart by Gibra [30] are closely related. Ryan [71], Williams et al. [98], Lorenzen and Vance [52], and Woodall [99, 100] considered in particular the economic designs of np-control chart in their works. Economic design of attribute np-control chart using a variable sampling policy was proposed by Kooli and Limam [46]. A joint-adaptive scheme for np-control chart with multiple dependent state sampling was given by Zhou et al. [108]. Fallahnezhad et al. [28] developed the economic-statistical design of the np-control chart with variable sample size and sampling interval. Katebi and Moghadam [42] provided the economic statistical designs of attribute np-control chart using a full adaptive approach. The optimal economic statistical design of an adaptive attribute control chart for monitoring three-level products was proposed by Katebi and Rahim [43]. The robust economic design of np-control chart under different scenarios comprising different processes and economic parameters was given by Attia and Abdel-Aal [6]. It may be worthwhile to mention here that while determining $h^\star$, Taguchi et al. [89] did not assume a process failure mechanism. However, in the economic design of the Shewhart control chart, it has been taken into account by Duncan [25]. The approach in this Chapter has attempted to integrate the economic design of the control chart proposed by Duncan [25] with optimizing h for on-line quality control methods proposed by Taguchi et al. [89] along with other cost and production parameters like C_a and P_r.

In this Chapter, apart from improvising the existing models, methods given by Taguchi [88] and Taguchi et al. [89] are discussed and compared with Gibra [30] and Barlow et al. [8] in Section 2.2.

Parameters like the rate of production (P_r), the loss due to false alarm (C_f), and the loss due to non-detection of process abnormality (C_a) have the potential to play an important role in determining the optimal value of h, i.e., $h^\star$. Taguchi et al. [89] had given the methods for finding $h^\star$ but they did not consider the parameters P_r, C_f, and C_a in their proposed methodologies. Also, a pragmatic break-up of the diagnosis cost ($\tilde{y}$) has been considered from a practical manufacturing engineering standpoint. The novelty of this work is the consideration of these additional pertinent variables C_f, C_a, and P_r as well as the pragmatic break-up of $\tilde{y}$ for determining $h^\star$ pertaining to online quality control methods. Taking inspiration from their work, a modified approach has thus been given in Section 2.3 of this Chapter narrating the online quality control methods to be adopted for yielding $h^\star$. For the sake of clarity, it is to be noted that the modified L has been arrived at containing h. Subsequently, L has been differentiated with respect to h and equated to zero for determining $h^\star$ corresponding to which L is minimum. Further, in order to cross-check the condition of sufficiency for minimizing L, the second derivative of L is found too and the value of $h^\star$ thus found is appropriately substituted to yield a strictly positive value. The pertinent necessary and sufficiency conditions for arriving at $h^\star$ are also shown in this Chapter.

In Section 2.4, a comparison has been made and illustrated for Case I with corresponding Taguchi's approach using a numerical example. It has been found that the proposed model is cost-effective compared to other available methods proposed by Ho et al. [37] and Ho and Da Costa Quinino [36]. Numerical examples for both approaches proposed in Section 2.3 are provided in Sections 2.4.1, 2.4.2, and 2.4.3. The distinguishing feature of this Chapter is the demonstration of the effectiveness of the proposed methodology through a real-life case study in Section 2.4.4 in the context of an Indian manufacturing plant that produces iron pipes for the transportation of drinking water across the globe.

In Section 2.5, we have given the results of the sensitivity analysis that has been conducted to evaluate the robustness of the proposed model for further strengthening the methodology. Sensitivity analysis allows readers to understand the degree to which changes in input parameter configurations across a wide range of possible situations would change the output parameters impacting the methodology's performance.

In Section 2.6 of this Chapter, the conclusion has been drawn in a generic manner about the usage of both Case I and Case II from the perspective of practical application. The content of this Chapter can be found in Sandeep and Mukhopadhyay [74].

2.2 Taguchi's Monitoring Method

In this Section, we are considering the two cases. In Case I, the process shifts to produce all (100)% defective items from no defective item initially. In Case II, the process shifts to produce $\pi(100)\%$ defective items from no defective item initially as given by Taguchi et al. [89].

2.2.1 The Total Cost Per Unit [Case I]

Here we are considering the case in which the process shifts to produce all (100)% defective items from no defective item initially.

Taguchi et al. [89] considered essentially three costs, namely, diagnosis cost ($\tilde{y}$), adjustment cost (z), and cost of producing the defective item (x) for minimizing the total cost per unit (L).

The adjustment cost includes the recovery cost and loss caused by halting the production process in order to recover the process. Thus, the adjustment cost is calculated as follows:

$$z = z_1 \times t_0 + z_2 \ . \tag{2.1}$$

where $z_1$is the cost of stopping the process for one unit of time, t is the average recovery time, and z_2 is the direct cost to recover the process, including labor cost, material cost, and equipment cost

$\bar{\eta}$ is the average trouble occurrence interval. If the number of such trouble occurrences is zero, then $\bar{\eta} = 2\times$ total production of that duration. Otherwise,

$$\bar{\eta} = \frac{\text{total production in a duration}}{\text{number of troubles during that duration}} \ . \tag{2.2}$$

The total cost per unit (L) given by Taguchi et al. [89] is as follows:

$$L = \frac{\tilde{y}}{h} + \frac{z}{\bar{\eta}} + \frac{l \times x}{\bar{\eta}} + \frac{h+1}{2} \times \frac{x}{\bar{\eta}} \ . \tag{2.3}$$

where l is the number of items produced from the time an out-of-control item is found during the checking process until the time of stopping the production process for recovery (time lag), and h is the diagnosis interval (number of items manufactured between two consecutive diagnoses). These notations can also be found in Chapter 1 under Notations and Abbreviations.

Taguchi et al. [89] considered the average number of defective items produced between two consecutive diagnoses as $(\frac{h+1}{2})$. The pertinent justification, as put forward by Taguchi [88], is given in Section 2.3. Also, the consecutive items in the process under monitoring are independent and identically distributed in order to derive the above formula.

2.2.2 Optimal Diagnosis Interval between Two Consecutive Diagnoses (ODICD)

Taguchi et al. [89] found $h^\star$ by arriving at the derivative of equation (2.3) with respect to h and then solved for h by equating the derivative with zero. It is given in the following:

$$h^\star = \sqrt{\frac{2\tilde{y}\bar{\eta}}{x}} \ . \tag{2.4}$$

Since $h^\star$ is independent of l and z, Taguchi et al. [89] stated that the optimal value of $h^\star$ won't be accurate until the average trouble occurrence interval $\bar{\eta}$ is much greater than the time lag l, i.e., $\bar{\eta} \gg l$ and the cost of defective item x is much more than the cost of adjustment per item $\frac{z}{\bar{\eta}}$, i.e., $x \gg \frac{z}{\bar{\eta}}$. If any of these two conditions is not satisfied by L, then the value of $h^\star$ will not be accurate. In such situations, an alternate expression for L is needed. The alternate expression of L is defined in Section 2.2.3.

2.2.3 Alternate Approach for Optimal Diagnosis Interval (AAODI)

The reasoning given by Taguchi [88] for alternate approach is as follows: Since we diagnose the process with interval h, the average number of items produced up to the time of finding trouble is $(\bar{\eta} + \frac{h}{2})$. This is due to the reason that when a defective item is found at the r^{th} diagnosis, the defect must have occurred between the $(r-1)^{th}$ and the r^{th} diagnosis. So, the average number of defective items produced after the trouble occurred would be $\frac{h}{2}$, making $\bar{\eta}$ longer by $\frac{h}{2}$. Then an accurate equation of L will be:

$$L = \frac{\tilde{y}}{h} + \frac{z}{\bar{\eta} + \frac{h}{2}} + \frac{l \times x}{\bar{\eta} + \frac{h}{2}} + \frac{h+1}{2} \times \frac{x}{\bar{\eta} + \frac{h}{2}} \; . \tag{2.5}$$

After differentiating equation (2.5) with respect to h, equating it with zero and using appropriate approximations, Taguchi et al. [89] found the following equation for determining $h^{\star}$:

$$h^{\star} = \sqrt{\frac{2\left(\bar{\eta} + l\right)\tilde{y}}{x - \frac{z}{\bar{\eta}}}} \; . \tag{2.6}$$

Nayebpour and Woodall [64] provided the detailed derivation of equation (2.6) from equation (2.5) in the technical report at University of Houston.

There can be a significant difference in the calculation of $h^{\star}$ between ODICD and AAODI. This can be illustrated with the help of the following example.

The data for a production process is given as: x = \$6, $\tilde{y}$ = \$200, z = \$5,000, l = 1,250 units, $\bar{\eta}$ = 5,000 units. Using equation (2.4), we get $h^{\star}$ as 577 units, and the corresponding L is \$3.19. The use of equation (2.6) gives $h^{\star}$= 707 units and the corresponding L is \$3.21 using the equation (2.3). It can be clearly seen that there is no significant difference in the L yet there is a significant difference between $h^{\star}$ of ODICD and AAODI. This difference diminishes if $\bar{\eta} \gg l$ and $x \gg \frac{z}{\bar{\eta}}$.

2.2.4 Optimal Diagnosis Interval With Fraction of Defective Items (ODIFDI) [Case II]

Here we are considering the case in which the process shifts to produce $\pi(100)\%$ defective items from no defective item initially. Recall that x_d is the loss per unit incurred due to generating a defective item and the defective item is sent to the next stage of production or to the consumer and $x_d \gg x$. π is the probability of noticing the defective item at the time of diagnosis and $(1-\pi)$ is the probability of corresponding non-detection of the defective item.

Taguchi et al. [89] provided the following formula for the average loss due to defective items when the production process deviates from normal conditions.

$$= \left[\frac{h+1}{2}\pi^2 + h\pi\,(1-\pi)\right]x + \left[\frac{h+1}{2}\pi\,(1-\pi) + h\,(1-\pi)^2\right]x_d\ . \tag{2.7}$$

If we put $\pi = 0$ in equation (2.7), the average loss turns out to be hx_d. Taguchi et al. [89] stated that the loss hx_d corresponds to the loss $\frac{1}{2}(h+1)x$ for $\pi = 1$. Taguchi et al. [89] substituted $2x_d$ in place of x in equation (2.3) and obtained the following formula for L.

$$L = \frac{\tilde{y}}{h} + \frac{z}{\bar{\eta}} + \frac{l \times x}{\bar{\eta}} + (h+1) \times \frac{x_d}{\bar{\eta}}\ . \tag{2.8}$$

$h^\star$ obtained by Taguchi et al. [89] is

$$h^\star = \sqrt{\frac{2(\bar{\eta}+l)\tilde{y}}{2x_d - \frac{z}{\bar{\eta}}}}\ . \tag{2.9}$$

If $\pi \neq 1$, Taguchi et al. [89] stated that it is possible to detect all the defective items by tracing back to the point when the production process starts to deviate from the normal conditions. Taguchi et al. [89] also stated if this is done then there are no undetected defective items. The new formula for L is

$$L = \frac{\tilde{y}}{h} + \frac{z}{\bar{\eta}} + \frac{l \times x}{\bar{\eta}} + (h+1) \times \frac{x}{\bar{\eta}}\ . \tag{2.10}$$

$h^\star$ obtained by Taguchi et al. [89] is

$$h^\star = \sqrt{\frac{2(\bar{\eta} + l)\tilde{y}}{2x - \frac{z}{\bar{\eta}}}} \ . \tag{2.11}$$

2.2.5 Comparison With Other Approaches for Finding the Optimal Diagnosis Interval

Gibra [30] developed a model for the economic design of the np-control chart. He assumed the process with a single assignable cause, where the fraction of defectives shifts to out-of-control value p_1 from the in-control value p_0 $(p_1 > p_0)$. In this process, a sample of size n is taken after every h unit of time. If the number of defective items found in the sample is more than the upper control limit, then a search for an assignable cause is initiated. The approximate value of $h^\star$ calculated by Gibra [30] is as follows:

$$h^\star \quad = \quad \sqrt{\frac{2P_r\tilde{y}}{\lambda x}} \ . \tag{2.12}$$

We can clearly see that the difference between Taguchi et al. [89] and Gibra [30] is nothing but the replacement of $\bar{\eta}$ by $\frac{P_r}{\lambda}$.

Barlow et al. [8] have considered only two costs, diagnosis cost consisting essentially of fixed cost and cost incurred due to time passed between the system failure and its discovery. If the time to failure follows an exponential distribution, then $h^\star$ computed by Barlow et al. [8] is given in equation (2.13).

$$h^\star \quad = \quad \sqrt{\frac{2\tilde{y}}{\lambda x}} \ . \tag{2.13}$$

2.3 Determination of Optimal Diagnosis Interval

In this Section, we are considering the two cases. In Case I, the process shifts to produce all (100)% defective items from no defective item initially. In Case II, the process shifts to produce $\pi(100)\%$ defective items from no defective item initially.

2.3.1 Optimal Diagnosis Interval [Case I] (Method I)

Here, we have considered some parameters hitherto not given due consideration by Taguchi et al. [89], Gibra [30], and Barlow et al. [8]. These parameters are the sampling cost per unit (y_1), sample preparation cost per unit (y_2), cost per unit of testing an item of the product (y_3), the energy consumption cost per unit (y_4), the consumables cost, if any, per unit (y_5), the reporting cost per unit (y_6), the rate of production(P_r), the loss due to false alarm(C_f), and the loss due to non-detection of process abnormality (C_a).

Here, the cost parameter $\tilde{y}$ considered by Taguchi et al. [89] has been appropriately replaced by $\sum_{i=1}^{6} y_i$. The other important parameter for our model such as P_r has been considered by Gibra [30] but not considered by Taguchi et al. [89]. Also, it is to be noted that, the parameter λ considered by Gibra [30] and Barlow et al. [8] has not been considered in our model similar to Taguchi et al. [89] since it is the context of np-control chart whereas the intent of this treatise is a generic way to determine $h^{\star}$ for the diagnosis of production process abnormalities. The remaining parameters identical with Taguchi et al. [89] are x, z, $\bar{\eta}$, l, h, and L.

Diagnosis cost includes the costs y_1, y_2, y_3, y_4, y_5, and y_6. Since, diagnosis is done after every h items, therefore, diagnosis cost per item $= \frac{\sum_{i=1}^{6} y_i}{h}$.

Also, the recovery cost for h items is z. So, recovery cost per item $= \frac{z}{\bar{\eta}}$.

Since the loss due to false alarm is C_f, this implies, the loss per item due to false alarm $= \frac{C_f}{\bar{\eta}}$.

Loss due to non-detection of process abnormality for h items is C_a. This implies, the loss per item due to non-detection of process abnormality $= \frac{C_a}{h}$.

Table 2.1: Pattern of troubles.

Pattern	Diagnosis i						Diagnosis (i+1)
No. 1	○	○	○	···	○	○	×
No. 2	○	○	○	···	○	×	×
⋮	⋮						
No. h - 1	○	○	×	···	×	×	×
No. h	○	×	×	···	×	×	×
items		(1)	(2)	···	(h - 2)	(h - 1)	(h)

Taking a cue from Taguchi [88], it may be relevant to put the rationale for considering the average number of defective items as $\frac{(h+1)}{2}$. In Table 2.1, let $\circ$ means the item produced is non-defective, and $\times$ means the item produced is defective. It is assumed that once a process begins to produce defective units, every unit produced after that will be defective until the problem is set right.

It emerges from Table 2.1 that the total number of defective units for all h patterns is $\frac{h(h+1)}{2}$. So, the average number of defective items is $\frac{h(h+1)}{2h} = \frac{(h+1)}{2}$.

The cost per item because of the generation of defective items = ((average number of defective items produced $\times$ cost of a defective item) / $\bar{\eta}$) $=\frac{h+1}{2} \times \frac{x}{\bar{\eta}}$.

Since the diagnosis takes place as long as l items are produced. It implies, the cost due to time lag per item = $\frac{l \times x}{\bar{\eta}}$.

Hence, we propose L as the following:

L = (diagnosis cost per item) + (recovery cost per item) + (loss due to time lag per item) + (the loss per item due to the production of the defective items) + (loss per item due to false alarm) + (loss due to non-detection of process abnormality per item).

Thus, mathematically L is expressed as:

$$L = \frac{\sum_{i=1}^{6} y_i}{h} + \frac{z}{\bar{\eta}} + \frac{l \times x}{\bar{\eta}} + \frac{h+1}{2} \times \frac{x}{\bar{\eta}} + \frac{C_f}{\bar{\eta}} + \frac{C_a}{h} \ . \tag{2.14}$$

As we know $l \propto P_r$ and $P_r \propto \frac{1}{h}$

$$P_r = K' \frac{l}{h} \ . \tag{2.15}$$

This implies,

$$l = K P_r h \ . \tag{2.16}$$

where, $K = \frac{1}{K'}$ and thus L gets transformed as:

$$L = \frac{\sum_{i=1}^{6} y_i + C_a}{h} + \frac{z + C_f}{\bar{\eta}} + \frac{KP_r h \times x}{\bar{\eta}} + \frac{h+1}{2} \times \frac{x}{\bar{\eta}} \ . \tag{2.17}$$

Differentiating equation (2.17) with respect to h, we get:

$$\frac{\partial L}{\partial h} = -\frac{\sum_{i=1}^{6} y_i + C_a}{h^2} + \frac{KP_r x}{\bar{\eta}} + \frac{x}{2\bar{\eta}} \ . \tag{2.18}$$

Equating equation (2.18) with zero,we obtain

$$\begin{aligned}
-\frac{\sum_{i=1}^{6} y_i + C_a}{h^2} + \frac{KP_r x}{\bar{\eta}} + \frac{x}{2\bar{\eta}} &= 0 \\
\Rightarrow \frac{\sum_{i=1}^{6} y_i + C_a}{h^2} &= \frac{KP_r x}{\bar{\eta}} + \frac{x}{2\bar{\eta}} \\
\Rightarrow (1 + 2KP_r)\, x h^2 &= 2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta} \\
\Rightarrow h^\star &= \sqrt{\frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}}{(1 + 2KP_r)\, x}} \ .
\end{aligned} \tag{2.19}$$

To show that $h^\star$ given in equation (2.19) minimizes L, we need to show that $\frac{\partial^2 L}{\partial h^2}$ is positive. Differentiating equation (2.18) with respect to h will gives us

$$\frac{\partial^2 L}{\partial h^2} = \frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)}{h^3} \ . \tag{2.20}$$

Substituting equation (2.19) in equation (2.20).

$$\frac{\partial^2 L}{\partial h^2} = \frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)}{\left(\frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}}{(1+2KP_r)x}\right)^{\frac{3}{2}}} . \tag{2.21}$$

Since all the quantities in the above mathematical expression for $\frac{\partial^2 L}{\partial h^2}$ are positive, this implies that the sufficiency condition in terms of $\frac{\partial^2 L}{\partial h^2} > 0$ is satisfied. Hence, $h^\star$ given in equation (2.19 minimizes L.

2.3.2 Alternate approach [Case I] (Method II)

In a process with diagnosis interval h, the average number of items produced up to the time of finding the problem is $(\bar{\eta} + \frac{h}{2})$.

The corresponding reason given by Taguchi [88] can be put forward like this, if a defective item is found at the r^{th} cycle, then it can be safely said that till the $(r-1)^{th}$ cycle every item produced was non-defective in nature. Hence, the defect must have occurred between the r^{th} and the $(r-1)^{th}$ cycle. Consequently, the average number of defective items produced subsequent to the emergence of the problem turns out to be $(\bar{\eta} + \frac{h}{2})$. The equation of L, thus, can be re-written in a transformed manner as follows:

$$L = \frac{\sum_{i=1}^{6} y_i + C_a}{h} + \frac{z + C_f}{\bar{\eta} + \frac{h}{2}} + \frac{KP_r h \times x}{\bar{\eta} + \frac{h}{2}} + \frac{h+1}{2} \times \frac{x}{\bar{\eta} + \frac{h}{2}} . \tag{2.22}$$

Equation (2.22) can be further transformed for mathematical convenience by having a binomial expansion of the term $\frac{1}{\bar{\eta}+\frac{h}{2}}$.

$$\frac{1}{\bar{\eta} + \frac{h}{2}} = \frac{1}{\bar{\eta}\left(1 + \frac{h}{2\bar{\eta}}\right)} = \frac{1}{\bar{\eta}}\left(1 + \frac{h}{2\bar{\eta}}\right)^{-1} \approx \frac{1}{\bar{\eta}}\left(1 - \frac{h}{2\bar{\eta}}\right) . \tag{2.23}$$

By substituting equation (2.23) in equation (2.22), we eventually obtain the following mathematical expression for L.

$$L = \frac{\sum_{i=1}^{6} y_i + C_a}{h} + \frac{z + C_f}{\bar{\eta}} \times \left(1 - \frac{h}{2\bar{\eta}}\right) + \frac{KP_r h \times x}{\bar{\eta}} \times \left(1 - \frac{h}{2\bar{\eta}}\right) + \frac{(h+1)\,x}{2\bar{\eta}} \times \left(1 - \frac{h}{2\bar{\eta}}\right) . \tag{2.24}$$

Differentiating equation (2.24) with respect to h.

$$\frac{\partial L}{\partial h} = -\frac{\sum_{i=1}^{6} y_i + C_a}{h^2} - \frac{z + C_f}{2\bar{\eta}^2} + \frac{KP_r x}{\bar{\eta}} - \frac{KP_r h x}{\bar{\eta}^2} + \frac{x}{2\bar{\eta}} - \frac{(2h+1)\,x}{4\bar{\eta}^2} . \tag{2.25}$$

Since, for a standard production process $\bar{\eta}$ (average trouble occurrence interval) can be presumed to be reasonably high, its square term turns out to be larger. As the fourth term is much smaller than the third term and also, and the sixth term is much smaller than the fifth term. After neglecting these terms, we get

$$\frac{\partial L}{\partial h} = -\frac{\sum_{i=1}^{6} y_i + C_a}{h^2} - \frac{z + C_f}{2\bar{\eta}^2} + \frac{KP_r x}{\bar{\eta}} + \frac{x}{2\bar{\eta}} . \tag{2.26}$$

Equating equation (2.26) zero, we obtain

$$-\frac{\sum_{i=1}^{6} y_i + C_a}{h^2} - \frac{z + C_f}{2\bar{\eta}^2} + \frac{KP_r x}{\bar{\eta}} + \frac{x}{2\bar{\eta}} = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{6} y_i + C_a}{h^2} = \frac{KP_r x}{\bar{\eta}} + \frac{x}{2\bar{\eta}} - \frac{z + C_f}{2\bar{\eta}^2}$$

$$\Rightarrow \frac{\sum_{i=1}^{6} y_i + C_a}{h^2} = \frac{(1 + 2KP_r)\,x}{2\bar{\eta}} - \frac{z + C_f}{2\bar{\eta}^2}$$

$$\Rightarrow h^2 \left((1 + 2KP_r)\,x - \frac{z + C_f}{\bar{\eta}} \right) = 2 \left(\sum_{i=1}^{6} y_i + C_a \right) \bar{\eta}$$

$$\Rightarrow h^{\star} = \sqrt{\frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}}{\left(1 + 2KP_r\right)x - \frac{z+C_f}{\bar{\eta}}}}\ . \tag{2.27}$$

It may be pertinent to note here that according to Taguchi, $h^{\star} = \sqrt{\frac{2(\tilde{y}+l)\bar{\eta}}{x-\frac{z}{\bar{\eta}}}}$.

The detailed mathematical derivation with the condition for sufficiency for minimizing L is given hereunder.

Differentiate equation (2.26) with respect to h.

$$\frac{\partial^2 L}{\partial h^2} = \frac{2\left(\sum_{i=1}^{6} y_i + q\right)}{h^3}\ . \tag{2.28}$$

Substituting equation (2.27) in equation (2.28).

$$\begin{aligned}
\frac{\partial^2 L}{\partial h^2} &= \frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)}{\left(\frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}}{\left((1+2KP_r)x-\frac{z+C_f}{\bar{\eta}}\right)}\right)^{\frac{3}{2}}} \\
\Rightarrow \frac{\partial^2 L}{\partial h^2} &= \frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}}{\left(\frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)}{(1+2KP_r)x-\frac{z+C_f}{\bar{\eta}}}\right)^{\frac{3}{2}}} \\
\Rightarrow \frac{\partial^2 L}{\partial h^2} &= \frac{\left(\left(1+2KP_r\right)x-\frac{z+C_f}{\bar{\eta}}\right)^{\frac{3}{2}}}{\left(2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}\right)^{\frac{1}{2}}}\ .
\end{aligned} \tag{2.29}$$

The reason thereof is given hereunder.

- The denominator is positive being a sum of positive quantities.
- For the numerator, $x > \frac{z}{\bar{\eta}}$.
- Similarly, $x > \frac{C_f}{\bar{\eta}}$.
- $(1 + 2KP_r)$ is a positive quantity.
- Hence, $\left|(1 + 2KP_r)x\right| > \left|\frac{z+C_f}{\bar{\eta}}\right|$.

Thus, $h^\star$ given in equation (2.27) minimizes L.

The mathematical equations thus derived in (2.19) and (2.27) for obtaining $h^\star$ are provided for two methods under two conditions - Method I: when $x \gg \frac{z+C_f}{\bar{\eta}}$ and Method II: when $\frac{z+C_f}{\bar{\eta}}$ approach to the close vicinity of x. While encountering a practical situation, it is to be ascertained first which method between I and II fulfills the required criteria for the appropriate application.

2.3.3 Optimal Diagnosis Interval With Fraction of Defective Items [Case II]

Taguchi et al. [89] provided the formula given in equation (2.7) for the average loss due to defective items when the production process deviates from normal conditions. After rearranging the term given in equation (2.7), we obtain the average loss as

$$= \left[\frac{\pi + h(2-\pi)}{2}\right]\left[x\pi + (1-\pi)x_d\right] . \tag{2.30}$$

Loss due to time lag is given by

$$= l\pi x . \tag{2.31}$$

Substituting $l = KP_rh$ in equation (2.31), we obtain

$$= KP_rh\pi x . \tag{2.32}$$

Substituting equations (2.30) and (2.32) in equation (2.24) at appropriate places, we obtain the following formula for L:

$$L = \frac{\sum_{i=1}^{6} y_i}{h} + \frac{z + C_f}{\bar{\eta}} \times \left(1 - \frac{h}{2\bar{\eta}}\right) + \frac{KP_r h\pi x}{\bar{\eta}} \times \left(1 - \frac{h}{2\bar{\eta}}\right) + \frac{[\pi + h(2-\pi)][x\pi + (1-\pi)x_d]}{2\bar{\eta}} \times \left(1 - \frac{h}{2\bar{\eta}}\right) . \tag{2.33}$$

Differentiating equation (2.33) with respect to h.

$$\frac{\partial L}{\partial h} = -\frac{\sum_{i=1}^{6} y_i}{h^2} - \frac{z + C_f}{2\bar{\eta}^2} + \frac{KP_r x\pi}{\bar{\eta}} - \frac{KP_r hx\pi}{\bar{\eta}^2} + \frac{(2-\pi)[x\pi + (1-\pi)x_d]}{2\bar{\eta}} - \frac{[\pi + 2h(2-\pi)][x\pi + (1-\pi)x_d]}{4\bar{\eta}^2} . \tag{2.34}$$

Since, for a standard production process $\bar{\eta}$ (average trouble occurrence interval) can be presumed to be reasonably high, its square term turns out to be larger. As the fourth term is much smaller than the third term and also, the sixth term is much smaller than the fifth term, neglecting these terms, we obtain,

$$\frac{\partial L}{\partial h} = -\frac{\sum_{i=1}^{6} y_i}{h^2} - \frac{z + C_f}{2\bar{\eta}^2} + \frac{KP_r x\pi}{\bar{\eta}} + \frac{(2-\pi)[x\pi + (1-\pi)x_d]}{2\bar{\eta}} . \tag{2.35}$$

Equating equation (2.35) with zero yields,

$$-\frac{\sum_{i=1}^{6} y_i}{h^2} - \frac{z + C_f}{2\bar{\eta}^2} + \frac{KP_r x\pi}{\bar{\eta}} + \frac{(2-\pi)[x\pi + (1-\pi)x_d]}{2\bar{\eta}} = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{6} y_i}{h^2} = \frac{(2-\pi)[x\pi + (1-\pi)x_d] + 2KP_r x\pi}{2\bar{\eta}} - \frac{z + C_f}{2\bar{\eta}^2}$$

$$\Rightarrow \frac{\sum_{i=1}^{6} y_i}{h^2} = -\frac{z + C_f}{2\bar{\eta}^2} + \frac{2KP_r x\pi + (2-\pi)[x\pi + (1-\pi)x_d]}{2\bar{\eta}}$$

$$\Rightarrow h^2 \left(2KP_r x\pi + (2-\pi)[x\pi + (1-\pi)x_d] - \frac{z + C_f}{\bar{\eta}}\right) = 2\bar{\eta} \sum_{i=1}^{6} y_i$$

$$\Rightarrow h^{\star} = \sqrt{\frac{2\bar{\eta}\sum_{i=1}^{6} y_i}{\Big((2-\pi)\left[x\pi + (1-\pi)\,x_d\right] + 2KP_r x\pi\Big) - \frac{z+C_f}{\bar{\eta}}}}\ . \tag{2.36}$$

The detailed mathematical derivation with the condition for sufficiency for minimizing L is given hereunder. Differentiating equation (2.35) with respect to h yields,

$$\frac{\partial^2 L}{\partial h^2} = \frac{2\sum_{i=1}^{6} y_i}{h^3}\ . \tag{2.37}$$

Substituting equation (2.36) in equation (2.37), we obtain

$$\frac{\partial^2 L}{\partial h^2} = \frac{2\sum_{i=1}^{6} y_i}{\left(\frac{2\bar{\eta}\sum_{i=1}^{6} y_i}{\left(2KP_r x\pi+(2-\pi)\left[x\pi+(1-\pi)x_d\right]\right)-\frac{z+C_f}{\bar{\eta}}}\right)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{\partial^2 L}{\partial h^2} = \frac{\left(\frac{2KP_r x\pi+(2-\pi)\left[x\pi+(1-\pi)x_d\right]}{\bar{\eta}} - \frac{z+C_f}{\bar{\eta}^2}\right)^{\frac{3}{2}}}{\left(2\sum_{i=1}^{6} y_i\right)^{\frac{1}{2}}}\ . \tag{2.38}$$

The reasoning for $\frac{\partial^2 L}{\partial h^2} > 0$ is the same as given in the sufficiency condition in Section 2.3.2.

2.4 Numerical Examples

In this Section, numerical examples are provided for both approaches (i.e. the process shifts to produce all (100)% defective items from no defective item initially (Case I) and the process shifts to produce $\pi(100)\%$ defective items from no defective item initially (Case II) along with a real-life case study for better understanding.

2.4.1 Numerical Examples Demonstrating the Results of Process With All Defective Items

A further illustration has been made with the help of the following numerical example. A comparison has also been made with corresponding Taguchi's approach. The numerical values given below have been considered as input variables, wherever applicable, for enumerating the $h^\star$ values as shown in Table 2.2.

x = \$6, y_1 = \$30, y_2 = \$40, y_3 = \$60, y_4 = \$50, y_5 = \$40, y_6 = \$30, C_f = \$5,000, C_a = \$1,000, z = \$5,000, P_r = 100 units per hour, $K = 0.025$, $\bar{\eta}$ = 1,000 units and l = 500 units.

Table 2.2: The numeric values of the optimal diagnosis interval.

Taguchi (Equation 2.4)	Proposed Method I (Equation 2.19)	Taguchi (Equation 2.6)	Proposed Method II (Equation 2.27)
$h^\star = \sqrt{\frac{2\tilde{y}\bar{\eta}}{x}}$	$h^\star = \sqrt{\frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}}{(1+2KP_r)x}}$	$h^\star = \sqrt{\frac{2(\tilde{y}+l)\bar{\eta}}{x-\frac{z}{\bar{\eta}}}}$	$h^\star = \sqrt{\frac{2\left(\sum_{i=1}^{6} y_i + C_a\right)\bar{\eta}}{\left((1+2KP_r)x - \left(\frac{z+C_f}{\bar{\eta}}\right)\right)}}$
289	264	1225	310

It can be concluded from Table 2.2 that $h^\star$ is less for both Method I and Method II compared to corresponding Taguchi's approach. For Method II, it is substantially less, possibly due to taking into account P_r in lieu of l. Therefore, the process abnormality, if any, can be diagnosed more quickly by adopting the prescribed formulae.

Thus, L for the above parameters using Method I is \$19.50 and L using Method II is \$17.52. As L obtained by using Method II is \$1.98 less than L obtained by Method I, it would be appropriate to use $h^\star$ as 310 based on Method II.

Here it would be appropriate to say that we can not compare L obtained by our approaches with L obtained by Taguchi's approaches since we have introduced some new cost parameters while constructing L like C_f and C_a which were not considered by Taguchi et al. [89]

2.4.2 Numerical Example Demonstrating the Results of the Process With Fraction of Defective Items

The numerical values given below have been considered as input variables.
$x = \$6$, $x_d = \$10$, $y_1 = \$30$, $y_2 = \$40$, $y_3 = \$60$, $y_4 = \$50$, $y_5 = \$40$, $y_6 = \$30$, $C_f = \$5{,}000$, $z = \$5{,}000$, $P_r = 100$ units per hour, $K = 0.025$, $\bar{\eta} = 1{,}000$ units and $\pi = 0.05$.

Using equation (2.36), we obtain $h^\star$ as 217 and the corresponding value of L is \$12.19, which can be obtained using equation (2.33).

If the fraction of defective items is changed from $\pi = 0.05$ to $\pi = 0.20$, $h^\star$ would be 200 and the corresponding L would be \$12.39.

However, for $\pi = 0.30$, $h^\star$ turns out to be 189 and L turns out to be \$12.53.

We can see that as the generation of the fraction of defective items increases, $h^\star$ decreases. Therefore, one can detect faster and in a shorter time span if the production worsens with the generation of more defective items.

2.4.3 Comparison of the Proposed Cost Model with Other Newer Cost Models Vis-a-Vis Taguchi's Original Cost Model

It is worthwhile to note the distinction between other new cost models proposed by authors other than Taguchi and the cost models proposed by us. The distinctions are as follows.

(1) The other cost models have considered a particular amount of shift in the process. However, our proposed method has considered whether a defective item has been generated or not due to the process shift.

(2) The other cost models have not taken into account the parameters like y_1, y_2, y_3, y_4, y_5, y_6, C_a, P_r and in some cases even C_f. Notwithstanding this difference in the cost models, a comparison has been made between these newer cost models and our proposed cost model with reference to Taguchi's original cost model.

The values of $h^\star$ and L are provided in Table 2.3 to facilitate the comparison between different cost models.

The numerical example presented here is adapted from Taguchi et al. [89]. The input variables considered by Ho et al. [37] are as follows: x = \$20, y_1 = \$0.04, y_2 = \$0.05, y_3 = \$0.03, y_4 = \$0.05, y_5 = \$0.05, y_6 = \$0.03, C_f = \$0.80, C_a = \$6, z = \$900, $l = 3$ units, $P_r = 10$ units per hour, $K = 0.025$ and $\bar{\eta} = 1{,}500$.

Table 2.3 furnishes the value of $h^\star$ as well as L of the proposed model against Ho et al. [37] and Ho and Da Costa Quinino [36].

Table 2.3: The Numeric values of optimal diagnosis interval and the corresponding total cost per unit for different cost models.

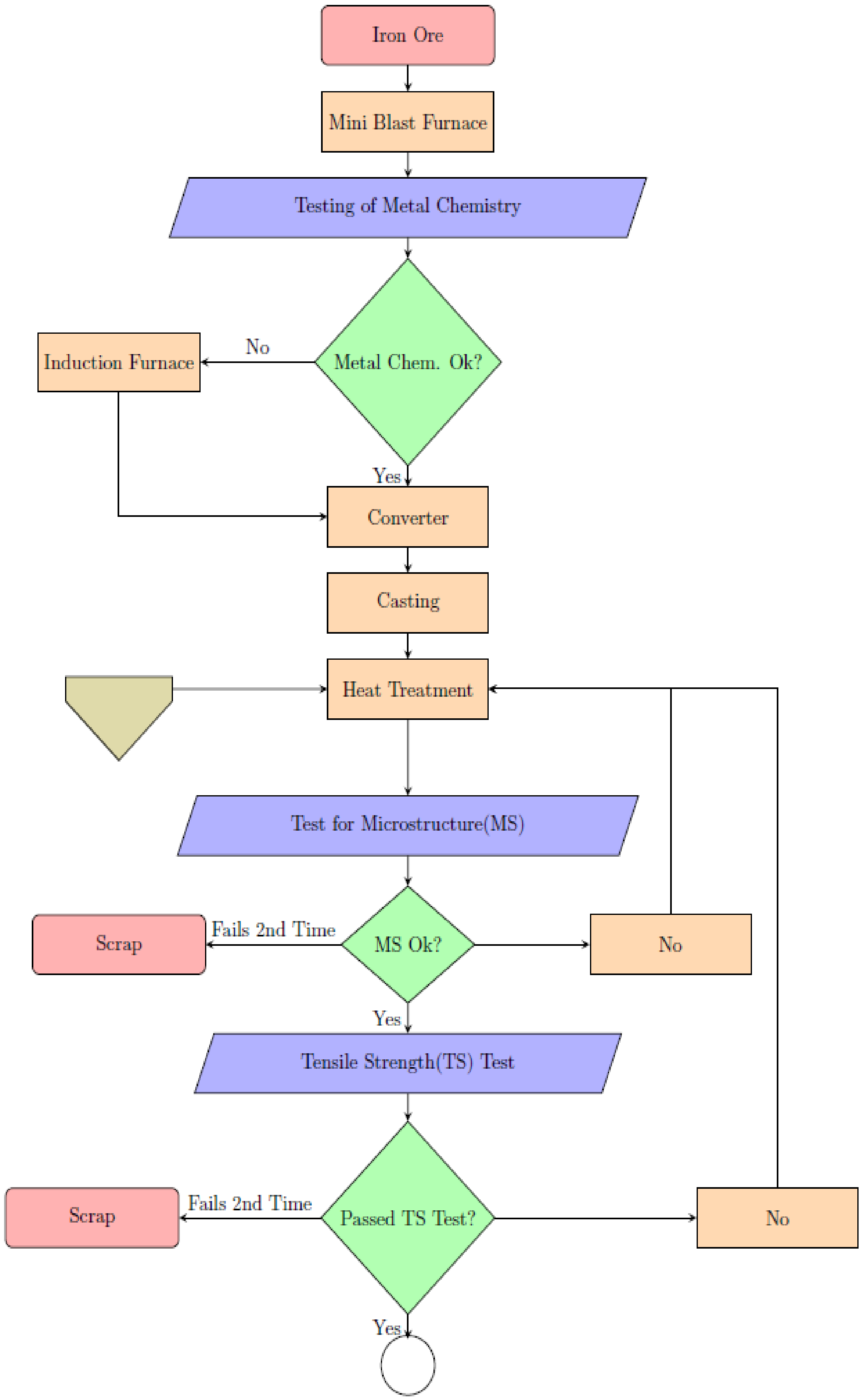
Method	Optimal diagnosis interval ($h^\star$)	Total cost per unit (L)
Proposed	25	\$1.1505
Ho et al. [37]	32	\$1.445
Ho and Da Costa Quinino [36]	40	\$1.2877

Note - The proposed model is 20.38% cheaper than the model presented by Ho et al. [37] and 10.65 % cheaper than the model presented by Ho and Da Costa Quinino [36].

2.4.4 A Numerical Example Based on Real Data

A reputed Indian company produces iron pipes of size 500mm in diameter with an extra feature of ductility for the transportation of drinking water. The corresponding high-level process map is given in Figure 2.1. The following explanations will pave the way for a clearer comprehension of the high-level process map.

The pipes are checked for several quality characteristics like Tensile Strength, Microstructure, Inner Diameter, Outer Diameter, and Hydro Leakage. The hot metal prepared with the help of a mini blast furnace is transferred to the converter for magnesium treatment to incorporate the ductile behavior through the formation of nodules instead of flakes in the microstructure. Thereafter, the converted hot metal is cast in the horizontal centrifugal casting machine for the formation of pipes followed by heat treatment and finishing operations before dispatching the pipes to the customers. As


Iron Ore
Mini Blast Furnace
Testing of Metal Chemistry
Metal Chem. Ok?
No
Induction Furnace
Yes
Converter
Casting
Heat Treatment
Test for Microstructure(MS)
MS Ok?
Fails 2nd Time
Scrap
No
Yes
Tensile Strength(TS) Test
Passed TS Test?
Fails 2nd Time
Scrap
No
Yes

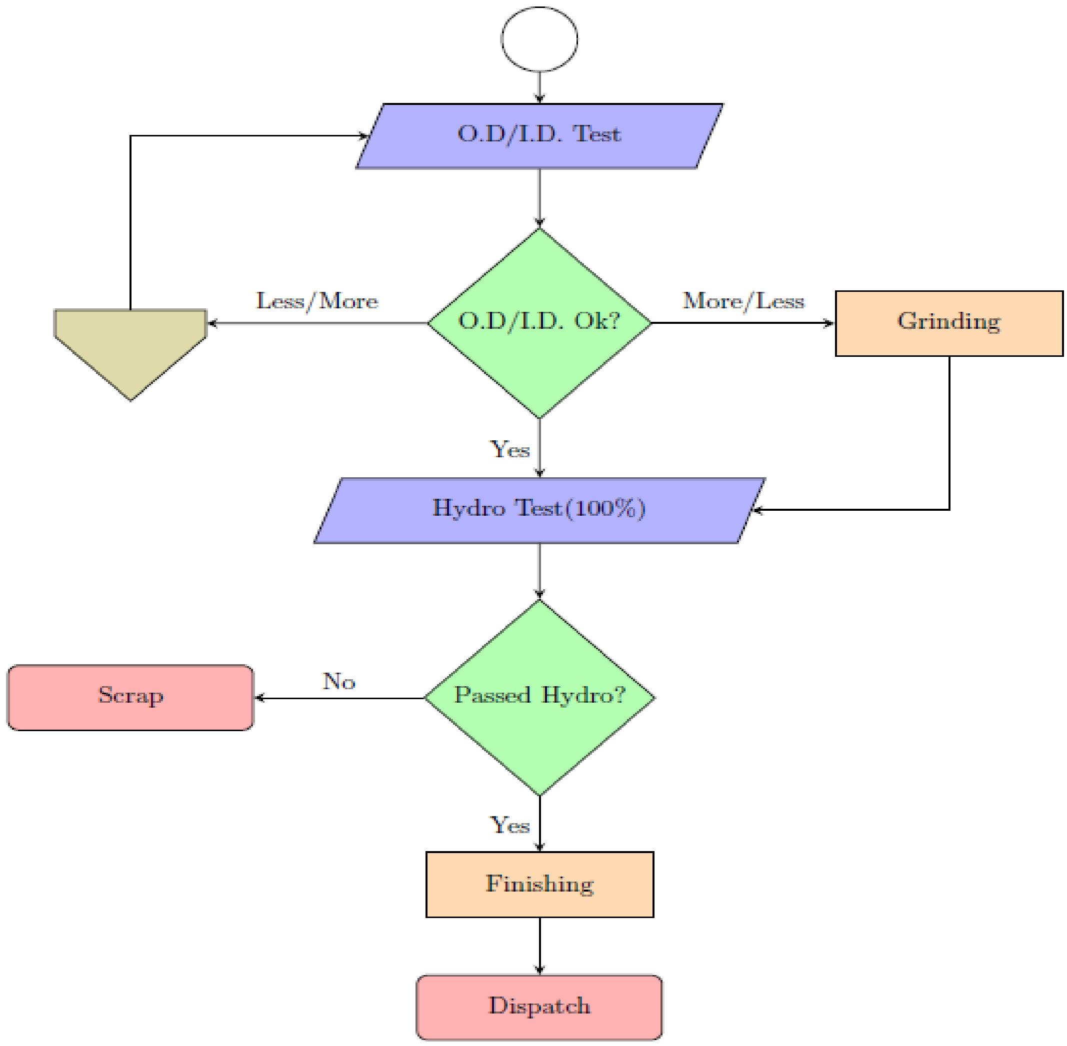


Figure 2.1: The high-level process map depicting iron pipe production.

far as the quality characteristic Tensile Strength is concerned, it is a crucially important characteristic for the transportation of drinking water through ductile iron pipes and it is tested for the pipes after heat treatment operation by a machine named as Universal Testing Machine (UTM). UTM measures percentage elongation along with Tensile Strength. y_3 is the cost component that is incurred for testing by UTM. y_3 has been estimated as Rs 20 per sample. In order to take the sample, a small section is extracted from the spigot end of the pipe. The corresponding cost of taking a sample is y_1. The pertinent cost component y_1 has been estimated to be Rs 20 per sample. After extracting a small portion from the spigot end of the pipe, it is prepared for testing by reshaping it into a dumbbell-like configuration. The corresponding cost for re-shaping the collected sample is y_2 and it has been estimated to be Rs 40 per sample. y_4 is the cost of electrical energy that is consumed for cutting the sample from the spigot end of the pipe, re-shaping the sample, and testing the sample. y_4 has been estimated to be Rs 4 per sample. y_6 is the cost component associated with feeding the tensile data in the computer and communicating it appropriately to the pertinent process owners in processes like heat treatment, converter, casting, mini blast furnace, etc. y_6 has been estimated to be Rs 10 per sample. The cost component C_f over $\bar{\eta}$ with regard to Tensile Strength arises when there is a failure on the basis of the first sample. After corrective action, the second sample is always found to be conforming. The estimate of C_f to the extent of Rs 1900 over $\bar{\eta}$ is based on a failure rate of 10% on the first sample. Considering rejection to the extent of 2% after dispatching to the customers' end, the cost of producing the rejected pipes along with the corresponding shipment cost is estimated to be Rs 1800 for the cost component C_a over h. The cost component z over $\bar{\eta}$ includes not only the cost of adjustment of machines used for taking the sample, preparing the sample, and testing the sample but also the cost of fine-tuning the oil chambers and gas chambers at heat treatment. It is estimated to be Rs 42000.

The following data captured by us are of fundamental importance and can be seen at a glance for convenience. x = Rs 400, y_1 = Rs 20, y_2 = Rs 40, y_3 = Rs 20, y_4 = Rs 4, y_5 = Rs 0, y_6 = Rs 10, C_f = Rs 1,900, C_a = Rs 1,800, z = Rs 42,000, $K = 0.020$, P_r = 55 units per hour and $\bar{\eta}$ = 19,800 units.

Using equation (2.27), the obtained values of $h^\star$ and L respectively are 242 and Rs 17.88. The current practice of the Indian company is to test every 100th ductile iron pipe for Tensile Strength. L using $h^\star$ as 100 is Rs 26.90, which is 50.45% more than L when $h^\star$ is 242, which emerged from this study of ours. If one follows the approach given by Taguchi et al. [89], the obtained value of $h^\star$ will be 107 items and L will be Rs 4.5.

2.5 Sensitivity Analysis

The following observations help us understand the major influence of input parameters on values of $h^\star$ and L.

(1) Parameter x: Runs 1 and 2 consist of changing the level of x from Rs 400 to Rs 800 keeping the levels of other factors like $\sum_{i=1}^{6} y_i$, C_f, C_a, z, P_r, and $\bar{\eta}$ constant to observe the effect of x on $h^\star$ and L. It is evident from Table 2.4 that an increase in x to the tune of Rs 400 results in a decrease in $h^\star$ from 242 items to 171 items and the corresponding increase in L from Rs 17.88 to Rs 24.37 per item.

(2) Parameter C_a: From Table 2.4 one can understand the influence of C_a on $h^\star$ and L when other factors remain constant through comparison of runs 1 and 5. It demonstrates clearly that when q increases from Rs 1800 to Rs 3600, $h^\star$ increases from 242 items to 338 items, and L increases from Rs 17.88 to Rs 24.08 per item.

(3) Parameter P_r: Comparison between run 1 and run 7 helps us to see the effect of P_r on $h^\star$ and L when the level of P_r changes from 55 to 110 items per hour. It can be observed from Table 2.4 that $h^\star$ decreases from 242 to 186 items and the corresponding loss L increases from Rs 17.88 to Rs 22.56 per item.

(4) Parameter $\bar{\eta}$: Comparison between runs 1 and 8 helps one understand the effect of $\bar{\eta}$ on $h^\star$ and L. Table 2.4 reveals that change in $\bar{\eta}$ from 19800 to 39600 results in an increase in $h^\star$ from 242 to 342 items and a decrease in L from Rs 17.88 to Rs 12.18 per item.

The following observations help us understand the minor influence of input parameters on values of $h^\star$ and L.

(1) Sum of Parameters y_1, y_2, y_3, y_4, y_5, y_6 or $\sum_{i=1}^{6} y_i$: Comparison between runs 1 and 3 reveals that change in $\sum_{i=1}^{6} y_i$ from Rs 94 to Rs 188 per item results in a change in $h^\star$ from 242 to 248 items and the L changes from Rs 17.88 to Rs 18.26 when other factors remain constant.

(2) Parameter C_f: Runs 1 and 4 facilitate us to compare the change in C_f from Rs 1900 to Rs 3800 when other factors remain constant. It can be seen from Table 2.4 that $h^\star$ remains the same at 242 items. However, L increases slightly from Rs 17.88 to Rs 17.97 per item.

(3) Parameter z: Table 2.4 clearly demonstrates that when z increases from Rs 42000 to Rs 84000 (vide run 1 and run 6) and other factors remain constant, there is no change in $h^\star$. However, L changes slightly from Rs 17.88 to Rs 20.00 per item.

Table 2.4: Effect of one-factor-at-a-time changes in input parameters on $h^\star$ and L

Run	x	$\sum_{i=1}^{6} y_i$	C_f	C_a	z	P_r	$\bar{\eta}$	$h^\star$	L
1	400	94	1900	1800	42000	55	19800	242	17.88
2	800	94	1900	1800	42000	55	19800	171	24.37
3	400	188	1900	1800	42000	55	19800	248	18.26
4	400	94	3800	1800	42000	55	19800	242	17.97
5	400	94	1900	3600	42000	55	19800	338	24.08
6	400	94	1900	1800	84000	55	19800	242	20.00
7	400	94	1900	1800	42000	110	19800	186	22.56
8	400	94	1900	1800	42000	55	39600	342	12.18

It emerges from Tables 2.4, 2.5, and 2.6 that change in all the input parameters except C_f from lower to higher levels influence $h^\star$ as per equations (2.19), (2.27), and (2.36). For the one-factor-at-a-time change in C_f from low level to high level has no effect on $h^\star$ (both resulting in 242 items). However, for simultaneous change in the levels of multiple factors, $h^\star$ reduces from 272 items to 260 items. It might have been caused on account of the interaction between P_r and C_f or x and C_f.

Table 2.5: Effect of simultaneous changes in input parameters on $h^\star$ and L

Run	x	$\sum_{i=1}^{6} y_i$	C_f	C_a	z	P_r	$\bar{\eta}$	$h^\star$	L
1	400	94	1900	1800	42000	55	19800	242	17.88
2	400	94	1900	3600	84000	110	39600	368	22.25
3	400	188	3800	1800	42000	110	39600	270	15.89
4	400	188	3800	3600	84000	55	19800	343	26.58
5	800	94	3800	1800	84000	55	39600	242	17.88
6	800	94	3800	3600	42000	110	19800	184	42.48
7	800	188	1900	1800	84000	110	19800	135	33.81
8	800	188	1900	3600	42000	55	39600	342	23.25

Table 2.6: Summary of changes in $h^\star$ and L triggered by simultaneous changes in input parameters

Input Parameters	Levels of Input Parameters	$h^\star$	L
x	1	306	20.65
	2	226	29.36
$\sum_{i=1}^{6} y_i$	1	259	25.12
	2	273	24.88
C_f	1	272	24.30
	2	260	25.71
C_a	1	222	21.37
	2	309	28.46
z	1	260	24.88
	2	272	25.13
P_r	1	292	21.40
	2	239	28.61
$\bar{\eta}$	1	226	30.19
	2	306	19.82

* Here 1 and 2 denote the low and high levels of the pertinent input parameters.

It emerges conspicuously from the sensitivity analysis that the parameters x and P_r within their respective domains have a strong influence on $h^\star$. To be more precise, increases in the values of x as well as P_r result in a decrease in $h^\star$ convincingly. By and large, the increase in the values of other parameters within their respective domains influences $h^\star$ on the higher side.

2.6 Conclusion

It is to be noted that $h^\star$ may vary from one situation to another situation depending upon the numerical estimates of the input parameters in the prescribed equations (2.19), (2.27) and (2.36).

For example, if $\sum_{i=1}^{6} y_i$, which lies in the numerator of the formulae, is high then $h^\star$ will also turn out to be large.

On the contrary, if P_r, which lies in the denominator of the formulae, is high then $h^\star$ will turn out to be small.

Thus, $h^\star$ is directly proportional to the input parameters of the numerator for the equations (2.19), (2.27) and (2.36), as appropriate. These input parameters are $\sum_{i=1}^{6} y_i$, C_a, $\bar{\eta}$, z, and C_f.

It is also to be noted that $h^\star$ is inversely proportional to the parameters x and P_r for the equations (2.19), (2.27) and (2.36), as appropriate.

It is worth noting from the comparative analysis given in Table 2.2 of the proposed methodology with Taguchi's approach that $h^\star$ is less in the proposed methodology. Particularly, when x is in the closed proximity of $z/\bar{\eta}$, the proposed value of $h^\star$ is significantly less (310) than Taguchi's approach (1225). However, when x is much higher than $z/\bar{\eta}$, the proposed value of $h^\star$ is marginally less (264) than Taguchi's approach (289). This implies that the cost component C_a would be less. Consequently, customer satisfaction would be on the higher side. However, $\sum_{i=1}^{6} y_i$ is likely to increase and it would be primarily influenced by y_4 and y_5. The other components of the diagnosis cost in the form of y_1, y_2, y_3, and y_6 would hardly increase on account of the fact that in an organization these costs are primarily dependent on the salaries and wages of the pertinent workforce working essentially in the laboratories. Salaries and wages of the workforce for the pertinent workers are normally decided on a monthly basis and have no correlation with the frequency of taking samples. From the customer-centric point of view, particularly with the advent of JIT and/or lean management in the short-run production processes, the proposed methodology would be quite advantageous.

Chapter 3

Evolving Parameters of Shewhart's $\bar{X}$ Control Chart From Present-day Industrial Engineering Perspective[1]

3.1 Introduction

As discussed in Chapter 1, control charts were invented essentially to differentiate between the assignable or the special causes of variation and the chance or the common causes of variation. The motive was that if there is an assignable cause in the process, one should be able to detect it and then eliminate it. While we try to find these assignable causes, there are some costs associated with the endeavor like the cost of sampling, the cost of testing, the cost for removal of the assignable cause, the cost of generating a defective item, etc. Due to the association of these costs, it is imperative to consider the economic consequences of the design of a control chart.

The economic design of $\bar{X}$ control chart has been explored immensely after its introduction by Duncan [25]. Some remarks were given by Chiu [18] on the economical design of the $\bar{X}$-chart provided by Duncan [25]. Both Montgomery [58] and Vance [93] have provided an excellent list of works done in the area of the economic design of control charts. Von Collani [95] used a simple procedure where he assumed that the production speed is constant to determine the optimal design of $\bar{X}$-chart. Saniga [78] developed a method to determine the economical statistical design for Shewhart-type control charts and applied that method to determine the parameters of $\bar{X}$ and R charts.

[1] Evolving Parameters of Shewhart's $\bar{X}$ Control Chart From Present-day Industrial Engineering Perspective. *Communications in Statistics - Theory and Methods*, **54**(**16**), 5257-5283, 2025.

A literature review for the economic design of control charts was also given by Ho and Case [35]. The economic design of the $\bar{X}$-chart using Taguchi's loss function was studied by Ben-Daya and Duffuaa [10]. Economic design of $\bar{X}$ control charts with multiple assignable causes under the Burr XII shock model was given by Saadatmelli et al. [72]. Safaei et al. [73] provided the multi-objective economic statistical design of the $\bar{X}$-chart considering Taguchi's loss function. Amiri et al. [5] and Abolmohammadi et al. [1] developed the economical design of the $\bar{X}$-chart using Taguchi's loss function for the variable parameters.

In this Chapter, a modified model has been proposed for minimizing the loss function, i.e., $\mathcal{L}$. In our model, we have introduced some additional parameters considering the aspect of granularity for estimating the diagnosis cost of a product in a pragmatic manner through the sampling cost per unit (y_1), sample preparation cost per unit (y_2), cost per unit of testing an item of the product (y_3), the energy consumption cost per unit (y_4), the consumables cost, if any, per unit (y_5), the reporting cost per unit (y_6), and also the average income difference per hour per piece of a product between an in-control and out-of-control process (V) along with the rate of production (P_r). The additional cost parameters have been considered over and above the cost of finding an assignable cause, the loss incurred due to false alarm, hourly cost for exercising control over a process through employing a control chart, average income difference per unit of a product between an in-control and out-of-control process based on the available literature although the concept of cost per unit of a product was not considered by Duncan [25] in his original work. These cost components have been taken into account for obtaining the optimal values of the sample size (n), the sampling interval (h), and the control limits' multiplier (k). Note that the optimal parameters of an $\bar{X}$-chart have been determined using two different approaches to minimize $\mathcal{L}$ with appropriate cost components from the perspective of industry-friendly implementation. In the first methodology, we have differentiated $\mathcal{L}$ with respect to n, as done by Duncan [25] although n is an integer. Consequently, for mathematical soundness, a second method has been envisaged by us for minimizing $\mathcal{L}$ by finding partial derivatives of $\mathcal{L}$ with respect to h and k only unlike Duncan [25]. In this second method, we have provided an iterative algorithmic approach for the determination of the optimal parameters keeping in view, from the present-day perspective, the parameters y_1, y_2,..., y_6, and P_r, which were not taken into account by Duncan [25], the originator of the economic design of the control chart. The algorithms and the pertinent mathematica codes for both approaches are also given for determining the optimal values of n, h, and k. It has amply been demonstrated with the help of numerical examples for both the approaches that P_r and V play a major role in determining the optimal value of h. There is an appreciable difference in the optimal values of h and $\mathcal{L}$ thus obtained between Duncan's methodology and the proposed methodology, which has resulted in significantly less loss.

The rest of the Chapter is organized like this. Section 3.2 presents the construction of the economic design model for the $\bar{X}$-chart. Section 3.3 contains a brief discussion of our contribution. Section 3.4 explains how one can find the optimal parameters of the $\bar{X}$-chart adopting two different approaches. Section 3.5 demonstrates the algorithm for finding the optimal parameters and the corresponding mathematica codes are given in Section 3.6. Section 3.7 contains numerical examples along with the sensitivity analysis from related literature. Section 3.8 is dedicated to a case example based on real-life data. Finally, Section 3.9 provides the concluding remarks. The content of this Chapter can be found in Sandeep and Ranjan Mukhopadhyay [76].

3.2 The Economic Design of $\bar{X}$-chart

The first economic model for the design of an $\bar{X}$-chart was proposed by Duncan [25] and he provided the methodology to find the parameters of the control chart as mentioned earlier. By taking the work of Girshick and Rubin [31] into consideration, a design was proposed by Duncan [25] for maximizing the net income occurred per unit of time (hour), i.e., I. It is assumed that initially, the process is in-control with mean μ_0 and process standard deviation σ_0 (where σ_0 is assumed to remain the same throughout the process). However, due to the presence of a single assignable cause, there is a shift of δ amount in the process centering. This assignable cause occurs randomly, due to which mean shifts on either side to $\mu_0 - \delta\sigma_0$ or to $\mu_0 + \delta\sigma_0$. The process is monitored using an $\bar{X}$-chart having a central line at μ_0 and the control limits at $\mu_0 \pm k(\sigma_0/\sqrt{n})$. Samples are taken after every h hours. Whenever the search takes place for the presence of an assignable cause, the process is allowed to operate. It is also assumed that the parameters μ_0, δ, and σ_0 of a stable process are known and the parameters n, h, and k are to be determined.

The time required for an assignable cause to occur can reasonably be assumed to be exponentially distributed with parameter λ per unit of operating time (hour). Thus, considering that an assignable cause has occurred in the interval between r^{th} and $(r+1)^{th}$ sample, the average time for the occurrence of an assignable cause is

$$\begin{aligned}
& \frac{\int_{rh}^{(r+1)h} e^{-\lambda t}\lambda(t-rh)dt}{\int_{rh}^{(r+1)h} e^{-\lambda t}\lambda dt} \\
\Rightarrow\ = \ & \frac{e^{-\lambda rh}\int_0^h e^{-\lambda T}\lambda T dT}{e^{-\lambda rh}\int_0^h e^{-\lambda T}\lambda dT} \\
\Rightarrow\ = \ & \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})} \\
\Rightarrow\ = \ & \frac{h}{2}-\frac{\lambda h^2}{12} + \text{terms of order } \lambda^3 h^4 \text{ or higher} \\
\Rightarrow\ \approx \ & \frac{h}{2}-\frac{\lambda h^2}{12} \quad \text{(ignoring the higher-order terms).} \qquad (3.1)
\end{aligned}$$

Therefore, the average time taken by an assignable cause to occur is $(h/2-\lambda h^2/12)$. This result has been used to work out the formula for computing I.

Recall that P is the probability of detecting an assignable cause when an assignable cause has already occurred and Q is the probability of an assignable cause remaining undetected. In other words, we can say that the probability of a sample point falling outside the control limits is P and the probability of a point falling inside the control limits is Q when the process mean has already shifted by an amount of δ. Thus P is given by

$$P=\int_{-\infty}^{-k-\delta\sqrt{n}}\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz+\int_{k-\delta\sqrt{n}}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz\ . \qquad (3.2)$$

If δ is strictly positive, i.e., $\delta>0$ in the above expression for P in equation (3.2), then the first component of the equation would turn out to be virtually zero since it is the area under the standard normal curve from $-\infty$ to $-k-\delta\sqrt{n}$. For $\delta<0$, the second component of P would turn out to be virtually zero since it is the area under the standard normal curve from $k-\delta\sqrt{n}$ to ∞. The probability that a sample point will fall outside the control limit when the process is in-control is given by

$$\alpha = 2\int_{k}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz \ . \tag{3.3}$$

To design the economic model of a control chart, a certain number of assumptions are required about the production process behavior. Here, we are assuming that during the search for an assignable cause, the process is allowed to operate in an unhindered manner. The cost of repairing or adjustment is not taken into consideration against the net income for determining the parameters of $\bar{X}$-chart. From the start of production (initially the process can reasonably be assumed to be in-control state) to the detection and the elimination of an assignable cause is known as a production cycle. The production cycle has four states and these are (a) In-control state (b) Time required to take the sample and illustrate the results (c) Out-of-control state and (d) Time required to find an assignable cause. By considering these states we can determine the average net income per hour by adopting the following steps:

(1) The average time a process will be in-control or the average time taken by an assignable cause to occur is $1/\lambda$.

(2) The probability of detecting an assignable cause at the i^{th} sample given that it had already occurred is $Q^{i-1}P$. Therefore, the average number of samples required to detect the shift would be $1/P$ because it is the expected value of the underlying geometric distribution. Since h is the sampling interval, the average duration of the out-of-control state is $h/P - (h/2 - \lambda h^2/12) = (1/P - 1/2 + \lambda h/12)h$ since from equation (3.1) it can be found that the average time for the occurrence of an assignable cause is $(h/2 - \lambda h^2/12)$.

(3) As already mentioned, the time required to take a sample and obtain the results is t and is directly proportional to n. Consequently, the total time required on this behalf is tn.

(4) Recall that T_1 is the average time required to discover an assignable cause. Therefore, the proportion of time for which a process would be in-control state is

$$\rho = \frac{1/\lambda}{1/\lambda + gh + tn + T_1} = \frac{1}{1 + \lambda D} \ . \tag{3.4}$$

where $g = (1/P - 1/2 + \lambda h/12)$ and $D = (gh + tn + T_1)$. Consequently, the proportion of time for which the process would be out-of-control state is

$$\rho_1 = 1 - \rho = 1 - \frac{1}{1 + \lambda D} = \frac{\lambda D}{1 + \lambda D} \ . \tag{3.5}$$

From a practical standpoint, one may choose a suitable value of λ (the average rate of occurrence of an assignable cause per unit of the operating time) to examine its influence on ρ (proportion of time for which the process will be in-control). Figure 3.1 depicts the values of ρ against λ corresponding to the optimal values of $n = 4$, $h = 0.5835$ hours and $k = 2.87$ obtained for $\lambda = 0.01$ per hour. It is important to note here that from Figure 3.1 one can also obtain the value λ for a given value of ρ. For example, corresponding to $\rho = 0.8$, the λ from Figure 3.1 can be obtained as 0.09578 per hour. This can simply be found by drawing a line parallel to the abscissa, say, from $\rho = 0.8$ on the ordinate. From the point of intersection of the parallel line with the graph, a perpendicular is drawn on the abscissa to arrive at the value of λ as 0.09578 per hour.

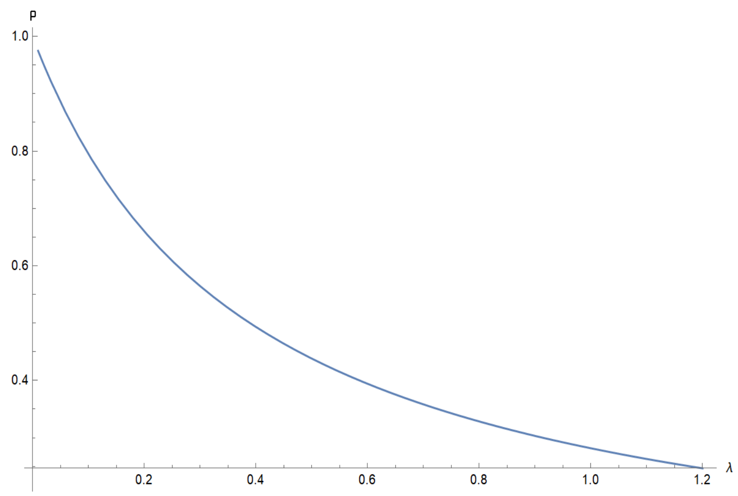


Figure 3.1: Graph of ρ as a function of λ.

(5) If the process mean gets shifted from μ_0 to $\mu_0 \pm \delta\sigma_0$, nonconforming items will discernably be generated. Recall that the average income per hour per piece of a product is I_0 when a process is in-control state and the average income per hour per piece of a product is I_1 when the process mean gets shifted anywhere between $\mu_0 - \delta\sigma_0$ and $\mu_0 + \delta\sigma_0$. Also, $V = I_0 - I_1$ is the average income difference per piece per hour of

a product between in-control and out-of-control state. It is assumed that initially the process is centered between the control limits so that V remains the same for $-\delta$ as well as $+\delta$ amount of shift.

(6) The expected number of false alarms before a process actually goes out-of-control would be the average number of samples taken in that period times α.

$$\begin{aligned}
\text{Expected number of false alarms} &= \alpha \sum_{r=0}^{\infty} \int_{rh}^{(r+1)h} r\lambda e^{-\lambda t} dt \\
\Rightarrow \text{Expected number of false alarms} &= \alpha \sum_{r=0}^{\infty} r \left(e^{-rh\lambda} - e^{-(r+1)h\lambda} \right) \\
\Rightarrow \text{Expected number of false alarms} &= \alpha \left(1 - e^{-\lambda h} \right) \sum_{r=0}^{\infty} r e^{-rh\lambda} \\
\Rightarrow \text{Expected number of false alarms} &= -\alpha \left(1 - e^{-\lambda h} \right) \frac{\partial}{\partial \lambda} \frac{1}{h} \sum_{r=0}^{\infty} e^{-rh\lambda} \\
\Rightarrow \text{Expected number of false alarms} &= \frac{\alpha e^{-\lambda h}}{1 - e^{-\lambda h}} \ . \qquad (3.6)
\end{aligned}$$

Neglecting the terms of order $\lambda^2 h^2$ or higher, equation (3.6) would turn out to be approximately $\alpha/\lambda h$. Therefore, the expected number of false alarms per hour would be

$$\frac{\alpha/\lambda h}{1/\lambda + D} = \frac{\rho\alpha}{h} \qquad \text{[substituting the value of } \rho \text{ from (3.4)].} \qquad (3.7)$$

As already mentioned that C_2 is the cost for finding an assignable cause when there is none, then the expected loss per hour due to the false alarms will be $\rho\alpha C_2/h$.

(7) Since the average length of the cycle is

$$1/\lambda + D = \frac{1 + \lambda D}{\lambda} \ . \qquad (3.8)$$

Thus, the average number of times per hour a process remains out-of-control would be

$$\upsilon = \frac{1}{1/\lambda + D} = \frac{\lambda}{1 + \lambda D} \ . \qquad (3.9)$$

Recall that the cost of correctly finding an assignable cause is C_3. Then, the corresponding average cost per hour on this behalf turns out to be υC_3 for correctly discovering an assignable cause.

(8) The hourly cost for exercising control over a process through employing the technique of control chart is given by a linear function

$$\frac{d}{h} + \frac{n\sum_{i=1}^{6} y_i}{h} \quad . \tag{3.10}$$

where d is the cost per sample for maintaining the process through $\bar{X}$-chart that is independent of n. The cost components y_1, y_2,..., y_5, and y_6 have a direct bearing on n.

Hence, the net income (I) per hour can be computed as

$$I = \rho P_r I_0 + \rho_1 P_r I_1 - \frac{\rho\alpha C_2}{h} - \upsilon C_3 - \frac{d}{h} - \frac{n\sum_{i=1}^{6} y_i}{h} \quad . \tag{3.11}$$

Using $\rho + \rho_1 = 1$ from equations (3.4 and 3.5) and $I_1 = I_0 - V$ from step (5), equation (3.11) turns out to be

$$I = P_r I_0 - \rho_1 P_r V - \frac{\rho\alpha C_2}{h} - \upsilon C_3 - \frac{d}{h} - \frac{n\sum_{i=1}^{6} y_i}{h} \quad . \tag{3.12}$$

Substituting the expression of ρ from equation (3.4), ρ_1 from equation (3.5) and υ from equation (3.9), in equation (3.12), we obtain

$$\begin{aligned} I &= P_r I_0 - \frac{\lambda D P_r V}{1+\lambda D} - \frac{\alpha p/h}{1+\lambda D} - \frac{\lambda q}{1+\lambda D} - \frac{d}{h} - \frac{n\sum_{i=1}^{6} y_i}{h} \\ \Rightarrow I &= P_r I_0 - \frac{P_r \lambda V D + \alpha C_2/h + \lambda C_3}{1+\lambda D} - \frac{d}{h} - \frac{n\sum_{i=1}^{6} y_i}{h} \quad . \end{aligned} \tag{3.13}$$

Thus, modified $\mathcal{L}$ is

$$\mathcal{L} = \frac{P_r \lambda V D + \alpha C_2/h + \lambda C_3}{1 + \lambda D} + \frac{d}{h} + \frac{n \sum_{i=1}^{6} y_i}{h} \,. \tag{3.14}$$

We can maximize I by minimizing $\mathcal{L}$. Henceforth, the focus will be given to minimizing $\mathcal{L}$ for determining the optimal values of n, h, and k.

3.3 Contribution to Knowledge

In this Chapter, we have considered some parameters which have not been given due consideration by Duncan [25] and other authors. These parameters are y_1, y_2, y_3, y_4, y_5, y_6, P_r, I_0, I_1, and V. The novelty of the proposed methodology lies in adopting a combination of partial derivative and iterative algorithmic approach with a suitable stopping rule to find out n, an integer, and h and k, the continuous variables, by minimizing the modified $\mathcal{L}$ considered from the present-day perspective of manufacturing engineering. Note that Duncan [25] in his pioneering work adopted the partial derivative approach only while finding optimum n, h, and k although n is an integer.

A comparison has been made between the two approaches through numerical examples to delineate the effectiveness of the proposed approach in terms of loss minimization besides its mathematical sanctity.

3.4 Determination of the Optimal Parameters

In this Section, the optimal parameters of $\bar{X}$-chart are determined using two different approaches, namely, the original approach and the alternate approach. In the original approach, we have partially differentiated $\mathcal{L}$ with respect to n, h, and k, as done by Duncan [25] although n is an integer. Consequently, for mathematical soundness, an alternate approach has been envisaged by us for minimizing $\mathcal{L}$ by finding partial derivatives of $\mathcal{L}$ with respect to h and k only unlike Duncan [25].

3.4.1 Original Approach to Determine the Optimal Parameters

For determining the optimal values of n, h, and k, we need to partially differentiate $\mathcal{L}$ with respect to n, h, and k. Thereafter, we need to equate these partial derivatives with zero. Since the underlying pre-condition for the differentiability of a variable is its existence as a continuous variable, n, which by default is an integer in nature, is assumed as a continuous variable for mathematical convenience. Subsequently, after finding the optimal value of n, it is appropriately rounded off to the nearest integer value, and the corresponding values of h and k are found by plugging in the rounded integer value of n in the appropriate equations for h and k. Further, it is to be noted that P is a function of n & k, and α is a function of k only. Thus, we obtain the principal equations (3.15), (3.16) and (3.17) along with the corresponding supplementary equations (3.18), (3.19) and (3.20) hereunder.

$$\lambda h \frac{\partial D}{\partial n}\left(P_r V - \frac{\alpha C_2}{h} - \lambda C_3\right) + \sum_{i=1}^{6} y_i \left(1+\lambda D\right)^2 = 0 \; . \tag{3.15}$$

$$\lambda h^2 \frac{\partial D}{\partial h}\left(P_r V - \frac{\alpha C_2}{h} - \lambda C_3\right) - \alpha C_2 \left(1+\lambda D\right) - \left(d + n\sum_{i=1}^{6} y_i\right)\left(1+\lambda D\right)^2 = 0 \; . \tag{3.16}$$

$$\lambda \frac{\partial D}{\partial k}\left(P_r V - \frac{\alpha C_2}{h} - \lambda C_3\right) + \frac{C_2}{h}\frac{\partial \alpha}{\partial k}\left(1+\lambda D\right) = 0 \; . \tag{3.17}$$

where,

$$\frac{\partial D}{\partial n} = -\frac{h \partial P/\partial n}{P^2} + t \; . \tag{3.18}$$

$$\frac{\partial D}{\partial h} = \left(\frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{6}\right) \; . \tag{3.19}$$

$$\frac{\partial D}{\partial k} = -\frac{h \partial P/\partial k}{P^2} \; . \tag{3.20}$$

There does not exist a simple straight-forward procedure for solving the equations (3.15), (3.16) and (3.17). So, we shall try to find an approximate solution by assuming λ to be small since it essentially denotes the average rate of occurrence of an assignable cause per unit of operating time. It is to be noted that the operating time can be a shift of eight hours or even a day of twenty-four hours. Neglecting all the terms of magnitude smaller than the principal term influenced by λ and substituting equations (3.18), (3.19) and (3.20) respectively in equations (3.15), (3.16), and (3.17), we obtain,

$$-\frac{\lambda h^2 P_r V \partial P/\partial n}{P^2} + \sum_{i=1}^{6} y_i \doteq 0 \ . \tag{3.21}$$

$$\lambda h^2 P_r V \left(\frac{1}{P} - \frac{1}{2}\right) - \alpha C_2 - \left(d + n\sum_{i=1}^{6} y_i\right) \doteq 0 \ . \tag{3.22}$$

$$-\frac{\lambda h^2 P_r V \partial P/\partial k}{P^2} + C_2 \frac{\partial \alpha}{\partial k} \doteq 0 \ . \tag{3.23}$$

Equation (3.22) yields

$$h \doteq \sqrt{\frac{\alpha C_2 + d + n\sum\limits_{i=1}^{6} y_i}{\lambda P_r V(1/P - 1/2)}} \ . \tag{3.24}$$

Plugging in the value of h from equation (3.24) in equation (3.21), we get

$$\begin{aligned}
-\frac{(\alpha C_2 + d + n\sum_{i=1}^{6} y_i)\lambda P_r V \partial P/\partial n}{\lambda P_r V(1/P - 1/2)P^2} + \sum_{i=1}^{6} y_i = 0 \\
\Rightarrow -(\alpha C_2 + d + n\sum_{i=1}^{6} y_i) + \frac{P^2(1/P - 1/2)\sum_{i=1}^{6} y_i}{\partial P/\partial n} = 0 \\
\Rightarrow -n\sum_{i=1}^{6} y_i + \frac{P^2(1/P - 1/2)\sum_{i=1}^{6} y_i}{\partial P/\partial n} = \alpha C_2 + d \\
\Rightarrow -n + \frac{P^2(1/P - 1/2)}{\partial P/\partial n} = \frac{\alpha C_2 + d}{\sum\limits_{i=1}^{6} y_i} \ .
\end{aligned} \tag{3.25}$$

From equation (3.21), we achieve

$$-\frac{\lambda h^2 P_r V}{P^2} = -\frac{\sum_{i=1}^{6} y_i}{\partial P/\partial n} \; . \tag{3.26}$$

Substituting the value of $-\frac{\lambda h^2 P_r V}{P^2}$ in equation (3.23), we obtain

$$-\frac{\partial P/\partial k \sum_{i=1}^{6} y_i}{\partial P/\partial n} + C_2 \frac{\partial \alpha}{\partial k} = 0 \; . \tag{3.27}$$

Taking the partial derivative of P for $\delta > 0$ given in equation (3.2) with respect to n and k, we get

$$\begin{aligned} \frac{\partial P}{\partial n} &= \frac{e^{-(k-\delta\sqrt{n})^2/2}}{\sqrt{2\pi}} \frac{\delta}{2\sqrt{n}} \; . \\ \frac{\partial P}{\partial k} &= -\frac{e^{-(k-\delta\sqrt{n})^2/2}}{\sqrt{2\pi}} \; . \end{aligned} \tag{3.28}$$

Substituting the values of $\frac{\partial P}{\partial n}$ and $\frac{\partial P}{\partial k}$ in equation (3.27), we attain

$$\begin{aligned} -\frac{-\left(\frac{e^{-(k-\delta\sqrt{n})^2/2}}{\sqrt{2\pi}}\right) \sum_{i=1}^{6} y_i}{\frac{e^{-(k-\delta\sqrt{n})^2/2}}{\sqrt{2\pi}} \frac{\delta}{2\sqrt{n}}} + C_2 \frac{\partial \alpha}{\partial k} &= 0 \\ \Rightarrow \frac{\partial \alpha}{\partial k} &= -\frac{2\sqrt{n} \sum_{i=1}^{6} y_i}{\delta C_2} \; . \end{aligned} \tag{3.29}$$

Note that $\delta < 0$ too will yield the same expression for $\frac{\partial \alpha}{\partial k}$ given above.

Taking the partial derivative of α given in equation (3.3) with respect to k, we get

$$\frac{\partial \alpha}{\partial k} = -\frac{2e^{-k^2/2}}{\sqrt{2\pi}} \; . \tag{3.30}$$

Equating the right hand sides of the equations (3.29) and (3.30), we obtain

$$\begin{aligned} -\frac{2e^{-k^2/2}}{\sqrt{2\pi}} &= -\frac{2\sqrt{n} \sum_{i=1}^{6} y_i}{\delta C_2} \\ \Rightarrow e^{-k^2/2} &= \frac{\sqrt{2\pi n} \sum_{i=1}^{6} y_i}{\delta C_2} \; . \end{aligned} \tag{3.31}$$

Taking log on both sides of the equation (3.31), we get

$$
\begin{aligned}
-k^2/2 &= \log\left(\sqrt{2\pi n}\sum_{i=1}^{6} y_i/\delta C_2\right) \\
\Rightarrow k &= \sqrt{-2\log\left(\sqrt{2n\pi}\sum_{i=1}^{6} y_i/\delta C_2\right)}\ . \qquad (3.32)
\end{aligned}
$$

The six costs y_1, y_2,..., y_5, and y_6, although considered as a sum, help estimate the sum properly by giving due focus on the six individual components from the perspective of manufacturing engineering.

Thus, if one knows the values of δ, λ, P_r, V, t, T_1, C_2, d, y_1, y_2, y_3, y_4, y_5 and y_6 then the appropriate method to find the approximate values of n, h and k is given through equations (3.24), (3.25) and (3.32).

3.4.2 An Alternate Approach to Determine the Optimal Parameters

For obtaining the optimal values of n, h, and k, $\mathcal{L}$ is partially differentiated with respect to h and k. Subsequently, these partial derivatives are equated to zero. Further, notice that P is a function of n and k. Additionally, α is a function of k only. Thus, we obtain the primary equations (3.33) and (3.34) along with the corresponding supplementary equations (3.35) and (3.36) provided hereunder.

$$
\lambda h^2 \frac{\partial D}{\partial h}\left(P_r V - \frac{\alpha C_2}{h} - \lambda C_3\right) - \alpha C_2\left(1+\lambda D\right) - \left(d + n\sum_{i=1}^{6} y_i\right)\left(1+\lambda D\right)^2 = 0\ . \qquad (3.33)
$$

$$
\lambda \frac{\partial D}{\partial k}\left(P_r V - \frac{\alpha C_2}{h} - \lambda C_3\right) + \frac{C_2}{h}\frac{\partial \alpha}{\partial k}\left(1+\lambda D\right) = 0\ . \qquad (3.34)
$$

where

$$
\frac{\partial D}{\partial h} = \left(\frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{6}\right)\ . \qquad (3.35)
$$

$$\frac{\partial D}{\partial k} = -\frac{h\partial P/\partial k}{P^2} \ . \tag{3.36}$$

There is no simple procedure for solving the above system of primary and supplementary equations. Therefore, we try to find out an approximate solution by presuming λ to be small. Neglecting all terms of smaller magnitude than the principal term affected by λ and substituting equations (3.35) and (3.36) respectively in equations (3.33) and (3.34), we get

$$\lambda h^2 P_r V\left(\frac{1}{P}-\frac{1}{2}\right) - \alpha C_2 - \left(d + n\sum_{i=1}^{6} y_i\right) \doteq 0 \ . \tag{3.37}$$

$$-\frac{\lambda h^2 P_r V \partial P/\partial k}{P^2} + C_2\frac{\partial \alpha}{\partial k} \doteq 0 \ . \tag{3.38}$$

From equation (3.37), we achieve

$$h \doteq \sqrt{\frac{\alpha C_2 + d + n\sum_{i=1}^{6} y_i}{\lambda P_r V(1/P - 1/2)}} \ . \tag{3.39}$$

Plugging in the value of $\partial P/\partial k$ from equation (3.28) in equation (3.38), we attain

$$\frac{\partial \alpha}{\partial k} = -\frac{\lambda h^2 P_r V}{P^2 C_2}\frac{e^{-(k-\delta\sqrt{n})^2/2}}{\sqrt{2\pi}} \ . \tag{3.40}$$

Now, plugging in the value of $\partial\alpha/\partial k$ from equation (3.30) on the left-hand side in equation (3.40), we secure,

$$-\frac{2e^{-k^2/2}}{\sqrt{2\pi}} = \frac{\lambda h^2 P_r V}{P^2 C_2}\left(-\frac{e^{-(k-\delta\sqrt{n})^2/2}}{\sqrt{2\pi}}\right)$$

$$\Rightarrow e^{-\frac{\delta^2 n}{2}+k\delta\sqrt{n}} = \frac{2P^2 C_2}{\lambda h^2 P_r V} \ . \tag{3.41}$$

Taking log on both sides of the equation (3.41), we get

$$\begin{aligned} -\frac{\delta^2 n}{2} + k\delta\sqrt{n} &= \log\left(\frac{2P^2C_2}{\lambda h^2 P_r V}\right) \\ \Rightarrow k &= \frac{\delta\sqrt{n}}{2} + \frac{1}{\delta\sqrt{n}} \log\left(\frac{2P^2C_2}{\lambda h^2 P_r V}\right) . \end{aligned} \quad (3.42)$$

Equations (3.39) and (3.42) provide respectively the optimal values of h and k for the given values of δ, λ, P_r, V, t, T_1, C_2, d, y_1, y_2, y_3, y_4, y_5 and y_6.

3.5 Algorithm for Determining n, h, and k for $\bar{X}$-chart

Algorithms that need to be adopted to determine the optimal parameters for both approaches are given in this Section.

3.5.1 Algorithm for Original Approach

In order to demonstrate succinctly the methodology for arriving at the values of n, h, and k, the following steps need to be carried out.

(1) Find the value of α using equation (3.3) for a given value of k.

(2) For the same value of k as used in step (1) and for a given value of δ and n, determine the value of $-n + \frac{P^2(1/P-1/2)}{\partial P/\partial n}$. The value of P can be calculated from equation (3.2).

(3) Compare $(\alpha C_2 + d)/\sum_{i=1}^{6} y_i$ with the value of $-n + \frac{P^2(1/P-1/2)}{\partial P/\partial n}$ found in step (2). This comparison provides us the first approximate integer value of n after appropriately rounding-off.

(4) Substituting the approximate integer value of n found in step (3), obtain the first approximate value of k using equation (3.32).

(5) Iterate steps (1), (2), (3), and (4) again to obtain the revised values of n and k.

(6) Stopping rule for iterations: when the integer value of n obtained from step (3) yields the same value of k considered up to one place of decimal, the iterations will be stopped.

(7) Finally, determine h by plugging-in the revised values of n and k in equation (3.24).

(8) Subsequently, evaluate $\mathcal{L}$ using equation (3.14) for the revised values of n, h and k.

(9) The values of n, h and k for which $\mathcal{L}$ is minimum, are the optimal values.

3.5.2 Algorithm for the Alternate Approach

In order to find the values of h and k, the following steps need to be carried out.

(1) For an assumed integer value of n, and an assumed value of k (irrespective of whether it is an integer or not), find P by using equation (3.2).

(2) Obtain the value of α from equation (3.3) for the same assumed value of k used in step (1).

(3) Determine h using equation (3.39).

(4) Subsequently, determine the revised value of k using equation (3.42).

(5) Repeat steps (1) to (4) for the revised values of P, α, h, and k.

(6) Then evaluate $\mathcal{L}$ using equation (3.14) for the revised values of k, P, and h.

(7) Finally, repeat steps (1) to (6) for another integer value of n, incremented by 1, for the value of n assumed in step (1).

(8) The values of n, h and k for which $\mathcal{L}$ is minimum, are the optimal values.

3.6 Code for Determining n, h and k for $\bar{X}$-chart in Mathematica

Mathematica codes are provided in this Section for both approaches.

3.6.1 Mathematica Code for Original Approach

f[z_]:=Exp[$-z$^2/2]/Sqrt[2 * Pi]

$f[z]$

P[n_, k_, δ_]:=Integrate[$f[z]$, {z, $k-\delta$ * Sqrt[n], ∞}]

$P[n, k, \delta]$

S[n_, k_, δ_]:=D[$P[n,k,\delta]$, n]

$S[n, k, \delta]$

t[n_, k_, δ_]:=$\frac{e^{-\frac{1}{2}(k-\sqrt{n}\delta)^2}\delta}{2\sqrt{n}\sqrt{2\pi}}$

$t[n, k, \delta]$

Q[n_, k_, δ_]:=(((($P[n,k,\delta]$)^2) * (1/$P[n,k,\delta]$ − 1/2))/$t[n,k,\delta]$) − n

$Q[n, k, \delta]$

g[y_1_, y_2_, y_3_, y_4_, y_5_, y_6_]:=$y_1 + y_2 + y_3 + y_4 + y_5 + y_6$

$g[y_1, y_2, y_3, y_4, y_5, y_6]$

α[k_]:=2Integrate[$f[z]$, {z, k, ∞}]

$\alpha[k]$

Q[n_, k_, δ_]:=(((($P[n,k,\delta]$)^2) * (1/$P[n,k,\delta]$ − 1/2))/$t[n,k,\delta]$) − n

$Q[n, k, \delta]$

H[α_, p_, d_, y_1_, y_2_, y_3_, y_4_, y_5_, y_6_]:=($\alpha[k]$ * $p + d$)/$g[y_1, y_2, y_3, y_4, y_5, y_6]$

$H[\alpha, p, d, y_1, y_2, y_3, y_4, y_5, y_6]$

$k[\mathrm{n\_}, y_1\_, y_2\_, y_3\_, y_4\_, y_5\_, y_6\_, \delta\_, \mathrm{p\_}]=$

$\mathrm{Sqrt}[-2 * \mathrm{Log}[(\mathrm{Sqrt}[2 * n * \pi] * g[y_1, y_2, y_3, y_4, y_5, y_6])/(\delta * p)]]$

$k[n, y_1, y_2, y_3, y_4, y_5, y_6, \delta, p]$

$h\,[\alpha\_, \mathrm{p\_}, \mathrm{d\_}, P_r\_, \lambda\_, \mathrm{V\_}, y_1\_, y_2\_, y_3\_, y_4\_, y_5\_, y_6\_, \mathrm{n\_}, \mathrm{k\_}, \delta\_, \mathrm{p\_}] :=$

$\mathrm{Sqrt}\left[(\alpha[k] * p + d + n * g[y_1, y_2, y_3, y_4, y_5, y_6]) \Big/ (\lambda * P_r * V * (1/P - 1/2))\right]$

$h\,[\alpha, p, d, P_r, \lambda, V, y_1, y_2, y_3, y_4, y_5, y_6, n, k, \delta, p]$

$D[\mathrm{n\_}, \mathrm{P\_}, \lambda\_, \mathrm{h\_}, \mathrm{E\_}, \mathrm{t\_}] = (1/P - 1/2 + (\lambda * h)/12) * h + t * n + E$

$D[n, P, \lambda, h, E, t]$

$L\,[\alpha\_, \mathrm{p\_}, \mathrm{d\_}, y_1\_, y_2\_, y_3\_, y_4\_, y_5\_, y_6\_, \mathrm{n\_}, \mathrm{D\_}, \delta\_, \mathrm{q\_}, P_r\_, \lambda\_, \mathrm{h\_}, \mathrm{V\_}] =$

$(P_r * \lambda * V * D + (\alpha * p/h) + \lambda * q) \Big/ (1 + \lambda * D) + \ (d + n * g\,[y_1, y_2, y_3, y_4, y_5, y_6]) \Big/ h$

$L\,[\alpha, p, d, y_1, y_2, y_3, y_4, y_5, y_6, n, D, \delta, q, P_r, \lambda, h, V]$

3.6.2 Mathematica Code for Alternate Approach

$f[\mathrm{z\_}] := \mathrm{Exp}[-z\char`^2/2]/\mathrm{Sqrt}[2 * \mathrm{Pi}]$

$f[\mathrm{z}]$

$P[\mathrm{n\_}, \mathrm{k\_}, \delta\_] := \mathrm{Integrate}[f[z], \{z, k - \delta * \mathrm{Sqrt}[n], \infty\}]$

$P[n, k, \delta]$

$\alpha[\mathrm{k\_}] := 2\mathrm{Integrate}[f[z], \{z, k, \infty\}]$

$\alpha[k]$

$g\,[y_1\_, y_2\_, y_3\_, y_4\_, y_5\_, y_6\_] := y_1 + y_2 + y_3 + y_4 + y_5 + y_6$

$g\,[y_1, y_2, y_3, y_4, y_5, y_6]$

$h\,[\alpha\_, \mathrm{p\_}, \mathrm{b\_}, y_1\_, y_2\_, y_3\_, y_4\_, y_5\_, y_6\_, \mathrm{n\_}, \mathrm{k\_}, \delta\_, \mathrm{p\_}, P_r\_] =$

$\mathrm{Sqrt}\left[(\alpha[k] * p + b + n * g\,[y_1, y_2, y_3, y_4, y_5, y_6]) \,/\, (\lambda * P_r * V * (1/P[n, k, \delta] - 1/2))\right]$

$h\,[\alpha, p, b, y_1, y_2, y_3, y_4, y_5, y_6, n, k, \delta, p, P_r]$

$k\,[\mathrm{n\_}, \mathrm{p\_}, \lambda\_, \mathrm{h\_}, P_r\_, \mathrm{V\_}, \delta\_] =$

$\delta * \mathrm{Sqrt}[n]/2 + (1/(\delta * \mathrm{Sqrt}[n]))\mathrm{Log}\left[(2 * P\char`^(2) * p) \Big/ (\lambda * h\char`^2 * P_r * V)\right]$

$k\,[n, p, \lambda, h, P_r, V, \delta]$

$D[\mathrm{n\_}, \mathrm{P\_}, \lambda\_, \mathrm{h\_}, \mathrm{E\_}, \mathrm{t\_}] = (1/P - 1/2 + (\lambda * h)/12) * h + t * n + E$

$D[n, P, \lambda, h, E, t]$

$L[\alpha\_, \mathrm{p\_}, \mathrm{d\_}, y1\_, y2\_, y3\_, y4\_, y5\_, y6\_, \mathrm{n\_}, \mathrm{D\_}, \delta\_, \mathrm{q\_}, P_r\_, \lambda\_, \mathrm{h\_}, \mathrm{V\_}] =$
$(P_r * \lambda * V * D + (\alpha * p/h) + \lambda * q) \Big/ (1 + \lambda * D) + (d + n * g[y_1, y_2, y_3, y_4, y_5, y_6]) \Big/ h$

$L[\alpha, p, d, y_1, y_2, y_3, y_4, y_5, y_6, n, D, \delta, q, P_r, \lambda, h, V]$

3.7 Numerical Examples using Mathematica

Numerical examples are provided in this Section to demonstrate both approaches.

3.7.1 Numerical Examples for Original Approach

Example 3.1. To demonstrate the first procedure, let us consider $\delta = 2$, $\lambda = 0.01$ per hour, $V = \$5$, $t = 0.05$ hours, $T_1 = 2$ hours, $C_2 = \$50$, $C_3 = \$25$, $d = \$0.50$, $\sum_{i=1}^{6} y_i =$ \$0.10 and $P_r = 100$ units per hour.

(1) We obtain $\alpha = 0.0124$ for $k = 2.5$ from equation (3.3).

(2) Thus, $-n + \frac{P^2(1/P-1/2)}{\partial P/\partial n} \approx (\alpha C_2 + d)/\sum_{i=1}^{6} y_i = 11.2$.

(3) We obtain n from Figure 3.2 by drawing a line parallel to the abscissa from 11.2 on the ordinate. The parallel line intersects with the corresponding graph for $k = 2.5$ and $\delta = 2$. From this point of intersection, a perpendicular is drawn on the abscissa to obtain n as 4.79. The rounded-off integer value of n is thus 5.

(4) The value of P obtained from equation (3.2) for $n = 5$, $k = 2.5$ and $\delta = 2$ is 0.9757. It yields $(1/P - 1/2) = 0.5249$.

(5) The value of $h = 0.7857$ hours from equation (3.24). Thus, we get $\mathcal{L} = \$15.25$ using equation (3.14).

(6) Subsequently, for $n = 5$, we get $\frac{\sum_{i=1}^{6} y_i \sqrt{n}}{\delta C_2} = 0.00223$ which, in turn, produces $k = 3.2$ from equation (3.32).

(7) For $k = 3.2$, we get $\alpha = 0.00137$ from equation (3.3).

(8) For $\alpha = 0.00137$, we get $(\alpha C_2 + d)/\sum_{i=1}^{6} y_i = 5.687$.

(9) We get $n = 5.70$ with suitable approximation from Figure 3.2. So, the approximate integer value of n is 6.

(10) The value of P for $n = 6$, $k = 3.2$ and $\delta = 2$ is 0.9553 which gives, $(1/P - 1/2)$ $= 0.5468$.

(11) The value of h thus obtained is 0.6537 hours from equation (3.24). Finally, we get $\mathcal{L} = \$14.97$ from equation (3.14). Thus, the optimal value of $\mathcal{L}$ is \$14.97.

Example 3.2. To demonstrate the first procedure, let us consider $\delta = 2$, $\lambda = 0.01$ per hour, $V = \$5$, $t = 0.05$ hours, $T_1 = 2$ hours, $C_2 = \$50$, $C_3 = \$25$, $d = \$0.50$, $\sum_{i=1}^{6} y_i =$ \$0.10 and $P_r = 200$ units per hour.

(1) The steps (1) to (10) are identical with Example 3.1.

(2) The optimal values of n, h, and k thus obtained are 6, 0.4623 hours, and 3.2, respectively. The corresponding minimal value of $\mathcal{L}$ is \$27.66.

In Table 3.1 comparative values of n, h, and k are provided for the proposed methodology.

Table 3.1: Comparative values of n, h and k between 1st approximation and optimal values for Example - 3.2.

Variable	1st Approximation	Optimal Values
n	5	6
k	2.5	3.2
h	0.5556	0.4623
$\mathcal{L}$	27.92	27.66

Table 3.2 provides the values of the sample size n for $k = 2(0.5)3$ and $\delta = 2$. However, for the other combinations of the pairs of the values of $(k,\ \delta)$ and the ordinate value of $-n + \frac{P^2(1/P-1/2)}{\partial P/\partial n}$, the corresponding value of n can be obtained from the abscissa of Figure 3.2.

Table 3.2: Values of $-n + \frac{P^2(1/P-1/2)}{\partial P/\partial n}$ for $\delta = 2$.

n	$k = 2$	$k = 2.5$	$k = 3$
2	0.39	—	—
3	3.31	0.37	—
4	14.51	3.69	0.03
5	54.51	14.58	3.24
6	199.14	48.55	12.61
7	739.76	156.20	38.79
8	2832.69	509.17	112.9
9	11199.2	1709.84	329.46

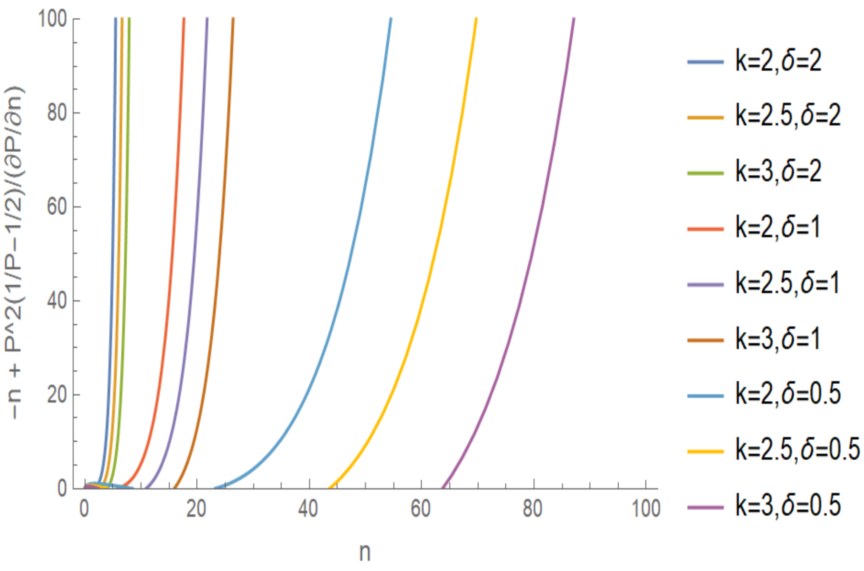


Figure 3.2: Graph of $-n + \frac{P^2(1/P-1/2)}{\partial P/\partial n}$ as a function of n.

Example 3.3. After modifying the estimate of $\sum_{i=1}^{6} y_i$ from \$0.1 to \$1 in the backdrop of today's perspective and keeping the other input parameters intact, the values of n, h and k have been enumerated as mentioned below.

We have $\delta = 2$, $\lambda = 0.01$ per hour, $V = \$100$, $t = 0.05$ hours, $T_1 = 2$ hours, $C_2 = \$50$, $C_3 = \$25$, $d = \$0.50$, $\sum_{i=1}^{6} y_i = \$1$ and $P_r = 100$ units per hour. We presume $k = 2.5$ at the outset.

(1) The steps (1) to (10) are similar to those in Example 3.1.

(2) The optimal values found for n, h, and k respectively are 3, 0.2224 hours, and 2.5. The corresponding optimal value of $\mathcal{L}$ thus obtained is \$243.92.

Table 3.3 provides the comparative values of n, h, and k for the proposed methodology.

Table 3.3: Comparative values of n, h and k between 1st approximation and optimal values for Example - 3.3.

Variable	1st Approximation	Optimal Values
n	3	3
k	2.5	2.5
h	0.2424	0.2424
$\mathcal{L}$	243.92	243.92

Table 3.4 gives the association between n and the kind of variable Shewhart control chart to be employed

Table 3.4: Control chart to be employed for different n.

Value of n	Control Chart
1	$I - MR$ Chart
2 - 6	$\bar{X} - R$ Chart
> 6	$\bar{X} - S$ Chart

A Pertinent Remark

For more sensitive processes like zinc coating, gold plating or filling-in processes in pharmaceutical or chemical industries, it is of vital importance to catch the shift in the mean in a prompt manner to avoid the increase in cost due to over-coating/over-filling or to avoid customer dissatisfaction on account of under-coating/under-filling. In these sensitive processes with underlying usage of costly materials, it is imperative to have a relatively lower value of h.

3.7.2 Numerical Example for the Alternate Approach

To demonstrate the second method, let us consider $\delta = 2$, $\lambda = 0.01$ per hour, $V = \$5$, $t = 0.05$ hours, $T_1 = 2$ hours, $C_2 = \$50$, $C_3 = \$25$, $d = \$0.50$, $\sum_{i=1}^{6} y_i = \$0.10$ and $P_r = 100$ units per hour.

Table 3.5 contains the optimal values of the parameters h, k, and $\mathcal{L}$ for different integer values of n. From Table 3.5 one can clearly see that for $n = 4$, $\mathcal{L} = \$14.70$ is minimum and the corresponding optimal values of h and k are respectively 0.5835 hours and 2.87.

Table 3.5: Optimal values of h, k, and $\mathcal{L}$ for integer values of n.

n	h	k	$\mathcal{L}$
1	0.2860	2.53	18.22
2	0.4208	2.65	15.64
3	0.5191	2.75	14.88
4	0.5835	2.87	14.70
5	0.6276	3.00	14.77
6	0.6621	3.14	14.98

The pertinent detailed explanation for obtaining these optimal values provided in Table 3.5, is given here. Steps carried out to find out the optimal values for $n = 1$ are as follows:

(1) For $n = 1$, $k = 2.5$, we obtain $P = 0.3085$.

(2) Then, for $k = 2.5$, we get $\alpha = 0.0124$.

(3) For $n = 1$, $k = 2.5$, $P = 0.3085$ and $\alpha = 0.0124$, we obtain $h = 0.2984$ hours.

(4) For $P = 0.3085$ and $h = 0.2984$ hours, the revised value of k is 2.53.

(5) Now for $n = 1$ and $k = 2.53$, we obtain revised $P = 0.2976$.

(6) Then, for $k = 2.53$, we get revised $\alpha = 0.0114$.

(7) For $k = 2.53$, $P = 0.2976$ and $\alpha = 0.0114$, we obtain revised $h = 0.286$ hours.

(8) For $n = 1$, $k = 2.53$ and $h = 0.286$ hours, we achieve $\mathcal{L} = \$18.22$. Similarly, it is done for other values of n.

3.7.3 Sensitivity Analysis

It emerges from Table 3.6 that as the values of λ, P_r, V, and t increase, the optimal value of h decreases. The corresponding domain of h lies between 0.1762 (hours) to 1.4678 (hours) and that of the optimal value of $\mathcal{L}$ ranges between \$13.87 to \$119.52. It is to be noted that the change in $\mathcal{L}$ is impacted to a large extent on account of 10 times the change in V.

Also, an increase in the values of d and $\sum_{i=1}^{6} y_i$ results in an increase in the optimal value of h and a decrease in the optimal value of k. The domain of k ranges between 2.21 to 2.96 only.

Table 3.6: Effect of input parameters on the optimal values for the economic design of $\bar{X}$-chart.

Run	δ	λ	P_r	V	t	T_1	C_2	C_3	d	$\sum_{i=1}^{6} y_i$	n	h	k	$\mathcal{L}$
1	2.0	0.01	100	5	0.05	2	50	25	0.5	0.1	4	0.5835	2.87	\$14.70
2	2.0	0.02	100	5	0.05	2	50	25	0.5	0.1	4	0.3940	2.96	\$26.63
3	2.0	0.03	100	5	0.05	2	50	25	0.5	0.1	4	0.3217	2.96	\$37.75
4	2.0	0.01	200	5	0.05	2	50	25	0.5	0.1	4	0.3940	2.96	\$26.97
5	2.0	0.01	100	50	0.05	2	50	25	0.5	0.1	4	0.1762	2.96	\$119.52
6	2.0	0.01	100	5	0.50	2	50	25	0.5	0.1	2	0.3967	2.70	\$19.86
7	2.0	0.01	100	5	0.05	20	50	25	0.5	0.1	4	0.5572	2.96	\$87.37
8	2.0	0.01	100	5	0.05	2	5	2.5	0.5	0.1	3	0.5497	2.21	\$13.87
9	2.0	0.01	100	5	0.05	2	50	25	5.0	0.1	5	1.4678	2.72	\$18.99
10	2.0	0.01	100	5	0.05	2	50	25	0.5	1.0	3	1.1157	2.44	\$18.16
11	1.5	0.01	100	5	0.05	2	50	25	0.5	0.1	6	0.6024	2.81	\$15.83
12	1.0	0.01	100	5	0.05	2	50	25	0.5	0.1	4	0.3785	2.40	\$29.31

3.8 A Case Example Based on Real Life Data

An Indian organization produces cast iron pipes of size 500 mm diameter with the additional feature of ductility incorporated through an appropriate converter in the process. The purpose of the product is to transport drinking water from one place to another place, sometimes, even transcending the geographical borders of nations. These pipes are examined for several quality characteristics like Tensile Strength, Outer Diameter, Inner Diameter, Microstructure, and Hydro Leakage prior to dispatching them to customers. The quality characteristic Tensile Strength of iron pipes is of crucial importance from the perspective of transportation of drinking water from one place to another on account of being a relevant measure of load-bearing capacity particularly when the pipelines are laid beneath the roads and highways through which heavy vehicles continually move. The following set of parameters, published in Sandeep and Mukhopadhyay [74], was collected from the said Indian organization:

δ = 2, λ = 0.01 per hour, V = Rs 50, t = 0.1 hours, T_1 = 1.5 hours, C_2 = Rs 188 , C_3 = Rs 94, d = Rs 150, y_1 = Rs 20, y_2 = Rs 40, y_3 = Rs 20, y_4 = Rs 4, y_5 = Rs 0, y_6 = Rs 10, which gives $\sum_{i=1}^{6} y_i$= Rs 94 and P_r = 55 units per hour. The optimal parameters obtained are n = 2, h = 5.04 hours, and k = 1.60 through the iterative approach developed by us. The corresponding value of $\mathcal{L}$ thus obtained is Rs 194.99. It may be pertinent to note here that hitherto the organization was using n = 1 and h = 1.82 hours. Thus, considering the suggested value of k = 1.60, $\mathcal{L}$ turns out to be Rs 238.13.

If we use the approach given by Duncan [25], it fails to provide the first iterative value itself of k because the formula for k based on the approach provided by Duncan [25] is $k = \sqrt{-2\log\left(\sqrt{2n\pi}\sum_{i=1}^{6} y_i/\delta C_2\right)}$, which may yield a complex value and for the above set of parameters it has yielded a complex value that lacks physical interpretation. It is to be noted that in the first approach, we have resorted to finding the partial derivative of the associated loss function with respect to n along with h and k following the footsteps of Duncan [25], nonetheless, in the present-day industrial engineering context. However, it is also to be noted that n is an integer. In order to avoid this mathematical lacuna of finding the partial derivative of the loss function with respect to an integer, an alternate approach has been adopted essentially relying on a combination of partial derivative and iterative procedure. The difference in the optimal results emanates from it. Since the alternate approach is mathematically more sound, it would be always better to adopt it by theoreticians as well as practitioners.

The usefulness of the suggested model can be well understood through this real-life case example.

3.9 Concluding Remarks

Duncan [25] partially differentiated $\mathcal{L}$ with respect to n which is an integer quantity but here we have also provided an alternate iterative algorithmic approach to facilitate comparison between two mathematical approaches taking into account the cost components from the present-day perspective along with the production rate, which was not considered by Duncan [25]. If one follows the footsteps of the methodology proposed by Duncan [25], the optimal values of the parameters thus obtained are $n = 6$, $h = 0.6537$ hours, $k = 3.2$ and the corresponding $\mathcal{L} = \$14.97$.

However, if one follows the suggested alternate iterative algorithmic approach, the optimal values to be obtained are $n = 4$, $h = 0.5835$ hours, $k = 2.87$, and the corresponding $\mathcal{L} = \$14.70$. Thus, it is imperative to follow the iterative algorithmic approach to derive the optimal parameters of the Shewhart control chart from a mathematical standpoint as well as a contemporary perspective on manufacturing engineering.

This Chapter has dealt with the determination of n, h, and k based on economic considerations. Of course, one can not be oblivious to the fact that essentially in a control chart, one is interested in finding out the between subgroup variation when assignable or special causes of variation are present in a process. Consequently, one needs to adhere to the principles of rational subgrouping so that within-subgroup or within-sample variation is kept at a bare minimum level with the underlying presumption that within a subgroup or sample of size n hardly is there any change in process parameters during a short span of time. However, it is of crucial importance to determine n, h, and k from economic considerations. It goes without saying that the inherent variation estimated from within subgroup or within-sample variation ($\hat{\sigma} = \frac{\bar{R}}{d_2}$) forms the basis for determining the control limits of an $\bar{X}$-chart apart from n and k.

It is to be noted that the optimal parameters of the Shewhart control chart (n, h, and k) are also applicable to control the within subgroup variation through an R-chart apart from controlling the between subgroup variation through an $\bar{X}$-chart. The analogous usual procedure for determining the control limits' multipliers for an R-chart is to be adopted in lieu of D_3 and D_4. Note that D_3 and D_4 are essentially the 3σ control limits for range. However, suitable changes need to be incorporated now considering $k\sigma$ control limits for range. Thus, UCL_R and LCL_R respectively turn out to be $\bar{R}(1+k\frac{d_3}{d_2})$ and $\bar{R}(1-k\frac{d_3}{d_2})$. It is easy to see that for $k = 3$, $D_4 = (1+3\frac{d_3}{d_2})$ and $D_3 = (1-3\frac{d_3}{d_2})$. As usual, the constants d_3 and d_2 are to be considered as factors for computing control chart lines and are dependent on the optimal sample size n. It goes without saying that $\bar{X}$-chart must be accompanied by an R-chart resorting to the optimal n, h, and k given in this treatise.

3.9.1 A Pivotal Point About the Advantage of the Proposed Methodology Compared to Duncan's Methodology

In Duncan's methodology, mathematical expressions within a square root turn up for h and k, given respectively in equations (3.24) and (3.32). Note that for h, the denominator in equation (3.24) will be always positive along with other parameters in the equation since $1/P$ would turn out more than 1 as P would be less than 1 being a probability measure for an out-of-control process. However, there is no such guarantee of obtaining a positive value within the square root term for the expression of k in equation (3.32). In our proposed methodology of iterative algorithmic approach, the expression for k in equation (3.42) is obviously free from the occurrence of this puzzling situation since it is devoid of any such square root term except for n that holds a positive value always.

It is crucially important to adopt the proposed iterative algorithmic approach unlike the partial derivative approach of Duncan from a mathematical standpoint since n is an integer.

3.9.2 Arguments and Counter-arguments on Weaknesses of Economic Design of the Control Charts

It is worthwhile to mention here the arguments and counter-arguments with respect to an economic design model for the Shewhart control charts. While the arguments against these types of economic models were given by Woodall [99], the corresponding counter-arguments highlighting the difference of views in favor of the economic design of control charts were given by Lorenzen and Vance in the same article given by Woodall [99].

Arguments by Woodall

1. Any control chart which generates a large number of false alarms brings extra variability in the process by overadjustment. Also, due to a large number of false alarms,

the optimal cost will be misleading.

2. The cost and time parameters of the economic models are beyond the control of the workers or the management. If these parameters are fixed, then the economic model is a barrier in the process improvement.

3. For the economic models, the cost of poor quality is balanced by the cost of repairing the process and the cost of sampling. Due to this reason, economic models are not effective in producing charts that can detect small shifts in the process before substantial losses can occur.

Counter-arguments by Lorenzen and Vance

1. There is no doubt that the economic design can cause a large number of false alarms. This occurs when the cost of a false alarm is very low as compared to the loss occurred due to operating in an out-of-control state. There will be no variability in the process due to the false alarms since there will be no cause for out-of-control signal. Hence, no cause will be found, a false alarm will be declared and no action will be taken.

2. Even when the minimal cost is high, then the process must be changed. The management or the workers then propose these changes and estimate the new cost and the time parameters. Then determine the minimal cost and compare the savings to the cost of the process change and if the cost does not decrease then there is no need to change the process.

3. In general, the minimal cost model is optimal for a reasonable change in any of the parameters. But sometimes, the cost can change drastically due to a small change in some parameters. So, one can perform sensitivity analysis for those parameters.

Chapter 4

Optimal Parameters of the Economic Statistical Design of the u-chart and the Multi-objective Economic Statistical Design of the u-chart[1]

4.1 Introduction

As mentioned earlier, control charts serve as a primary tool for detecting and addressing assignable causes of variation while allowing the process to continue undisturbed when governed solely by common causes. These charts are utilized to detect both sudden shifts and gradual trends in process behavior for specific quality characteristics.

A control chart typically features CL, UCL, and LCL corresponding to the quality characteristic being analyzed. The vertical axis of the chart represents the quality characteristic, while the horizontal axis represents samples or subgroups collected at regular intervals. According to Shewhart, a minimum of 24 to 25 samples or subgroups should be collected periodically for meaningful analysis of process behavior.

In many cases, the quality characteristic is of the attribute type, such as the number of defects per inspection unit. The process is deemed to be in control if the number of defects falls within the control limits. However, if it exceeds these limits, the process is considered out of control, prompting an investigation to identify and eliminate assignable causes. If no assignable cause is found, the process is allowed to operate without any intervention.

[1]Double-objective Economic Statistical Design of the u-chart: NSGA II Approach. *Journal of Statistical Computation and Simulation*, **95**(**5**), 1091-1109, 2025.

It is worth noting that the number of defects per unit is a "lower-the-better" quality characteristic. Consequently, it may appear that only the upper control limit is necessary for controlling the process. However, in addition to the upper control limit, a lower control limit is also added to prevent any inspection bias. Figure 4.1 illustrates a schematic representation of an attribute control chart displaying the sample statistic u_i, representing the number of defects per inspection unit for the i^{th} sample.

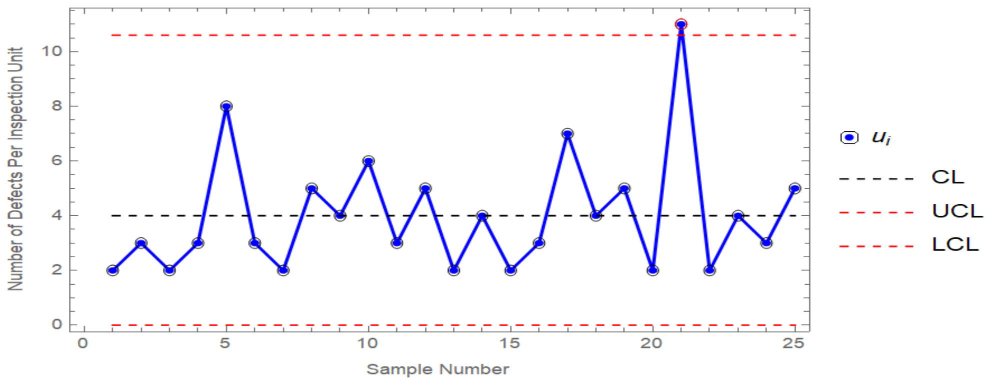


Figure 4.1: Schematic Diagram of *u*-Chart.

The appearance of control limits on a control chart depends on whether the number of inspection units remains constant across samples or subgroups. If the number of inspection units remains constant, the control limits will be straight lines. However, if the number of inspection units varies, the control limits will no longer be straight lines. In fact, they will be jig-jag lines.

Products or components are often categorized as conforming or non-conforming based on the presence or absence of defects, particularly critical or major defects. While critical defects significantly impact functionality, minor defects, like blemishes in paint, primarily affect aesthetics. Control charts dealing with defectives and defects are known as attribute control charts, whereas those focusing on measurable quality characteristics are called variable control charts. Among attribute control charts, commonly used ones include the *p*-chart, *np*-chart, *c*-chart, and *u*-chart. This Chapter specifically discusses the *u*-chart, designed for situations where the number of inspection units for defect detection varies across samples or subgroups. An inspection unit can be one item, such as one water pump, or it can be multiple items such as ten water pumps. The *u*-chart's key parameters include the sample size (n), the sampling interval (h), and the control limits' multiplier (k). Considering the economic aspects, it's crucial to consider the cost factors associated with control chart design, including sampling, testing, eliminating assignable causes, and producing defective items.

The first economic design for an $\bar{X}$ control chart, a variable control chart, was given by Duncan [25]. Yu and Chen [105], Koo and Case [44], da Costa Quinino et al. [21], Vommi and Seetala [94], Carolan et al. [14] developed an economic model for a variable sampling interval of an $\bar{X}$-chart for a continuous flow process. A more generic economic model which can be applied to most of the control charts was developed by Lorenzen and Vance [52]. Some limitations over the use of the economic design of control charts were given by Woodall [99]. To overcome these limitations, Saniga [78] proposed an economic statistical design of $\bar{X}$-chart. On the other hand, there are a few works done in the area of economic designs for attributes. Lupo [55], Lupo [54] considered the economic design of the c-chart for determining the variable sample size and sampling interval. The multi-objective economic-statistical design of c-charts for monitoring the number of non-conformities was given by Amiri and Jafarian-Namin [3].

Since there is no article that deals with the economic and statistical aspects of the u-chart, we are proposing the economic statistical design and multi-objective economic statistical design of the u-chart in this Chapter. The optimal parameters for the economic statistical design of the u-chart are then found with the help of the Genetic Algorithm and optimal parameters for the multi-objective economic statistical design of the u-chart are then found with the help of the NSGA II.

The GA is chosen for finding the optimal parameters of the economic statistical design of the u-chart because it is well-suited for finding global optima in complex search spaces. It maintains a population of candidate solutions and uses selection, crossover, and mutation operators to explore the search space more effectively than traditional gradient-based methods, which can get trapped in local optima. It can easily accommodate constraints in the optimization problem formulation, such as bounds on decision variables or non-linear constraints as given in the model proposed in this study. It can easily handle complex optimization problems like the economic statistical design of u-chart containing a large number of input variables. The pseudocode that is used to implement the Genetic Algorithm is given in Section 1.3.2 of Chapter 1.

The optimal parameters for the multi-objective economic statistical design are then found using the NSGA II approach in such a way that both expected cost per cycle (C_E) as well as out-of-control Average Run Length (ARL_δ) are minimized while maintaining a reasonably large in-control Average Run Length (ARL_0). NSGA II is widely known for its capability to enhance solution quality by discovering diverse solution sets across the Pareto front. It is an often-used multi-objective evolutionary algorithm introduced by Deb [23]. We are using the NSGA II since it is specifically designed to handle problems with multiple objectives. It uses a genetic algorithm approach to search for Pareto-optimal solutions. These Pareto-optimal solutions cannot be improved in one objective without sacrificing another. This allows decision-makers to choose from a range of alternatives, considering the trade-offs between different objectives. It utilizes evolutionary search techniques, such as crossover and mutation, to explore the solution space efficiently. It can handle complex optimization problems with many decision variables and constraints as we have in our proposed model. It can adapt to changes in problem formulations or objective functions without significantly modifying the algorithm. It offers flexibility in incorporating additional objectives or constraints as the problem evolves. The pseudocode that is used to implement the NSGA II is given in Section 1.3.3 of Chapter 1.

The rest of the Chapter is organized like this. The reason for choosing the *u*-chart for this study and some basic properties of the *u*-chart are given in Section 4.2. Section 4.3 deals with the economic statistical design of the *u*-chart. The economic statistical design of the *u*-chart has been proposed in Section 4.3.3 by taking into account properties of the *u*-chart given in Section 4.2, assumptions for the model in Section 4.3.1, the expected cost per cycle in Section 4.3.2. In Section 4.3.4, the applicability of the economic statistical design is exemplified through a numerical example. The results of the sensitivity analysis are provided in Section 4.3.5. The conclusion based on the economic statistical design is given in Section 4.3.6. Section 4.4 deals with the multi-objective economic statistical design of the *u*-chart. The multi-objective economic statistical design has been proposed in Section 4.4.1. The aspects of applicability for the multi-objective economic statistical design are demonstrated by a numerical example in Section 4.4.2. The results of the sensitivity analysis are provided in Section 4.4.3. An outline is given in Section 4.4.4 about the parameters of u-chart to be implemented in practical situations. The conclusion based on the multi-objective economic statistical design is given in Section 4.4.5. The organizations or individuals can choose any of the two approaches based on the proposition provided in Section 4.5. The content related to the multi-objective economic statistical design of the u-chart can be found in Sandeep and Ranjan Mukhopadhyay [77].

4.2 Reason for Choosing the u-chart for This Study

In quality control, the application of Poisson distribution is often made as a model of the number of nonconformities or defects in a unit of product. In fact, any random phenomenon that occurs on a per unit (per unit area, per unit volume, per unit time, etc.) basis is often well approximated by the Poisson distribution. One can easily derive the Poisson distribution as a limiting form of the Binomial distribution. In a Binomial distribution with parameters n and p, if n approaches infinity and p approaches zero in such a way that np is a constant, then the Poisson distribution results in (Montgomery [59]). For example, the number of pinholes per unit of surface area of a drinking water pipe, the number of spark faults per unit length of a power cable, or the number of coding faults per thousand lines of a code is Poisson distributed.

However, when the area of opportunity for nonconformities varies from one subgroup to another, the conventional c-chart, which tracks the total number of nonconformities, becomes impractical. For instance, when n differs across subgroups, the defects per inspection unit (c/n) may serve as a suitable sample statistic. Here, u represents the nonconformities per unit, i.e., c/n, where c denotes the count of nonconformities found in n inspection units within a sample or subgroup. Notably, the number of inspection units can vary across samples or subgroups, making the u-chart more suitable compared to the c-chart, which is limited to situations with a constant number of inspection units.

Unlike the statistic c in a c-chart, the u statistic in a u-chart does not strictly adhere to a Poisson distribution. While the expected value ($E(u) = \bar{u}$) and variance ($V(u) = \bar{u}/n$) properties differ from those of c, for practical purposes, u can be approximated to follow a Poisson distribution as suggested by Grant and Leavenworth [33]. Subsequently, the probabilities of Type I error (α) and Type II error (β) can be computed as

$$\alpha = 1 - \sum_{x=LCL}^{UCL} \frac{u_0^x e^{-u_0}}{x!} . \tag{4.1}$$

$$\beta = \sum_{x=LCL}^{UCL} \frac{u_1^x e^{-u_1}}{x!} . \tag{4.2}$$

$$\begin{aligned} Power\ (P) &= 1 - \beta \\ &= 1 - \sum_{x=LCL}^{UCL} \frac{u_1^x e^{-u_1}}{x!} . \end{aligned} \tag{4.3}$$

where $LCL = max(0, [(u_0 - k\sqrt{u_0/n}) + 1])$, $UCL = [u_0 + k\sqrt{u_0/n}]$, and [z] is the greatest integer less than or equal to z.

In determining the control limits for the u-chart, the three-sigma approach is commonly employed as suggested by Shewhart and reinforced by Deming [24], in his book "Out of the Crisis". This method, based on three-sigma limits, offers a rational and economical approach to minimize errors while giving the signal for the presence of assignable causes of variation, if any. The choice of three-sigma limits is widely accepted due to its practical effectiveness, especially in cases where the true distribution of the quality characteristic is unknown.

Montgomery [59] also notes that despite the unknown distribution of the quality characteristic, three-sigma limits are commonly adopted in practice. These limits are determined as a multiple of the standard deviation, typically chosen as three, for the plotted statistic. This approach is justified by its practical efficacy, especially when exact probability limits are difficult to compute. In this Chapter, we consider the $\pm k$-sigma limits to determine the optimal value of k, along with n and h, based on an economic statistical model outlined in Section 4.3 and a multi-objective economic statistical design outlined in Section 4.4.

4.3 An Economic Statistical Design of u-chart

In this Section, based on certain assumptions the expected cost per cycle has been considered for proposing an economic statistical design of u-chart.

4.3.1 Assumptions for the Model

The following assumptions are deemed valid for the proposed model:

(1) The quality characteristic u follows Poisson distribution.

(2) The process is initially assumed to be in an in-control state, i.e., $u = u_0$.

(3) Whenever there is an occurrence of an assignable cause amounting to a shift of δ amount, the process mean shifts from u_0 to u_1, where $u_1 = u_0 + \delta u_0$.

(4)The occurrence of an assignable cause follows an exponential distribution with a mean of $1/\lambda$.

4.3.2 Expected Cost Per Cycle

In this Section, the well-known Lorenzen and Vance [52] cost model is extended to the economical statistical design and the multi-objective economic statistical design of a u-chart since it is one of the widely used statistically constrained economic models. The expected cost per cycle, i.e., C_E is:

$$\begin{aligned} C_E &= \frac{\frac{C_0}{\lambda} + C_1(-\tau + nt + h(ARL_\delta) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{sC_2}{ARL_0} + C_3}{\frac{1}{\lambda} + \frac{(1-\gamma_1)sT_0}{ARL_0} - \tau + nt + h(ARL_\delta) + T_1 + T_2} \\ &+ \frac{(\frac{d+ny}{h})(\frac{1}{\lambda} - \tau + nt + h(ARL_\delta) + \gamma_1 T_1 + \gamma_2 T_2)}{\frac{1}{\lambda} + \frac{(1-\gamma_1)sT_0}{ARL_0} - \tau + nt + h(ARL_\delta) + T_1 + T_2}\ . \end{aligned} \tag{4.4}$$

where C_0 is the quality cost per hour when the process is in-control, C_1 is the quality cost per hour when the process is out-of-control, T_0 is the expected time associated with a false alarm, T_1 is the expected time required to discover an assignable cause, T_2 is the expected time required to eliminate an assignable cause, t is the expected time to take a sample and obtain the results, C_2 is the cost for searching an assignable cause when there is none, C_3 is the average cost of identifying and eliminating an assignable cause, d is the cost per sample for maintaining u-chart in a process, y is the variable cost of sampling an inspection unit, γ_1 is a binary variable that takes the value 1 if the production continues during the search for an assignable cause and 0 otherwise, and γ_2 is a binary variable that takes the value 1 if the production continues during the elimination of an assignable cause through intervening in the process and 0 otherwise.

τ is the expected time taken by an assignable cause to occur and can be computed using equation (4.5)

$$\begin{aligned} \tau &= \frac{\int_{rh}^{(r+1)h} e^{-\lambda t}\lambda(t-rh)dt}{\int_{rh}^{(r+1)h} e^{-\lambda t}\lambda dt} \\ \Rightarrow \tau &= \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})} \\ \Rightarrow \tau &= \frac{1}{\lambda} - \frac{h}{e^{\lambda h}-1} \ . \end{aligned} \tag{4.5}$$

ARL_0 is the in-control Average Run Length and can be obtained using equation (4.6)

$$ARL_0 = \frac{1}{\alpha} \ . \tag{4.6}$$

ARL_δ is the out-of-control Average Run Length and can be computed using equation (4.7)

$$ARL_\delta = \frac{1}{P} \ . \tag{4.7}$$

where P is the power of the test given in equation (4.3).

s: Average number of samples taken when the process is in-control. It can be determined using equation (4.8)

$$\begin{aligned} s &= \sum_{r=0}^{\infty} \int_{rh}^{(r+1)h} r\lambda e^{-\lambda t} dt \\ \Rightarrow s &= \frac{1}{e^{\lambda h} - 1} \ . \end{aligned} \tag{4.8}$$

4.3.3 The Economic Statistical Design

In this Section, we are proposing an economic statistical design for a u-chart. We aim to minimize C_E taking into account ARL_0, ARL_δ, and ATS. The economic statistical design is given as follows:

$$\begin{aligned} & Min \ C_E \\ s.t. \quad & ARL_0 \geq ARL_L \\ & ARL_\delta \leq ARL_u \\ & ATS \leq ATS_u \\ & and \ n \in Z^+ \ . \end{aligned} \tag{4.9}$$

where ATS equals h/P. The ARL_L, ARL_u, and ATS_u are the respective lower and upper bounds of ARL_0, ARL_δ and ATS.

The GA has been used for solving the mathematical formulation (4.9). Although GA has been used to find the relevant optimal parameters of various other control charts, however, it has not been used so far for the economic statistical design in the realm of u-chart.

4.3.4 A Numerical Example

In this Section, the application of the proposed model for finding the optimal parameters of the economic statistical design of u-chart using GA is shown via a numerical example.

For illustrating the results of the economic statistical design of u-chart, we require input parameters along with constraints. These are mentioned now. The average number of nonconformities per inspection unit is u_0 = 6.36 and the rate of occurrence of assignable cause is $\lambda = 0.01$ per hour. Once an assignable cause has occurred, the mean shifts with an amount of $\delta = 2$. The rest of the parameters considered are: C_0 = \$40, C_1 = \$60, C_2 = \$25, C_3 = \$12.5, d = \$1, y = \$0.5, t = 0.05 hours, T_0 = 0.5 hours, T_1 = 0.5 hours, T_2 = 1.5 hours, ARL_L = 370, ARL_u = 5 and ATS_u = 5. Some of these parameters are taken from Amiri and Jafarian-Namin [3] and the rest are speculated to match accordingly. Furthermore, the process is allowed to operate during the detection and elimination of the assignable causes. The optimal values of parameters thus obtained are $n = 1$, $h = 3.76$ hours, $k = 2.16$, and C_E is \$ 42.13. The convergence curve is given in Figure 4.2.

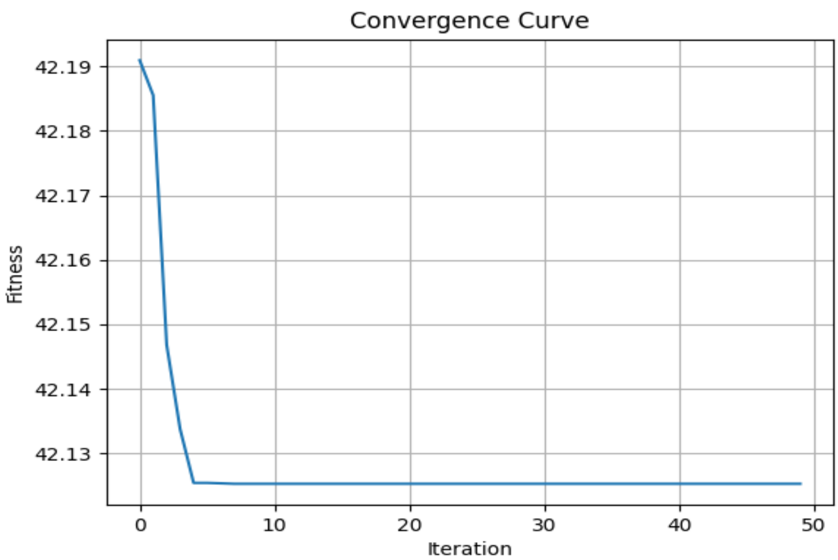


Figure 4.2: Convergence curve.

4.3.5 Sensitivity Analysis

The low and high levels of input parameters are given in Table 4.1. These levels of input parameters are arbitrarily chosen which is a common approach used in research articles that deal with the economic design of the control charts.

Table 4.1: Low and High Levels of Input Parameters

Input Parameter	Low Level	High Level
C_0	10	30
C_1	20	60
C_2	15	45
C_3	7.5	22.5
d	0.5	2.5
y	1	10
δ	1	2
λ	0.01	0.05
t	0.05	1
T_0	0.05	2.5
T_1	0.05	2.5
T_2	0.25	7.5
γ_1	0	1
γ_2	0	1

Table 4.2 contains the optimal values of output parameters with a one-factor-at-a-time change in input parameters given in Table 4.1. The corresponding convergence curves for each parameter given in Table 4.2 are provided in Appendix A.

Table 4.2: Effect of one-factor-at-a-time change in input parameters on C_E

Input Parameter	C_E	n	h	k
None	11.61	1	4.45	1.77
C_0	26.35	2	8.00	2.75
C_1	13.66	1	1.90	1.83
C_2	12.14	1	4.19	2.17
C_3	11.74	1	4.49	1.77
d	11.89	2	7.96	1.38
y	12.76	2	8.00	1.37
δ	11.11	2	4.74	2.17
λ	13.45	2	3.34	1.43
t	11.68	1	4.48	1.74
T_0	11.61	2	4.45	1.83
T_1	11.61	1	4.45	1.79
T_2	11.61	1	4.45	1.82
γ_1	11.61	1	4.45	1.82
γ_2	11.63	1	4.46	1.78

* "None" represents the state when each input variable is at its pertinent low level. The value of C_E against "None" is pivotal for comparison when input parameters are changed from their respective low level to high level.

Based on Table 4.2, which illustrates the effect of one-factor-at-a-time changes in input parameters on C_E, following conclusions can be drawn:

(1) Different input parameters have varying degrees of impact on the output C_E.

For instance, changing the parameter C_0 from its low-level to high-level results in a substantial increase in C_E from \$11.61 to \$26.35, indicating a high sensitivity to C_0.

(2) Other than C_0, the input parameters like C_1, λ, y, C_2, δ, and d exhibit a significant change in C_E.

(3) Input parameters like C_3, t, T_0, T_1, T_2, γ_1, and γ_2 show negligible variations in C_E. This suggests that marginal changes in these input parameters may not significantly affect C_E.

4.3.6 Conclusion

In this Section, we have considered only the economic aspects of the u-chart for the first time. We have found the optimal parameters n, h, and k while minimizing C_E. The corresponding algorithm for finding the optimal parameters of the pertinent economic statistical design of u-chart is given in Section 1.3.2.

4.4 Multi-objective Economic Statistical Design of u-chart

The assumptions and expected cost per cycle required for proposing the multi-objective economic statistical design of the u-chart are the same as mentioned in Sections 4.3.1 and 4.3.2, respectively.

4.4.1 Multi-objective Economic Statistical Design

In this Section, we are proposing a multi-objective design for a u-chart with two objective functions. We aim to minimize C_E as well as ARL_δ simultaneously. The multi-objective design is given as follows:

$$Min\ C_E$$

$$Min\ ARL_\delta$$

$$\begin{aligned} s.t. \quad ARL_0 &\geq ARL_L \\ ATS &\leq ATS_u \\ and\ n &\in Z^+ \ . \end{aligned} \tag{4.10}$$

where h/P is ATS when an assignable cause occurs. ARL_L and ATS_u are the respective lower and upper bounds of ARL and ATS.

It has already been mentioned that to design a control chart, one requires three decision variables, namely, n, h, and k. NSGA II method has been used to determine the optimal parameters for the multi-objective design of the u-chart given in equation (4.10).

4.4.2 A Numerical Example

In this Section, the application of the proposed model for finding the optimal parameters of the multi-objective economic statistical design of u-chart is shown through a numerical example.

To illustrate the results of the proposed model, the required parameters and constraints used in the multi-objective economic statistical design of u-chart are provided here. The mean number of nonconformities per unit is $u_0 = 6.36$ and the rate of occurrence of assignable cause is $\lambda = 0.01$ per hour. Once there is an assignable cause in the process, the mean shifts with a size of $\delta = 2$. The rest of the parameters are: $C_0 = \$40$, $C_1 = \$60$, $C_2 = \$25$, $C_3 = \$12.5$, $d = \$1$, $y = \$0.1$, $t = 0.05$ hours, $T_0 = 0.5$ hours, $T_1 = 0.5$ hours, $T_2 = 1.5$ hours, $ARL_L = 370$ and $ATS_u = 5$. Many of these parameters are taken from Amiri and Jafarian-Namin [3] and the rest are adjusted accordingly. Furthermore, it is assumed that the production continues during the detection and elimination of the assignable causes for the process. The optimal set of solutions is given in Table 4.3. The corresponding optimal Pareto front is given in Figure 4.3.

Table 4.3: The Non-dominated Set of Optimal Parameters

C_E	ARL_δ	n	h	k
42.13	1.70	1	3.76	2.20
42.19	1.70	1	3.93	2.11
42.23	1.42	1	5.28	1.79
42.38	1.42	1	5.78	1.65
42.46	1.25	1	6.98	1.40
42.68	1.24	1	7.36	1.16
42.98	1.13	1	6.75	1.01
43.33	1.12	2	7.33	1.18
43.81	1.05	3	7.19	1.09
44.42	1.04	5	7.19	1.16
47.33	1.00	17	5.28	1.03

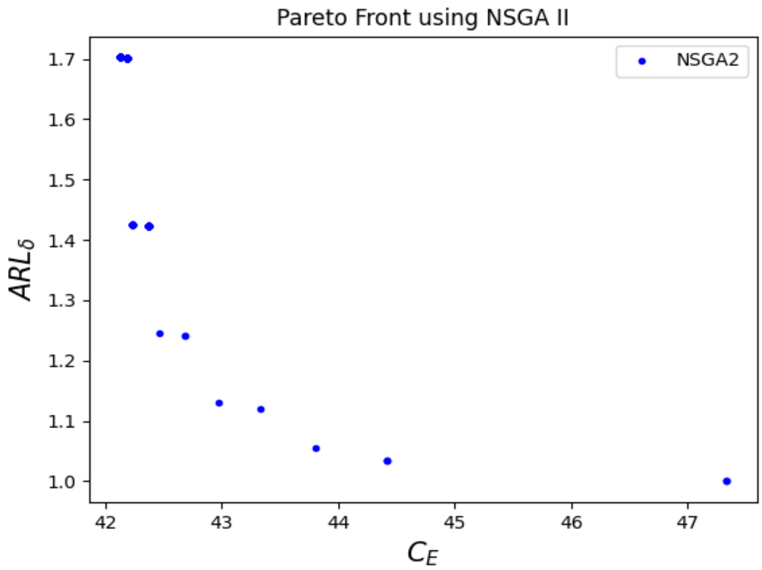


Figure 4.3: Optimal Pareto Front.

4.4.3 Sensitivity Analysis

The low and high levels of input parameters are the same as given in Table 4.1.

Table 4.4 contains three different sets of values of C_E and ARL_δ and the corresponding optimal values of n, h, and k when each input variable given in Table 4.1 is changed one-factor-at-a-time from low level to high level and the optimal Pareto fronts corresponding to each parameter provided in Table 4.4 are given in Appendix B. These three sets are (i) when C_E is minimum (ii) median values of C_E and ARL_δ, and (iii) when ARL_δ is minimum.

Table 4.4: Effect of one-factor-at-a-time change in input parameters on C_E and ARL_δ

Input Parameter	C_E	ARL_δ	n	h	k
	11.61	2.51	1	4.45	1.79
None	11.94	1.43	2	7.54	1.38
	15.17	1.00	18	6.92	1.06
	26.35	8.64	1	7.85	2.67
C_0	29.66	1.87	1	7.95	1.37
	33.63	1.00	17	7.79	1.04
	13.66	2.51	1	1.89	1.74
C_1	14.19	1.43	2	4.80	1.35
	18.02	1.00	20	5.67	1.13
	12.14	3.57	1	4.19	2.17
C_2	13.06	1.64	2	7.74	1.38
	19.12	1.00	18	6.97	1.06
	11.74	2.51	1	4.49	1.83
C_3	12.07	1.43	2	7.60	1.31
	15.71	1.00	18	6.36	1.06
	11.89	1.87	1	7.96	1.40
d	12.23	1.39	2	7.96	1.28
	14.80	1.00	17	7.79	1.03
	12.76	1.87	1	7.99	1.36
y	15.63	1.19	3	7.96	1.07
	34.11	1.00	17	7.81	1.03
	11.11	1.70	1	4.74	2.13
δ	11.51	1.19	2	7.69	1.38
	15.17	1.00	19	7.15	1.11
	13.45	1.87	1	3.34	1.39
λ	13.75	1.43	2	5.68	1.29
	17.05	1.00	18	5.75	1.05
	11.64	2.51	1	4.47	1.74
t	11.94	1.43	2	7.31	1.17
	16.38	1.00	18	6.01	1.06
	11.61	2.51	1	4.45	1.79
T_0	11.93	1.43	2	7.62	1.34
	15.30	1.00	17	6.50	1.02
	11.61	2.51	1	4.45	1.81
T_1	11.93	1.43	2	7.59	1.34
	16.36	1.00	19	5.66	1.09
	11.61	1.87	1	6.64	1.36
T_2	12.00	1.39	2	7.76	1.32
	15.63	1.00	16	5.88	1.01
	11.61	2.51	1	4.45	1.83
γ_1	12.00	1.43	2	7.05	1.33
	16.02	1.00	20	6.21	1.11
	11.63	1.87	1	6.65	1.41
γ_2	12.04	1.39	2	7.58	1.31
	14.70	1.00	17	7.52	1.03

* “None” represents the state when each input variable is at its pertinent low level.

The detailed analysis based on the values given in Table 4.4 reveals that whenever the input parameters are changed from their respective low to high levels, given in Table 4.1, the minimum C_E either increases or remains same for most of the input parameters (93%) and the maximum ARL_δ either decreases or remains same for most of the input parameters (93%).

The maximum change in the minimum C_E is observed when the parameter C_0 is increased from its pertinent low level to high level leading to an increase of 127% in the C_E value.

Similarly, the maximum change in the maximum ARL_δ is observed for C_0 leading to an increase of 244%.

The maximum decrease in the minimum C_E is observed when δ is changed from its pertinent low level to high level leading to a decrease of 4.3%.

Last but certainly not the least, the maximum decrease in the maximum ARL_δ is observed for δ leading to a decrease of 32.3% for the corresponding change δ from low level to high level.

In summary, the systematic exploration of one-factor-at-a-time change in input variables in Table 4.4 provides a clear understanding of how adjustments to each input parameter influence or impact both C_E and ARL_δ. It consolidates the findings on the minimum, median, and maximum values of C_E and the corresponding maximum, median, and minimum values of ARL_δ obtained from Optimal Pareto front emerged through the application of NSGA II.

4.4.4 Parameters Emerged for u-chart

In this Section, three sets of central lines and control limits corresponding to the optimal sets of parameters have been recommended with respect to the numerical example given in Section 4.4.2.

(1) When C_E is minimum:- For minimum C_E, the optimum values are: $n = 1$, $h = 3.76$ hours, and $k = 2.20$. Consequently,

$$UCL = \bar{u} + 2.20\sqrt{\bar{u}/1}$$
$$CL = \bar{u}$$
$$LCL = \bar{u} - 2.20\sqrt{\bar{u}/1}$$

where $\bar{u} = \frac{\sum_{i=1}^{m} u_i}{m}$; m is the number of samples or subgroups (preferably 24 or 25) that need to be collected maintaining h = 3.76 hours between consecutive subgroups. The pertinent ARL_δ is 1.70.

(2) Median values of C_E and ARL_δ:- For median values of C_E and ARL_δ, the optimum values are: n = 1, h = 7.36 hours, and k = 1.16. Consequently,

$$UCL = \bar{u} + 1.16\sqrt{\bar{u}/1}$$
$$CL = \bar{u}$$
$$LCL = \bar{u} - 1.16\sqrt{\bar{u}/1}$$

where $\bar{u} = \frac{\sum_{i=1}^{m} u_i}{m}$; m is the number of samples or subgroups that need to be collected maintaining h = 7.36 hours between consecutive subgroups. The pertinent ARL_δ is 1.24.

(3) When ARL_δ is minimum:- For minimum ARL_δ, the optimum values are: n = 17, h = 5.28 hours, and k = 1.03. Consequently,

$$UCL = \bar{u} + 1.03\sqrt{\bar{u}/17}$$
$$CL = \bar{u}$$
$$LCL = \bar{u} - 1.03\sqrt{\bar{u}/17}$$

where $\bar{u} = \frac{\sum_{i=1}^{m} u_i}{m}$; m is the number of samples or subgroups that need to be collected maintaining h = 5.28 hours between consecutive subgroups. The pertinent ARL_δ is 1.00.

4.4.5 Conclusion

Considering the significance of both statistical aspects (ARL_δ) and economic aspects (C_E), the selection of the optimal set of parameters should be based on the specific requirements. Depending on the particular objectives and priorities, individuals or organizations can choose the optimal set of parameters that aligns most closely with their needs. If the organization gives preference to economic considerations, then it should use that set of optimal parameters for which C_E is minimum. On the contrary, if the organization gives preference to statistical aspects, then it should choose the set of optimal parameters for which ARL_δ is minimum. The corresponding algorithm for finding the optimal parameters of the pertinent design of u-chart is given in Section 1.3.3.

4.5 Proposition

Organizations or individuals should use the economic statistical design approach when their principal objective is cost minimization. However, in this case, statistical performances like ARL_0, ARL_δ, ATS, Type I error, and Type II error need to be considered as constraints rather than principal objectives.

Contrarily, when organizations or individuals require a strict balance between economic efficiency and statistical performance to meet quality requirements, they need to use the multi-objective economic statistical design approach. In this case, both cost minimization and statistical performance measures simultaneously need to be considered as principal objectives.

Chapter 5

Optimal Parameters of the Economic Statistical Design of the p-chart and the Multi-objective Economic Statistical Design of the p-chart[1]

5.1 Introduction

As discussed in previous Chapters, control charts are statistical process control techniques that monitor the relevant characteristics of a process and detect any shifts or drifts over time of the sample statistics being plotted. Based on the type of data being monitored, the control charts are divided into two categories: attribute control charts and variable control charts. Attribute control charts are those control charts that are designed to deal with defects and defectives. On the other hand, variable control charts are those control charts that treat measurable product characteristics. Several attribute control charts are available encompassing p-chart, np-chart, c-chart, u-chart, and demerit control chart. Among these control charts, the p-chart has substantial importance in statistical process control and it is used to monitor the proportion or percentage of non-conforming items or defectives within a sample over a period of time for different samples. The primary purpose of a p-chart is to identify whether a process is in control or out of control, based on the proportion of non-conforming items or defectives. This aids organizations in identifying the main cause of the concerned quality problem and

[1]The Multi-objective Economic Statistical Design of the p-chart. *Communications in Statistics - Simulation and Computation*, 1-18, 2024.

taking necessary corrective action to restore the status quo. The p-chart can be particularly useful when the sample size varies from sample to sample, wherein the calculation of control limits is based on the sample size and proportion of non-conforming items or defectives. The p-chart is often found to be a valuable tool in industries such as healthcare, manufacturing, and other service sectors, where appropriate control over the pertinent process is called for to maximize the yield.

Six Sigma is a way in quality management to improve statistical methods within a structured methodology in problem-solving in a business organization to achieve better, faster, and less expensive products in a competitive environment. The popularly used structured methodology consists of a sequence of activities in projects like Define, Measure, Analyse Improve, and Control. The control chart, in general, and the attribute control chart called p-chart, in particular, are useful tools in the control phase while implementing Six Sigma. The important questions for implementing a control chart including the p-chart are associated with the determination of optimal parameters like the sample size (n), the sampling interval (h), and the control limits' multiplier (k). This Chapter would help one from this broader perspective of industrial engineering and its implementation. In order to determine the values of these parameters for a control chart, the economic design of various types of control charts has been studied by many researchers. For instance, Duncan [25] proposed the economic design of the $\bar{X}$-chart. Similarly, the economic design of the np-chart was studied by Ladany [47], which was further investigated by several researchers, like Heikes et al. [34], Duncan [26], Gibra [30], Williams et al. [98], Kooli and Limam [45], and Kooli and Limam [46]. The robust economic design of np-chart under different scenarios comprising different processes and economic parameters was given by Attia and Abdel-Aal [7].

The studies pertaining to np-chart had aimed at determining the optimal values for the control chart parameters, like the sample size, sampling interval, and acceptance number. However, Duncan [25] obtained the value of the control limits' multiplier instead of the acceptance number, in order to minimize the corresponding cost. The objective was to strike a balance between the cost of inspection and sampling with the cost of false alarms enabling to reduction of the cost of producing non-conforming products. In fact, by adopting the economic design approach, the optimum levels of the important control chart parameters are obtained so that the corresponding overall cost associated with exercising statistical process control procedure gets minimized.

Although economic design is cost-effective, it has certain limitations related to statistical properties. Notably, it has not adequately considered the Type I error probability, Type II error probability (or its complement, power), and ARL while determining the desired levels of control chart parameters. Consequently, it may lead to an increase in the probability of Type I error resulting in more number of false alarms than expected, as observed by Woodall [99]. This increased Type I error probability would cause an over-adjustment of the process resulting in increased dispersion of the concerned quality characteristic. To tackle these issues, Saniga [78] proposed an approach, referred to as the economic statistical design, that imposes bounds on Type I and Type II error probabilities and ATS. This approach preserves the cost-effectiveness of economic designs and ensures a very small probability of false alarms along with incorrect adjustments. Thus, it would satisfy the industry's long-cherished demands for low process variability as well as long-term quality. Based on the approach introduced by Saniga [78], an economic statistical design for the S control chart using Taguchi's loss function was proposed by Yang [103].

The single objective economic statistical design and multi-objective economic statistical design of control charts other than the p-chart have already been studied by many researchers, some of which are mentioned here. The economic design of a moving average control chart for non-normal data using variable sampling intervals was studied by Patil and Shirke [69]. Huang [40] proposed an economic design of max charts using Taguchi's loss function. An economic-statistical design of $\bar{X}$ control charts with multiple assignable causes was put forth by Yu et al. [106]. A general model for the economic-statistical design of adaptive control charts for processes with multiple assignable causes was introduced by Nenes et al. [67]. An optimum variable-dimension EWMA chart for multivariate statistical process control was studied by Epprecht et al. [27]. An economic design of residuals MEWMA control chart with variable sampling intervals and sample size was proposed by Xue et al. [102]. The economic design approach for an SPC inspection procedure implementing the adaptive c-chart was given by Lupo [54]. The economic design of the CUSUM control charts for monitoring a process with correlated samples was given by Lee [48]. Najafi et al. [63] introduced the economic-statistical design of the CUSUM control chart under the exponential shock model. Torng et al. [91] proposed an economic-statistical design of the joint $\bar{X}$ and S control charts with double sampling and variable sampling intervals. The economic-statistical design of the control chart with runs rules for correlation within the sample was proposed by Lee and Khoo [50]. The economic-statistical design of the np control chart with variable sample size and sampling interval was given by Fallahnezhad et al. [28]. Yang et al. [104] used a multi-objective particle swarm optimization algorithm to develop a multi-objective model for the optimal design of $\bar{X}$ and S control charts. Safaei et al. [73] studied the multi-objective economic statistical design of $\bar{X}$ control chart considering Taguchi's loss function. Faraz and Saniga [29] examined a bi-objective optimization model for the economic-statistical design of the $\bar{X}$ and S^2 control charts. Lupo [56] developed a multi-objective optimization model for the optimal design of a c-chart. Bashiri et al. [9] proposed a multi-objective economic-statistical design for the cumulative count of conforming control chart, while Amiri et al. [4] considered a multi-objective economical-statistical design of the EWMA chart using two meta-heuristics. Morabi et al. [61] presented a multi-objective optimization model for designing an $\bar{X}$ control chart with fuzzy parameters to monitor the process mean.

It is worth noting here that while arriving at the p-chart parameters for the economic statistical and multi-objective economic statistical designs, the parameter k has been considered by us but hitherto not considered by others in the case of np-chart. What is more important is that the discourse had been restricted to the control chart for number defective, i.e., np-chart, where without loss of generality sample size essentially remains constant. However, there is no guarantee that the sample size would remain constant from sample to sample or subgroup to subgroup. Therefore, in order to come out of these lacunae of dealing with the different kinds of designs of attribute control charts, the following aspects have been addressed in this Chapter:

(1) An economic statistical design of a p-chart has been considered and its relevant

parameters are determined using a GA. The pseudocode used to implement the GA is given in Section 1.3.2 of Chapter 1

(2) k has also been considered in addition to the bounds given on ARL_0, ARL_δ, and ATS. C_E has been considered as the only objective function in the case of the economic statistical design of the p-chart.

(3) A multi-objective economic statistical design of a p-chart has been considered and relevant parameters of the multi-objective design are determined using the NSGA II. The pseudocode used to implement the NSGA II is given in Section 1.3.3 of Chapter 1.

(4) C_E and ARL_δ are considered as the objective functions in the case of the multi-objective economic statistical design of the p-chart in addition to the bounds given on ARL_0 and ATS.

The objective of this Chapter is to determine the optimal values of parameters n, h, and k of the economic statistical and multi-objective economic statistical designs of the p-chart. GA and NSGA II are used to find the optimal values of the parameters of both designs of the p-chart since GA and NSGA II both are powerful tools on account of their ability to handle complex optimization problems like the economic statistical design and multi-objective economic statistical designs of the p-chart taking into account many variables together as well as the non-linear constraints.

The rest of the Chapter is organized like this. Some assumptions and basic properties are given in Section 5.2. Section 5.3 deals with the economic statistical design of the p-chart. The economic statistical design of the p-chart has been proposed in Section 5.3.3 by taking into account assumptions for the model given in Section 5.2, the production cycle given in Section 5.3.1, the expected cost per cycle given in Section 5.3.2. In order to demonstrate the applicability, numerical examples for the proposed approach and comparison with the approach proposed by Fallahnezhad et al. [28] are provided in Section 5.3.4. The results of the sensitivity analysis are exhibited in Section 5.3.5. The conclusion based on the economic statistical design is given in Section 5.3.6. Section 5.4 deals with the multi-objective economic statistical design of the p-chart. The multi-objective economic statistical design has been proposed in Section 5.4.1. The aspects of applicability for the multi-objective economic statistical design are demonstrated through the numerical examples in Section 5.4.2. The results of the sensitivity analysis are provided in Section 5.4.3. The conclusion based on the multi-objective economic statistical design is given in Section 5.3.4. The decision-makers need to use the proposition given in Section 5.5 whenever they need to choose one out of the two approaches. The content related to the multi-objective economic statistical design of the p-chart can be found in Sandeep and Mukhopadhyay [75].

5.2 The Reason for Choosing p-chart

When an item or unit is produced, it will either be defective or non-defective. Suppose, the probability of producing a defective item is p_0. Since the underlying random variable considers only binary values, 0 or 1, it will follow a Bernoulli distribution. Its mean will be p_0 and standard deviation will be $\sqrt{p_0(1-p_0)}$. For a shift of δ amount in the process, the probability of generating a defective item increases to $p_1 = p_0 + \delta\sqrt{p_0(1-p_0)}$.

In fact, p is the ratio of the non-conforming items in a population to the total number of items in that population. The items may have several quality characteristics that are being examined simultaneously. If the item or unit does not conform to the standard on one or more of these characteristics, it is declared as non-conforming. The underlying statistical principles for a fraction non-conforming chart or p-chart are based on the binomial distribution. In a stable production process, the probability that any item will not conform to specifications is p, and that successive items or units produced may reasonably be assumed to be independent. Then each unit or item produced can be conceived to be a realization of a Bernoulli random variable with parameter p. If a random sample of n units or items of a product is selected, and if x is the number of units or items that are non-conforming then x has a binomial distribution with parameters n and p such that $P(x) = \binom{n}{x}p^x(1-p)^{n-x}$ where $x = 0, 1, 2, ..., n$. Consequently, Type I error (α), and Type II error (β) can be expressed as:

$$\begin{aligned}
\alpha &= 1 - P(LCL \leq \hat{p} \leq UCL | p = p_0) \\
\Rightarrow \alpha &= 1 - P(LCL \leq \frac{x}{n} \leq UCL | p = p_0) \\
\Rightarrow \alpha &= 1 - P(nLCL \leq x \leq nUCL | p = p_0)
\end{aligned}$$

$$\Rightarrow \alpha = 1 - \sum_{x=nLCL}^{nUCL} \binom{n}{x} p_0^x (1-p_0)^{n-x} \ . \tag{5.1}$$

$$\begin{aligned}
\beta &= 1 - P(nLCL \leq x \leq nUCL | p = p_1) \\
\Rightarrow \beta &= \sum_{x=nLCL}^{nUCL} \binom{n}{x} p_1^x (1-p_1)^{n-x} \ .
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\text{Thus, power } (P) &= 1 - \beta \\
\Rightarrow P &= 1 - \sum_{x=nLCL}^{nUCL} \binom{n}{x} p_1^x (1-p_1)^{n-x} \ .
\end{aligned} \tag{5.3}$$

where $LCL = max(0, [p_0 - k\sqrt{p_0(1-p_0)/n} + 1])$, $UCL = [p_0 + k\sqrt{p_0(1-p_0)/n}]$, and [x] is the greatest integer less than or equal to x.

It is crucial to note that both p-chart and np-chart deal with lower-the-better characteristics that exhibit stochastic fluctuations, with the desired value of zero. Consequently, the upper control limit plays a major role in determining an assignable cause if a sample point exceeds it. However, to avoid the potential presence of inspection bias, which can be considered as an assignable or special cause of variation in certain instances, it is advised to have a positive lower control limit for the p-chart depending on n, which further depends on p and k. This work has kept this issue in its perspective during the formulation of the algorithm and the corresponding code.

The implementation of control charts involves several costs like the cost of collection and inspection of samples, the cost of detecting an assignable cause, the cost of producing non-conforming products, and the cost associated with false alarms by wrongly declaring a process out-of-control when in reality it remains in-control in a production system.

Additionally, when the production process is halted while searching for an assignable cause, the downtime cost of the production system is also taken into account. A solid approach to compute such costs was given by Duncan [25]. Lorenzen and Vance [52] provided an economic model which can be applied to several control charts.

In this Chapter, we have considered a pragmatic break-up of the cost of collection and inspection of samples from the contemporary industrial engineering perspective as consisting of sampling cost, sample preparation cost, testing cost, energy cost, consumables cost, and reporting cost. It is to be noted that for a particular situation, if any component of this break-up is not applicable, it is to be presumed as zero.

5.3 An Economic Statistical Design of p-chart

In this Section, certain assumptions given in Section 5.2, the production cycle, and the expected cost per cycle have been considered for proposing an economic statistical design of p-chart.

5.3.1 The Production Cycle

There are four states in the production cycle of a system, namely, the in-control state, i.e., P_{in}, the out-of-control state, i.e., P_{out}, the time required to find the assignable cause (T_1), and the time required to remove the assignable cause (T_2). Subsequent to the removal of an assignable cause of variation, the expected length of the production cycle, i.e., T_δ is the sum of the states P_{in}, P_{out}, T_1, and T_2.

The duration of an in-control state is given by:

$$P_{in} = \frac{1}{\lambda} + \frac{(1-\gamma_1).s.T_0}{ARL_0}\ . \tag{5.4}$$

The expected number of occurrences of the assignable cause per unit of time is denoted by λ. The effect of the false alarm in the in-control state is represented by the second term in equation (5.4). The expected number of samples taken when the process is in-control state is denoted by s, and T_0 is the expected time associated with a false alarm. Considering the r^{th} and $(r+1)^{th}$ samples and corresponding sampling interval as h hours, the expression of s is given as follows:

$$\begin{aligned} s &= \sum_{r=0}^{\infty} \int_{rh}^{(r+1)h} r\lambda e^{-\lambda t} dt \\ \Rightarrow s &= \frac{1}{e^{\lambda h} - 1} \ . \end{aligned} \tag{5.5}$$

As mentioned earlier, τ denotes the average time taken by an assignable cause to occur in the interval between r^{th} and $(r+1)^{th}$ sample. Thus, τ can be expressed as follows:

$$\tau = \frac{\int_{rh}^{(r+1)h} e^{-\lambda t} \lambda (t - rh) dt}{\int_{rh}^{(r+1)h} e^{-\lambda t} \lambda dt} = \frac{1 - (1 + \lambda h) e^{-\lambda h}}{\lambda (1 - e^{-\lambda h})} \ . \tag{5.6}$$

Consequently, the duration of an out-of-control state is given in equation (5.7):

$$P_{out} = (h - \tau) + h.(ARL_{\delta} - 1) + nt \ . \tag{5.7}$$

where h is the sampling interval, n is the sample size, and t is the time required to take a sample and obtain the results.

Thus, the expected production cycle time (T_{δ}) can be obtained using equations (5.4) and (5.7) along with the parameters T_1 and T_2 as given below in equation (5.8):

$$T_{\delta} = \frac{1}{\lambda} + \frac{(1 - \gamma_1).s.T_0}{ARL_0} + (h - \tau) + h.(ARL_{\delta} - 1) + nt + T_1 + T_2$$

$$\Rightarrow T_{\delta} = \frac{1}{\lambda} + \frac{(1 - \gamma_1).s.T_0}{ARL_0} - \tau + h.ARL_{\delta} + nt + T_1 + T_2 \ . \tag{5.8}$$

As usual, in-control Average Run Length (ARL_0) and out-of-control Average Run Length (ARL_{δ}) are expressed respectively through equations (5.9) and (5.10).

$$ARL_0 = \frac{1}{\alpha} \ . \tag{5.9}$$

$$ARL_\delta = \frac{1}{P} \ . \tag{5.10}$$

The values of α and P can be plugged in respectively from equations (5.1) and (5.3).

5.3.2 The Expected Cost Per Cycle

The cost components that yield the expected cost per cycle, i.e., C_E are furnished below:

(1) the cost of employing the p-chart for exercising control over a process:

$$\left(\frac{d + n\sum_{i=1}^{6} y_i}{h}\right)\left(\frac{1}{\lambda} - \tau + h.ARL_\delta + nt + \gamma_1 T_1 + T_2\right) \ . \tag{5.11}$$

(2) the cost incurred due to false alarm and the cost for detecting and eliminating the assignable cause:

$$\frac{s.C_2}{ARL_0} + C_3 \ . \tag{5.12}$$

(3) the cost incurred on account of increased fraction defectives from p_0 to p_1 when the process goes out-of-control:

$$C_p P_r (p_1 - p_0)(nt - \tau + h.ARL_\delta + \gamma_1 T_1) \ . \tag{5.13}$$

Therefore, C_E is given in equation (5.14). It is essentially the sum of cost components given in the equations (5.11), (5.12), and (5.13) with a divisor T_δ.

$$\begin{aligned} C_E = &\left[\left(\frac{d + \sum_{i=1}^{6} ny_i}{h}\right)\left(\frac{1}{\lambda} - \tau + h.ARL_\delta + nt + \gamma_1 T_1 + T_2\right)\right. \\ &\left. + \frac{s.C_2}{ARL_0} + C_3 + C_p P_r (p_1 - p_0)(nt - \tau + h.ARL_\delta + \gamma_1 T_1)\right] / T_\delta \ . \end{aligned} \tag{5.14}$$

where C_2 is the cost for finding an assignable cause when there is none, C_3 is the average cost for finding and removing an assignable cause, C_p is the penalty incurred per defective item, d is the cost per sample for maintaining p-chart in a process, y_1 is the sampling cost per unit, y_2 is the sample preparation cost per unit, y_3 is the testing cost per unit, y_4 is the energy cost per unit, y_5 is the consumables cost per unit, y_6 is the reporting cost per unit, P_r is the rate of production, and γ_1 is a binary dummy variable.

5.3.3 The Economic Statistical Design of p-chart

The formulation of the economic statistical design, thus conceived by us, for arriving at the levels of important decision variables, namely, n, h, and k, is delineated hereunder:

$$\begin{aligned} & Min\ C_E \\ s.t. \quad & ARL_0 \geq ARL_L \\ & ARL_\delta \leq ARL_u \\ & ATS \leq ATS_u \\ & and\ n \in Z^+ \ . \end{aligned} \tag{5.15}$$

where ATS equals h/P. The ARL_L, ARL_u, and ATS_u are the respective lower and upper bounds of ARL_0, ARL_δ and ATS.

The GA has been used for solving the mathematical formulation (5.15). Although GA has been used to find the relevant optimal parameters of various other control charts, however, it has not been used so far for the economic statistical design in the realm of p-chart. In this endeavor of ours, GA has been used to explore the optimal parameters for the economic statistical design of p-chart.

5.3.4 Numerical Examples

In this section, the application of the formulated model to determine the optimal parameters of the economic statistical design of the p-chart has been illustrated through a few examples.

Example 5.1. A foundry operates a production line, churning out 84 castings every hour. They periodically collect samples of the molten iron and analyze the cooling curve to gauge the carbon-silicate content in the castings. There's a quality standard in place to ensure the carbon-silicate level doesn't exceed a certain threshold, which could lead to castings with low tensile strength. The process of taking these samples costs \$4.22 per sample and consumes around five minutes for each sample. If a casting doesn't meet the quality standard (a non-conforming item is produced), it costs the company at an average of \$100. Based on historical data, when the production process is in control, about 1.36% of the produced items are non-conforming. On average, the process remains under control for approximately 50 hours. When the production process goes out of control, it necessitates a complete system flush and restart, which takes about 45 minutes. This involves the cost of a repair crew, charged at a rate of \$22.80 per hour, and a downtime cost of \$21.34 per minute. The parameters used here are taken from Fallahnezhad et al. [28]. Fallahnezhad et al. [28] obtained these parameters from Lorenzen and Vance [52]. The parameters are as follows: $p_0 = 0.0136$, $p_1 = 0.0715$, $P_r = 84$ per hour, $C_p = \$100$, $t = 0.0833$ hours, $\lambda = 0.05$ per hour, $d = \$0$, $y_1 = \$0.50$, $y_2 = \$0.50$, $y_3 = \$0.90$, $y_4 = \$0.90$, $y_5 = \$0.90$, $y_6 = \$0.52$, $C_2 = \$977.4$, $C_3 = \$977.4$, $T_1 = 0.0833$ hours, $T_2 = 0.75$ hours, $ARL_L = 370$, $ARL_u = 2$ and $ATS_u = 4$. The optimal values of parameters thus obtained are $n = 30$, $h = 1.94$ hours, $k = 2.55$ and the corresponding value of C_E is \$153.71. The relevant convergence curve is given in Figure 5.1.

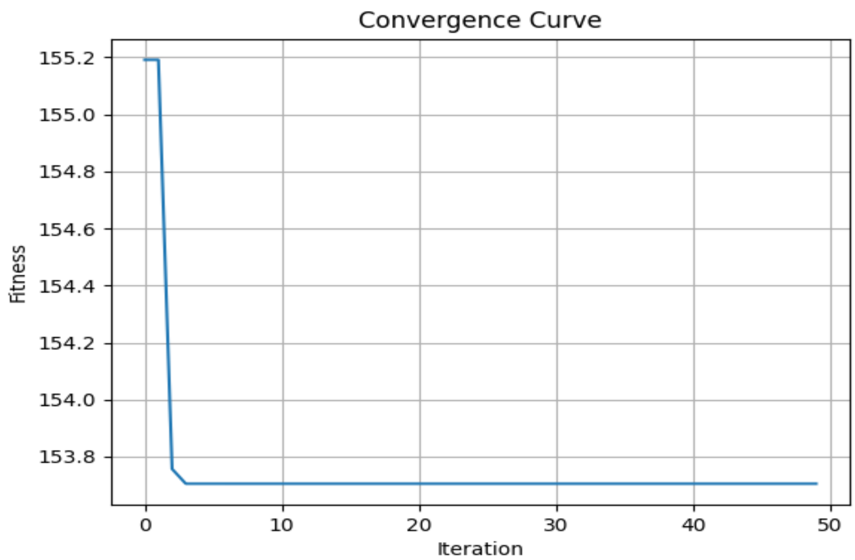


Figure 5.1: Convergence curve for Example 5.1.

Example 5.2. All the parameters of Example 5.1 are kept identical in this case. However, as a distinction, it has been considered that the process remains stopped while hunting for the assignable causes. The optimal values of parameters thus obtained are $n = 30$, $h = 8.00$ hours, $k = 1.39$, and C_E is \$217.12. The pertinent convergence curve is given in Figure 5.2.

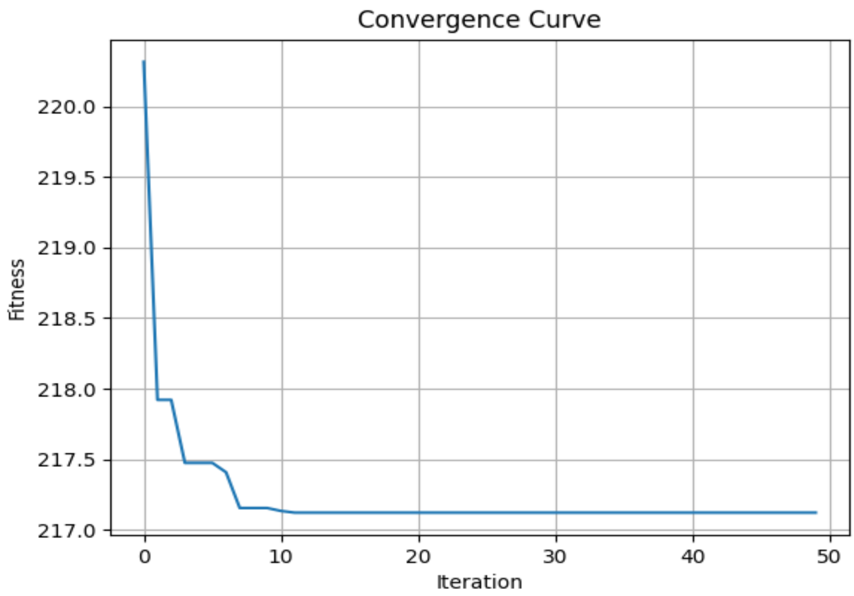
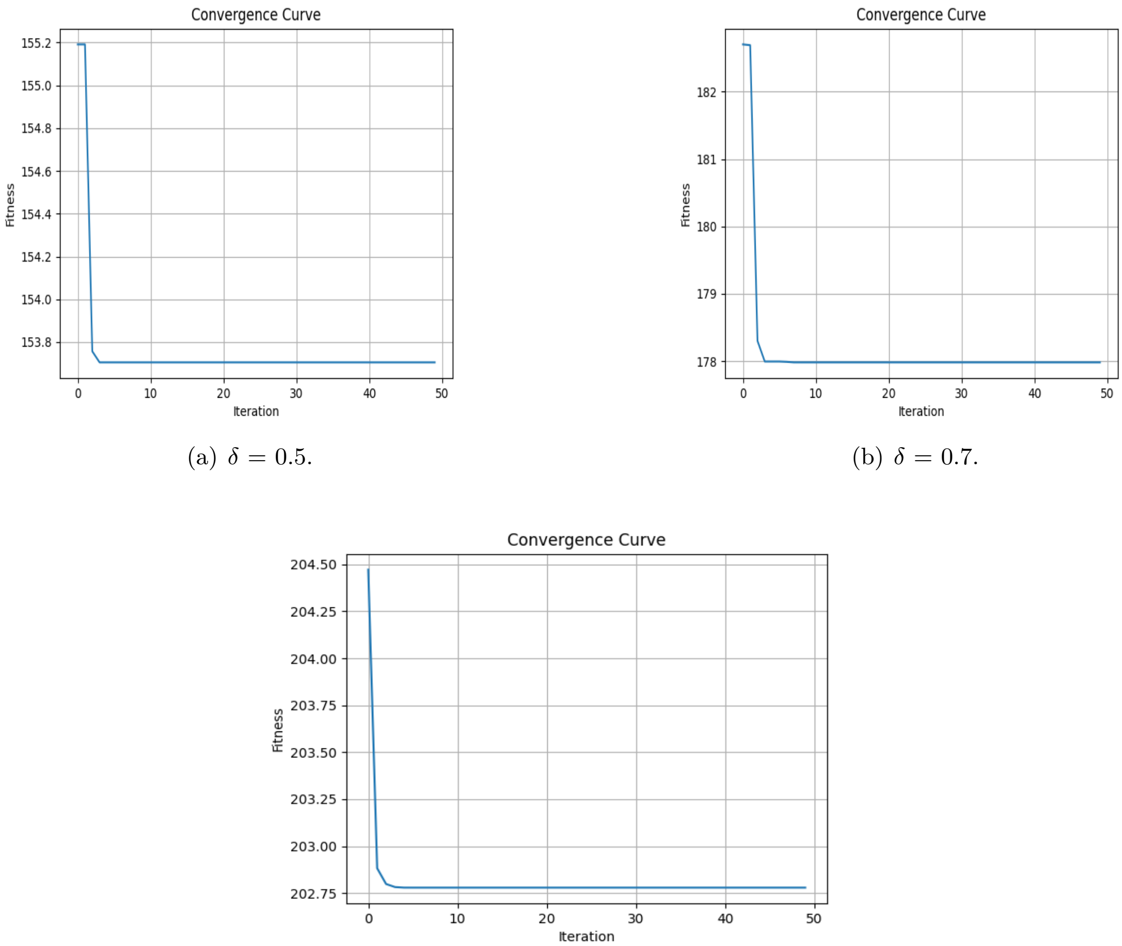


Figure 5.2: Convergence curve for Example 5.2.

Table 5.1 contains the values of C_E for different values of δ along with n, h, and k. A comparison has been made in Table 5.1 between the proposed approach and the approach given in Fallahnezhad et al. [28] with respect to the expected cost per cycle. The corresponding convergence curves for different values of δ are provided in Figure 5.3.

Table 5.1: Comparison between the proposed approach and the approach given in Fallahnezhad et al. [28].

δ	n	h	k	Proposed C_E	Fallahnezhad et al. [28] C_E
0.5	30	1.94	2.55	153.71	370.99
0.7	31	1.85	2.59	177.98	339.87
0.9	30	1.76	2.71	202.78	318.53



(a) $\delta = 0.5$.

(b) $\delta = 0.7$.

(c) $\delta = 0.9$.

Figure 5.3: Convegence curve for different values of δ.

From Table 5.1 one can clearly observe that C_E is significantly less for each value of δ for the proposed approach when compared to the approach given in Fallahnezhad et al. [28].

5.3.5 Sensitivity Analysis

The low and high levels of input parameters are given in Table 5.2. These levels for the input parameters are chosen by and large in an arbitrary manner in the backdrop of the common approach adopted in various research papers in the concerned literature dealing with the economic design of control charts.

Table 5.2: The ranges of the input parameters.

Parameter	Low level	High level
p_o	0.025	0.095
δ	0.7	1.8
λ	0.03	0.07
C_2	200	1500
C_3	100	800
C_p	100	200
d	1	8
$\sum_{i=1}^{6} y_i$	10	100
T_0	0.05	3
T_1	0.05	3
T_2	0.30	10
P_r	84	168
t	0.06	1
γ_1	0	1

Table 5.3 contains the optimal values of output parameters with a one-factor-at-a-time change in input parameters. The corresponding convergence curves for each parameter given in Table 5.3 are provided in Appendix C.

Table 5.3: Effect of one-factor-at-a-time change in input parameters on C_E and ARL_δ

Input parameter	C_E	n	h	k
None	82.64	30	0.94	2.65
p_0	131.64	30	0.63	2.82
δ	170.06	30	0.71	2.71
λ	165.12	30	0.85	2.95
C_2	113.95	30	1.91	2.69
C_3	101.97	30	0.95	2.65
C_p	143.34	31	0.66	2.72
d	82.84	31	0.93	3.00
$\sum_{i=1}^{6} y_i$	126.12	31	3.01	2.46
T_0	82.64	30	0.94	2.65
T_1	143.74	30	0.98	2.71
T_2	65.18	31	0.93	2.70
P_r	143.34	31	0.66	2.72
t	437.30	30	1.25	2.98
γ_1	179.90	30	5.21	2.44

* “None” represents the state when each input variable is at its pertinent low level. The value of C_E against “None” is pivotal for comparison when input parameters are changed from their respective low level to high level.

It is worth noting from Table 5.3 that C_E remains almost same for a one-factor-at-a-time change in the input variables d and T_0. However, C_E increases for the rest of the input variables like p_0, δ, λ, C_2, C_3, C_p, $\sum_{i=1}^{6} y_i$, T_1, P_r, t, γ_1 when they are changed from their respective low level to high level essentially on account of remaining in the numerator of equation (5.14). On the contrary, C_E decreases when T_2 is changed from low level to high level as it lies in the denominator of equation (5.14).

5.3.6 Conclusion

The proposed approach has lower values of C_E for different values of the shift assumed in a process, i.e., δ, when it is compared to the approach given in Fallahnezhad et al. [28]. Therefore, the proposed approach is more cost-effective. The optimum values of n, h, and k for different values of δ and the corresponding values of C_E are provided in Table 5.4 for ready comprehension.

Table 5.4: The optimal values of n, h, and k for different δ and corresponding C_E for p-chart.

δ	n	h	k	Proposed C_E	Fallahnezhad et al. [28] C_E
0.5	30	1.94	2.55	153.71	370.99
0.7	31	1.85	2.59	177.98	339.87
0.9	30	1.76	2.71	202.78	318.53

5.4 The Multi-objective Economic Statistical Design of p-chart

The assumptions and expected cost per cycle required for proposing the multi-objective economic statistical design of the p-chart are respectively the same as mentioned in Sections 5.2 and 5.3.2.

5.4.1 The Multi-objective Economic Statistical Design

The primary objective of multi-objective economic statistical design is to identify a solution that can achieve a balance among all the objectives. In this Section, we are proposing a multi-objective design for a p-chart with two objective functions. One of them is to minimize C_E and the other is to minimize ARL_δ. The multi-objective design is given as follows:

$$\begin{aligned} & Min\ C_E \\ & Min\ ARL_\delta \\ s.t.\quad & ARL_0 \geq ARL_L \\ & ATS \leq ATS_u \\ & and\ n \in Z^+\ . \end{aligned} \tag{5.16}$$

where ATS when an assignable cause occurs is h/P. ARL_L and ATS_u are the respective lower and upper bounds of ARL_0 and ATS.

To design a control chart, three decision variables need to be specified, namely, n, h, and k. The NSGA II method is a powerful approach for solving the equation (5.16). Although NSGA II has been utilized for various control charts, it has not yet been applied to the multi-objective economic statistical design of the p-chart. In this study, the NSGA II method is employed to search for the optimal solutions for the multi-objective design of the p-chart.

5.4.2 Numerical Examples

In this Section, we illustrate the application of the formulated model to determine the statistical parameters of the p-chart through a series of examples, showcasing the optimal approach.

Example 5.3. The required parameters are taken from Fallahnezhad et al. [28] where he used the data obtained by Lorenzen and Vance [52]. Lorenzen and Vance [52] took the data from a factory where castings are produced by melting metal, pouring liquid metal into a mold, and then allowing it to solidify. This factory operates one production line, generating 84 castings per hour. To ensure the quality of these castings, they periodically sample the molten iron and analyze the cooling curve, which is an indicator of the carbon-silicate content in the castings. The sampling process incurs a cost of \$4.22 per sample and takes approximately five minutes per sample.

In this context, it's crucial to maintain the carbon-silicate content within specified limits to prevent the production of castings with low tensile strength. When the process is under control, historical data shows that the nonconforming items are produced at a rate of 1.36%. On average, the process remains under control for approximately 50 hours before any deviations occur. In the event that the process goes out of control, it requires a system flush and restart, which takes roughly 45 minutes. The repair crew required for this task costs $22.80 per hour, and the downtime costs are incurred at a rate of $21.34 per minute. The parameters thus obtained are as follows:

$p_0 = 0.0136$, $p_1 = 0.0715$, $P_r = 84$ per hour, $C_p = \$100$, $t = 0.0833$ hours, $\lambda = 0.05$ per hours, d $= \$0$, $y_1 = \$0.50$, $y_2 = \$0.50$, $y_3 = \$0.90$, $y_4 = \$0.90$, $y_5 = \$0.90$, $y_6 = \$0.52$, $C_2 = \$977.4$, $C_3 = \$977.4$, $T_1 = 0.0833$ hours, $T_2 = 0.75$ hours, $ARL_L = 370$, and $ATS_u = 4$. The optimal set of solutions is given in Table 5.5. The corresponding optimal Pareto front using NSGA II is given in Figure 5.4.

Table 5.5: The non-dominated set of optimal parameters for Example 5.3.

C_E	ARL_δ	n	h	k
153.71	1.56	30	1.94	2.60
155.15	1.51	31	2.04	2.78
156.59	1.48	32	2.14	2.45
156.59	1.48	32	2.14	2.45
158.02	1.44	33	2.24	2.75
159.45	1.41	34	2.34	2.32
160.88	1.38	35	2.44	2.57
162.31	1.35	36	2.54	2.33
163.73	1.33	37	2.64	2.19
165.14	1.31	38	2.74	2.59
166.55	1.28	39	2.85	2.56
167.96	1.27	40	2.94	2.35
169.36	1.25	41	3.04	2.47
170.76	1.23	42	3.14	2.42
172.15	1.22	43	3.24	2.01
173.54	1.20	44	3.33	2.13
174.92	1.19	45	3.43	1.89
176.30	1.18	46	3.52	2.50
177.67	1.16	47	3.62	2.39
179.04	1.15	48	3.71	2.21
180.40	1.14	49	3.81	2.10
181.76	1.13	50	3.90	2.10
186.67	1.12	30	5.76	1.78

187.96	1.11	31	5.90	1.42
189.24	1.10	32	6.01	1.83
190.53	1.09	33	6.15	1.70
191.81	1.09	34	6.24	1.47
193.08	1.08	35	6.34	1.07
194.35	1.07	36	6.48	0.75
195.62	1.07	37	6.58	0.97
196.87	1.06	38	6.69	0.83
198.13	1.06	39	6.80	0.78
199.37	1.05	40	6.89	1.37
200.61	1.05	41	7.01	0.96
203.05	1.04	43	7.21	1.11
204.27	1.04	44	7.21	1.11
205.47	1.04	45	7.27	1.11
206.78	1.03	46	7.06	1.31
207.94	1.03	47	7.18	1.48
209.16	1.03	48	7.18	1.48
210.38	1.03	49	7.18	1.48
211.55	1.03	50	7.23	1.46
239.35	1.00	32	6.84	0.82

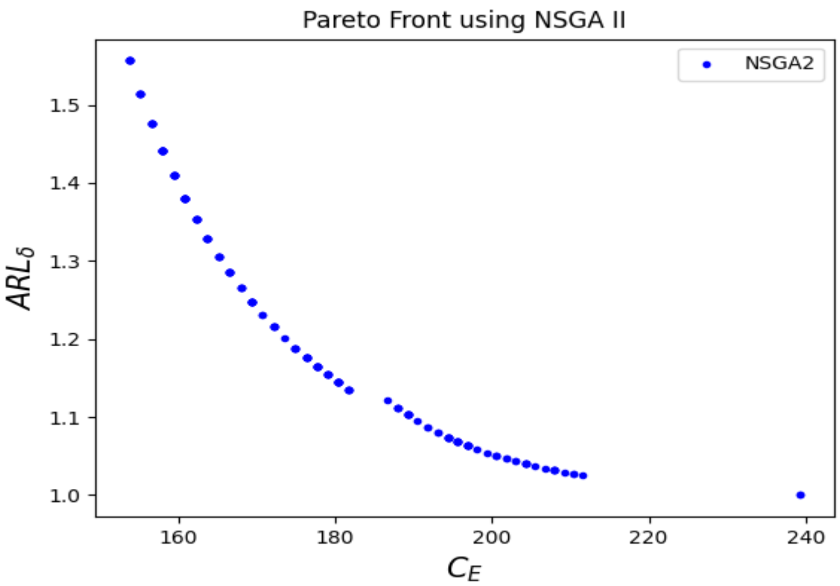


Figure 5.4: Optimal Pareto front for Example 5.3.

Example 5.4. All the parameters of Example 5.3 are kept identical in this case. However, as a distinction, it has been considered that the process remains stopped while hunting for the assignable causes. The corresponding optimal set of parameters and Pareto front using NSGA II are respectively given in Table 5.6 and Figure 5.5.

Table 5.6: The non-dominated set of optimal parameters for Example 5.4.

C_E	ARL_δ	n	h	k
217.13	1.12	30	7.96	1.50
219.19	1.11	31	7.96	1.01
221.32	1.10	32	7.97	1.24
223.54	1.09	33	7.97	1.14
225.83	1.09	34	7.97	1.23
225.83	1.09	34	7.97	1.23
228.17	1.08	35	7.98	1.33
230.59	1.07	36	7.97	1.19
233.03	1.07	37	7.97	1.35
235.55	1.06	38	7.97	1.28
238.10	1.06	39	7.98	1.10
240.70	1.05	40	7.97	1.25
243.33	1.05	41	7.97	1.25
246.00	1.05	42	7.97	1.21
248.71	1.04	43	7.97	1.46
251.44	1.04	44	7.97	1.57
254.20	1.04	45	7.97	1.73
254.29	1.00	30	7.94	0.85

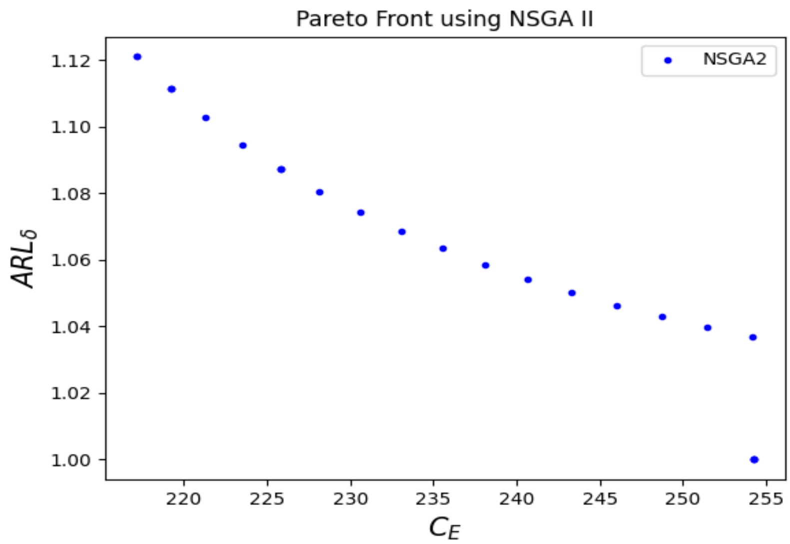
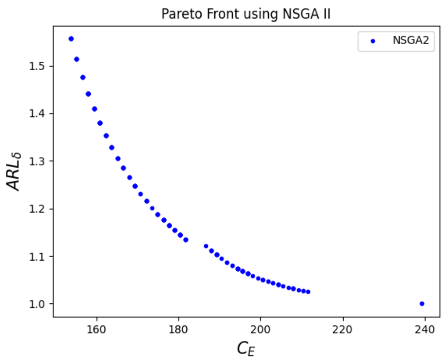


Figure 5.5: Optimal Pareto front for Example 5.4.

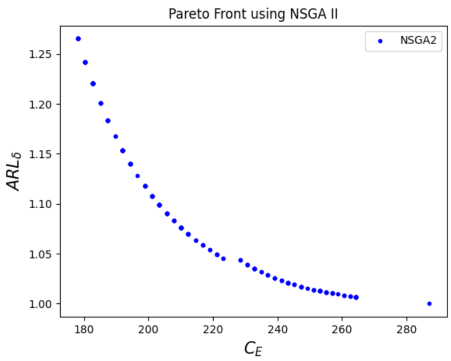
Table 5.7 contains the first quartile, i.e., Q_1, median, i.e., Q_2, third quartile, i.e., Q_3 and the values of C_E & ARL_δ for different values of δ along with n, h, and k. A comparison has been made in Table 5.7 between the proposed approach and the approach given in Fallahnezhad et al. [28] with respect to both C_E and ARL_δ. Since there is no work in literature for the multi-objective design of the p-chart, the comparison is made with the results given in Fallahnezhad et al. [28] for the multi-objective design of *np*-chart. The corresponding convergence curves for different values of δ are provided in Figure 5.6.

Table 5.7: Comparison between the proposed approach and the approach given in Fallahnezhad et al. [28].

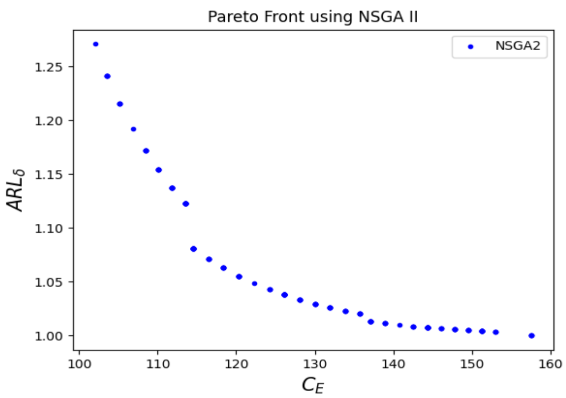
δ	Quartile	n	h	k	Proposed (ARL_δ)	Proposed (C_E)	Fallahnezhad et al. [28] (ARL_δ)	Fallahnezhad et al. [28] (C_E)
0.5	Q_1	39	2.85	2.56	1.28	166.55	2.39	370.99
	Q_2	50	3.90	2.10	1.13	181.76		
	Q_3	40	6.89	1.37	1.05	199.37		
0.7	Q_1	40	2.65	2.60	1.12	199.98	2.08	339.87
	Q_2	50	3.44	2.31	1.05	223.24		
	Q_3	40	5.85	1.23	1.02	246.26		
0.9	Q_1	39	2.39	2.42	1.05	231.76	1.87	318.53
	Q_2	49	3.01	2.49	1.02	262.14		
	Q_3	42	5.22	1.46	1.01	294.93		



(a) $\delta = 0.5$.



(b) $\delta = 0.7$.



(c) $\delta = 0.9$.

Figure 5.6: Convegence curve for different values of δ.

From Table 5.7 it is clear that both C_E and ARL_δ are less for the proposed approach when compared with the approach given in Fallahnezhad et al. [28].

5.4.3 Sensitivity Analysis

The low and high levels of input parameters are the same as given in Table 5.1.

Table 5.8 contains three different sets of values of C_E and ARL_δ and the corresponding optimal values of n, h, and k when each input variable given in Table 5.2 is changed one-factor-at-a-time from low level to high level and the corresponding optimal Pareto Fronts are given in Appendix. These three sets are (i) when C_E is minimum (ii) median values of C_E and ARL_δ, and (iii) when ARL_δ is minimum. The corresponding optimal Pareto fronts for every input value of Table 5.8 are given in Appendix D.

Table 5.8: Effect of one-factor-at-a-time change in input parameters on C_E and ARL_δ

Input parameter	C_E	ARL_δ	n	h	k
None	82.64	1.27	30	0.94	2.66
	109.20	1.03	37	2.13	1.93
	139.04	1.00	30	4.21	0.80
p_0	131.64	1.19	30	0.63	2.91
	175.11	1.01	30	1.85	1.19
	219.97	1.00	30	2.99	0.53
δ	171.00	1.00	30	0.87	2.68
	241.73	1.00	39	1.39	1.42
	276.11	1.00	30	2.55	0.83
λ	165.15	1.27	30	0.82	2.72
	196.22	1.04	36	1.61	1.61
	227.81	1.00	30	2.97	0.64
C_2	113.00	1.22	41	1.55	2.99
	162.19	1.03	49	4.06	1.95
	301.48	1.00	32	6.66	0.86
C_3	101.97	1.27	30	0.95	2.69
	128.09	1.03	37	2.16	2.06
	157.64	1.00	30	4.26	0.78
C_p	143.34	1.27	30	0.66	2.73
	187.26	1.03	37	1.50	1.50
	225.73	1.00	30	2.94	0.57
d	82.84	1.27	30	0.95	2.73
	109.29	1.03	37	2.14	1.27
	139.09	1.00	30	4.21	0.63
$\sum_{i=1}^{6} y_i$	126.12	1.08	30	3.01	2.20
	139.76	1.01	31	4.07	1.28
	154.57	1.00	30	4.99	0.76

T_0	82.64 109.20 139.04	1.27 1.03 1.00	30 37 30	0.94 2.13 4.21	2.66 1.93 0.80
T_1	143.74 167.04 193.87	1.27 1.03 1.00	30 37 30	0.98 2.22 4.36	2.70 1.64 0.62
T_2	65.18 87.34 110.54	1.27 1.03 1.00	30 38 30	0.93 2.13 4.13	2.95 1.87 0.83
P_r	143.34 187.26 225.73	1.27 1.03 1.00	30 37 30	0.66 1.50 2.94	2.73 1.50 0.57
t	437.30 450.98 461.00	1.27 1.04 1.00	30 31 30	1.25 3.23 5.52	2.69 1.69 0.36
γ_1	179.90 186.45 191.34	1.08 1.01 1.00	30 32 30	5.21 6.23 6.74	1.71 1.12 0.81

"None" represents the values of output parameters when all the input parameters are maintained at their corresponding low levels.

It is worth noting from Table 5.8 that ARL_δ remains same when the values of the input parameters are changed from low level to high level for almost every variable except p_0, δ, and C_2. C_E also remains almost same for a one-factor-at-a-time change in the variables d and T_0. However, C_E increases for the rest of the variables like p_0, δ, λ, C_2, C_3, C_p, $\sum_{i=1}^{6} y_i$, T_1, P_r, t, γ_1 as they remain in the numerator of the equation (5.14). On the contrary, C_E decreases when T_2 is changed from low level to high level as it lies in the denominator of equation (5.14).

5.4.4 Conclusion

The proposed approach has lower ARL_δ values when compared to the approach presented in Fallahnezhad et al. [28]. This suggests without loss of generality that the proposed approach detects the change more rapidly. The proposed approach also has lower values of C_E, when it is compared to the approach given in Fallahnezhad et al. [28]. Therefore, the proposed approach is more cost-effective. Combining the above two phenomena it can be concluded that the proposed approach yields a dual benefit in the form of a lower out-of-control Average Run Length for rapid detection of process abnormalities at a lower expected cost per cycle when it is compared to the approach described in Fallahnezhad et al. [28].

5.5 Proposition

When the decision-makers are primarily focused on the minimization of the cost, they should choose the economic statistical design approach and consider the statistical performances like ARL_0, ARL_δ, ATS, Type I error, and Type II error as constraints. However, the decision-makers should use the multi-objective economic statistical design approach whenever they are required to strike a balance between economic aspects and statistical aspects simultaneously.

Chapter 6

Optimal Parameters of the Multi-objective Economic Statistical Design of the CUSUM Control Chart

6.1 Introduction

Control charts are known to be a pivotal tool for exercising control over a process in statistical process control, with their origins dating back to Shewhart's pioneering work in the 1920s. The design of the control chart encompasses the critical task of selecting parameters such as sample size, sampling interval, and control limits' multiplier. These design parameters are crucial as they directly influence the quality of the control over the underlying process being monitored.

The CUSUM control chart, pioneered by Page in 1954, is a widely adopted method for exercising control over the mean of a quality characteristic in production processes. The usefulness of the techniques of the CUSUM control chart in practical applications in industry is to identify the trend of the process. If the trend is found to worsen, the concerned quality characteristic needs to be analyzed for possible assignable or special causes with or without halting the process. One can refer to the article written by Mukhopadhyay [62] where the CUSUM chart has been used for routine bias correction on the shop floor for measuring and controlling moisture content in tobacco. When compared with the $\bar{X}$-chart, the CUSUM chart demonstrates superior efficiency in detecting small to moderate shifts in the process mean. Similar to the Shewhart control chart the CUSUM control chart also needs to be implemented by collecting samples at regular intervals. The chart depicts a cumulative sum of deviations between sample means and a target value over time. As long as the computed CUSUM statistic remains within a predefined decision interval, the process is said to be in control. However, if the CUSUM statistic exceeds the decision interval, it serves as an indication that the process went out of control. Consequently, it requires further investigation to find the potential causes (assignable causes) of this change. However, there are some costs associated with finding the assignable causes, like the costs of sampling, the costs of eliminating the assignable causes, and the costs of producing nonconforming units. It is interesting to note that the cost of producing nonconforming units increases with less sampling and less effort or lower costs of eliminating assignable causes. This underscores the importance of the trade-off between sampling cost followed by the cost of eliminating assignable causes and the cost of producing nonconforming units. From an economic viewpoint, it is always better to consider the economic statistical design of the control chart.

The economic design of control charts has always been a significant area to focus on in statistical process control. The objective of economic design is to determine optimal design parameters that minimize the expected cost per hour, thereby achieving control over a process in a cost-effective manner. Duncan [25] introduced the first cost model for $\bar{X}$ chart, paving the way for subsequent research in the economic design of other control charts. A few of these are mentioned here. Lorenzen and Vance [52] proposed the unified economic design of control charts. The economic design of the control chart for sustainable operations under the gamma shock model was proposed by Wang and Li [96].

Four parameters are of paramount importance in designing a CUSUM chart. These parameters are the sample size (n), the sampling interval (h), the reference value (K), and the decision interval (H). Taylor [90] was the first researcher to explore the economic design of the CUSUM chart, laying the foundation for subsequent investigations in this area. Since then various methodologies have been proposed for the economic design of the CUSUM chart. For instance, Goel and Wu [32] developed a procedure specifically tailored for controlling the mean of a process that follows a normal distribution using a CUSUM chart.

Chiu [17] introduced a production model centered on quality surveillance, employing the CUSUM chart with decision interval criteria. Chung [20] devised a search algorithm utilizing the one-dimensional H-pattern search technique given by Hooke and Jeeves [39] for the economic design of the CUSUM chart. Simpson and Keats [82] utilized two-level fractional factorial designs to find the optimal parameters using the economic model for control charts given by Lorenzen and Vance [52] under CUSUM conditions.

Pan and Su-Tsu [68] proposed an innovative approach for monitoring and evaluating environmental performance through the economic design of the CUSUM chart. Additionally, an economic model of the CUSUM chart for controlling the process mean in short production runs was proposed by Nenes and Tagaras [66]. Lee [49] proposed the economic design of a CUSUM control chart for non-normally correlated data. Overall, the economic design of the CUSUM chart has drawn significant attention and continues to be a subject of active research, with various methodologies and applications contributing to its advancement in the field of statistical process control.

Along with the economic aspects associated with control charts, sometimes it is necessary to consider the statistical aspects like detecting the shift as early as possible. The multi-objective economic statistical design of control charts is one way to study the economic aspects along with the statistical aspects. In the literature, there are several papers dealing with the multi-objective economic design of other control charts, some of which are included here. Celano and Fichera [15] studied the multi-objective economic design of an $\bar{X}$ control chart. Yang et al. [104] used a multi-objective particle swarm optimization algorithm to develop a multi-objective model for the optimal design of $\bar{X}$ and S control charts. Safaei et al. [73] studied the multi-objective economic statistical design of $\bar{X}$ control chart considering Taguchi's loss function. Faraz and Saniga [29] examined a bi-objective optimization model for the economic-statistical design of $\bar{X}$ and S^2 control charts. Lupo [56] developed a multi-objective optimization model for the optimal design of a c-chart. Bashiri et al. [9] proposed a multi-objective economic-statistical design for the cumulative count of conforming control chart. Morabi et al. [61] presented a multi-objective optimization model for designing an $\bar{X}$ control chart with fuzzy parameters to monitor the process mean.

However, the multi-objective economic statistical design of the CUSUM chart is yet to be explored. So, in this Chapter, a multi-objective economic statistical design of the CUSUM chart is proposed using the cost model given by Lorenzen and Vance [52]. In this model, we would try to minimize C_E and ARL_δ simultaneously while maintaining a reasonably large ARL_0. Along with minimizing the two objectives, we would find out the optimal parameters n, h, and H. The optimal value of K is half of the magnitude of the shift given in σ units, as mentioned in Section 6.2. The proposed multi-objective model has been solved with the help of NSGA II which was introduced by Deb [23]. The pseudocode used to implement the NSGA II is given in Section 1.3.3 of Chapter 1.

Amiri et al. [4] considered a multi-objective economical-statistical design of the EWMA chart and solved it using NSGA II and MOGA algorithms. Mobin et al. [57] used NSGA-II algorithm to solve multi-objective design of $\bar{X}$ control chart. Zandieh et al. [107] proposed the economic-statistical design of the c-chart with multiple assignable causes and solved it using a hybrid NSGA-II approach.

The rest of the Chapter is organized like this. The sample statistic of the CUSUM chart is given in Section 6.2. The formulation of the multi-objective economic statistical design of the CUSUM chart is given in Section 6.3. The aspects of applicability for the proposed approach are demonstrated by a numerical example in Section 6.4. The results of the sensitivity analysis are provided in Section 6.5. The conclusion for the proposed approach is given in Section 6.6.

6.2 The Statistic of CUSUM Chart

Let's consider a scenario where the variation of a quality characteristic x follows the normal distribution with mean μ and standard deviation σ in the in-control state, denoted as $x \sim N(\mu, \sigma^2)$, where both μ and σ are known. However, over time, the process may transit to an out-of-control state, resulting in a shift in the mean of the quality characteristic from μ_0 (where $\mu_0 = \mu$) to $\mu_0 \pm \delta\sigma_0$, where μ_0 and σ_0 represent respectively the sample mean and the sample standard deviation. Here, δ signifies the magnitude of the shift in mean, while the standard deviation is assumed to remain constant. Samples of size n are taken after every h hours of production, with reference value K and decision interval H. Subsequently, the gathered sample information is plotted on a CUSUM chart. In the event of CUSUM statistic surpassing the decision interval, an investigation to identify and eliminate the assignable cause is initiated. The CUSUM statistic can be calculated as

$$\begin{aligned} C_i^- &= max(0, (\mu_o - K) - x_i + C_{i-1}^-) \ . \\ C_i^+ &= max(0, x_i - (\mu_o + K) + C_{i-1}^+) \ . \end{aligned} \tag{6.1}$$

where i denote the i^{th} sample and $C_0^- = C_0^+ = 0$.

The effectiveness of a control chart can be assessed using $ARLs$, which represents the average number of samples needed to detect an out-of-control condition or trigger a false alarm. The in-control ARL, i.e., ARL_0 is used for calculating the false alarm rate whereas the out-of-control ARL, i.e., ARL_δ is an indicator of the power (or effectiveness) of the control chart. One of the major difficulties in the economic design of the CUSUM chart is the evaluation of average run lengths. In this Chapter, we are utilizing approximation for calculating ARL given by Siegmund [81]. The main reason for using Siegmund's approximation is its simplicity. Woodall and Adams [101] also recommended to use approximation given by Siegmund [81]. For one-sided CUSUM (i.e., C_i^- or C_i^+) Siegmund's approximation for ARL is

$$ARL = \frac{e^{-2\Delta b} + 2\Delta b - 1}{2\Delta^2} \ . \tag{6.2}$$

For $\Delta \neq 0$, where $\Delta = -\delta - K$ for the lower one-sided CUSUM C_i^-, $\Delta = \delta - K$ for the upper one-sided CUSUM C_i^+, δ being the magnitude of process shift in σ units for which ARL needs to be calculated, $K = \frac{\delta}{2}$, and $b = H + 1.166$. For $\Delta = 0$, ARL can be calculated by b^2. Hence, the formula given in equation (6.2) can be used to determine ARL_0 when $\delta = 0$ and when $\delta \neq 0$ it can be used to calculate ARL_δ.

For $\delta = 0$, equation (6.2) can be used for calculating one-sided in-control ARL, i.e., ARL_0^- and ARL_0^+:

$$ARL_0^- = ARL_0^+ = \frac{e^{2Kb} - 2Kb - 1}{2K^2} \ . \tag{6.3}$$

Whereas for $\delta \neq 0$, equation (6.2) can be used for calculating one-sided out-of-control ARL, i.e., ARL_{δ}^{-}, and ARL_{δ}^{+}:

$$\begin{aligned} ARL_{\delta}^{-} &= \frac{e^{2\delta b+2Kb} - 2\delta b - 2Kb - 1}{2(\delta + K)^2} \; . \\ ARL_{\delta}^{+} &= \frac{e^{-2\delta b+2Kb} + 2\delta b - 2Kb - 1}{2(\delta - K)^2} \; . \end{aligned} \tag{6.4}$$

For calculating ARL of a two-sided CUSUM, one can use the formula given in equation (6.5) using two one-sided $ARLs$, i.e., ARL^{-}, and ARL^{+}:

$$\frac{1}{ARL} = \frac{1}{ARL^{-}} + \frac{1}{ARL^{+}} \; . \tag{6.5}$$

The above values of ARL_0 and ARL_{δ} are used in the multi-objective economic statistical design of the CUSUM chart defined in Section 6.3.

6.3 A Multi-objective Economic Statistical Design of CUSUM Chart

In this Section, based on certain assumptions the expected cost per cycle has been considered for proposing a multi-objective economic statistical design of the CUSUM chart.

6.3.1 Assumptions for the Model

The following assumptions are deemed valid for the proposed model:

(1) The mean of the quality characteristics is assumed to follow Normal distribution.

(2) The mean of the quality characteristic shifts from μ_0 to $\mu_0 + \delta\sigma_0$

(3) The occurrence of an assignable cause follows an exponential distribution with a mean of $1/\lambda$.

6.3.2 Expected Cost Per Cycle

In this Section, Lorenzen and Vance [52] cost model has been extended to a multi-objective economic statistical design of a CUSUM chart. The reason is that it is the most widely used statistically constrained economic model. The expected cost per cycle, i.e., C_E is:

$$C_E = \frac{\frac{C_0}{\lambda} + C_1(-\tau + nt + h(ARL_\delta) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{sC_2}{ARL_0} + C_3}{\frac{1}{\lambda} + \frac{(1-\gamma_1)sT_0}{ARL_0} - \tau + nt + h(ARL_\delta) + T_1 + T_2} + \frac{(\frac{d+ny}{h})(\frac{1}{\lambda} - \tau + nt + h(ARL_\delta) + \gamma_1 T_1 + \gamma_2 T_2)}{\frac{1}{\lambda} + \frac{(1-\gamma_1)sT_0}{ARL_0} - \tau + nt + h(ARL_\delta) + T_1 + T_2} . \tag{6.6}$$

where C_0 is the quality cost per hour when the process is in-control, C_1 is the quality cost per hour when the process is out-of-control, T_0 is the expected time associated with a false alarm, T_1 is the expected time required to discover an assignable cause, T_2 is the expected time required to eliminate an assignable cause, t is the expected time to take a sample and obtain the results, C_2 is the cost for searching an assignable cause when there is none, C_3 is the average cost of identifying and eliminating an assignable cause, d is the cost per sample for maintaining u-chart in a process, y is the variable cost of sampling an inspection unit, γ_1 is a binary variable that takes the value 1 if the production continues during the search for an assignable cause and 0 otherwise, and γ_2 is a binary variable that takes the value 1 if the production continues during the elimination of an assignable cause through intervening in the process and 0 otherwise.

τ is the expected time taken by an assignable cause to occur and can be computed using equation (6.7)

$$\begin{aligned} \tau &= \frac{\int_{rh}^{(r+1)h} e^{-\lambda t}\lambda(t - rh)dt}{\int_{rh}^{(r+1)h} e^{-\lambda t}\lambda dt} \\ \Rightarrow \tau &= \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})} \\ \Rightarrow \tau &= \frac{1}{\lambda} - \frac{h}{e^{\lambda h} - 1} . \end{aligned} \tag{6.7}$$

s: Average number of samples taken when the process is in-control. It can be computed using equation (6.8)

$$\begin{aligned} s &= \sum_{r=0}^{\infty} \int_{rh}^{(r+1)h} r\lambda e^{-\lambda t} dt \\ \Rightarrow s &= \frac{1}{e^{\lambda h} - 1} \ . \end{aligned} \tag{6.8}$$

6.3.3 Multi-objective Economic Statistical Design

A multi-objective economic statistical design of the CUSUM chart with two objectives has been proposed in this Section. Our aim here is to minimize C_E and ARL_δ simultaneously. The multi-objective economic statistical design of the CUSUM chart is given as follows:

$$\begin{aligned} Min\ C_E \\ Min\ ARL_\delta \\ s.t. \quad ARL_0 \geq ARL_L \\ ARL_\delta \leq ARL_u \\ and\ n \in Z^+ \ . \end{aligned} \tag{6.9}$$

ARL_L and ARl_u are the respective lower and upper bounds of ARL_0 and ARL_δ.

It has already been mentioned that to design a CUSUM chart, one requires three decision variables, namely, n, h, and H. NSGA II has been used to determine the optimal parameters of the multi-objective design for the CUSUM chart given in equation (6.9).

6.4 Numerical Example

In this Section, an application of the economic design of the CUSUM chart is shown with the help of an example taken from Lee [49] where he considered a hypothetical set of process and cost parameters.

He considered a factory in which yogurt drinks are produced and contained in bottles. The target quantity of yogurt drink for each bottle is 0.02 liters. The produced yogurt drink is then inserted into fifteen bottles at a time. Now, those fifteen bottles are packed in a box later. Suppose that the hourly in-control quality cost is $C_0 = \$10$ and that of an out-of-control state is $C_1 = \$100$. Since the inter-occurrence time of the assignable causes was assumed to follow an exponential distribution, let's assume that assignable causes occur with a frequency of about one every hundred hours of operation. Thus, $\lambda = 0.01$ per hour. The cost per sample for maintaining the CUSUM chart and the variable cost of sampling respectively are $d = \$0.5$ and $y = \$0.1$. The cost of investigating a false alarm, i.e., C_2 is \$50. The average cost of identifying and eliminating an assignable cause, i.e., C_3 is \$25. It takes on an average of three minutes ($t = 0.05$ hours) to take a sample and obtain the results. It requires about $T_1 = 2$ hours to discover an assignable cause, and it requires about $T_2 = 2$ hours to eliminate the assignable cause. The lower and upper control limits on ARL_0 and ARL_δ respectively are 200 and 14 units and constraints on n, h, and H respectively range from 2 to 20, 0.01 to 2, and 0.0001 to 5. Further, it is assumed that the process continues to operate while searching and elimination of an assignable cause are going on. There are 82 non-dominated solutions, so it will be difficult to give all solutions in the form of a Table. Due to this reason, Table 6.1 contains the solution in terms of percentile (i.e. 5th, 10th, 15th,....,100th) with an increment of 5 percentile beginning with the 1st percentile. The corresponding optimal Pareto front is shown in Figure 6.1.

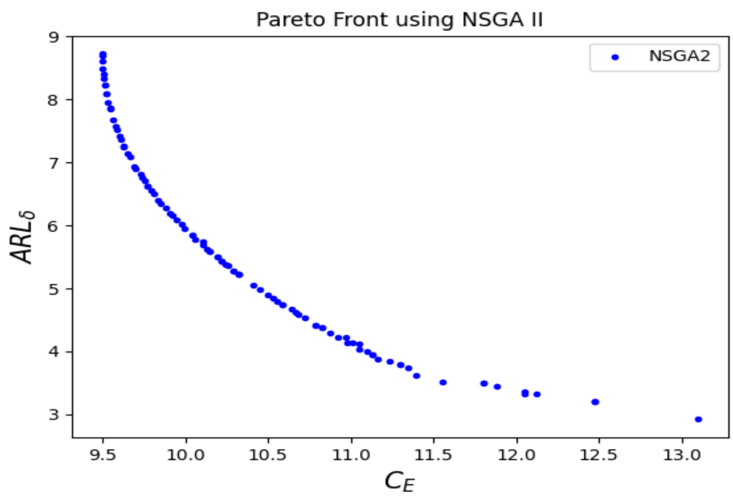


Figure 6.1: Optimal Pareto Front

Table 6.1: The non-dominated set of optimal parameters along with C_E and ARL_δ.

C_E	ARL_δ	n	h	H
9.50	8.72	2	0.36	4.19
9.50	8.49	2	0.37	4.07
9.52	8.10	2	0.40	3.88
9.56	7.69	2	0.43	3.67
9.61	7.37	2	0.44	3.51
9.69	6.94	2	0.48	3.29
9.75	6.71	2	0.53	3.18
9.75	6.71	2	0.53	3.18
9.84	6.39	2	0.55	3.02
9.92	6.15	2	0.56	2.89
10.04	5.84	2	0.65	2.74
10.13	5.63	2	0.64	2.63
10.24	5.38	2	0.71	2.50
10.41	5.05	2	0.78	2.33
10.55	4.80	2	0.82	2.20
10.69	4.58	2	0.94	2.09
10.92	4.23	2	1.07	1.90
11.05	4.13	2	0.94	1.85
11.16	3.88	2	1.15	1.73
11.40	3.62	2	1.19	1.59
12.05	3.37	2	1.03	1.46
13.10	2.92	2	1.07	1.22

The minimum value of C_E is \$9.50 and the pertinent optimal parameters are $n = 2$, $h = 0.36$ hours, and $H = 4.19$ units. Lee [49] obtained the corresponding optimal values of C_E, n, h, and H under the normality condition respectively as \$16.78, 2, 0.85 hours, and 1.69 units where he used Markov chain-based approach given by Prabhu et al. [70] to find ARL_0 and ARL_δ. If one uses the proposed multi-objective economic statistical approach, C_E decreases by 43.39%.

6.5 Sensitivity Analysis

Table 6.2 contains two sets of values along with optimal parameters n, h, and H for a given δ and these are (i) C_E is minimum and ARL_δ is maximum (ii) C_E is maximum and ARL_δ is minimum. As the magnitude of the shift (δ) increases, both C_E and ARL_δ decrease. However, n remains the same. As the magnitude of shift increases, h decreases, and H increases.

Table 6.2: Optimal values of C_E and ARL_δ along with respective optimal parameters for different values of δ.

δ	C_E	ARL_δ	n	h	H
1.0	9.50 13.10	8.72 2.92	2 2	0.36 1.07	4.19 1.22
1.5	7.97 10.91	4.78 2.03	2 2	0.48 1.09	3.08 1.00
2.0	7.09 9.34	2.96 1.48	2 2	0.64 1.19	2.30 0.80
2.5	6.50 8.34	2.05 1.08	2 2	0.79 1.46	1.80 0.57

Table 6.3 also contains two sets of values along with optimal parameters n, h, and H when cost parameters are increased from their respective low levels to high levels with a change of one-factor-at-a-time and these are (i) C_E is minimum and ARL_δ is maximum (ii) C_E is maximum and ARL_δ is minimum. All the low levels of input parameters are kept the same as given in the numerical example Section and high levels are chosen at random since it is a common practice in research articles that deal with the economic design of control charts. For $\delta = 1.0$, Table 6.2 presents the corresponding values of the optimal parameters when each input variable is at its pertinent low level. The minimum change in the minimum C_E is obtained corresponding to the parameter C_0 and the maximum change in the minimum C_E is obtained corresponding to the parameter C_1. Similarly, The minimum change in the maximum C_E is also obtained corresponding to the parameter C_0, and the maximum change in the maximum C_E is obtained corresponding to the parameter C_1. Also, the minimum change in the minimum ARL_δ is obtained corresponding to the parameter C_0, and the maximum change in the minimum ARL_δ is obtained corresponding to the parameter d. Similarly, The minimum change in the maximum ARL_δ is also obtained corresponding to the parameter C_0, and the maximum change in the maximum ARL_δ is obtained corresponding to the parameter d.

Table 6.3: Optimal values of C_E and ARL_δ along with respective optimal parameters for cost parameters.

Input Parameters	Low Level	High Level	C_E	ARL_δ	n	h	H
C_0	10	20	9.59 13.02	8.83 3.12	2 2	0.36 0.96	4.25 1.32

C_1	100	200	15.64	8.92	2	0.24	4.29
			20.96	3.42	2	0.52	1.48
C_2	50	100	9.91	9.92	2	0.34	4.79
			20.77	3.18	2	0.77	1.36
C_3	25	50	9.73	8.52	2	0.37	4.09
			13.80	3.05	2	0.89	1.29
d	0.5	5	14.15	3.57	2	1.87	1.56
			19.68	1.69	2	1.60	0.54
y	0.1	1	12.38	5.17	2	1.02	2.39
			15.74	2.17	2	1.45	0.81

Table 6.4 contains two sets of values along with optimal parameters n, h, and H when time-related parameters are increased from their respective low levels to high levels with one-factor-at-a-time change. The minimum change in the minimum C_E is obtained corresponding to the parameter t and the maximum change in the minimum C_E is obtained corresponding to the parameter λ. Similarly, the minimum change in the maximum C_E is also obtained corresponding to the parameter t, and the maximum change in the maximum C_E is obtained corresponding to the parameter λ. Also, the minimum change in the minimum ARL_δ is obtained corresponding to the parameters T_1 and T_2, and the maximum change in the minimum ARL_δ is obtained corresponding to the parameter λ. Similarly, the minimum change in the maximum ARL_δ is obtained corresponding to the parameter t, and the maximum change in the maximum ARL_δ is obtained corresponding to the parameter λ. All of the corresponding optimal Pareto fronts are given in the Appendix E.

Table 6.4: Optimal values of C_E and ARL_δ along with respective optimal parameters for time parameters.

Input Parameters	Low Level	High Level	C_E	ARL_δ	n	h	H
λ	0.01	0.05	28.10	8.23	2	0.20	3.94
			34.45	2.40	2	0.71	0.94
t	0.05	0.25	9.84	8.82	2	0.36	4.24
			12.27	3.26	2	1.18	1.40
T_0	2	5	9.50	8.72	2	0.36	4.19
			13.10	2.92	2	1.07	1.22
T_1	2	5	12.03	8.26	2	0.40	3.96
			14.84	3.14	2	1.08	1.34
T_2	2	5	12.03	8.26	2	0.40	3.96
			14.84	3.14	2	1.08	1.34

6.6 Conclusions

In this Chapter, a multi-objective economic statistical design of the CUSUM chart is proposed. C_E and ARL_δ are considered as two objectives. This multi-objective problem is then solved with the help of the NSGA II algorithm. Since, as far as we know, there is no research article on the multi-objective design of the CUSUM chart, results are compared with the results of a single objective design proposed by Lee [49]. The minimum value of C_E obtained by the proposed approach is \$9.50 and the corresponding optimal parameters are $n = 2$, $h = 0.36$ hours, and $H = 4.19$ units. The respective optimal values of C_E, n, h, and H under the normality condition obtained by Lee [49] are \$16.78, 2, 0.85 hours, and 1.69 units. The proposed multi-objective economic statistical approach reduces C_E by 43.39%. In a practical situation pertaining to an industry, if the cost and time parameters are adequately estimated, the demonstrated algorithm and pseudocode given in Section 1.3.3 can suitably be used for the effective implementation of the CUSUM control chart.

Chapter 7

Conclusion and Scope for Future Research

We summarize the results of this thesis and also provide scope for future research in this Chapter.

7.1 Conclusion

In Chapter 2 of this thesis, taking inspiration from the study done by Taguchi et al. [89], we proposed an improvised total cost per unit, i.e., L by considering additional parameters C_f, C_a, P_r, and the pragmatic break-up of the diagnosis cost. These additional parameters have the potential to play a major role in determining the optimal diagnosis interval pertaining to online quality control methods. The effectiveness of the proposed study is then demonstrated with the help of a real-life case study. The values of $h^\star$ and L thus obtained using the proposed approach respectively are 242 and Rs 17.88. The current practice of the Indian company is to test every 100th ductile iron pipe for Tensile Strength. L using $h^\star$ as 100 is Rs 26.90, which is 50.45% more than L when $h^\star$ is 242, which emerged from this study of ours.

The first economic design of the $\bar{X}$ control chart was proposed by Duncan [25]. Taking a cue from Duncan [25], in Chapter 3, we proposed an improvised loss function, i.e. $\mathcal{L}$ for determining the optimal parameters of the $\bar{X}$ control chart by introducing addition parameters like y_1, y_2, y_3, y_4, y_5, y_6, and P_r. Two different approaches have also been proposed for determining the optimal parameters for the proposed loss function. In the first methodology, we have differentiated $\mathcal{L}$ with respect to n like Duncan [25] along with h, and k, although n is an integer. Consequently, for mathematical soundness, an iterative algorithmic approach has also been proposed for minimizing $\mathcal{L}$ by finding partial derivatives of $\mathcal{L}$ with respect to h and k only. In the case of the real-life case study, using Duncan [25] approach, i.e., differentiating with respect to n, failed to provide a real value of the optimal sampling interval. Instead, it yielded an imaginary value of the optimal sampling interval. However, there was no problem while determining the optimal sampling interval with the help of the iterative algorithmic approach. Also, the value of the modified loss function thus obtained using an iterative algorithmic approach turned out to be less than the value of the loss function obtained by an Indian organization, considered for this study, using their own sampling interval.

In the literature, there are articles that deal with finding the optimal parameters of the economic statistical design and the multi-objective economic statistical design of c-chart. However, when the area of opportunities for nonconformities varies from sample to sample or subgroup to subgroup, the use of c-chart becomes impractical. In those cases, the use of u-chart becomes a suitable option. However, to the best of our knowledge, there is no research article in the literature that deals with determining the optimal parameters of the economic statistical design of the u-chart or the multi-objective economic statistical design of the u-chart. For the first time, in Chapter 4, we proposed the economic statistical and the multi-objective economic statistical designs of the u-chart. The GA has been used for finding the optimal parameters of the economic statistical design of the u-chart. On the other hand, NSGA II has been used to determine the optimal parameters of the multi-objective economic statistical design of the u-chart.

A substantial amount of literature is devoted to the study of finding the optimal parameters of the economic design, the economic statistical design, and the multi-objective economic statistical design of np-chart. Also, these articles have given more importance to finding the optimal value of the acceptance number instead of the control limits' multiplier, i.e., k, while finding the optimal parameters. When the sample size varies from sample to sample or subgroup to subgroup, the use of p-chart becomes a suitable option. As far as our knowledge goes, the studies given in the literature have not dealt with the economic statistical design and the multi-objective economic statistical design of the p-chart. Thus, in Chapter 5, we proposed the economic statistical design and the multi-objective economic statistical design of the p-chart while giving due consideration to the control limits' multiplier. Similar to the treatise on u-chart given in Chapter 4, GA and NSGA II have respectively been used for finding the optimal parameters of the economic statistical design and the multi-objective economic statistical design of the p-chart.

The economic design and the economic statistical design of the CUSUM chart have been extensively studied in the literature. However, we could not observe any article that deals with determining the optimal parameters of the multi-objective economic statistical design of the CUSUM chart. Therefore, the multi-objective economic statistical design of the CUSUM chart has been proposed in Chapter 6. The corresponding optimal parameters of the multi-objective economic statistical design of the CUSUM chart have been determined using the NSGA II approach.

7.2 Scope for Future Research

Researchers can explore the following research problems in the future:

1. One possible area of exploration could involve developing advanced optimization algorithms for finding the optimal sampling interval and one could also try to find a way to determine the optimal value of n for on-line quality control methods along with the diagnosis interval.

2. Further, one could extend the work done on the multi-objective economic statistical design of the u-chart by using other algorithms in addition to NSGA II. In that case, one can consider the NSGA II approach as a baseline for comparing the results obtained using other algorithms.

3. We have assumed that the sample statistic u follows the approximate Poisson distribution in order to determine the corresponding optimal parameters. One can look for other suitable distributions for the statistic u while finding the corresponding optimal parameters.

4. Comparative studies concerning the economic statistical design of the u-chart under the influence of different kinds of loss functions (like Taguchi's loss function, Linex loss function etc.) to find the optimal values of the relevant parameters can turn out to be quite interesting for the researchers.

5. The multi-objective economic statistical design of the p-chart could be explored using the other methodologies (like MOGA, Goal programming, PSO etc.) given in the literature for finding the optimal parameters. Results obtained by using those methodologies can then be compared with the results obtained using the NSGA II approach leading to a comprehensive comparative analysis.

6. Other dimensions of the work may encompass exploration of the parameters of the multi-objective economic statistical design of the CUSUM chart making use of other algorithms apart from NSGA II to facilitate comparison.

7. Also, one can try to find the relevant control chart parameters for the CUSUM chart under the influence of different kinds of loss functions.

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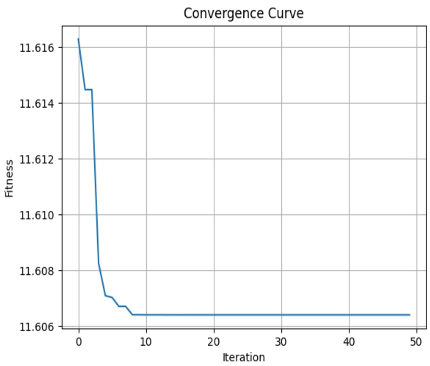
Appendix

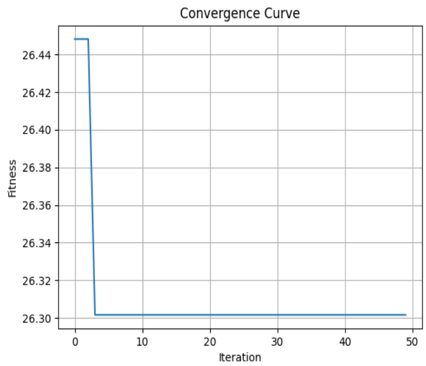
All corresponding convergence curves and optimal Pareto fronts are given in this Chapter.

Appendix A

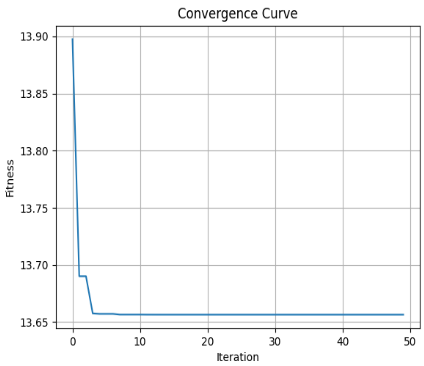
The corresponding convergence curves for every parameter of Table 4.2 for the economic statistical design of the u-chart are given hereunder.



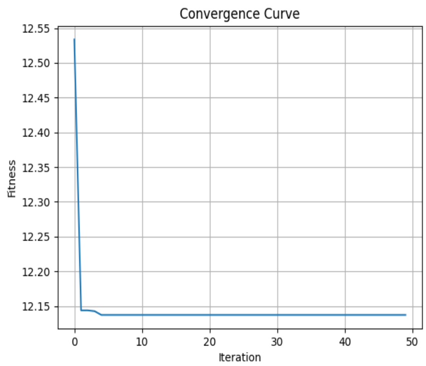
(a) Each parameter with low level.



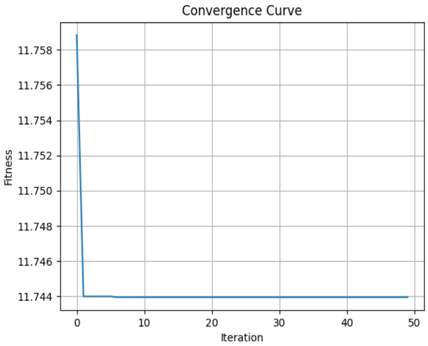
(b) C_0 with high level.



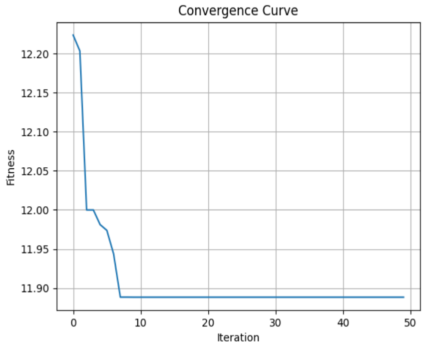
(c) C_1 with high level.



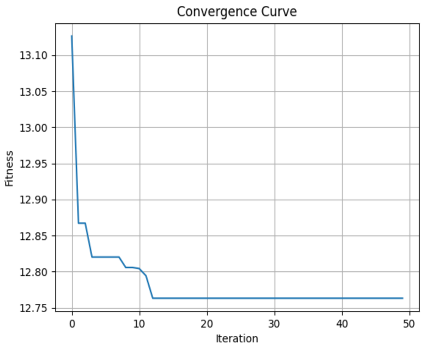
(d) C_2 with high level.



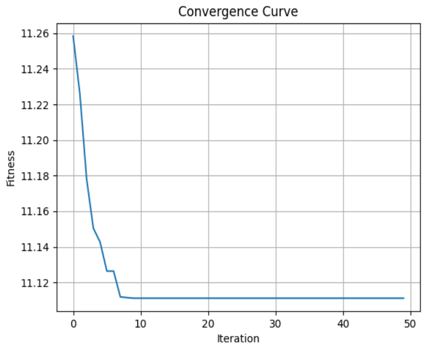
(e) C_3 with high level.



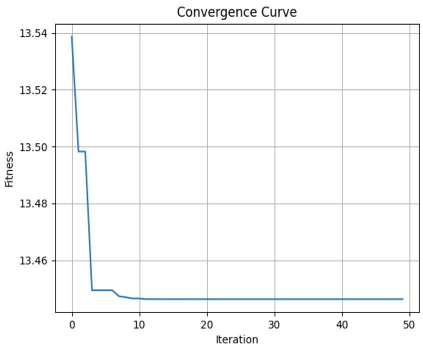
(f) d with high level.



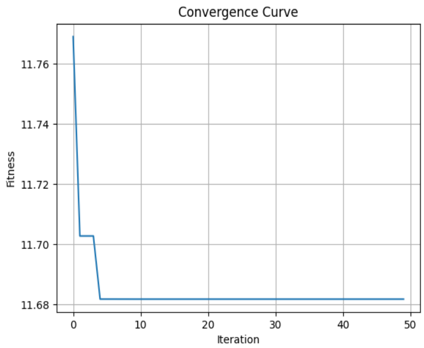
(g) y with high level.



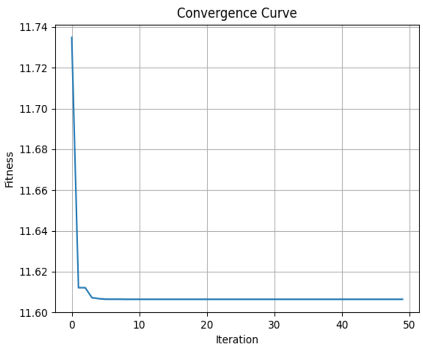
(h) δ with high level.



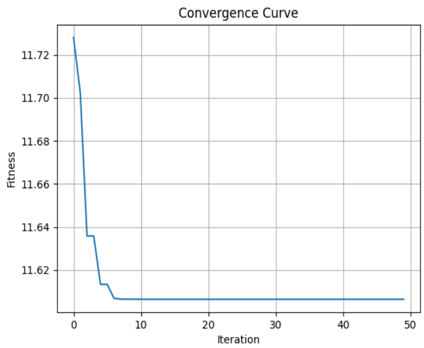
(i) λ with high level.



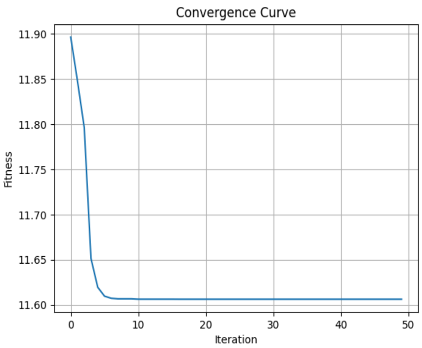
(j) t with high level.



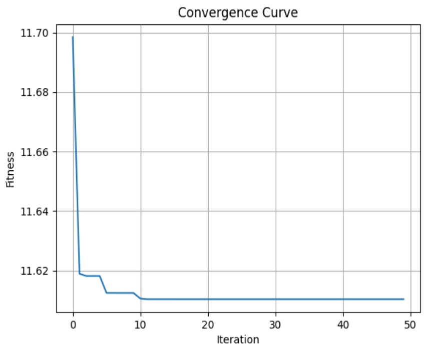
(k) T_0 with high level.



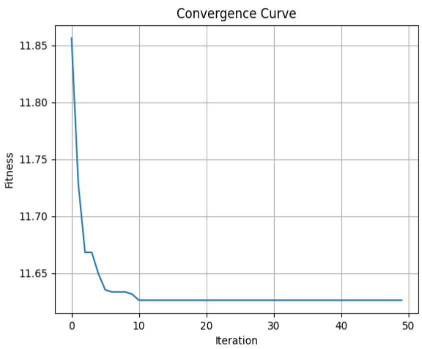
(l) T_1 with high level.



(m) T_2 with high level.



(n) γ_1 with high level.

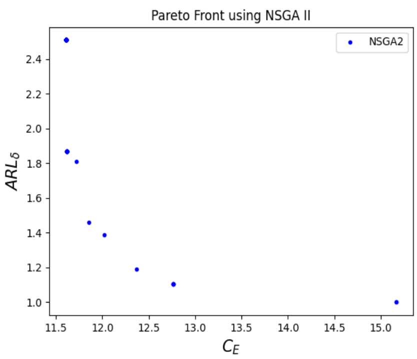


(o) γ_2 with high level.

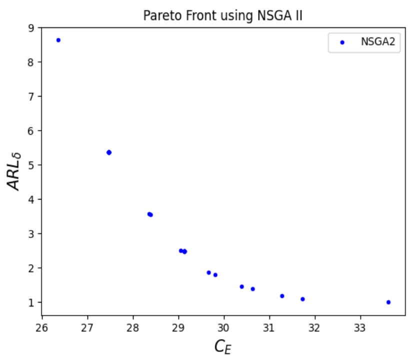
Convergence curves with low levels and high levels of input parameters.

Appendix B

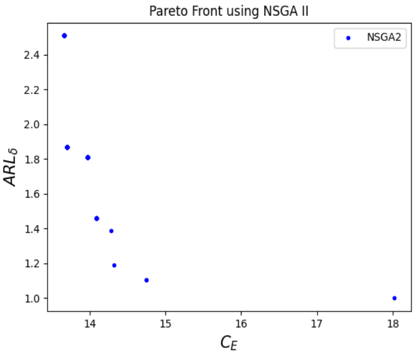
The corresponding optimal Pareto fronts for every parameter of Table 4.4 for the multi-objective economic statistical design of the u-chart are given hereunder.



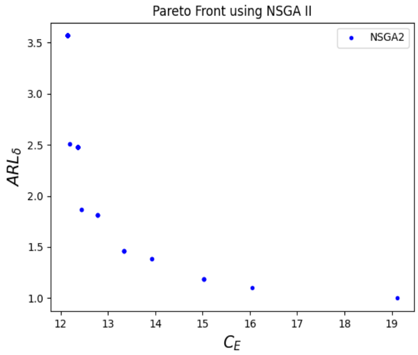
(a) Each parameter with low level.



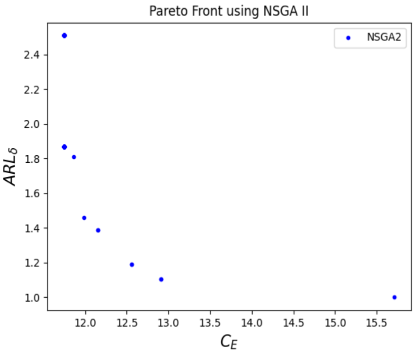
(b) C_0 with high level.



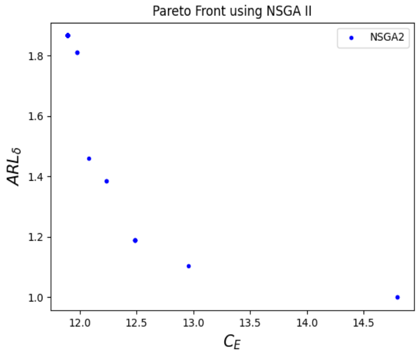
(c) C_1 with high level.



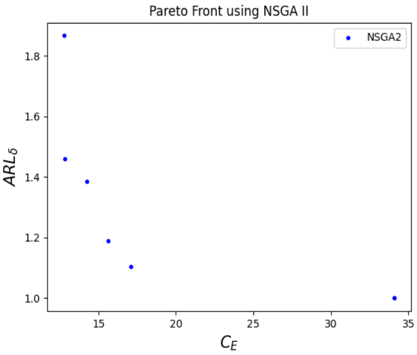
(d) C_2 with high level.



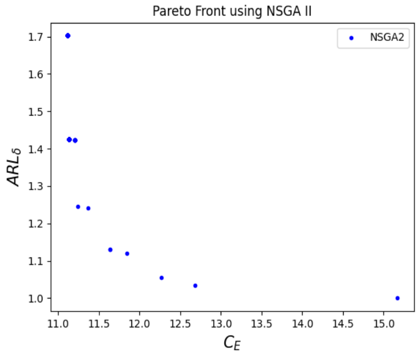
(e) C_3 with high level.



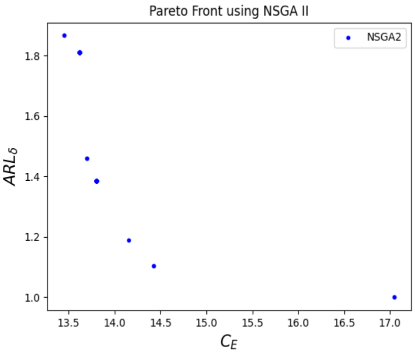
(f) d with high level.



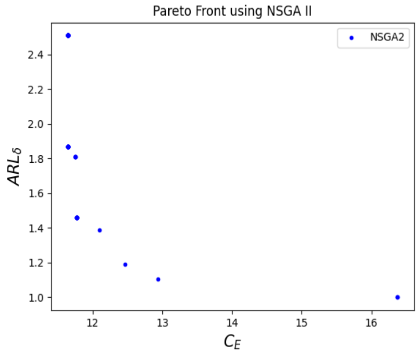
(g) y with high level.



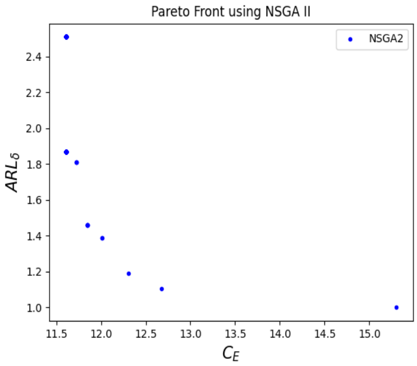
(h) δ with high level.



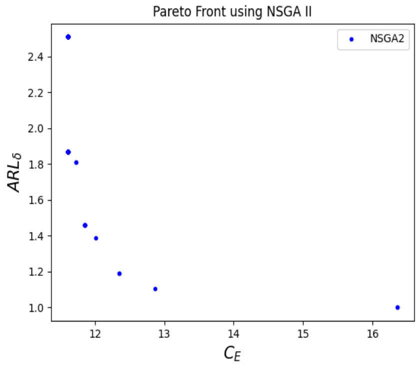
(i) λ with high level.



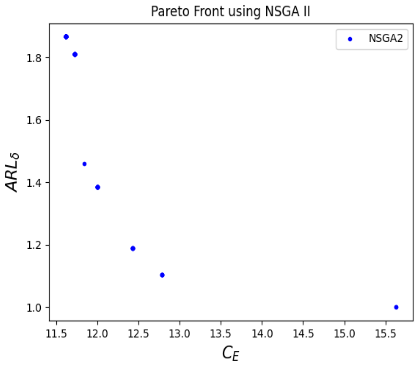
(j) t with high level.



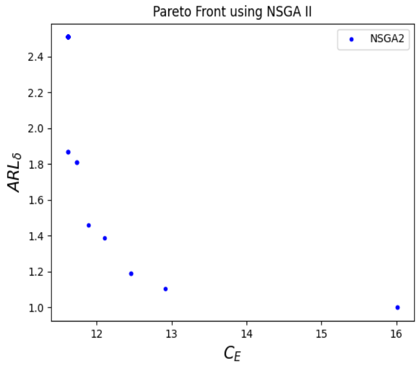
(k) T_0 with high level.



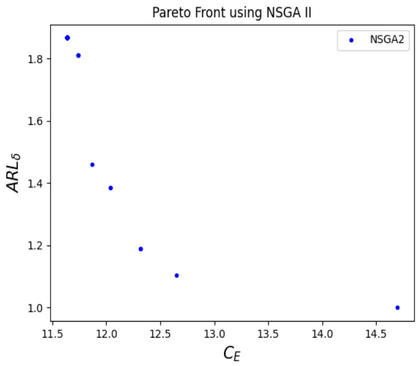
(l) T_1 with high level.



(m) T_2 with high level.



(n) γ_1 with high level.

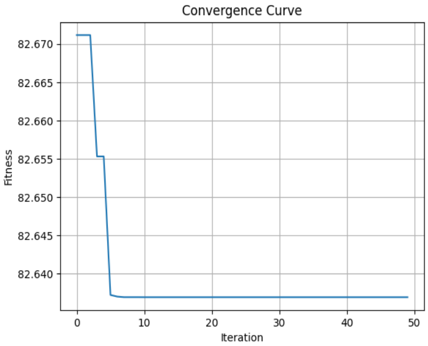


(o) γ_2 with high level.

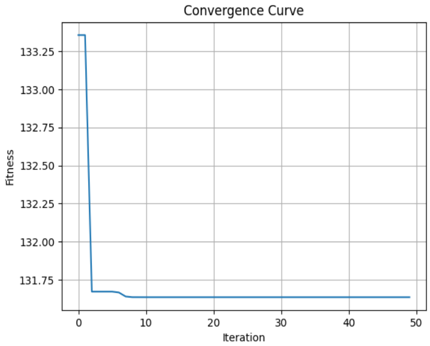
Optimal Pareto fronts with low levels and high levels of input parameters.

Appendix C

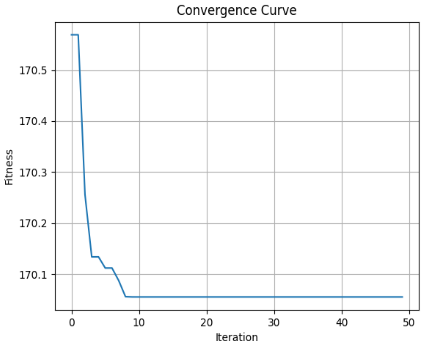
The corresponding convergence curves for each parameter given in Table 5.2 for the economic statistical design of the p-chart are provided hereunder.



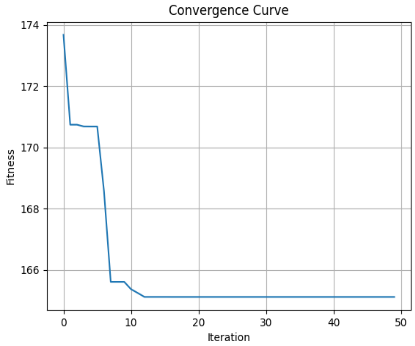
(a) Each parameter with low level.



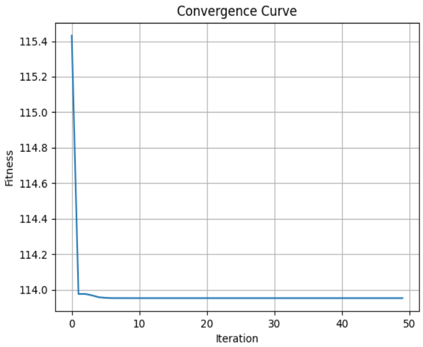
(b) p_0 with high level.



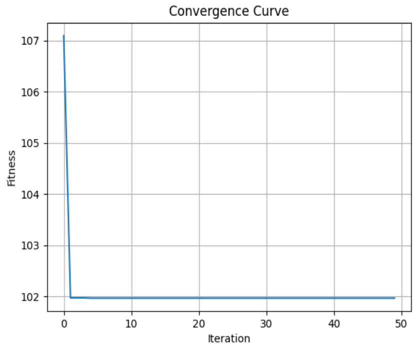
(c) δ with high level.



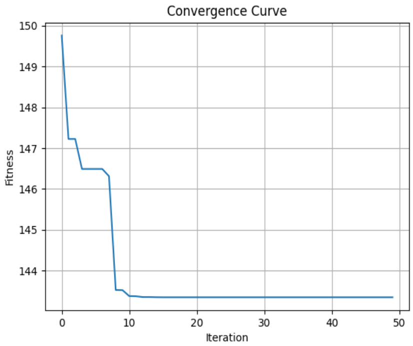
(d) λ with high level.



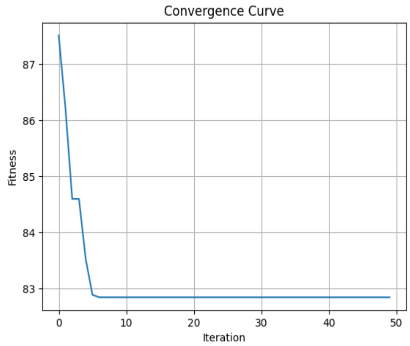
(e) C_1 with high level.



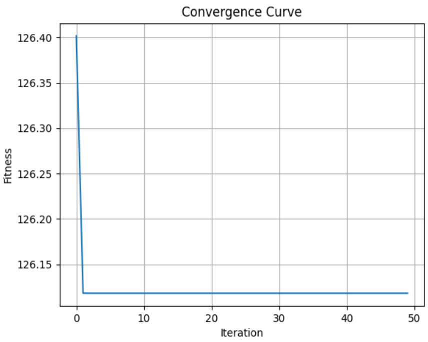
(f) C_2 with high level.



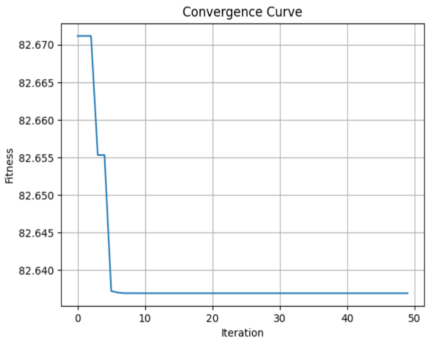
(g) V with high level.



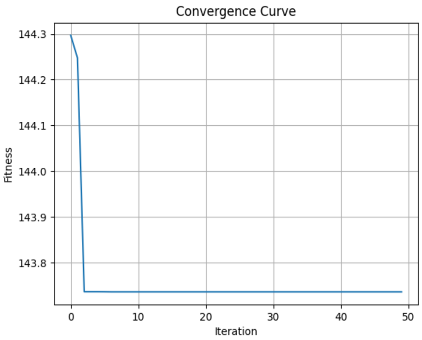
(h) d with high level.



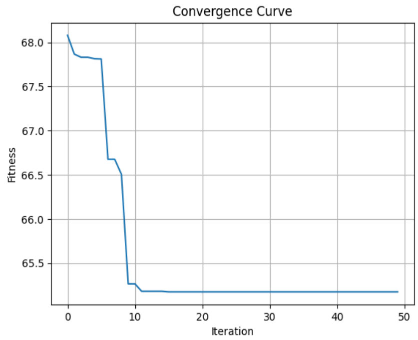
(i) $\sum_{i=1}^{6} y_i$ with high level.



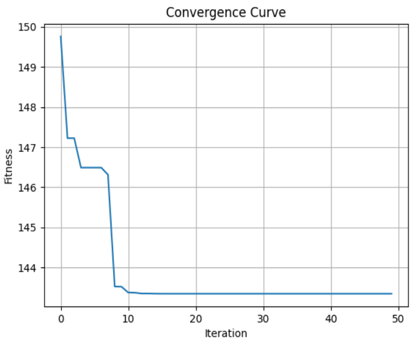
(j) T_0 with high level.



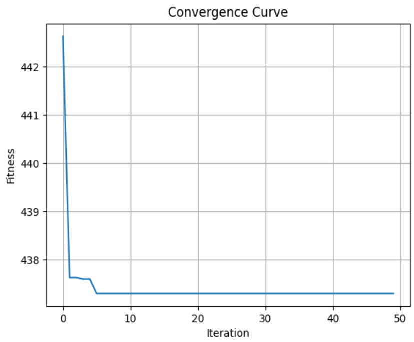
(k) T_1 with high level.



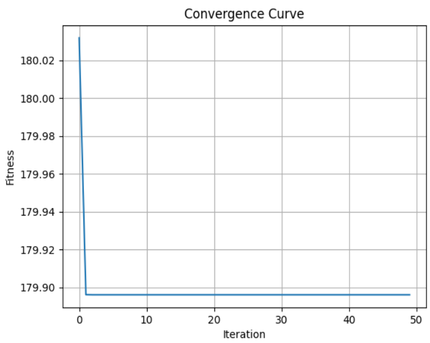
(l) T_2 with high level.



(m) P_r with high level.



(n) t with high level.

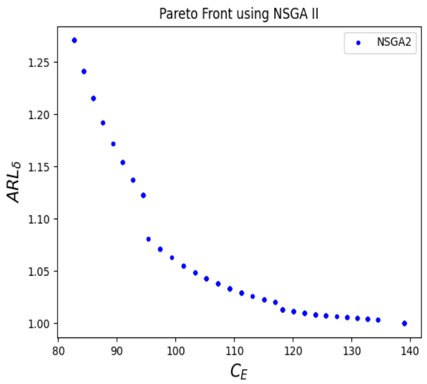


(o) γ_1 with high level.

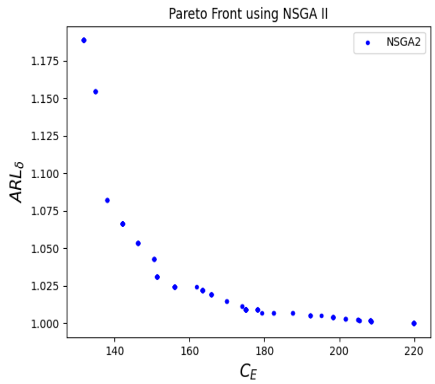
Convergence curves with low and high levels of the input parameters.

Appendix D

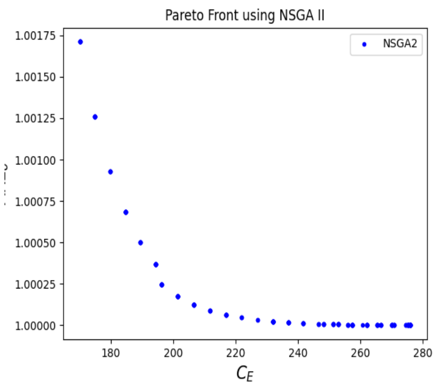
The corresponding optimal Pareto fronts for each parameter provided in Table 5.8 for the multi-objective economic statistical design of the p-chart are given hereunder.



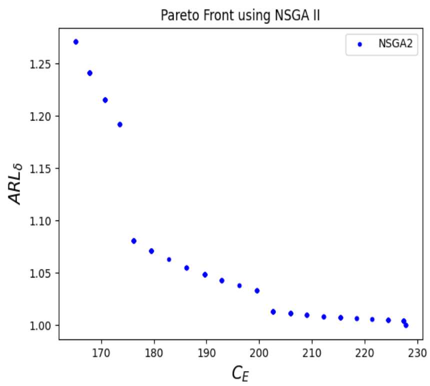
(a) Each parameter with low level.



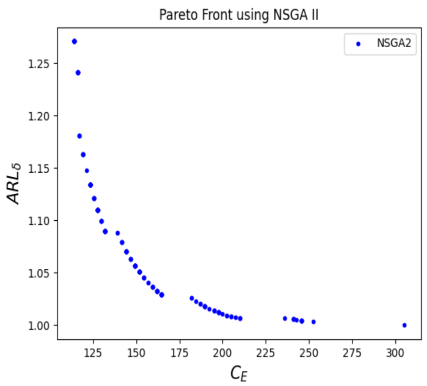
(b) p_0 with high level.



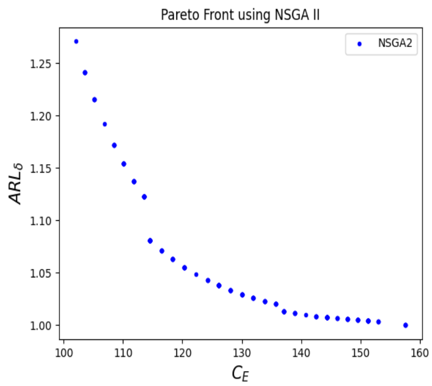
(c) δ with high level.



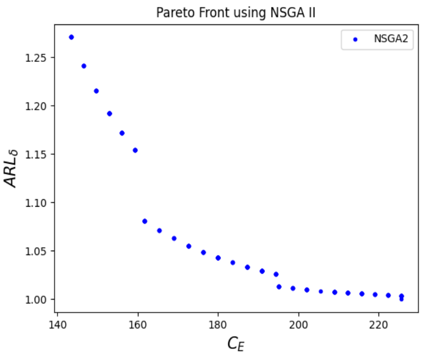
(d) λ with high level.



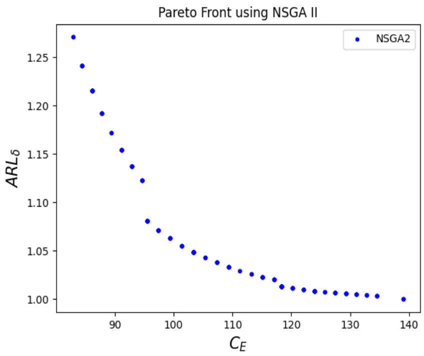
(e) C_1 with high level.



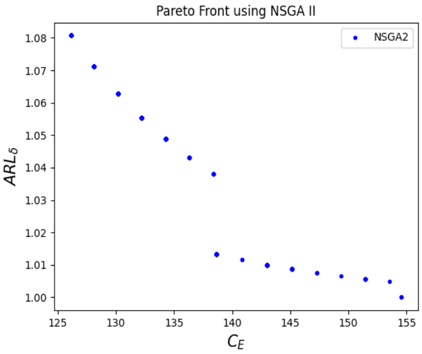
(f) C_2 with high level.



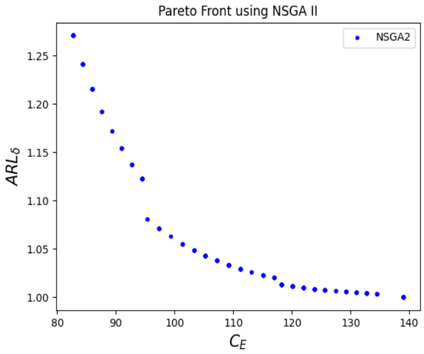
(g) V with high level.



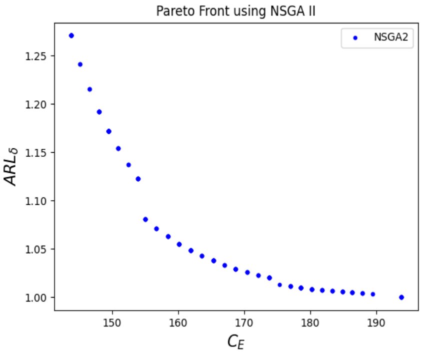
(h) d with high level.



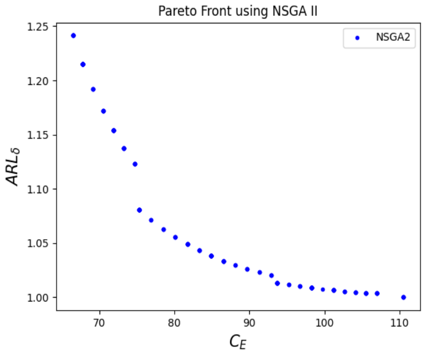
(i) $\sum_{i=1}^{6} y_i$ with high level.



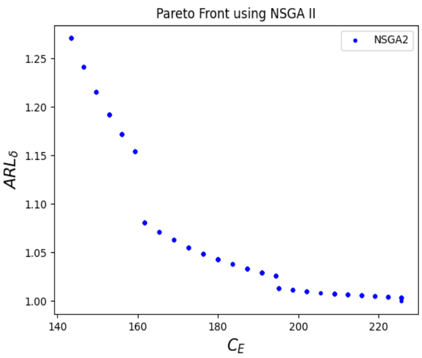
(j) T_0 with high level.



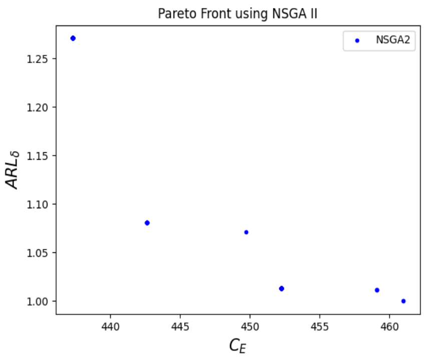
(k) T_1 with high level.



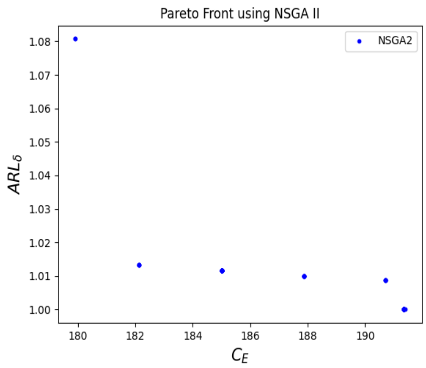
(l) T_2 with high level.



(m) P_r with high level.



(n) t with high level.

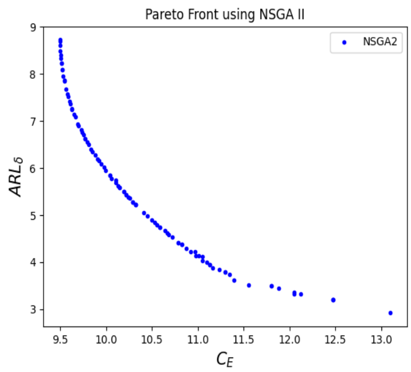


(o) γ_1 with high level.

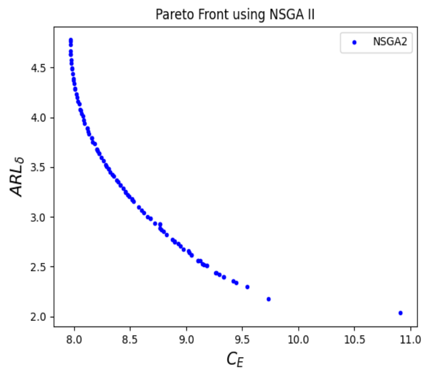
Optimal Pareto fronts with low levels and high levels of input parameters.

Appendix E

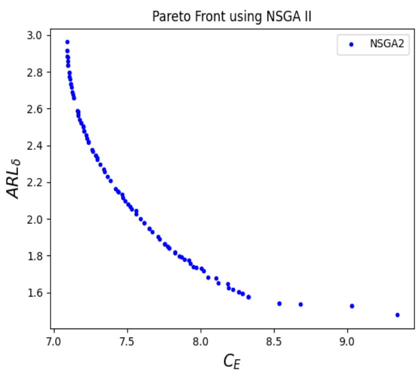
The corresponding optimal Pareto fronts for different values of δ provided in Table 6.2 for the multi-objective economic statistical design of the CUSUM chart are given hereunder.



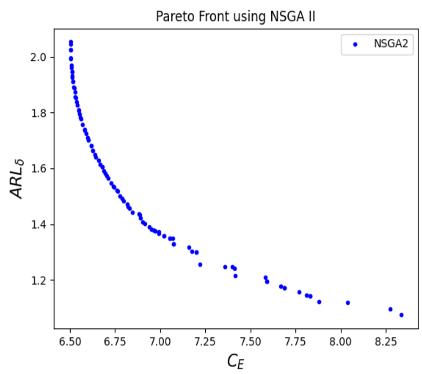
(a) Each parameter with low level i.e. for δ = 1.0.



(b) For δ = 1.5.



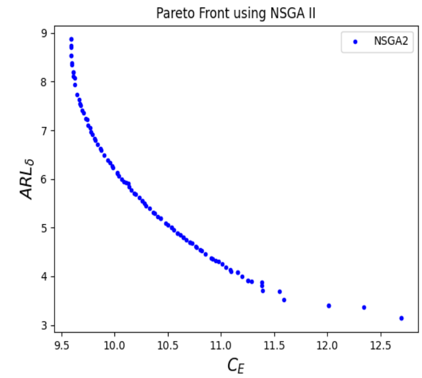
(c) For δ = 2.0.



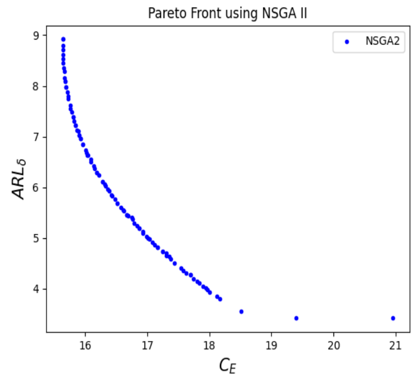
(d) For δ =2.5.

Optimal Pareto fronts for different values of δ.

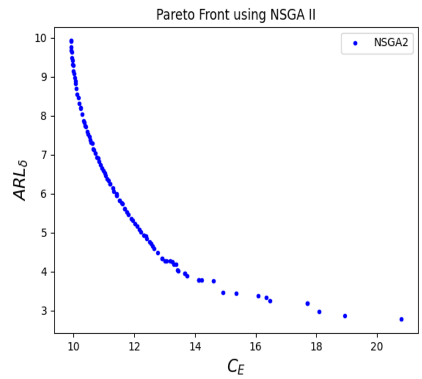
The corresponding optimal Pareto fronts for high levels of cost parameters provided in Table 6.3 for the multi-objective economic statistical design of the CUSUM chart are given hereunder.



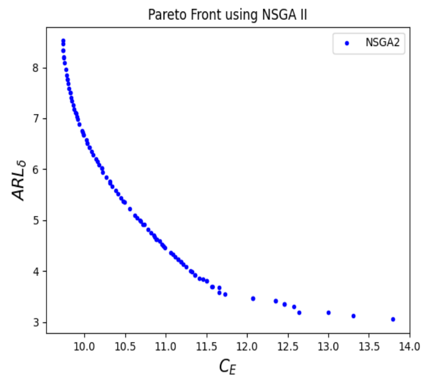
(a) C_0 with high level.



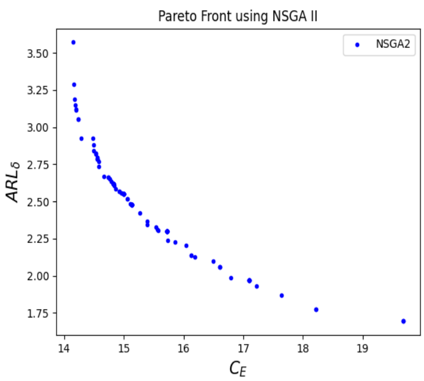
(b) C_1 with high level.



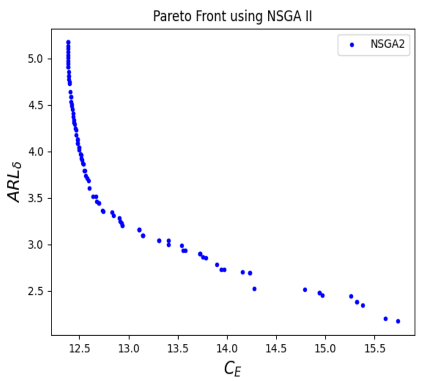
(c) W with high level.



(d) Y with high level.



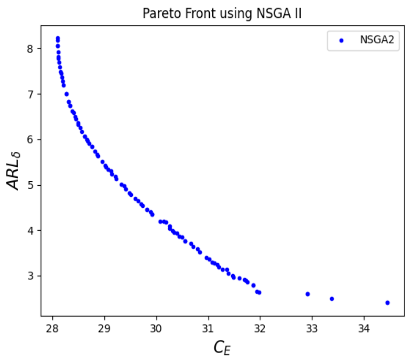
(e) d with high level.



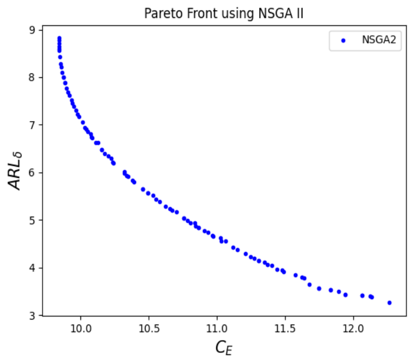
(f) y with high level.

Optimal Pareto fronts with high levels of cost parameters.

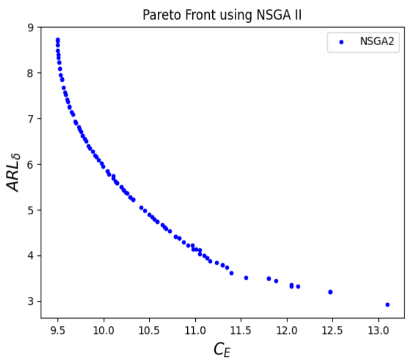
The corresponding optimal Pareto fronts for high levels of time-related parameters provided in Table 6.4 for the multi-objective economic statistical design of the CUSUM chart are given hereunder.



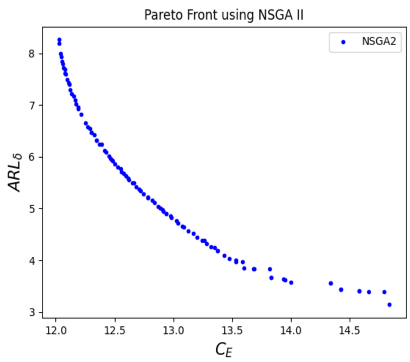
(a) λ with high level.



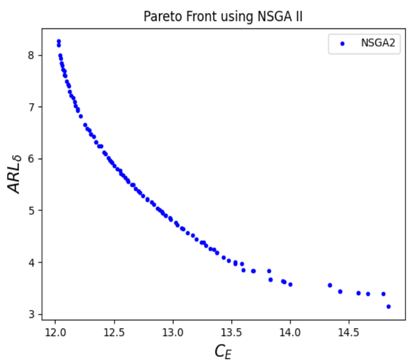
(b) t with high level.



(c) T_0 with high level.



(d) T_1 with high level.



(e) T_2 with high level.

Optimal Pareto fronts with high levels of time-related input parameters.