INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination
M. Tech. CS I year, 1st Sem, AY 2025–2026
Probability and Stochastic Processes

Date: 11. 09. 2025 Time: 2:30 Hours Total Marks: 60

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1. Answer **all** the questions from **Group-A**.
2. Answer **any three** from **Group-B**. If you answer more than three questions:
   (a) Only the first three answers will be considered.
   (b) The rest will be ignored.

Group-A

1. Prove the *Bonferroni's inequality* to n events. That is, show that

$$\Pr\left(\bigcap_{i=1}^{n} E_i\right) \geq 1 - n + \sum_{i=1}^{n} \Pr(E_i).$$

[5]

2. In Laplace's rule of succession, show that if the first n flips all result in heads, then the conditional probability that the next m flips also result in all heads is $(n+1)/(n+m+1)$. [5]

3. Suppose that we roll twice a fair k-sided die with the numbers 1 through k on the die's faces, obtaining values X_1 and X_2 .
   (a) What is $\mathbf{E}[\max(X_1, X_2)]$? What is $\mathbf{E}[\min(X_1, X_2)]$?
   (b) Also show that $\mathbf{E}[\max(X_1, X_2)] + \mathbf{E}[\min(X_1, X_2)] = \mathbf{E}[X_1] + \mathbf{E}[X_2]$.

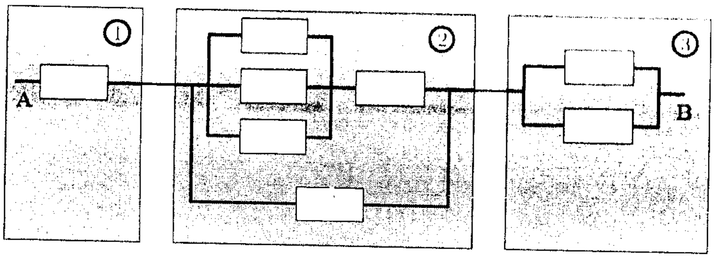
[10]

4. A coin that has probability of heads equal to p is tossed successively and independently until a head comes twice in a row or a tail comes twice in a row. Find the expected value of the number of tosses. [10]

Group-B

5. You are given that at least one of the events E_i, $i = 1, \ldots, n$, is certain to occur, but certainly no more than two occur. If $\Pr(E_i) = p$, and $\Pr(E_i \cap E_j) = q$, $i \neq j$, show that $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n}$. [10]

6. (a) An electrical system consists of identical components that are operational with probability p independently of other components. The components are connected in three subsystems, as shown in the figure. The system is operational if there is a path that starts at point **A**, ends at point **B**, and consists of operational components. This is the same as requiring that all three subsystems are operational. What are the probabilities that the three subsystems, as well as the entire system, are operational?



(b) As a simplified model for weather forecasting, suppose that the weather (either wet or dry) tomorrow will be the same as the weather today with probability p. Show that the weather is dry on January 1, then P_n, the probability that it will be dry n days later, satisfies

$$
\begin{aligned}
P_n &= (2p-1)P_{n-1} + (1-p), n \geq 1 \\
P_0 &= 1
\end{aligned}
$$

Prove that

$$P_n = \frac{1}{2} + \frac{1}{2}(2p-1)^n, n \geq 0.$$

[3+7=10]

7. A permutation on the numbers $[1, n]$ can be represented as a function $\pi : [1, n] \to [1, n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. Let $\mathsf{Perm}[n]$ be the set of all possible permutations π. A *fixed point* of a permutation π is a value for which $\pi(x) = x$. Now suppose we pick a permutation ρ from $\mathsf{Perm}[n]$ uniformly at random. Find the probability that

(a) $P_n = \Pr(\{\rho \text{ does not has any fixed point}\})$;

(b) ρ has exactly k fixed points, where $0 < k < n$.

[7+3=10]

8. There are n urns of which the i-th urn contains $i - 1$ red balls and $n - i$ blue balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that

(a) the second ball is blue;

(b) the second ball is blue given the first ball is blue.

[4+6 = 10]

9. Let $X_1, \ldots, X_n$ be independent random variables and let $Z = X_1 + \cdots + X_n$. Suppose that each X_i, $i = 1, \ldots, n$ assumes value 1 with probability p_i and 0 with probability $1 - p_i$, and that $p_1, \ldots, p_n$ are chosen so that the mean of Z is a given $\mathbf{E}[Z] > 0$. Deduce what p_is should be so that the variance of Z is maximized. [10]

10. (a) Consider n independent tosses of a coin with probability of a head equal to p. Let X and Y be the number of heads and of tails, respectively. Compute the correlation coefficient of X and Y.

(b) Show that

$$\mathbf{cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_i \sum_j \mathbf{cov}(X_i, Y_j).$$

[5+5 = 10]