On moments of certain families of L-functions

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On moments of certain families of L-functions

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To
My Parents and Sisters.

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Notations & Abbreviations

$\mathbb{C}$	The set of complex numbers.
$\mathbb{H}$	The upper half plane $\{x+iy \in \mathbb{C} : y > 0\}$.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of non-negative real numbers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Z}$	The set of integers.
$\mathbb{N}$	The set of natural numbers.
GRH	The Grand/Generalized Riemann Hypothesis.
RH	The Riemann Hypothesis.
$f = O(g)$	This means $\|f(x)\| \leq c\|g(x)\|$ for some $c > 0$.
$f = O_\alpha(g)$	This means the above implied constant c depends only on some parameter α.
$f = o(g)$	This means $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
$f \ll g$	This means $f = O(g)$.
$f \sim g$	This means $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$,
$n \asymp X$	This means $c_1 X \leq n \leq c_2 X$ for some c_1, $c_2 > 0$.
$e(z)$	The exponential function $e(z) = e^{2\pi i z}$.
$\square$	This means perfect square.
$\int_{(c)}$	This means $\int_{(c)} = \int_{c-i\infty}^{c+i\infty}$.
$\sum^{\star}$	This means a sum over square-free integers.
$\sum^{\flat}$	This means a sum over fundamental discriminants.
J_ν	The Bassel functions of first kind.
K_ν	The Bassel functions of second kind.
$\mathbb{F}_q$	Finite field with q elements.
$\mathbb{F}_q[t]$	Polynomial ring over $\mathbb{F}_q$.

$\deg(f)$ or $d(f)$	Degree of a polynomial $f \in \mathbb{F}_q[t]$.
$\mathcal{M}_n$	Set of all monic polynomials of degree n.
$\mathcal{H}_n$	Set of all monic, square-free polynomials of degree n.
$\mathcal{P}_n$	Set of all monic irreducible polynomials of degree n.
$\mathcal{M}$	Set of all monic polynomials $\mathcal{M} := \cup_{n\geq 1}\mathcal{M}_n$.
$\mathcal{P}$	Set of all monic irreducible polynomials $\mathcal{P} := \cup_{n\geq 1}\mathcal{P}_n$.
$\|f\|$	Norm of a polynomial $f(\neq 0) \in \mathbb{F}_q[t]$, $\|f\| = q^{\deg(f)}$.

Introduction

L-functions are one of the most studied objects in number theory. Many problems in analytic number theory can be studied by the theory of L-functions. For example, the prime number theorem and the prime number theorem in arithmetic progressions have been studied via non-vanishing results of the Riemann zeta function and the Dirichlet L-functions, respectively. In addition, many well-known unresolved problems in Number Theory, such as the Riemann Hypothesis, the Lindelof hypothesis, and the Birch-Swinnerton-Dyer conjecture, are concerned with the study of L-functions.

We give some examples of the L-functions over number fields as well as function fields. We refer the reader to see [26], [42] and [71] for more details.

1. The Riemann zeta function.

   The most famous example of L-functions is the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}, \qquad \text{for } \Re(s) > 1.$$

   In 1859, Riemann proved that $\zeta(s)$ has an analytic continuation to the entire complex plane with a simple pole at $s = 1$ and satisfies the functional equation

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \xi(1-s).$$

   He conjectured that all the zeros of $\zeta(s)$ in the critical strip $0 \leq \Re(s) \leq 1$ lie on the line $\Re(s) = \frac{1}{2}$, and this is known as the Riemann hypothesis (RH). He realized that the question about the distribution of prime numbers can be studied by analyzing the zeros of the Riemann-zeta function. For example, the Riemann Hypothesis (RH) implies that $\pi(X) = \int_2^X \frac{\mathrm{dt}}{\log t} + O(X^{1/2+\varepsilon})$, where $\pi(X) = \#\{p : p \text{ prime} \leq X\}$.

2. Dirichlet L-functions.

Let χ be a primitive Dirichlet character modulo q. The Dirichlet L-function $L(s,\chi)$ is defined by

$$L(s,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^s}=\prod_p\left(1-\chi(p)p^{-s}\right)^{-1}, \qquad \text{for } \Re(s)>1.$$

If χ_0 is the trivial character modulo q, $L(s,\chi_0)=\zeta(s)\prod_{p|q}\Big(1-p^{-s}\Big)$, so it behaves like the Riemann zeta function. Otherwise, $L(s,\chi)$ is an entire function on the whole complex plane. Also, it satisfies the functional equation

$$\Lambda(s,\chi)=\left(\frac{\pi}{q}\right)^{\frac{s}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi)=\epsilon(\chi)\Lambda(1-s,\bar{\chi}),$$

where $\epsilon(\chi)$ is called the root number which is given by

$$\epsilon(\chi)=\frac{i^{\mathfrak{a}}}{\sqrt{q}}\sum_{b \bmod q}\chi(b)e\left(\frac{b}{q}\right)$$

with $\mathfrak{a}=0$, if $\chi(-1)=1$ and $\mathfrak{a}=1$, if $\chi(-1)=-1$.

3. Modular L-functions.

Let f be a holomorphic Hecke eigenform of weight k for the full modular group $SL(2,\mathbb{Z})$. The Fourier expansion of f at ∞ is given by

$$f(z)=\sum_{n\geq 1}\lambda_f(n)n^{(k-1)/2}e(nz), \quad z\in\mathbb{H},$$

where $\lambda_f(1)=1$ and $|\lambda_f(n)|\leq\tau(n)$ for $n\geq 1$. A modular L-function associated with the Hecke eigenform f is defined by

$$L(s,f)=\sum_{n\geq 1}\frac{\lambda_f(n)}{n^s}=\prod_p\Big(1-\lambda_f(p)p^{-s}+p^{-2s}\Big)^{-1}, \quad \text{for } \Re(s)>1.$$

It has an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(s,f)=\pi^{-s}\Gamma\left(\frac{s+(k-1)/2}{2}\right)\Gamma\left(\frac{s+(k+1)/2}{2}\right)L(s,f)=\epsilon(f)\Lambda(1-s,\bar{f}),$$

where $\epsilon(f)$ is the root number and $\bar{f}$ is the dual form of f. For more general L-functions, one can look at Chapter 5 of [42].

4. Zeta function over function fields.

Let $\mathbb{A}=\mathbb{F}_q[t]$ be the ring of polynomials over a finite field $\mathbb{F}_q$, where $q=p^r$, $r\geq 1$, and p is a prime. For a polynomial f in $\mathbb{F}_q[t]$, its degree will be denoted by $\deg(f)$. The set of all monic polynomials and monic irreducible polynomials of degree n are denoted by $\mathcal{M}_n$ and $\mathcal{P}_n$ respectively. Let $\mathcal{M}=\cup_{n\geq 1}\mathcal{M}_n$ and $\mathcal{P}=\cup_{n\geq 1}$ $\mathcal{P}_n$. We also denote the set of all monic polynomials and monic irreducible polyno mials of degree less than or equal to n by $\mathcal{M}_{\leq n}$ and $\mathcal{P}_{\leq n}$ respectively. Let $\mathcal{H}_n$ denote the set of monic square-free polynomials of degree n. Observe that for $n\geq 1$, $|\mathcal{M}_n|=q^n$ and

$$|\mathcal{H}_n|=\begin{cases} q, & \text{if } n=1,\\ q^{n-1}(q-1), & \text{if } n\geq 2.\end{cases}$$

If f is a non-zero polynomial in $\mathbb{A}$, we define the norm of f to be $|f|=q^{\deg(f)}$. If $f=0$, we set $|f|=0$. The prime polynomial theorem (see [71], Theorem 2.2) states that

$$|\mathcal{P}_n|=\frac{q^n}{n}\ +\ O\Big(\frac{q^{\frac{n}{2}}}{n}\Big).$$

Let us note that if we denote $X=q^n$, then we have $|\mathcal{P}_n|=\frac{X}{\log_q X}\ +\ O\Big(\frac{X^{\frac{1}{2}}}{\log_q X}\Big)$. This is analogous to the prime number theorem under the Riemann Hypothesis over number fields.

The zeta function of $\mathbb{A}$ is denoted by $\zeta_{\mathbb{A}}(s)$ and defined by

$$\zeta_{\mathbb{A}}(s) := \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - |P|^{-s}\right)^{-1}, \qquad \text{for } \Re(s) > 1.$$

Let us observe that

$$\zeta_{\mathbb{A}}(s) = \sum_{n=0}^{\infty} \frac{1}{q^{ns}} \left(\sum_{f \in \mathcal{M}_n} 1 \right) = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}.$$

By the above identification, it is clear that the zeta function $\zeta_{\mathbb{A}}(s)$ has an analytic continuation to the complex plane with a simple pole at $s = 1$ having residue $\frac{1}{\log q}$. It also satisfies the functional equation

$$\xi_{\mathbb{A}}(s) = q^{-s}(1 - q^{-s})^{-1} \zeta_{\mathbb{A}}(s) = \xi_{\mathbb{A}}(1 - s).$$

Using the change of variable $u = q^{-s}$, $\zeta_{\mathbb{A}}(s)$ can be rewritten as

$$\mathcal{Z}(u) = \sum_{f \in \mathcal{M}} u^{\deg(f)} = \sum_{n=0}^{\infty} (uq)^n = \frac{1}{1 - qu}, \quad \text{for } |u| < \frac{1}{q}.$$

Clearly, $\mathcal{Z}(u)$ has a simple pole at $u = \frac{1}{q}$ with residue $-\frac{1}{q}$.

5. Dirichlet L-functions over function fields.

Let $Q \in \mathbb{A}$ be a non-zero polynomial and $\mathfrak{D} := \deg(Q) - 1$. Let χ be a non-trivial Dirichlet character modulo Q. The Dirichlet L-function $L(s, \chi)$ is defined by

$$L(s, \chi) = \sum_{f \in \mathcal{M}} \frac{\chi(f)}{|f|^s}, \quad \text{for } \Re(s) > 1.$$

Using $u = q^{-s}$, we rewrite $L(s, \chi)$ as

$$\mathcal{L}(u, \chi) = \sum_{f \in \mathcal{M}} \chi(f) u^{\deg(f)} = \sum_{n \geq 0} a_n(\chi) u^n, \quad \text{for } |u| < \frac{1}{q}.$$

Here $a_n(\chi) = \sum_{f \in \mathcal{M}_n} \chi(f)$. Let us assume that $n \geq \mathfrak{D} + 1 = \deg(Q)$. If $\deg(f) = n$, we can write $f = gQ + r$, where $\deg(r) \leq \mathfrak{D}$ or $r = 0$. Here g is a polynomial of degree $n - \deg(Q) \geq 0$. Since χ is periodic modulo Q and g can be chosen in $q^{n-\deg(Q)}$ ways, we have

$$a_n(\chi) = \sum_{f \in \mathcal{M}_n} \chi(f) = q^{n-\deg(Q)} \sum_{\deg(r) \leq \mathfrak{D}} \chi(r) = 0.$$

Hence, we have

$$L(s, \chi) = \sum_{\deg(f) \leq \mathfrak{D}} \frac{\chi(f)}{|f|^s}, \quad \text{or} \quad \mathcal{L}(u, \chi) = \sum_{n=0}^{\mathfrak{D}} a_n(\chi) u^n.$$

It follows that $\mathcal{L}(u, \chi)$ is a polynomial in u of degree $\mathfrak{D}$. Let $\Lambda(u, \chi)$ be the complete L-function which is defined by

$$\Lambda(u, \chi) = (1 - \mathfrak{a}_\chi u)^{-1} \mathcal{L}(u, \chi).$$

It satisfies the functional equation (see Rosen [71], Theorem 9.24A.)

$$\Lambda(u, \chi) = \epsilon(\chi) \left(q^{\frac{1}{2}} u\right)^{\mathfrak{D} - \mathfrak{a}_\chi} \Lambda\left(\frac{1}{uq}, \bar{\chi}\right),$$

where $|\epsilon(\chi)| = 1$ and $\mathfrak{a}_\chi$ equal to 1, if χ is an even character, and 0, otherwise. A Dirichlet character χ is called even if for all $c \in \mathbb{F}_q^*$ and $f \in \mathbb{A}$, $\chi(cf) = \chi(f)$, otherwise it's called odd character. We refer to Section 3.2 of Chapter 3 for details on quadratic Dirichlet L-function over function fields.

In general, it is difficult to study an individual L-function. It's usually easier to study L-functions in a certain family where they have some common properties and structures. This method usually gives information about each member of the family. So computing moments of L-functions is one of the important tools to study them. The moments methods have applications in many areas such as non-vanishing of L-functions at the central point $s = \frac{1}{2}$, zeros of L-functions on the critical line $s = \frac{1}{2} + it$ and the subconvexity problems. Although there is no precise definition of a family of L-functions, articles by Conrey, Farmer, Keating, Rubinstein, and Snaith [21] & [20], Diacoun, Goldfeld and Hoffstein [23], Iwaniec and Sarnak [45], and Sarnak [75] provide a variety of perspectives on this topic. Now we give some examples of moments of L-functions in certain families.

1. Moments of the Riemann zeta function on the critical line $s = \frac{1}{2} + it$.

   One of the main aims of studying the moments of the Riemann zeta function is to prove the Lindelöf hypothesis, which states that for each $\varepsilon > 0$

$$\zeta\Big(\frac{1}{2} + it\Big) \ll_{\varepsilon} t^{\varepsilon}.$$

   In 1916, Hardy-Littlewood [34] showed that

$$\frac{1}{T}\int_T^{2T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t \sim \log T, \quad \text{as} \quad T \to \infty.$$

   Ingham [41] proved an asymptotic formula for the fourth moment

$$\frac{1}{T}\int_T^{2T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 \mathrm{d}t \sim \frac{1}{2\pi^2}(\log T)^4,$$

   and in 1979, Heath-Brown [36] improved Ingham's result with explicit lower order terms. So far, we do not know the asymptotic formula for any higher moments of the Riemann zeta function. Using random matrix theory, Keating and Snaith [49] conjectured that

$$\frac{1}{T}\int_T^{2T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \mathrm{d}t \sim a_k g_k (\log T)^{k^2}, \quad \text{as} \quad T \to \infty.$$

   Here

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \binom{k-1}{j}^2 p^{-j},$$

   and

$$g_k = \prod_{j=1}^{k-1} \frac{j!}{(k+j)!}.$$

   In 2005, Conrey, Farmer, Keating, Rubinstein, and Snaith [21] refined the above conjecture, providing a more precise asymptotic expressions for the lower-order terms.

Ramachandra [69] showed that

$$\frac{1}{T}\int_{T}^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}\mathrm{d}t \gg (\log T)^{k^2},$$

for any $k \in \mathbb{N}$. Assuming the Riemann hypothesis (RH), Soundararajan [81] showed that for every positive real number k and $\varepsilon > 0$,

$$\frac{1}{T}\int_{T}^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}\mathrm{d}t \ll_{k,\varepsilon} c_k(\log T)^{k^2+\varepsilon},$$

and Harper [35] removed the exponent ε in the above bound.

2. Moments of the quadratic Dirichlet L-function at the central point $s = \frac{1}{2}$.

   Let d be a fundamental discriminant and $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$ denote the primitive quadratic Dirichlet character of conductor $|d|$. For the quadratic Dirichlet L-function $L\left(\frac{1}{2}, \chi_d\right)$, Jutila [47] proved that

$$\frac{1}{X}\sideset{}{^\flat}\sum_{|d|\le X} L\left(\frac{1}{2},\chi_d\right) = (a_1 \log X + a_0) + O(X^{-\frac{1}{4}+\varepsilon}),$$

   and

$$\frac{1}{X}\sideset{}{^\flat}\sum_{|d|\le X} L\left(\frac{1}{2},\chi_d\right)^2 = a_3(\log X)^3 + O((\log X)^{\frac{5}{2}+\varepsilon}),$$

   where a_i's are explicit constants and the summations are over the fundamental discriminants. For $k = 2, 3$, Soundararajan [80] obtained

$$\frac{1}{X}\sideset{}{^\flat}\sum_{|d|\le X} L\left(\frac{1}{2},\chi_d\right)^k \sim P_k(\log X),$$

   where P_k is a polynomial of degree $\frac{k(k+1)}{2}$. In the same paper, he also proved that for at least 87.5% of the odd square-free integers $d > 0$, $L\left(\frac{1}{2}, \chi_{8d}\right) \neq 0$. Let us note that, according to the Chowla conjecture [17] it is believed that $L\left(\frac{1}{2}, \chi_{8d}\right)$ never vanishes.

In 2000, Keating and Snaith [50] conjectured that, for k fixed with $\Re(k) \geq 0$

$$\frac{1}{X^\flat}\sideset{}{^\flat}\sum_{|d|\leq X} L\left(\frac{1}{2},\chi_d\right)^k \sim a_k' g_k' (\log X)^{\frac{1}{2}k(k+1)},$$

where

$$a_k' = 2^{-k(k+2)/2}\prod_{p\geq 3}\frac{(1-1/p)^{\frac{1}{2}k(k+1)}}{(1+1/p)}\left(\frac{1}{2}\left(1-\frac{1}{\sqrt{p}}\right)^{-k}+\frac{1}{2}\left(1+\frac{1}{\sqrt{p}}\right)^{-k}+\frac{1}{p}\right),$$

$$g_k' = \frac{G(k+1)\sqrt{\Gamma(k+1)}}{\sqrt{G(2k+1)\Gamma(2k+1)}},$$

$X^\flat$ is the number of fundamental discriminants in the sum, and G is the Barnes's G-function (see [50], [1]). Conrey $et.\, al.$ [21] revised the above conjecture in 2005, yielding more precise asymptotic formulations for the lower-order terms and the conjectured lower bound was proved by Rudnick and Soundararajan [73] for every even $k \in \mathbb{N}$

$$\frac{1}{X}\sideset{}{^\flat}\sum_{|d|\leq X} L\left(\frac{1}{2},\chi_d\right)^k \gg_k (\log X)^{\frac{1}{2}k(k+1)}.$$

3. Moments of the Dirichlet L-function at the central point $s = \frac{1}{2}$.

Let χ be a primitive Dirichlet character modulo q and $L\left(\frac{1}{2},\chi\right)$ denotes the Dirichlet L-function associated to χ. For the Dirichlet L-functions, Paley (in 1931) [67] proved that, as $q \to \infty$,

$$\frac{1}{\phi^*(q)}\sum_{\substack{\chi \bmod q\\ \chi \text{ primitive}}}\left|L\left(\frac{1}{2},\chi\right)\right|^2 \sim \frac{\phi(q)}{q}\log q,$$

where $\phi^*(q)$ denotes number of primitive characters modulo q. In 2011, Young [86] established

$$\frac{1}{\phi^*(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = R(\log q) + O\left(q^{-\frac{5}{512}+\varepsilon}\right),$$

as q (prime) $\to \infty$ and R is a polynomial of degree 4. Iwaniec and Sarnak [43] showed that at least 33.33% of the L-functions in the family of primitive characters modulo q do not vanish at $s = \frac{1}{2}$. This proportion was improved by Bui [11] to 34.11%. Bui and Keating [14] conjectured the main term of all even moments of $L(\frac{1}{2}, \chi)$:

$$\frac{1}{\phi^*(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim a_k g_k \prod_{p|q} \left(\sum_{m \geq 0} \frac{\tau_k(p^m)^2}{p^m} \right)^{-1} (\log q)^{k^2},$$

where $\tau_k(n) = \sum_{n_1 \ldots n_k = n} 1$. In 2005, Rudnik and Soundararajan [74] proved the lower bound for the above conjecture. They showed for all large prime q,

$$\frac{1}{\phi^*(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \gg_k (\log q)^{k^2}.$$

For the upper bounds, Huxley [40] obtained the following results

$$\frac{1}{Q} \sum_{q \leq Q} \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}} \left| L\left(\frac{1}{2}, \chi\right) \right|^6 \ll Q(\log Q)^9,$$

and

$$\frac{1}{Q} \sum_{q \leq Q} \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}} \left| L\left(\frac{1}{2}, \chi\right) \right|^8 \ll Q(\log Q)^{16},$$

which agree with the order of magnitude suggested above. Das and Khan [25] proved that for a sufficiently large prime p and for a fixed Hecke-Maass form f for $SL(2, \mathbb{Z})$

$$\sum_{\substack{\chi \bmod p \\ \chi \text{ even, primitive}}} L(\tfrac{1}{2}, f \otimes \chi) \overline{L(\tfrac{1}{2}, \chi)} = \tfrac{p-2}{2} L(1, f) + O\left(p^{\frac{7}{8}+\theta+\varepsilon}\right),$$

where θ represents the best bound towards the Ramanujan-Petersson conjecture for f, which can currently be taken to be $\theta = 7/64$. As a corollary they showed for every large p, there exists a primitive Dirichlet character χ modulo p such that $L(\frac{1}{2}, f \otimes \chi)L(\frac{1}{2}, \chi) \neq 0$.

This thesis consists of three projects, which are discussed below.

In Chapter 1, we study the problem of computing the first moment of quadratic twists of $GL(2)$ Hecke L-functions and L-functions of quadratic characters at the central point $s = \frac{1}{2}$. More precisely, let f be a holomorphic Hecke eigenform of weight $k \equiv 0 \pmod 4$ for $SL(2, \mathbb{Z})$ and h be a compactly supported smooth function. Then we have (see Theorem 1.1.1)

$$\sum_{(d,2)=1}^{\star} L(\tfrac{1}{2}, \chi_{8d})L(\tfrac{1}{2}, f \otimes \chi_{8d})h(\tfrac{8d}{D}) = c_1\hat{h}(0)D\log D + c_2 D \ + O(k^{\frac{1}{2}}D^{\frac{7}{8}+\varepsilon}),$$

where c_1, c_2 and $\hat{h}(0)$ are some absolute constants depending on f and the summation is over square-free integers.

We also prove an upper bound for the second moment of $L(\frac{1}{2}, \chi_{8d})L(\frac{1}{2}, f \otimes \chi_{8d})$. For any $\varepsilon > 0$, we have (see Theorem 1.1.2)

$$\sum_{\substack{0<8d\leq D\\(d,2)=1}}^{\star} \left| L(\tfrac{1}{2}, \chi_{8d})L(\tfrac{1}{2}, f \otimes \chi_{8d}) \right|^2 \ll D^{\frac{4}{3}+\varepsilon}.$$

As a corollary of the above results we obtain that (see Corollary 1.1.4)

$$\#\left\{\chi_{8d},\ 0 < 8d \leq D,\ d\,(\text{odd square-free}):\ L(\tfrac{1}{2}, \chi_{8d})L(\tfrac{1}{2}, f \otimes \chi_{8d}) \neq 0\right\} \gg D^{\frac{2}{3}-\varepsilon}.$$

Furthermore under the GRH we show that there are infinitely many primes p such that $L(\frac{1}{2}, \chi_8)L(\frac{1}{2}, f \otimes \chi_{8p}) \neq 0$, which follows from the result given below.

Assuming the GRH, we have (see Theorem 1.1.5)

$$\sum_{p>2} (\log p)\, L(\tfrac{1}{2}, \chi_{8p})L(\tfrac{1}{2}, f \otimes \chi_{8p})h(\tfrac{8p}{D}) = C_1\hat{h}(0)D\log D + C_2 D \ + O(D^{\frac{7}{8}+\varepsilon}),$$

where C_1, C_2 and $\hat{h}(0)$ are some absolute constants and the summation is over odd primes.

In Chapter 2, we compute the first moment of $L(1/2, F)L(1/2, g \otimes F)$ L-functions where g is a fixed holomorphic Hecke eigenform for $SL(2,\mathbb{Z})$ of weight $k \equiv 0 \,(\mathrm{mod}\, 4)$ and $\{F\}$ runs over an orthogonal basis of Hecke-Maass cusp forms for $SL(3,\mathbb{Z})$.

Let us denote $R = T^{\theta}$ for any fixed θ in $(0,1)$. We choose the test function $h(\boldsymbol{\mu})$ such that it has the localizing effect at a ball of radius R about $w(\boldsymbol{\mu}_0)$, where w are elements in the Weyl group $\mathfrak{W}$ of $GL(3,\mathbb{R})$. For a precise definition of $h(\boldsymbol{\mu})$, $\Lambda'_{1/2}$ and $\mathfrak{W}$ we refer to Chapter 2. Let $d_{\mathrm{spec}}\boldsymbol{\mu} := \mathrm{spec}(\boldsymbol{\mu})d\boldsymbol{\mu}$ with

$$\mathrm{spec}(\boldsymbol{\mu}) = \prod_{j=1}^{3}\left(3\nu_j \tan\left(\frac{3\pi}{2}\nu_j\right)\right) \quad \text{and} \quad d\boldsymbol{\mu} = d\mu_1\, d\mu_2 = d\mu_2\, d\mu_3 = d\mu_3\, d\mu_1.$$

Let us define

$$\mathcal{N}_F := \|F\|^2 \prod_{j=1}^{3} \cos\left(\frac{3\pi}{2}\nu_j\right)$$

to be the normalizing factor. Then we have (see Theorem 2.1.1)

$$\sum_F \frac{h(\boldsymbol{\mu}_F)}{\mathcal{N}_F} L(\tfrac{1}{2}, F)L(\tfrac{1}{2}, g \otimes F) = \frac{1}{192\,\pi^5} \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} M(\boldsymbol{\mu}, k)h(\boldsymbol{\mu})\mathrm{spec}(\boldsymbol{\mu})d\boldsymbol{\mu} + O\left(T^{\frac{17}{6}+\varepsilon}R^2\right),$$

where $M(\boldsymbol{\mu}, k)$ is defined by (2.1.1). As a corollary we prove that there exist infinitely many Hecke-Maass cusp forms F for $SL(3,\mathbb{Z})$ such that $L(\frac{1}{2}, F)L(\frac{1}{2}, g \otimes F) \neq 0$.

In Chapter 3, we obtain an analogue of a result of Chandee [16] on shifted moments of the Riemann zeta function over function fields for quadratic Dirichlet L-functions.

A generalized shifted moments of $\zeta(s)$ is defined as

$$M_{\boldsymbol{k}^{(m)}}(T, \boldsymbol{\alpha}^{(m)}) := \int_0^T |\zeta(\tfrac{1}{2} + it + i\alpha_1)|^{2k_1} \ldots |\zeta(\tfrac{1}{2} + it + i\alpha_m)|^{2k_m}\, dt,$$

where $\boldsymbol{k}^{(m)} = (k_1, \ldots, k_m)$ is a sequence of non-negative real numbers and $\boldsymbol{\alpha}^{(m)} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ with $\alpha_i \neq \alpha_j$ for $i \neq j$. In [16], Chandee obtained lower and upper bounds of $M_{\boldsymbol{k}^{(m)}}(T, \boldsymbol{\alpha}^{(m)})$ for some special choices of $\boldsymbol{\alpha}^{(m)}$ and for large values of T.

More precisely, assuming the RH, she proved that

$$M_{\boldsymbol{k}^{(m)}}(T,\boldsymbol{\alpha}^{(m)}) \ll_{\boldsymbol{k}^{(m)},\varepsilon} T\,(\log T)^{k_1^2+\ldots+k_m^2+\varepsilon} \prod_{i<j} \left(\min\left\{\tfrac{1}{|\alpha_i-\alpha_j|},\log T\right\}\right)^{2k_ik_j}$$

and unconditionally

$$M_{\boldsymbol{k}^{(m)}}(T,\boldsymbol{\alpha}^{(m)}) \gg_{\boldsymbol{k}^{(m)},\boldsymbol{\alpha}^{(m)}} T\,(\log T)^{k_1^2+\ldots+k_m^2} \prod_{i<j} \left(\min\left\{\frac{1}{|\alpha_i-\alpha_j|},\log T\right\}\right)^{2k_ik_j},$$

for sufficiently large T.

Let $D \in \mathbb{F}_q[t]$ be a monic square-free polynomial. The quadratic character χ_D attached to D is defined using the quadratic residue symbol for $\mathbb{F}_q[t]$ by $\chi_D(f) = \left(\frac{D}{f}\right)$ and the corresponding Dirichlet L-function is denoted by $L(s,\chi_D) = \mathcal{L}(u,\chi_D)$ with $u = q^{-s}$.

Define the hyperelliptic ensemble $\mathcal{H}_{n,q}$ or simply $\mathcal{H}_n$ as

$$\mathcal{H}_n = \{D \in \mathbb{F}_q[t] : D \text{ is monic, square-free, and } \deg(D) = n\}.$$

For each D in the hyperelliptic ensemble $\mathcal{H}_n$, there is an associated hyperelliptic curve given by $C_D : y^2 = D(t)$. This curve is non-singular and of genus g given by

$$2g = n - 1 - \lambda, \tag{0.0.1}$$

where $\lambda = 1$, if n even, and 0, otherwise. Let us note that, $g \to \infty$ as n does so. For more details, see Section 3.2.

Let $\boldsymbol{k}^{(m)} = (k_1,\ldots,k_m) \in \mathbb{R}_+^m$ be a fixed m-tuple ($m \geq 1$) and $\boldsymbol{v}^{(m)} = (v_1,\ldots,v_m) \in \mathbb{C}^m$ with $v_j = e^{i\theta_j}$, $\theta_j \in [0,\pi)$ and $\alpha_j \in [0,\frac{1}{2})$ for $j = 1,\ldots,m$. Also, $\theta_j = \theta_j(g)$ is a real valued function of g such that $\lim_{g\to\infty} g|\theta_j|$ and $\lim_{g\to\infty} g|\theta_i-\theta_j|$, $i \neq j$ exist or equal ∞. Let us note that one can obtain the moments of $\mathcal{L}\left(\frac{v}{\sqrt{q}},\chi_D\right)$ by allowing the shifts α_j to tend to 0.

From now onwards, we assume that n and g are connected via (0.0.1). Let us define

$$\mu(\boldsymbol{v}^{(m)},g) = \sum_{j=1}^{m} k_j \log\left(\min\left\{\frac{1}{2|\theta_j|},g\right\}\right), \tag{0.0.2}$$

$$\sigma(\boldsymbol{v}^{(m)},g) = 2\left(\sum_{j=1}^{m} k_j^2\right)\log g \,+\, 2\sum_{j=1}^{m} k_j^2 \log\left(\min\left\{\frac{1}{2|\theta_j|},g\right\}\right)$$

$$+\ 4\sum_{i<j} k_i k_j \left(\log\left(\min\left\{\tfrac{1}{|\theta_i-\theta_j|}, g\right\}\right) + \log\left(\min\left\{\tfrac{1}{|\theta_i+\theta_j|}, g\right\}\right)\right). \tag{0.0.3}$$

Let $\boldsymbol{k}^{(m)} \in \mathbb{R}_+^m$ and $\boldsymbol{v}^{(m)}$ be as before. Assume that $\alpha_j = O\left(\frac{1}{g}\right)$ for $j = 1, 2, \dots, m$. We prove that (see Theorem 3.1.13) for n large, and for any $\epsilon > 0$,

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\right|^{2k_1}\cdots\left|\mathcal{L}\Big(\frac{v_m}{q^{\frac{1}{2}+\alpha_m}}, \chi_D\Big)\right|^{2k_m}$$
$$\ll_{\boldsymbol{k}^{(m)},\varepsilon} n^{\varepsilon}\exp\left(\mu\left(\boldsymbol{v}^{(m)}, g\right) + \frac{1}{2}\sigma\left(\boldsymbol{v}^{(m)}, g\right)\right),$$

where $\mu\left(\boldsymbol{v}^{(m)}, g\right)$ and $\sigma\left(\boldsymbol{v}^{(m)}, g\right)$ are defined by (0.0.2) and (0.0.3) respectively.

Let us define $\mathcal{L}^{(l)}(u, \chi_D)$ as the l-th derivative of $\mathcal{L}(u, \chi_D)$. As an important consequence of the above result, we have the following upper bound. Let $l \in \mathbb{N}$ and $\varepsilon > 0$. For n large, we have (see Theorem 3.1.14)

$$\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}^{(l)}\left(q^{-1/2}, \chi_D\right)\right|^k \ll_{k,l,\varepsilon} |\mathcal{H}_n| g^{\frac{1}{2}k(k+1)+lk+\varepsilon}.$$

We also obtain a lower bound of the same order of magnitude as the upper bound. For $m = 2$, we set

$$W = \{j \in \{1,2\} : \lim_{g\to\infty} g|\theta_j| < \infty\} \text{ and } W^c = \{j \in \{1,2\} : \lim_{g\to\infty} g|\theta_j| = \infty\}. \tag{0.0.4}$$

We also define a constant depending on W and W^c as

$$c_{\boldsymbol{v}^{(2)}} = \max\left\{\lim_{g\to\infty} g|\theta_1|,\ \lim_{g\to\infty} g|\theta_2|,\ \lim_{g\to\infty} g|\theta_1-\theta_2|,\ \lim_{g\to\infty} g|\theta_1+\theta_2|\right\}, \tag{0.0.5}$$

where the maximum is taken only over the finite entries of the set.

Let us note that, if $|W| = 2$ and $|W^c| = 0$ then

$$c_{\boldsymbol{v}^{(2)}} = \max\left\{\lim_{g\to\infty} g|\theta_1|,\ \lim_{g\to\infty} g|\theta_2|,\ \lim_{g\to\infty} g|\theta_1-\theta_2|,\ \lim_{g\to\infty} g|\theta_1+\theta_2|\right\}.$$

If $W = \{1\}$ and $W^c = \{2\}$ then $c_{\boldsymbol{v}^{(2)}} = \lim_{g\to\infty} g|\theta_1|$. If $W = \emptyset$ and $\lim\limits_{g\to\infty} g|\theta_1-\theta_2| < \infty$, then $c_{\boldsymbol{v}^{(2)}} = \lim\limits_{g\to\infty} g|\theta_1-\theta_2|$.

Assume that $\alpha_j = O\left(\frac{1}{g}\right)$ for $j = 1, 2$ and $\boldsymbol{k}^{(2)} \in \mathbb{N}^2$. Then for n large, we show that (see Theorem 3.1.11)

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\right|^{2k_1}\left|\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big)\right|^{2k_2} \gg_{\boldsymbol{k}^{(2)}, \boldsymbol{v}^{(2)}} \exp\left(\mu\left(\boldsymbol{v}^{(2)}, g\right) + \frac{1}{2}\sigma\left(\boldsymbol{v}^{(2)}, g\right)\right),$$

where $\mu\left(\boldsymbol{v}^{(2)}, g\right)$, $\sigma\left(\boldsymbol{v}^{(2)}, g\right)$ and $c_{\boldsymbol{v}^{(2)}}$ are defined by (0.0.2), (0.0.3) and (0.0.5) respectively. For simplicity, we will provide the complete proof of the above result in Section 3.4. One can easily extend the result for any $m \geq 3$ (see Theorem 3.1.12). Let $\boldsymbol{k}^{(m)}$ and $\boldsymbol{v}^{(m)}$ be as earlier. Assume that $\alpha_j = O\left(\frac{1}{g}\right)$ for all j and $\boldsymbol{k}^{(m)} \in \mathbb{N}^m$. Then for n large,

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\right|^{2k_1}\cdots\left|\mathcal{L}\Big(\frac{v_m}{q^{\frac{1}{2}+\alpha_m}}, \chi_D\Big)\right|^{2k_m} \gg_{\boldsymbol{k}^{(m)}, \boldsymbol{v}^{(m)}} \exp\left(\mu\left(\boldsymbol{v}^{(m)}, g\right) + \frac{1}{2}\sigma\left(\boldsymbol{v}^{(m)}, g\right)\right),$$

where $\mu\left(\boldsymbol{v}^{(m)}, g\right)$ and $\sigma\left(\boldsymbol{v}^{(m)}, g\right)$ are defined by (0.0.2) and (0.0.3) respectively.

In fact we can say that $\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)$ and $\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big)$ are essentially correlated when $|\theta_j| \asymp \frac{1}{g}$ for $j = 1, 2$ and independent when one of θ_j's is much larger than $\frac{1}{g}$. More precisely, we prove the following bounds (see Corollary (3.1.17) and (3.1.18)) for every $\varepsilon > 0$, $k \in \mathbb{R}_+$ and n large

$$\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big)\right|^{2k}$$
$$\ll_{k,\varepsilon}\begin{cases} |\mathcal{H}_n|\ g^{2k(4k+1)+\varepsilon} & \text{if } \ |W| = 2,\\ \frac{|\mathcal{H}_n|\ g^{3k^2+k+\varepsilon}}{|\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if } \ W = \{1\}, W^c = \{2\},\\ \frac{|\mathcal{H}_n|\ g^{4k^2+\varepsilon}}{|\theta_1\theta_2|^{k^2+k}|\theta_1+\theta_2|^{2k^2}} & \text{if } \ |W^c| = 2,\ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| < \infty,\\ \frac{|\mathcal{H}_n|\ g^{2k^2+\varepsilon}}{|\theta_1\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if } \ |W^c| = 2,\ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| = \infty, \end{cases}$$

and for $k \in \mathbb{N}$,

$$\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big)\right|^{2k}$$

$$\gg_{k,\boldsymbol{v}^{(2)}} \begin{cases} |\mathcal{H}_n| \ g^{2k(4k+1)} & \text{if} \quad |W| = 2, \\ \frac{|\mathcal{H}_n| \ g^{3k^2+k}}{|\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if} \quad W = \{1\}, W^c = \{2\}, \\ \frac{|\mathcal{H}_n| \ g^{4k^2}}{|\theta_1\theta_2|^{k^2+k}|\theta_1+\theta_2|^{2k^2}} & \text{if} \ |W^c| = 2, \ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| < \infty \\ \frac{|\mathcal{H}_n| \ g^{2k^2}}{|\theta_1\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if} \ |W^c| = 2, \ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| = \infty. \end{cases}$$

where W and W^c are defined by (0.0.4).

Note that the first Chapter 1 is based on [1]. The second Chapter 2 is based on [2], a collaboration with Kummari Mallesham. The third Chapter 3 is based on [3], a collaboration with Pranendu Darbar.

Chapter 1

First moment of quadratic twist of $GL(2)$ L-functions and quadratic L-functions at $s = 1/2$

1.1 Introduction

The central values of L-functions are related to several deep problems with great significance in analytic number theory and therefore, it is of profound interest to understand when the central value of an L-function is non-zero. Moreover, it is an interesting question to understand whether two or more L-functions are simultaneously non-vanishing at the central point and this type of questions have been studied by several authors (see for example [44], [70], [60], [8], [80], [85], [55], [56], [10], [25], [63], [76]). Let S_k denote the space of holomorphic Hecke eigenforms of weight k (even integer ≥ 4) for the full modular group $SL(2,\mathbb{Z})$. Soundararajan and Young [82] proved that for $f \in S_k$ with $k \equiv 0 \pmod 4$,

$$\sideset{}{^\star}\sum_{\substack{0<8d\leq D\\(d,2)=1}} |L(\tfrac{1}{2}, f \otimes \chi_{8d})|^2 \geq (c + o(1)) D \log D, \quad (c \text{ is a positive constant})$$

where the lower bound is unconditional and the equality holds under the GRH. In 2010, Hoffstein and Kontorovich [38] proved by the multiple Dirichlet series method that there exists some $|d| \ll (N_1 N)^{2+\varepsilon}$ such that $L(\frac{1}{2}, \chi_d \chi_{N_1}) L(\frac{1}{2}, f \otimes \chi_d) \neq 0$, where $L(s,f)$ has analytic conductor N and χ_{N_1} is a quadratic Dirichlet character of conductor N_1. The method of Soundrarajan's earlier work [79] allows us to treat the case of quadratic twists of two distinct GL_1 L-functions; i.e., one can show $L(\frac{1}{2}, \chi_1 \otimes \chi_d)L(\frac{1}{2}, \chi_2 \otimes \chi_d) \neq 0$ for infinitely many quadratic characters χ_d, where χ_1, χ_2 are fixed characters modulo q. However, the case of quadratic twists of two distinct GL_2 L-functions; i.e., proving the existence of a single quadratic character χ_d such that for fixed $f, g \in S_k$, $L(\frac{1}{2}, f \otimes \chi_d)L(\frac{1}{2}, g \otimes \chi_d) \neq 0$ is completely out of reach by current techniques. Even under the GRH, the method of Soundararajan-Young [82] does not work in this case.

In this chapter, we study first moment of quadratic Dirichlet L-functions and twist of modular L-functions for $SL(2, \mathbb{Z})$ at the central point.

Every holomorphic Hecke eigenform f in S_k admits a Fourier expansion given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \quad z \in \mathbb{H},$$

where $e(nz) = e^{2\pi i n z}$ with $\lambda_f(1) = 1$ and $|\lambda_f(n)| \leq \tau(n)$ for all n, ($\tau(n)$ is the divisor function of n). The coefficient $\lambda_f(n)$ is called the normalized Fourier coefficient. Let d be a fundamental discriminant[1] and $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$ denote the primitive quadratic Dirichlet character of conductor $|d|$. Define the Dirichlet L-function $L(s, \chi_d)$ by

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s},$$

and the twisted L-function $L(s, f \otimes \chi_d)$ by

$$L(s, f \otimes \chi_d) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \chi_d(n).$$

Both the L-functions converge absolutely in the half plane $\Re(s) > 1$. Moreover, they admit analytic continuation to the entire complex plane and satisfy the following functional equations. We set

$$\Lambda(s, \chi_d) = (\tfrac{|d|}{\pi})^{(\frac{s+\mathfrak{a}}{2})} \Gamma(\tfrac{s+\mathfrak{a}}{2}) L(s, \chi_d)$$

and

$$\Lambda(s, f \otimes \chi_d) = (\tfrac{|d|}{2\pi})^{s} \Gamma(s + \tfrac{k-1}{2}) L(s, f \otimes \chi_d).$$

---

[1] If d is a fundamental discriminant, then $d \equiv 1 \pmod 4$ and d is square-free, or $d = 4d_0$, where $d_0 \equiv 2, 3 \pmod 4$ and d_0 is square-free.

where $\mathfrak{a}$ is 0 if χ_d is even and 1 if χ_d is odd. Then the L-functions satisfy the functional equations

$$\Lambda(1-s,\chi_d)=\epsilon(\chi_d)\Lambda(s,\chi_d) \ \text{ and } \ \Lambda(s,f\otimes\chi_d)=\epsilon(f\otimes\chi_d)\Lambda(1-s,f\otimes\chi_d),$$

where $\epsilon(\chi_d)=\frac{i^{\mathrm{a}}\sqrt{|d|}}{\tau(\chi_d)}$ and $\epsilon(f\otimes\chi_d)=i^k\epsilon(d)$ with $\epsilon(d)=\left(\frac{d}{-1}\right)$ is 1 or -1 depending on whether d is positive or negative. Note that the sign of the functional equation of the twisted L-function $\epsilon(f\otimes\chi_d)$ is negative if $k\equiv 2\,(\mathrm{mod}\,4)$ and d is positive, or if $k\equiv 0\,(\mathrm{mod}\,4)$ and d is negative, and in these cases the central L-value is zero.

We assume that ε is an arbitrarily small positive quantity which is not necessarily the same at each occurrence, and the implied constants in $\ll$ and O depend on f or ε or on both f & ε. We denote $e(z)=\exp(2\pi iz)$ and $\int_{(c)}=\int_{c-i\infty}^{c+i\infty}$. The symbols $\square$, $\sum^{\star}$ and $\sum^{\flat}$ denote a perfect square, a sum over square-free integers and a sum over fundamental discriminants respectively, throughout the chapter. Also we reserve the letter p for primes throughout the chapter. Now we state the main theorem of this chapter.

Theorem 1.1.1. *Let $f\in S_k$ with $k\equiv 0\,(\mathrm{mod}\,4)$. Then for a smooth function h with compact support in $[0,1]$ satisfying the condition $t^{(i)}h^{(i)}(t)\ll_i 1$ for every $i\geq 0$, we have*

$$\sum_{(d,2)=1}^{\star} L(\tfrac{1}{2},\chi_{8d})L(\tfrac{1}{2},f\otimes\chi_{8d})h(\tfrac{8d}{D}) = c_1\hat{h}(0)D\log D + c_2 D \ + O(k^{\frac{1}{2}}D^{\frac{7}{8}+\varepsilon}), \tag{1.1.1}$$

where $\hat{h}(0)=\int_{-\infty}^{\infty}h(t)dt$, $c_1=\frac{4}{\pi^2}L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{L}_2(0,0)$ and c_2 are constants arising from $\mathcal{L}_2(u,v)$ defined in (1.3.11) and (1.3.12) with $c_1,c_2=O(k^{\varepsilon})$.

We also prove an upper bound for the second moment of $L(\frac{1}{2},\chi_{8d})L(\frac{1}{2},f\otimes\chi_{8d})$.

Theorem 1.1.2. *For any $\varepsilon>0$, we have*

$$\sum_{\substack{0<8d\leq D\\(d,2)=1}}^{\star} \left|L(\tfrac{1}{2},\chi_{8d})L(\tfrac{1}{2},f\otimes\chi_{8d})\right|^2 \ll D^{\frac{4}{3}+\varepsilon}. \tag{1.1.2}$$

As a corollary of Theorem 1.1.1, we have the following result.

Corollary 1.1.3. *Let f be a holomorphic Hecke eigenform of weight $k\equiv 0\,(\mathrm{mod}\,4)$ for the full modular group $SL(2,\mathbb{Z})$. Then there exist infinitely many primitive quadratic*

Dirichlet characters χ_{8d} *(where* d *runs over odd square-free positive integers) such that* $L(\frac{1}{2},\chi_{8d})L(\frac{1}{2},f\otimes\chi_{8d})\neq 0$.

Moreover, we will prove a stronger result than Corollary 1.1.3. Using Theorem 1.1.1, Theorem 1.1.2 and the Cauchy-Schwarz inequality, we have the following corollary.

Corollary 1.1.4. *For any* $\varepsilon>0$*, we have*

$$\#\left\{\chi_{8d},\ 0<8d\leq D,\ d\,(\textit{odd square-free}):\ L(\tfrac{1}{2},\chi_{8d})L(\tfrac{1}{2},f\otimes\chi_{8d})\neq 0\right\}\gg D^{\frac{2}{3}-\varepsilon}.$$

Furthermore, under the GRH, we can get the same results if we replace d (odd square-free) with p (prime). More precisely, we prove the following theorem.

Theorem 1.1.5. *Assuming the GRH, we have*

$$\sum_{p\,prime>2}(\log p)\,L(\tfrac{1}{2},\chi_{8p})L(\tfrac{1}{2},f\otimes\chi_{8p})h(\tfrac{8p}{D})=C_1\hat{h}(0)D\log D+C_2D\ +O(D^{\frac{7}{8}+\varepsilon}),\tag{1.1.3}$$

where $\hat{h}(0)=\int_{-\infty}^{\infty}h(t)dt$, $C_1=L(1,\mathrm{sym}^2f)L(1,f)\mathcal{Z}_2(0,0)$ *and* C_2 *are constants arising from* $\mathcal{Z}_2(u,v)$ *defined in* (1.5.4) *and* (1.5.5).

As a corollary of Theorem 1.1.5, we get the following result.

Corollary 1.1.6. *Let* f *be a holomorphic Hecke eigenform of weight* $k\equiv 0\,(\mathrm{mod}\,4)$ *for the full modular group* $SL(2,\mathbb{Z})$*. Assuming GRH, there exist infinitely many quadratic Dirichlet characters* χ_{8p} *(where* p *runs over odd primes) such that* $L(\frac{1}{2},\chi_{8p})L(\frac{1}{2},f\otimes\chi_{8p})\neq 0$.

We now make some remarks on Theorem 1.1.1.

Remark 1.1.7. *Our method is inspired by that of [82]. First, we express the central value of the L-function as a short sum using the approximate functional equation. Next, we sum over* d *using the Poisson summation formula. Then we extract the main term from the contribution of zero frequency on the dual side and estimate the rest. The large sieve inequality of Heath-Brown [37] plays a crucial role in the estimation.*

Remark 1.1.8. *In Theorem 1.1.1 and Theorem 1.1.2, we take* f *to be a modular form of full level and the sum over fundamental discriminants of the form* $8d$ *(*d *odd square-free) for simplicity. However, our results can be extended to congruence subgroups and* *to the sum over all discriminants. More precisely, one can consider the following: Let* a*,* q*,* d_1*,* d_2 *be fixed integers. Assume that* $d \equiv a \,(\mathrm{mod}\, q)$ *runs over square-free integers such that* $\chi_{d_1 d}$ *and* $\chi_{d_2 d}$ *are primitive quadratic characters. Let* f *be a fixed holomorphic Hecke eigenform of weight* k *and level* N*. Also assume,* $\epsilon(\chi_{d_1 d}) = \epsilon(f \otimes \chi_{d_2 d}) = 1$*. With some additional work, the method used here yields a result similar to Theorem 1.1.1 for the sum*

$$\sideset{}{^\star}\sum_{\substack{d \equiv a \bmod q \\ (d,l)=1}} L(\tfrac{1}{2}, \chi_{d_1 d}) L(\tfrac{1}{2}, f \otimes \chi_{d_2 d}) h(\tfrac{d}{D}),$$

where $l =$ *product of all distinct primes* p *such that* $p \mid d_1 d_2$*, but* $p \nmid q$*. For example, if* $f \in S_k$ *with* $k \equiv 2 \,(\mathrm{mod}\, 4)$*, one can study the sum*

$$\sideset{}{^\star}\sum_{(d,2)=1} L(\tfrac{1}{2}, \chi_{8d}) L(\tfrac{1}{2}, f \otimes \chi_{-8d}) h(\tfrac{8d}{D})\,;$$

and if $f \in S_k$ *with* $k \equiv 0 \,(\mathrm{mod}\, 4)$*, one can study the sums*

$$\sideset{}{^\star}\sum_{d \equiv 1 \bmod 4} L(\tfrac{1}{2}, \chi_{d}) L(\tfrac{1}{2}, f \otimes \chi_{8d}) h(\tfrac{d}{D})$$

and

$$\sideset{}{^\star}\sum_{\substack{d \equiv 3 \bmod 4 \\ (d,p)=1}} L(\tfrac{1}{2}, \chi_{pd}) L(\tfrac{1}{2}, f \otimes \chi_{4d}) h(\tfrac{d}{D}),$$

where $p \equiv 3 \,(\, mod 4)$ *a fixed prime, by this method and obtain asymptotic formula as in Theorem 1.1.1.*

1.2 Preliminaries

In this section we recall some standard formulae and estimates that will be used later. We shall work with discriminants of the form $8d$ where d is an odd, square-free positive integer, so that $\chi_{8d}(n) = \left(\frac{8d}{n}\right)$ is an even primitive character of conductor $8d$.

1.2.1 Approximate functional equations

For ξ, $c > 0$, define

$$U(\xi) := \frac{1}{2\pi i}\int_{(c)} g_1(s)\xi^{-s}\frac{ds}{s} \text{ and } V(\xi) = \frac{1}{2\pi i}\int_{(c)} \frac{g_2(s)}{s}\xi^{-s}ds, \tag{1.2.1}$$

where

$$g_1(s) = \pi^{-s/2}\left(\frac{\Gamma(\frac{s}{2}+\frac{1}{4})}{\Gamma(\frac{1}{4})}\right) \text{ and } g_2(s) = (2\pi)^{-s}\left(\frac{\Gamma(\frac{k}{2}+s)}{\Gamma(\frac{k}{2})}\right).$$

Then U and V are real-valued, smooth on $(0,+\infty)$, bounded as ξ approaches 0 and decays exponentially as $\xi \to +\infty$.

Lemma 1.2.1. *For an odd, positive, square-free integer d and $f \in S_k$ with $k \equiv 0 \pmod 4$, we have*

$$L(\tfrac{1}{2}, \chi_{8d}) = 2\sum_{n=1}^{\infty}\frac{\chi_{8d}(n)}{\sqrt{n}}U\left(\frac{n}{\sqrt{8d}}\right),$$

and

$$L(\tfrac{1}{2}, f\otimes\chi_{8d}) = 2\sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi_{8d}(n)}{\sqrt{n}}V\left(\frac{n}{8d}\right).$$

Proof. This is a special case of the approximate functional equation (See §5.2 of [42]). ☐

1.2.2 Poisson summation formula

Here we quote Lemma 2.6 of [80] which is an application of the Poisson summation formula.

Lemma 1.2.2. *Let Φ be a smooth function on $[0,\infty)$ with compact support. Let n be an odd integer. Then*

$$\sum_{(d,2)=1}\left(\frac{d}{n}\right)\Phi\left(\frac{d}{D}\right) = \frac{D}{2n}\left(\frac{2}{n}\right)\sum_{k\in\mathbb{Z}}(-1)^k G_k(n)\hat{\Phi}\left(\frac{kD}{2n}\right),$$

where

$$G_k(n) := \left(\frac{1-i}{2}+\left(\frac{-1}{n}\right)\frac{1+i}{2}\right)\sum_{a \bmod n}\left(\frac{a}{n}\right)e\left(\frac{ak}{n}\right),$$

and

$$\hat{\Phi}(y) := \int_{-\infty}^{+\infty} \left(\cos(2\pi xy) + \sin(2\pi xy)\right) \Phi(x) dx$$

is a Fourier-type transform of Φ.

The precise value of the Gauss-type sum $G_k(n)$ has been calculated in Lemma 2.3 of [80] as follows.

Lemma 1.2.3. *If m and n are relatively prime odd integers, then $G_k(mn) = G_k(m)G_k(n)$. Moreover, if p^α is the largest power of p dividing k (setting $\alpha = \infty$ if $k = 0$), then*

$$G_k(p^\beta) = \begin{cases} 0 & \text{if } \beta \le \alpha \text{ is odd,} \\ \phi(p^\beta) & \text{if } \beta \le \alpha \text{ is even,} \\ -p^\alpha & \text{if } \beta = \alpha + 1 \text{ is even,} \\ \left(\frac{kp^{-\alpha}}{p}\right) p^\alpha \sqrt{p} & \text{if } \beta = \alpha + 1 \text{ is odd,} \\ 0 & \text{if } \beta \ge \alpha + 2. \end{cases}$$

1.2.3 The large sieve inequality for quadratic characters

Heath-Brown [37] proved the following large sieve inequality for quadratic characters.

Theorem 1.2.4 (Heath-Brown)**.** *For any $M, N \ge 1$ and any sequence of complex numbers a_n, we have*

$$\sideset{}{^\star}\sum_{\substack{m\le M \\ (m,2)=1}} \left| \sideset{}{^\star}\sum_{\substack{n\le N \\ (n,2)=1}} a_n \left(\frac{n}{m}\right) \right|^2 \ll (MN)^\varepsilon (M+N) \sideset{}{^\star}\sum_{\substack{n\le N \\ (n,2)=1}} |a_n|^2,$$

for any $\varepsilon > 0$, where the implied constant depends only on ε.

As a applications of the above theorem we have following results.

Lemma 1.2.5. *For any fixed $\sigma \in \left[\frac{1}{2}, 1\right]$, for every $t \in \mathbb{R}$ and for every $\varepsilon > 0$, we have the bound*

$$\sideset{}{^\flat}\sum_{|d|\le D} |L(\sigma + it, \chi_d)|^4 \ll \left(D(1+|t|)\right)^{1+\varepsilon},$$

$$\sideset{}{^\flat}\sum_{|d|\le D} |L(\sigma + it, f \otimes \chi_d)|^2 \ll k(D(1+|t|))^{1+\varepsilon},$$

and

$$\sum_{|d|\le D}^{\flat} |L(\sigma+it,\chi_d)|^2 \ll D^{1+\varepsilon}\,(1+|t|)^{\frac{1}{2}+\varepsilon}\,,$$

where the implied constants depend only on ε.

Proof. See [Theorem 2, [37]] and [Corollary 2.5, [82]] for the proof of the first and the second results respectively. The third result follows from the Cauchy-Schwarz inequality and the first result. □

1.3 Proof of the Theorem 1.1.1

We define

$$S(D) = \sum_{(d,2)=1}^{\star} L(\tfrac{1}{2},\chi_{8d})L(\tfrac{1}{2}, f\otimes\chi_{8d})h(\tfrac{8d}{D}).$$

Then by Lemma 1.2.1, we have

$$S(D) = 4\sum_{(d,2)=1}^{\star}\sum_{n}\sum_{m}\frac{\lambda_f(m)\chi_{8d}(nm)}{\sqrt{nm}}h(\tfrac{8d}{D})U(\tfrac{n}{\sqrt{8d}})V(\tfrac{m}{8d}). \tag{1.3.1}$$

By using the Möbius inversion formula we remove the square-free condition in the d-sum in (1.3.1) and obtain

$$S(D) = 4\sum_{(a,2)=1}\mu(a)\sum_{(d,2)=1}\sum_{n}\sum_{m}\frac{\lambda_f(m)}{\sqrt{nm}}\chi_{8a^2d}(nm)h(\tfrac{8a^2d}{D})U(\tfrac{n}{\sqrt{8a^2d}})V(\tfrac{m}{8a^2d}). \tag{1.3.2}$$

Since h is compactly supported, the contributing range of the variable a is essentially up to $\ll \sqrt{D}$. For a fixed a, by the definition of U, V in (1.2.1), we obtain that the inner sum over d, n and m in the above expression (1.3.2) is given by

$$\begin{aligned}&\frac{1}{(2\pi i)^2}\int_{(3)}\frac{\Gamma(\frac{u}{2}+\frac{1}{4})}{\Gamma(\frac{1}{4})}\int_{(3)}\frac{\Gamma(\frac{k}{2}+v)}{\Gamma(\frac{k}{2})}\\ &\quad\times\sum_{(d,2)=1}L(\tfrac{1}{2}+u,\chi_{8a^2d})L(\tfrac{1}{2}+v,f\otimes\chi_{8a^2d})\left(\frac{8a^2d}{\sqrt{\pi}}\right)^{u}\left(\frac{8a^2d}{2\pi}\right)^{v}h(\tfrac{8a^2d}{D})\frac{du}{u}\frac{dv}{v}.\end{aligned} \tag{1.3.3}$$

We move the contour to the lines $\Re(u) = \Re(v) = \varepsilon$ without encountering a pole and apply the bound

$$L(\tfrac{1}{2}+u,\chi_{8a^2d})L(\tfrac{1}{2}+v,f\otimes\chi_{8a^2d}) \ll (ad)^{\varepsilon}|L(\tfrac{1}{2}+u,\chi_{d_0})L(\tfrac{1}{2}+v,f\otimes\chi_{d_0})|,$$

where d_0 is the square-free part of $2d$. Now we apply the Cauchy-Schwarz inequality which yields the following bound for (1.3.3)

$$\ll D^{\varepsilon} \int_{(\varepsilon)} \left| \frac{\Gamma(\frac{u}{2}+\frac{1}{4})}{\Gamma(\frac{1}{4})} \right| \left\{ \sideset{}{^\star}\sum_{d_0 \le D/a^2} |L(\tfrac{1}{2}+u, \chi_{d_0})|^2 \right\}^{1/2} \frac{|du|}{|u|}$$

$$\times \int_{(\varepsilon)} \left| \frac{\Gamma(\frac{k}{2}+v)}{\Gamma(\frac{k}{2})} \right| \left\{ \sideset{}{^\star}\sum_{d_0 \le D/a^2} |L(\tfrac{1}{2}+v, f \otimes \chi_{d_0})|^2 \right\}^{1/2} \frac{|dv|}{|v|}.$$

By Lemma 1.2.5, we have

$$\sideset{}{^\star}\sum_{d_0 \le D/a^2} |L(\tfrac{1}{2}+u, \chi_{d_0})|^2 \ll (1+|u|)^{1/2} a^{-2} D^{1+\varepsilon},$$

and

$$\sideset{}{^\star}\sum_{d_0 \le D/a^2} |L(\tfrac{1}{2}+v, f \otimes \chi_{d_0})|^2 \ll k(1+|v|) a^{-2} D^{1+\varepsilon}.$$

Using Stirling asymptotics, we also have $|\Gamma(\frac{u}{2}+\frac{1}{4})\Gamma(\frac{1}{4})^{-1}| \ll (1+|u|)^{-1}$ and $|\Gamma(\frac{k}{2}+v)\Gamma(\frac{k}{2})^{-1}| \ll k(|v|+k)^{-1}$. It follows that the quantity in (1.3.3) is bounded above by $O(a^{-2}k^{1/2}D(kD)^{\varepsilon}$. Therefore, we have

$$\sum_{\substack{a>Y\\(a,2)=1}} \mu(a) \sum_{(d,2)=1} \sum_{n} \sum_{m} \frac{\lambda_f(m)}{\sqrt{nm}} \chi_{8a^2d}(nm) h(\tfrac{8a^2d}{D}) U(\tfrac{n}{\sqrt{8a^2d}}) V(\tfrac{m}{8a^2d})$$

$$\ll \frac{k^{1/2}D}{Y}(kD)^{\varepsilon}, \tag{1.3.4}$$

for an appropriate parameter Y to be chosen later.

Now we analyse contribution of $a \le Y$ in the sum (1.3.2). Let

$$S_Y(D) = 4 \sum_{\substack{a\le Y\\(a,2)=1}} \mu(a) \sum_{(n,2a)=1} \sum_{(m,2a)=1} \frac{\lambda_f(m)}{\sqrt{nm}} \sum_{(d,2)=1} \chi_{8d}(nm) h(\tfrac{8a^2d}{D}) U(\tfrac{n}{\sqrt{8a^2d}}) V(\tfrac{m}{8a^2d}). \tag{1.3.5}$$

By applying Lemma 1.2.2 (Poisson summation formula), we first execute d sum in the above expression to obtain

$$S_Y(D) = 2D \sum_{\substack{a\le Y\\(a,2)=1}} \frac{\mu(a)}{a^2} \sum_{(n,2a)=1} \sum_{(m,2a)=1} \left(\frac{2}{nm}\right) \frac{\lambda_f(m)}{\sqrt{nm}} \sum_{l\in\mathbb{Z}} (-1)^l \frac{G_l(nm)}{nm} I(l,n,m,a), \tag{1.3.6}$$

where the integral $I(l, n, m, a)$ equals

$$\begin{aligned} & \int_{\mathbb{R}} H(x)\left(\cos\left(\frac{2\pi lxD}{2nma^2}\right) + \sin\left(\frac{2\pi lxD}{2nma^2}\right)\right)dx && (1.3.7)\\ = \; & \frac{(1+i)}{2}\widehat{H}(\frac{lD}{2nma^2}) + \frac{(1-i)}{2}\widehat{H}(-\frac{lD}{2nma^2}) && (1.3.8)\\ = \; & I_1 + I_2\,, \quad \text{say.} && (1.3.9)\end{aligned}$$

Here $\widehat{H}(y) = \int\limits_{\mathbb{R}} H(x)e(-xy)dx$ is the Fourier transform of $H(x) = h(x)U(\frac{n}{\sqrt{xD}})V(\frac{m}{xD})$.

1.3.1 The main term

In the expression (1.3.6), the main contribution comes from the zero frequency $l = 0$. This we denote by $S_Y^0(D)$. Note that $G_0(nm) = \phi(mn)$ if $nm = \Box$, and is 0 otherwise. Also,

$$\sum_{\substack{a \le Y\\(a,2nm)=1}} \frac{\mu(a)}{a^2} = \frac{1}{\zeta(2)}\prod_{p|2nm}\left(1-\frac{1}{p^2}\right)^{-1} + O(Y^{-1}) = \frac{8}{\pi^2}\prod_{p|nm}\left(1-\frac{1}{p^2}\right)^{-1} + O(Y^{-1}).$$

Now, using the definition of U and V in (1.2.1), we have

$$\begin{aligned}\widehat{H}(0) &= \int_{\mathbb{R}} h(x)U(\tfrac{n}{\sqrt{xD}})V(\tfrac{m}{xD})dx\\ &= \frac{1}{(2\pi i)^2}\int_{(3)}\int_{(3)} g_1(u)g_2(v)\left(\frac{\sqrt{D}}{n}\right)^u\left(\frac{D}{m}\right)^v F(\tfrac{u}{2}+v)\frac{du}{u}\frac{dv}{v},\end{aligned}$$

where $F(s) = \int\limits_{\mathbb{R}} h(x)x^s dx$. Hence, by the above observation we get

$$\begin{aligned} S_Y^0(D) = \frac{16D}{\pi^2}\frac{1}{(2\pi i)^2}\int_{(3)}\int_{(3)} & \frac{g_1(u)g_2(v)}{uv}D^{\frac{u}{2}}D^v F(\tfrac{u}{2}+v)\mathcal{L}(u,v)dv\ du\\ & + O\left(\frac{D}{Y}\sum_{\substack{(nm,2)=1\\nm=\Box}}\frac{|\lambda(m)|}{\sqrt{nm}}\left|\int_{-\infty}^{\infty} h(x)U(\tfrac{n}{\sqrt{xD}})V(\tfrac{m}{xD})dx\right|\right), \quad (1.3.10)\end{aligned}$$

where

$$\mathcal{L}(u,v) = \sum_{\substack{(nm,2)=1\\nm=\Box}}\frac{\lambda_f(m)}{n^{\frac{1}{2}+u}m^{\frac{1}{2}+v}}\prod_{p|nm}\left(\frac{p}{p+1}\right). \quad (1.3.11)$$

Let $\mathcal{L}_p(u,v)$ denote the local Euler factor of $\mathcal{L}(u,v)$ at the prime p. For $p>2$

$$\mathcal{L}_p(u,v) \quad = \quad 1+\left(\frac{p}{p+1}\right)\sum_{\substack{i,j\geq 0\\ i+j\geq 1\\ i+j=even}}\frac{\lambda_f(p^j)}{p^{i(\frac{1}{2}+u)+j(\frac{1}{2}+v)}}.$$

In the region $\Re(u)$, $\Re(v)>-\frac{1}{4}+\varepsilon$ this Euler factor equals

$$1+\left(\frac{p}{p+1}\right)\left\{\sum_{i\geq 1}\frac{1}{p^{i(1+2u)}}+\sum_{j\geq 1}\frac{\lambda_f(p^{2j})}{p^{j(1+2v)}}+\sum_{\substack{i\geq 1\\ i \text{ odd}}}\frac{\lambda_f(p)}{p^{i(\frac{1}{2}+u)+(\frac{1}{2}+v)}}+O\left(\frac{1}{p^{2(1+u+v)}}\right)\right\}.$$

Thus we write

$$\mathcal{L}(u,v)=\zeta(1+2u)L(1+2v,\mathrm{sym}^2(f))L(1+u+v,f)\mathcal{L}_2(u,v), \tag{1.3.12}$$

where $\mathcal{L}_2(u,v)$ is a Dirichlet series which converges absolutely for $\Re(u)$, $\Re(v)>-\frac{1}{4}+\varepsilon$ and is uniformly bounded. Now we evaluate the double integral in the first term on the right side of the equation (1.3.10). We first shift the contour to $\Re(u)=\Re(v)=\frac{1}{10}$ without encountering a pole. Then we move the contour $\Re(v)=\frac{1}{10}$ to $\Re(v)=-\frac{1}{5}$, in doing so we encounter a simple pole at $v=0$. Since $F(s)\ll(1+|t|)^{-A}$ for any $A>0$, by integration by parts, the integrals on $\Re(u)=\frac{1}{10}$, $\Re(v)=-\frac{1}{5}$ are bounded by $O(D^{-\frac{3}{20}+\varepsilon})$. The contribution from the residue at $v=0$ is

$$\frac{1}{2\pi i}\int_{(\frac{1}{10})}\frac{g_1(u)}{u}D^{\frac{u}{2}}\zeta(1+2u)L(1,\mathrm{sym}^2f)L(1+u,f)\mathcal{L}_2(u,0)F(\frac{u}{2})du. \tag{1.3.13}$$

Now we shift the contour in (1.3.13) to $\Re(u)=-\frac{1}{4}+2\varepsilon$ encountering a double pole at $u=0$. The integral on the line $\Re(u)=-\frac{1}{4}+2\varepsilon$ is bounded by $O(D^{-\frac{1}{8}+\varepsilon})$. The residue at $u=0$ is

$$\begin{aligned}&\frac{1}{4}F(0)L(1,\mathrm{sym}^2f)L(1,f)\mathcal{L}_2(0,0)\log D+\frac{1}{2}F(0)g_1'(0)L(1,\mathrm{sym}^2f)L(1,f)\mathcal{L}_2(0,0)\\ &\quad+\frac{1}{2}F(0)L(1,\mathrm{sym}^2f)L'(1,f)\mathcal{L}_2(0,0)+\frac{1}{2}F(0)L(1,\mathrm{sym}^2f)L(1,f)\mathcal{L}_2'(0,0)\\ &\quad\quad+\gamma\ F(0)L(1,\mathrm{sym}^2f)L(1,f)\mathcal{L}_2(0,0)+\frac{1}{4}F'(0)L(1,\mathrm{sym}^2f)L(1,f)\mathcal{L}_2(0,0)\end{aligned}$$

(γ is the Euler constant)

$$=\frac{1}{4}\widehat{h}(0)L(1,\mathrm{sym}^2f)L(1,f)\mathcal{L}_2(0,0)\log D+\frac{c_2}{16}, \tag{1.3.14}$$

where c_2 is sum of the constant terms. Therefore, we get the following

$$S_Y^0(D) = c_1\widehat{h}(0)D\log D + c_2D + O\left(D^{1+\varepsilon}Y^{-1} + D^{\frac{17}{20}+\varepsilon}\right), \tag{1.3.15}$$

where $c_1 = \frac{4}{\pi^2}L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{L}_2(0,0)$.

1.3.2 The error term

It remains to estimate the contribution of the non-zero frequencies $l \neq 0$ in (1.3.6). Denote this by $S_Y^\star(D)$. The l-sum in (1.3.6) is essentially of the form

$$\sum_{l\in\mathbb{Z}} G_l(nm)\int_{\mathbb{R}} h(x)U(\tfrac{n}{\sqrt{xD}})V(\tfrac{m}{xD})e\left(-\frac{xlD}{2nma^2}\right)dx. \tag{1.3.16}$$

By integration by parts, one can see that the integral is negligibly small if $|l| \geq Y^2D^{1/2+\varepsilon} \geq \frac{nma^2}{D}$. To study the remaining part in (1.3.6) we obtain a different expression for I_1 (and I_2) using the Mellin inversion formula. Let $\widetilde{g}(s)$ be the Mellin transform of the function $g(x)$. Note that if $\psi(x) = \sin x$ (or $\cos x$), then $\widetilde{\psi}(s) = \Gamma(s)\sin(\frac{1}{2}\pi s)$ $\left(\text{or}\,\Gamma(s)\cos(\frac{1}{2}\pi s)\right)$ for $0 < \Re(s) < 1$. Suppose $g(x)$ is a smooth compactly supported function on $\mathbb{R}$. Let $G(y) := \int_0^\infty g(x)\sin(2\pi xy)dx$, $y \neq 0$. Then by Mellin inversion, we get

$$\begin{aligned} G(y) &= \int_0^\infty \sin(2\pi xy)\,\frac{1}{2\pi i}\int_{(-1/2)}\widetilde{g}(s+1)x^{-s}ds\,\frac{dx}{x} \\ &= \int_0^\infty \sin(\mathrm{sgn}(y)x)\,\frac{1}{2\pi i}\int_{(1/2)}\widetilde{g}(1-s)(x/2\pi|y|)^s ds\,\frac{dx}{x}. \end{aligned}$$

Now by the above observation and interchanging the order of integration we have

$$\begin{aligned} G(y) &= \frac{1}{2\pi i}\int_{(1/2)}\widetilde{g}(1-s)\Gamma(s)\sin\left(\tfrac{\mathrm{sgn}(y)\pi s}{2}\right)(2\pi|y|)^{-s}ds \\ &= \frac{1}{2\pi i}\int_{(1/2+\varepsilon)}\widetilde{g}(1-s)\Gamma(s)\sin\left(\tfrac{\mathrm{sgn}(y)\pi s}{2}\right)(2\pi|y|)^{-s}ds. \end{aligned}$$

Let $W(x,y,z) = h(x/D)U(\frac{y}{\sqrt{x}})V(\frac{z}{x})$. By the above expression of $G(y)$, we get

$$I_1 = \frac{D^{-1}}{2\pi i}\int_{((1/2+\varepsilon)}\widetilde{W}\left(1-s,n,m\right)\left(\frac{nma^2}{\pi|l|}\right)^s\Gamma(s)\cos\left(\frac{\pi s}{2}\right)ds,$$

where $\widetilde{W}(s;y,z) = \int_0^\infty W(x,y,z)x^s\frac{dx}{x}$. Applying Mellin inversion on the variables n and m in the above expression, we get

$$I_1 = \frac{1}{D}\left(\frac{1}{2\pi i}\right)^3 \int_{(\frac{1}{2}+\varepsilon)}\int_{(\frac{1}{2}+\varepsilon)}\int_{(\frac{1}{2}+\varepsilon)} W^{\#}(1-s,u,v)\times \frac{1}{n^u m^v}\left(\frac{nma^2}{\pi|l|}\right)^s \Gamma(s)\cos\left(\frac{\pi s}{2}\right) ds\,du\,dv, \quad (1.3.17)$$

where $W^{\#}(s,u,v) = \int_0^\infty\int_0^\infty\int_0^\infty W(x,y,z)x^s y^u z^v \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$. We get a similar expression for I_2. Since $W(x,y,z)$ is smooth weight function, by integration by parts we have

$$|W^{\#}(s,u,v)| \ll \frac{D^{\Re(s)}D^{\Re(u/2)}D^{\Re(v)}}{|uv|(1+|s|)^{98}(1+|u|)^{49}(1+|v|)^{98}} \quad (1.3.18)$$

for $\Re(u),\Re(v) > 0$. Note that $G_l(n) = G_{4l}(n)$ for odd n. Therefore,

$$\begin{aligned} S_Y^{\star}(D) &= 2D \sum_{\substack{a\le Y\\(a,2)=1}} \frac{\mu(a)}{a^2} \sum_{(n,2a)=1}\sum_{(m,2a)=1}\left(\frac{2}{nm}\right)\frac{\lambda_f(m)}{\sqrt{nm}}\sum_{l\ne 0}\frac{G_l(nm)}{nm}(I_1+I_2) \\ &= S_1 + S_2, \qquad \text{say.} \end{aligned}$$

We now estimate S_1 and note that a similar estimation holds for S_2. We have

$$S_1 = 2\sum_{\substack{a\le Y\\(a,2)=1}}\frac{\mu(a)}{a^2}\sum_{l\ne 0}\sum_{(n,2a)=1}\sum_{(m,2a)=1}\frac{\lambda_f(m)}{\sqrt{nm}}\frac{G_{4l}(nm)}{nm}\times$$
$$\left(\frac{1}{2\pi i}\right)^3\int_{(\frac{1}{2}+\varepsilon)}\int_{(\frac{1}{2}+\varepsilon)}\int_{(\frac{1}{2}+\varepsilon)} W^{\#}(1-s,u,v)\frac{1}{n^u m^v}\left(\frac{nma^2}{\pi|l|}\right)^s\Gamma(s)\cos\left(\frac{\pi s}{2}\right) dsdudv. \quad (1.3.19)$$

Note that we can write $4l = l_1 l_2^2$, where l_1 is square-free and l_2 is positive, so that the sum over l in (1.3.19) becomes now a sum over l_1 (fundamental discriminant) and positive integers l_2. Consider the Dirichlet series

$$\mathcal{L}(\alpha,\beta,\gamma;a,l_1) = \sum_{l_2=1}^{\infty}\sum_{(n,2a)=1}\sum_{(m,2a)=1}\frac{\lambda_f(m)}{n^\alpha m^\beta |l_2|^{2\gamma}}\frac{G_{l_1 l_2^2}(nm)}{nm}, \quad (1.3.20)$$

which converges absolutely if $\Re(\alpha)$, $\Re(\beta)$, $\Re(\gamma) > \frac{1}{2}$. So the expression of S_1 in (1.3.19) becomes

$$S_1 = 2 \sum_{\substack{a \le Y \\ (a,2)=1}} \frac{\mu(a)}{a^2} \sideset{}{^\flat}\sum_{l_1} \left(\frac{1}{2\pi i}\right)^3 \int_{(\varepsilon)} \int_{(\varepsilon)} \int_{(\frac{1}{2}+\varepsilon)} W^{\#}(1-s, u+s, v+s)\Gamma(s) \\ \times \cos\left(\frac{\pi s}{2}\right) \left(\frac{a^2}{\pi |l_1|}\right)^s \mathcal{L}(\tfrac{1}{2}+u, \tfrac{1}{2}+v, s; a, l_1) ds\, du\, dv. \quad (1.3.21)$$

We now analyse the Dirichlet series $\mathcal{L}(\alpha,\beta,\gamma;a,l_1)$ in (1.3.20). Let $\mathcal{L}_p(\alpha,\beta,\gamma;a,l_1)$ denote the local Euler factor of the Dirichlet series $\mathcal{L}(\alpha,\beta,\gamma;a,l_1)$ at the prime p. First consider the case when $p \nmid 2al_1$. Then

$$\mathcal{L}_p(\alpha,\beta,\gamma;a,l_1) = \sum_{k,i,j \ge 0} \frac{\lambda_f(p^j))}{p^{i\alpha + j\beta + 2k\gamma}} \frac{G_{l_1 p^{2k}}(p^{i+j})}{p^{i+j}}.$$

Using the property of $G_k(n)$ (see Lemma 1.2.3), we analyze the local Euler factor $\mathcal{L}_p(\alpha,\beta,\gamma;a,l_1)$. The terms $k \ge 1$ contribute $\ll p^{-(1+2\varepsilon)}$, when $\Re(\gamma) \ge \frac{1}{2}+\varepsilon$ and $\Re(\alpha)$, $\Re(\beta) \ge 0$. The contribution of the term $k=0$ is $1 + \chi_{k_1}(p) p^{-\frac{1}{2}-\alpha} + \chi_{k_1}(p)\lambda_f(p) p^{-\frac{1}{2}-\beta}$. Similarly if $p | l_1$ but $p \nmid 2a$, then this local Euler factor is equal to

$$1 - \frac{1}{p^{1+2\alpha}} - \frac{\lambda_f(p^2)}{p^{1+2\beta}} - \frac{\lambda_f(p)}{p^{1+\alpha+\beta}} + O\left(\frac{1}{p^{1+\varepsilon}}\right),$$

in the region $\Re(\gamma) \ge \frac{1}{2}+\varepsilon$ and $\Re(\alpha)$, $\Re(\beta) \ge 0$. Finally, if $p | 2a$ the corresponding local Euler factor is $1 + O\left(\frac{1}{p^{1+2\varepsilon}}\right)$ in the same region as above. Therefore, we get

$$\mathcal{L}(\alpha,\beta,\gamma;a,l_1) = \frac{L_a(\frac{1}{2}+\alpha, \chi_{l_1}) L_a(\frac{1}{2}+\beta, f \otimes \chi_{l_1})}{\zeta_a(1+2\alpha) L_a(1+2\beta, \mathrm{sym}^2 f) L_a(1+\alpha+\beta, f)} \mathcal{L}_2(\alpha,\beta,\gamma;a,l_1),$$

where $\mathcal{L}_2(\alpha,\beta,\gamma;a,l_1)$ is a function uniformly bounded in the above region and L_a is given by the Euler product defining $L(s,f)$ but omitting the primes dividing a. We now restrict the l_1-sum in (1.3.21) up to $l_1 \le Y^2 D^{1/2+\varepsilon}$ by the observation in (1.3.16). We move the contour to $\Re(s) = 3/4$, $\Re(u) = \Re(v) = -\frac{1}{2} + \frac{1}{\log D}$ in (1.3.21). On the lines of integration we have

$$|\mathcal{L}(\tfrac{1}{2}+u, \tfrac{1}{2}+v, s; a, l_1)| \ll D^{\varepsilon} \left(|L(1+u, \chi_{l_1})| |L(1+v, f \otimes \chi_{l_1})| \right). \quad (1.3.22)$$

After using the bound for $\mathcal{L}(\frac{1}{2}+u, \frac{1}{2}+v, s; a, l_1)$, $W^{\#}(1-s, s+u, s+v)$ in (1.3.18), and the bound $|\Gamma(s)\cos(\pi s/2)| \ll |s|^{\Re(s)-\frac{1}{2}}$ on the contour $\Re(s) = 3/4$, $\Re(u) = \Re(v) = -\frac{1}{2} + \frac{1}{\log D}$, we apply Cauchy-Schwarz inequality in the l_1 sum followed by Lemma 1.2.5 to obtain the estimation $S_1 \ll k^{1/2} D^{3/4+\varepsilon} Y$.

Hence the theorem follows with the optimal choice $Y = D^{1/8}$.

1.4 Proof of Theorem 1.1.2

Using the Conrey-Iwaniec bound for one $L(\frac{1}{2}, f\otimes\chi_{8d}) \ll d^{1/3}$ [see Corollary 1.2, [19]] and then applying Cauchy-Schwarz inequality we get

$$\left(\sideset{}{^\star}\sum_{\substack{0<8d\leq D\\(d,2)=1}} \left|L(\tfrac{1}{2},\chi_{8d})L(\tfrac{1}{2},f\otimes\chi_{8d})\right|^2\right)^2 \ll \left(\sideset{}{^\star}\sum_{\substack{0<8d\leq D\\(d,2)=1}} d^{1/3}|L(\tfrac{1}{2},\chi_{8d})|^2\,|L(\tfrac{1}{2},f\otimes\chi_{8d})|\right)^2$$
$$\ll D^{2/3}\sideset{}{^\star}\sum_{\substack{0<8d\leq D\\(d,2)=1}} |L(\tfrac{1}{2},\chi_{8d})|^4 \sideset{}{^\star}\sum_{\substack{0<8d\leq D\\(d,2)=1}} |L(\tfrac{1}{2},f\otimes\chi_{8d})|^2.$$

Therefore, by Lemma 1.2.5 we have

$$\sideset{}{^\star}\sum_{\substack{0<8d\leq D\\(d,2)=1}} \left|L(\tfrac{1}{2},\chi_{8d})L(\tfrac{1}{2},f\otimes\chi_{8d})\right|^2 \ll D^{4/3+\varepsilon}.$$

This complete the proof of the theorem.

1.5 Proof of Theorem 1.1.5

We define

$$\mathcal{G}(D) = \sum_{p>2} (\log p)\, L(\tfrac{1}{2},\chi_{8p})L(\tfrac{1}{2},f\otimes\chi_{8p})h(\tfrac{8p}{D}).$$

Then by Lemma 1.2.1, we have

$$\mathcal{G}(D) = 4\sum_{p>2}\sum_{n}\sum_{m}\frac{(\log p)\lambda_f(m)\chi_{8p}(nm)}{\sqrt{nm}}h(\tfrac{8p}{D})U(\tfrac{n}{\sqrt{8p}})V(\tfrac{m}{8p}).$$

Using the rapid decay of h, we obtain

$$\mathcal{G}(D) = 4\sum_{n}\sum_{m}\frac{\lambda_f(m)}{\sqrt{nm}}\sum_{l}\Lambda(l)\chi_{8l}(nm)h(\tfrac{8l}{D})U(\tfrac{n}{\sqrt{8l}})V(\tfrac{m}{8l}) + O(X^{5/8+\varepsilon}). \tag{1.5.1}$$

Applying Mellin inversion on the variable l in the above expression, we get

$$4\sum_{n}\sum_{m}\frac{\lambda_f(m)\chi_8(nm)}{\sqrt{nm}}\frac{1}{2\pi i}\int_{(3)}\sum_{l}\frac{\Lambda(l)}{l^s}\chi_{mn}(l)\,\tilde{h}(s)D^s\mathrm{d}s$$

$$= -4\sum_{n}\sum_{m}\frac{\lambda_f(m)\chi_8(nm)}{\sqrt{nm}}\frac{1}{2\pi i}\int_{(3)}\frac{L'}{L}(s,\chi_{nm})\,\tilde{h}(s)D^s\mathrm{d}s, \tag{1.5.2}$$

where

$$\tilde{h}(s) = \int_0^{\infty} h(x)U\left(\frac{n}{\sqrt{Dx}}\right)V\left(\frac{m}{Dx}\right)x^{s-1}\mathrm{d}x.$$

Since h, U, and V are smooth functions, integration by parts yields the bound

$$\tilde{h}(s) \ll (1+|s|)^{-A}\,(1+m/D)^{-A}\left(1+n/\sqrt{D}\right)^{-A/2},$$

for any $\Re(s) > 0$ and any $A \in \mathbb{N}$.

To evaluate the integral in (1.5.2), we shift the line of integration $\Re(s) = 3$ to $\Re(s) = \frac{1}{2}+\varepsilon$. In doing so we encounter a simple pole at $s = 1$ with residue $-\tilde{h}(1)D$ only if χ_{mn} is a trivial character, which means mn is a perfect square.

Note that under the GRH, for $\Re(s) \geq \frac{1}{2}+\varepsilon$ (see [42], Theorem 5.17), we have

$$\frac{L'}{L}(s,\chi_{nm}) \ll \log\log\left((1+|s|)(2+mn)\right).$$

So the integral on $\Re(s) = \frac{1}{2}+\varepsilon$ is bounded by $O\left(D^{\frac{1}{2}+\varepsilon}\log\log\left((1+|s|)(2+mn)\right)\right)$. The contribution from the residue at $s = 1$ is $-\tilde{h}(1)D = -D\int_0^{\infty} h(x)U\left(\frac{n}{\sqrt{Dx}}\right)V\left(\frac{m}{Dx}\right)\mathrm{d}x$. Using the definition of U and V in (1.2.1), we have

$$-\tilde{h}(1)D = -D\frac{1}{(2\pi i)^2}\int_{(2)}\int_{(2)} g_1(u)g_2(v)\left(\frac{\sqrt{D}}{n}\right)^{u}\left(\frac{D}{m}\right)^{v}\xi(v+\frac{u}{2})\frac{du}{u}\frac{dv}{v},$$

where $\xi(s) = \int_0^{\infty} h(x)x^s dx$. Therefore

$$\mathcal{G}(D) = \frac{4D}{(2\pi i)^2}\int_{(2)}\int_{(2)} g_1(u)g_2(v)D^{\frac{u}{2}+v}\xi(v+\frac{u}{2})\mathcal{Z}(u,v)\frac{du}{u}\frac{dv}{v} + O(X^{5/8+\varepsilon}), \tag{1.5.3}$$

where

$$\mathcal{Z}(u,v) = \sum_{\substack{(nm,2)=1\\ nm=\square}}\frac{\lambda_f(m)}{n^{\frac{1}{2}+u}m^{\frac{1}{2}+v}}. \tag{1.5.4}$$

By (1.3.11) and (1.3.12), one can easily get

$$\mathcal{Z}(u,v) = \zeta(1+2u)L(1+2v,\mathrm{sym}^2(f))L(1+u+v,f)\mathcal{Z}_2(u,v), \tag{1.5.5}$$

where $\mathcal{Z}_2(u,v)$ is a Dirichlet series which converges absolutely for $\Re(u)$, $\Re(v) > -\frac{1}{4}+\varepsilon$ and is uniformly bounded.

Now we evaluate the double integral in first term on the right side of the equation (1.5.3). We first shift contour to $\Re(u) = \Re(v) = \varepsilon$ without encountering a pole. Then we move the contour $\Re(v) = \varepsilon$ to $\Re(v) = -\frac{1}{4}+\varepsilon$, in doing so we encounter a simple pole at $v = 0$. Since $F(s) \ll (1+|t|)^{-A}$ for any $A > 0$, by integration by parts, the integrals on $\Re(u) = \varepsilon$, $\Re(v) = -\frac{1}{4}+\varepsilon$ are bounded by $O(D^{-\frac{1}{4}+\varepsilon})$. The contribution from the residue at $v = 0$ is

$$\frac{1}{2\pi i}\int_{(\varepsilon)} \frac{g_1(u)}{u} D^{\frac{u}{2}} \zeta(1+2u) L(1, \mathrm{sym}^2 f) L(1+u, f) \mathcal{L}_2(u,0) F(\frac{u}{2}) du. \tag{1.5.6}$$

Now we shift the contour in (1.5.6) to $\Re(u) = -\frac{1}{4}+2\varepsilon$ encountering a double pole at $u = 0$. The integral on the line $\Re(u) = -\frac{1}{4}+2\varepsilon$ is bounded by $O(D^{-\frac{1}{8}+\varepsilon})$. The residue at $u = 0$ is

$$\begin{aligned}
&\frac{1}{4}F(0)L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{Z}_2(0,0)\log D + \frac{1}{2}F(0)g_1'(0)L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{Z}_2(0,0)\\
&\quad + \frac{1}{2}F(0)L(1,\mathrm{sym}^2 f)L'(1,f)\mathcal{Z}_2(0,0) + \frac{1}{2}F(0)L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{Z}_2'(0,0)\\
&\qquad + \gamma\ F(0)L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{Z}_2(0,0) + \frac{1}{4}F'(0)L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{Z}_2(0,0)
\end{aligned}$$

(γ is the Euler constant)

$$= \frac{1}{4}\widehat{h}(0)L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{Z}_2(0,0)\log D + \frac{C_2}{4}, \tag{1.5.7}$$

where C_2 is sum of the constant terms. Therefore, we get the following

$$\mathcal{G}(D) = C_1\widehat{h}(0)D\log D + C_2 D + O\left(D^{\frac{7}{8}+\varepsilon}\right), \tag{1.5.8}$$

where $C_1 = L(1,\mathrm{sym}^2 f)L(1,f)\mathcal{Z}_2(0,0)$.

Chapter 2

First moment of $GL(2) \times GL(3)$ and $GL(3)$ L-functions at $s = 1/2$

2.1 Introduction

Like the Birch and Swinnerton-Dyer conjecture which relates the order of vanishing of the Hasse-Weil L-function at the central point to the rank of an elliptic curve, the vanishing or non-vanishing of an automorphic L-function at the special points are related to several deep problems of great significance in number theory. Therefore, it is a profoundly interesting question to understand whether the product of two or more L-functions are simultaneous non-vanishing at the central point. This type of questions has been studied by many authors (see for example [44], [60], [52], [64], [56], [39],[57]). In 2014, Das & Khan [25] proved that $GL(2) \times GL(1)$ and $GL(1)$ L-functions are simultaneous non-vanishing. Ramakrishnan & Rogawski [70] in 2005, showed a simultaneous non-vanishing result for $GL(2) \times GL(1)$ and $GL(2)$ L-functions. Similar types of non-vanishing results for $GL(2) \times GL(2)$ and $G(2)$ L-functions were proved by Xu [85] and Liu [55] for Maass forms and holomorphic Hecke eigenforms in the weight aspect respectively. The non-vanishing problem for $GL(3) \times GL(2)$ and $GL(2)$ L-functions was first studied by Li [54] in 2009. More precisely, let f be a fixed Hecke-Maass cusp form for $SL(3)$. Li Proved that there are infinitely many $SL(2)$ Hecke-maass cusp forms u_j such that $L(\frac{1}{2}, f \times u_j)\, L(\frac{1}{2}, u_j) \neq 0$. In the level aspect, for the Hecke eigenform case a similar result was obtained by Khan in [51].

In this chapter, we consider the first moment of the product of $GL(2) \times GL(3)$ and

$GL(3)$ L-functions. More precisely, we fix a holomorphic Hecke eigenform g for $SL(2,\mathbb{Z})$. We study the first moment of $L(s,\mathcal{F})L(s,g\otimes\mathcal{F})$ at the central point as $\mathcal{F}$ runs over an orthogonal basis of the space of Hecke-Maass cusp forms for $SL(3,\mathbb{Z})$.

Let $\boldsymbol{\mu}_{\mathcal{F}} = (\mu_1,\mu_2,\mu_3)$ be the Langlands parameter and $\boldsymbol{\nu}_{\mathcal{F}} = (\nu_1,\nu_3,\nu_3)$ be the spectral parameter of a Hecke-Maass cusp form $\mathcal{F}$ for $SL(3,\mathbb{Z})$. Let $\boldsymbol{\mu}_0 = (\mu_{0,1},\mu_{0,2},\mu_{0,3})$ be a fixed point in $\Lambda'_{1/2}$ (see (2.2.1)). So $\boldsymbol{\nu}_0 = (\nu_{0,1},\nu_{0,2},\nu_{0,3})$ satisfies the relations (2.2.2) and (2.2.3). As in [7] (or see [39], [68]), we also consider

$$|\mu_{0,j}| \asymp |\nu_{0,j}| \asymp \|\boldsymbol{\mu}_0\| \asymp \|\boldsymbol{\nu}_0\| := T, \quad 1 \le j \le 3,$$

i.e. $\boldsymbol{\mu}_0$ is in generic position. Let us denote $R = T^{\theta}$ for any fixed θ in $(0,1)$. We choose the test function $h(\boldsymbol{\mu})$ so that it has the localizing effect at a ball of radius R about $w(\boldsymbol{\mu}_0)$, where w are elements in the Weyl group $\mathfrak{W}$ of $GL(3,\mathbb{R})$. It is defined by

$$h(\boldsymbol{\mu}) := P(\boldsymbol{\mu})^2 \left(\sum_{w\in\mathcal{W}} \psi\left(\frac{w(\boldsymbol{\mu}) - \boldsymbol{\mu}_0}{R} \right) \right)^2,$$

where $\psi(\boldsymbol{\mu}) = \exp\left(-(\mu_1^2+\mu_2^2+\mu_3^2)\right)$ and

$$P(\boldsymbol{\mu}) = \prod_{1\le n\le A_0} \prod_{j=1}^{3} \frac{\left(\nu_j - \frac{1}{3}(1+2n)\right)\left(\nu_j + \frac{1}{3}(1+2n)\right)}{|\nu_{0,k}|^2}$$

for some fixed large $A_0 > 0$. Here

$$\mathfrak{W} := \left\{ I, w_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, w_3 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, w_4 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, w_5 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, w_6 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \right\}$$

is the Weyl group for $SL(3,\mathbb{R})$. Let $\mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu} = \mathrm{spec}(\boldsymbol{\mu})\mathrm{d}\boldsymbol{\mu}$ with

$$\mathrm{spec}(\boldsymbol{\mu}) = \prod_{\mathrm{j}=1}^{3} \left(3\nu_{\mathrm{j}} \tan\left(\frac{3\pi}{2}\nu_{\mathrm{j}} \right) \right) \quad \text{and} \quad \mathrm{d}\boldsymbol{\mu} = \mathrm{d}\mu_1\,\mathrm{d}\mu_2 = \mathrm{d}\mu_2\,\mathrm{d}\mu_3 = \mathrm{d}\mu_3\,\mathrm{d}\mu_1.$$

Let us define

$$\mathcal{N}_{\mathcal{F}} = \|\mathcal{F}\|^2 \prod_{j=1}^{3} \cos\left(\frac{3\pi}{2}\nu_j \right)$$

to be the normalizing factor. Now we state the main theorem of this chapter.

Theorem 2.1.1. *Let g be a holomorphic Hecke eigenform for $SL(2,\mathbb{Z})$ of weight $k \equiv 0 \,(\mathrm{mod}\, 4)$. Let $\{\mathcal{F}\}$ be a basis of the space of Hecke-Maass cusp forms for $SL(3,\mathbb{Z})$.*

Then we have

$$\sum_{F} \frac{h(\boldsymbol{\mu}_F)}{\mathcal{N}_F} L(\tfrac{1}{2}, F) L(\tfrac{1}{2}, g \otimes F) = \frac{1}{192\,\pi^5} \iint_{\Re(\boldsymbol{\mu})=0} M(\boldsymbol{\mu}, k) h(\boldsymbol{\mu}) \mathrm{spec}(\boldsymbol{\mu}) \mathrm{d}\boldsymbol{\mu} + O\left(T^{\frac{17}{6}+\varepsilon} R^2\right),$$

where

$$M(\boldsymbol{\mu}, k) = \zeta(\tfrac{3}{2}) + L(1, g) \prod_{j=1}^{3} \frac{\Gamma(\frac{1}{4} + \frac{\mu_j}{2})}{\Gamma(\frac{1}{4} - \frac{\mu_j}{2})} + L(1, g) \prod_{j=1}^{3} \frac{\Gamma(\frac{k}{2} + \mu_j)}{\Gamma(\frac{k}{2} - \mu_j)} + \zeta(\tfrac{3}{2}) \prod_{j=1}^{3} \frac{\Gamma(\frac{k}{2} + \mu_j)\Gamma(\frac{1}{4} + \frac{\mu_j}{2})}{\Gamma(\frac{k}{2} - \mu_j)\Gamma(\frac{1}{4} - \frac{\mu_j}{2})}. \tag{2.1.1}$$

Note that, $\iint_{\Re(\boldsymbol{\mu})=0} M(\boldsymbol{\mu}, k) h(\boldsymbol{\mu}) \mathrm{spec}(\boldsymbol{\mu}) d\boldsymbol{\mu} \asymp T^3 R^2$.

Remark 2.1.2. *We have used the bounds $\zeta(1/2+it) \ll t^{1/6}$ and $L(1/2+it, g \otimes f) \ll t$ in the estimation of the error terms (see section 2.3.1.6). Note that using best know bounds for $\zeta(1/2+it)$ and $L(1/2+it, g \otimes f)$ one can slightly improve the error term. Since we are only focusing on simultaneous non-vanishing results, we did not elaborate on such a small improvement in the error term.*

As a corollary of Theorem 2.1.1, we have the following result.

Corollary 2.1.3. *Let g be a holomorphic Hecke eigenform of weight $k \equiv 0 \pmod 4$ for $SL(2, \mathbb{Z})$. Then there exist infinitely many Hecke-Maass cusp forms F for $SL(3, \mathbb{Z})$ such that $L(\frac{1}{2}, F) L(\frac{1}{2}, g \otimes F) \neq 0$.*

Remark 2.1.4. *If g is a holomorphic Hecke eigenform for $SL(2, \mathbb{Z})$ with $k \equiv 2 \pmod 4$, then $L(\frac{1}{2}, g \otimes F) = 0$. In that case one can consider the following sum*

$$\sum_{F} \frac{h(\boldsymbol{\mu}_F)}{\mathcal{N}_F} L(\tfrac{1}{2}, F) L'(\tfrac{1}{2}, g \otimes F)$$

and get similar results.

2.2 Preliminaries

In this section we review some definitions, essential facts and tools that will be used in later development.

2.2.1 Automorphic forms for $SL(3,\mathbb{Z})$ and their L-functions.

Let

$$\mathbb{H}_3 = GL(3,\mathbb{R})/O(3,\mathbb{R})\mathbb{R}^*$$

be the generalized upper half plane. For $0 \le c \le \infty$, let

$$\Lambda'_c = \left\{ \boldsymbol{\mu} = (\mu_1,\mu_2,\mu_3) \in \mathbb{C}^3, \quad \begin{matrix} |\Re(\mu_j)| \le c, \quad \mu_1+\mu_2+\mu_3 = 0, \\ \{-\mu_1,-\mu_2,-\mu_3\} = \{\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_3\} \end{matrix} \right\}. \tag{2.2.1}$$

Consider $\boldsymbol{\mu}$ to be the Langlands parameter of a Hecke-Maass form $\mathcal{F}$ in $L^2\left(SL(3,\mathbb{Z})\backslash\mathbb{H}_3\right)$. Let us define

$$\nu_1 = \frac{1}{3}(\mu_1-\mu_2), \quad \nu_2 = \frac{1}{3}(\mu_2-\mu_3), \quad \nu_3 = -\nu_1-\nu_2 = \frac{1}{3}(\mu_3-\mu_1) \tag{2.2.2}$$

where $\boldsymbol{\nu} = (\nu_1,\nu_2,\nu_3)$ is known as the spectral parameter of $\mathcal{F}$. So

$$\mu_1 = 2\nu_1+\nu_2, \quad \mu_2 = \nu_2-\nu_1, \quad \mu_3 = -\nu_1-2\nu_2. \tag{2.2.3}$$

Let $A_{\mathcal{F}}(m_1,m_2)$ be the normalised Fourier coefficients of a $GL(3)$ Hecke Maass cusp form $\mathcal{F}$ with Langlands parameters $\boldsymbol{\mu}_{\mathcal{F}} = (\mu_1,\mu_2,\mu_3)$. The standard L-function associated to $\mathcal{F}$ is given by

$$L(s,\mathcal{F}) = \sum_{n\ge1} \frac{A_{\mathcal{F}}(1,n)}{n^s} \quad \text{for } \Re(s) > 1.$$

The dual form of $\mathcal{F}$ is denoted by $\widetilde{\mathcal{F}}$ with the Langlands parameter $\boldsymbol{\mu}_{\tilde{\mathcal{F}}} = (-\mu_1,-\mu_2,-\mu_3)$ and the coefficients $A_{\mathcal{F}}(n,1) = \overline{A_{\mathcal{F}}(1,n)} = A_{\tilde{\mathcal{F}}}(1,n)$. Let us define

$$\Lambda(s,\mathcal{F}) := \gamma(s,\mathcal{F})\, L(s,\mathcal{F}),$$

where $\gamma(s,\mathcal{F}) = \prod_{j=1}^{3} \Gamma_{\mathbb{R}}(s-\mu_j)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. $\Lambda(s,\mathcal{F})$ is called the completed L-function, which is an entire function and satisfies the functional equation

$$\Lambda(s,\mathcal{F}) = \Lambda(1-s,\tilde{\mathcal{F}}).$$

2.2.2 The maximal Eisenstein series.

Let $u \in \mathbb{C}$ have sufficiently large real part. Let f be a Hecke-Maass cusp form for $SL(2,\mathbb{Z})$ with the spectral parameter it_f, Hecke eigenvalues $\lambda_f(m)$ and $\|f\| = 1$. The maximal Eisenstein series and its Hecke eigenvalue at (m,n) are denoted by $E_{u,f}^{\max}(z)$ and $B_{u,f}^{\max}(m,n)$, respectively. The Hecke eigenvalue $B_{u,f}^{\max}(1,m)$ is defined by (see Subsection 10.9 of Goldfeld [30])

$$B_{u,f}^{\max}(1,m) = \sum_{d_1 d_2 = m} \lambda_f(d_1) d_1^{-u} d_2^{-2u}$$

and satisfies the following Hecke relations

$$B_{u,f}^{\max}(m,1) = \overline{B_{u,f}^{\max}(1,m)}, \quad B_{u,f}^{\max}(m,n) = \sum_{d|(m,n)} \mu(d) B_{u,f}^{\max}(\tfrac{m}{d},1) B_{u,f}^{\max}(1,\tfrac{n}{d}).$$

The L-function associated to $E_{u,f}^{\max}(z)$ is given by

$$L(s, E_{u,f}^{\max}) = \sum_{m\geq 1} \frac{B_{u,f}^{\max}(1,m)}{m^s} = \zeta(s-2u)L(s+u,f), \tag{2.2.4}$$

for sufficiently large $\Re(s)$. It satisfies the functional equation

$$\Lambda(s, E_{u,f}^{\max}) = \prod_{j=1}^{3} \Gamma_{\mathbb{R}}(s+\mu_j') L(s, E_{u,f}^{\max}) = \Lambda(1-s, E_{-u,f}^{\max}),$$

where $\mu_1' = u + it_f$, $\mu_2' = u - it_f$ and $\mu_3' = -2u$. The normalized factor for the maximal Eisenstein series is defined by

$$\mathcal{N}_{u,f}^{\max} := 8L(1, \mathrm{Ad}^2 f)|L(1+3u,f)|^2.$$

2.2.3 The minimal Eisenstein series.

Let $\nu_1, \nu_2 \in \mathbb{C}$ and (μ_1, μ_2, μ_3) be the Langlands parameter given by (2.2.3). We denote the minimal Eisenstein series by $E_{\nu_1,\nu_2}^{\min}(z)$. The Hecke eigenvalue $B_{\nu_1,\nu_2}^{\min}(m,n)$ of $E_{\nu_1,\nu_2}^{\min}(z)$ at (m,n) is defined by (see Subsection 10.8 of Goldfeld [30])

$$B_{\boldsymbol{\mu}}^{\min}(1,m) = B_{\nu_1,\nu_2}^{\min}(1,m) := \sum_{d_1 d_2 d_3 = n} d_1^{-\mu_1} d_2^{-\mu_2} d_3^{-\mu_3}$$

and satisfies the following Hecke relations

$$B^{\min}_{\nu_1,\nu_2}(m,1) = \overline{B^{\min}_{\nu_1,\nu_2}(1,m)}, \quad B^{\min}_{\nu_1,\nu_2}(m,n) = \sum_{d|(m,n)} \mu(d) B^{\min}_{\nu_1,\nu_2}(\tfrac{m}{d},1) B^{\min}_{\nu_1,\nu_2}(1,\tfrac{n}{d}).$$

The L-function associated to $E^{\max}_{\nu_1,\nu_2}(z)$ is given by

$$L(s, E^{\min}_{\nu_1,\nu_2}) = \sum_{m \geq 1} \frac{B^{\min}_{\nu_1,\nu_2}(1,m)}{m^s} = \zeta(s+\mu_1)\zeta(s+\mu_2)\zeta(s+\mu_3), \tag{2.2.5}$$

for $\Re(s) > 1$. The normalized factor for the minimal Eisenstein series is defined by

$$\mathcal{N}^{\min}_{\boldsymbol{\mu}} = \mathcal{N}^{\min}_{\nu_1,\nu_2} := \frac{1}{16} \prod_{j=1}^{3} |\zeta(1+3\nu_j)|^2.$$

2.2.4 The Rankin-Selberg L-function on $GL(2) \times GL(3)$.

Let $\mathcal{F}$ be a $GL(3)$ Hecke Maass cusp with Langlands parameters $\mu = (\mu_1, \mu_2, \mu_3)$. Let g be a Hecke eigenform for $SL(2,\mathbb{Z})$ of weight k and $\lambda_g(n)$ be the n-th Hecke eigenvalue. The Rankin-Selberg L-function of g and $\mathcal{F}$ is defined by

$$L\left(s, g \otimes \mathcal{F}\right) = \sum_{m,n \geq 1}\sum \frac{\lambda_g(n)\overline{A_{\mathcal{F}}(m,n)}}{(nm^2)^s} \quad, \Re(s) > 1.$$

It is entire and satisfies the functional equation

$$\Lambda(s, g \otimes \mathcal{F}) = i^{3k}\, \Lambda(1-s, g \otimes \tilde{\mathcal{F}}),$$

where

$$\Lambda(s, g \otimes \mathcal{F}) = \gamma(s, g \otimes \mathcal{F})\, L(s, g \otimes \mathcal{F})$$

and

$$\gamma(s, g \otimes \mathcal{F}) = \prod_{j=1}^{3} \Gamma_{\mathbb{R}}\left(s + \frac{k-1}{2} - \mu_j\right) \Gamma_{\mathbb{R}}\left(s + \frac{k+1}{2} - \mu_j\right).$$

Let $E^{\max}_{u,f}$ be the maximal Eisenstein series as in §2.2.2. The Rankin-Selberg L-function $L(s, g \otimes E^{\max}_{u,f})$ is defined by

$$L(s, g \otimes E^{\max}_{u,f}) = \sum_{m,n \geq 1}\sum \frac{\lambda_g(n)\overline{B^{\max}_{u,f}(m,n)}}{(nm^2)^s}$$

$$= L(s+2u,g)L(s-u,g\otimes f), \tag{2.2.6}$$

for sufficiently large $\Re(s)$.

Let $E^{\min}_{\nu_1,\nu_2}$ be the minimal Eisenstein series as in §2.2.3. The Rankin-Selberg L-function $L(s,g\otimes E^{\min}_{\nu_1,\nu_2})$ is defined by

$$\begin{aligned} L(s,g\otimes E^{\min}_{\nu_1,\nu_2}) &= \sum\sum_{m,n\geq 1} \frac{\lambda_g(n)\overline{B^{\min}_{\nu_1,\nu_2}(m,n)}}{(nm^2)^s} \\ &= L(s-\mu_1,g)L(s-\mu_2,g)L(s-\mu_3,g) \end{aligned} \tag{2.2.7}$$

for $\Re(s)>1$.

From the above functional equation we deduce the approximate functional equation for $L(s,\digamma)$ and $L(s,g\otimes\digamma)$ at the central point $s=\frac{1}{2}$ (See §5.2 of [42]). Let us define $G_1(s)=Q_1(s)e^{s^2}$ and $G_2(s)=Q_2(s)e^{s^2}$ where $Q_i(s)$'s are polynomial such that $Q_i(0)=1$ for $i=1,2$. Also assume $Q_1(s)$ and $Q_2(s)$ vanishes at the poles of $\frac{\gamma(\frac{1}{2}+u,\digamma)}{\gamma(\frac{1}{2},\digamma)}$ and $\frac{\gamma(\frac{1}{2}+u,g\otimes\digamma)}{\gamma(\frac{1}{2},g\otimes\digamma)}$ respectively.

We define

$$V_\digamma(y) := \frac{1}{2\pi i}\int_{(3)} y^{-u}\frac{\gamma(\frac{1}{2}+u,\digamma)}{\gamma(\frac{1}{2},\digamma)}G_1(u)\frac{\mathrm{d}u}{u},$$

$$\tilde{V}_\digamma(y) := \frac{1}{2\pi i}\int_{(3)} y^{-u}\frac{\gamma(\frac{1}{2}+u,\tilde{\digamma})}{\gamma(\frac{1}{2},\digamma)}G_1(u)\frac{\mathrm{d}u}{u},$$

and

$$W_\digamma(y) := \frac{1}{2\pi i}\int_{(3)} y^{-u}\frac{\gamma(\frac{1}{2}+u,g\otimes\digamma)}{\gamma(\frac{1}{2},g\otimes\digamma)}G_2(u)\frac{\mathrm{d}u}{u},$$

$$\tilde{W}_\digamma(y) := \frac{1}{2\pi i}\int_{(3)} y^{-u}\frac{\gamma(\frac{1}{2}+u,g\otimes\tilde{\digamma})}{\gamma(\frac{1}{2},g\otimes\digamma)}G_2(u)\frac{\mathrm{d}u}{u}.$$

Lemma 2.2.1. *We have*

$$L(\tfrac{1}{2},\digamma) = \sum_{n\geq 1}\frac{A_\digamma(1,l)}{l^{1/2}}V_\digamma(l) + \sum_{l\geq 1}\frac{\overline{A_\digamma(1,l)}}{l^{1/2}}\tilde{V}_\digamma(l).$$

Moreover, for $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)$ with $\mu_j\asymp T$, one has

$$y^{j_1}\frac{\partial^{j_1}}{\partial y^{j_1}}V_\digamma(y) \ll \left(\frac{y}{T^{3/2}}\right)^{-A}, \quad y^{j_1}\frac{\partial^{j_1}}{\partial y^{j_1}}\tilde{V}_\digamma(y) \ll \left(\frac{y}{T^{3/2}}\right)^{-A}$$

for any $A>0$ *and any* $j_1\in\mathbb{N}\cup\{0\}$*. Also, for* $y\gg T^{3/2}$

$$V_{F}(y)=1+O_B\left(\frac{T^{3/2}}{y}\right)^{-B},\quad \tilde{V}_{F}(y)=\prod_{j=1}^{3}\frac{\Gamma\left(\frac{1}{4}+\frac{\mu_j}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{\mu_j}{2}\right)}+O_B\left(\frac{T^{3/2}}{y}\right)^{-B}$$

for any $B>0$.

Lemma 2.2.2. *We have*

$$L(\tfrac{1}{2},g\otimes F)=\sum\sum_{m,n\geq1}\frac{\lambda_g(n)A_{F}(m,n)}{(nm^2)^{1/2}}W_{F}(nm^2)+i^{3k}\sum\sum_{m,n\geq1}\frac{\lambda_g(n)\overline{A_{F}(m,n)}}{(nm^2)^{1/2}}\tilde{W}_{F}(nm^2),$$

Moreover, for $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)$ *with* $\mu_j\asymp T$*, one has*

$$y^{j_1}\frac{\partial^{j_1}}{\partial y^{j_1}}W_{F}(y)\ll_k\left(\frac{y}{T^3}\right)^{-A},\quad y^{j_1}\frac{\partial^{j_1}}{\partial y^{j_1}}\tilde{W}_{F}(y)\ll_k\left(\frac{y}{T^3}\right)^{-A}$$

for any $A>0$ *and any* $j_1\in\mathbb{N}\cup\{0\}$*. Also, for* $y\gg T^3$

$$W_{F}(y)=1+O_{B,k}\left(\frac{T^3}{y}\right)^{-B},\quad \tilde{W}_{F}(y)=\prod_{j=1}^{3}\frac{\Gamma\left(\frac{k}{2}+\mu_j\right)}{\Gamma\left(\frac{k}{2}-\mu_j\right)}+O_{B,k}\left(\frac{T^3}{y}\right)^{-B}$$

for any $0<B<\frac{k-1}{2}$.

Now we state the Voronoi summation formula for a Hecke eigenform g of weight k for the full modular group $SL(2,\mathbb{Z})$.

Lemma 2.2.3. *Let* $\lambda_g(n)$ *be the* n*-th Fourier coefficient of* g *and* ψ *be compactly supported smooth function on* $(0,\infty)$*. Let* $a,q\in\mathbb{Z}$ *with* $(a,q)=1$*. Then we have*

$$\sum_{n\geq1}\lambda_g(n)e\left(\frac{an}{q}\right)\psi(n)=\sum_{l\geq1}\lambda_g(l)e\left(-\frac{dl}{q}\right)\omega(n),$$

where $ad\equiv1\pmod q$ *and*

$$\omega(y)=\int_0^\infty\psi(x)J_{k-1}\left(\frac{4\pi\sqrt{xy}}{q}\right)\mathrm{d}x.$$

Proof. See [[53], Theorem A.4]. □

2.2.5 The Kloostermam sums.

For $n_1, n_2, m_1, m_2, D_1, D_2 \in \mathbb{N}$, we define the following Kloosterman sums.

$$\tilde{S}(n_1, n_2, m_1; D_1, D_2) := \sum_{\substack{C_1 (\operatorname{mod} D_1),\, C_2 (\operatorname{mod} D_2) \\ (C_1, D_1) = (C_2, D_2/D_1) = 1}} \sum e\left(n_2 \frac{\bar{C}_1 C_2}{D_1} + m_1 \frac{\bar{C}_2}{D_2/D_1} + n_1 \frac{C_1}{D_1} \right)$$

for $D_1 \mid D_2$, and

$$\begin{aligned} & S(n_1, m_2, m_1, n_2; D_1, D_2) \\ := & \sum_{\substack{B_1, C_1 (\operatorname{mod} D_1);\, B_2, C_2 (\operatorname{mod} D_2) \\ D_1 C_2 + B_1 B_2 + D_2 C_1 \equiv 0 \, (\operatorname{mod} D_1 D_2) \\ (B_j, C_j, D_j) = 1}} \sum\sum\sum e\left(\frac{n_1 B_1 + m_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1)}{D_2} \right), \end{aligned}$$

where $B_j Y_j + C_j Z_j \equiv 1 \, (\operatorname{mod} D_j)$ for $j = 1, 2$.

2.2.6 Integral Kernels.

Following [[15], Theorem 2 & 3], we define the integral kernel in terms of Mellin-Barnes representations. For $s \in \mathbb{C}$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ define the meromorphic function

$$\tilde{G}^{\pm}(s, \boldsymbol{\mu}) := \frac{\pi^{-3s}}{12288 \pi^{7/2}} \left(\prod_{j=1}^{3} \frac{\Gamma(\frac{1}{2}(s - \mu_j))}{\Gamma(\frac{1}{2}(1 - s + \mu_j))} \pm i \prod_{j=1}^{3} \frac{\Gamma(\frac{1}{2}(1 + s - \mu_j))}{\Gamma(\frac{1}{2}(2 - s + \mu_j))} \right),$$

and for $\boldsymbol{s} = (s_1, s_2) \in \mathbb{C}^2$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ define the meromorphic function

$$G(\boldsymbol{s}, \boldsymbol{\mu}) := \frac{1}{\Gamma(s_1 + s_2)} \prod_{j=1}^{3} \Gamma(s_1 - \mu_j) \Gamma(s_2 + \mu_j).$$

We also define the following trigonometric functions

$$\begin{aligned} S^{++}(\boldsymbol{s}; \boldsymbol{\mu}) &:= \frac{1}{24\pi^2} \prod_{j=1}^{3} \cos\left(\tfrac{3}{2}\pi\nu_j\right), \\ S^{+-}(\boldsymbol{s}; \boldsymbol{\mu}) &:= -\frac{1}{32\pi^2} \frac{\cos\left(\frac{3}{2}\pi\nu_2\right) \sin\left(\pi(s_1 - \mu_1)\right) \sin\left(\pi(s_2 + \mu_2)\right) \sin\left(\pi(s_2 + \mu_3)\right)}{\sin\left(\frac{3}{2}\pi\nu_1\right) \sin\left(\frac{3}{2}\pi\nu_3\right) \sin\left(\pi(s_1 + s_2)\right)}, \\ S^{-+}(\boldsymbol{s}; \boldsymbol{\mu}) &:= -\frac{1}{32\pi^2} \frac{\cos\left(\frac{3}{2}\pi\nu_1\right) \sin\left(\pi(s_1 - \mu_1)\right) \sin\left(\pi(s_1 - \mu_2)\right) \sin\left(\pi(s_2 + \mu_3)\right)}{\sin\left(\frac{3}{2}\pi\nu_2\right) \sin\left(\frac{3}{2}\pi\nu_3\right) \sin\left(\pi(s_1 + s_2)\right)}, \end{aligned}$$

$$S^{--}(\boldsymbol{s};\boldsymbol{\mu}) := \frac{1}{32\pi^2}\frac{\cos\left(\frac{3}{2}\pi\nu_3\right)\sin\left(\pi(s_1-\mu_2)\right)\sin\left(\pi(s_2+\mu_2)\right)}{\sin\left(\frac{3}{2}\pi\nu_2\right)\sin\left(\frac{3}{2}\pi\nu_1\right)}. \tag{2.2.8}$$

For $y\in\mathbb{R}^*$ with $\operatorname{sgn}(y)=\epsilon$, let

$$K_{w_4}(y;\boldsymbol{\mu}) := \int_{-i\infty}^{+i\infty}|y|^{-s}\tilde{G}^{\epsilon}(s,\boldsymbol{\mu})\frac{\mathrm{d}s}{2\pi i}. \tag{2.2.9}$$

For $\boldsymbol{y}=(y_1,y_2)\in(\mathbb{R}^*)^2$ with $\operatorname{sgn}(y_1)=\epsilon_1$, $\operatorname{sgn}(y_2)=\epsilon_2$, let

$$K_{w_6}^{\epsilon_1,\epsilon_2}(\boldsymbol{y};\boldsymbol{\mu}) := \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}|4\pi^2y_1|^{-s_1}|4\pi^2y_2|^{-s_2}G(\boldsymbol{s},\boldsymbol{\mu})S^{\epsilon_1,\epsilon_2}(\boldsymbol{s};\boldsymbol{\mu})\frac{\mathrm{d}s_1\mathrm{d}s_2}{(2\pi i)^2}. \tag{2.2.10}$$

2.2.7 The Kuznetsov formula

Define the spectral measure on the hyperplane $\mu_1+\mu_2+\mu_3=0$ by $\mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu}=\operatorname{spec}(\boldsymbol{\mu})\mathrm{d}\boldsymbol{\mu}$, where

$$\operatorname{spec}(\boldsymbol{\mu})=\prod_{\mathrm{j}=1}^{3}\left(3\nu_{\mathrm{j}}\tan\left(\frac{3\pi}{2}\nu_{\mathrm{j}}\right)\right) \quad \text{and} \quad \mathrm{d}\boldsymbol{\mu}=\mathrm{d}\mu_1\,\mathrm{d}\mu_2=\mathrm{d}\mu_2\,\mathrm{d}\mu_3=\mathrm{d}\mu_3\,\mathrm{d}\mu_1.$$

Now we state the Kuznetsov trace formula in the version of Buttcane [[15], Theorem 2, 3, 4].

Lemma 2.2.4. *Let $n_1,n_2,m_1,m_2\in\mathbb{N}$ and h be a holomorphic function on*

$$\Lambda_{\frac{1}{2}+\delta}=\left\{\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)\in\mathbb{C}^3,\mu_1+\mu_2+\mu_3=0,\Re(\mu_j)\le\frac{1}{2}+\delta\right\}$$

for some $\delta>0$, symmetric under the Weyl group $\mathfrak{W}$, of rapid decay when $|\Im(\mu_j)|\to\infty$ and satisfies

$$h(3\nu_1\pm1,3\nu_2\pm1,3\nu_3\pm1)=0.$$

Then we have

$$\mathcal{C}+\mathcal{E}_{\min}+\mathcal{E}_{\max}=\Delta+\Sigma_4+\Sigma_5+\Sigma_6,$$

where

$$\mathcal{C} := \sum_{F}\frac{h(\boldsymbol{\mu}_F)}{\mathcal{N}_F}\overline{A_F(m_1,m_2)}A_F(n_1,n_2),$$

$$\mathcal{E}_{\min} := \frac{1}{24(2\pi i)^2}\iint_{\Re(\boldsymbol{\mu})=0}\frac{h(\boldsymbol{\mu})}{\mathcal{N}_{\boldsymbol{\mu}}^{\min}}\overline{B_{\boldsymbol{\mu}}^{\min}(m_1,m_2)}B_{\boldsymbol{\mu}}^{\min}(n_1,n_2)\,\mathrm{d}\mu_1\mathrm{d}\mu_2,$$

$$\mathcal{E}_{\max} := \sum_{f}\frac{1}{2\pi i}\int_{\Re(u)=0}\frac{h(u+it_f,u-it_f,-2u)}{\mathcal{N}_{u,f}^{\max}}\overline{B_{u,f}^{\max}(m_1,m_2)}B_{u,f}^{\max}(n_1,n_2)\mathrm{d}u,$$

and

$$\begin{aligned}
\Delta &:= \delta_{m_1,n_1}\delta_{m_2,n_2}\frac{1}{192\pi^5}\iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})\,\mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu},\\
\Sigma_4 &:= \sum_{\epsilon=\pm1}\sum_{\substack{D_2|D_1\\ m_2D_1=n_1{D_2}^2}} \frac{\tilde{S}(-\epsilon n_2, m_2, m_1; D_2, D_1)}{D_1D_2}\Phi_{w_4}\left(\frac{\epsilon m_1m_2n_2}{D_1D_2}\right),\\
\Sigma_5 &:= \sum_{\epsilon=\pm1}\sum_{\substack{D_1|D_2\\ m_1D_2=n_2{D_1}^2}} \frac{\tilde{S}(\epsilon n_1, m_1, m_2; D_1, D_2)}{D_1D_2}\Phi_{w_5}\left(\frac{\epsilon m_1m_2n_1}{D_1D_2}\right),\\
\Sigma_6 &:= \sum_{\epsilon_1,\epsilon_2=\pm1}\sum_{D_1,D_2} \frac{S(\epsilon_2n_2, \epsilon_1n_1, m_1, m_2; D_1, D_2)}{D_1D_2}\Phi_{w_6}\left(-\frac{\epsilon_2m_1n_2D_2}{{D_1}^2}, -\frac{\epsilon_1m_2n_1D_1}{{D_2}^2}\right)
\end{aligned}$$

with

$$\begin{aligned}
\Phi_{w_4}(y) &:= \iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})K_{w_4}(y;\boldsymbol{\mu})\mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu},\\
\Phi_{w_5}(y) &:= \iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})K_{w_4}(-y;-\boldsymbol{\mu})\mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu}, \qquad (2.2.11)\\
\Phi_{w_6}(\boldsymbol{y}) &:= \iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})K_{w_6}^{\mathrm{sgn(y_1),sgn(y_2)}}(\boldsymbol{y};\boldsymbol{\mu})\mathrm{d}_{\mathrm{spce}}\boldsymbol{\mu}.
\end{aligned}$$

Here we quote Lemma 8 and Lemma 9 of [7] which are used in truncating summation in geometric terms after the application of the Kuznetsov formula.

Lemma 2.2.5. *Let* $0 < |y| \le T^{3-\varepsilon}$*. Then for any constant* $B > 0$*, we have*

$$\Phi_{w_4}(y) \ll_{\varepsilon,B} T^{-B}.$$

If $|y| > T^{3-\varepsilon}$*, then*

$$|y|^j\frac{\mathrm{d}^{\mathrm{j}}}{\mathrm{d}y^j}\Phi_{w_4}(y) \ll_{\varepsilon,j} T^{3+\varepsilon}R^2\left(T+|y|^{1/3}\right)^j$$

for any $j \in \mathbb{N}\cup\{0\}$*.*

Lemma 2.2.6. *Let* $\Upsilon := \min\left\{|y_1|^{1/3}|y_2|^{1/6}, |y_2|^{1/3}|y_1|^{1/6}\right\}$*. If* $\Upsilon \ll T^{1-\varepsilon}$*, then for any constant* $B > 0$

$$\Phi_{w_6}(y_1, y_2) \ll_{\varepsilon,B} T^{-B}.$$

If $\Upsilon \gg T^{1-\varepsilon}$, then we have

$$\begin{aligned}&|y_1|^{j_1}|y_2|^{j_2}\frac{\partial^{j_1}}{\partial y_1^{j_1}}\frac{\partial^{j_2}}{\partial y_2^{j_2}}\Phi_{w_6}(y_1,y_2)\\ &\ll_{\varepsilon,j_1,j_2} T^3R^2\left(T+|y_1|^{1/2}+|y_1|^{1/3}|y_2|^{1/6}\right)^{j_1}\left(T+|y_2|^{1/2}+|y_2|^{1/3}|y_1|^{1/6}\right)^{j_2}\end{aligned}$$

for any $j_1, j_2 \in \mathbb{N}\cup\{0\}$.

2.3 Proof of Theorem 2.1.1

Let us define

$$\mathcal{S}(T) := \sum_{F}\frac{h(\boldsymbol{\mu}_F)}{\mathcal{N}_F}\,L(1/2, g\otimes F)\,L(1/2,F).$$

By using the approximate functional equations of $L(1/2, g\otimes F)$ and $L(1/2,F)$ (Lemma 2.2.2 and Lemma 2.2.1), we get

$$\begin{aligned}\mathcal{S}(T) &= \sum\sum_{m,n\geq 1}\sum_{l\geq 1}\frac{\lambda_g(n)}{m\,(nl)^{1/2}}\sum_{F}\frac{h_1(\boldsymbol{\mu}_F)}{\mathcal{N}_F}A_F(m,n)\,\overline{A_F(1,l)}\\ &+\sum\sum_{m,n\geq 1}\sum_{l\geq 1}\frac{\lambda_g(n)}{m\,(nl)^{1/2}}\sum_{F}\frac{h_2(\boldsymbol{\mu}_F)}{\mathcal{N}_F}A_F(m,n)A_F(1,l)\\ &+\sum\sum_{m,n\geq 1}\sum_{l\geq 1}\frac{\lambda_g(n)}{m\,(nl)^{1/2}}\sum_{F}\frac{h_3(\boldsymbol{\mu}_F)}{\mathcal{N}_F}\overline{A_F(m,n)}A_F(1,l)\\ &+\sum\sum_{m,n\geq 1}\sum_{l\geq 1}\frac{\lambda_g(n)}{m\,(nl)^{1/2}}\sum_{F}\frac{h_4(\boldsymbol{\mu}_F)}{\mathcal{N}_F}\overline{A_F(m,n)}\,\overline{A_F(1,l)}\\ &= \mathcal{S}_1(T)+\mathcal{S}_2(T)+\mathcal{S}_3(T)+\mathcal{S}_4(T),\ \text{(say)}.\end{aligned}$$

Here

$$\begin{aligned}h_1(\boldsymbol{\mu}_F) = h(\boldsymbol{\mu}_F)W_F(nm^2)\tilde{V}_F(l),\ \ h_2(\boldsymbol{\mu}_F) = h(\boldsymbol{\mu}_F)W_F(nm^2)V_F(l),\\ h_3(\boldsymbol{\mu}_F) = h(\boldsymbol{\mu}_F)\tilde{W}_F(nm^2)V_F(l)\text{ and }h_4(\boldsymbol{\mu}_F) = h(\boldsymbol{\mu}_F)\tilde{W}_F(nm^2)\tilde{V}_F(l).\end{aligned}$$

Now, Theorem 2.1.1 follows immediately by the proposition given below.

Proposition 2.3.1. *We have*

$$\mathcal{S}_1(T) = L(1,g)\frac{1}{192\,\pi^5}\iint_{\Re(\boldsymbol{\mu})=0}h(\boldsymbol{\mu})\prod_{j=1}^{3}\frac{\Gamma(\frac{1}{4}+\frac{\mu_j}{2})}{\Gamma(\frac{1}{4}-\frac{\mu_j}{2})}\mathrm{spec}(\boldsymbol{\mu})\mathrm{d}\boldsymbol{\mu}+\mathrm{O}(\mathrm{T}^{\frac{17}{6}+\varepsilon}\mathrm{R}^2),$$

$$\mathcal{S}_2(T) = \zeta(\tfrac{3}{2})\frac{1}{192\pi^5}\iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})\mathrm{spec}(\boldsymbol{\mu})\mathrm{d}\boldsymbol{\mu} + \mathrm{O}(\mathrm{T}^{\frac{17}{6}+\varepsilon}\mathrm{R}^2),$$

$$\mathcal{S}_3(T) = L(1,g)\frac{1}{192\,\pi^5}\iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})\prod_{j=1}^{3}\frac{\Gamma(\frac{k}{2}+\mu_j)}{\Gamma(\frac{k}{2}-\mu_j)}\mathrm{spec}(\boldsymbol{\mu})\mathrm{d}\boldsymbol{\mu} + \mathrm{O}(\mathrm{T}^{\frac{17}{6}+\varepsilon}\mathrm{R}^2),$$

$$\mathcal{S}_4(T) = \zeta(\tfrac{3}{2})\frac{1}{192\pi^5}\iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})\prod_{j=1}^{3}\frac{\Gamma(\frac{k}{2}+\mu_j)\Gamma(\frac{1}{4}+\frac{\mu_j}{2})}{\Gamma(\frac{k}{2}-\mu_j)\Gamma(\frac{1}{4}-\frac{\mu_j}{2})}\mathrm{spec}(\boldsymbol{\mu})\mathrm{d}\boldsymbol{\mu} + \mathrm{O}(\mathrm{T}^{\frac{17}{6}+\varepsilon}\mathrm{R}^2).$$

2.3.1 Proof of the Proposition 2.3.1

We only prove the first identity ($S_1(T)$) of the Proposition 2.3.1, as the proof of other identities is the same as the first identity.

Applying the Kuznetsov's trace formula (Lemma 2.2.4), one has

$$S_1(T) = \sum_{m,n\geq 1}\sum\sum_{l\geq 1}\frac{\lambda_g(n)}{m\,(nl)^{1/2}}\left(\Delta^{(1)} + \Sigma_4^{(1)} + \Sigma_5^{(1)} + \Sigma_6^{(1)} - \mathcal{E}_{\min}^{(1)} - \mathcal{E}_{\max}^{(1)}\right), \tag{2.3.1}$$

where

$$\Delta^{(1)} := \delta_{m,1}\,\delta_{n,l}\frac{1}{192\pi^5}\iint_{\Re(\boldsymbol{\mu})=0} h_1(\boldsymbol{\mu})\mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu},$$

$$\Sigma_4^{(1)} := \sum_{\epsilon=\pm 1}\sum_{\substack{D_2|D_1\\ lD_1=m{D_2}^2}}\frac{\tilde{S}(-\epsilon n,l,1;D_2,D_1)}{D_1D_2}\Phi_{w_4}^{(1)}\left(\frac{\epsilon ln}{D_1D_2}\right),$$

$$\Sigma_5^{(1)} := \sum_{\epsilon=\pm 1}\sum_{\substack{D_1|D_2\\ D_2=n{D_1}^2}}\frac{\tilde{S}(\epsilon m,1,l;D_1,D_2)}{D_1D_2}\Phi_{w_5}^{(1)}\left(\frac{\epsilon lm}{D_1D_2}\right),$$

$$\Sigma_6^{(1)} := \sum_{\epsilon_1,\epsilon_2=\pm 1}\sum_{D_1,D_2}\frac{S(\epsilon_2 n,\epsilon_1 m,1,l;D_1,D_2)}{D_1D_2}\Phi_{w_6}^{(1)}\left(-\frac{\epsilon_2 nD_2}{{D_1}^2},-\frac{\epsilon_1 lmD_1}{{D_2}^2}\right),$$

with $\Phi_{w_4}^{(1)}(y)$, $\Phi_{w_5}^{(1)}(y)$ and $\Phi_{w_6}^{(1)}(\boldsymbol{y})$ defined as in (2.2.11) by using the new test function $h_1(\boldsymbol{\mu}_F) = h(\boldsymbol{\mu}_F)W_F(nm^2)\tilde{V}_F(l)$ respectively; and

$$\mathcal{E}_{\min}^{(1)} := \frac{1}{24(2\pi i)^2}\iint_{\Re(\boldsymbol{\mu})=0}\frac{h_1(\boldsymbol{\mu})}{\mathcal{N}_{\boldsymbol{\mu}}^{\min}}\overline{B_{\boldsymbol{\mu}}^{\min}(1,l)}B_{\boldsymbol{\mu}}^{\min}(m,n)\mathrm{d}\mu_1\mathrm{d}\mu_2,$$

$$\mathcal{E}_{\max}^{(1)} := \sum_f\frac{1}{2\pi i}\int_{\Re(u)=0}\frac{h_1(u+it_f,u-it_f,-2u)}{\mathcal{N}_{u,f}^{\max}}\overline{B_{u,f}^{\max}(1,l)}B_{u,f}^{\max}(m,n)\mathrm{d}u.$$

2.3.1.1 The diagonal term

Let us denote $\mathcal{D}^{(1)}$ to be contribution of $\Delta^{(1)}$ to the sum in the equation (2.3.1). Thus we have

$$\mathcal{D}^{(1)} = \frac{1}{192\pi^5}\iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})D(\boldsymbol{\mu})\mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu},$$

where

$$D(\boldsymbol{\mu}) := \frac{1}{(2\pi i)^2}\int_{(3)}\int_{(3)} L(1+u_1+u_2,g)\frac{\gamma(\frac{1}{2}+u_1,\tilde{F})}{\gamma(\frac{1}{2},F)}\frac{\gamma(\frac{1}{2}+u_2,g\otimes F)}{\gamma(\frac{1}{2},g\otimes F)} \times G_1(u_1)G_2(u_2)\frac{\mathrm{d}u_1}{u_1}\frac{\mathrm{d}u_2}{u_2} \tag{2.3.2}$$

with $G_i(u)=Q_i(u)e^{u^2}$ for $i=1,2$, $\gamma(u,\tilde{F})=\prod_{j=1}^{3}\Gamma_{\mathbb{R}}(u+\mu_j)$, $\gamma(u,F)=\prod_{j=1}^{3}\Gamma_{\mathbb{R}}(u-\mu_j)$ and

$$\gamma(u,g\otimes F)=\prod_{j=1}^{3}\Gamma_{\mathbb{R}}(u+\tfrac{k-1}{2}-\mu_j)\Gamma_{\mathbb{R}}(u+\tfrac{k+1}{2}-\mu_j).$$

Now we evaluate the double integral in the first term on the right side of the equation (2.3.2). We first shift the contour to $\Re(u_1)=\epsilon$, $\Re(u_2)=\epsilon$ without encountering a pole, for any $\epsilon>0$. Then we move the contour $\Re(u_2)=\epsilon$ to $\Re(u_2)=-\frac{1}{2}$, in doing so we encounter a simple pole at $u_2=0$. Since, $G_2(u)\ll e^{-t^2}$ and $L(1/2+it,g)\ll t^{1/3}$, the integral on $\Re(u_1)=\epsilon$, $\Re(u_2)=-\frac{1}{2}$ is bounded by $O\left(\prod_{j=1}^{3}|\mu_j|^{-1/2}\right)$. The contribution from the residue at $u_2=0$ is

$$\frac{1}{2\pi i}\int_{(\epsilon)} L(1+u_1,g)\frac{\gamma(\frac{1}{2}+u_1,\tilde{F})}{\gamma(\frac{1}{2},F)}G_1(u_1)\frac{\mathrm{d}u_1}{u_1}. \tag{2.3.3}$$

Now shift the contour in (2.3.3) to $\Re(u_1)=-\frac{1}{2}$ encountering a simple pole at $u_1=0$. The integral on the line $\Re(u_1)=-\frac{1}{2}$ is bounded by $O\left(\prod_{j=1}^{3}|\mu_j|^{-1/4}\right)$. The contribution from the residue at $u_1=0$ is $L(1,g)\frac{\gamma(\frac{1}{2},\tilde{F})}{\gamma(\frac{1}{2},F)}$. Note that, $\iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})\mathrm{spec}(\boldsymbol{\mu})d\boldsymbol{\mu}\asymp T^3R^2$. Therefore,

$$\mathcal{D}^{(1)} = L(1,g)\frac{1}{192\pi^5}\iint_{\Re(\boldsymbol{\mu})=0} h(\boldsymbol{\mu})\prod_{j=1}^{3}\frac{\Gamma(\frac{1}{4}+\frac{\mu_j}{2})}{\Gamma(\frac{1}{4}-\frac{\mu_j}{2})}\mathrm{spec}(\boldsymbol{\mu})\mathrm{d}\boldsymbol{\mu}+\mathrm{O}(\mathrm{T}^{\frac{9}{4}+\varepsilon}\mathrm{R}^2).$$

2.3.1.2 Contribution of $\Sigma_4^{(1)}$

Let $E_4^{(1)}$ be the contribution of Σ_4 to the sum in the equation (2.3.1). So

$$E_4^{(1)} = \sum_{m,n\geq 1}\sum\sum_{l\geq 1} \frac{\lambda_g(n)}{m\,(nl)^{1/2}} \sum_{\epsilon=\pm 1} \sum_{\substack{D_2|D_1 \\ lD_1=m{D_2}^2}} \frac{\tilde{S}(-\epsilon n, l, 1; D_2, D_1)}{D_1 D_2} \Phi_{w_4}^{(1)}\left(\frac{\epsilon l n}{D_1 D_2}\right).$$

Let $U_i(x)$ $(i=1,2)$ be smooth functions which are compactly supported in $[1,2]$, satisfy $x^j U_i^{(j)}(x) \ll 1$. By partition of unity, to get a bound for $E_4^{(1)}$ it is enough to estimate the following sum

$$\sum_{m,n\geq 1}\sum\sum_{l\geq 1} \frac{\lambda_g(n) U_1(\frac{nm^2}{N}) U_2(\frac{l}{L})}{(nm^2)^{1/2}\,(l)^{1/2}} \sum_{\substack{\delta, D \\ \delta l = mD}} \frac{\tilde{S}(\mp n, l, 1; D, \delta D)}{\delta D^2} \Phi_{w_4}^{(1)}\left(\frac{\pm l n}{\delta D^2}\right), \tag{2.3.4}$$

where $1 \leq N \leq T^{3+\varepsilon}$ and $1 \leq L \leq T^{3/2+\varepsilon}$. By Lemma 2.2.5, $\Phi_{w_4}^{(1)}\left(\frac{\pm ln}{\delta D^2}\right)$ is negligibly small unless

$$\frac{ln}{\delta D^2} \gg T^{3-\varepsilon}. \tag{2.3.5}$$

Note that $\delta l = mD$. By (2.3.5), we deduce that

$$1 \leq l\delta^3 \leq \frac{nm^2}{T^{3-\varepsilon}} \leq T^{\varepsilon},$$

which implies, $\delta, L, mD \leq T^{\varepsilon}$ and $T^{3-\varepsilon} \leq n \leq N \leq T^{3+\varepsilon}$.

We recall the Kloosterman sum

$$\tilde{S}(\mp n, l, 1; D, \delta D) = \sum_{\substack{C_1 (\bmod D), C_2 (\bmod \delta D) \\ (C_1, D) = (C_2, \delta) = 1}}\sum e\left(l \frac{\bar{C}_1 C_2}{D} + \frac{\bar{C}_2}{\delta} \mp n \frac{C_1}{D}\right).$$

In (2.3.4), the only non-trivial sum is the sum over n, which is given by

$$\sum_{n\geq 1} \lambda_g(n)\, e\left(\mp n \frac{C_1}{D}\right) \theta(n),$$

where

$$\theta(y) = \frac{1}{\sqrt{y}} U_1\left(\frac{m^2 y}{N}\right) \Phi_{w_4}^{(1)}\left(\frac{\pm l y}{\delta D^2}\right).$$

Now the $GL(2)$-Voronoi summation formula transforms the above sum into

$$\frac{1}{D}\sum_{n\geq 1}\lambda_g(n)\, e\left(\pm n\frac{\bar{C_1}}{D}\right)\Theta(n),$$

where

$$\Theta(y) = 2\pi i^k \int_0^\infty \theta(x) J_{k-1}\left(\frac{4\pi\sqrt{xy}}{D}\right) \mathrm{d}x.$$

We now analyse the integral transform $\Theta(n)$. Using the properties of Bessel functions we arrive at the following expression

$$\frac{N^{1/4}D^{1/2}}{m^{1/2}n^{1/4}}\int_0^\infty \frac{1}{x^{3/4}}U_1(x)\Phi^{(1)}_{w_4}\left(\frac{\pm lNx}{\delta(mD)^2}\right) e\left(\pm\frac{2\pi\sqrt{nNx}}{mD}\right)\mathrm{d}x. \tag{2.3.6}$$

By Lemma 2.2.5, we have

$$|x|^j \frac{\mathrm{d}^{\mathrm{j}}}{\mathrm{d}x^j}\Phi^{(1)}_{w_4}(x) \ll_{\varepsilon,j} T^{3+\varepsilon}R^2\left(T+|x|^{1/3}\right)^j.$$

First, we change the variable $x \to x^2$ in the integral of equation (2.3.6). Then by repeated integration by parts we see that

$$\Theta(n) \ll_{\varepsilon,j} \frac{N^{1/4}D^{1/2}T^{3+\varepsilon}R^2}{m^{1/2}n^{1/4}}\left(T+\left(\frac{lN}{\delta(mD)^2}\right)^{1/3}\right)^j\left(\frac{mD}{\sqrt{nN}}\right)^j.$$

Since $1 \leq \delta, L, m, D \leq T^\varepsilon$ and $T^{3-\varepsilon} \leq N \leq T^{3+\varepsilon}$.

Therefore,

$$E_4^{(1)} \ll_A T^{-A},$$

for any $A > 0$.

2.3.1.3 Contribution of $\Sigma_5^{(1)}$

Let $E_5^{(1)}$ be the contribution of Σ_5 to the sum in the equation (2.3.1). As in the previous case it is enough to estimate the following sum

$$\sum\sum_{m,n\geq 1}\sum_{l\geq 1}\frac{\lambda_g(n)U_1(\frac{nm^2}{N})U_2(\frac{l}{L})}{(nm^2)^{1/2}\,(l)^{1/2}}\sum_{\substack{\delta,D\\ \delta=nD}}\frac{\tilde{S}(\pm m,1,l;D,\delta D)}{\delta D^2}\Phi^{(1)}_{w_5}\left(\frac{\pm lm}{\delta D^2}\right) \tag{2.3.7}$$

to get a bound for $E_5^{(1)}$. Here $1 \leq nm^2 \leq N \leq T^{3+\varepsilon}$ and $1 \leq l \leq L \leq T^{3/2+\varepsilon}$. Since $\Phi_{w_5}^{(1)}\left(\frac{\pm lm}{\delta D^2}\right)$ is negligibly small unless

$$T^{3-\varepsilon} \ll \frac{lm}{\delta D^2}.$$

Note that $\delta = nD$, together with above inequality we get

$$nD^3 \leq \frac{lm}{T^{3-\varepsilon}} \leq T^{\varepsilon}.$$

Therefore $n, \delta, D \leq T^{\varepsilon}$ and $T^{3/2-\varepsilon} \leq m, l \leq T^{3/2+\varepsilon}$. Recall the Kloosterman sum

$$\tilde{S}(\pm m, 1, l; D, \delta D) = \sum_{\substack{C_1 (\operatorname{mod} D),\, C_2 (\operatorname{mod} \delta D) \\ (C_1, D) = (C_2, \delta) = 1}} \sum e\left(\frac{\bar{C}_1 C_2}{D} + l\frac{\bar{C}_2}{\delta} \mp m\frac{C_1}{D}\right).$$

Now, the l-sum in the equation (2.3.7) is given by

$$\sum_{l \geq 1} e\left(\frac{l\bar{C}_2}{\delta}\right) \frac{1}{\sqrt{l}} U_2\left(\frac{l}{L}\right) \Phi_{w_5}^{(1)}\left(\frac{\pm lm}{\delta D^2}\right).$$

An application of the Poisson summation formula to the above sum yields

$$\sqrt{L} \sum_{\substack{l \in \mathbb{Z} \\ l \equiv -C_2 (\operatorname{mod} \delta)}} \int_{\mathbb{R}} \frac{1}{\sqrt{y}} U_2(y) \Phi_{w_5}^{(1)}\left(\frac{\pm Lmy}{\delta D^2}\right) e\left(\frac{-lLy}{\delta}\right) \mathrm{d}y. \tag{2.3.8}$$

Note that, for $l \neq 0$ (non-zero frequency), by repeated integration by parts we have the inner integral in (2.3.8) is

$$\ll_{\varepsilon, j} T^{3+\varepsilon} R^2 \left(T + \left(\frac{mL}{\delta D^2}\right)^{1/3}\right)^j \left(\frac{\delta}{|l| L}\right)^j.$$

Thus we need to have $\delta = 1$, otherwise there will be no zero frequency. Therefore, we have $D = 1$ and $n = 1$. In this case the contribution of zero frequency is given by

$$\sqrt{L} \sum_{m \geq 1} \frac{1}{m} U_1\left(\frac{m^2}{N}\right) \int_{\mathbb{R}} \frac{1}{\sqrt{y}} U_2(y) \Phi_{w_5}^{(1)}(\pm Lmy) \,\mathrm{d}y.$$

By the definition of $\Phi_{w_5}^{(1)}(y)$ and $K_{w_4}(y; \boldsymbol{\mu})$, the above y-integral becomes

$$\int_{\mathbb{R}} \frac{1}{\sqrt{y}} U_2(y) \iint_{\Re(\boldsymbol{\mu}) = 0} h_1(\boldsymbol{\mu}) K_{w_4}(\mp Lmy; -\boldsymbol{\mu}) \,\mathrm{d}_{\mathrm{spec}} \boldsymbol{\mu} \,\mathrm{d}y$$

$$\begin{aligned}
&= \iint_{\Re(\boldsymbol{\mu})=0} h_1(\boldsymbol{\mu}) \int_{-i\infty}^{i\infty} \left(\int_{\mathbb{R}} y^{-s-\frac{1}{2}} U_2(y) \mathrm{d}y \right) |mL|^{-s} \tilde{G}^{\pm}(s,-\boldsymbol{\mu}) \frac{\mathrm{d}s}{2\pi i}\, \mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu} \\
&= \iint_{\Re(\boldsymbol{\mu})=0} h_1(\boldsymbol{\mu}) \int_{-i\infty}^{i\infty} |mL|^{-s} \widehat{U}_2\left(\frac{1}{2}-s\right) \tilde{G}^{\pm}(s,-\boldsymbol{\mu}) \frac{\mathrm{d}s}{2\pi i}\, \mathrm{d}_{\mathrm{spec}}\boldsymbol{\mu}, \qquad (2.3.9)
\end{aligned}$$

where $\widehat{U}_2(s)$ is the Mellin transform of $U_2(y)$, which is entire and rapidly decaying. Then we can restrict the s-integral to $|\mathrm{Im}(s)| \le T^{\varepsilon}$. Recall the definition of $\tilde{G}^{\pm}(s,\boldsymbol{\mu})$

$$\tilde{G}^{\pm}(s,\boldsymbol{\mu}) := \frac{\pi^{-3s}}{12288\pi^{7/2}} \left(\prod_{j=1}^{3} \frac{\Gamma(\frac{1}{2}(s-\mu_j))}{\Gamma(\frac{1}{2}(1-s+\mu_j))} \pm i \prod_{j=1}^{3} \frac{\Gamma(\frac{1}{2}(1+s-\mu_j))}{\Gamma(\frac{1}{2}(2-s+\mu_j))} \right).$$

Since $\boldsymbol{\mu}_0 \asymp \mu_{0,j} \asymp T$, then the $\boldsymbol{\mu}$-integral in (2.3.9) is bounded by $T^{3/2+\varepsilon}R^2$.

Thus,

$$E_5^{(1)} \ll T^{\frac{9}{4}+\varepsilon} R^2.$$

2.3.1.4 Contribution of $\Sigma_6^{(1)}$

In this case we have to bound the following sum

$$\sum_{m,n\ge 1}\sum\sum_{l\ge 1} \frac{\lambda_g(n) U_1(\frac{nm^2}{N}) U_2(\frac{l}{L})}{(nm^2)^{1/2}\,(l)^{1/2}} \sum_{D_1,D_2} \frac{S(\pm n, \pm m, 1, l; D_1, D_2)}{D_1 D_2} \Phi_{w_6}^{(1)}\left(\frac{\mp n D_2}{{D_1}^2}, \frac{\mp l m D_1}{{D_2}^2} \right). \qquad (2.3.10)$$

Using the property Φ_{w_6} (Lemma 2.2.6), we have

$$T^{1-\varepsilon} \le \frac{(lmn^2)^{1/6}}{D_1^{1/2}} \quad \text{and} \quad T^{1-\varepsilon} \le \frac{(nl^2m^2)^{1/6}}{D_2^{1/2}}.$$

From these conditions we infer that $1 \le D_2 \le T^{\varepsilon}, 1 \le D_1 \le T^{1/2+\varepsilon}/m$ and $1 \le m \le T^{1/2+\varepsilon}$, also $T^{3-\varepsilon} \le N \le T^{3+\varepsilon}$ and $T^{3/2-\varepsilon} \le L \le T^{3/2+\varepsilon}$.

The Kloosterman sum $S(\pm n, \pm m, 1, l; D_1, D_2)$ is given by

$$\sum\sum\sum\sum_{\substack{B_1,C_1 (\mathrm{mod}\, D_1);\, B_2,C_2 (\mathrm{mod}\, D_2) \\ D_1C_2+B_1B_2+D_2C_1 \equiv 0\, (\mathrm{mod}\, D_1D_2) \\ (B_j,C_j,D_j)=1}} e\left(\frac{\pm nB_1 + (Y_1D_2 - Z_1B_2)}{D_1} + \frac{\pm mB_2 + l(Y_2D_1 - Z_2B_1)}{D_2} \right),$$

where $B_jY_j + C_jZ_j \equiv 1\, (\mathrm{mod}\, D_j)$ for $j=1,2$. Note that $(B_1,D_1) \mid D_2$, so $(B_1,D_1) \le T^{\varepsilon}$ as $D_2 \le T^{\varepsilon}$. Let $B_1 = B_1^{'}(B_1,D_1)$ and $D_1 = D_1^{'}(B_1,D_1)$ with $(B_1^{'},D_1^{'})=1$.

Now we apply $GL(2)$-Voronoi summation and Poisson summation formulae on the n and l-sums in the equation (2.3.10) respectively. Then the n and l-sums in (2.3.10) transform into

$$2\pi i^k \frac{\sqrt{NL}}{mD_1^{'}} \sum_{n\geq 1} \sum_{\substack{l\in\mathbb{Z} \\ l\equiv -(Y_2D_1-Z_2B_1) \bmod D_2}} \lambda_g(n) e\left(\frac{\mp \bar{B}_1^{'} n}{D_1^{'}}\right) \times \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi_{w_6}^{(1)}\left(\frac{\mp ND_2x}{{D_1}^2m^2}, \frac{\mp mLD_1y}{{D_2}^2}\right) J_{k-1}\left(\frac{4\pi\sqrt{Nnx}}{mD_1^{'}}\right) e\left(\frac{-lLy}{D_2}\right) \frac{U_1(x)}{\sqrt{x}} \frac{U_2(y)}{\sqrt{y}} \mathrm{d}x\,\mathrm{d}y. \tag{2.3.11}$$

By repeated integration by parts, the y-integral in (2.3.11) is

$$\ll_{\varepsilon,j} T^{3+\varepsilon}R^2\left(T+\left(\frac{mD_1L}{D_2^2}\right)^{1/2}+\left(\frac{mD_1L}{D_2^2}\right)^{1/3}\left(\frac{ND_2}{D_1^2m^2}\right)^{1/6}\right)^j\left(\frac{D_2}{|l|L}\right)^j,$$

for $l\neq 0$. Therefore the non-zero frequency ($l\neq 0$) in (2.3.11) contributes $O(T^{-A})$. In the zero frequency ($l=0$) case, we must have $D_2 \mid (Y_2D_1-Z_2B_1)$. In this case equation (2.3.11) becomes

$$2\pi i^k \frac{\sqrt{NL}}{mD_1^{'}} \sum_{n\geq 1} \lambda_g(n) e\left(\frac{\mp \bar{B}_1^{'} n}{D_1^{'}}\right) \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi_{w_6}^{(1)}\left(\frac{\mp ND_2x}{{D_1}^2m^2}, \frac{\mp mLD_1y}{{D_2}^2}\right) \times J_{k-1}\left(\frac{4\pi\sqrt{Nnx}}{mD_1^{'}}\right) \frac{U_1(x)}{\sqrt{x}} \frac{U_2(y)}{\sqrt{y}} \mathrm{d}x\,\mathrm{d}y. \tag{2.3.12}$$

Recall the standard properties of Bessel function (see [83], p. 206)

$$J_r(2\pi x) = \frac{1}{\sqrt{x}}e(x)G_r(x) + \frac{1}{\sqrt{x}}e(-x)\bar{G}_r(x) \tag{2.3.13}$$

with

$$x^jG_r^{(j)}(x) \ll 1.$$

Inserting the properties of Bessel function (2.3.13) in the above expression (2.3.12) we get (at the cost of changing the smooth function $U_1^{*}(x) = U_1(x)G_{k-1}\left(\frac{2\pi\sqrt{Nnx}}{mD_1^{'}}\right)$)

$$\frac{N^{1/4}\sqrt{L}}{\sqrt{m}{D_1^{'}}^{1/2}} \sum_{n\geq 1} \frac{\lambda_g(n)}{n^{1/4}} e\left(\frac{\mp \bar{B}_1^{'} n}{D_1^{'}}\right) \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi_{w_6}^{(1)}\left(\frac{\mp ND_2x}{{D_1}^2m^2}, \frac{\mp mLD_1y}{{D_2}^2}\right) \times e\left(\frac{2\pi\sqrt{Nnx}}{mD_1^{'}}\right) \frac{U_1^{*}(x)}{x^{3/4}} \frac{U_2(y)}{\sqrt{y}} \mathrm{d}x\,\mathrm{d}y. \tag{2.3.14}$$

Integrating by parts we see that the x-integral in (2.3.14) is negligibly small unless $n \le T^{\varepsilon}$ (dual length). Therefore the sum in (2.3.10) is bounded above by

$$T^{\varepsilon} N^{1/4}\sqrt{L} \sum_{n\le T^{\varepsilon}} \frac{\lambda_g(n)}{n^{1/4}} \sum\sum_{\substack{D_1\le T^{1/2+\varepsilon}\\ D_2\le T^{\varepsilon}}} \frac{1}{D_1^{3/2}D_2} \sum_{m\le \frac{T^{1/2+\varepsilon}}{D_1}} \frac{1}{m^{3/2}}\mathcal{C}(D_1,D_2,n,m)\mathcal{I}(D_1,D_2,n,m),$$

where

$$\mathcal{C}(D_1,D_2,n,m) = \sum\sum_{\substack{B_1,C_1 (\mathrm{mod}\, D_1);\, B_2,C_2 (\mathrm{mod}\, D_2)\\ D_1C_2+B_1B_2+D_2C_1\equiv 0\, (\mathrm{mod}\, D_1D_2)\\ (B_j,C_j,D_j)=1\\ D_2 | (Y_2D_1-Z_2B_1)}} e\left(\frac{\mp \bar{B}_1' n}{D_1'} + \frac{Y_1D_2-Z_1B_2}{D_1} \pm \frac{mB_2}{D_2}\right),$$

$$\mathcal{I}(D_1,D_2,n,m) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi_{w_6}^{(1)}\left(\frac{\mp ND_2x}{{D_1}^2m^2}, \frac{\mp mLD_1y}{{D_2}^2}\right) e\left(\frac{2\pi\sqrt{Nnx}}{mD_1'}\right) \frac{U_1^*(x)}{x^{3/4}}\frac{U_2(y)}{\sqrt{y}}\mathrm{d}x\,\mathrm{d}y.$$

Using (2.2.11), we have

$$\begin{aligned}\mathcal{I}(D_1,D_2,n,m) = \int_{\mathbb{R}}\int_{\mathbb{R}}\iint_{\Re(\boldsymbol{\mu})=0} h_1(\boldsymbol{\mu}) \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} |4\pi^2\frac{ND_2x}{{D_1}^2m^2}|^{-s_1} |4\pi^2\frac{mLD_1y}{{D_2}^2}|^{-s_2}\\ \times\frac{1}{\Gamma(s_1+s_2)}\prod_{j=1}^{3}\Gamma(s_1-\mu_j)\Gamma(s_2+\mu_j)S^{\mp,\mp}(\boldsymbol{s};\boldsymbol{\mu})\,\frac{\mathrm{d}s_1\mathrm{d}s_2}{(2\pi i)^2}\,\mathrm{d}_{\mathrm{spce}}\boldsymbol{\mu}\\ \times e\left(\frac{2\pi\sqrt{Nnx}}{mD_1'}\right)\frac{U_1^*(x)}{x^{3/4}}\frac{U_2(y)}{\sqrt{y}}\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where $S^{\mp,\mp}(\boldsymbol{s};\boldsymbol{\mu})$ is defined as in (2.2.8).

$$\begin{aligned}= \iint_{\Re(\boldsymbol{\mu})=0} h_1(\boldsymbol{\mu}) \int_{-i\infty}^{+i\infty} \left|4\pi^2\frac{ND_2}{{D_1}^2m^2}\right|^{-s_1}\left(\int_{\mathbb{R}} x^{-3/4-s_1} e\left(\frac{2\pi\sqrt{Nnx}}{mD_1'}\right)U_1^*(x)\mathrm{d}x\right)\\ \times\int_{-i\infty}^{+i\infty}\left|4\pi^2\frac{mLD_1}{{D_2}^2}\right|^{-s_2}\left(\int_{\mathbb{R}} y^{-1/2-s_2}U_2(y)\mathrm{d}y\right)\\ \times\frac{1}{\Gamma(s_1+s_2)}\prod_{j=1}^{3}\Gamma(s_1-\mu_j)\Gamma(s_2+\mu_j)S^{\mp,\mp}(\boldsymbol{s};\boldsymbol{\mu})\,\frac{\mathrm{d}s_1\mathrm{d}s_2}{(2\pi i)^2}\,\mathrm{d}_{\mathrm{spce}}\boldsymbol{\mu},\end{aligned}$$

$$\begin{aligned}= \iint_{\Re(\boldsymbol{\mu})=0} h_1(\boldsymbol{\mu}) \int_{-i\infty}^{+i\infty} \left|4\pi^2\frac{ND_2}{{D_1}^2m^2}\right|^{-s_1}\hat{U}_1^*(\tfrac{1}{4}-s_1)\int_{-i\infty}^{+i\infty}\left|4\pi^2\frac{mLD_1}{{D_2}^2}\right|^{-s_2}\hat{U}_2(\tfrac{1}{2}-s_2)\\ \times\frac{1}{\Gamma(s_1+s_2)}\prod_{j=1}^{3}\Gamma(s_1-\mu_j)\Gamma(s_2+\mu_j)S^{\mp,\mp}(\boldsymbol{s};\boldsymbol{\mu})\,\frac{\mathrm{d}s_1\mathrm{d}s_2}{(2\pi i)^2}\,\mathrm{d}_{\mathrm{spce}}\boldsymbol{\mu}.\end{aligned}$$

Note that $\hat{U}_2$ is the Mellin transform of U_2, which is entire and rapidly decaying. So we can restrict the s_2-integral to $|\Im(s_2)| \leq T^{\varepsilon}$. Similarly we can restrict the s_1-integral to $|\Im(s_1)| \leq \frac{\sqrt{Nn}}{mD_1'} \asymp T^{1+\varepsilon}$.

Recall that $\mu_1 + \mu_2 + \mu_3 = 0$. For fixed σ, $t \in \mathbb{R}$ and $|t| \geq 1$, we have the Stirling formula

$$\Gamma(\sigma + it) = \sqrt{2\pi} e^{-\pi|t|/2} |t|^{\sigma - 1/2} e^{it(\log|t| - 1)} e^{i(\sigma - 1/2)\lambda\pi/2} \left(1 + O(|t|^{-1})\right).$$

So we have,

$$\frac{1}{\Gamma(s_1+s_2)} \prod_{j=1}^{3} \Gamma(s_1 - \mu_j)\Gamma(s_2 + \mu_j) S^{\mp,\mp}(\boldsymbol{s};\boldsymbol{\mu}) \ll \prod_{j=1}^{3} |\mu_j|^{-1}.$$

Therefore, the integral $\mathcal{I}(D_1, D_2, n, m)$ is bounded by $O\Big(T^{1+\varepsilon}R^2\Big)$. Moreover, by the standard (Weil-type) bounds we get (see [[7], 3.1 and 3.2])

$$\mathcal{C}(D_1, D_2, n, m) \ll (nm)^{\varepsilon} (D_1 D_2)^{1/2+\varepsilon}$$

.

Hence the contribution of $\Sigma_6^{(1)}$ is $O(T^{5/2+\varepsilon}R^2)$.

2.3.1.5 Contribution of $\mathcal{E}_{\min}^{(1)}$

Let us denote

$$E_{\min}^{(1)} := \frac{1}{24(2\pi i)^2} \iint_{\Re(\boldsymbol{\mu})=0} \frac{h(\boldsymbol{\mu})}{\mathcal{N}_{\boldsymbol{\mu}}^{\min}} \sum\sum_{m,n\geq 1} \frac{\lambda_g(n)}{mn^{1/2}} B_{\boldsymbol{\mu}}^{\min}(m,n) W_F(nm^2) \\ \times \sum_{l\geq 1} \frac{1}{l^{1/2}} \overline{B_{\boldsymbol{\mu}}^{\min}(1,l)} \tilde{V}_F(l)\, \mathrm{d}\mu_1 \mathrm{d}\mu_2.$$

Inserting the definition of $V_F(y)$, $\tilde{W}(y)$ and using (2.2.7), (2.2.5) we have

$$E_{\min}^{(1)} = \frac{1}{24(2\pi i)^2} \iint_{\Re(\boldsymbol{\mu})=0} \frac{h(\boldsymbol{\mu})}{\mathcal{N}_{\boldsymbol{\mu}}^{\min}} \mathcal{I}_{\min}^{(1)}(\boldsymbol{\mu}) \mathrm{d}\mu_1 \mathrm{d}\mu_2,$$

where

$$\mathcal{I}^{(1)}_{\min}(\boldsymbol{\mu}) = \frac{1}{(2\pi i)^2}\int_{(3)}\int_{(3)} G(s_2)\prod_{j=1}^{3}\frac{\Gamma_{\mathbb{R}}\left(s_2+\frac{1}{2}+\mu_j\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2}-\mu_j\right)}\zeta(s_2+\tfrac{1}{2}-\mu_j)$$
$$\times G(s_1)\prod_{j=1}^{3}\frac{\Gamma_{\mathbb{R}}\left(s_1+\frac{1}{2}+\frac{k-1}{2}-\mu_j\right)\Gamma_{\mathbb{R}}\left(s_1+\frac{1}{2}+\frac{k+1}{2}-\mu_j\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k-1}{2}-\mu_j\right)\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k+1}{2}-\mu_j\right)}L(s_1+\tfrac{1}{2}+\mu_j,g)\frac{\mathrm{d}s_1}{s_1}\frac{\mathrm{d}s_2}{s_2}.$$

Now we move the line of integration from $\Re(s_1)=\Re(s_2)=3$ to $\Re(s_1)=\Re(s_2)=\varepsilon$. Since $L(1/2+it,g)\ll t^{1/3}$, $\zeta(1/2+it)\ll t^{1/6}$ and $G(s)\ll e^{-t^2}$, we have

$$\mathcal{I}^{(1)}_{\min}(\boldsymbol{\mu}) \ll \prod_{j=1}^{3}(1+|\mu_j|)^{1/2}.$$

Finally, using the bound

$$\mathcal{N}^{\min}_{\boldsymbol{\mu}} = \mathcal{N}^{\min}_{\nu_1,\nu_2} := \frac{1}{16}\prod_{j=1}^{3}|\zeta(1+3\nu_j)|^2 \gg \prod_{j=1}^{3}\left(\frac{1}{\log(1+3|\Im(\nu_j)|)}\right)^2,$$

we obtain

$$E^{(1)}_{\min} \ll T^{\frac{3}{2}+\varepsilon}R^2.$$

2.3.1.6 Contribution of $\mathcal{E}^{(1)}_{\max}$

Let us define

$$E^{(1)}_{\max} := \sum_f \frac{1}{2\pi i}\int_{\Re(u)=0}\frac{h(u+it_f,u-it_f,-2u)}{\mathcal{N}^{\max}_{u,f}}$$
$$\times \sum\sum_{m,n\geq 1}\frac{\lambda_g(n)}{mn^{1/2}}B^{\max}_{u,f}(m,n)W_{\mathcal{F}}(nm^2)\sum_{l\geq 1}\frac{1}{l^{1/2}}\overline{B^{\max}_{u,f}(1,l)}\tilde{V}_{\mathcal{F}}(l)\mathrm{d}u.$$

By similar arguments as above one has

$$E^{(1)}_{\max} = \sum_f \frac{1}{2\pi i}\int_{\Re(u)=0}\frac{h(u+it_f,u-it_f,-2u)}{\mathcal{N}^{\max}_{u,f}}\mathcal{I}(u+it_f,u-it_f,-2u)\mathrm{d}u,$$

where $\mathcal{I}^{(1)}_{\max}(u+it_f,u-it_f,-2u)$ is given by

$$\frac{1}{(2\pi i)^2}\int_{(3)}\int_{(3)} G(s_1)G(s_2)L(s_1+\tfrac{1}{2}-2u,g)L(s_1+\tfrac{1}{2}+u,g\otimes f)L(s_2+\tfrac{1}{2}-u,g)\times$$
$$\zeta(s_2+\tfrac{1}{2}+2u)\prod_{j=1}^{3}\frac{\Gamma_{\mathbb{R}}\left(s_1+\frac{1}{2}+\frac{k-1}{2}-\alpha_j\right)\Gamma_{\mathbb{R}}\left(s_1+\frac{1}{2}+\frac{k+1}{2}-\alpha_j\right)\Gamma_{\mathbb{R}}\left(s_2+\frac{1}{2}+\alpha_j\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k-1}{2}-\alpha_j\right)\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k+1}{2}-\alpha_j\right)\Gamma_{\mathbb{R}}\left(\frac{1}{2}-\alpha_j\right)}\frac{\mathrm{d}s_1}{s_1}\frac{\mathrm{d}s_2}{s_2}.$$

Here $\alpha_1 = u + it_f$, $\alpha_2 = u - it_f$ and $\alpha_3 = -2u$. By the definition of h we have $h(u + it_f, u - it_f, -2u)$ is negligibly small unless

$$|u + it_f - \mu_{0,1}| \leq R, \;\; |u - it_f - \mu_{0,2}| \leq R, \;\; |-2u - \mu_{0,3}| \leq R.$$

Note that $\mu_{0,j} \asymp T$ and $\mathcal{N}_{u,f}^{\max} := 8L(1, \mathrm{Ad}^2 f)|L(1 + 3u, f)|^2 \gg (1 + \log|u|)^{-1}$.

We first shift the line of integration to $\Re(s_1) = \Re(s_2) = \varepsilon$ and use the bounds

$$\zeta(1/2 + it) \ll |t|^{1/6}, \;\; L(1/2 + it, f) \ll |t|^{1/3}, \;\; L(1/2 + it, g) \ll |t|^{1/3}$$

and $L\left(1/2 + it, g \otimes f\right) \ll |t|$ to get

$$E_{\max}^{(1)} \ll T^{11/6+\varepsilon} R \sum_{T-R \leq t_f \leq T+R} 1.$$

Using the Weyl law for the number of eigenvalues associated to $GL(2)$ eigenforms we have

$$E_{\max}^{(1)} \ll T^{17/6+\varepsilon} R^2.$$

Chapter 3

Shifted moments of quadratic L-functions over function fields

3.1 Introduction

The correlation of L-functions i.e., study of the mean values of products of shifted values of L-functions near the critical line has become central to number theory. Random matrix theory has recently become a fundamental tool for understanding the correlation of L-functions. Montgomery [59] showed that two-point correlations between the non-trivial zeros of the Riemann ζ-function, on the scale of the mean zero spacing, are similar to the corresponding correlations between the eigenvalues of random unitary matrices in the limit of large matrix size and conjectured that these correlations are, in fact, identical to each other.

Keating and Snaith [50] suggested that the value distribution of the Riemann zeta function on its critical line is related to that the characteristic polynomials of random unitary matrices. Conjectures for the moments of L-functions have been attempted for many decades, with very little progress until the random matrix theory came into the subject.

The main observation is that the structure of the mean values of L-functions is more clearly revealed if one considers the average of a product of L-functions, where each L-function is evaluated at a location slightly shifted from the critical point.

In this chapter, we discus about the moments and correlation of Riemann zeta function (belonging to unitary family), Dirichlet L-functions (contained in the symplectic family) and quadratic Dirichlet L-functions associated with hyperelliptic curves of large genus over a fixed finite field which are also members of the symplectic family. These families and their random matrix analogs have been discussed from the perspective of the leading terms in the asymptotic expressions by several authors (see [18], [21], [20], [4], [49], [50], [22] and [48]).

Our main goal of this chapter is to establish lower and upper bounds for the correlation of shifted values of quadratic Dirichlet L-functions near the critical line associated to the hyperelliptic curves of large genus over a fixed finite field.

3.1.1 Moments of the Riemann Zeta function

A classical question in the theory of Riemann zeta function is to determine the asymptotic behaviour of

$$M_k(T) := \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\, dt,$$

where $k \in \mathbb{C}$, as $T \to \infty$. It is believed that for a given positive real number k,

$$M_k(T) \sim c_k T (\log T)^{k^2},$$

where c_k is a positive constant. Ramachandra [69] showed that

$$M_k(T) \gg T (\log T)^{k^2},$$

for any $k \in \mathbb{N}$. Using moments of characteristic polynomials of random matrices, Keating and Snaith [49], conjectured an exact value of c_k for $\Re(k) > -\frac{1}{2}$. Assuming the Riemann hypothesis (RH), Soundararajan [81] showed that for every positive real number k and $\varepsilon > 0$,

$$M_k(T) \ll_{k,\varepsilon} c_k T (\log T)^{k^2+\varepsilon} \tag{3.1.1}$$

and Harper [35] removed the exponent ϵ in the bound of (3.1.1).

3.1.1.1 Shifted Moments of the Riemann Zeta function

A generalization of the moments of $\zeta(s)$ are the shifted moments, defined as

$$M_{\boldsymbol{k}^{(m)}}(T, \boldsymbol{\alpha}^{(m)}) := \int_0^T |\zeta(\tfrac{1}{2} + it + i\alpha_1)|^{2k_1} \dots |\zeta(\tfrac{1}{2} + it + i\alpha_m)|^{2k_m} \, dt,$$

where $\boldsymbol{k}^{(m)} = (k_1, \dots, k_m)$ is a sequence of non-negative real numbers ($k_i \in \mathbb{R}_+$) and $\boldsymbol{\alpha}^{(m)} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ with $\alpha_i \neq \alpha_j$ for $i \neq j$, $|\alpha_i - \alpha_j| = O(1)$, $\alpha_i = O(\log T)$ also $\alpha_i = \alpha_i(T)$ is real valued function in terms of T such that $\lim\limits_{T \to \infty} \alpha_i \log T$ and $\lim\limits_{T \to \infty} |\alpha_i - \alpha_j| \log T$ exits or equals to $\pm\infty$. In [16], Chandee obtained the lower and the upper bounds of $M_{\boldsymbol{k}^{(m)}}(T, \boldsymbol{\alpha}^{(m)})$ for some special choices of $\boldsymbol{k}^{(m)}$ and for a large values of T. More precisely, she proved the following theorems.

Theorem 3.1.1. *Assume RH. Let $\boldsymbol{k}^{(m)} \in \mathbb{R}_+^m$ and $\boldsymbol{\alpha}^m$ be defined as above. Then for T large,*

$$M_{\boldsymbol{k}^{(m)}}(T, \boldsymbol{\alpha}^{(m)}) \ll_{\boldsymbol{k}^{(m)}, \varepsilon} T \, (\log T)^{k_1^2 + k_2^2 + \ldots + k_m^2 + \varepsilon} \prod_{i<j} \left(\min \left\{ \tfrac{1}{|\alpha_i - \alpha_j|}, \log T \right\} \right)^{2k_i k_j}.$$

Theorem 3.1.2. *Unconditionally, for large T, $\boldsymbol{k}^{(m)} \in \mathbb{N}^m$ and $\boldsymbol{\alpha}^{(m)}$ be defined as above with $\alpha_i = O(\log \log T)$,*

$$M_{\boldsymbol{k}^{(m)}}(T, \boldsymbol{\alpha}^{(m)}) \gg_{\boldsymbol{k}^{(m)}, \beta^{(m)}} T \, (\log T)^{k_1^2 + k_2^2 + \ldots + k_m^2} \prod_{i<j} \left(\min \left\{ \frac{1}{|\alpha_i - \alpha_j|}, \log T \right\} \right)^{2k_i k_j},$$

where

$$\beta^{(m)} = \max_{\{(i,j) : |\alpha_i - \alpha_j| = O(1/\log T)\}} \left\{ \lim_{T \to \infty} |\alpha_i - \alpha_j| \log T \right\}.$$

As a corollary of the above Theorem 3.1.1 and 3.1.2, we have the following results.

Corollary 3.1.3. *Assume RH. Let $k \in \mathbb{R}_+$. Then for T large, we have*

$$M_{(k,k)}(T, (\alpha_1, \alpha_2)) \begin{cases} \ll_{k,\varepsilon} T \, (\log T)^{4k^2 + \varepsilon}, & \text{if } \lim\limits_{T \to \infty} |\alpha_1 - \alpha_2| \log T < \infty \\ \ll_{k,\varepsilon} \frac{1}{|\alpha_1 - \alpha_2|^{2k^2}} T (\log T)^{2k^2 + \varepsilon}, & \text{if } \lim\limits_{T \to \infty} |\alpha_1 - \alpha_2| \log T = \infty. \end{cases}$$

Corollary 3.1.4. *Unconditionally, for T large and $k \in \mathbb{N}$, we have*

$$M_{(k,k)}(T,(\alpha_1,\alpha_2)) \begin{cases} \gg_k T\,(\log T)^{4k^2}\,, & \text{if } \lim\limits_{T\to\infty} |\alpha_1-\alpha_2|\log T = 0 \\ \gg_{k,\beta^{(2)}} T\,(\log T)^{4k^2}\,, & \text{if } \lim\limits_{T\to\infty} |\alpha_1-\alpha_2|\log T < \infty \\ \gg_k \frac{1}{|\alpha_1-\alpha_2|^{2k^2}} T(\log T)^{2k^2}, & \text{if } \lim\limits_{T\to\infty} |\alpha_1-\alpha_2|\log T = \infty. \end{cases}$$

The above corollary implied

$$T\,(\log T)^{4k^2} \ll M_{(k,k)}(T,(\alpha_1,\alpha_2)) \ll T\,(\log T)^{4k^2+\varepsilon}$$

when $\lim\limits_{T\to\infty} |\alpha_1-\alpha_2|\log T < \infty$, which means that the correlation of $|\zeta\left(\frac{1}{2}+it+i\alpha_1\right)|$ and $|\zeta\left(\frac{1}{2}+it+i\alpha_2\right)|$ transition at $|\alpha_1-\alpha_2| \approx \frac{1}{\log T}$. Furthermore

$$\frac{1}{|\alpha_1-\alpha_2|^{2k^2}} T\,(\log T)^{2k^2} \ll M_{(k,k)}(T,(\alpha_1,\alpha_2)) \ll \frac{1}{|\alpha_1-\alpha_2|^{2k^2}} T\,(\log T)^{2k^2+\varepsilon}$$

when $\lim\limits_{T\to\infty} |\alpha_1-\alpha_2|\log T = \infty$, which means that, these distribution appear independent when $|\alpha_1-\alpha_2|$ is much large than $\frac{1}{\log T}$.

The moments of the derivatives of the Riemann zeta function were studied by several mathematicians. An analog of Soundararajan's estimate (3.1.1) for the derivatives of the Riemann zeta function was obtained by Milinovich [62]. Under the RH, he showed that for every $\varepsilon > 0$,

$$\int_0^T |\zeta^{(l)}(\tfrac{1}{2}+it)|^{2k}\,dt \ll_{k,l,\varepsilon} T\,(\log T)^{k^2+2kl+\varepsilon},$$

where $k, l \in \mathbb{N}$ and $\zeta^{(l)}$ is the l-th derivative of ζ.

3.1.2 Moments of quadratic Dirichlet L-functions

Let χ_d be a real primitive Dirichlet character modulo d given by the Kronecker symbol $\chi_d(n) = \left(\frac{d}{n}\right)$. It is interesting to determine the asymptotic behaviour of $\sum\limits_{0<d\le X} L(\frac{1}{2},\chi_d)^k$ as $X \to \infty$. Extending their approach to the zeta function, using random matrix theory, Keating and Snaith [50] made the following conjecture about the asymptotic behaviour of moments of Dirichlet L-functions $L(\frac{1}{2},\chi_d)$.

Conjecture 3.1.5 (Keathing & Snaith)**.** *For k fixed with $\Re(k) \geq 0$, as $X \to \infty$*

$$\frac{1}{X} \sideset{}{^\flat}\sum_{|d| \leq X} L(\tfrac{1}{2}, \chi_d)^k \sim c_k \; (\log X)^{\frac{k(k+1)}{2}},$$

and the precise values of c_k follows from work of Keathing and Snaith [50].

In [73], Rudnick and Soundararajan obtained that for any rational number $k \geq 1$,

$$\frac{1}{X} \sideset{}{^\flat}\sum_{|d| \leq X} L(\tfrac{1}{2}, \chi_d)^k \gg_k \; (\log X)^{\frac{k(k+1)}{2}}.$$

Assuming the GRH, Soundararajan established that for any positive real number k and $\epsilon > 0$,

$$\frac{1}{X} \sideset{}{^\flat}\sum_{|d| \leq X} L(\tfrac{1}{2}, \chi_d)^k \ll_{k,\epsilon} \; (\log X)^{\frac{k(k+1)}{2}+\varepsilon}.$$

In general, it is important to find the asymptotic behaviour of the following correlation of shifted values of Dirichlet L-functions:

$$S_{\boldsymbol{k}^{(m)}}(\boldsymbol{\alpha}^{(m)}, X) := \sideset{}{^\flat}\sum_{d \leq X} L(\tfrac{1}{2} + \alpha_1, \chi_d)^{k_1} \ldots L(\tfrac{1}{2} + \alpha_m, \chi_d)^{k_m},$$

where $\boldsymbol{k}^{(m)} = (k_1, \ldots, k_m)$ be a sequence of real numbers and $\boldsymbol{\alpha}^{(m)} = (\alpha_1, \ldots, \alpha_m)$ be a sequence of complex numbers with $\alpha_i \neq \alpha_j$ for $i \neq j$.

Conrey *et. al.* [20, 21] gave a conjecture on the asymptotic behaviour for the moments of $L\left(\frac{1}{2}, \chi_d\right)$.

Conjecture 3.1.6. *[Conrey, Farmer, Keating, Rubinstein and Snaith] Suppose $g(u)$ is a suitable weight function with support in either $(0, \infty)$ or $(-\infty, 0)$ and let $\gamma_d(s) = |d|^{\frac{1}{2}-s}\gamma(s, a)$ where $a = 0$ if $d > 0$, and $d = 1$ if $d < 0$, and*

$$\gamma(s, a) = \pi^{s-\frac{1}{2}} \Gamma\left(\frac{1+a-s}{2}\right) / \Gamma\left(\frac{s+a}{2}\right).$$

That is, the factor in the functional equation

$$L(s, \chi_d) = \varepsilon_d \gamma_d(s) L(1-s, \chi_d).$$

For $k \in \mathbb{N}$ we have

$$\sum_{d}^{\flat} L\left(\tfrac{1}{2}, \chi_d\right)^k g(|d|) = \sum_{d}^{\flat} Q_k\left(\log|d|\right) g(|d|)\left(1 + O(|d|^{-\frac{1}{2}+\varepsilon})\right), \tag{3.1.2}$$

where Q_k is the polynomial of degree $k(k+1)/2$ given by k-fold residue

$$\begin{aligned} Q_k(x) = & \frac{(-1)^{\frac{k(k-1)}{2}} 2^k}{k!} \frac{1}{(2\pi i)^k} \\ & \times \oint \dots \oint \frac{G(z_1, \dots, z_k) \Delta(z_1^2, \dots, z_k^2)^2}{\prod_{j=1}^{k} z_j^{2k-1}} e^{\frac{x}{2}\sum_{j=1}^{k} z_j} dz_1 \dots dz_k, \end{aligned}$$

with

$$G(z_1, \dots, z_k) = A_k(z_1, \dots, z_k) \prod_{j=1}^{k} \gamma(\tfrac{1}{2} + z_j, a)^{-\frac{1}{2}} \prod_{1 \le i \le j \le k} \zeta\left(1 + \zeta_i + z_j\right),$$

$\Delta(z_1^2, \dots, z_k^2)$ the Vandermonde determinant given by

$$\Delta(z_1^2, \dots, z_k^2) = \prod_{1 \le i < j \le k} (z_j - z_i), \tag{3.1.3}$$

and A_k is the Euler product, which is absolutely convergent for $|\Re(z_j)| < 1/2$, for $j = 1, \dots, k$, defined by

$$\begin{aligned} A_k(z_1, \dots, z_k) = & \prod_{p} \prod_{1 \le i \le j \le k} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right) \\ & \times \left(\frac{1}{2} \prod_{j=1}^{k} \left(1 - \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1} + \frac{1}{2} \prod_{j=1}^{k} \left(1 + \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1} + \frac{1}{p}\right) \left(1 + \frac{1}{p}\right)^{-1}. \end{aligned}$$

Analogous questions for higher degree L-functions have been studied by Milinovich and Turnage-Butterbaugh [66].

3.1.3 Moments of L-functions in the hyperelliptic ensemble

Let $\mathbb{F}_q$ be a finite field of odd cardinality and $\mathbb{F}_q[t]$ be the polynomial ring over $\mathbb{F}_q$ in variable t. Let $D \in \mathbb{F}_q[t]$ be a monic square-free polynomial. The quadratic character χ_D attached to D is defined using the quadratic residue symbol for $\mathbb{F}_q[t]$ by $\chi_D(f) = \left(\frac{D}{f}\right)$ and the corresponding Dirichlet L-function is denoted by $L(s, \chi_D)$. It is often convenient to work with the equivalent L-function $\mathcal{L}(u, \chi_D)$ written in terms of the variable $u = q^{-s}$.

Define the hyperelliptic ensemble $\mathcal{H}_{n,q}$ or simply $\mathcal{H}_n$ as

$$\mathcal{H}_n = \{D \in \mathbb{F}_q[t] : D \text{ is monic, square-free, and } \deg(D) = n\}.$$

For each D in the hyperelliptic ensemble $\mathcal{H}_n$, there is an associated hyperelliptic curve given by $C_D : y^2 = D(t)$. This curve is non-singular and of genus g given by

$$2g = n - 1 - \lambda, \tag{3.1.4}$$

where

$$\lambda = \begin{cases} 1, & \text{if } n \text{ even,} \\ 0, & \text{if } n \text{ odd.} \end{cases}$$

Note that, $g \to \infty$ as n does so. See section 3.2 for more details about the properties of Dirichlet L-function $L(s, \chi_D)$ and their spectral interpretation.

Now we present the function fields version of Conjecture 3.1.6 from [1].

Conjecture 3.1.7. *Suppose* $q \equiv 1 \pmod 4$ *is the fixed cardinality of the finite field* $\mathbb{F}_q$ *and let* $\mathfrak{X}_D(s) = |D|^{\frac{1}{2}-s}\mathfrak{X}(s)$ *and* $\mathfrak{X}(s) = q^{-\frac{1}{2}+s}$. *That is,* $\mathfrak{X}_D(s)$ *is the factor in the functional equation*

$$L\left(s, \chi_D\right) = \mathfrak{X}_D(s) L\left(s, \chi_D\right).$$

For $k \in \mathbb{N}$, *we have*

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\tfrac{1}{2}, \chi_D\right)^k = \sum_{D \in \mathcal{H}_n} Q_k\left(\log_q |D|\right)\left(1 + O(|D|^{-\frac{1}{2}+\varepsilon})\right), \tag{3.1.5}$$

where Q_k *is the polynomial of degree* $k(k+1)/2$ *given by* k*-fold residue*

$$Q_k(x) = \frac{(-1)^{\frac{k(k-1)}{2}} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{G(z_1, \ldots, z_k) \Delta(z_1^2, \ldots, z_k^2)^2}{\prod_{j=1}^{k} z_j^{2k-1}} q^{\frac{x}{2}\sum_{j=1}^{k} z_j} dz_1 \ldots dz_k,$$

where $\Delta(z_1^2, \ldots, z_k^2)$ *is defined as in* (3.1.3),

$$G(z_1, \ldots, z_k) = A_k(z_1, \ldots, z_k) \prod_{j=1}^{k} \mathfrak{X}(\tfrac{1}{2} + z_j)^{-\frac{1}{2}} \prod_{1 \le i \le j \le k} \zeta_{\mathbb{A}}\left(1 + z_i + z_j\right),$$

and A_k is the Euler product, which is absolutely convergent for $|\Re(z_j)| < 1/2$, for $j = 1, \dots, k$, defined by

$$A_k(z_1, \dots, z_k) = \prod_{P \in \mathcal{P}} \prod_{1 \le i \le j \le k} \left(1 - \frac{1}{|P|^{1+z_i+z_j}}\right) \times \left(\frac{1}{2} \prod_{j=1}^{k} \left(1 - \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1} + \frac{1}{2} \prod_{j=1}^{k} \left(1 + \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1} + \frac{1}{|P|}\right) \left(1 + \frac{1}{|P|}\right)^{-1}.$$

Andrade and Keating [4] conjectured that as $g \to \infty$,

$$\sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D)^k = q^{2g+1} \left(P_k(2g+1) + o(1)\right), \tag{3.1.6}$$

where P_k is a polynomial of degree $\frac{k(k+1)}{2}$. Assuming $q \equiv 1 \pmod 4$, the conjecture (3.1.6) is known for $k = 1$ from the work of Andrade [3], and the error term in the asymptotic formula was improved by Florea [27]. In [28, 29], Florea also proved the conjecture (3.1.6) for $k = 2, 3$ and 4 assuming $q \equiv 1 \pmod 4$. For $n = 2g + 2$, Jung [46] obtained that

$$\frac{1}{|\mathcal{H}_{2g+2}|} \sum_{D \in \mathcal{H}_{2g+2}} L(\tfrac{1}{2}, \chi_D) = P(1)(g+1) + \frac{P'(1)}{\log q} - P(1)\zeta_{\mathbb{A}}\left(\frac{1}{2}\right) + O\left(2^{g+1} q^{-\frac{g}{2}}\right),$$

where $P(s) = \prod_P \left(1 - (1+|P|)^{-1}|P|^{-s}\right)$ and $\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)$ is defined in Section 2.

Andrade [2] established the following lower bound:

Theorem 3.1.8 (Andrade)**.** *For every even natural number k, we have*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} L(\tfrac{1}{2}, \chi_D)^k \gg_k n^{\frac{k(k+1)}{2}}.$$

On the other hand, A. Florea [[29], Theorem 2.7] found the following upper bound for a single shifted L-function associated with hyperelliptic curves:

Theorem 3.1.9 (Florea)**.** *Let $v = e^{i\theta}$, with $\theta \in [0, \pi)$. Then for every positive k and any $\epsilon > 0$,*

$$\sum_{D \in \mathcal{H}_{2g+1}} \left|\mathcal{L}\left(\frac{v}{\sqrt{q}}, \chi_D\right)\right|^k \ll_{k,\varepsilon} q^{2g+1} g^{\varepsilon} \exp\left(k\,\mathcal{M}(v,g) + \frac{k^2}{2}\mathcal{V}(v,g)\right),$$

where $\mathcal{M}(v,g)=\frac{1}{2}\log\left(\min\{g,\frac{1}{2\theta}\}\right)$ *and* $\mathcal{V}(v,g)=\mathcal{M}(v,g)+\frac{1}{2}\log g$.

3.1.3.1 Shifted moments and main results

For a fixed m-tuple $\boldsymbol{k}^{(m)}=(k_1,\dots,k_m)\in\mathbb{R}_+^m(\ m\geq 1)$, we shall investigate the following mean values of the product of m-shifted quadratic Dirichlet L-functions:

$$\mathcal{S}_n(\boldsymbol{v}^{(m)},\boldsymbol{k}^{(m)}):=\sum_{D\in\mathcal{H}_n}\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}},\chi_D\Big)^{2k_1}\dots\mathcal{L}\Big(\frac{v_m}{q^{\frac{1}{2}+\alpha_m}},\chi_D\Big)^{2k_m}, \tag{3.1.7}$$

where $\boldsymbol{v}^{(m)}=(v_1,\dots,v_m)\in\mathbb{C}^m$ with $v_j=e^{i\theta_j}$, $\theta_j\in[0,\pi)$ and $\alpha_j\in[0,\frac{1}{2})$ for $j=1,\dots,m$. Also, $\theta_j=\theta_j(g)$ is a real valued function of g such that $\lim\limits_{g\to\infty}g|\theta_j|$ and for $i\neq j$, $\lim\limits_{g\to\infty}g|\theta_i-\theta_j|$ exists or equals ∞. Note that one can obtain the moments of $\mathcal{L}\Big(\frac{v}{\sqrt{q}},\chi_D\Big)$ by allowing the shifts α_j to tend to 0.

Throughout the chapter, we follow the convention that n and g are connected via (3.1.4). Before stating our main results, let us define

$$\mu(\boldsymbol{v}^{(m)},g)=\sum_{j=1}^{m}k_j\log\left(\min\left\{\frac{1}{2|\theta_j|},g\right\}\right), \tag{3.1.8}$$

$$\sigma(\boldsymbol{v}^{(m)},g)=2\left(\sum_{j=1}^{m}k_j^2\right)\log g\;+\;2\sum_{j=1}^{m}k_j^2\log\left(\min\left\{\frac{1}{2|\theta_j|},g\right\}\right) \tag{3.1.9}$$

$$+\;4\sum_{i<j}k_ik_j\left(\log\left(\min\left\{\tfrac{1}{|\theta_i-\theta_j|},g\right\}\right)+\log\left(\min\left\{\tfrac{1}{|\theta_i+\theta_j|},g\right\}\right)\right).$$

For $m=2$, we set

$$W=\{j\in\{1,2\}:\lim_{g\to\infty}g|\theta_j|<\infty\}\;\text{ and }\;W^c=\{j\in\{1,2\}:\lim_{g\to\infty}g|\theta_j|=\infty\}. \tag{3.1.10}$$

We also define a constant depending on W and W^c as

$$c_{\boldsymbol{v}^{(2)}}=\max\left\{\lim_{g\to\infty}g|\theta_1|,\;\lim_{g\to\infty}g|\theta_2|,\;\lim_{g\to\infty}g|\theta_1-\theta_2|,\;\lim_{g\to\infty}g|\theta_1+\theta_2|\right\}, \tag{3.1.11}$$

where maximum is taken only over the finite entries of the set.

Remark 3.1.10. *If* $|W|=2$ *and* $|W^c|=0$ *then*

$$c_{\boldsymbol{v}^{(2)}}=\max\left\{\lim_{g\to\infty}g|\theta_1|,\;\lim_{g\to\infty}g|\theta_2|,\;\lim_{g\to\infty}g|\theta_1-\theta_2|,\;\lim_{g\to\infty}g|\theta_1+\theta_2|\right\}.$$

If $W = \{1\}$ *and* $W^c = \{2\}$ *then* $c_{\boldsymbol{v}^{(2)}} = \lim_{g\to\infty} g|\theta_1|$. *If* $W = \emptyset$ *and* $\lim\limits_{g\to\infty} g|\theta_1 - \theta_2| < \infty$, *then* $c_{\boldsymbol{v}^{(2)}} = \lim\limits_{g\to\infty} g|\theta_1 - \theta_2|$. *When none of the limits is finite or when* $c_{\boldsymbol{v}^{(2)}} = 0$, *then the implied constant in Theorem 3.1.11, Theorem 3.1.12 and Corollary 3.1.18 is independent of* $\boldsymbol{v}^{(m)}$.

We obtain a lower bound (of the conjectured order of magnitude[1]) for $\mathcal{S}_n(\boldsymbol{v}^{(2)}, \boldsymbol{k}^{(2)})$ in the large degree limit i.e. when n is sufficiently large and q is fixed.

Theorem 3.1.11. *Let* $\boldsymbol{k}^{(2)} \in \mathbb{N}^2$ *and* $\boldsymbol{v}^{(2)}$ *be as earlier. Assume that* $\alpha_j = O\left(\frac{1}{g}\right)$ *for* $j = 1, 2$. *Then for* n *large,*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D\in\mathcal{H}_n} \left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\right|^{2k_1} \left|\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big)\right|^{2k_2} \gg_{\boldsymbol{k}^{(2)},\boldsymbol{v}^{(2)}} \exp\left(\mu\left(\boldsymbol{v}^{(2)}, g\right) + \frac{1}{2}\sigma\left(\boldsymbol{v}^{(2)}, g\right)\right),$$

where $\mu\left(\boldsymbol{v}^{(2)}, g\right)$ *and* $\sigma\left(\boldsymbol{v}^{(2)}, g\right)$ *are defined by* (3.1.8) *and* (3.1.9) *respectively.*

In this chapter, we will provide the complete proof of the Theorem 3.1.11 and from observations in the footnotes [2, 5, 7], one can easily extend Theorem 3.1.11 to the following form.

Theorem 3.1.12. *Let* $\boldsymbol{k}^{(m)} \in \mathbb{N}^m$ *and* $\boldsymbol{v}^{(m)}$ *be as earlier. Assume that* $\alpha_j = O\left(\frac{1}{g}\right)$ *for all* j. *Then for* n *large,*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D\in\mathcal{H}_n} \left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\right|^{2k_1} \cdots \left|\mathcal{L}\Big(\frac{v_m}{q^{\frac{1}{2}+\alpha_m}}, \chi_D\Big)\right|^{2k_m} \gg_{\boldsymbol{k}^{(m)},\boldsymbol{v}^{(m)}} \exp\left(\mu\left(\boldsymbol{v}^{(m)}, g\right) + \frac{1}{2}\sigma\left(\boldsymbol{v}^{(m)}, g\right)\right),$$

where $\mu\left(\boldsymbol{v}^{(m)}, g\right)$ *and* $\sigma\left(\boldsymbol{v}^{(m)}, g\right)$ *are defined by* (3.1.8) *and* (3.1.9) *respectively.*

We also establish an upper bound of nearly the conjectured order of magnitude for the sum $\mathcal{S}_n(\boldsymbol{v}^{(m)}, \boldsymbol{k}^{(m)})$.

Theorem 3.1.13. *Let* $\boldsymbol{k}^{(m)} \in \mathbb{R}_+^m$ *and* $\boldsymbol{v}^{(m)}$ *be as earlier. Assume that* $\alpha_j = O\left(\frac{1}{g}\right)$ *for all* j. *Then for* n *large, and for any* $\epsilon > 0$,

$$\frac{1}{|\mathcal{H}_n|} \sum_{D\in\mathcal{H}_n} \left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\right|^{2k_1} \cdots \left|\mathcal{L}\Big(\frac{v_m}{q^{\frac{1}{2}+\alpha_m}}, \chi_D\Big)\right|^{2k_m} \ll_{\boldsymbol{k}^{(m)},\varepsilon}$$

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[1] The conjectural order of magnitude of these L-functions in the hyperelliptic ensemble can be compared with the autocorrelation of the random matrix polynomials (for example, see [[20], Eqs. (3.6) and (4.19)]).

$$n^{\varepsilon} \exp\left(\mu\left(\boldsymbol{v}^{(m)}, g\right) + \frac{1}{2}\sigma\left(\boldsymbol{v}^{(m)}, g\right)\right),$$

where $\mu\left(\boldsymbol{v}^{(m)}, g\right)$ *and* $\sigma\left(\boldsymbol{v}^{(m)}, g\right)$ *are defined by* (3.1.8) *and* (3.1.9) *respectively.*

Let us define $\mathcal{L}^{(l)}(u, \chi_D)$ as the l-th derivative of $\mathcal{L}(u, \chi_D)$. As an important consequence of Theorem 3.1.13, we have the following upper bound.

Theorem 3.1.14. *Let* $l \in \mathbb{N}$ *and* $\varepsilon > 0$. *For* n *large, we have*

$$\sum_{D \in \mathcal{H}_n} \left|\mathcal{L}^{(l)}\left(q^{-1/2}, \chi_D\right)\right|^k \ll_{k,l,\varepsilon} |\mathcal{H}_n| g^{\frac{1}{2}k(k+1)+lk+\varepsilon}.$$

3.1.3.2 Applications

From the Theorem 3.1.13 and in light of (3.2.3) if we specialize n as $n = 2g+1$ and the α_j's are zero then we recover Theorem 3.1.9.

The case of the mean value for $\mathcal{L}(q^{-1/2}, \chi_D)$ taken over $\mathcal{H}_{2g+2}$ was investigated by Jung [46]. Taking $n = 2g+2$, we have the following corollary which generalizes the Theorem 3.1.9 :

Corollary 3.1.15. *Let* $\varepsilon > 0$. *For* n *large, we have*

$$\frac{1}{|\mathcal{H}_{2g+2}|} \sum_{D \in \mathcal{H}_{2g+2}} |\mathcal{L}(q^{-1/2}, \chi_D)|^k \ll_{k,\varepsilon} g^{\frac{1}{2}k(k+1)+\varepsilon}.$$

Similarly, Theorem 3.1.14 provides an upper bound for the k-th moment of $\mathcal{L}^{(m)}(q^{-1/2}, \chi_D)$ with $D \in \mathcal{H}_{2g+1}$ and $D \in \mathcal{H}_{2g+2}$. More precisely,

Corollary 3.1.16. *Let* $l \in \mathbb{N}$ *and* $\varepsilon > 0$. *For* n *large, we have*

$$\sum_{D \in \mathcal{H}_{2g+1}} \left|\mathcal{L}^{(l)}\left(q^{-1/2}, \chi_D\right)\right|^k \ll_{k,l,\varepsilon} q^{2g+1} g^{\frac{1}{2}k(k+1)+lk+\varepsilon},$$

$$\sum_{D \in \mathcal{H}_{2g+2}} \left|\mathcal{L}^{(l)}\left(q^{-1/2}, \chi_D\right)\right|^k \ll_{k,l,\varepsilon} q^{2g+2} g^{\frac{1}{2}k(k+1)+lk+\varepsilon}.$$

Corollary 3.1.17. *Let* W *and* W^c *be defined by* (3.1.10). *For every* $\varepsilon > 0$, n *large and* $k \in \mathbb{R}_+$,

$$\sum_{D \in \mathcal{H}_n} \left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big)\right|^{2k}$$

$$\ll_{k,\varepsilon} \begin{cases} |\mathcal{H}_n| \ g^{2k(4k+1)+\varepsilon} & \text{if} \quad |W| = 2, \\ \frac{|\mathcal{H}_n| \ g^{3k^2+k+\varepsilon}}{|\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if} \quad W = \{1\}, W^c = \{2\}, \\ \frac{|\mathcal{H}_n| \ g^{4k^2+\varepsilon}}{|\theta_1\theta_2|^{k^2+k}|\theta_1+\theta_2|^{2k^2}} & \text{if } |W^c| = 2, \ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| < \infty, \\ \frac{|\mathcal{H}_n| \ g^{2k^2+\varepsilon}}{|\theta_1\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if } |W^c| = 2, \ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| = \infty. \end{cases}$$

Corollary 3.1.18. *Let W and W^c be defined by (3.1.10). For n large and $k \in \mathbb{N}$,*

$$\sum_{D\in\mathcal{H}_n} \left| \mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big) \right|^{2k}$$
$$\gg_{k,\boldsymbol{v}^{(2)}} \begin{cases} |\mathcal{H}_n| \ g^{2k(4k+1)} & \text{if} \quad |W| = 2, \\ \frac{|\mathcal{H}_n| \ g^{3k^2+k}}{|\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if} \quad W = \{1\}, W^c = \{2\}, \\ \frac{|\mathcal{H}_n| \ g^{4k^2}}{|\theta_1\theta_2|^{k^2+k}|\theta_1+\theta_2|^{2k^2}} & \text{if } |W^c| = 2, \ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| < \infty, \\ \frac{|\mathcal{H}_n| \ g^{2k^2}}{|\theta_1\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} & \text{if } |W^c| = 2, \ \lim\limits_{g\to\infty} g|\theta_1-\theta_2| = \infty. \end{cases}$$

Remark 3.1.19. *Theorem 3.1.12 and Theorem 3.1.13 give the lower and the upper bound of the same order of magnitude for the shifted moments of $\mathcal{L}$-function over all monic square-free polynomials near the critical line. In particular, from Theorem 3.1.12 and Theorem 3.1.13 we have*

$$g^{k(2k+1)} \ll_k \frac{1}{|\mathcal{H}_n|} \sum_{D\in\mathcal{H}_n} \mathcal{L}\left(q^{-\frac{1}{2}}, \chi_D\right)^{2k} \ll_{k,\varepsilon} g^{k(2k+1)+\varepsilon},$$

which give the lower and the upper bound of the same order of magnitude for the $2k$-th moment of $\mathcal{L}\left(q^{-\frac{1}{2}}, \chi_D\right)$. The order of magnitude matches that of the main terms of Conjecture 3.1.7 and Conjecture 3.1.6.

Remark 3.1.20. *Theorem 3.1.12 and Theorem 3.1.13 can be compared with Theorem 3.1.2 and Theorem 3.1.1 respectively of [16]. Also, from Corollary 3.1.18 and Corollary 3.1.17, we get*

$$g^{2k(4k+1)} \ll_{k,\boldsymbol{v}^{(2)}} \frac{1}{|\mathcal{H}_n|} \sum_{D\in\mathcal{H}_n} \left| \mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D\Big)\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}}, \chi_D\Big) \right|^{2k} \ll_{k,\varepsilon} g^{2k(4k+1)+\varepsilon},$$

when $\lim_{g\to\infty} g|\theta_1|,\ \lim_{g\to\infty} g|\theta_2|,\ \lim_{g\to\infty} g|\theta_1-\theta_2| < \infty$, *and*

$$\frac{g^{2k^2}}{|\theta_1\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}} \ll_{k,\boldsymbol{v}^{(2)}} \frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}},\chi_D\Big)\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}},\chi_D\Big)\right|^{2k} \ll_{k,\varepsilon} \frac{g^{2k^2+\varepsilon}}{|\theta_1\theta_2|^{k^2+k}|\theta_1-\theta_2|^{2k^2}|\theta_1+\theta_2|^{2k^2}},$$

when $\lim_{g\to\infty} g|\theta_1|,\ \lim_{g\to\infty} g|\theta_2|,\ \lim_{g\to\infty} g|\theta_1-\theta_2| = \infty$.

As we have already discussed when two zeta functions are correlated or independent in Subsection 3.1.1.1, similarly by we can say that the correlation of $\mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}},\chi_D\Big)$ *and* $\mathcal{L}\Big(\frac{v_2}{q^{\frac{1}{2}+\alpha_2}},\chi_D\Big)$ *transition at* $|\theta_1|,\ |\theta_2|,\ |\theta_1-\theta_2| \approx \frac{1}{g}$, *and these distribution appear independent when* $|\theta_1|,\ |\theta_2|,\ |\theta_1-\theta_2|$ *are much larger than* $\frac{1}{g}$.

3.2 Background for L-functions over function fields

We begin this section with some preliminaries of L-functions over function fields. We will use [71] as a general reference.

3.2.1 Basic facts on $\mathbb{F}_q[t]$

We start by fixing a finite field $\mathbb{F}_q$ of odd cardinality $q = p^r$, $r \geq 1$ with a prime p. We denote by $\mathbb{A} = \mathbb{F}_q[t]$ the polynomial ring over $\mathbb{F}_q$. For a polynomial f in $\mathbb{F}_q[t]$, its degree will be denoted by either $\deg(f)$ or $d(f)$.

The set of all monic polynomials and monic irreducible polynomials of degree n are denoted by $\mathcal{M}_{n,q}$ (or simply $\mathcal{M}_n$ as we fix q) and $\mathcal{P}_{n,q}$ (or simply $\mathcal{P}_n$) respectively. Let $\mathcal{M} = \cup_{n\geq 1}\mathcal{M}_n$ and $\mathcal{P} = \cup_{n\geq 1}\mathcal{P}_n$. We also denote the set of all monic polynomials and monic irreducible polynomials of degree less or equal to n by $\mathcal{M}_{\leq n,q}$ (or simply $\mathcal{M}_{\leq n}$) and $\mathcal{P}_{\leq n,q}$ (or simply $\mathcal{P}_{\leq n}$) respectively. Let $\mathcal{H}_n$ denote the set of monic square-free polynomials of degree n. Observe that for $n \geq 1$, $|\mathcal{M}_n| = q^n$ and

$$|\mathcal{H}_n| = \begin{cases} q, & \text{if } n = 1, \\ q^{n-1}(q-1), & \text{if } n \geq 2. \end{cases}$$

If f is is a non-zero polynomial $\mathbb{F}_q[t]$, we define the norm of f to be $|f| = q^{d(f)}$. If $f = 0$, we set $|f| = 0$. The prime polynomial theorem (see [71], Theorem 2.2) states that

$$|\mathcal{P}_{n,q}| = \frac{q^n}{n} \ + \ O\Big(\frac{q^{\frac{n}{2}}}{n}\Big). \tag{3.2.1}$$

The zeta function of $\mathbb{A}$, denoted by $\zeta_{\mathbb{A}}(s)$, is defined by

$$\zeta_{\mathbb{A}}(s) := \sum_{f\in\mathcal{M}} \frac{1}{|f|^s} = \prod_{P\in\mathcal{P}} \left(1 - |P|^{-s}\right)^{-1}, \qquad \Re(s) > 1.$$

One can easily prove that $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$, and this provides an analytic continuation of the zeta function to the complex plane with a simple pole at $s = 1$. Using the change of variable $u = q^{-s}$, the zeta function becomes

$$\mathcal{Z}(u) = \sum_{f\in\mathcal{M}} u^{d(f)} = \frac{1}{1-qu}, \qquad \text{if } |u| < \frac{1}{q}.$$

3.2.2 Quadratic Dirichlet characters and properties of their L-functions

For a monic irreducible polynomial P, the quadratic residue symbol $\left(\frac{f}{P}\right)$ is defined by

$$\left(\frac{f}{P}\right) = \begin{cases} 1, & \text{if } f \text{ is a square (mod } P),\ P \nmid f \\ -1, & \text{if } f \text{ is not a square (mod } P),\ P \nmid f \\ 0, & \text{if } \ P \mid f. \end{cases}$$

For monic square-free polynomials $D \in \mathbb{F}_q[t]$, the symbol $\left(\frac{D}{\cdot}\right)$ is defined by extending the above residue symbol multiplicatively. We denote the quadratic Dirichlet character χ_D by

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

The L-function associated to the quadratic Dirichlet character χ_D is defined by

$$L(s,\chi_D) = \sum_{f\in\mathcal{M}} \frac{\chi_D(f)}{|f|^s} = \prod_{P\in\mathcal{P}} \left(1 - \chi_D(P)\,|P|^{-s}\right)^{-1}, \ \ \Re(s) > 1.$$

Using the change of variable $u = q^{-s}$, this L-function turns into

$$\mathcal{L}(u,\chi_D) = \sum_{f\in\mathcal{M}} \chi_D(f)\, u^{d(f)} = \prod_{P\in\mathcal{P}} \Big(1 - \chi_D(P)\, u^{d(P)}\Big)^{-1}, \ \ |u| < \frac{1}{q}.$$

Now, if $n \geq d(D)$, for each monic $f \in \mathcal{A}$ such that $d(f) = n$, we can write uniquely $f = gD + R$ where $0 \leq d(R) \leq d(D) - 1$. Since χ_D is periodic modulo D

$$\sum_{d(f)=n} \frac{\chi_D(f)}{|M|S} = q^{-ns} \sum_{d(R)\leq d(D)-1} \chi_D(R) = 0.$$

It implies that $\mathcal{L}(u, \chi_D)$ is a polynomial of degree at most $d(D) - 1$. From [72], $\mathcal{L}(u, \chi_D)$ has a trivial zero at $u = 1$ if and only if $d(D)$ is even. This allows us to define the completed L-function as

$$L(s, \chi_D) = \mathcal{L}(u, \chi_D) = (1-u)^{\lambda} \mathcal{L}^*(u, \chi_D) = (1-q^{-s})^{\lambda} L^*(s, \chi_D),$$

where

$$\lambda = \begin{cases} 1, & \text{if } d(D) \text{ even}, \\ 0, & \text{if } d(D) \text{ odd}, \end{cases} \tag{3.2.2}$$

and $\mathcal{L}^*(u, \chi_D)$ is a polynomial of degree

$$2g = d(D) - 1 - \lambda \tag{3.2.3}$$

satisfying the functional equation

$$\mathcal{L}^*(u, \chi_D) = (qu^2)^g \mathcal{L}^* \left(\frac{1}{qu}, \chi_D \right).$$

Because $\mathcal{L}$ and $\mathcal{L}^*$ are polynomial in u, it is convenient to define

$$L^*(s, \chi_D) = \mathcal{L}^*(u, \chi_D)$$

so that the above functional equation can be rewritten as

$$L^*(s, \chi_D) = q^{(1-2s)g} L^*(1-s, \chi_D).$$

The Riemann hypothesis for curves over finite fields, established by Weil [84], asserts that all the non-trivial zero of $\mathcal{L}^*(u, \chi_D)$ lie on the circle $|u| = q^{-1/2}$, i.e,

$$\mathcal{L}^*(u, \chi_D) = \prod_{j=1}^{2g} (1 - u\,\nu_j\,) \quad \text{with } |\nu_j| = \sqrt{q} \text{ for all } j.$$

One can define the completed L-function in the following way. Set

$$X_D(s) = |D|^{\frac{1}{2}-s} X(s), \tag{3.2.4}$$

where

$$X(s) = \begin{cases} q^{s-\frac{1}{2}}, & \text{if } d(D) \text{ odd} \\ \frac{1-q^{-s}}{1-q^{-(1-s)}} q^{-1+2s}, & \text{if } d(D) \text{ even.} \end{cases}$$

Let us consider

$$\Lambda(s, \chi_D) = L(s, \chi_D) X_D(s)^{-\frac{1}{2}}. \tag{3.2.5}$$

Then $\Lambda(s, \chi_D)$ satisfies the symmetric functional equation

$$\Lambda(s, \chi_D) = \Lambda(1-s, \chi_D). \tag{3.2.6}$$

3.2.3 Spectral Interpretation

Let C be a non-singular projective curve over $\mathbb{F}_q$ of genus g. For each extension field of degree k of $\mathbb{F}_q$, denote by $N_k(C)$ the number of points of C in $\mathbb{F}_{q^k}$. Then, the zeta function associated to C defined as

$$Z_C(u) = \exp\left(\sum_{k=1}^{\infty} N_k(C) \frac{u^k}{k}\right), \quad |u| < \frac{1}{q},$$

is known to be a rational function of u of the form

$$Z_C(u) = \frac{P_C(u)}{(1-u)(1-qu)}.$$

Additionally, we know that $P_C(u)$ is a polynomial of degree $2g$ with integer coefficients, satisfying a functional equation

$$P_C(u) = (qu^2)^g P_C\left(\frac{1}{qu}\right).$$

The Riemann Hypothesis, proved by Weil [84], says that the zeros of $P_C(u)$ all lie on the circle $|u| = \frac{1}{\sqrt{q}}$. Thus one may give a spectral interpretation of $P_C(u)$ as the characteristic polynomial of a $2g \times 2g$ unitary matrix Θ_C:

$$P_C(u) = \det\left(I - u\sqrt{q}\Theta_C\right).$$

Thus the eigenvalues $e^{i\theta_j}$ of Θ_C correspond to the zeros, $q^{-1/2}e^{-i\theta_j}$, of $Z_C(u)$. The matrix Θ_C is called the unitarized Frobenius class of C.

To put this in the context of our case, note that, for a family of hyperelliptic curves $C_D : y^2 = D(t)$ of genus g, the numerator of the zeta function $Z_C(u)$ associated to C_D coincides with the L-function $\mathcal{L}^*(u, \chi_D)$, i.e., $P_C(u) = \mathcal{L}^*(u, \chi_D)$.

3.3 Preliminary Lemmas

We start with an analog of approximate functional equation for $L(s, \chi_D)$. Recall that $2g = n - 1 - \lambda$ where λ is defined as in (3.2.2).

Lemma 3.3.1 (Approximate functional equation)**.** *Let χ_D be a quadratic Dirichlet character, where $D \in \mathcal{H}_n$. Then for $1/2 \leq s < 1$,*

$$\begin{aligned} L(s, \chi_D) = &\sum_{f \in \mathcal{M}_{\leq g}} \frac{\chi_D(f)}{|f|^s} + X_D(s) \sum_{f \in \mathcal{M}_{\leq g-1}} \frac{\chi_D(f)}{|f|^{1-s}} \\ &- \lambda q^{-s(g+1)} \sum_{f \in \mathcal{M}_{\leq g}} \chi_D(f) - \lambda\, X_D(s)\, q^{-(1-s)g} \sum_{f \in \mathcal{M}_{\leq g-1}} \chi_D(f), \end{aligned}$$

where $X_D(s)$ is defined by (3.2.4).

Proof. The case $s = \frac{1}{2}$ is proved in [3] for $D \in \mathcal{H}_{2g+1}$ and [46] for $D \in \mathcal{H}_{2g+2}$. Their methods can be easily generalized for any $s \in (1/2, 1)$. □

The following lemma gives an asymptotic formula for a square polynomial in the hyperelliptic ensemble.

Lemma 3.3.2. *For $f \in \mathcal{M}$, we have*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \chi_D(f^2) = \prod_{\substack{P \in \mathcal{P} \\ P | f}} \left(1 + \frac{1}{|P|}\right)^{-1} + O(|\mathcal{H}_n|^{-1}).$$

Proof. See [[12], Lemma 3.7] for $n = 2g + 1$. To get the result for $n = 2g + 2$, it is a small adaptation of their proof. □

The following lemma is an analog of the Polya-Vinogradov inequality over function fields.

Lemma 3.3.3 (Polya-Vinogradov inequality)**.** *For $l \in \mathcal{M}$ not a perfect square, let $l = l_1{l_2}^2$ with l_1 square-free. Then for any $\epsilon > 0$,*

$$\Big|\sum_{D\in\mathcal{H}_n} \chi_D(l)\Big| \ll_\varepsilon \sqrt{|\mathcal{H}_n|}|l_1|^\epsilon.$$

Proof. A generalization of the above inequality was proved in [[13], Lemma 3.5] for $n = 2g+1$. Here we give a different proof in the above form for completeness.

First assume that $l_1 = P_1P_2\dots P_k$, where P_j's are distinct prime polynomials, and $\deg(l_1) \le n$. Similar to the proof of Lemma 3.5 in [12], which is the particular case $k = 2$, one can show:

$$\Big|\sum_{D\in\mathcal{H}_n} \chi_D(l)\Big| = \Big|\sum_{D\in\mathcal{H}_n} \chi_D(l_1)\Big| \le \frac{qg^{k-1}\left(d(P_1)+\ldots+d(P_k)\right)}{d(P_1)\ldots d(P_k)}|l_1|^{\frac{1}{2}} \ll_\epsilon \sqrt{|\mathcal{H}_n|}|l_1|^\epsilon.$$

Finally let $\deg(l_1) > n$. We combine Lemma 3.1 of [13] and Lemma 3.5 of [12] to obtain

$$\Big|\sum_{D\in\mathcal{H}_n} \chi_D(l_1)\Big| \ll_\epsilon \sqrt{|\mathcal{H}_n|}|l_1|^\epsilon.$$

□

The following lemma gives an upper bound for the logarithm of $\mathcal{L}(u, \chi_D)$ inside the critical region.

Lemma 3.3.4. *Let $0 \le \alpha \le \frac{1}{2}$, $v = e^{i\theta}$, $\theta \in [0, \pi)$ and N be a positive integer. Then for $D \in \mathcal{H}_n$,*

$$\log\Big|\mathcal{L}\Big(\frac{v}{q^{\frac{1}{2}+\alpha}}, \chi_D\Big)\Big| \le \tfrac{2g}{N+1}\log\left(\frac{1+q^{-\alpha(N+1)}}{1+q^{-2(N+1)}}\right) + \Re\sum_{d(f)\le N}\frac{a_\alpha\left(d(f)\right)\chi_D(f)\,\Lambda(f)v^{d(f)}}{|f|^{\frac{1}{2}}} + O(1),$$

where

$$a_\alpha(d(f)) = \frac{1}{d(f)|f|^\alpha} - \frac{1}{d(f)|f|^2} + O\left(\frac{1}{(N+1)q^{(N+1)\alpha}}\right), \qquad \textit{for } 1 \le d(f) \le N.$$

Proof. From the functional equation (3.2.6), we observe that

$$\Big|\Lambda(\alpha+it, \chi_D)\Big| = \left|\frac{\Lambda\left(\frac{5}{2}-it, \chi_D\right)\Lambda(1-\alpha-it, \chi_D)}{\Lambda\left(-\frac{3}{2}+it, \chi_D\right)}\right| = \frac{\left|\Lambda\left(\frac{5}{2}-it, \chi_D\right)\right|\left|\Lambda(1-\alpha+it, \chi_D)\right|}{\left|\Lambda\left(-\frac{3}{2}+it, \chi_D\right)\right|}.$$

Recall that

$$L(\alpha+it,\chi_D)=\left(1-q^{-\alpha-it}\right)^{\lambda}L^{*}(\alpha+it,\chi_D).$$

Note that $\left|L^{*}\left(\frac{5}{2}-it,\chi_D\right)\right|\sim 1$. Using the expression (3.2.5) for $\Lambda(s,\chi_D)$, we get

$$\left|L(\alpha+it,\chi_D)\right|=q^{g(5-2\alpha)}\left|1-q^{-\alpha-it}\right|^{\lambda}\prod_{j=1}^{2g}\left(\frac{q^{2\alpha-1}+1-2q^{\alpha-\frac{1}{2}}\cos(2\pi\theta_j-t\log q)}{q^4+1-2q^2\cos(2\pi\theta_j-t\log q)}\right)^{\frac{1}{2}}.$$

Since

$$q^{2\alpha-1}+1-2q^{\alpha-\frac{1}{2}}\cos(2\pi\theta_j-t\log q)=(q^{\alpha-\frac{1}{2}}-1)^2+4q^{\alpha-\frac{1}{2}}\sin^2\left(\pi\theta_j-\frac{t\log q}{2}\right)$$

with a similar expression holding for the denominator, it follows that

$$\log\left|L(\alpha+it,\chi_D)\right|=g\left(\frac{5}{2}-\alpha\right)\log q-\frac{1}{2}\sum_{j=1}^{2g}\log\left(\frac{a^2+\sin^2(\pi\theta_j-\frac{t\log q}{2})}{b^2+\sin^2(\pi\theta_j-\frac{t\log q}{2})}\right)+O(1),$$

where

$$a=\frac{q^2-1}{2q},\ b=\frac{q^{\alpha-\frac{1}{2}}-1}{2q^{\frac{\alpha}{2}-\frac{1}{4}}}.$$

The remaining part of the proof is the same as the proof of Lemma 8.1 in [29] proved by A. Florea. □

Lemma 3.3.5. *Let $\theta\in(-\pi,\pi)$, then we have* $\displaystyle\sum_{m=1}^{n}\frac{\cos(\theta m)}{m}\le\log\left(\min\left\{n,\frac{1}{|\theta|}\right\}\right)+O(1)$.

Proof. See [[29], Lemma 9.1]. □

Lemma 3.3.6. *Let k, y be integers such that $2ky\le n$. For any complex numbers $\{a(P)\}_{p\in\mathcal{P}}$, we have*

$$\sum_{D\in\mathcal{H}_n}\left|\sum_{d(P)\le y}\frac{a(P)\chi_D(P)}{|P|^{\frac{1}{2}}}\right|^{2k}\ll|\mathcal{H}_n|\,\frac{(2k)!}{k!\,2^k}\left(\sum_{d(P)\le y}\frac{|a(P)|^2}{|P|}\right)^{k}.$$

Proof. This is an easy generalization of the Lemma 8.4 of [29] and Lemma 6.3 of [82]. □

During the study of our main theorems it seems interesting to estimate the following bounds for the zeta function over function fields. This is an analog of bounding the Riemann zeta function near the 1-line.

Lemma 3.3.7. *Let $v = e^{i\theta}$, where $\theta \in (-\pi, \pi)$. Let C be a circle of radius $\frac{\widetilde{r}}{g}$ centred at $\frac{1}{q}$, where*

$$\widetilde{r} = \lim_{g\to\infty} g\,|\theta| < \infty.$$

For any u in C, we have

$$\mathcal{Z}(uv) \ll g \quad \textit{if } \lim_{g\to\infty} g\,|\theta| < \infty.$$

For any u such that $|u - \frac{1}{q}| = O(1/g)$, we have

$$\mathcal{Z}(uv) \ll \frac{1}{|\theta|} \quad \textit{if } \lim_{g\to\infty} g\,|\theta| = \infty.$$

Proof. First assume that $\theta \in (-\pi,\, \pi)$ be such that $\lim_{g\to\infty} g\,|\theta| = \infty$. Then using $|u - \frac{1}{q}| = O\left(1/g\right)$, we have the following estimates:

$$\left|\frac{v}{(1-v)}(1-qu)\right| = o(1)$$

and

$$|(1-v)^{-1}| \ll \frac{1}{|\theta|}.$$

Thus

$$|\mathcal{Z}(uv)| = |(1-quv)^{-1}| = \left|(1-v)^{-1}\left(1 + \frac{v}{(1-v)}(1-qu)\right)^{-1}\right| \ll \frac{1}{|\theta|}.$$

Finally let θ be such that $\lim_{g\to\infty} g\,|\theta| < \infty$. Then $|u - \frac{1}{q}| \le \frac{\widetilde{r}}{g}$. We use the change of variable $u = q^{-s}$ to get the hypothesis in the form $|s-1| \le \frac{\widetilde{r}}{g}$. Since

$$\mathcal{Z}(uv) = \sum_{f\in\mathcal{M}} (uv)^{\deg(f)},$$

it is enough to show that

$$\sum_{f\in\mathcal{M}} \frac{1}{|f|^{1+\widetilde{r}/g\,-\,i\theta/\log q}} = O(g).$$

Therefore using Lemma 3.3.5 and the prime polynomial theorem, we obtain

$$\log\left|\sum_{f\in\mathcal{M}} \frac{1}{|f|^{1+\widetilde{r}/g\,-\,i\theta/\log q}}\right| = \Re\sum_{P\in\mathcal{P}} \frac{1}{|P|^{1+\widetilde{r}/g\,-\,i\theta/\log q}} + O(1)$$

$$
\begin{aligned}
&=\Re\sum_{n}\frac{1}{nq^{n(\widetilde{r}/g\,-\,i\theta/\log q)}}+O(1)\\
&=\Re\sum_{n\le g}\frac{q^{\frac{in\theta}{\log q}}}{n}-\Re\sum_{n\le g}\left(\frac{1}{n}-\frac{1}{nq^{\frac{\widetilde{r}n}{g}}}\right)q^{\frac{in\theta}{\log q}}+\Re\sum_{n>g}\frac{q^{\frac{in\theta}{\log q}}}{nq^{\frac{\widetilde{r}n}{g}}}+O(1)\\
&=\sum_{n\le g}\frac{\cos(n\theta)}{n}+O(1)\le\ \log\left(\min\left\{g,\frac{1}{|\theta|}\right\}\right)\le\ \log g,
\end{aligned}
$$

and the lemma's proof is concluded. □

3.4 Proof of Theorem 3.1.11

Throughout this section, for the sake of simplicity, we write $\boldsymbol{v}^{(2)}$ and $\boldsymbol{k}^{(2)}$ simply as $\boldsymbol{v}$ and $\boldsymbol{k}$ respectively. For any $k_1,k_2\in\mathbb{N}$, we write

$$\mathcal{L}\Big(\frac{v_1}{q^{1/2+\alpha_1}},\chi_D\Big)^{k_1}\mathcal{L}\Big(\frac{v_2}{q^{1/2+\alpha_2}},\chi_D\Big)^{k_2}=\sum_{f\in\mathcal{M}}a_f\frac{\chi_D(f)}{|f|^{1/2}}, \tag{3.4.1}$$

where

$$a_f=\sum_{f_1f_2=f}\frac{\tau_{k_1}(f_1)\tau_{k_2}(f_2)}{|f_1|^{\alpha_1}|f_2|^{\alpha_2}}\,e^{i\big(\theta_1\,d(f_1)+\theta_2\,d(f_2)\big)}. \tag{3.4.2}$$

We start by defining the following truncated L-function which is an analog of Dirichlet polynomials over number fields:

$$\mathcal{L}_{\le(k_1+k_2)X}\big(\boldsymbol{v},\chi_D\big):=\sum_{f\in\mathcal{M}_{\le(k_1+k_2)X}}a_f\frac{\chi_D(f)}{|f|^{1/2}},$$

where a_f is defined by (3.4.2) and the parameter X will be chosen later. We call X as point of truncation of (3.4.1).

Using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left(\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{1/2+\alpha_1}},\chi_D\Big)^{k_1}\mathcal{L}\Big(\frac{v_2}{q^{1/2+\alpha_2}},\chi_D\Big)^{k_2}\overline{\mathcal{L}_{\le(k_1+k_2)X}\big(\boldsymbol{v},\chi_D\big)}\right|\right)^2\\
&\quad\le\left(\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{1/2+\alpha_1}},\chi_D\Big)^{k_1}\mathcal{L}\Big(\frac{v_2}{q^{1/2+\alpha_2}},\chi_D\Big)^{k_2}\right|^2\right)\left(\sum_{D\in\mathcal{H}_n}\left|\overline{\mathcal{L}_{\le(k_1+k_2)X}\big(\boldsymbol{v},\chi_D\big)}\right|^2\right).
\end{aligned}
$$

Therefore, we obtain

$$\sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{1/2+\alpha_1}},\chi_D\Big)^{k_1}\mathcal{L}\Big(\frac{v_2}{q^{1/2+\alpha_2}},\chi_D\Big)^{k_2}\right|^2 \geq \frac{S_1^2}{S_2}, \tag{3.4.3}$$

where

$$S_1 := \sum_{D\in\mathcal{H}_n}\left|\mathcal{L}\Big(\frac{v_1}{q^{1/2+\alpha_1}},\chi_D\Big)^{k_1}\mathcal{L}\Big(\frac{v_2}{q^{1/2+\alpha_2}},\chi_D\Big)^{k_2}\overline{\mathcal{L}_{\leq(k_1+k_2)X}\big(\boldsymbol{v},\chi_D\big)}\right|$$

and

$$S_2 := \sum_{D\in\mathcal{H}_n}\left|\overline{\mathcal{L}_{\leq(k_1+k_2)X}\big(\boldsymbol{v},\chi_D\big)}\right|^2.$$

Now we establish an asymptotic formula for S_2 and a lower bound for S_1.

3.4.1 Estimation of the sum S_2

Inserting the D-sum after expanding the square in S_2, we get

$$S_2 = \sum_{f\in\mathcal{M}_{\leq(k_1+k_2)X}}\ \sum_{f'\in\mathcal{M}_{\leq(k_1+k_2)X}}\frac{a_f\,\overline{a_{f'}}}{|ff'|^{1/2}}\sum_{D\in\mathcal{H}_n}\chi_D(ff').$$

Case 1. Assume that $ff'\neq\square$. Observe that $a_f \ll_\varepsilon |f|^\varepsilon$ and using Lemma 3.3.3, we obtain that

$$S_2 \ll \sqrt{|\mathcal{H}_n|}\sum_{f\in\mathcal{M}_{\leq2(k_1+k_2)X}}\frac{1}{|f|^{\frac{1}{2}-\varepsilon}} \ll \sqrt{|\mathcal{H}_n|}\,q^{2\left(\frac{1}{2}+\varepsilon\right)(k_1+k_2)X}.$$

Let us choose $X=\frac{g}{2(k_1+k_2)}$ [2]. So, we have

$$S_2 \ll q^{\left(\frac{3}{2}+\varepsilon\right)g}.$$

---

[2] For the Theorem 3.1.12, the point of truncation will be $(k_1+\ldots+k_m)X$ and the choice of X is equal to $\frac{g}{2(k_1+\ldots+k_m)}$.

Case 2. Assume that $ff' = \Box = l^2$, where $l \in \mathbb{F}_q[t]$. By using Lemma 3.3.2 and $\tau_k(f) \ll_\varepsilon |f|^\varepsilon$,

$$\begin{aligned} S_2 = {} & |\mathcal{H}_n| \sum_{l \in \mathcal{M}_{\leq (k_1+k_2)X}} \frac{1}{|l|} \sum_{f_1f_2f_3f_4 = l^2} \frac{\tau_{k_1}(f_1)\tau_{k_1}(f_2)\tau_{k_2}(f_3)\tau_{k_2}(f_4)}{|f_1f_2|^{\alpha_1}|f_3f_4|^{\alpha_2}} \\ & \times e^{i\theta_1(d(f_1)-d(f_2))+i\theta_2(d(f_3)-d(f_4))} \prod_{P|l}\left(1+\frac{1}{|P|}\right)^{-1} \\ & + O\left(\sum_{l \in \mathcal{M}_{\leq (k_1+k_2)X}} \frac{1}{|l|} \sum_{f_1f_2f_3f_4 = l^2} \frac{\tau_{k_1}(f_1)\tau_{k_1}(f_2)\tau_{k_2}(f_3)\tau_{k_2}(f_4)}{|f_1f_2|^{\alpha_1}|f_3f_4|^{\alpha_2}}\right) \\ = {} & |\mathcal{H}_n| \sum_{l \in \mathcal{M}_{\leq (k_1+k_2)X}} \frac{b(l)}{|l|} + O\left(q^{\varepsilon(k_1+k_2)X}\right), \end{aligned}$$

where

$$b(l) = \sum_{f_1f_2f_3f_4 = l^2} \frac{\tau_{k_1}(f_1)\tau_{k_1}(f_2)\tau_{k_2}(f_3)\tau_{k_2}(f_4)}{|f_1f_2|^{\alpha_1}|f_3f_4|^{\alpha_2}} e^{i\theta_1(d(f_1)-d(f_2))+i\theta_2(d(f_3)-d(f_4))} \prod_{P|l}\left(1+\frac{1}{|P|}\right)^{-1}.$$

We use the Perron's formula[3] to get

$$\sum_{l \in \mathcal{M}_{\leq (k_1+k_2)X}} \frac{b(l)}{|l|} = \frac{1}{2\pi i} \int_{|u|=r} B(u) \frac{(qu)^{-(k_1+k_2)X}}{(1-qu)} \frac{du}{u},$$

where

$$B(u) = \sum_{l \in \mathcal{M}} b(l) u^{d(l)} \quad \text{and} \quad r < \frac{1}{q}.$$

For an irreducible polynomial P, we observe that

$$\begin{aligned} b(P) & = \left(1+\frac{1}{|P|}\right)^{-1} \sum_{f_1f_2f_3f_4 = P^2} \frac{\tau_{k_1}(f_1)\tau_{k_1}(f_2)\tau_{k_2}(f_3)\tau_{k_2}(f_4)}{|f_1f_2|^{\alpha_1}|f_3f_4|^{\alpha_2}} e^{i\theta_1(d(f_1)-d(f_2))+i\theta_2(d(f_3)-d(f_4))} \\ & = \left(1+\frac{1}{|P|}\right)^{-1} \left(\sum_{\substack{j=1 \\ \epsilon_j \in \{\pm 1\}}}^{2} \frac{k_j(k_j+1)}{2} \frac{e^{2i\epsilon_j\theta_j\, d(P)}}{|P|^{2\alpha_j}} + \sum_{j=1}^{2} \frac{k_j^2}{|P|^{2\alpha_j}} + \sum_{\epsilon_j \in \{\pm 1\}} \frac{k_1k_2}{|P|^{\alpha_1+\alpha_2}} e^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)d(P)} \right), \end{aligned}$$

---

[3] Perron's formula in function fields comes through the Cauchy's integral formula. More precisely $\sum_{f \in \mathcal{M}_{\leq X}} a_f = \frac{1}{2\pi i} \int_{|u|=r} \left(\sum_{f \in \mathcal{M}} a_f\, u^{\deg(f)}\right) \frac{du}{u^{X+1}(1-u)}$, provided that the power series $\sum_{f \in \mathcal{M}} a_f\, u^{\deg(f)}$ is absolutely convergent in $|u| \leq r < 1$.

which allows us to write $B(u)$ as

$$B(u) = \prod_{j=1}^{2} \mathcal{Z}^{k_j^2}(u) \prod_{\substack{j=1 \\ \epsilon_j \in \{\pm 1\}}}^{2} \mathcal{Z}^{\frac{k_j(k_j+1)}{2}} \left(ue^{2i\epsilon_j\theta_j} \right) \prod_{\epsilon_j \in \{\pm 1\}} \mathcal{Z}^{k_1k_2} \left(ue^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)} \right) C(u).$$

Here $C(u)$ is absolutely convergent for $|u| < \frac{1}{\sqrt{q}}$. Therefore,

$$\begin{aligned} \sum_{l \in \mathcal{M}_{\leq (k_1+k_2)X}} \frac{b(l)}{|l|} &= \frac{1}{2\pi i} \int\limits_{|u|=r} B(u) \frac{(qu)^{-(k_1+k_2)X}}{(1-qu)} \frac{du}{u} \\ &= \frac{1}{2\pi i} \int\limits_{|u|=r} \prod_{j=1}^{2} \mathcal{Z}^{k_j^2}(u) \prod_{\substack{j=1 \\ \epsilon_j \in \{\pm 1\}}}^{2} \mathcal{Z}^{\frac{k_j(k_j+1)}{2}} \left(ue^{2i\epsilon_j\theta_j} \right) \prod_{\epsilon_j \in \{\pm 1\}} \mathcal{Z}^{k_1k_2} \left(ue^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)} \right) \\ &\qquad \times C(u) \frac{(qu)^{-(k_1+k_2)X}}{(1-qu)} \frac{du}{u}, \end{aligned} \tag{3.4.4}$$

where $r = \frac{1}{q^{1+\varepsilon}}$.

3.4.1.1 Calculating the main term of S_2

To get the main term we have to shift the contour of integration (3.4.4) over u to a circle of radius $|u| = R = \frac{1}{q^{1/2+\varepsilon}}$. The integrand has a pole at $u = \frac{1}{q}$ of order $k_1^2 + k_2^2 + 1$ and at $u = \frac{1}{qe^{2i\epsilon_j\theta_j}}$ of order $\frac{k_j(k_j+1)}{2}$ and at $u = \frac{1}{q\, e^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}}$ of order k_1k_2, where $\epsilon_j \in \{\pm 1\}$ and $j = 1, 2$.

We define

$$\begin{aligned} D(u) = &\prod_{j=1}^{2} \mathcal{Z}^{k_j^2}(u) \prod_{\substack{j=1 \\ \epsilon_j \in \{\pm 1\}}}^{2} \mathcal{Z}^{\frac{k_j(k_j+1)}{2}} \left(ue^{2i\epsilon_j\theta_j} \right) \\ &\times \prod_{\epsilon_j \in \{\pm 1\}} \mathcal{Z}^{k_1k_2} \left(ue^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)} \right) C(u) \frac{(qu)^{-(k_1+k_2)X}}{u(1-qu)}. \end{aligned}$$

Using the Cauchy's residue theorem[4], we obtain

$$\frac{1}{2\pi i}\int\limits_{|u|=r} D(u)\,du = \frac{1}{2\pi i}\int\limits_{|u|=R} D(u)\,du - \operatorname*{Res}_{u=1/q} D(u) - \sum_{\substack{j=1\\ \epsilon_j\in\{\pm 1\}}}^{2} \operatorname*{Res}_{u=1/qe^{2i\epsilon_j\theta_j}} D(u)$$

$$- \sum_{\epsilon_j\in\{\pm 1\}} \operatorname*{Res}_{u=1/qe^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}} D(u),$$

where $r = \frac{1}{q^{1+\varepsilon}}$ and $R = \frac{1}{q^{1/2+\varepsilon}}$.

On the circle $|u| = R = \frac{1}{q^{1/2+\varepsilon}}$, we see that the functions $\frac{1}{1-qu}$, $\frac{1}{1-que^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}}$ and $\frac{1}{1-que^{2i\epsilon_j\theta_j}}$ are bounded. This leads to

$$\frac{1}{2\pi i}\int\limits_{|u|=R} D(u)\,du \ll q^{-(\frac{1}{2}-\varepsilon)(k_1+k_2)X}.$$

Evaluation of the sum of residues

We claim that

$$\operatorname*{Res}_{u=1/q} D(u) + \sum_{\substack{j=1\\ \epsilon_j\in\{\pm 1\}}}^{2} \operatorname*{Res}_{u=1/qe^{2i\epsilon_j\theta_j}} D(u) + \sum_{\epsilon_j\in\{\pm 1\}} \operatorname*{Res}_{u=1/qe^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}} D(u)$$

$$\sim_{\boldsymbol{k},\widetilde{c}} g^{k_1^2+k_2^2} \prod_{j=1}^{2} \left(\min\left\{\frac{1}{2|\theta_j|}, g\right\}\right)^{k_j(k_j+1)} \left(\min\left\{\tfrac{1}{|\theta_1-\theta_2|}, g\right\}\right)^{2k_1k_2} \left(\min\left\{\tfrac{1}{|\theta_1+\theta_2|}, g\right\}\right)^{2k_1k_2},$$

where

$$\widetilde{c} := \widetilde{c}_{\boldsymbol{v}} + 1 \text{ with } \widetilde{c}_{\boldsymbol{v}} \text{ is defined as in (3.1.11).} \tag{3.4.5}$$

Let us define the following sets

$$W_1 = \{j \in \{1,2\} : \lim_{g\to\infty} g|\theta_j| < \infty\}, \quad W_1^c = \{j \in \{1,2\} : \lim_{g\to\infty} g|\theta_j| = \infty\},$$

$$W_2 = \{(1,2) : \lim_{g\to\infty} g|\theta_1-\theta_2| < \infty\}, \quad W_2^c = \{(1,2) : \lim_{g\to\infty} g|\theta_1-\theta_2| = \infty\},$$

$$W_{-2} = \{(1,2) : \lim_{g\to\infty} g|\theta_1+\theta_2| < \infty\}, \quad W_{-2}^c = \{(1,2) : \lim_{g\to\infty} g|\theta_1+\theta_2| = \infty\}.$$

---

[4] Cauchy's residue theorem says that if γ is a simple closed, positively oriented contour in the complex plane and f is analytic excepts for some points $z_1, \ldots, z_n$ inside γ, then $\oint_{\gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z)$.

We call the elements of the sets W_1 and W_1^c as finite and infinite "single shift" respectively. We also call the elements of the sets $W_{\epsilon 2}$ and $W_{\epsilon 2}^c$, $\epsilon \in \{1,-1\}$ as finite and infinite "pair shift" respectively[5].

Estimation of finite "single shift" and "pair shift"

Cauchy's residue theorem allows us to write

$$\operatorname*{Res}_{u=\frac{1}{q}} D(u) + \sum_{\substack{j\in W_1\\ \epsilon_j\in\{\pm 1\}}} \operatorname*{Res}_{u=\frac{1}{qe^{2i\epsilon_j\theta_j}}} D(u) + \sum_{\substack{(1,2)\in W_{\epsilon 2}\\ \epsilon,\epsilon_j\in\{\pm 1\}}} \operatorname*{Res}_{u=\frac{1}{qe^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}}} D(u) = \int_{\Gamma} D(u)\,du,$$

where Γ is a circle centered at $\frac{1}{q}$ of radius $\frac{\widetilde{c}}{g}$ and $\widetilde{c}$ is defined by (3.4.5). We apply the definitions of the sets W_1, W_1^c and $W_{\epsilon 2}$, $W_{\epsilon 2}^c$ to write

$$\begin{aligned} D(u) &= \left(\frac{1}{1-qu}\right)^{k_1^2+k_2^2} \prod_{\substack{j\in W_1\\ \epsilon_j\in\{\pm 1\}}} \left(\frac{1}{1-que^{2i\epsilon_j\theta_j}}\right)^{\frac{k_j(k_j+1)}{2}} \prod_{\substack{(1,2)\in W_{\epsilon 2}\\ \epsilon_j\in\{\pm 1\}}} \left(\frac{1}{1-que^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}}\right)^{k_1k_2} \\ &\quad\times \prod_{\substack{j\in W_1^c\\ \epsilon_j\in\{\pm 1\}}} \mathcal{Z}^{\frac{k_j(k_j+1)}{2}}\left(ue^{2i\epsilon_j\theta_j}\right) \prod_{\substack{(1,2)\in W_{\epsilon 2}^c\\ \epsilon_j\in\{\pm 1\}}} \mathcal{Z}^{k_1k_2}\left(ue^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}\right) C(u) \frac{(qu)^{-(k_1+k_2)X}}{u(1-qu)} \\ &= \left(\frac{1}{1-qu}\right)^{k_1^2+k_2^2+1} \prod_{\substack{j\in W_1\\ \epsilon_j\in\{\pm 1\}}} \left(\frac{1}{1-que^{2i\epsilon_j\theta_j}}\right)^{\frac{k_j(k_j+1)}{2}} \prod_{\substack{(1,2)\in W_{\epsilon 2}\\ \epsilon_j\in\{\pm 1\}}} \left(\frac{1}{1-que^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}}\right)^{k_1k_2} \\ &\qquad \times (qu)^{-(k_1+k_2)X}\, \widetilde{E}(u), \end{aligned}$$

where

$$\widetilde{E}(u) = \prod_{\substack{j\in W_1^c\\ \epsilon_j\in\{\pm 1\}}} \mathcal{Z}^{\frac{k_j(k_j+1)}{2}}\left(ue^{2i\epsilon_j\theta_j}\right) \prod_{\substack{(1,2)\in W_{\epsilon 2}^c\\ \epsilon,\epsilon_1,\epsilon_2\in\{\pm 1\}}} \mathcal{Z}^{k_1k_2}\left(ue^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}\right) \frac{C(u)}{u}.$$

Note that $\widetilde{E}(u)$ is analytic on and inside the circle Γ and it's radius of convergence is $\gg \frac{1}{g}$. Therefore for $|u-\frac{1}{q}| = O(\frac{1}{g})$,

$$\widetilde{E}(u) = \sum_{n=0}^{\infty} e_n\,(1-qu)^n\,.$$

---

[5] Note that for two dimensional correlations only one of the sets $W_{\epsilon 2}, W_{\epsilon 2}^c$, $\epsilon \in \{1,-1\}$ contains the "pair shift" $(1,2)$ but for higher dimensional correlations either of the sets $W_{\epsilon 2}, W_{\epsilon 2}^c$ may contain more than one "pair shift" which are of the form (j_1, j_2).

Next we evaluate the integral $\int_\Gamma D(u)\,du$ which is equal to

$$\int_\Gamma \frac{1}{(1-qu)^{V+1}}\left(1+\sum_{n=1}^{\infty}\frac{b_n}{(1-qu)^n}\right)\widetilde{E}(u)\,(qu)^{-(k_1+k_2)X}\,du, \tag{3.4.6}$$

where

$$V = k_1^2+k_2^2+\sum_{j\in W_1}k_j(k_j+1)+\sum_{\substack{(1,2)\in W_{\epsilon 2}\\ \epsilon\in\{\pm1\}}}2k_1k_2 \quad \text{and}$$

$$1+\sum_{n=1}^{\infty}\frac{b_n}{(1-qu)^n}=\prod_{\substack{j\in W_1\\ \epsilon_j\in\{\pm1\}}}\left(1+\sum_{n=1}^{\infty}(-1)^n\binom{\frac{k_j(k_j+1)}{2}+n}{\frac{k_j(k_j+1)}{2}}\frac{(e^{-2i\epsilon_j\theta_j}-1)^n}{(1-qu)^n}\right)$$
$$\times\prod_{\substack{(1,2)\in W_{\epsilon 2}\\ \epsilon,\epsilon_1,\epsilon_2\in\{\pm1\}}}\left(1+\sum_{m=1}^{\infty}(-1)^m\binom{k_1k_2+m}{k_1k_2}\frac{(e^{-i(\epsilon_1\theta_1+\epsilon_2\theta_2)}-1)^m}{(1-qu)^m}\right).$$

For $n\geq 0$, we deduce that

$$\int_\Gamma\frac{1}{(1-qu)^{V+1}}\frac{b_n}{(1-qu)^n}\widetilde{E}(u)(qu)^{-(k_1+k_2)X}\,du$$
$$=e_0b_n\frac{F_{V+n}((k_1+k_2)X)}{(V+n)!}+\sum_{l=1}^{V+n}e_lb_n\frac{F_{V+n-l}((k_1+k_2)X)}{(V+n-l)!},$$

where $F_n(x)=x(x+1)(x+2)\dots(x+n-1)$, for $n\geq 1$ and $F_0(x)=1$.

From the choice of $X=\frac{g}{2(k_1+k_2)}$, the right hand side of the above equation becomes

$$\frac{e_0\,b_n}{(V+n)!}\left(\frac{g}{2}\right)^{V+n}+\sum_{l=1}^{V+n}\frac{e_l\,b_n\,d_{V+n-l}}{(V+n-l)!}\left(\frac{g}{2}\right)^{V+n-l}, \tag{3.4.7}$$

where

$$d_l=1+\frac{l!}{e_{V+n-l}}\left(\frac{s_{l-1}^{(l)}e_{V+n-(l+1)}}{(l+1)!}+\frac{s_{l-1}^{(l+1)}e_{V+n-(l+2)}}{(l+2)!}+\dots+\frac{s_{l-1}^{(V+n-2)}e_1}{(V+n-1)!}+\frac{s_{l-1}^{(V+n-1)}e_0}{(V+n)!}\right)$$

with

$$s_{k-i}^{(k)}=\sum_{1\leq l_1<\dots<l_i\leq k}l_1\dots l_i\,,\quad i=1,2,\dots,k.$$

The coefficients $s_{k-i}^{(k)}$ are called the Stirling numbers of first kind and $s_{k-i}^{(k)}\leq(k+1)!$ (see [33], equation (6.9)). For more details about d_l see Appendix 3.7.2. Therefore, the integral (3.4.6) is equal to

$$\begin{aligned}\frac{e_0 g^V}{2^V}\sum_{n=0}^{\infty}\frac{b_n\, g^n}{2^n(V+n)!}+\sum_{l=1}^{V}\frac{e_l\, g^{V-l}}{2^{V-l}}\sum_{n=0}^{\infty}\frac{b_n\, d_{V+n-l}\, g^n}{2^n(V+n-l)!}&\\ +\sum_{l=1}^{\infty}e_{V+l}\sum_{n=0}^{\infty}\frac{b_{n+l}\, d_n\, g^n}{2^n n!}&.\end{aligned}\tag{3.4.8}$$

We claim that the main contribution comes from only the first term of the above expression. To prove this, we have to find an upper bound for the coefficients b_n, e_l and d_n.

Let us denote

$$M:=\max_{\substack{j\in W_1\\(1,2)\in W_{\epsilon 2}\\ \epsilon\in\{\pm 1\}}}\left\{\frac{k_j(k_j+1)}{2},k_1k_2\right\},\quad \beta:=\max_{\substack{j\in W_1\\(1,2)\in W_{\epsilon 2}\\ \epsilon\in\{\pm 1\}}}\left\{|1-e^{2i\theta_j}|,|1-e^{i(\theta_1\pm\theta_2)}|\right\},$$

and $2w:=\max_j\{\,|W_j|\,\}$. We can write b_n as

$$\begin{aligned}b_n=(-1)^n\sum_{\substack{\sum n_j+m_{12}=n\\ n_j,m_{12}\geq 0\\ j\in W_1,(1,2)\in W_{\epsilon 2}\\ \epsilon\in\{\pm1\}}}\prod_{\substack{j\in W_1\\ \epsilon_j\in\{\pm1\}}}&\binom{\frac{k_j(k_j+1)}{2}+n_j}{\frac{k_j(k_j+1)}{2}}(e^{-2i\epsilon_j\theta_j}-1)^{n_j}\\ &\times\prod_{\substack{(1,2)\in W_{\epsilon 2}\\ \epsilon_j\in\{\pm1\}}}\binom{k_1k_2+m_{12}}{k_1k_2}(e^{-i(\epsilon_1\theta_1+\epsilon_2\theta_2)}-1)^{m_{12}}.\end{aligned}$$

Note that the number of terms such that $\sum n_j+m_{12}=n$ with $n_j,m_{12}\geq 0$ and $j\in W_1$, $(1,2)\in W_{\epsilon 2}$ is $\binom{w+n-1}{w-1}$. Therefore, for large g and $n\geq 1$, we obtain

$$|b_n|\leq\binom{w+n-1}{w-1}\binom{M+n}{M}^w\beta^n\leq a_0 n^t\beta^n,\tag{3.4.9}$$

where a_0, t are constants depend on w and M.

Let r be the radius of convergence of $\widetilde{E}(u)$. Note that $\frac{1}{g}=o(r)$. Hence

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n}=\frac{1}{r}=o(g),$$

and this gives

$$|e_n| \leq e_0\, a_1 \left(\frac{2}{r}\right)^n . \tag{3.4.10}$$

where $a_1 \in \mathbb{R}$ depends on $\widetilde{E}$.

Note that,

$$\frac{e_l d_{V+n-l}}{(V+n-l)!} = \frac{e_l}{(V+n-l)!} + \left(\frac{s^{(V+n-l)}_{V+n-l-1} e_{l-1}}{(V+n-l+1)!} + \frac{s^{(V+n-l+1)}_{V+n-l-1} e_{l-2}}{(V+n-l+2)!} + \ldots + \frac{s^{(V+n-2)}_{V+n-l-1} e_1}{(V+n-1)!} + \frac{s^{(V+n-1)}_{V+n-l-1} e_0}{(V+n)!} \right),$$

which implies together with (3.4.10),

$$\frac{e_l d_{V+n-l}}{(V+n-l)!} \leq e_l + e_{l-1} + \ldots + e_1 + e_0 \leq e_0 a_1 \sum_{k=1}^{l} \left(\frac{2}{r}\right)^k \ll l \left(\frac{2}{r}\right)^l .$$

By using the above bounds, the fact that $b_0 = 1$ and $\frac{1}{r} = o(g)$, the second sum of (3.4.8) is bounded by

$$\begin{aligned} &\ll_{\boldsymbol{k}} \sum_{l=1}^{V} l \left(\frac{2}{r}\right)^l \left(\frac{g}{2}\right)^{V-l} \left(1 + \sum_{n=1}^{\infty} \frac{n^t (g\beta)^n}{2^n}\right) \\ &= o\left(g^V \prod_{j \in W_1^c} \frac{1}{|\theta_j|^{k_j(k_j+1)}} \prod_{\substack{(1,2)\in W^c_{\epsilon 2} \\ \epsilon \in \{\pm 1\}}} \frac{1}{|\theta_1 + \epsilon\theta_2|^{2k_1k_2}} \right). \end{aligned}$$

Since $|g\beta| < \widetilde{c} < 1$ as $g \to \infty$, the inside n-sum in the above expression is $O(1)$. For any $l \geq 1$, using (3.4.9), we get

$$\sum_{n=0}^{\infty} \frac{b_{n+l} d_n g^n}{2^n n!} \ll_{\boldsymbol{k}} l^t \beta^l \sum_{n=0}^{\infty} \frac{(n+l)^t}{l^t} \frac{(g\beta)^n}{2^n} \ll_{\boldsymbol{k}} l^t \beta^l \sum_{n=0}^{\infty} \frac{(n+1)^t (g\beta)^n}{2^n} \ll_{\boldsymbol{k}} l^t \beta^l .$$

From the fact $\beta/r = o(1)$ and (3.4.10), the third sum of the equation (3.4.8) is bounded above by

$$\ll_{\boldsymbol{k}} \frac{1}{r^V} \sum_{l=1}^{\infty} \left(\frac{2\beta}{r}\right)^l l^t$$

$$= o\left(g^V \prod_{j\in W_1^c} \frac{1}{|\theta_j|^{k_j(k_j+1)}} \prod_{\substack{(1,2)\in W_{\epsilon 2}^c\\ \epsilon\in\{\pm 1\}}} \frac{1}{|\theta_1+\epsilon\theta_2|^{2k_1k_2}} \right).$$

Finally, we consider the first sum of the equation (3.4.8). Using the bound of b_n (see (3.4.9)), we note that

$$1 \ll \sum_{n=0}^{\infty} \frac{b_n g^n}{2^n (V+n)!} \ll \frac{1}{V!} + \sum_{n=1}^{\infty} \frac{n^t (\beta\, g)^n}{2^n (V+n)!} = O(1)$$

where the implied constant depends on $\widetilde{c}$ and $\boldsymbol{k}$ ($i.e.$, V). Therefore, we conclude that

$$\int_{\Gamma} D(u)\, du \sim_{\boldsymbol{k},\widetilde{c}} g^V \prod_{j\in W_1^c} \frac{1}{|\theta_j|^{k_j(k_j+1)}} \prod_{\substack{(1,2)\in W_{\epsilon 2}^c\\ \epsilon\in\{\pm 1\}}} \frac{1}{|\theta_1+\epsilon\theta_2|^{2k_1k_2}}$$

$$\sim_{\boldsymbol{k},\widetilde{c}} g^{k_1^2+k_2^2} \prod_{j=1}^{2} \left(\min\left\{\frac{1}{|2\theta_j|}, g\right\}\right)^{k_j(k_j+1)} \left(\min\left\{\frac{1}{|\theta_1-\theta_2|}, g\right\}\right)^{2k_1k_2} \left(\min\left\{\frac{1}{|\theta_1+\theta_2|}, g\right\}\right)^{2k_1k_2},$$

as required.

Evaluation of infinite "single shift" and "pair shift"

We claim that

$$\sum_{\substack{j\in W_1\\ \epsilon_j\in\{\pm 1\}}} \operatorname*{Res}_{u=\frac{1}{qe^{2i\epsilon_j\theta_j}}} D(u) + \sum_{\substack{(1,2)\in W_{\epsilon 2}\\ \epsilon,\epsilon_j\in\{\pm 1\}}} \operatorname*{Res}_{u=\frac{1}{qe^{i(\epsilon_1\theta_1+\epsilon_2\theta_2)}}} D(u) \tag{3.4.11}$$

$$= o\Bigg(g^{k_1^2+k_2^2} \prod_{j=1}^{2} \left(\min\left\{\frac{1}{|2\theta_j|}, g\right\}\right)^{k_j(k_j+1)} \left(\min\left\{\frac{1}{|\theta_1-\theta_2|}, g\right\}\right)^{2k_1k_2} \times \left(\min\left\{\frac{1}{|\theta_1+\theta_2|}, g\right\}\right)^{2k_1k_2} \Bigg).$$

For the sake of simplicity, we will provide all the details of the proof of the claim (3.4.11) in the Appendix section.

3.4.2 Estimation of the sum S_1

We define

$$\widetilde{\mathcal{L}}(u,\chi_D) := \sum_{f\in\mathcal{M}} a_f \chi_D(f) \Big(\frac{u}{\sqrt{q}}\Big)^{d(f)},$$

where a_f is defined by (3.4.2). We begin with the integral

$$I = \frac{1}{2\pi i} \oint\limits_{|u|=r} \widetilde{\mathcal{L}}(u, \chi_D) \frac{u^{-(k_1+k_2)X}}{(1-u)} \frac{du}{u}, \quad r < 1.$$

Integrating term by term we get

$$I = \sum_{f \in \mathcal{M}_{\leq (k_1+k_2)X}} a_f \frac{\chi_D(f)}{|f|^{1/2}}.$$

On the other hand we move the contour of integration to $|u| = q^y$, encountering a simple pole at $u = 1$, $y > \frac{1}{2}$. In doing so, we obtain

$$I = \widetilde{\mathcal{L}}(1, \chi_D) + \frac{1}{2\pi i} \oint\limits_{|u|=q^y} \widetilde{\mathcal{L}}(u, \chi_D) \frac{u^{-(k_1+k_2)X}}{(1-u)} \frac{du}{u}.$$

We use the Lindelöf bound $\widetilde{\mathcal{L}}(u, \chi_D) \ll q^{\varepsilon n}$ [[5], Theorem 3.3][6] to obtain

$$\frac{1}{2\pi i} \oint\limits_{|u|=q^y} \widetilde{\mathcal{L}}(u, \chi_D) \frac{u^{-(k_1+k_2)X}}{(1-u)} \frac{du}{u} \ll \frac{q^{\varepsilon n}}{q^{y((k_1+k_2)X+1)}}. \tag{3.4.12}$$

It follows that

$$\widetilde{\mathcal{L}}(1, \chi_D) = \sum_{f \in \mathcal{M}_{\leq (k_1+k_2)X}} a_f \frac{\chi_D(f)}{|f|^{1/2}} + O_\varepsilon\left(\frac{q^{\varepsilon n}}{q^{y((k_1+k_2)X+1)}} \right). \tag{3.4.13}$$

From the approximation (3.4.13),

$$\begin{aligned} S_1 &\gg \left| \sum_{D \in \mathcal{H}_n} \mathcal{L}\Big(\frac{v_1}{q^{1/2+\alpha_1}}, \chi_D\Big)^{k_1} \mathcal{L}\Big(\frac{v_2}{q^{1/2+\alpha_2}}, \chi_D\Big)^{k_2} \overline{\mathcal{L}_{\leq X}(\boldsymbol{v}, \chi_D)} \right| \\ &= \left| \sum_{f \in \mathcal{M}_{\leq (k_1+k_2)X}} \sum_{f' \in \mathcal{M}_{\leq (k_1+k_2)X}} \frac{a_f \, \overline{a_{f'}}}{|ff'|^{1/2}} \sum_{D \in \mathcal{H}_n} \chi_D(ff') \right| + O_\varepsilon\left(q^{n\varepsilon} |\mathcal{H}_n| \frac{q^{(1/2+\varepsilon)(k_1+k_2)X}}{q^{((k_1+k_2)X+1)y}} \right) \\ &= |S_2| + O_\varepsilon\left(q^{n\varepsilon} \, |\mathcal{H}_n| \frac{q^{(1/2+\varepsilon)(k_1+k_2)X}}{q^{((k_1+k_2)X+1)y}} \right). \end{aligned}$$

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[6] One can use the Theorem 3.1.13 to get the better bound for the integral (3.4.12), but for our case Lindelöf bound is enough.

We choose $X = \frac{g}{2(k_1+k_2)}$ and $y = \frac{2}{3}$. Hence the estimate of S_2 gives us

$$S_1 \gg_{\boldsymbol{k}} |\mathcal{H}_n| \, g^{k_1^2+k_2^2} \prod_{j=1}^{2} \left(\min\left\{ \frac{1}{|2\theta_j|}, g \right\} \right)^{k_j(k_j+1)} \left(\min\left\{ \frac{1}{|\theta_1-\theta_2|}, g \right\} \right)^{2k_1k_2}$$
$$\times \left(\min\left\{ \frac{1}{|\theta_1+\theta_2|}, g \right\} \right)^{2k_1k_2} + O\left(|\mathcal{H}_n| q^{-\frac{g}{12}+\varepsilon n} \right).$$

Inserting the estimates of S_1 and S_2 in (3.4.3) finishes the proof of Theorem 3.1.11.

3.5 Proof of Theorem 3.1.13

To keep things simple we use the notation $\boldsymbol{v}$ instead of $\boldsymbol{v}^{(m)}$. Let $\boldsymbol{k}^{(m)} \in \mathbb{R}_+^m$. The proof of the Theorem 3.1.13 will rely on getting an upper bound of the cardinality

$$\Upsilon_n(\boldsymbol{v}, V) = \# \left\{ D \in \mathcal{H}_n : \sum_{j=1}^{m} 2k_j \log \left| \mathcal{L}\Big(\frac{v_j}{q^{\frac{1}{2}+\alpha_j}}, \chi_D \Big) \right| \geq \mu(\boldsymbol{v}, g) + V \right\},$$

for sufficiently large n and for all $V > 2$, where $\mu(\boldsymbol{v}, g)$ is defined by (3.1.8). Recall that

$$2g = n - 1 - \lambda,$$

where g and λ are defined by (3.2.3) and (3.2.2) respectively. We can write

$$\sum_{D \in \mathcal{H}_n} \left| \mathcal{L}\Big(\frac{v_1}{q^{\frac{1}{2}+\alpha_1}}, \chi_D \Big) \right|^{2k_1} \cdots \left| \mathcal{L}\Big(\frac{v_m}{q^{\frac{1}{2}+\alpha_m}}, \chi_D \Big) \right|^{2k_m} = \int_{-\infty}^{\infty} \Upsilon_n(\boldsymbol{v}, V) \exp\big(\mu(\boldsymbol{v}, g) + V \big) dV. \tag{3.5.1}$$

We will estimate an upper bound of $\Upsilon_n(\boldsymbol{v}, V)$ for different ranges of V. The Lemma 3.3.4 leads to

$$\sum_{j=1}^{m} 2k_j \log \left| \mathcal{L}\Big(\frac{v_j}{q^{\frac{1}{2}+\alpha_j}}, \chi_D \Big) \right| \leq \frac{4g}{N+1} \sum_{j=1}^{m} k_j \log \left(\frac{1+q^{-\alpha_j(N+1)}}{1+q^{-2(N+1)}} \right)$$
$$+ 2\Re \sum_{d(f) \leq N} \sum_{j=1}^{m} k_j \frac{a_{\alpha_j}\left(d(f)\right) \chi_D(f) \Lambda(f) {v_j}^{d(f)}}{|f|^{\frac{1}{2}}} + O\left(\sum_{j=1}^{m} k_j \right)$$
$$\leq \frac{4gK}{N+1} \log 2 + 2\Re \sum_{d(f) \leq N} \sum_{j=1}^{m} k_j \frac{a_{\alpha_j}\left(d(f)\right) \chi_D(f) \Lambda(f) {v_j}^{d(f)}}{|f|^{\frac{1}{2}}} + O(K),$$

where

$$K=\sum_{j=1}^{m}k_j \quad \text{and} \quad a_{\alpha_j}\left(d(f)\right)=\frac{1}{d(f)|f|^{\alpha_j}}-\frac{1}{d(f)|f|^2}+O\left(\frac{1}{(N+1)q^{(N+1)\alpha_j}}\right).$$

Applying the prime polynomial theorem, the contribution from square polynomials $f=P^2$ to the second term of the right hand side of the above inequality is

$$\begin{aligned}&2\Re\sum_{d(P)\le\frac{N}{2}}\sum_{j=1}^{m}k_j\frac{a_{\alpha_j}\left(2d(P)\right)\chi_D(P)d(P){v_j}^{2d(P)}}{|P|}+O(\log\log n)\\&\le\mu(\boldsymbol{v},g)+\frac{2gK}{N+1}+O(\log\log n),\end{aligned}$$

where the error term $O(\log\log n)$ comes from the sum over P such that $P|D$. Also it is easy to verify that the contribution from $f=P^r$ with $r\ge3$ is $O(1)$. Therefore, we deduce that

$$\sum_{j=1}^{m}2k_j\log\left|\mathcal{L}\Big(\frac{v_j}{q^{\frac{1}{2}+\alpha_j}},\chi_D\Big)\right|\le S_1(D)+S_2(D)+\mu(\boldsymbol{v},g)+\frac{5gK}{N+1}+O(\log\log n),$$

where

$$\begin{aligned}S_1(D)&=2\sum_{d(P)\le N_0}\frac{\chi_D(P)}{|P|^{1/2}}\sum_{j=1}^{m}k_ja_{\alpha_j}\left(d(P)\right)d(P)\cos(\theta_jd(P)),\\S_2(D)&=2\sum_{N_0<d(P)\le N}\frac{\chi_D(P)}{|P|^{1/2}}\sum_{j=1}^{m}k_ja_{\alpha_j}\left(d(P)\right)d(P)\cos(\theta_jd(P)).\end{aligned}$$

We rewrite $\sigma(\boldsymbol{v},g)$ as

$$\sigma(\boldsymbol{v},g)=2\left(\sum_{j=1}^{m}k_j^2\right)\log g+2\sum_{j=1}^{m}k_j^2F_j+4\sum_{i<j}k_ik_jF_{i,j},$$

where

$$F_j=\log\left(\min\left\{\frac{1}{2|\theta_j|},g\right\}\right) \quad \text{and} \quad F_{i,j}=\log\left(\min\left\{\tfrac{1}{|\theta_i-\theta_j|},g\right\}\right)+\log\left(\min\left\{\tfrac{1}{|\theta_i+\theta_j|},g\right\}\right).$$

From now onward, for the sake of simplicity we write $\sigma(\boldsymbol{v},g)$ simply as $\boldsymbol{\sigma}$. We consider various different ranges of V. The ranges $-\infty < V \leq \sqrt{\log g}$ yields

$$\int_{-\infty}^{\infty} \Upsilon_n(\boldsymbol{v},V)\exp\big(\mu(\boldsymbol{v},g)+V\big)dV \ll |\mathcal{H}_n|\exp\big(\sqrt{\log g}+\mu(\boldsymbol{v},g)\big) \ll |\mathcal{H}_n| g^{o(1)}\exp\left(\mu(\boldsymbol{v},g)\right).$$

Applying Lemma 3.3.1, it is enough to assume that $\sqrt{\log g} \leq V \leq \frac{Kg}{\log_q g}$. We define the quantity A by

$$A = \begin{cases} \frac{\log \boldsymbol{\sigma}}{2}, & \text{if} \sqrt{\log g} \leq V \leq \boldsymbol{\sigma}, \\ \frac{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}{2V}, & \text{if} \, \boldsymbol{\sigma} \leq V \leq \frac{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}{25K}, \\ 7K, & \text{if} \, V > \frac{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}{25K}. \end{cases}$$

Let us consider

$$\frac{g}{N+1} = \frac{V}{A} \quad \text{and} \quad N_0 = \frac{N}{\log_q g}.$$

Notice that, if $D \in \Upsilon_n(\boldsymbol{v},V)$ then we must have either

$$S_1(D) \geq V(1-\frac{6K}{A}) := V_1 \quad \text{or} \quad S_2(D) \geq \frac{KV}{A} := V_2.$$

To determine an upper bound of $\Upsilon_n(\boldsymbol{v},V)$, we will actually examine the set

$$\Upsilon_n(\boldsymbol{v},V_i) = \#\left\{D \in \mathcal{H}_n : S_i(D) \geq V_i\right\},$$

for $i=1,2$. We set $a_j(P) := a_{\alpha_j}\left(d(P)\right)d(P)\cos(\theta_j d(P))$. So

$$a_j(P) = \frac{\cos(\theta_j d(P))}{|P|^{\alpha_j}} - \frac{\cos(\theta_j d(P))}{|P|^2} + O\left(\frac{d(P)}{(N+1)q^{(N+1)\alpha_j}}\right) \ll 1.$$

Using Lemma 3.3.6, we obtain

$$\begin{aligned}\sum_{D\in\mathcal{H}_n} |S_2(D)|^{2l} &\ll |\mathcal{H}_n|\frac{(2l)!}{l!\,2^l}\left(\sum_{N_0<d(P)\leq N}\frac{(\sum_{j=1}^m 2k_j)^2}{|P|}\right)^l \\ &\ll |\mathcal{H}_n|\frac{(2l)!}{l!2^l}\left(4K^2\left(\log\log_q g+O(1)\right)\right)^l,\end{aligned}$$

for any l such that $2lN \leq n$, which implies that $l \leq \frac{g}{N}+\frac{1}{2N} \leq \frac{2V}{A}$.

Therefore, by using Markov's inequality and Stirling's formula, it follows that

$$\begin{aligned}\Upsilon_n(\boldsymbol{v}, V_2) &\le {V_2}^{-2l}\Big(\sum_{D\in\mathcal{H}_n}|S_2(D)|^{2l}\Big)\\ &\ll |\mathcal{H}_n|\Big(\frac{A}{KV}\Big)^{2l}\frac{(2l)!}{l!2^l}\left(4K^2\left(\log\log_q g+O(1)\right)\right)^l\\ &\ll |\mathcal{H}_n|\exp(-\frac{V}{2A}\log V).\end{aligned}$$

Again applying Lemma 3.3.6 and Stirling's formula, we get

$$\begin{aligned}\sum_{D\in\mathcal{H}_n}\left|S_1(D)\right|^{2l} \ll&|\mathcal{H}_n|\frac{(2l)!}{l!2^l}\left(\sum_{d(P)\le N_0}\frac{1}{|P|}\left(\sum_{j=1}^{m}\frac{2k_j\cos(\theta_j d(P))}{|P|^{\alpha_j}}\right)^2\right)^l\\ \ll&|\mathcal{H}_n|\frac{(2l)!}{l!2^l}\left(\sum_{d(P)\le N_0}4\left(\sum_{j=1}^{m}\frac{{k_j}^2\cos^2(\theta_j d(P))}{|P|^{1+2\alpha_j}}\right.\right.\\ &\left.\left.+2\sum_{i<j}k_ik_j\frac{\cos(\theta_i d(P))\cos(\theta_j d(P))}{|P|^{1+(\alpha_i+\alpha_j)}}\right)\right)^l\\ \ll&|\mathcal{H}_n|\left(\frac{l\boldsymbol{\sigma}}{e}\right)^l,\end{aligned}$$

for any l such that $2lN_0 \le n$, which implies that $l \le \frac{V}{A}\log_q g$. Markov's inequality gives us

$$\Upsilon_n(\boldsymbol{v}, V_1) \ll {V_1}^{-2l}\left(\sum_{D\in\mathcal{H}_n}|S_1(D)|^{2l}\right) \ll |\mathcal{H}_n|\left(\frac{l\boldsymbol{\sigma}}{e{V_1}^2}\right)^l.$$

It is now convenient to consider the case when $V \le \frac{\boldsymbol{\sigma}^2}{K^3}$ and the case $V > \frac{\boldsymbol{\sigma}^2}{K^3}$ separately.

Case 1. Assume that $V \le \frac{\boldsymbol{\sigma}^2}{K^3}$. We choose $l = \lfloor\frac{{V_1}^2}{\boldsymbol{\sigma}}\rfloor$. The definition of A and this choice of l implies that $l \le \frac{V}{A}\log_q g$. In this case, we find that

$$\Upsilon_n(\boldsymbol{v}, V_1) \ll |\mathcal{H}_n| \exp\left(l\log\left(\frac{l\boldsymbol{\sigma}}{e{V_1}^2}\right)\right) \ll |\mathcal{H}_n| \exp\left(-\frac{{V_1}^2}{\boldsymbol{\sigma}}\right).$$

Case 2. Assume that $V > \frac{\boldsymbol{\sigma}^2}{K^3}$. We choose $l = \lfloor 10V\rfloor$. Again from the definition of A, it is easy to see that this choice l satisfies $l \le \frac{V}{A}\log_q g$. Notice that $V > \frac{\boldsymbol{\sigma}^2}{K^3}$, implies $\log V > 2\log\boldsymbol{\sigma} - 3\log K$. So, we have

$$A = K \quad \text{and} \quad {V_1}^2 = 25V^2.$$

Hence, we conclude that

$$\begin{aligned}\Upsilon_n(\boldsymbol{v}, V_1) &\ll |\mathcal{H}_n| \exp\left(10V \log\left(\frac{10V\boldsymbol{\sigma}}{e{V_1}^2}\right)\right)\\ &\ll |\mathcal{H}_n| \exp\left(-4V \log V\right),\end{aligned}$$

for sufficiently large g.

Therefore combining the above estimates, we deduce that

$$\Upsilon_n(\boldsymbol{v}, V) \ll |\mathcal{H}_n| \left\{\exp\left(-\tfrac{V}{2A}\log V\right) + \exp\left(-\tfrac{{V_1}^2}{\boldsymbol{\sigma}}\right) + \exp\left(-4V\log V\right)\right\}. \tag{3.5.2}$$

We extract the value of V_1 for various ranges of V coming from the definition of A.

If $\sqrt{\log g} \leq V \leq \boldsymbol{\sigma}$, then

$$A = \frac{1}{2}\log\boldsymbol{\sigma} \quad \text{and} \quad V_1 = V\Big(1 - \frac{12K}{\log\boldsymbol{\sigma}}\Big).$$

So, for sufficiently large g, (3.5.2) implies that

$$\begin{aligned}\Upsilon_n(\boldsymbol{v}, V) &\ll |\mathcal{H}_n| \exp\left(-\tfrac{V^2}{\boldsymbol{\sigma}}\left(1 - \tfrac{12K}{\log\boldsymbol{\sigma}}\right)^2\right)\\ &\ll |\mathcal{H}_n| \exp\left(-\tfrac{V^2}{\boldsymbol{\sigma}}\left(1 - \tfrac{24K}{\log\boldsymbol{\sigma}}\right)\right).\end{aligned}$$

If $\boldsymbol{\sigma} \leq V \leq \frac{1}{25K}\boldsymbol{\sigma}\log\boldsymbol{\sigma}$, then

$$A = \frac{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}{2V} \quad \text{and} \quad V_1 = V\Big(1 - \tfrac{12KV}{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}\Big).$$

For this range of V, $\frac{\log V}{\boldsymbol{\sigma}\log\boldsymbol{\sigma}} > \frac{1}{\boldsymbol{\sigma}}$ and hence from (3.5.2) we obtain

$$\begin{aligned}\Upsilon_n(\boldsymbol{v}, V) \ll |\mathcal{H}_n| \Bigg\{ &\exp\left(-\frac{V^2\log V}{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}\right) + \exp\left(-4V\log V\right)\\ &+ \exp\left(-\tfrac{V^2}{\boldsymbol{\sigma}}\left(1 - \frac{12KV}{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}\right)^2\right)\Bigg\}\\ \ll |\mathcal{H}_n| &\exp\left(-\frac{V^2}{\boldsymbol{\sigma}}\left(1 - \frac{24KV}{\boldsymbol{\sigma}\log\boldsymbol{\sigma}}\right)\right).\end{aligned}$$

Finally, if $V > \frac{1}{25K}\boldsymbol{\sigma}\log\boldsymbol{\sigma}$, then

$$A = 7K \quad \text{and} \quad V_1 = \frac{V}{7}.$$

So from (3.5.2), we get that

$$\Upsilon_n(\boldsymbol{v}, V) \ll |\mathcal{H}_n| \exp\left(-\tfrac{V}{98K}\log V\right).$$

Adding these estimates in (3.5.2) for different ranges of V, we conclude that

$$\Upsilon_n(\boldsymbol{v}, V) \ll \begin{cases} |\mathcal{H}_n| n^{\varepsilon} \exp\left(-\frac{V^2}{\boldsymbol{\sigma}}\right) & \text{, if } 3 \leq V \leq 2021\boldsymbol{\sigma}, \\ |\mathcal{H}_n| n^{\varepsilon} \exp(-4V) & \text{, if } V > 2021\boldsymbol{\sigma}. \end{cases} \tag{3.5.3}$$

Inserting (3.5.3) in (3.5.1) finishes the proof of Theorem 3.1.13.

3.6 Proof of Theorem 3.1.14

Let $C_{1/g}$ be the circle in the complex plane whose center is origin and radius is $\frac{1}{g}$. By Cauchy's integral formula

$$\mathcal{L}^{(l)}(q^{-1/2}, \chi_D) = \frac{l!}{2\pi i} \oint_{C_{1/g}} L\left(\frac{1}{2} + \theta, \chi_D\right) \frac{d\theta}{\theta^{l+1}}.$$

Notice that if $\theta = \alpha - \frac{it}{\log q}$, then

$$L\Big(\frac{1}{2} + \theta, \chi_D\Big) = \mathcal{L}\Big(\frac{v}{q^{\alpha + 1/2}}, \chi_D\Big),$$

where $\alpha = O\left(\frac{1}{g}\right)$. Therefore, applying Hölder's inequality, we see that

$$\begin{aligned} \sum_{D \in \mathcal{H}_n} \left|\mathcal{L}^{(l)}(q^{-1/2}, \chi_D)\right|^k \leq \left(\frac{l!}{2\pi}\right)^k & \Big(\sum_{D \in \mathcal{H}_n} \oint_{C_{1/g}} \left|L\Big(\frac{1}{2} + \theta, \chi_D\Big)\right|^k |d\theta|\Big) \\ & \times \Big(\oint_{C_{1/g}} |\theta|^{-\frac{k(l+1)}{(k-1)}} |d\theta|\Big)^{(k-1)} \\ \ll \left(\frac{l!}{2\pi}\right)^k \left(\frac{2\pi}{g}\right)^{k-1} \left(\frac{g}{2\pi}\right)^{k(l+1)} & \Big(\sum_{D \in \mathcal{H}_n} \oint_{C_{1/g}} \left|L\Big(\frac{1}{2} + \theta, \chi_D\Big)\right|^k |d\theta|\Big) \\ \ll \left(\frac{l!}{2\pi}\right)^k \left(\frac{2\pi}{g}\right)^{k} \left(\frac{g}{2\pi}\right)^{k(l+1)} & \max_{|\theta| \leq \frac{1}{g}} \sum_{D \in \mathcal{H}_n} \left|L\Big(\frac{1}{2} + \theta, \chi_D\Big)\right|^k. \end{aligned}$$

As a direct application of Theorem 3.1.13, we obtain

$$\sum_{D \in \mathcal{H}_n} \left|\mathcal{L}\Big(\frac{v}{q^{\alpha + 1/2}}, \chi_D\Big)\right|^k \ll_{\varepsilon} |\mathcal{H}_n|\, g^{\frac{k(k+1)}{2} + \varepsilon}.$$

Using this upper bound to the above inequality, we conclude that

$$\sum_{D\in\mathcal{H}_n} \left|\mathcal{L}^{(l)}(q^{-1/2},\chi_D)\right|^k \ll_\varepsilon |\mathcal{H}_n|\, g^{\frac{k(k+1)}{2}+kl+\varepsilon}.$$

3.7 Appendix

3.7.1 Proof of claim (3.4.11)

We have to show that for $j \in W_1^c$,

$$\underset{u=\frac{1}{qe^{2i\theta_j}}}{Res} D(u) = \tag{3.7.1}$$

$$o\left(g^{k_1^2+k_2^2}\prod_{j=1}^{2}\left(\min\left\{\frac{1}{2|\theta_j|},g\right\}\right)^{k_j(k_j+1)}\left(\min\left\{\frac{1}{|\theta_1-\theta_2|},g\right\}\right)^{2k_1k_2}\left(\min\left\{\frac{1}{|\theta_1+\theta_2|},g\right\}\right)^{2k_1k_2}\right),$$

and for $(1,2)\in W_{2\epsilon}^c$, $\epsilon\in\{\pm1\}$[7],

$$\underset{u=\frac{1}{qe^{i(\theta_1+\epsilon\theta_2)}}}{Res} D(u) = \tag{3.7.2}$$

$$o\left(g^{k_1^2+k_2^2}\prod_{j=1}^{2}\left(\min\left\{\frac{1}{2|\theta_j|},g\right\}\right)^{k_j(k_j+1)}\left(\min\left\{\frac{1}{|\theta_1-\theta_2|},g\right\}\right)^{2k_1k_2}\left(\min\left\{\frac{1}{|\theta_1+\theta_2|},g\right\}\right)^{2k_1k_2}\right).$$

We will prove the claim (3.7.1) and the proof of the claim (3.7.2) follows in the similar way. We assume that $\epsilon,\epsilon_j\in\{1,-1\}$ for $j=1,2$. To prove the claim (3.7.1), without loss of generality, we assume that $1\in W_1^c$, so $(1,2)\in W_{-2}^c$. Note that if $2\in W_1^c$ and $(1,2)\in W_2^c$, then they are not close to each other i.e., $|\theta_1-\theta_2|\gg\frac{1}{g}$, $|\theta_1-(\theta_1-\theta_2)|=|\theta_2|\gg\frac{1}{g}$ and $|\theta_2-(\theta_1+\theta_2)|=|\theta_1|\gg\frac{1}{g}$, otherwise they will contained be in the sets W_1 and W_2 respectively. By Cauchy's theorem, we obtain

$$\underset{u=1/qe^{2i\theta_j}}{Res} D(u) = \oint_{\widetilde{C}} D(u)\,du, \quad j=1,2,$$

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[7] To estimate infinite pair shift for Theorem 3.1.13, one can follow the article of V. Chande [[16], Appendix].

where $\widetilde{C}$ is the circle centered at $u = 1/qe^{2i\theta_j}$ with radius $\frac{\widetilde{c}}{g}$ and $\widetilde{c}$ is defined by (3.4.5). Note that, $\frac{1}{g} = o(|\theta_j|)$ and $\theta_j = o(1)$ for all j. For u on the circle $\widetilde{C}$, we write

$$\mathcal{Z}(u) = (1-qu)^{-1} = (1-e^{-2i\theta_1})^{-1}\left(1 + \frac{e^{-2i\theta_1}(1-que^{2i\theta_1})}{1-e^{-2i\theta_1}}\right)^{-1}.$$

Therefore, we get

$$|\mathcal{Z}(u)| \ll \frac{1}{|\theta_1|}.$$

If $2 \in W_1$, then it is easy to see that $|\theta_1 \pm \theta_2| \sim |\theta_1|$. For u on the circle $\widetilde{C}$,

$$\mathcal{Z}(ue^{2i\epsilon_j\theta_2}) = (1-que^{2i\epsilon_j\theta_2})^{-1} = (1-e^{-2i\theta_1})^{-1}\left(1 + \frac{e^{-2i\theta_1}(1-que^{2i(\epsilon_j\theta_2-\theta_1)})}{1-e^{-2i\theta_1}}\right)^{-1},$$

which implies that

$$|\mathcal{Z}(ue^{2i\epsilon_j\theta_2})| \ll \frac{1}{|\theta_1|}.$$

Also, if $(1,2) \in W_2$, then for u on the circle $\widetilde{C}$, we see that

$$|\mathcal{Z}(ue^{2i(\epsilon_1\theta_1+\epsilon_2\theta_2)})| \ll \frac{1}{|\theta_1|}.$$

For elements in the infinite single shift and pair shift, we have to partition the sets W_1^c, $W_{\epsilon 2}^c$ into three different subsets to estimate bounds for the corresponding zeta functions. For $2 \in W_1^c$, we divide the set W_1^c into three subsets. First we define

$$W_{11}^c := \left\{2 \in W_1^c : \lim_{g\to\infty} \frac{|\theta_1|}{|\theta_2|} < +\infty \text{ and } \lim_{g\to\infty} \frac{\theta_1}{\theta_2} \neq 1\right\}.$$

If $2 \in W_{11}^c$, then for u on the circle $\widetilde{C}$,

$$|\mathcal{Z}(ue^{2i\epsilon_j\theta_2})| \ll \frac{1}{|\theta_2|}.$$

Next, we consider

$$W_{12}^c = \left\{2 \in W_1^c : \lim_{g\to\infty} \frac{|\theta_1|}{|\theta_2|} = \infty\right\}.$$

For $2 \in W_{12}^c$ and u on the circle $\widetilde{C}$, we obtain

$$|\mathcal{Z}(ue^{2i\epsilon_j\theta_2})| \ll \frac{1}{|\theta_1|}.$$

Lastly, let

$$W_{13}^c = \left\{ 2 \in W_1^c : \lim_{g\to\infty} \frac{\theta_1}{\theta_2} = 1 \right\}.$$

For $2 \in W_{13}^c$ and u on the circle $\widetilde{C}$,

$$|\mathcal{Z}(ue^{2i\epsilon_j\theta_2})| \ll \frac{1}{|\theta_1 - \theta_2|}.$$

Similarly, for $(1,2) \in W_{\epsilon 2}^c$, we define

$${}^1W_{\epsilon 2}^c = \left\{ (1,2) \in W_{\epsilon 2}^c \,:\, \lim_{g\to\infty} \frac{|\theta_1|}{|\theta_1 - \epsilon\theta_2|} \,<\, +\infty \text{ and } \lim_{g\to\infty} \frac{\theta_1}{(\theta_1 - \epsilon\theta_2)} \neq 1 \right\}.$$

In this case, for u on the circle $\widetilde{C}$,

$$|\mathcal{Z}(ue^{i(\theta_1 - \epsilon\theta_2)})| \ll \frac{1}{|\theta_1 - \theta_2|}.$$

Let

$${}^2W_{\epsilon 2}^c = \left\{ (1,2) \in W_{\epsilon 2}^c : \lim_{g\to\infty} \frac{|\theta_1|}{|\theta_1 - \epsilon\theta_2|} \;=\; +\infty \right\}.$$

Inside the set ${}^2W_{\epsilon 2}^c$, for u on the circle $\widetilde{C}$,

$$|\mathcal{Z}(ue^{i(\theta_1 - \epsilon\theta_2)})| \ll \frac{1}{|\theta_1|}.$$

Lastly, we consider

$${}^3W_{\epsilon 2}^c = \left\{ (1,2) \in W_{\epsilon 2}^c : \lim_{g\to\infty} \frac{\theta_1}{(\theta_1 - \epsilon\theta_2)} = 1 \right\}.$$

For u on the circle $\widetilde{C}$,

$$|\mathcal{Z}(ue^{i(\theta_1 - \epsilon\theta_2)})| \ll \frac{1}{|\theta_2|}.$$

Using these bounds for the zeta functions, we conclude that

$$\oint_{\widetilde{C}} D(u)\,du \;\ll\; g^{\frac{k_1(k_1+1)}{2}-1}\, |\theta_1|^{-\left(k_1^2 + k_2^2 + \frac{k_1(k_1+1)}{2}\right)} \min\left\{ \frac{1}{|\theta_1|}, \frac{1}{|\theta_2|}, \frac{1}{|\theta_1 - \theta_2|} \right\}^{k_2(k_2+1)}$$
$$\times \min\left\{ \frac{1}{|\theta_1|}, \frac{1}{|\theta_2|}, \frac{1}{|\theta_1 - \theta_2|} \right\}^{2k_1k_2} \min\left\{ \frac{1}{|\theta_1|}, \frac{1}{|\theta_2|}, \frac{1}{|\theta_1 + \theta_2|} \right\}^{2k_1k_2}.$$

Using the fact $\frac{1}{|\theta_j|} = o(g)$, one can easily check that the integral $\oint\limits_{\widetilde{C}} D(u)\,du$ is equal to

$$o\left(g^{k_1^2+k_2^2}\prod_{j=1}^{2}\left(\min\left\{\frac{1}{|2\theta_j|},g\right\}\right)^{k_j(k_j+1)}\left(\min\left\{\frac{1}{|\theta_1-\theta_2|},g\right\}\right)^{2k_1k_2}\right.$$
$$\left.\times\left(\min\left\{\frac{1}{|\theta_1+\theta_2|},g\right\}\right)^{2k_1k_2}\right),$$

and we obtain the claim (3.7.1).

3.7.2 Deduction of d_n

We start with the expression (3.4.7), i.e.,

$$e_0 b_n \frac{F_{V+n}((k_1+k_2)X)}{(V+n)!} + \sum_{l=1}^{V+n} e_l b_n \frac{F_{V+n-l}((k_1+k_2)X)}{(V+n-l)!}$$

where $F_n(x) = x(x+1)(x+2)\ldots(x+n-1)$, for $n \geq 2$ and $F_0(x) = 1$, $F_1(x) = x$. We expand $F_n(x)$ to get

$$F_n(x) = x\left(x^{n-1} + s_{n-2}^{(n-1)}x^{n-2} + s_{n-3}^{(n-1)}x^{n-3} + s_{n-4}^{(n-1)}x^{n-4} + \ldots + s_0^{(n-1)}\right)$$

with

$$s_{k-i}^{(k)} = \sum_{1\leq l_1<\ldots<l_i\leq k} l_1\ldots l_i\,, \quad i = 1,2,\ldots,k.$$

This gives us

$$\begin{aligned}
& e_0 b_n \frac{F_{V+n}(x)}{(V+n)!} + e_1 b_n \frac{F_{V+n-1}(x)}{(V+n-1)!} + e_2 b_n \frac{F_{V+n-2}(x)}{(V+n-2)!} + e_3 b_n \frac{F_{V+n-3}(x)}{(V+n-3)!} \\
&\qquad + \ldots + e_{V+n-1} b_n F_1(x) + e_{V+n} b_n F_0(x) \\
&= \frac{e_0 b_n}{(V+n)!}\left(x^{V+n} + s_{V+n-2}^{(V+n-1)}x^{V+n-1} + s_{V+n-3}^{(V+n-1)}x^{V+n-2} + \ldots + s_0^{(V+n-1)}x\right) \\
&+ \frac{e_1 b_n}{(V+n-1)!}\left(x^{V+n-1} + s_{V+n-3}^{(V+n-2)}x^{V+n-2} + s_{V+n-4}^{(V+n-2)}x^{V+n-3} + \ldots + s_0^{(V+n-2)}x\right) \\
&+ \frac{e_2 b_n}{(V+n-2)!}\left(x^{V+n-2} + s_{V+n-4}^{(V+n-3)}x^{V+n-3} + s_{V+n-5}^{(V+n-3)}x^{V+n-4} + \ldots + s_0^{(V+n-3)}x\right) \\
&\qquad + \ldots + \\
&+ \frac{e_{V+n-3} b_n}{3!}\left(x^3 + s_1^{(2)}x^2 + s_0^{(2)}x\right) + \frac{e_{V+n-2} b_n}{2!}\left(x^2 + s_0^{(1)}x\right) + e_{V+n-1} b_n x + e_{V+n} b_n.
\end{aligned}$$

$$:= \frac{e_0 b_n}{(V+n)!}x^{V+n} + \frac{e_1 b_n d_{V+n-1}}{(V+n-1)!}x^{V+n-1} + \frac{e_2 b_n d_{V+n-2}}{(V+n-2)!}x^{V+n-2} + \frac{e_3 b_n d_{V+n-3}}{(V+n-3)!}x^{V+n-3}$$
$$+ \ldots + \frac{e_{V+n-3} b_n \, d_3}{3!}x^3 + \frac{e_{V+n-2} b_n \, d_2}{2!}x^2 + e_{V+n-1} b_n \, d_1 x + e_{V+n} b_n \, d_0,$$

where $d_0 = 1$, and $1 \leq l \leq V+n-1$,

$$d_l = 1 + \frac{l!}{e_{V+n-l}} \left(\frac{s_{l-1}^{(l)} e_{V+n-l-1}}{(l+1)!} + \frac{s_{l-1}^{(l+1)} e_{V+n-l-2}}{(l+2)!} + \ldots + \frac{s_{l-1}^{(V+n-2)} e_1}{(V+n-1)!} + \frac{s_{l-1}^{(V+n-1)} e_0}{(V+n)!} \right).$$

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List of Publications

1. G. Maiti. *Simultaneous Non-vanishing of Quadratic Dirichlet L-functions and twists of Hecke L-functions*, Mathematische Zeitschrift, (2022).
2. G. Maiti, K. Mallesham. *Simultaneous non-vanishing of central values of $GL(2) \times GL(3)$ and $GL(3)$ L-functions.* https://arxiv.org/abs/2108.03985, (2021), (Submitted after minor revision).
3. G. Maiti, P. Darbar. *Correlation of shifted values of L-functions in the hyperelliptic ensemble.* Finite Fields and Their Applications, 76, (2021). ISSN 1071-5797. https://www.sciencedirect.com/science/article/pii/S1071579721001222.